Simplicial structures and normal forms for mapping class groups and braid groups

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In this paper we show that the mapping class groups of any surface with nonempty boundary form a simplicial group as the number of marked points varies. This extends the simplicial structure on braid groups of surfaces found by Berrick, Cohen, Wong and Wu. We use the simplicial maps to construct compatible normal forms for elements of the braid groups and mapping class groups of surfaces with boundary.

1 Introduction

Over the last decade, a number of results have revealed previously unsuspected relationships among mapping class groups of surfaces, their braid groups and homotopy theory. In particular, Berrick, Cohen, Wong and Wu [2] showed a linkage between the braid groups of the disc and sphere and homotopy groups of spheres. This was obtained by a study of the face maps between braid groups corresponding to the forgetting of strands in a braid. In [3; 4], the present authors, together with J Wu, showed that generalising such Δ–structures both to braid groups on all surfaces and to their mapping class groups leads to more general relations involving homotopy theory.

Any discussion of Δ–structures naturally prompts the question as to whether the face maps are part of a more comprehensive simplicial structure; that is, whether there are also degeneracy maps present. In the most familiar case, of pure braid groups on the disc, such a structure gives rise to semidirect products of the pure braid groups, and so the important Artin combing, from which one deduces solubility of the word problem in the braid groups. In this paper we explore the existence of degeneracies and normal forms.

For a surface M (always connected, but not necessarily oriented or closed) and $k \geq 0$, recall that the configuration space

$$\text{Conf}_k(M) = \{(x_0, \ldots, x_{k-1}) \in M^k \mid x_i \neq x_j \text{ when } i \neq j\},$$

consists of all ordered configurations of k points in M. Permuting the points gives a natural action of Σ_k (the symmetric group on k letters) on $\mathrm{Conf}_k(M)$; the resulting quotient space $\Sigma_k \backslash \mathrm{Conf}_k(M)$ is the space of all *unordered* configurations in M. The pure and full braid groups of M are given respectively by

$$P_k(M) = \pi_1(\mathrm{Conf}_k(M)),$$
$$B_k(M) = \pi_1(\Sigma_k \backslash \mathrm{Conf}_k(M)).$$

Thus, elements of $P_k(M)$ are homotopy classes of loops in the configuration space, based at some chosen configuration, and elements of $B_k(M)$ can be considered as homotopy classes of paths in the configuration space that start at the chosen configuration and end at some permutation of it. There is a natural map $\pi\colon B_k(M) \to \Sigma_k$ given by recording the permutation of the chosen configuration at which a path ends. Evidently $\mathrm{Ker}(\pi) = P_k(M)$.

In [2] the authors study the sequences

$$\underline{B}(M) = \{B_1(M), B_2(M), \ldots\},$$
$$\underline{P}(M) = \{P_1(M), P_2(M), \ldots\}.$$

They describe functions $d_i\colon B_{k+1}(M) \to B_k(M)$, $0 \leq i \leq k$, known as face maps, given by deleting the ith strand of a braid. These are not group homomorphisms but they satisfy the crossed identity $d_i(\beta_1 \beta_2) = d_i(\beta_1) d_{i \cdot \pi(\beta_1)}(\beta_2)$ for all $\beta_1, \beta_2 \in B_{k+1}(M)$. Here $i \cdot \pi(\beta_1)$ denotes the result of performing the permutation $\pi(\beta_1)$ on i. Since π is the identity on the pure braid group, then d_i restricts to a group homomorphism $d_i\colon P_{k+1}(M) \to P_k(M)$.

In the case that M admits a nonvanishing vector field (equivalently, M has nonempty boundary or zero Euler characteristic) there are also maps in the other direction, the so-called degeneracy maps $s_j\colon B_k(M) \to B_{k+1}(M)$. The map s_j is given by "doubling up" the jth strand of a braid; that is, we take another copy of the path in the jth component and use the vector field to translate it a small distance from the original. The nonvanishing vector field enables us to do this in a coherent fashion. Again, the functions s_j are not group homomorphisms but they do satisfy a crossed identity analogous to the one above and when we restrict to the pure braid groups, we get a group homomorphism.

The face maps and degeneracy maps just described satisfy the so-called simplicial identities. Thus, for surfaces M admitting a nonvanishing vector field, $\underline{B}(M)$ is a *crossed simplicial group* and $\underline{P}(M)$ is a *simplicial group*. Simplicial groups can be considered as combinatorial models of spaces; any simplicial group $\underline{G}$ has an associated sequence of homotopy sets $\pi_n(\underline{G})$. One of the main results of [2] was that

each $\pi_n(\underline{P}(S^2))$ is a group and there are isomorphisms $\pi_n(\underline{P}(S^2)) \cong \pi_n(S^2)$ for all $n \geq 4$. This gives a new description and a possible new method of calculation for the homotopy groups of spheres.

The braid groups $P_k(M)$ and $B_k(M)$ are closely related to (and indeed, with the exception of a few low degree cases, are subgroups of) the important mapping class groups $\Gamma^k(M)$ and $\Gamma^{(k)}(M)$. We have

$$\Gamma^{(k)}(M) = \pi_0(\mathrm{Diff}(M, [\boldsymbol{m}_k])),$$

where $\mathrm{Diff}(M, [\boldsymbol{m}_k])$ is the group of self-diffeomorphisms of M that leave invariant a set $[\boldsymbol{m}_k]$ of k marked points. If M has nonempty boundary, the diffeomorphisms are required to fix it pointwise, and if M is orientable the diffeomorphisms are assumed to be orientation-preserving. The group $\Gamma^k(M) \leq \Gamma^{(k)}(M)$ is the subgroup corresponding to those diffeomorphisms that also fix each of the marked points. These definitions also make sense when $k = 0$, in which case we denote the mapping class group by $\Gamma(M)$.

In [4] it was shown that for any surface M, the face maps on the braid groups of M extend to maps $d_i \colon \Gamma^{(k+1)}(M) \to \Gamma^{(k)}(M)$. Geometrically, the map d_i can be described as forgetting the ith marked point. Again, it is not a group homomorphism, but it satisfies a crossed identity and when we restrict to $\Gamma^{k+1}(M)$, we get a group homomorphism.

For the sequence $\underline{\Gamma}(M) = \{\Gamma^1(M), \Gamma^2(M), \ldots\}$, the existence of face homomorphisms makes it possible to define homotopy sets $\pi_n(\underline{\Gamma}(M))$. Thus, one of the main results of [4] was that the homotopy sets of $\underline{\Gamma}(M)$ are isomorphic to those of $\underline{P}(M)$, except in a few low-degree cases.

We extend the results of [2; 4] by considering degeneracy maps on mapping class groups. We show that for surfaces with nonempty boundary, it is possible to extend the degeneracy maps for braid groups to maps of mapping class groups. We do this by considering splittings

$$\Gamma^{(k)}(M) \cong \Gamma(M) \ltimes B_k(M) \quad \text{and} \quad \Gamma^k(M) \cong \Gamma(M) \ltimes P_k(M)$$

which hold for all surfaces M with nonempty boundary. Our main result is the following.

Theorem A *For a surface M with nonempty boundary, $\underline{\Gamma}^{\mathrm{sym}}(M)$ forms a crossed simplicial group and $\underline{\Gamma}(M)$ forms a simplicial group. These simplicial structures extend the known simplicial structures for braid groups.*

Li and Wu [10] used the crossed simplicial group structure on the classical braid groups (on the disc) to show that the centre of $B_k = B_k(D^2)$ modulo the boundary Brunnian braids is $\mathbb{Z}$ times the group of exponent 2 elements in $\pi_k(S^2)$. On the other hand, the centre of its associated pure braid group P_k modulo the boundary Brunnian braids is $\mathbb{Z} \times \pi_k(S^2)$, as reflected in the conjugation action of B_k and P_k. A similar situation occurs in Wu [11], where the crossed simplicial group structure is essentially used for distinguishing the actions of B_k and P_k via the Artin representation on the group whose centre is the homotopy group. Accordingly, the crossed simplicial group structure of the mapping class group might aid understanding of its conjugation action on Brunnian mapping classes.

As pointed out by the referee, there is a possible general categorical application of crossed simplicial groups. One can consider a crossed simplicial group $\{G_k\}_{k \geq 0}$ as a category $\mathcal{G}$ where the morphisms are endomorphisms of the G_k generated by group translations and face and degeneracy operations. In the case of the crossed simplicial group of finite cyclic groups, functors to an abelian category give rise to cyclic homology theory; other crossed simplicial groups lead to their own associated homology theories; see Fiedorowicz and Loday [8].

We use the simplicial structures on braid groups and mapping class groups to construct semidirect product decompositions for these groups and normal forms for their elements. In Section 3 we show that $P_{k+1}(M)$ is an iterated semidirect product of $\pi_1(M)$ and k finitely generated free groups, and this allows us to read off the following normal form for braids.

Theorem B Let M be a surface admitting a nonvanishing vector field. For $k \geq 0$, every element of $P_{k+1}(M)$ can be written in a unique way as $\alpha_0 \cdots \alpha_k$, where α_i is represented by a path in $\mathrm{Conf}_{k+1}(M) \subseteq M^{k+1}$ in which the first i points are stationary and the remaining points move in parallel (with respect to the vector field), tracing a path in M.

By considering the decomposition $\Gamma^{k+1}(M) \cong \Gamma(M) \ltimes P_{k+1}(M)$, we get the following normal form for mapping classes.

Theorem C Let M be a surface with boundary. For $k \geq 0$, every element of $\Gamma^{k+1}(M)$ can be written in a unique way as $\theta\phi_0 \cdots \phi_k$, where θ is the identity on a collar neighbourhood of a boundary component that contains the marked points and ϕ_i is given by pushing the marked points around some α_i as in Theorem B.

Alternative normal forms, analogous to Artin combing for braids over the disc, are given in Section 3. Such forms derive from coface maps rather than simplicial maps, where

coface maps are an aspect of the bi-Δ–group structure studied in Wu [12]. This suggests interesting speculations on the relationships between simplicial group structures and bi-Δ–group structures that enrich the same Δ–group structure. Developing a theme that appears to originate in Cohen [5], Wu [12, Example 1.2.8] relates results in homotopy theory concerning Hopf invariants to the theory of braids. In particular, one can use normal forms on braids to determine the Cohen braids; see Bardakov, Mikhailov, Vershinin and Wu [1].

The paper is organised as follows. In Section 2 we recall the simplicial structure on the symmetric groups and the definition of a crossed simplicial group. We review the simplicial group structure on the braid groups, as given in [2], and establish a preferred set of generators for braids. Finally, we explain how to extend the degeneracy maps to the mapping class groups, in the case of surfaces with nonempty boundary. In Section 3 we look at how these algebraic structures can be used to obtain semidirect product decompositions for braid groups and mapping class groups of surfaces and we display normal forms for elements of these groups.

2 Simplicial structures

A *simplicial set* is a sequence of sets $\underline{X} = \{X_0, X_1, \ldots\}$ together with face maps $d_i\colon X_k \to X_{k-1}$ and degeneracy maps $s_j\colon X_k \to X_{k+1}$ for each $i, j = 0, \ldots, k$. The face maps and degeneracy maps must satisfy the *simplicial identities*

(1)
$$d_j d_i = d_i d_{j+1} \qquad \text{whenever } i \leq j,$$
$$s_i s_j = s_{j+1} s_i \qquad \text{whenever } i \leq j,$$
$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{whenever } i < j, \\ \text{id} & \text{whenever } i = j \text{ or } j+1, \\ s_j d_{i-1} & \text{whenever } i > j+1. \end{cases}$$

If each X_k is a group and the face maps and degeneracy maps are group homomorphisms, then $\underline{X}$ is said to be a *simplicial group*. A sequence of sets equipped with only face maps (that is, no degeneracy maps) is called a Δ–*set*. Similarly we have the notion of Δ–*group*.

2.1 Simplicial structure on the symmetric groups

The collection of symmetric groups $\underline{\Sigma} = \{\Sigma_{k+1}\}_{k \geq 0}$ is an important example of a simplicial set. We regard Σ_k as the group of permutations of $[k] = \{0, \ldots, k-1\}$ and write $i \cdot \sigma$ for the result of evaluating σ on i. Thus $\sigma\tau$ denotes the permutation obtained by first applying σ and then τ.

The face maps $d_i \colon \Sigma_k \to \Sigma_{k-1}$ $(i = 0, 1, \ldots, k-1)$ are defined by commutative diagrams

(2)
$$
\begin{array}{ccc}
[n-1] & \xrightarrow{\ d^i\ } & [n] \\
{\scriptstyle d_i\tau}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\
[n-1] & \xrightarrow[\ d^{i\cdot\tau}\ ]{} & [n]
\end{array}
$$

where $d^i \colon [n-1] \to [n]$ is the unique order-preserving map that misses i. If we represent a permutation by a diagram of arrows as in Figure 1, then d_i removes the arrow that starts at i.
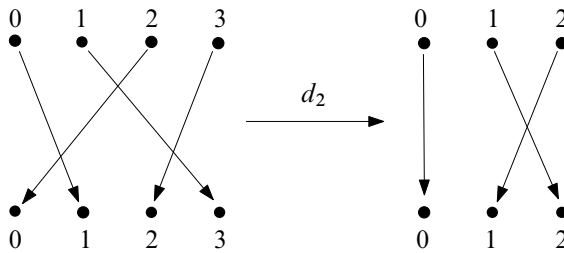

Figure 1: The face maps for the symmetric groups

On the other hand, we can define a degeneracy map $s_j \colon \Sigma_k \to \Sigma_{k+1}$ $(j = 0, 1, \ldots, k-1)$ by doubling up the arrow that starts at j; see Figure 2. This can be defined algebraically by requiring that

$$
j \cdot s_j(\sigma) = j \cdot \sigma, \quad (j+1) \cdot s_j(\sigma) = j \cdot \sigma + 1
$$

and that the following diagram commutes:

(3)
$$
\begin{array}{ccc}
[n+1] & \xrightarrow{\ s^j\ } & [n] \\
{\scriptstyle s_j\sigma}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\
[n+1] & \xrightarrow[\ s^{j\cdot\sigma}\ ]{} & [n]
\end{array}
$$

Here $s^j \colon [n+1] \to [n]$ is the unique order preserving map that hits j twice. It can easily be checked that the simplicial identities hold.
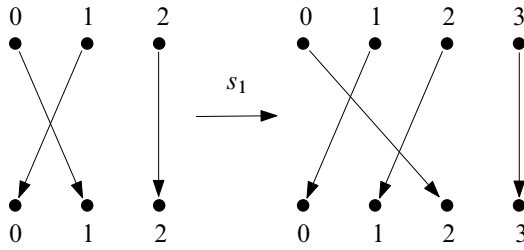
Figure 2: The degeneracy maps for the symmetric groups

2.2 Definition of a crossed simplicial group

There is an intermediate notion between simplicial set and simplicial group, that of a *crossed simplicial group*. This is a simplicial set $\underline{X} = \{X_k\}_{k \geq 0}$ for which each X_i is a group and there are homomorphisms $\pi \colon X_k \to \Sigma_{k+1}$ that commute with the face and degeneracy maps and satisfy

$$d_i(x_1 x_2) = d_i(x_1) d_{i \cdot \pi(x_1)}(x_2),$$
$$s_j(x_1 x_2) = s_j(x_1) s_{j \cdot \pi(x_1)}(x_2).$$

(Note that there are several choices to be made with regards to which side the symmetric group acts on $\{0, 1, \dots, k-1\}$ and how the face maps for symmetric groups are defined. Different choices give rise to different versions of these identities.)

The symmetric groups with π as the identity map form the prototypical crossed simplicial group. If $\underline{X}$ is any other crossed simplicial group then the homomorphisms $\pi \colon X_k \to \Sigma_{k+1}$ fit together to give a *morphism of crossed simplicial groups $\pi \colon \underline{X} \to \underline{\Sigma}$*, that is, a collection of maps respecting the crossed simplicial group structure.

Any simplicial group is a crossed simplicial group with π taken to be the trivial homomorphism. In the converse direction, a crossed simplicial group need not be a simplicial group. There is a corresponding notion of *crossed Δ–group* when we have only the face maps.

A useful fact about crossed simplicial groups is as follows.

Lemma 2.1 *If $\underline{X}$ is a crossed simplicial (resp. Δ–) group and $\pi \colon \underline{X} \to \underline{\Sigma}$ is the associated morphism, then $\operatorname{Ker} \pi \leq \underline{X}$ is a simplicial (resp. Δ–) group.* □

2.3 Simplicial structure on braid groups

Let M be any surface. Select marked points $m_0, m_1, \dots$ in M. We write $\boldsymbol{m}_k$ for the ordered set of marked points $(m_0, \dots, m_{k-1})$ and $[\boldsymbol{m}_k]$ for the corresponding unordered set of points.

To define face maps and degeneracy maps for the braid groups, we think of $B_k(M)$ as the set of path-homotopy classes $[\lambda]$ of paths λ in $\mathrm{Conf}_k(M)$ that start at the distinguished configuration $\boldsymbol{m}_k$ and end at some permutation of it. To describe the multiplication on $B_k(M)$, we use the map

$$\pi\colon B_k(M) \to \Sigma_k$$

that sends $[\lambda] \in B_k(M)$ to the permutation $\pi[\lambda]$ defined by

$$\lambda(1) = (m_{0\cdot\pi[\lambda]}, \ldots, m_{(k-1)\cdot\pi[\lambda]}).$$

For $[\lambda] \in B_k(M)$, let $(\lambda^0, \ldots, \lambda^{k-1})$ be the coordinates of λ. There is a left action of Σ_k on $\mathrm{Conf}_k(M)$ given by

$$(4) \qquad\qquad \sigma \cdot (x_0, \ldots, x_{k-1}) = (x_{0\cdot\sigma}, \ldots, x_{(k-1)\cdot\sigma})$$

and this induces a left action of Σ_k on $B_k(M)$ given by

$$\sigma \cdot [\lambda] = [\sigma \cdot \lambda],$$

where $\sigma \cdot \lambda$ denotes the path $(\lambda^{0\cdot\sigma}, \ldots, \lambda^{(k-1)\cdot\sigma})$ in $\mathrm{Conf}_k(M)$. Now the product in $B_k(M)$ is given by

$$[\lambda_1][\lambda_2] = [\lambda_1 * (\pi[\lambda_1]\cdot\lambda_2)],$$

where $*$ denotes the usual concatenation product of paths (read from left to right). With this multiplication, the map π is a group homomorphism. The kernel of π is $P_k(M) = \pi_1(\mathrm{Conf}_k(M), \boldsymbol{m}_k)$, the kth pure braid group of M.
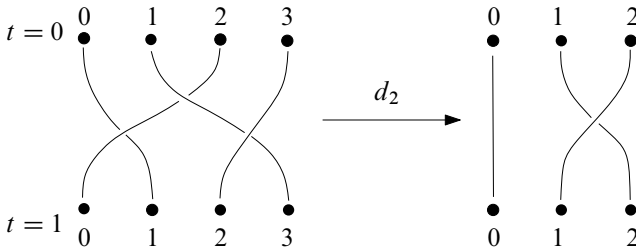

Figure 3: The face maps for the braid groups

In [2] it is shown that for any surface M, the braid groups of M form a crossed Δ–group $\underline{B}(M)$ with

$$\underline{B}(M)_k = B_{k+1}(M).$$

The homomorphism $\underline{B}(M)_k \to \Sigma_{k+1}$ is provided by the homomorphism π defined above. The face map $d_i\colon B_{k+1}(M) \to B_k(M)$ removes the ith component of a path in $\mathrm{Conf}_{k+1}(M)$. Geometrically, we think of d_i as removing the ith strand of a braid,

as in Figure 3. A slight modification to the endpoints is necessary so that the resulting path represents an element of $B_k(M)$; see [4, Section 2.3] for details.

In the case that M admits a nonvanishing vector field, there are also degeneracy maps for $\underline{B}(M)$, defined using the vector field. The idea is that the degeneracy map $s_j\colon B_k(M)\to B_{k+1}(M)$ is given by doubling up the jth string in a braid; see Figure 4.
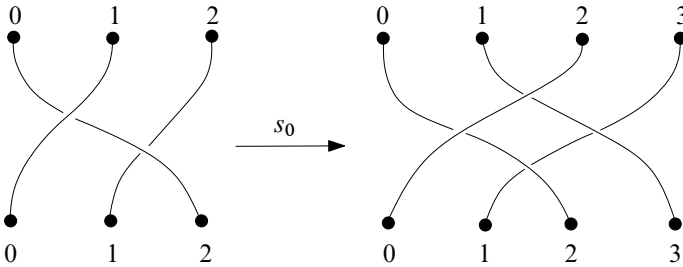

Figure 4: The degeneracy maps for the braid groups

This can be described explicitly on the configuration space level by sending a path $(\lambda^0,\dots,\lambda^{k-1})$ in $\mathrm{Conf}_k(M)$ to

$$(\lambda^0,\dots,\lambda^j,\lambda^j+\epsilon,\dots,\lambda^{k-1})$$

where $\lambda^j+\epsilon$ is obtained by sliding λ^j a small distance in the direction of the vector field. Again, a small modification regarding the basepoints is necessary to make this properly defined; for full details see [2, Section 3.2].

Now recall that the kernel of $\pi\colon\underline{B}(M)\to\underline{\Sigma}$ is $\underline{P}(M)$ where $\underline{P}(M)_k=P_{k+1}(M)$. The situation as to when $\underline{B}(M)$ and $\underline{P}(M)$ admit simplicial structures may be summarised as follows.

Theorem 2.2 *Let M be an arbitrary surface.*

  (a) *The collection $\underline{B}(M)=\{B_{k+1}(M)\}_{k\geq0}$ forms a crossed Δ–group, and*

$$\underline{P}(M)=\{P_{k+1}(M)\}_{k\geq0}$$

    forms a Δ–group.

  (b) *The following are equivalent:*
    (i) *$\underline{B}(M)$ is a crossed simplicial group.*
    (ii) *$\underline{P}(M)$ is a simplicial group.*
    (iii) *M admits a nonvanishing vector field.*

Proof Part (a) is simply [2, Proposition 4.2.1 (1), (2)]. For (b), [2, Corollary 3.2.7, Theorem 3.2.12] show that (iii) $\Rightarrow$ (i), while (i) $\Rightarrow$ (ii) by Lemma 2.1 above. It remains to show that the existence of a nonvanishing vector field is also a necessary condition for the existence of degeneracies. For surfaces other than S^2, this follows from the fact (see Gonçalves and Guaschi [9, Theorem 2]) that there is no section of $d_k\colon P_{k+1}(M) \to P_k(M)$ when $k \geq 3$ (by means of the simplicial identities in Section 2 above). For S^2, further analysis [2, Proposition 3.2.14] gives the result. □

To conclude this section, we explain how the face and degeneracy maps can be used to construct preferred generating sets for $P_k(M)$ and $B_k(M)$. We begin with a generating set $\mathcal{X}$ for $\pi_1(M)$. First, fix a basepoint $x_0 \in M$ away from all the marked points m_i, and take representatives for the elements of $\mathcal{X}$ that are loops based at x_0 that avoid all points m_i. Now, for each $i \geq 0$, let us choose a generating set $\mathcal{X}^i$ for $\pi_1(M - \boldsymbol{m}_i, m_i)$. If $i = 0$, $\boldsymbol{m}_i$ is assumed to be empty; the set $\mathcal{X}^0$ is obtained by conjugating elements of $\mathcal{X}$ by a path from m_0 to x_0. If $i \geq 1$, the set $\mathcal{X}^i$ consists of loops of the following two types. Those of the first type are given by concatenating a fixed path δ from m_i to x_0, a generator in $\mathcal{X}$ for $\pi_1(M, x_0)$ and δ^{-1}. The second type are small, simple loops, based at m_i, each of which encircles one of the marked points $m_0, \ldots, m_{i-1}$. See Figure 5.

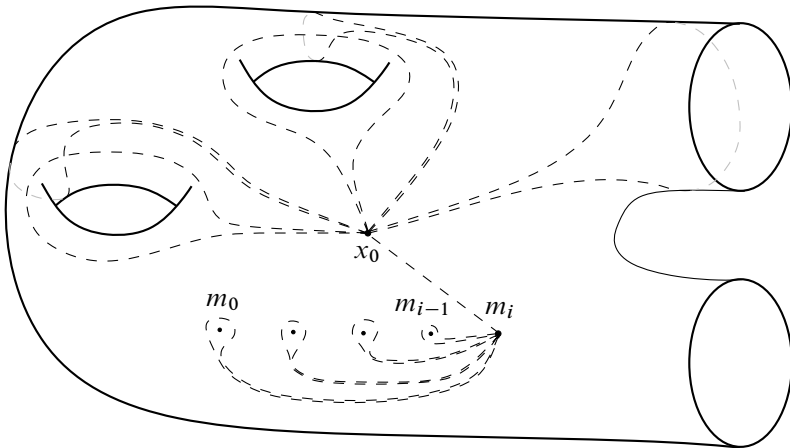Figure 5: Generating set $\mathcal{X}^i$ for $\pi_1(M - \boldsymbol{m}_i, m_i)$

Next, recall from [3, Lemma 3.11] that for each $j \geq 0$ the map

$$M - \boldsymbol{m}_j \longrightarrow \mathrm{Conf}_{j+1}(M),$$
$$x \longmapsto (m_0, \ldots, m_{j-1}, x),$$

induces the injection ι_j in the short exact sequence

$$(5) \qquad 1 \longrightarrow \pi_1(M - \boldsymbol{m}_j, m_j) \xrightarrow{\iota_j} P_{j+1}(M) \xrightarrow{d_j} P_j(M) \longrightarrow 1.$$

Let c_{m_i} denote a constant path at m_i. On the group level, ι_j sends a loop γ in $M - \boldsymbol{m}_j$ to the loop $(c_{m_0}, \ldots, c_{m_{j-1}}, \gamma)$ in the configuration space. We take $\mathrm{Conf}_0(M)$ to be a point so that $P_0(M)$ is trivial. In the case $j = 0$, ι_j is the identity map from $\pi_1(M, m_0)$ to $P_1(M)$.

For $0 \le i \le k - 2$, define

$$\mathcal{Y}_k^i = s_{k-2} \circ \cdots \circ s_i \circ \iota_i(\mathcal{X}^i) \subseteq P_k(M),$$

while $\mathcal{Y}_k^{k-1} = \iota_{k-1}(\mathcal{X}^{k-1}) \subseteq P_k(M)$. By considering the action of ι and s, we can see that every element of $\mathcal{Y}_k^i$ is a path in $\mathrm{Conf}_k(M)$ of the form

$$(c_{m_0}, \ldots, c_{m_{i-1}}, \gamma, \gamma + \epsilon, \ldots, \gamma + (k-1)\epsilon),$$

where $\gamma \in \mathcal{X}^i$. Thus, elements of $\mathcal{Y}_k^i$ are represented by paths in which the first i points remain stationary and the remaining $k - i$ points move "in parallel", tracing an element of $\mathcal{X}^i$.

Lemma 2.3 *For any surface M that admits a nonvanishing vector field, the set $\mathcal{Y}_k = \bigcup_{i=0}^{k-1} \mathcal{Y}_k^i$ is a finite generating set for $P_k(M)$.*

Proof The result certainly holds for $k = 1$ when $P_k(M) = \pi_1(M, m_0)$ and $\mathcal{Y}_k = \mathcal{Y}_1^0 = \mathcal{X}^0$. Using the exact sequence (5) with k substituted for j, we proceed by induction on k. Suppose that $k \ge 1$.

First observe that because $d_k \colon P_{k+1}(M) \to P_k(M)$ has $d_k \circ s_{k-1} = \mathrm{id}_{P_k(M)}$, we have

$$d_k(\mathcal{Y}_{k+1}^i) = \mathcal{Y}_k^i \quad \text{for each } i \in \{0, \ldots, k\}.$$

Thus, by the induction hypothesis, $d_k(\bigcup_{i=0}^{k-1} \mathcal{Y}_{k+1}^i) = \mathcal{Y}_k$ generates $P_k(M)$. Finally, since $\mathcal{Y}_{k+1}^k = \iota_k(\mathcal{X}^k)$ generates $\mathrm{Im}\, \iota_k = \mathrm{Ker}\, d_k$, the set $\mathcal{Y}_{k+1}$ does after all generate $P_{k+1}(M)$. $\qquad\square$

Let $\mathcal{S}_k \subseteq B_k(M)$ be a set of elements such that $\pi(\mathcal{S}_k)$ generates Σ_k. We may assume that the paths representing the elements of $\mathcal{S}_k$ are contained in a small neighbourhood of the marked points.

Lemma 2.4 *For any surface M that admits a nonvanishing vector field, the set $\mathcal{S}_k \cup \mathcal{Y}_k$ is a finite generating set for $B_k(M)$.*

Proof This is immediate from Lemma 2.3 since $B_k(M)$ fits into the short exact sequence

$$1 \longrightarrow P_k(M) \longrightarrow B_k(M) \xrightarrow{\pi} \Sigma_k \longrightarrow 1. \qquad \square$$

2.4 Simplicial structure on mapping class groups

We use $x \cdot \phi$ (respectively $\boldsymbol{x} \cdot \phi$) to denote the result of evaluating a diffeomorphism ϕ on the point $x \in M$ (resp. $\boldsymbol{x} \in \mathrm{Conf}_{k+1}(M)$) and $\phi \odot \psi$ to denote the diffeomorphism given by first performing ϕ and then ψ. The right action of the diffeomorphism group on the surface is thereby compatible with our other choices of actions.

In [4], it is shown that $\underline{\Gamma}^{\mathrm{sym}}(M)$ defined by

$$\underline{\Gamma}^{\mathrm{sym}}(M)_k = \Gamma^{(k+1)}(M)$$

is a crossed Δ–group and that $\underline{\Gamma}(M)$ given by

$$\underline{\Gamma}(M)_k = \Gamma^{k+1}(M)$$

is a Δ–group. The face map $d_i\colon \Gamma^{(k+1)}(M) \to \Gamma^{(k)}(M)$ is given by forgetting the ith marked point, and similarly for $d_i\colon \Gamma^{k+1}(M) \to \Gamma^k(M)$. This definition must be modified slightly to ensure that for a mapping class ϕ, both $d_i\phi$ and $d_j\phi$ fix the same set of marked points; see [4, Section 2.4] for details. The map to the symmetric group $\mu\colon \Gamma^{(k+1)}(M) \to \Sigma_{k+1}$ is defined by

$$(m_0, \ldots, m_k) \cdot \psi = (m_{0 \cdot \mu(\psi)}, \ldots, m_{k \cdot \mu(\psi)})$$

for $[\psi] \in \Gamma^{(k+1)}(M)$. Thus, μ records how the diffeomorphisms permute the marked points. Full details are given in [4, Section 2.4].

In this section we deal with surfaces with nonempty boundary; we extend the degeneracy maps for braid groups to give degeneracies for mapping class groups and thus prove that $\underline{\Gamma}^{\mathrm{sym}}(M)$ (resp. $\underline{\Gamma}(M)$) is in fact a crossed simplicial group (resp. simplicial group).

In what follows we make use of the so-called push-map $p\colon B_k(M) \to \Gamma^{(k)}(M)$; see for example Farb and Margalit [7, Section 4.2]. For a path $\lambda = (\lambda^0, \ldots, \lambda^{k-1})$ in the configuration space, define $p([\lambda]) = [\Psi_\lambda(1)]$, where Ψ_λ is the diffeotopy of M given by simultaneously pushing a small disc containing m_i around the path λ^i for $i = 0, \ldots, k-1$ (this extends the homomorphism $\partial^P\colon P_k(M) \to \Gamma^k(M)$ given in [4, Section 3]).

Lemma 2.5 *For all surfaces M with boundary and for all $k \geq 1$, there are split short exact sequences*

$$1 \longrightarrow B_k(M) \xrightarrow{\ p\ } \Gamma^{(k)}(M) \longrightarrow \Gamma(M) \longrightarrow 1,$$

$$1 \longrightarrow P_k(M) \xrightarrow{\ p\ } \Gamma^k(M) \longrightarrow \Gamma(M) \longrightarrow 1.$$

Hence $\Gamma^{(k)}(M) \cong \Gamma(M) \ltimes B_k(M)$ and $\Gamma^k(M) \cong \Gamma(M) \ltimes P_k(M)$.

Proof There is a fibration sequence

$$\mathrm{Diff}(M, [\boldsymbol{m}_k]) \hookrightarrow \mathrm{Diff}(M) \xrightarrow{\ \mathrm{ev}_{[\boldsymbol{m}_k]}\ } \Sigma_k \backslash \mathrm{Conf}_k(M).$$

Here $\mathrm{Diff}(M)$ is the group of (orientation-preserving) diffeomorphisms of M that fix the boundary pointwise and $\mathrm{ev}_{[\boldsymbol{m}_k]}$ is given by evaluating a diffeomorphism at the distinguished (unordered) configuration $[\boldsymbol{m}_k]$. The first exact sequence in the theorem is part of the long exact sequence associated to this fibration, using the fact that $\pi_1(\mathrm{Diff}(M)) = 1$ for all surfaces M with boundary; see Earle and Schatz [6, Theorem 1D]. A splitting $c\colon \Gamma(M) \to \Gamma^{(k)}(M)$ is given by gluing a cylinder with k marked points to one of the boundary components of M, extending diffeomorphisms by the identity, and then smoothing. The proof in the case of the pure groups is entirely analogous. $\qquad\square$

Remark 2.6 When M has genus greater than or equal to 2, the lemma above requires that M have boundary. For otherwise, [7, Corollary 5.11] shows that the epimorphsm

$$\Gamma^1(M) \twoheadrightarrow \Gamma(M)$$

fails to split.

Implicit in our notation for the semidirect product is the fact that the mapping class group is acting on the right of the braid groups; the action determined by the split exact sequence is

$$(6) \qquad\qquad \beta \cdot \phi = p^{-1}(c(\phi)^{-1} \odot p(\beta) \odot c(\phi))$$

for a braid β and a mapping class ϕ. Here $c\colon \Gamma(M) \to \Gamma^k(M) \hookrightarrow \Gamma^{(k)}(M)$ is the splitting map introduced in the proof of Lemma 2.5. Also, our convention for the multiplication in the semidirect product is

$$(\phi_1, \beta_1)(\phi_2, \beta_2) = (\phi_1 \odot \phi_2, (\beta_1 \cdot \phi_2)\beta_2).$$

Another way to describe the decompositions in Lemma 2.5 is to say that

$$(7) \quad \Gamma^{(k)}(M) = c(\Gamma(M)) \odot p(B_k(M)) \quad \text{and} \quad \Gamma^k(M) = c(\Gamma(M)) \odot p(P_k(M))$$

and every element of the mapping class group can be written in a unique way as an element of the product. Thus, the decomposition tells us that any mapping class in $\Gamma^{(k)}(M)$ can be written in a unique way as a product $\phi_1 \odot \phi_2$, where ϕ_1 is fixed on a collar neighbourhood of the chosen boundary component that contains the marked points and ϕ_2 is given by pushing the marked points around paths in M.

Let us now give a geometric description of the right action (6) of $\Gamma(M)$ on $B_k(M)$ that is used to form the semidirect product $\Gamma(M) \ltimes B_k(M)$. Consider the action of $\mathrm{Diff}(M)$ on the right of $\mathrm{Conf}_k(M)$ given by

$$(x_0, \ldots, x_{k-1}) \cdot \phi = (x_0 \cdot c(\phi), \ldots, x_{k-1} \cdot c(\phi)).$$

Recall that we think of $B_k(M)$ as the set of homotopy classes of paths that start at the configuration $(m_0, \ldots, m_{k-1})$ and end at some permutation of it. Because $c(\phi)$ is the identity on a neighbourhood of the boundary that contains the marked points, the action just described induces an action of $\mathrm{Diff}(M)$, and subsequently of $\Gamma(M)$, on $B_k(M)$. We denote this action of $\Gamma(M)$ on the right of $B_k(M)$ by ϕ sending β to $\beta^{c(\phi)}$. The notation is intended to remind us that $\beta^{c(\phi)}$ is the image of β under the diffeomorphism $c(\phi)$.

Lemma 2.7 *The action of $\Gamma(M)$ on $B_k(M)$ arising from the split exact sequence is the natural geometric one; that is, $p^{-1}(c(\phi)^{-1} \odot p(\beta) \odot c(\phi)) = \beta^{c(\phi)}$.*

In view of the lemma, we later denote both of these quantities by $\beta \cdot \phi$.

Proof It is enough to look at the actions of $\Gamma(M)$ on the set $\mathcal{S}_k \cup \bigcup_{i=0}^{k-1} \mathcal{Y}_k^i$ of generators for $B_k(M)$. First suppose that $\beta \in \mathcal{S}_k$; recall from the text preceding Lemma 2.4 that the elements of $\mathcal{S}_k$ are represented by paths that are contained in a neighbourhood of the marked points. Thus $\beta^{c(\phi)} = \beta$ since $c(\phi)$ is the identity on a collar neighbourhood of the boundary that contains the marked points. Also, $p(\beta)$ is supported on a neighbourhood of the marked points so it commutes with $c(\phi)$ and we have that

$$p^{-1}(c(\phi)^{-1} \odot p(\beta) \odot c(\phi)) = \beta = \beta^{c(\phi)}.$$

Now suppose that $\beta \in \bigcup_{i=0}^{k-1} \mathcal{Y}_k^i$ so β is represented by a path in the configuration space in which, for some $0 \leq i \leq k-1$, the first i points are stationary and the remaining points move in parallel, tracing a generator γ of $\pi_1(M - \boldsymbol{m}_i, m_i)$. We claim that $c(\phi)^{-1} \odot p(\beta) \odot c(\phi) = p(\beta^{c(\phi)})$ from which it is immediate that $p^{-1}(c(\phi)^{-1} \odot p(\beta) \odot c(\phi)) = \beta^{c(\phi)}$.

To see the claim, let us first assume that $\beta \in \mathcal{Y}_k^{k-1} = \iota_{k-1}(\mathcal{X}^{k-1})$, say $\beta = \iota_{k-1}(\gamma)$ for $\gamma \in \mathcal{X}^{k-1}$. Thus, in the braid β, the first $k-1$ points are stationary and the last point

traces the path γ. Hence $p(\beta) = p(\gamma)$ and for similar reasons $p(\gamma^{c(\phi)}) = p(\beta^{c(\phi)})$. Furthermore, $p(\gamma) = \tau_{\gamma_1} \odot \tau_{\gamma_2}^{-1}$ where γ_1 and γ_2 form the boundary of a tubular neighbourhood of γ. Here we use τ_δ to denote the (positive) Dehn twist around δ for a simple loop δ in M. For this last equality, see [7, Fact 4.7].

If $\phi\colon M \to M$ is a diffeomorphism and δ is a simple curve in M then $\phi^{-1} \odot \tau_\delta \odot \phi = \tau_{\delta^\phi}$ (see for example [7, Fact 3.7]). Here δ^ϕ denotes the image of the curve δ under the diffeomorphism ϕ. Now we have that

$$
\begin{aligned}
c(\phi)^{-1} \odot p(\beta) \odot c(\phi) &= c(\phi)^{-1} \odot p(\gamma) \odot c(\phi) \\
&= c(\phi)^{-1} \odot \tau_{\gamma_1} \odot \tau_{\gamma_2}^{-1} \odot c(\phi) \\
&= \tau_{\gamma_1^{c(\phi)}} \odot \tau_{\gamma_2^{c(\phi)}}^{-1} = p(\gamma^{c(\phi)}) = p(\beta^{c(\phi)})
\end{aligned}
$$

as claimed. In the case that $\beta \in \mathcal{Y}_k^i$ for $0 \leq i < k - 1$, the diffeomorphism $p(\beta)$ is given by pushing the last $k - i$ marked points in parallel along some path $\gamma \in \mathcal{X}^i$. Similar reasoning shows that $c(\phi)^{-1} \odot p(\beta) \odot c(\phi) = p(\beta^{c(\phi)})$ in this case as well. $\square$

We are now in a position to define the degeneracy maps for mapping class groups. From [2] we have degeneracy maps $s_j\colon B_k(M) \to B_{k+1}(M)$ given by doubling the jth string, and we know that these maps satisfy the simplicial identities (1). This gives us a map

$$\mathrm{id} \ltimes s_j \colon \Gamma(M) \ltimes B_k(M) \longrightarrow \Gamma(M) \ltimes B_{k+1}(M)$$

and using Lemma 2.5 we get a map $s_j\colon \Gamma^{(k)}(M) \to \Gamma^{(k+1)}(M)$. More explicitly, if we write an element of $\Gamma^{(k)}(M)$ in the form $c(\phi) \odot p(\beta)$ for $\phi \in \Gamma(M)$ and $\beta \in B_k(M)$ then we define

$$s_j(c(\phi) \odot p(\beta)) = c(\phi) \odot p(s_j(\beta)).$$

This construction restricts to give a degeneracy map on the pure mapping class group $\Gamma^k(M)$.

Before checking that the degeneracy maps for mapping class groups behave as desired, we prove the following lemma.

Lemma 2.8 *If we define $\tilde{\pi}\colon \Gamma(M) \ltimes B_k(M) \to \Sigma_k$ by $\tilde{\pi}(\phi, \beta) = \pi(\beta)$ then the following diagram commutes:*

$$
\begin{array}{ccc}
\Gamma(M) \ltimes B_k(M) & \xrightarrow{\tilde{\pi}} & \Sigma_k \\
{\scriptstyle\cong}\big\downarrow & \nearrow{\scriptstyle\mu} & \\
\Gamma^{(k)}(M) & &
\end{array}
$$

Proof The isomorphism is given by $(\phi, \beta) \mapsto c(\phi) \odot p(\beta)$, and it is evident from the definitions that $\mu(c(\phi) \odot p(\beta)) = \mu(p(\beta)) = \pi(\beta) = \tilde{\pi}(\phi, \beta)$. $\square$

Proposition 2.9 *The degeneracy map* $s_j \colon \Gamma^{(k)}(M) \to \Gamma^{(k+1)}(M)$ *satisfies the crossed identity, that is, for all* $\theta_1, \theta_2 \in \Gamma^{(k)}(M)$ *we have*

$$s_j(\theta_1 \odot \theta_2) = s_j(\theta_1) \odot s_{j \cdot \mu(\theta_1)}(\theta_2).$$

Hence, the degeneracy map $s_j \colon \Gamma^k(M) \to \Gamma^{k+1}(M)$ *is a homomorphism.*

Proof By the definition of the degeneracy map and Lemma 2.8, it is enough to show that

$$(8) \qquad (\mathrm{id} \ltimes s_j)\big((\phi_1, \beta_1)(\phi_2, \beta_2)\big) = (\mathrm{id} \ltimes s_j)(\phi_1, \beta_1)(\mathrm{id} \ltimes s_{j \cdot \pi(\beta_1)})(\phi_2, \beta_2).$$

The left-hand side of (8) is

$$\begin{aligned}
(\mathrm{id} \ltimes s_j)\big(\phi_1 \odot \phi_2, (\beta_1 \cdot \phi_2)\beta_2\big) &= \big(\phi_1 \odot \phi_2, s_j((\beta_1 \cdot \phi_2)\beta_2)\big) \\
&= \big(\phi_1 \odot \phi_2, s_j(\beta_1 \cdot \phi_2)s_{j \cdot \pi(\beta_1 \cdot \phi_2)}(\beta_2)\big) \\
&= \big(\phi_1 \odot \phi_2, s_j(\beta_1 \cdot \phi_2)s_{j \cdot \pi(\beta_1)}(\beta_2)\big).
\end{aligned}$$

Here we have used the fact that the degeneracy map for the braid groups satisfies the crossed identity and the fact that for $\phi \in \Gamma(M)$ and $\beta \in B_k(M)$, we have that $\pi(\beta \cdot \phi) = \pi(\beta)$, which is immediate from Lemma 2.7.

The right-hand side of (8) is

$$\big(\phi_1, s_j(\beta_1)\big)\big(\phi_2, s_{j \cdot (\pi(\beta_1))}(\beta_2)\big) = \big(\phi_1 \odot \phi_2, (s_j(\beta_1) \cdot \phi_2)s_{j \cdot \pi(\beta_1)}(\beta_2)\big).$$

Thus we are reduced to checking that $s_j(\beta_1 \cdot \phi_2) = s_j(\beta_1) \cdot \phi_2$; this is immediate when we consider the description of the action given in Lemma 2.7. $\square$

We are now ready for the main theorem.

Theorem 2.10 *For a surface* M *with nonempty boundary,* $\underline{\Gamma}^{\mathrm{sym}}(M)$ *forms a crossed simplicial group and* $\underline{\Gamma}(M)$ *forms a simplicial group. These simplicial structures extend the known simplicial structures for braid groups.*

Proof In [4], it is shown that $\underline{\Gamma}^{\mathrm{sym}}(M)$ forms a Δ–group and that the Δ–group structure extends the one on $\underline{B}(M)$ that is given in [2]. We also know from Proposition 2.9 that the degeneracy maps for the mapping class groups satisfy the crossed identity, and it is immediate from the construction that they restrict to those on the braid groups. Thus, to complete the proof we just need to check that the simplicial identities hold.

Let us define an alternative face map $\widetilde{d}_i\colon \Gamma^{(k)}(M) \to \Gamma^{(k-1)}(M)$. From [2] we have face maps $d_i\colon B_k(M) \to B_{k-1}(M)$ given by deleting the ith string. This gives us a map

$$\text{id} \ltimes d_i\colon \Gamma(M) \ltimes B_k(M) \longrightarrow \Gamma(M) \ltimes B_{k-1}(M)$$

and using Lemma 2.5 we get a map $\widetilde{d}_i\colon \Gamma^{(k)}(M) \to \Gamma^{(k-1)}(M)$. Since the face maps and degeneracy maps for braid groups satisfy the simplicial identities, so do the maps s_j and $\widetilde{d}_i$ for mapping class groups. We claim that $\widetilde{d}_i$ is the same as the face map $d_i\colon \Gamma^{(k)}(M) \to \Gamma^{(k-1)}(M)$ defined by forgetting the ith marked point. Given the claim, the proof that $\underline{\Gamma}^{\text{sym}}(M)$ is a crossed simplicial group is complete. The statement for $\underline{\Gamma}(M)$ follows by Lemma 2.1.

To see that the claim holds, observe that

$$\widetilde{d}_i(c(\phi) \odot p(\beta)) := c(\phi) \odot p(d_i(\beta)) = c(\phi) \odot d_i(p(\beta)) = d_i(c(\phi) \odot p(\beta)).$$

For the third equality we have used the fact that the face map for the mapping class group restricts to that for the braid group. $\qquad\square$

Observe that in the proof of Lemma 2.5 there are different choices of splitting maps corresponding to the different choices of the boundary component to which we glue the cylinder. These different splittings give rise to different semidirect product decompositions and thus different degeneracy maps. Hence the degeneracy maps are actually bi-indexed.

A relationship between the different splitting maps can be described as follows. Let $c_i\colon \Gamma(M) \to \Gamma^{(k)}(M)$ be the splitting map given by gluing a cylinder with k marked points to the ith boundary component of M and let $M(i)$ denote the surface obtained by doing this gluing.

Let $\theta\colon M \to M$ be a diffeomorphism that carries the ith boundary component to the jth boundary component and $\overline{\theta}\colon M(i) \to M(j)$ the natural extension of θ. The diffeomorphism θ does not represent an element of $\Gamma(M)$ but it does represent an element of a larger mapping class group in which $\Gamma(M)$ is normal (namely the group of isotopy classes of all diffeomorphisms of M). Define $\text{conj}_\theta\colon \Gamma(M) \to \Gamma(M)$ by $\text{conj}_\theta(\phi) = \theta \odot \phi \odot \theta^{-1}$. Similarly we can define $\text{conj}_{\overline{\theta}}\colon \Gamma^k(M) \to \Gamma^k(M)$ if we think of the domain as the mapping class group of $M(j)$ and the codomain as the mapping class group of $M(i)$.

Lemma 2.11 *The splitting maps c_i and c_j fit into the following commutative diagram:*

$$
\begin{array}{ccc}
\Gamma(M) & \xrightarrow{\ c_j\ } & \Gamma^k(M) \\
{\scriptstyle \mathrm{conj}_\theta}\downarrow & & \downarrow{\scriptstyle \mathrm{conj}_{\bar\theta}} \\
\Gamma(M) & \xrightarrow{\ c_i\ } & \Gamma^k(M)
\end{array}
$$

An entirely analogous result holds in the case of the full mapping class groups $\Gamma^{(k)}(M)$.

3 Normal forms

3.1 Algebraic background

Recall that if we have a split exact sequence

$$
1 \longrightarrow N \xrightarrow{\ \iota\ } G \underset{s}{\overset{d}{\rightleftarrows}} Q \longrightarrow 1,
$$

then G is isomorphic to the semidirect product $Q \ltimes N$. Alternatively, we can say that $G = s(Q)\cdot\iota(N)$ and every element of G can be written in a unique way as an element of this product.

Suppose that we have two sequences of groups $N_0, N_1,\dots$ and $G_0, G_1,\dots$ and for each $k \geq 1$ there is a split exact sequence

$$
1 \longrightarrow N_{k-1} \xrightarrow{\ \iota_k\ } G_k \underset{s_{k-1}}{\overset{d_k}{\rightleftarrows}} G_{k-1} \longrightarrow 1.
$$

By iterating, we obtain that for each $k \geq 1$, G_k is isomorphic to the semidirect product

$$
(\cdots((G_0 \ltimes N_0) \ltimes N_1) \ltimes \cdots) \ltimes N_{k-1}.
$$

Alternatively, we may express this by saying that G_k is the product

$$
(9)\qquad G_k = s_{k-1}\cdots s_0(G_0)\cdot\left(\prod_{i=1}^{k-1} s_{k-1}\cdots s_i \iota_i(N_{i-1})\right)\cdot \iota_k(N_{k-1})
$$

and every element of G_k can be written in a unique way as an element of this product.

3.2 Normal forms for braid groups

Fix a surface M. For each $k \geq 1$, there are short exact sequences (see for example [3, Lemma 3.11])

$$(10) \qquad 1 \longrightarrow \pi_1(M - \mathbf{m}_k) \xrightarrow{\iota_k} P_{k+1}(M) \xrightarrow{d_k} P_k(M) \longrightarrow 1.$$

If M admits a nonvanishing vector field, these sequences are split: the degeneracy map $s_{k-1} \colon P_k(M) \to P_{k+1}(M)$ provides a splitting. Thus, from the general theory in Section 3.1 we have the following.

Theorem 3.1 *For every surface M that admits a nonvanishing vector field and for every $k \geq 0$, there is a semidirect product decomposition*

$$P_{k+1}(M) \cong \big(\cdots ((\pi_1(M) \ltimes \pi_1(M - \mathbf{m}_1)) \ltimes \pi_1(M - \mathbf{m_2})) \ltimes \cdots \big) \ltimes \pi_1(M - \mathbf{m}_k).$$

In particular, the pure braid group is a semidirect product of k finitely generated free groups and $\pi_1(M)$. □

Recall that the map ι_k sends a path γ in $M - \mathbf{m}_k$ to the path in the configuration space that is constant in the first k coordinates with the $(k+1)^{\text{st}}$ point tracing the path γ. The map s_j is given by introducing a new string, close to the jth, that moves in parallel with the jth string with respect to the vector field. Taking into account the behaviour of the maps ι_j and s_j, in the present case the expression in (9) gives Theorem B of the Introduction, as follows.

Theorem 3.2 *Let M be a surface admitting a nonvanishing vector field. For $k \geq 0$, every element of $P_{k+1}(M)$ can be written in a unique way in the form $\alpha_0 \cdots \alpha_k$ where α_i is represented by a path in $\mathrm{Conf}_{k+1}(M)$ in which the first i points are stationary and the remaining points move in parallel (with respect to the vector field), tracing a path in M.* □

Note that different decompositions and normal forms can be obtained by using different face maps and degeneracy maps in the split exact sequences (10). In particular, using d_0, s_0 and the map $x \mapsto (x, m_1, m_2, \ldots, m_k)$ yields a "left-weighted" version of the normal form in Theorem 3.2.

If the surface M has nonempty boundary, an alternative splitting for the sequence (10) is provided by the map $\widetilde{s}_{k-1} \colon P_k(M) \to P_{k+1}(M)$ given by $\alpha \mapsto (\alpha, c_{m_k})$. Here c_{m_k} is the map which is constant at the marked point m_k, and m_k is considered as being close to a chosen boundary component $\partial_0 M$. (In the terminology of [12, page 24], $\widetilde{s}_{k-1}$ is the final coface map d^k.) These splittings yield another semidirect decomposition of the form in Theorem 3.1 and the following normal forms.

Theorem 3.3 *Let M be a surface with boundary and $\partial_0 M$ one of its boundary components. For $k \geq 0$, every element of $P_{k+1}(M)$ can be written in a unique way in the form $\beta_0 \cdots \beta_k$, where β_i is represented by a path in $\mathrm{Conf}_{k+1}(M) \subseteq M^{k+1}$ in which every coordinate is constant, except for the ith coordinate which follows a path in M avoiding a neighbourhood of $\partial_0 M$ that contains $m_{i+1}, \ldots, m_k$.* $\qquad\square$

In the case of the disc, this is the well-known Artin combing of braids, as has previously been noted by J Wu, who used coface maps to obtain a bi-Δ–group structure on the collection of pure braid groups $P_{k+1}(M)$. He showed [12, Proposition 1.2.9] that every bi-Δ–group decomposes into iterated semidirect products in a way that parallels the arguments above.

We can also describe normal forms for elements of the full braid group. Let $\mathcal{P} \subseteq B_n(M)$ be a collection of braids such that $\pi|_{\mathcal{P}}\colon \mathcal{P} \to \Sigma_n$ is a bijection.

Corollary 3.4 *Any element of $B_k(M)$ can be written in a unique way as $\rho\alpha$ for α of the form given in Theorem 3.2 and $\rho \in \mathcal{P}$. Alternatively, any element can be written in a unique way as $\rho\beta$ for β of the form given in Theorem 3.3 and $\rho \in \mathcal{P}$.*

Proof This is an immediate consequence of the fact that $B_n(M)$ fits into an exact sequence

$$1 \longrightarrow P_n(M) \longrightarrow B_n(M) \longrightarrow \Sigma_n \longrightarrow 1. \qquad\square$$

3.3 Normal forms for mapping class groups

We can use Lemma 2.5 to create semidirect product decompositions and normal forms for $\Gamma^{k+1}(M)$.

Theorem 3.5 *Let M be a surface with boundary. For $k \geq 0$, $\Gamma^{k+1}(M)$ has the semidirect product decomposition*

$$\Gamma(M) \ltimes \left((\cdots ((\pi_1(M) \ltimes \pi_1(M - \boldsymbol{m}_1)) \ltimes \pi_1(M - \boldsymbol{m}_2)) \ltimes \cdots) \ltimes \pi_1(M - \boldsymbol{m}_k) \right).$$

Proof This is an immediate consequence of Lemma 2.5 and Theorem 3.1. $\qquad\square$

This leads to Theorem C of Section 1, as follows.

Theorem 3.6 *Let M be a surface with boundary. For $k \geq 0$, every element of $\Gamma^{k+1}(M)$ can be written in a unique way as $\theta \odot \phi_0 \odot \cdots \odot \phi_k$, where $\phi_i = p(\alpha_i)$ for α_i as in Theorem 3.2 and θ is the identity on a collar neighbourhood of the chosen boundary component that contains the marked points.*

Proof This follows from Equation (7) and Theorem 3.2. $\qquad\square$

Theorem 3.7 *Let M be a surface with boundary. For $k \geq 0$, every element of $\Gamma^{k+1}(M)$ can be written in a unique way as $\theta \odot \psi_0 \odot \cdots \odot \psi_k$ where $\psi_i = p(\beta_i)$ for β_i as in Theorem 3.3 and θ is fixed on a collar neighbourhood of the chosen boundary component that contains the marked points.*

Proof This is a consequence of Equation (7) and Theorem 3.3. $\qquad\square$

These normal forms can also be obtained independently of the results on braid groups. One considers the following exact sequences, valid for $k \geq 1$ (see [3, Lemma 3.13]):

$$1 \longrightarrow \pi_1(M - \boldsymbol{m}_k) \xrightarrow{\partial^\pi} \Gamma^{k+1}(M) \xrightarrow{d_k} \Gamma^k(M) \longrightarrow 1.$$

Here ∂^π is the connecting homomorphism in the long exact sequence associated to the fibration $\mathrm{Diff}(M, \boldsymbol{m}_k) \to M - \boldsymbol{m}_k$ given by evaluation at m_k. As for braid groups, there are splitting maps given either by the degeneracy $s_{k-1}: \Gamma^k(M) \to \Gamma^{k+1}(M)$ or by $\widetilde{s}_{k-1}: \Gamma^k(M) \to \Gamma^{k+1}(M)$. Here $\widetilde{s}_{k-1}$ is defined by gluing a cylinder with one marked point to $\partial_0 M$ and extending diffeomorphisms by the identity, and may be considered as a final coface map, as was done for the braid groups.

We can also describe normal forms for elements of the full mapping class group $\Gamma^{(k)}(M)$. Let $\Theta \subseteq \Gamma^{(k)}(M)$ be a collection of mapping classes such that $\mu|_\Theta: \Theta \to \Sigma_k$ is a bijection, where $\mu: \Gamma^{(k)}(M) \to \Sigma_k$ is as in Section 2.4 above.

Corollary 3.8 *Any element of $\Gamma^{(k)}(M)$ can be written in a unique way as $\theta \odot \phi$ for ϕ of the form given in Theorem 3.6 and $\theta \in \Theta$. Alternatively, any element can be written in a unique way as $\theta \odot \phi'$ for ϕ' of the form given in Theorem 3.7 and $\theta \in \Theta$.*

Proof This is an immediate consequence of the fact that $\Gamma^{(k)}(M)$ fits into the exact sequence

$$1 \longrightarrow \Gamma^k(M) \longrightarrow \Gamma^{(k)}(M) \xrightarrow{\mu} \Sigma_k \longrightarrow 1. \qquad\square$$

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