Algebraic & Geometric Topology 14 (2014) 3419–3491

Segal-type algebraic models of n–types

DAVID BLANC
SIMONA PAOLI

For each $n \geq 1$, we introduce two new Segal-type models of n–types of topological spaces: *weakly globular n–fold groupoids* and their lax version. We show that any n–type can be represented up to homotopy by such models via an explicit *algebraic fundamental n–fold groupoid* functor. We compare these models to Tamsamani's weak n–groupoids, and extract from them a model for $(k-1)$–connected n–types.

55S45; 18G50, 18B40

1 Introduction and summary

Many homotopy invariants of a topological space T, such as its homotopy, homology and cohomology groups, are graded by dimension so that we do not need to know all of T to determine $\pi_n T$, $H_n T$ or $H^n(T;G)$, but only a skeleton or Postnikov section of T. Thus, for many purposes a good first approximation to homotopy theory is the study of *n–types*: spaces T whose homotopy groups $\pi_k(T,t_0)$ vanish for $k > n$.

One advantage of such approximations is that they have algebraic models: the classical example is the homotopy category of connected 1–types, which is equivalent to the category of groups. More generally, all 1–types are modeled by groupoids via the fundamental groupoid functor $\widehat{\pi}_1\colon \mathsf{Top} \to \mathsf{Gpd}$.

The arrows of $\widehat{\pi}_1 T$ are homotopy classes of paths, so higher-order approximations should encode higher homotopies (see Grothendieck [32]), and thus involve higher categorical structures.

Many such structures have been shown to model the homotopy category $\mathrm{ho}\mathsf{P}^n\mathsf{Top}$ of n–types of topological spaces: in the path-connected case, these include the cat^n–groups of Loday [38], the crossed n–cubes of Ellis and Steiner [28] and Porter [43], the n–hypercrossed complexes of Carrasco and Cegarra [21], and the weakly globular cat^n–groups of Paoli [42]. Special models exist for $n = 2, 3$, starting with the crossed modules of Mac Lane and Whitehead [39], and including the homotopy double groupoids of Brown, Hardie, Kamps and Porter [16], the homotopy bigroupoids of Hardie, Kamps and Kieboom [33], the strict 2–groupoids of Moerdijk and Svensson [41],

Published: 15 January 2015 DOI: 10.2140/agt.2014.14.3419

the double groupoids of Cegarra, Heredia and Remedios [22], the double groupoids with connections of Brown and Spencer [18], the Gray groupoids of Leroy [37], Berger [10] and Joyal and Tierney [36], and the quadratic modules of Baues [4]. In the general case, such models include Batanin's higher groupoids (see Batanin [3] and Cisinski [23]), the n–hypergroupoids of Glenn [30], and Tamsamani's weak n–groupoids (see Tamsamani [48] and Simpson [47]).

In this paper we discuss three algebraically defined categories of Segal-type objects which can be used to model all n–types of topological spaces. All three are full subcategories of the category $[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$ of $(n-1)$–fold simplicial objects in groupoids.

1.1 The three models The first is the known category Tam^n of *Tamsamani weak n–groupoids*. The second is a new category $\mathsf{Gpd}^n_{\mathrm{wg}}$ of *weakly globular n–fold groupoids*. This is a full subcategory of the category Gpd^n of n–fold groupoids (iteratively defined as groupoids internal to Gpd^{n-1}). The third is another new category $\mathsf{PsGpd}^n_{\mathrm{wg}}$ of *weakly globular pseudo n–fold groupoids*.

To grasp the idea behind these notions, it is useful to consider another higher categorical structure which embeds in all three of the above, the category n–Gpd of strict n–groupoids (iteratively defined as groupoids enriched in $(n-1)$–Gpd).

There are full and faithful inclusions:

(1-2)
$$\begin{tikzcd}
 & \mathsf{PsGpd}^n_{\mathrm{wg}} & \\
\mathsf{Tam}^n \arrow[ur, hook] & & \mathsf{Gpd}^n_{\mathrm{wg}} \arrow[ul, hook'] \\
 & n\text{–}\mathsf{Gpd} \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

The category n–Gpd admits a multi-simplicial description as the full subcategory of those $(n-1)$–fold simplicial objects $X \in [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$ satisfying the following:

(i) $X^{(1)}_0 \in [\Delta^{n-2^{\mathrm{op}}}, \mathsf{Gpd}]$ and $X^{(1,\ldots,r+1)}_{1\overset{r}{\cdots}1 0} \in [\Delta^{n-r-2^{\mathrm{op}}}, \mathsf{Gpd}]$ are discrete – that is, constant multi-simplicial sets – for all $1 \le r \le n-2$. Here we use the notation of Section 2.6(b).

(ii) The Segal maps (see Definition 2.3 below) in all directions are isomorphisms.

In addition, we require that after applying $\pi_0\colon \mathsf{Gpd} \to \mathsf{Set}$ in each simplicial dimension we obtain a strict $(n-1)$–groupoid.

The sets in (i) corresponds to the set of r–cells $(1 \le r \le n-2)$ in the strict n–groupoid. By (ii), their composition is associative and unital.

Condition (i) is also called the *globularity condition*, since it determines the globular shape of the cells in a strict n–groupoid. For instance, when $n = 2$ we can picture the 2–cells as globes:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\zeta"] & \bullet \end{tikzcd}$$

Although strict n–groupoids have applications in homotopy theory, especially in their equivalent form of crossed n–complexes (see Brown, Higgins and Sivera [17]), they cannot model all n–types of topological spaces (see Simpson [46, Section 5] for a counterexample in dimension 3). Therefore, we must relax the strict structure in order to recover all n–types.

We consider three approaches to this:

(a) In the first approach, we preserve condition (i) and relax (ii), by allowing the Segal maps to be suitable iteratively defined equivalences. The composition of cells is then no longer strictly associative and unital. This leads to the category Tam^n of Tamsamani weak n–groupoids (Definition 5.1).

(b) In this paper we offer a second approach, in which condition (ii) is preserved, while (i) is replaced by weak globularity, so that the multi-simplicial objects in (i) are no longer required to be discrete, but only "homotopically discrete" (in a way that allows iteration). This leads to the category $\mathsf{Gpd}^n_{\mathrm{wg}}$ of weakly globular n–fold groupoids (Definition 3.19).

(c) We also describe a third approach, in which both (i) and (ii) are relaxed. This yields the category $\mathsf{PsGpd}^n_{\mathrm{wg}}$ of weakly globular pseudo n–fold groupoids (Definition 6.4).

Moreover, we have a *realization* functor $B\colon \mathsf{PsGpd}^n_{\mathrm{wg}} \to \mathsf{Top}$; the composite

$$\mathsf{PsGpd}^n_{\mathrm{wg}} \hookrightarrow [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}] \xrightarrow{N} [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}] \xrightarrow{\mathrm{Diag}_{(n)}} [\Delta^{\mathrm{op}}, \mathsf{Set}] \xrightarrow{\|-\|} \mathsf{Top}, \tag{1-3}$$

where $\mathrm{Diag}_{(n)}$ is the n–fold diagonal. The same is therefore true of the two subcategories Tam^n and $\mathsf{Gpd}^n_{\mathrm{wg}}$. In all three categories, maps which induce weak homotopy equivalences on realizations are called *geometric weak equivalences*.

The precise definitions of these three categories appear as cited above; here we will only highlight some key features common to all three:

(1) The construction of each category is by induction, starting in all three cases from the category of groupoids for $n=1$. A weakly globular pseudo n–fold groupoid is in particular a simplicial object X in $\mathsf{PsGpd}^{n-1}_{\mathrm{wg}}$, (and similarly for the other two categories).

(2) Moreover, X_0 is a homotopically discrete weakly globular pseudo $(n{-}1)$–fold groupoid (Definition 6.1) and similarly for $\mathsf{Gpd}^{n}_{\mathrm{wg}}$, while in the case of a Tamsamani weak n–groupoid

$$X \in [\Delta^{\mathrm{op}}, \mathsf{Tam}^{n-1}],$$

X_0 is actually discrete.

(3) Since $\mathsf{PsGpd}^{n}_{\mathrm{wg}}$ is a subcategory of $[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$, we can apply the functor π_0 to each groupoid of any weakly globular pseudo n–fold groupoid X to obtain

$$\overline{\pi}_0^{(n)} X \in [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Set}].$$

In each of our three categories the functor $\overline{\pi}_0^{(n)}$ lifts to functors

$$\begin{aligned}
\Pi_0^{(n)}&\colon \mathsf{PsGpd}^{n}_{\mathrm{wg}} \to \mathsf{PsGpd}^{n-1}_{\mathrm{wg}},\\
\Pi_0^{(n)}&\colon \mathsf{Gpd}^{n}_{\mathrm{wg}} \to \mathsf{Gpd}^{n-1}_{\mathrm{wg}},\\
\Pi_0^{(n)}&\colon \mathsf{Tam}^{n} \to \mathsf{Tam}^{n-1}.
\end{aligned}$$

These serve as algebraic $(n{-}1)$–Postnikov section functors, so we have a natural Postnikov tower

$$\mathsf{PsGpd}^{n}_{\mathrm{wg}} \xrightarrow{\Pi_0^{(n)}} \mathsf{PsGpd}^{n-1}_{\mathrm{wg}} \xrightarrow{\Pi_0^{(n-1)}} \cdots \longrightarrow \mathsf{Gpd} \xrightarrow{\pi_0} \mathsf{Set}.$$

and similarly for the other two categories.

(4) In all three categories, let $\gamma\colon X_0 \to X_0^d$ denote the weak equivalence from the homotopically discrete object X_0 to its discretization (so γ is the identity for $X \in \mathsf{Tam}^{n}$). For each $k \geq 2$ the composite of the maps

$$X_k \xrightarrow{\mu_k} X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \xrightarrow{\gamma_*} X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \tag{1-4}$$

(see Definition 2.3) is called the k^{th} *induced Segal map*. We require these maps to be geometric weak equivalences.
When $X \in \mathsf{Tam}^{n}$, the second map in (1-4) is the identity, while when $X \in \mathsf{Gpd}^{n}_{\mathrm{wg}}$, the first map is an isomorphism.

1.5 Main results The process of discretizing the homotopically discrete sub-objects in $\mathsf{Gpd}^n_{\mathrm{wg}}$ and $\mathsf{PsGpd}^n_{\mathrm{wg}}$ gives rise to discretization functors D_n that make the diagram

$$\begin{tikzcd} & \mathsf{PsGpd}^n_{\mathrm{wg}} \arrow[dl, "D_n"'] & \\ \mathsf{Tam}^n & & \mathsf{Gpd}^n_{\mathrm{wg}} \arrow[ul, hook'] \arrow[ll, "D_n"] \end{tikzcd} \tag{1-6}$$

commute. All three categories $\mathsf{PsGpd}^n_{\mathrm{wg}}$, $\mathsf{Gpd}^n_{\mathrm{wg}}$ and Tam^n share some useful features.

First, the realization functor $B\colon \mathsf{PsGpd}^n_{\mathrm{wg}} \to \mathsf{Top}$ actually lands in the category $\mathsf{P}^n\mathsf{Top}$ of n–types, so the same is true of the categories $\mathsf{Gpd}^n_{\mathrm{wg}}$ and Tam^n.

Furthermore, all three models have *algebraic homotopy groups* $\omega_k(X,x)$ (see also Section 3.26), which allow one to extract $\pi_k BX$ directly from the model X. In addition, we have higher-dimensional analogues of the categorical notion of an equivalence of groupoids. Together, these two notions allow to define *algebraic weak equivalences* in each of the categories, and show that these are the same as the geometric weak equivalences (see Corollary 4.8, [45, Section 6], Section 6.35, and Remark 6.37). Thus each of these models is entirely algebraic.

Our main results are as follows:

Theorem A *For each $n \geq 1$:*

(a) *The functor $\widehat{Q}_{(n)}$ induces a faithful embedding*

$$\mathrm{ho}\mathsf{P}^n\mathsf{Top} \hookrightarrow \mathrm{ho}\,\mathsf{Gpd}^n_{\mathrm{wg}}\,,$$

so for each $T \in \mathsf{P}^n\mathsf{Top}$ there is an isomorphism in $\mathrm{ho}\mathsf{P}^n\mathsf{Top}$ *between T and $B\widehat{Q}_{(n)}T$.*

(b) *There exists a functor $\Pi_0^{(n)}\colon \mathsf{Gpd}^n_{\mathrm{wg}} \to \mathsf{Gpd}^{n-1}_{\mathrm{wg}}$ with a natural isomorphism $\Pi_0^{(n)}\widehat{Q}_{(n)} \cong \widehat{Q}_{(n-1)}$, so we can extract the model for the $(n-1)^{\mathrm{st}}$ Postnikov section $P^{n-1}T$ from $\widehat{Q}_{(n)}T$ algebraically.*

(c) *There are* algebraic homotopy group *functors $\omega_k\colon \mathsf{Gpd}^n_{\mathrm{wg}} \to \mathsf{Gp}$ such that $\pi_k(BG;x_0) \cong \omega_k(G;x_0)$ $(0 \leq k \leq n)$.*

(See Theorem 4.32, Proposition 4.28 and Theorem 4.6.)

Theorem B *The functors $\widehat{Q}_{(n)}$ and B induce equivalences of categories*

$$\mathrm{ho}\mathsf{P}^n\mathsf{Top} \approx \mathrm{ho}\,\mathsf{PsGpd}^n_{\mathrm{wg}}.$$

(See Theorem 6.28.)

Furthermore, every object of $\mathsf{PsGpd}^n_{\mathrm{wg}}$ is weakly equivalent through a zig-zag to an object of $\mathsf{Gpd}^n_{\mathrm{wg}}$ as well as to an object of Tam^n (see Remark 6.32). Thus we can regard Tam^n and $\mathsf{Gpd}^n_{\mathrm{wg}}$ as two different types of partial strictifications of the category $\mathsf{PsGpd}^n_{\mathrm{wg}}$ which preserve the homotopy type. The passage from $\mathsf{PsGpd}^n_{\mathrm{wg}}$ to Tam^n strictifies the globularity condition, while the passage from $\mathsf{PsGpd}^n_{\mathrm{wg}}$ to $\mathsf{Gpd}^n_{\mathrm{wg}}$ strictifies the Segal maps.

The fundamental n–fold groupoid functor $\widehat{Q}_{(n)}\colon \mathsf{Top} \to \mathsf{Gpd}^n_{\mathrm{wg}}$ provides an explicit form for the algebraic model of an n–type. This is a desirable feature of an algebraic model, especially in view of applications.

We discuss an application to the modeling of $(k-1)$–connected n–types in Section 7.A. To this end we identify suitable subcategories $\mathsf{PsGpd}^{(n,k)}_{\mathrm{wg}}$ and $\mathsf{Gpd}^{(n,k)}_{\mathrm{wg}}$ of $\mathsf{PsGpd}^n_{\mathrm{wg}}$ and $\mathsf{Gpd}^n_{\mathrm{wg}}$ respectively, which are algebraic models of $(k-1)$–connected n–types, and we also establish a connection with iterated loop spaces (Proposition 7.9).

1.7 Organization of the paper In Section 2 we describe the construction of the *fundamental n–fold groupoid* functor $\widehat{Q}_{(n)}$: to obtain a multi-simplicial algebraic model from a space, we first take a fibrant simplicial set model using the singular functor $\mathcal{S}\colon \mathsf{Top} \to [\Delta^{\mathrm{op}}, \mathsf{Set}]$. We can associate to any simplicial set X an "n–fold resolution" $\mathrm{Or}_{(n)}\, X$, which is an object of $[\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$ representing the same homotopy type (Lemma 2.13). We then obtain an n–fold groupoid by applying the left adjoint $P_{(n)}\colon [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}] \to \mathsf{Gpd}^n$ to the n–fold nerve $N_{(n)}\colon \mathsf{Gpd}^n \to [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$. Thus $\widehat{Q}_{(n)}$ is the composite

$$\mathsf{Top} \xrightarrow{\mathcal{S}} [\Delta^{\mathrm{op}}, \mathsf{Set}] \xrightarrow{\mathrm{Or}_{(n)}} [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}] \xrightarrow{P_{(n)}} \mathsf{Gpd}^n$$

(see Definition 2.30).

For a general n–fold simplicial set Y, $P_{(n)}Y$ does not have a simple and explicitly computable expression. However, we show that the fibrancy of $\mathcal{S}T$ induces a property of $\mathrm{Or}_{(n)}\,\mathcal{S}T$, which we call $(n,2)$*–fibrancy* (see Definition 2.31 and Proposition 2.39). We then show that to apply $P_{(n)}$ to an $(n,2)$–fibrant n–simplicial set, we need only apply the usual fundamental groupoid functor in each of the $n-1$ simplicial directions. We thus have (see Theorem 2.40)

$$\widehat{Q}_{(n)}T = \hat{\pi}_1^{(1)}\hat{\pi}_1^{(2)}\cdots\hat{\pi}_1^{(n)}\,\mathrm{Or}_{(n)}\,\mathcal{S}T.$$

In Section 3, we describe certain features of those n–fold groupoids which are in the image of the functor $\widehat{Q}_{(n)}$ (and thus will be used to represent n–types). These are encoded in the notion of *weakly globular n–fold groupoids*. As explained in Section 1.1, we first identify a suitable subcategory of *homotopically discrete* objects (Section 3.A), which are needed for the weak globularity condition in the definition of weakly globular n–fold groupoid (Section 3.B).

In Section 4 we show that the n–Postnikov section P^nT and $B\widehat{Q}_{(n)}T$ have the same homotopy type (Proposition 4.28), so $\mathsf{Gpd}^n_{\mathrm{wg}}$ represents all n–types. In Section 4.A we show that the realization of a weakly globular n–fold groupoid is an n–type (an alternative proof using a comparison with Tamsamani's model is given in Section 5). Section 4.B provides a new iterative description of the fundamental n–fold groupoid functor $Q_{(n)}$. This is used in Proposition 4.28 of Section 4.C, where we show that the functor $\widehat{Q}_{(n)}$ lands in the category $\mathsf{Gpd}^n_{\mathrm{wg}}$. This leads to one of our main results, Theorem 4.32, saying that B and $\widehat{Q}_{(n)}$ induce functors

$$\mathrm{ho}\mathsf{P}^n\mathsf{Top} \underset{\widehat{Q}_{(n)}}{\overset{B}{\rightleftarrows}} \mathrm{ho}\,\mathsf{Gpd}^n_{\mathrm{wg}} \tag{1-8}$$

with $B\circ\widehat{Q}_{(n)}\cong \mathrm{Id}$.

In Section 5 we provide an equivalent definition of Tamsamani's weak n–groupoids (see Section 5.A), and in Section 5.B we construct a *discretization* functor

$$D_n\colon \mathsf{Gpd}^n_{\mathrm{wg}} \to \mathsf{Tam}^n$$

that replaces a weakly globular n–fold groupoid $G\in\mathsf{Gpd}^n_{\mathrm{wg}}$ by a Tamsamani weak n–groupoid D_nG of the same homotopy type (Theorem 5.19).

In Section 6 we consider the wider context of *weakly globular pseudo n–fold groupoids*. These are defined in Section 6.A, and compared to Tamsamani's model in Section 6.B, where we again construct a discretization functor

$$D_n\colon \mathsf{PsGpd}^n_{\mathrm{wg}} \to \mathsf{Tam}^n,$$

and in Theorem 6.23 we show that for any $X\in\mathsf{PsGpd}^n_{\mathrm{wg}}$, there is a zig-zag of weak equivalences in $\mathsf{PsGpd}^n_{\mathrm{wg}}$ between X and D_nX. Our main Theorem 6.28 then follows.

In Section 7 we describe an application, and indicate some directions for future work: In Section 7.A, we show how to extract from our results an algebraic model for $(k-1)$–connected n–types (Proposition 7.7), and thus for iterated loop spaces. In Section 7.B we define *n–track categories* (one of the original motivations for our work), with possible future applications.

Appendix A proves some technical facts about $\mathrm{Or}_{(n)}$ needed in Section 2.

1.A Index of terminology and notation

BG	realization of an n–fold (pseudo) groupoid G	Definition 2.21
$\mathrm{c}X$	discrete groupoid on a set X	Definition 2.23
$\overline{c}^{(n)}$, $c^{(n)}$	discrete groupoid functor applied to an $(n-1)$–fold simplicial groupoid or an $(n-1)$–fold groupoid	Definition 3.12, Remark 3.13
D_n	discretization functors for $\mathsf{Gpd}^n_{\mathrm{wg}}$ and $\mathsf{PsGpd}^n_{\mathrm{wg}}$	Definition 5.18, Definition 6.22
Disc_0	0–discretization functor on weakly globular n–fold groupoids	Definition 5.13
Dec, Dec$'$	décalage functors on simplicial sets	Section 2.7
$\mathrm{Diag}_{(n)}$	n–fold diagonal functor	Section 2.6
$[\Delta^{n^{\mathrm{op}}}, \mathscr{C}]$	category of n–fold simplicial objects in $\mathscr{C}$	Section 2.5
F^n_{Tm}	Tamsamani's Poincaré n–groupoid functor	Theorem 5.7
Gpd	category of groupoids	Section 2.5
n–Gpd	category of strict n–groupoids	Section 1.1
$\mathsf{Gpd}(\mathscr{V})$	category of internal groupoids in $\mathscr{V}$	Definition 2.16
Gpd^n	category of n–fold groupoids	Definition 2.16
$\mathsf{Gpd}^n_{\mathrm{wg}}$	category of weakly globular n–fold groupoids.	Definition 3.19
$\mathsf{Gpd}^{(n,k)}_{\mathrm{wg}}$	category of (n,k)–weakly globular pseudo n–fold groupoids	Definition 7.3
$\mathsf{Gpd}^n_{\mathrm{hd}}$	category of homotopically discrete n–fold groupoids	Definition 3.3
$L_k X$	simplicial "bar-path construction"	Definition 4.10
μ_k	k^{th} Segal map	Equation (2-4)
$\widehat{\mu}_k$	k^{th} induced Segal map	Equation (1-4)

$N^{(i)}$	nerve functor of an n–fold groupoid in the i^{th} direction	Notation 2.18
$N_{(n)}$	multinerve functor on n–fold groupoids	Definition 2.19
$\mathrm{Or}_{(n)}$	n–fold ordinal sum of a simplicial set	Section 2.9
$P^{(i)}$	left adjoint to $N^{(i)}$	Proposition 2.38
$P_{(n)}$	left adjoint to $N_{(n)}$	Definition 2.30
$\mathsf{PsGpd}^n_{\mathrm{hd}}$	category of homotopically discrete pseudo n–fold groupoids	Definition 6.1
$\mathsf{PsGpd}^{(n,k)}_{\mathrm{wg}}$	category of (n,k)–weakly globular pseudo n–fold groupoids	Definition 7.3
$\mathsf{P}^n\mathsf{Top}$	full subcategory of n–Postnikov sections in Top	Definition 3.28
$\widehat{\pi}_1$	fundamental groupoid of a topological space	Definition 2.17
$\Pi_0^{(n)}$	algebraic $(n-1)^{\text{st}}$ Postnikov section functor	Lemma 3.15, Definition 3.19, Definition 5.1, Definition 6.1
$Q_{(n)}$, $\widehat{Q}_{(n)}$	fundamental n–fold groupoid functors	Definition 2.30
$T^{\mathrm{wg}}_{(n)}$	fundamental groupoid functor for $\mathsf{Gpd}^n_{\mathrm{wg}}$	Notation 5.15
$T^{\mathrm{Tm}}_{(n)}$	Tamsamani fundamental groupoid functor	Tamsamani's original definition 5.2
$T^{\mathrm{ps}}_{(n)}$	fundamental groupoid functor for $\mathsf{PsGpd}^n_{\mathrm{wg}}$	Notation 6.18
Tam^n, $\underline{\mathsf{Tam}}^n$	two equivalent formulations of the category of Tamsamani weak n–groupoids	Definition 5.1, Tamsamani's original definition 5.2
Top	category of topological spaces.	
$\mathcal{W}_{(n,k)}$	k–fold object of arrows of an n–fold groupoid	Definition 3.23
$\omega_k(G;x_0)$	k^{th} algebraic homotopy group	Section 3.26, Section 6.35

See also the list of special notations for n–fold simplicial objects in Section 2.6, in particular for the notation $\overline{F}$ for any functor F.

Acknowledgements This research was supported by the first author's Israel Science Foundation grant number 47377, and the second author's Marie Curie International Reintegration grant number 256341 and London Mathematical Society grant number 41134. The second author would also like to thank the Department of Mathematics at the University of Haifa for its hospitality during visits in March–April 2012, and June and August 2013. We thank Dorette Pronk for many helpful comments. Finally, we would like to thank the referee for his or her detailed and pertinent remarks.

2 The fundamental n–fold groupoid of a space

As noted above, the fundamental groupoid $\hat{\pi}_1 T$ of a (not necessarily connected) space T is an algebraic model for its 1–type. We now show how the notion of the fundamental 2–typical double groupoid defined in our paper [11, Section 2.21] generalizes to all n. We consider the standard model structure on Top, so that $\mathrm{ho}\mathsf{Top}$ means its localization with respect to the class of weak homotopy equivalences.

2.A Simplicial constructions

Given a topological space T, we construct its fundamental n–fold groupoid from a fibrant simplicial set model for T, such as its singular set $X := \mathcal{S}T \in [\Delta^{\mathrm{op}}, \mathsf{Set}]$. We therefore first recall some notation and constructions for simplicial sets.

2.1 Definition For any category $\mathcal{C}$, $[\Delta^{\mathrm{op}}, \mathcal{C}]$ is the category of simplicial objects in $\mathcal{C}$, where Δ denotes the category of finite ordered sets: $[0], [1]$ and so on. As usual, we write X_n for $X([n])$. If $\mathcal{C}$ is concrete, the *n–skeleton* $\mathrm{sk}_n X \in [\Delta^{\mathrm{op}}, \mathcal{C}]$ of any $X \in [\Delta^{\mathrm{op}}, \mathcal{C}]$ is generated under the degeneracy maps by $X_0, \dots, X_n$. The *n–coskeleton* functor $\mathrm{csk}_n\colon [\Delta^{\mathrm{op}}, \mathcal{C}] \to [\Delta^{\mathrm{op}}, \mathcal{C}]$ is left adjoint to sk_n. We say that X is *n–coskeletal* if the natural map $X \to \mathrm{csk}_n X$ is an isomorphism.

2.2 Remark There is an order-reversing involution $I\colon \Delta \to \Delta$, which induces a functor $I^*\colon [\Delta^{\mathrm{op}}, \mathcal{C}] \to [\Delta^{\mathrm{op}}, \mathcal{C}]$ (sending $d_i\colon X_n \to X_{n-1}$ to d_{n-i}). This functor I^* is not generally an isomorphism, but for a Kan complex $X \in [\Delta^{\mathrm{op}}, \mathsf{Set}]$ we have a natural isomorphism of fundamental groupoids $(\hat{\pi}_1 I^* X)^{\mathrm{op}} \cong \hat{\pi}_1 X$ (see Goerss and Jardine [31, I.8]).

2.3 Definition Let $X \in [\Delta^{\mathrm{op}}, \mathcal{C}]$ be a simplicial object in any category $\mathcal{C}$ with pullbacks. For each $1 \le j \le k$, let $\nu_j\colon X_k \to X_1$ be induced by the map $[1] \to [k]$ in

Δ sending 0 to $j-1$ and 1 to j. Then the diagram

$$\begin{tikzcd}[column sep=small]
&&&& X_k \arrow[dlll, "\nu_1"'] \arrow[dl, "\nu_2"] \arrow[drrr, "\nu_k"] &&&& \\
& X_1 \arrow[dl, "d_1"'] \arrow[dr, "d_0"] && X_1 \arrow[dl, "d_1"'] \arrow[dr, "d_0"] && \cdots && X_1 \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\
X_0 && X_0 && X_0 && \cdots\ X_0 && X_0
\end{tikzcd} \tag{2-4}$$

commutes. If we let $X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ denote the limit of the lower part of Diagram (2-4), the k^{th} *Segal map* for X is the unique map

$$\mu_k\colon X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$$

such that $\mathrm{pr}_j\ \mu_k = \nu_j$, where pr_j is the j^{th} projection (see Segal [45]).

Note that X is the nerve of an internal category in $\mathcal{C}$ if and only if all the Segal maps are isomorphisms.

2.5 *n*–fold simplicial objects An *n–fold simplicial object* in $\mathcal{C}$ is a functor $\Delta^{n^{\mathrm{op}}} \to \mathcal{C}$, and we denote the category of such by $[\Delta^{n^{\mathrm{op}}}, \mathcal{C}]$. Thus $X \in [\Delta^{n^{\mathrm{op}}}, \mathcal{C}]$ consists of objects $X_{i_1 i_2 \cdots i_n}$ in $\mathcal{C}$ for each n–fold multi-index $i_1, i_2, \dots, i_n \in \mathbb{N}$, along with face and degeneracy maps in each of the n directions, satisfying the usual simplicial identities. We assume a fixed ordering of these directions as first, second and so on.

2.6 Notation and conventions

(a) We can identify $[\Delta^{n^{\mathrm{op}}}, \mathcal{C}]$ with $[\Delta^{\mathrm{op}}, [\Delta^{n-1^{\mathrm{op}}}, \mathcal{C}]]$ in n different ways: thus, given an n–fold simplicial object $X \in [\Delta^{n^{\mathrm{op}}}, \mathcal{C}]$, for each $1 \le i \le n$ we write $X^{(i)} \in [\Delta^{\mathrm{op}}, [\Delta^{n-1^{\mathrm{op}}}, \mathcal{C}]]$ to indicate that the primary simplicial direction is the i^{th} one of the original X.

(b) More generally, if we choose k of the n directions $1 \le j_1 < j_2 < \cdots < j_k \le n$, we obtain a k–fold simplicial object $X^{(j_1, j_2, \dots, j_k)}$ in $[\Delta^{n-k^{\mathrm{op}}}, \mathcal{C}]$. Thus

$$X^{(j_1, j_2, \dots, j_k)} \in [\Delta^{k^{\mathrm{op}}}, [\Delta^{n-k^{\mathrm{op}}}, \mathcal{C}]]$$

is a diagram of objects $X^{(j_1, j_2, \dots, j_k)}_{i_1 \cdots i_k}$ in $[\Delta^{n-k^{\mathrm{op}}}, \mathcal{C}]$. For example, $X^{(1, \dots, k)}_{i_1 \cdots i_k} = X([i_1], \dots [i_k], -)$ in the notation of Definition 2.1.
Equivalently, for each object $a \in \Delta^{n-k}$, $X^{(j_1, j_2, \dots, j_k)}(a) \in [\Delta^{k^{\mathrm{op}}}, \mathcal{C}]$ is a k–fold simplicial object in $\mathcal{C}$, natural in a.

(c) In particular,

$$X^{(\widehat{i})} = X^{(1,\dots,i-1,i+1,\dots,n)} \in [\Delta^{n-1^{\mathrm{op}}}, [\Delta^{\mathrm{op}}, \mathcal{C}]]$$

is an $(n-1)$–fold simplicial object in $[\Delta^{\mathrm{op}}, \mathcal{C}]$ (in the i^{th} direction).

(d) Given $X \in [\Delta^{n^{\mathrm{op}}}, \mathcal{C}]$ and a functor $F\colon [\Delta^{\mathrm{op}}, \mathcal{C}] \to \mathcal{C}$, we denote by

$$F^{(k)}X \in [\Delta^{n-1^{\mathrm{op}}}, \mathcal{C}]$$

the object obtained by applying F objectwise to $X^{(\widehat{k})}$ (thought of as a $\Delta^{n-1^{\mathrm{op}}}$–indexed diagram in $[\Delta^{\mathrm{op}}, \mathcal{C}]$).
Thus for every $i_1, \dots, i_{n-1} \in \mathbb{N}$, we have

$$(F^{(k)}X)_{i_1,\dots,i_{n-1}} = FX^{(1,\dots,k-1,k+1,\dots,n)}_{i_1,\dots,i_{n-1}}.$$

(e) The composite $F^{(1)}F^{(2)}\dots F^{(n-1)}F^{(n)}$ will be denoted by $F_{(n)}\colon [\Delta^{n^{\mathrm{op}}}, \mathcal{C}] \to \mathcal{C}$.

(f) In particular, the n–fold diagonal functor $\mathrm{Diag}_{(n)}\colon [\Delta^{n^{\mathrm{op}}}, \mathcal{C}] \to [\Delta^{\mathrm{op}}, \mathcal{C}]$ is given by $(\mathrm{Diag}_{(n)}\, X)_m := X_{m,m,\dots,m}$. (In this case, the order does not matter.)

(g) For any functor $F\colon \mathcal{C} \to \mathcal{D}$, the prolongation of F to simplicial objects is denoted by $\bar{F}\colon [\Delta^{\mathrm{op}}, \mathcal{C}] \to [\Delta^{\mathrm{op}}, \mathcal{D}]$.

(h) In particular, for a functor $G\colon [\Delta^{n-1^{\mathrm{op}}}, \mathcal{C}] \to \mathcal{D}$, the result of applying G to an n–fold simplicial object $X \in [\Delta^{n^{\mathrm{op}}}, \mathcal{C}]$ in each simplicial dimension in the k^{th} direction will be denoted by $\bar{G}^{(k)}X \in [\Delta^{\mathrm{op}}, \mathcal{D}]$. Thus for every $j \in \mathbb{N}$ we have

$$(\bar{G}^{(k)}X)_j = GX^{(k)}_j.$$

2.7 Décalage Recall from Duskin [25, Section 2.6] the comonad $\mathrm{Dec}\colon [\Delta^{\mathrm{op}}, \mathsf{Set}] \to [\Delta^{\mathrm{op}}, \mathsf{Set}]$ on simplicial sets, where $(\mathrm{Dec}\, X)_n = X_{n+1}$, forgetting the *last* face and degeneracy operators in each dimension (see also Illusie [34]). The counit $\varepsilon\colon \mathrm{Dec}\, X \to X$ is given by $d_n\colon X_{n+1} \to X_n$ in simplicial dimension n. It has a section $\sigma\colon X \to \mathrm{Dec}\, X$, given by $s_n\colon X_n \to X_{n+1}$.

There is also a version forgetting the *first* face and degeneracy operators, which we denote by $\mathrm{Dec}'\colon [\Delta^{\mathrm{op}}, \mathsf{Set}] \to [\Delta^{\mathrm{op}}, \mathsf{Set}]$. In the notation of Remark 2.2, $\mathrm{Dec}'\, X := I^*\,\mathrm{Dec}\, I^*X$.

The comonad Dec yields a simplicial resolution $Y_\bullet \in [\Delta^{\mathrm{op}}, [\Delta^{\mathrm{op}}, \mathsf{Set}]]$ for any $X \in [\Delta^{\mathrm{op}}, \mathsf{Set}]$, with

$$Y_{k-1} := \mathrm{Dec}^k\, X := \underbrace{\mathrm{Dec}(\mathrm{Dec}\cdots\mathrm{Dec}\, X\cdots)}_{k} \quad \text{in } [\Delta^{\mathrm{op}}, \mathsf{Set}],$$

and the counit ε for Dec induces a map of bisimplicial sets $\varepsilon\colon Y \to c^{(2)} X$, where $c^{(2)} X$ is the constant simplicial object on X in $[\Delta^{\mathrm{op}}, [\Delta^{\mathrm{op}}, \mathsf{Set}]]$ (thinking of the outer simplicial direction of $[\Delta^{\mathrm{op}}, [\Delta^{\mathrm{op}}, \mathsf{Set}]]$ as second). The bisimplicial set $Y_\bullet$ is depicted in Figure 1, viewed as a horizontal simplicial object over $[\Delta^{\mathrm{op}}, \mathsf{Set}]$ (degeneracy maps and ε are not shown).

The corresponding resolution using Dec' is also depicted in Figure 1, viewed as a vertical simplicial object over $[\Delta^{\mathrm{op}}, \mathsf{Set}]$.

$$
\begin{tikzcd}[column sep=large, row sep=large]
\cdots \arrow[r] & X_5 \arrow[r, shift left=2, "d_5"] \arrow[r, "d_4" description] \arrow[r, shift right=2, "d_3"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, "d_1" description] \arrow[d, shift left=2, "d_2"] & X_4 \arrow[r, shift left=1, "d_4"] \arrow[r, shift right=1, "d_3"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, "d_1" description] \arrow[d, shift left=2, "d_2"] & X_3 \arrow[d, shift right=2, "d_0"'] \arrow[d, "d_1" description] \arrow[d, shift left=2, "d_2"] \\
\cdots \arrow[r] & X_4 \arrow[r, shift left=2, "d_4"] \arrow[r, "d_3" description] \arrow[r, shift right=2, "d_2"'] \arrow[d, shift right=1, "d_0"'] \arrow[d, shift left=1, "d_1"] & X_3 \arrow[r, shift left=1, "d_3"] \arrow[r, shift right=1, "d_2"'] \arrow[d, shift right=1, "d_0"'] \arrow[d, shift left=1, "d_1"] & X_2 \arrow[d, shift right=1, "d_0"'] \arrow[d, shift left=1, "d_1"] \\
\cdots \arrow[r] & X_3 \arrow[r, shift left=2, "d_3"] \arrow[r, "d_2" description] \arrow[r, shift right=2, "d_1"'] & X_2 \arrow[r, shift left=1, "d_2"] \arrow[r, shift right=1, "d_1"'] & X_1
\end{tikzcd}
$$

Figure 1: Corner of $\mathrm{Or}_{(2)} X$

2.8 Remark Note that if X is a fibrant simplicial set, then so is $\mathrm{Dec}\, X$, and the augmentation $\varepsilon\colon \mathrm{Dec}\, X \to X$ is a fibration (with section $\sigma\colon X \to \mathrm{Dec}\, X$). Similarly for Dec'.

2.9 Ordinal sum In order to produce an n–fold simplicial set out of a Kan complex $X \in [\Delta^{\mathrm{op}}, \mathsf{Set}]$, with the same homotopy type (that is, an n–fold resolution of X), we shall use the functor $\mathrm{Or}_{(n)} := \mathrm{or}_n^*\colon [\Delta^{\mathrm{op}}, \mathsf{Set}] \to [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$, induced by the ordinal sum $\mathrm{or}_n\colon \Delta^n \to \Delta$ (see Ehlers and Porter [27, Section 2]). Thus

$$(\mathrm{Or}_{(n)} X)_{p_1 \cdots p_n} := X_{n-1+p_1+\cdots+p_n}. \tag{2-10}$$

If we define $\overline{\mathrm{Or}}^{(i)}_{(n-1)}\colon [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}] \to [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$ for a bisimplicial set X by applying $\mathrm{Or}_{(n-1)}$ to X in each simplicial dimension in the i^{th} direction ($i = 1, 2$) (see Section 2.6(h)), we have

$$\mathrm{Or}_{(n)} X = \overline{\mathrm{Or}}^{(2)}_{(n-1)} \mathrm{Or}_{(2)} X. \tag{2-11}$$

See Figure 2 for a depiction of $\mathrm{Or}_{(3)} X$, where the vertical direction is first, the diagonal is second and the horizontal is third.

The bisimplicial set $\mathrm{Or}_{(2)}\, X$ appears in Figure 1: this means that if we choose the vertical direction to be first and the horizontal to be second, then

$$(\mathrm{Or}_{(2)}\, X)^{(1)}_i = \mathrm{Dec}^{i+1}\, X \quad\text{and}\quad (\mathrm{Or}_{(2)}\, X)^{(2)}_i = (\mathrm{Dec}')^{i+1} X. \tag{2-12}$$

2.13 Lemma *For any simplicial set $X \in [\Delta^{\mathrm{op}}, \mathsf{Set}]$, there is a natural weak equivalence $\varepsilon_{(n)}\colon \mathrm{Diag}_{(n)}\, \mathrm{Or}_{(n)}\, X \to X$.*

Proof By induction on $n \geq 2$.

For $n = 2$, as noted in Section 2.7, the counit $\varepsilon\colon \mathrm{Dec}\, X \to X$ induces a map of bisimplicial sets $\hat{\varepsilon}\colon \mathrm{Or}_{(2)}\, X \to \mathrm{c}^{(2)} X$ which is a weak equivalence of horizontal simplicial sets $(\mathrm{Dec}')^{i+1} X \to \mathrm{c}X_i$ (where $\mathrm{c}X_i$ is the constant simplicial set on the set X_i), using (2-12). Thus by Duskin [25, Section 2.6] it induces a weak equivalence

$$\varepsilon_{(2)}\colon \mathrm{Diag}_{(2)}\, \mathrm{Or}_{(2)}\, X \to \mathrm{Diag}_{(2)}\, \mathrm{c}^{(2)} X = X.$$

In the induction stage we have a weak equivalence

$$\varepsilon_{(n-1)}\colon \mathrm{Diag}_{(n-1)}\, \mathrm{Or}_{(n-1)}\, Y \to Y,$$

natural in Y. Using (2-11), and applying $\varepsilon_{(n-1)}$ to $\mathrm{Or}_{(n)}\, X$ in each simplicial dimension (in direction 2), we obtain a map of bisimplicial sets

$$\overline{\mathrm{Diag}}^{(2)}_{(n-1)}\, \mathrm{Or}_{(n)}\, X = \overline{\mathrm{Diag}}^{(2)}_{(n-1)} \overline{\mathrm{Or}}^{(2)}_{(n-1)}\, \mathrm{Or}_{(2)}\, X \xrightarrow{\bar{\varepsilon}^{(2)}_{(n-1)}} \mathrm{Or}_{(2)}\, X$$

which is a weak equivalence in each simplicial dimension in direction 2, by the induction hypothesis. Therefore, after applying $\mathrm{Diag}_{(2)}$ we obtain a weak equivalence of simplicial sets

$$\mathrm{Diag}_{(2)}\, \bar{\varepsilon}^{(2)}_{(n-1)}\colon \mathrm{Diag}_{(n)}\, \mathrm{Or}_{(n)}\, X \to \mathrm{Diag}_{(2)}\, \mathrm{Or}_{(2)}\, X.$$

Post-composing with $\varepsilon_{(2)}\colon \mathrm{Diag}_{(2)}\, \mathrm{Or}_{(2)}\, X \to X$ yields the required weak equivalence $\varepsilon_{(n)}\colon \mathrm{Diag}_{(n)}\, \mathrm{Or}_{(n)}\, X \to X$. □

2.B n–fold groupoids

Recall that a *groupoid* is a small category G in which all morphisms are isomorphisms. It can thus be described by a diagram of sets

$$\begin{tikzcd} G_1 \times_{G_0} G_1 G_1 \arrow[r, shift left=2, "d_0"] \arrow[r, "m"] \arrow[r, shift right=2, "d_2"'] & G_1 \arrow[l, bend right=50, "s_0"'] \arrow[l, bend left=50, "s_1"] \arrow[r, "s"] & G_0, \arrow[l, bend right=30, "i"'] \arrow[l, bend left=30, "t"] \end{tikzcd} \tag{2-14}$$

where G_0 is the set of objects of G and G_1 the set of arrows. Here s and t are the source and target functions, i associates to an object its identity map, d_0 and d_2 are the respective projections, with sections s_0 and s_1, and m is the composition; all satisfying appropriate identities. Let Gpd denote the category of small groupoids (a full subcategory of the category Cat of small categories).

We can think of (2-14) as the 2–skeleton of a simplicial set (with $G_2 := G_1 \times_{G_0} G_1$, and $d_1 = m\colon G_2 \to G_1$). The *nerve* functor $N\colon \mathsf{Gpd} \to [\Delta^{\mathrm{op}}, \mathsf{Set}]$ (see Segal [45]) assigns to G the corresponding 2–coskeletal simplicial set NG, so

$$(NG)_n := G_1 \times_{G_0} G_1 \times_{G_0} \overset{n}{\cdots} \times_{G_0} G_1 \times_{G_0} G_1 \tag{2-15}$$

for all $n \geq 2$, with face and degeneracy maps determined by the associativity and unit laws for the composition m.

2.16 Definition If $\mathcal{V}$ is any category with pullbacks, an *internal groupoid* in $\mathcal{V}$ is a diagram in $\mathcal{V}$ of the form (2-14), satisfying the same axioms (see Borceux [12, Section 8.1]). The category of internal groupoids in $\mathcal{V}$ is denoted by $\mathsf{Gpd}(\mathcal{V})$. Thus an (ordinary) groupoid is an internal groupoid in Set.

When $\mathcal{V}$ is locally finitely presentable, the nerve functor $N\colon \mathsf{Gpd}(\mathcal{V}) \to [\Delta^{\mathrm{op}}, \mathcal{V}]$ has a left adjoint, the fundamental internal groupoid (see Borceux [13, Sections 5.5–5.6]).

For each $n \geq 1$, an *n–fold groupoid* is defined inductively to be an internal groupoid in the category $\mathcal{V} = \mathsf{Gpd}^{n-1}$ of $(n-1)$–fold groupoids (where $\mathsf{Gpd}^0 := \mathsf{Set}$), so

$$\mathsf{Gpd}^n := \mathsf{Gpd}(\mathsf{Gpd}^{n-1}).$$

2.17 Definition Let $\hat{\pi}_1\colon [\Delta^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{Gpd}$ denote the *fundamental groupoid* functor. See Goerss and Jardine [31, Section I.8] and Definition 2.16 when $\mathcal{V} = \mathsf{Set}$. When X is fibrant, $\hat{\pi}_1 X$ has the simple form described in [31, Section I.8]. If $X \in [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$ is an n–fold simplicial set, then for each $1 \leq i \leq n$, $\hat{\pi}_1^{(i)} X$ is the $(n-1)$–fold simplicial object in Gpd obtained by applying the fundamental groupoid functor $\hat{\pi}_1$ in the i^{th} direction, that is, objectwise to the $\Delta^{n-1^{\mathrm{op}}}$–indexed diagram $X^{(i)}$.

2.18 Notation As in Section 2.6(d), for each $1 \leq i \leq n$, let

$$N^{(i)}\colon \mathsf{Gpd}^n \to [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-1}]$$

denote the nerve functor in the i^{th} direction. More generally, for any k of the n indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, $N^{(i_1, i_2, \ldots, i_k)}\colon \mathsf{Gpd}^n \to [\Delta^{k^{\mathrm{op}}}, \mathsf{Gpd}^{n-k}]$ takes an n–fold groupoid G to a k–fold simplicial object in $(n-k)$–fold groupoids by applying the nerve functor in the indicated k directions. Thus $N^{(\hat{i})}$ means that we take nerves in all but the i^{th} direction.

2.19 Definition The *multinerve*

$$N_{(n)}\colon \mathsf{Gpd}^n \to [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$$

is defined by applying $N^{(i)}$ for $1 \le i \le n$ to obtain the n–fold simplicial set $N_{(n)}G := N^{(1)}N^{(2)}\cdots N^{(n)}G$. We say that an n–fold groupoid G is *discrete* if $N_{(n)}G$ is a constant n–fold simplicial set. It is readily verified that we have an adjoint pair $P_{(n)} \dashv N_{(n)}$

(2-20) $$[\Delta^{n^{\mathrm{op}}}, \mathsf{Set}] \underset{P_{(n)}}{\overset{N_{(n)}}{\rightleftarrows}} \mathsf{Gpd}^n ,$$

where $P_{(n)}$ is the left adjoint to $N_{(n)}$ as in Definition 2.16 with $\mathcal{V} = \mathsf{Gpd}^{n-1}$.

2.21 Definition The composite of $N_{(n)}$ with $\mathrm{Diag}_{(n)}$ (see Section 2.6(f)) yields the *diagonal nerve* functor $dN := \mathrm{Diag}_{(n)} N_{(n)}$, and its geometric realization $BG := \|dNG\| \in \mathsf{Top}$ is called the *classifying space* of G.

A map of n–fold groupoids $f\colon G \to G'$ is called a *geometric weak equivalence* if it induces a weak equivalence of simplicial sets $dNf\colon dNG \to dNG'$ (that is, a homotopy equivalence of topological spaces on geometric realizations $Bf\colon BG \to BG'$).

2.22 Remark Since the diagonal of a bisimplicial set is its homotopy colimit, a map $f\colon X \to Y$ in $[\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]$ which is a weak equivalence $f_k\colon X_k \to Y_k$ in each simplicial dimension $k \ge 0$ is a geometric weak equivalence (see [31, IV, Proposition 1.7]). Thus by induction the same is true for a map $f\colon X \to Y$ in $[\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$ which is a geometric weak equivalence in each simplicial dimension.

2.23 Definition If $G \in \mathsf{Gpd}^{n-1}$ is an $(n-1)$–fold groupoid, then $\mathrm{c}^{(n)}G$ denotes the n–fold groupoid which, as a groupoid object in Gpd^{n-1}, is discrete on G. In particular, if A is a set, $A^d_{(n)}$ denotes the discrete n–fold groupoid $\mathrm{c}^{(1)}\cdots\mathrm{c}^{(n)}A$ on A. For an n–fold groupoid G we let G^d denote the discrete n–fold groupoid $(\pi_0 BG)^d_{(n)}$.

2.24 Notation If $G \in \mathsf{Gpd}^n$ is an n–fold groupoid for $n \ge 2$, it is a groupoid object in $(n-1)$–fold groupoids (see Definition 2.16): that is, it is described by a diagram $G^{(1)}_1 \rightrightarrows G^{(1)}_0$ in Gpd^{n-1}, as in (2-14). Thus it has an $(n-1)$–fold groupoid of objects denoted by $G^{(1)}_0$, in the notation of Section 2.6(a) (which in turn has its $(n-2)$–fold groupoid of objects $G^{(1,2)}_{00}$ and $(n-2)$–fold groupoid of morphisms $G^{(1,2)}_{01}$). Similarly, the $(n-1)$–fold groupoid of morphisms of G (in the first direction) is denoted $G^{(1)}_1$.

More explicitly, G may be described by a diagram in Gpd^{n-2} of the form:

$$\begin{tikzcd}
 & G_{11}\times_{G_{10}}G_{11} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "c^{1*}"] & G_{01}\times_{G_{00}}G_{01} \arrow[d, "c^{0*}"] \\
G_{11}\times_{G_{01}}G_{11} \arrow[r, "c^{*1}"] \arrow[d, shift left] \arrow[d, shift right] & G_{11} \arrow[r, shift left, "d_0^{*1}"] \arrow[r, shift right, "d_1^{*1}"'] \arrow[d, shift right, "d_1^{1*}"'] \arrow[d, shift left, "d_0^{1*}"] & G_{01} \arrow[l, bend right] \arrow[d, shift right, "d_1^{0*}"'] \arrow[d, shift left, "d_1^{0*}"] \\
G_{10}\times_{G_{00}}G_{10} \arrow[r, "c^{*0}"] & G_{10} \arrow[u, bend left] \arrow[r, shift left, "d_0^{*0}"] \arrow[r, shift right, "d_1^{*0}"'] & G_{00} \arrow[l, bend left] \arrow[u, bend right]
\end{tikzcd} \tag{2-25}$$

Here we omit throughout the upper index $(1,2)$, which indicates that we are showing only the first two directions of G.

More generally, for each $i \geq 2$ we let

$$G_{i1} := G_{11}\times_{G_{01}} \overset{i}{\cdots} \times_{G_{01}} G_{11} \quad \text{and} \quad G_{i0} := G_{10}\times_{G_{00}} \overset{i}{\cdots} \times_{G_{00}} G_{10} \tag{2-26}$$

as limits of $(n-2)$–fold groupoids, with $d_0^{i,*}, d_1^{i,*}\colon G_{i1} \to G_{i0}$ induced by the source and target maps.

2.27 Remark Using this convention, an n–fold groupoid G may be thought of as a diagram of sets with objects $G_{i_1,\dots,i_n}$ for each $(i_1,\dots,i_n) \in \mathbb{N}^n = \mathrm{Obj}\,(\Delta^n)$, where all the maps in the diagram are induced by those of (2-14) and the structure maps for the limits (2-26) (in each of the n directions).

The following technical fact about $\mathrm{Or}_{(n)}$ will be used in Section 4.A below.

2.28 Lemma *For any fibrant simplicial set $X \in [\Delta^{\mathrm{op}}, \mathsf{Set}]$ and $n \geq 2$, we have*

$$\overline{\mathrm{Or}}^{(2)}_{(n-1)} N^{(2)} \hat{\pi}_1^{(2)} \,\mathrm{Or}_{(2)}\, X = N^{(n)} \hat{\pi}_1^{(n)}\, \mathrm{Or}_{(n)}\, X. \tag{2-29}$$

Proof By induction on $n \geq 2$.

When $n = 2$, $\mathrm{Or}_{(n-1)}$ is the identity, so both sides of (2-29) are the same.

For $n = 3$, $\hat{\pi}_1^{(3)}\,\mathrm{Or}_{(3)}\, X$ is obtained from Figure 2 by replacing the left-hand square by

$$\begin{tikzcd}
X_5/{\sim} \arrow[r, shift left, "d_3"] \arrow[r, shift right, "d_2"'] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & X_4/{\sim} \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
X_4/{\sim} \arrow[r, shift left, "d_2"] \arrow[r, shift right, "d_1"'] & X_3/{\sim}
\end{tikzcd}$$

$$\begin{tikzcd}
X_5 \arrow[rr, shift left, "d_5"] \arrow[rr, shift right, "d_4"'] \arrow[dr, shift left, "d_3"] \arrow[dr, shift right, "d_2"'] \arrow[dd, shift right, "d_0"'] \arrow[dd, shift left, "d_1"] & & X_4 \arrow[dr, shift left, "d_3"] \arrow[dr, shift right, "d_2"'] \arrow[dd, shift right, "d_0"'] \arrow[dd, shift left, "d_1"] & \\
& X_4 \arrow[rr, shift left, "d_4"] \arrow[rr, shift right, "d_3"'] \arrow[dd, shift right, "d_0"'] \arrow[dd, shift left, "d_1"] & & X_3 \arrow[dd, shift right, "d_0"'] \arrow[dd, shift left, "d_1"] \\
X_4 \arrow[rr, shift left, "d_4"] \arrow[rr, shift right, "d_3"'] \arrow[dr, shift left, "d_2"] \arrow[dr, shift right, "d_1"'] & & X_3 \arrow[dr, shift left, "d_2"] \arrow[dr, shift right, "d_1"'] & \\
& X_3 \arrow[rr, shift left, "d_3"] \arrow[rr, shift right, "d_2"'] & & X_2
\end{tikzcd}$$

Figure 2: Corner of $\operatorname{Or}_{(3)} X$

and from Figure 1 we see this is the same as first applying $N\widehat{\pi}_1$ to $\operatorname{Or}_{(2)} X$ horizontally in each vertical dimension (which is $N^{(2)}\widehat{\pi}_1^{(2)}$), and then taking $\operatorname{Or}_{(2)}$ vertically in each horizontal dimension (which is $\overline{\operatorname{Or}}^{(2)}_{(2)}$).

For $n \geq 4$, we see that

$$N^{(n)}\widehat{\pi}_1^{(n)} \operatorname{Or}_{(n)} X = N^{(n)}\widehat{\pi}_1^{(n)}\overline{\operatorname{Or}}^{(2)}_{(n-1)} \operatorname{Or}_{(2)} X = \overline{N^{(n-1)}\widehat{\pi}_1^{(n-1)} \operatorname{Or}_{(n-1)}}^{(2)} \operatorname{Or}_{(2)} X$$

using (2-11) and the convention of Section 2.6(d). Applying the induction hypothesis (2-29) for $n-1$, we see this is equal to

$$\overline{\overline{\operatorname{Or}}^{(2)}_{(n-2)} N^{(2)}\widehat{\pi}_1^{(2)} \operatorname{Or}_{(2)}}^{(2)} \operatorname{Or}_{(2)} X = \overline{\overline{\operatorname{Or}}^{(2)}_{(n-2)} N^{(2)}\widehat{\pi}_1^{(2)}}^{(2)} \overline{\operatorname{Or}}^{(2)}_{(2)} \operatorname{Or}_{(2)} X,$$

and using (2-11) for $n = 3$, we see this is

$$\overline{\overline{\operatorname{Or}}^{(2)}_{(n-2)} N^{(2)}\widehat{\pi}_1^{(2)}}^{(3)} \operatorname{Or}_{(3)} X = \overline{\overline{\operatorname{Or}}^{(2)}_{(n-2)}}^{(3)} N^{(3)}\widehat{\pi}_1^{(3)} \operatorname{Or}_{(3)} X,$$

where for a 3–fold simplicial object Z, we have

$$\overline{\overline{\operatorname{Or}}^{(2)}_{(n-2)} N^{(2)}\widehat{\pi}_1^{(2)}}^{(3)} Z = \overline{\overline{\operatorname{Or}}^{(2)}_{(n-2)} N^{(2)}\widehat{\pi}_1^{(2)}}^{(2)} Z$$

by our indexing convention (Section 2.6(h)).

Now applying (2-29) for $n = 3$, we see this equals

$$\overline{\overline{\mathrm{Or}}^{(2)}_{(n-2)}}^{(2)}\, \overline{\mathrm{Or}}^{(2)}_{(2)} N^{(2)} \hat{\pi}_1^{(2)} \mathrm{Or}_{(2)} X = \overline{\mathrm{Or}}^{(2)}_{(n-1)} N^{(2)} \hat{\pi}_1^{(2)} \mathrm{Or}_{(2)} X,$$

using (2-11) once more. □

2.C The fundamental n–fold groupoid of a space

We now introduce the central construction of our paper. Its internal analogue in the category of groups is the fundamental cat^n–group of a space, due to Bullejos, Cegarra and Duskin [19, Section 2].

2.30 Definition We define $Q_{(n)}\colon [\Delta^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{Gpd}^n$ to be the composite

$$[\Delta^{\mathrm{op}}, \mathsf{Set}] \xrightarrow{\mathrm{Or}_{(n)}} [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}] \xrightarrow{P_{(n)}} \mathsf{Gpd}^n,$$

for $P_{(n)}$ the left adjoint to $N_{(n)}$ of (2-20). We define $\hat{Q}_{(n)}\colon \mathsf{Top} \to \mathsf{Gpd}^n$ to be the composite

$$\mathsf{Top} \xrightarrow{\mathcal{S}} [\Delta^{\mathrm{op}}, \mathsf{Set}] \xrightarrow{Q_{(n)}} \mathsf{Gpd}^n,$$

where $\mathcal{S}\colon \mathsf{Top} \to [\Delta^{\mathrm{op}}, \mathsf{Set}]$ is the singular set functor (see Goerss and Jardine [31, Section I.1], and call $\hat{Q}_{(n)}T$ the *fundamental n–fold groupoid* of $T \in \mathsf{Top}$.

We shall show that if $Y \in [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$ satisfies certain fibrancy conditions, then $P_{(n)}Y$ has a particularly simple form. These require that a certain 2–dimensional notion of fibrancy (introduced in [11, Section 2]) hold in every bisimplicial bidirection. They hold for $Y = \mathrm{Or}_{(n)} X$ when X is fibrant (in particular, for $X = \mathcal{S}T$), leading to a simple expression for $\hat{Q}_{(n)}T$ in Theorem 2.40 below.

2.31 Definition Let $n \geq 2$. An n–fold simplicial set $X \in [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$ is called $(n,2)$–*fibrant* if for each $1 \leq i \neq j \leq n$ and $a \in \Delta^{n-2}$, the bisimplicial set Y obtained by applying the 2–coskeleton functor to each vertical simplicial set $X^{(i,j)}(a)_{k\bullet}$ – that is, $Y_{k\bullet} := \mathrm{csk}_2\, X^{(i,j)}(a)_{k\bullet}$ – is a Kan complex for $k = 0, 1, 2$, and the horizontal face map $d_0\colon Y_{1\bullet} \to Y_{0\bullet}$ is a fibration in $[\Delta^{\mathrm{op}}, \mathsf{Set}]$.

2.32 Definition Let $G \in [\Delta^{m^{\mathrm{op}}}, \mathsf{Gpd}^{n-m}]$ be an m–fold simplicial object in $(n-m)$–fold groupoids (see Definition 2.16). We say that G is $(n,2)$–*fibrant* if, after applying the nerve functor in each of the $n-m$ groupoid directions, the resulting n–fold simplicial set $N^{(1,2,\ldots,n-m)}G \in [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$ is $(n,2)$–fibrant in the sense of Definition 2.31.

We recall the following results from our earlier paper [11] where the left adjoint $P^{(1)}\colon [\Delta^{\mathrm{op}}, \mathsf{Gpd}] \to \mathsf{Gpd}^2$ to the nerve $N^{(1)}\colon \mathsf{Gpd}^2 \to [\Delta^{\mathrm{op}}, \mathsf{Gpd}]$, is as described in Definition 2.16 with $\mathcal{V} = \mathsf{Gpd}$.

2.33 Proposition [11, Proposition 2.10] *The left adjoint $P^{(1)}\colon [\Delta^{\mathrm{op}}, \mathsf{Gpd}] \to \mathsf{Gpd}^2$ to the nerve $N^{(1)}\colon \mathsf{Gpd}^2 \to [\Delta^{\mathrm{op}}, \mathsf{Gpd}]$, when applied to a $(2,2)$–fibrant simplicial groupoid $G_\bullet$, is $\widehat{\pi}_1^{(1)} G_\bullet$ (that is, the functor $\widehat{\pi}_1$ applied in the simplicial direction).*

2.34 Proposition [11, Proposition 2.11] *If $X \in [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]$ is a $(2,2)$–fibrant bisimplicial set, then $\widehat{\pi}_1^{(1)} X$ is a $(2,2)$–fibrant simplicial groupoid.*

2.35 Lemma *If $G_\bullet$ is a $(2,2)$–fibrant simplicial groupoid (with simplicial sets of objects $G_{\bullet 0}$ and morphisms $G_{\bullet 1}$), then $N^{(2)} \widehat{\pi}_1^{(1)} G_\bullet = \widehat{\pi}_1^{(1)} N^{(2)} G_\bullet$.*

Proof It suffices to show that, for each $k \geq 2$,

$$\widehat{\pi}_1\big(G_{\bullet 1} \times_{G_{\bullet 0}} \overset{k}{\cdots} \times_{G_{\bullet 0}} G_{\bullet 1}\big) \cong \widehat{\pi}_1(G_{\bullet 1}) \times_{\widehat{\pi}_1(G_{\bullet 0})} \overset{k}{\cdots} \times_{\widehat{\pi}_1(G_{\bullet 0})} \widehat{\pi}_1(G_{\bullet 1}). \tag{2-36}$$

Since both sides are groupoids, we evidently have equality on objects, and (2-36) holds on morphisms by [11, Appendix A, following (8.13)]. □

2.37 Lemma *If $X \in [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$ is $(n,2)$–fibrant, then $\widehat{\pi}_1^{(k)} X$ is $(n,2)$–fibrant.*

Proof By definition of $(n,2)$–fibrancy, for each $a \in \Delta^{n-2}$ and $1 \leq i \neq j \leq n$, the bisimplicial set $X^{(i,j)}(\,)$ satisfies the hypotheses of Proposition 2.34. Hence, applying $\widehat{\pi}_1^{(k)}$ to it yields an $(n,2)$–fibrant object of $[\Delta^{n-2^{\mathrm{op}}}, [\Delta^{\mathrm{op}}, \mathsf{Gpd}]]$. □

2.38 Proposition *For each $1 \leq i \leq n$, $P^{(i)}\colon [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-1}] \to \mathsf{Gpd}^n$, the left adjoint to $N^{(i)}\colon \mathsf{Gpd}^n \to [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-1}]$ of (2-20), when applied to an $(n,2)$–fibrant simplicial $(n{-}1)$–fold groupoid X, is given by $P^{(i)} X = \widehat{\pi}_1^{(i)} X$.*

Proof We think of the simplicial direction of X as being the i^{th}, and let $1 \leq j \leq n$ be one of the groupoidal directions (so $i \neq j$). Applying the $(n{-}2)$–fold iterated nerve functor

$$N^{(\widehat{i}\widehat{j})}\colon [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-1}] \to [\Delta^{\mathrm{op}}, [\Delta^{n-2^{\mathrm{op}}}, \mathsf{Gpd}]] \cong [\Delta^{n-2^{\mathrm{op}}}, [\Delta^{\mathrm{op}}, \mathsf{Gpd}]]$$

of Notation 2.18 (in all but the i and j directions) to X yields an $(n{-}2)$–fold simplicial object in simplicial groupoids $\widetilde{X}$. Since X is $(n,2)$–fibrant, for each $a \in \Delta^{n-2}$, the simplicial groupoid $\widetilde{X}(a)$ (see Section 2.6(b)) satisfies the hypotheses of Proposition 2.33, where the simplicial direction is the original i and the groupoid direction is the original j. Using [11, (8.12)], we can therefore define a composition map

$$\big(N^{(i)}\hat{\pi}_1^{(i)}\tilde{X}(a)\big)_1 \times_{(N^{(i)}\hat{\pi}_1^{(i)}\tilde{X}(a))_0} \big(N^{(i)}\hat{\pi}_1^{(i)}\tilde{X}(a)\big)_1 \longrightarrow \big(N^{(i)}\hat{\pi}_1^{(i)}\tilde{X}(a)\big)_1.$$

As the construction is functorial in $a \in \Delta^{n-2^{\mathrm{op}}}$, it defines a map in Gpd^{n-1}, since it consists of maps in sets commuting with compositions in each of the different directions (see [11, Appendix A]). Thus $\hat{\pi}_1^{(i)}X$ is a groupoid object in Gpd^{n-1}, that is, $\hat{\pi}_1^{(i)}X \in \mathsf{Gpd}^n$.

It remains to show that $\hat{\pi}_1^{(i)}X = P^{(i)}X$. Since the (iterated) nerve functor is fully faithful, again using Proposition 2.33, we see that for any n–fold groupoid Y we have natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Gpd}^n}\big(\hat{\pi}_1^{(i)}X, Y\big) &\cong \mathrm{Hom}_{[\Delta^{n-2^{\mathrm{op}}},[\Delta^{\mathrm{op}},\mathsf{Gpd}]]}\big(\hat{\pi}_1^{(i)}\tilde{X}, \tilde{Y}\big)\\ &= \mathrm{Hom}_{[\Delta^{n-2^{\mathrm{op}}},[\Delta^{\mathrm{op}},\mathsf{Gpd}]]}\big(\tilde{X}, N^{(i)}\tilde{Y}\big)\\ &= \mathrm{Hom}_{[\Delta^{\mathrm{op}},\mathsf{Gpd}^{n-1}]}\big(X, N^{(i)}Y\big).\end{aligned}$$

Hence $\hat{\pi}_1^{(i)}$ is left adjoint to $N^{(i)}$, as required. □

2.39 Proposition *If $X \in [\Delta^{\mathrm{op}}, \mathsf{Set}]$ is a Kan complex, then $\mathrm{Or}_{(n)}\,X$ (see Section 2.9) is $(n,2)$–fibrant.*

See Appendix A for the proof.

2.40 Theorem *The functor $Q_{(n)}$ of Definition 2.30, applied to a Kan complex $X \in [\Delta^{\mathrm{op}}, \mathsf{Set}]$, is*

$$Q_{(n)}X = \hat{\pi}_1^{(1)}\hat{\pi}_1^{(2)}\cdots\hat{\pi}_1^{(n)}\,\mathrm{Or}_{(n)}\,X.$$

Proof We prove the theorem by induction on $n \geq 2$. For $n = 2$, see [11, Corollary 2.12]. Suppose the claim holds for $n-1$. The left adjoint $P_{(n)}\colon [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}] \to \mathsf{Gpd}^n$ to $N_{(n)}$ is the composite

$$[\Delta^{n^{\mathrm{op}}}, \mathsf{Set}] \cong [\Delta^{\mathrm{op}}, [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Set}]] \xrightarrow{\bar{P}^{(1)}_{(n-1)}} [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-1}] \xrightarrow{P^{(1)}} \mathsf{Gpd}^n,$$

where $\bar{P}^{(1)}_{(n-1)}$ is induced by applying $P_{(n-1)}$ in each dimension in the first simplicial direction, and $P^{(1)}$ is left adjoint to the nerve $N^{(1)}\colon \mathsf{Gpd}^n \to [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-1}]$. By the induction hypothesis and (2-11),

$$\begin{aligned}\bar{P}^{(1)}_{(n-1)}\,\mathrm{Or}_{(n)}\,X = \bar{P}^{(1)}_{(n-1)}\overline{\mathrm{Or}}^{(2)}_{(n-1)}\,\mathrm{Or}_{(2)}\,X = \bar{Q}_{(n-1)}\,\mathrm{Or}_{(2)}\,X &\\ = \hat{\pi}_1^{(2)}\cdots\hat{\pi}_1^{(n)}\,\overline{\mathrm{Or}}^{(2)}_{(n-1)}\,\mathrm{Or}_{(2)}\,X &\\ = \hat{\pi}_1^{(2)}\cdots\hat{\pi}_1^{(n)}\,\mathrm{Or}_{(n)}\,X.&\end{aligned}$$

Since X is a Kan complex, $\operatorname{Or}_{(n)} X$ is $(n,2)$–fibrant by Proposition 2.39. Therefore, by Lemma 2.37, $\widehat{\pi}_1^{(2)} \cdots \widehat{\pi}_1^{(n)} \operatorname{Or}_{(n)} X$ is $(n,2)$–fibrant. It follows by Proposition 2.38 that

$$P^{(1)} \widehat{\pi}_1^{(2)} \cdots \widehat{\pi}_1^{(n)} \operatorname{Or}_{(n)} X = \widehat{\pi}_1^{(1)} \widehat{\pi}_1^{(2)} \cdots \widehat{\pi}_1^{(n)} \operatorname{Or}_{(n)} X.$$

Therefore

$$\begin{aligned} Q_{(n)} X = P_{(n)} \operatorname{Or}_{(n)} X = P^{(1)}\, \overline{P}_{(n-1)} \operatorname{Or}_{(n)} X &= P^{(1)} \widehat{\pi}_1^{(2)} \cdots \widehat{\pi}_1^{(n)} \operatorname{Or}_{(n)} X \\ &= \widehat{\pi}_1^{(1)} \widehat{\pi}_1^{(2)} \cdots \widehat{\pi}_1^{(n)} \operatorname{Or}_{(n)} X, \end{aligned}$$

which concludes the induction step. □

2.41 Remark The functor $\operatorname{Or}_{(n)}\colon [\Delta^{\mathrm{op}}, \mathrm{Set}] \to [\Delta^{n^{\mathrm{op}}}, \mathrm{Set}]$ has a right adjoint, a generalized Artin–Mazur codiagonal (see Artin and Mazur [1, Section III], Bullejos, Cegarra and Duskin [19] and Cabello and Garzón [20]), so both $\operatorname{Or}_{(n)}$ and $P_{(n)}$ – and thus $Q_{(n)}$ – preserve colimits, and in particular coproducts.

On the other hand, clearly $\operatorname{Or}_{(n)}$ and $\widehat{\pi}_1$ preserve products when applied to Kan complexes, so $Q_{(n)}$ does, too. Therefore, $Q_{(n)}$ preserves fiber products over discrete simplicial sets.

3 Weakly globular n–fold groupoids

We now introduce the central notion of this paper: that of a weakly globular n–fold groupoid. We will show in the next section that the fundamental n–fold groupoid $\widehat{Q}_{(n)} T$ of a space T (see Definition 2.30) is such an object.

3.A Homotopically discrete n–fold groupoids

A homotopically discrete groupoid G is one in which there is at most one arrow between every two objects (that is, all automorphism groups are trivial). Hence its classifying space is homotopically trivial (that is, a disjoint union of contractible spaces; that is, a 0–type). For such a G, the set of arrows G_1 is simply $G_0 \times_{\pi_0 G} G_0$.

In order to provide a higher-dimensional analogue of this notion, we observe that this construction can be made in any category with suitable (co)limits, so we can iterate it. For this purpose we make the following definition.

3.1 Definition Let $f\colon A \to B$ be a morphism in a category $\mathcal{C}$ with finite limits. The diagonal map defines a unique section $s\colon A \to A \times_B A$ (so that $p_1 s = \mathrm{Id}_A = p_2 s$, where $A \times_B A$ is the pullback of

$$A \xrightarrow{f} B \xleftarrow{f} A$$

and $p_1, p_2\colon A{\times_B}A \to A$ are the two projections). The commutative diagram

$$\begin{tikzcd} A{\times_B}A \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"'] & A{\times_B}A \arrow[l, "p_2"'] \arrow[d, "p_1"] \\ A \arrow[r, "f"'] & B & A \arrow[l, "f"] \end{tikzcd}$$

defines a unique morphism $m\colon (A{\times_B}A){\times_A}(A{\times_B}A) \to A{\times_B}A$ such that $p_2 m = p_2\pi_2$ and $p_1 m = p_1\pi_1$, where π_1 and π_2 are the two projections. We denote by A^f the following object of $\mathsf{Cat}(\mathscr{C})$:

(3-2)
$$\begin{tikzcd} (A{\times_B}A){\times_A}(A{\times_B}A) \arrow[r, "m"] & A{\times_B}A \arrow[r, bend left, "p_2"] \arrow[r, "p_1"] & A \arrow[l, bend left, "s"] \end{tikzcd}$$

It is easy to see that A^f is an internal groupoid.

3.3 Definition We define a full subcategory $\mathsf{Gpd}^n_{\mathrm{hd}} \subset \mathsf{Gpd}^n$ of *homotopically discrete n–fold groupoids* by induction on $n \geq 1$:

A groupoid is called *homotopically discrete* if $G \cong A^f$ for some surjective map of sets $f\colon A \to B$. In general, an n–fold groupoid $G \in \mathsf{Gpd}^n$ is *homotopically discrete* if $G \cong A^f$ for some map $f\colon A \to B$ in $\mathsf{Gpd}^{n-1}_{\mathrm{hd}}$ with a section $f'\colon B \to A$ (that is, $f \circ f' = \mathrm{Id}_B$).

As noted above, for an (ordinary) groupoid G this just means that $\pi_1(BG, x) = 0$ for any $x \in G_0$.

3.4 Remark Note that the category $\mathsf{Gpd}^n_{\mathrm{hd}}$ is closed under pullbacks. We show this by induction on n. When $n = 1$, let $f\colon A \to B$, $f'\colon A' \to B'$ and $g\colon C \to D$ be surjections in Set. Then

$$A^f{\times_{C^g}}A'^{f'} = (A{\times_C}A')^{(f,f')}, \tag{3-5}$$

where $(f, f')\colon A{\times_C}A \to A'{\times_{C'}}A'$ is a surjection in Set. Thus $A^f{\times_{C^g}}A'^{f'} \in \mathsf{Gpd}_{\mathrm{hd}}$.

Suppose the statement holds for $n-1$, and let $f'\colon A' \to B'$ and $g\colon C \to D$ be maps with sections in $\mathsf{Gpd}^{n-1}_{\mathrm{hd}}$. Then $(f, f')\colon A{\times_C}A \to A'{\times_{C'}}A'$ is a map in $\mathsf{Gpd}^{n-1}_{\mathrm{hd}}$ with a section, by the inductive hypothesis, and (3-5) holds, showing that

$$A^f{\times_{C^g}}A'^{f'} \in \mathsf{Gpd}^n_{\mathrm{hd}}.$$

3.6 Example Given a commuting (inner) square of sets

$$
\begin{tikzcd}
A \arrow[r, "f"'] \arrow[d, "g"] & B \arrow[l, bend right, "f'"'] \arrow[d, "h"'] \\
C \arrow[r, "\ell"] \arrow[u, bend left, "g'"] & D \arrow[l, bend left, "\ell'"] \arrow[u, bend right, "h'"']
\end{tikzcd}
\tag{3-7}
$$

with $ff' = \mathrm{Id}_B$, $gg' = \mathrm{Id}_C$, $hh' = \ell\ell' = \mathrm{Id}_D$ and $fg' = h'\ell$, we obtain a morphism of homotopically discrete groupoids $v\colon A^f \to C^\ell$. The homotopically discrete double groupoid G associated to v is described in Figure 3, where we abbreviate $(A{\times_B}A){\times_{(C\times_D C)}}(A{\times_B}A)$ by $(A{\times_B}A){\times_{(g,g)}}(A{\times_B}A)$ and so on.

$$
\begin{tikzcd}[column sep=small]
 & (A{\times_B}A{\times_B}A){\times_{(g,g,g)}}(A{\times_B}A{\times_B}A) \arrow[r, shift left] \arrow[r, shift left=2] \arrow[d] & A{\times_B}A{\times_B}A \arrow[l, shift left] \arrow[d] \\
(A{\times_B}A){\times_{(g,g)}}(A{\times_B}A){\times_{(g,g)}}(A{\times_B}A) \arrow[r] \arrow[d, shift right] \arrow[d, shift right=2] & (A{\times_B}A){\times_{(g,g)}}(A{\times_B}A) \arrow[r, shift left] \arrow[r, shift left=2] \arrow[d, shift right] \arrow[d, shift right=2] & A{\times_B}A \arrow[l, shift left] \arrow[d, shift right] \arrow[d, shift right=2] \\
A{\times_C}A{\times_C}A \arrow[r] \arrow[u, shift right] & A{\times_C}A \arrow[r, shift left] \arrow[r, shift left=2] \arrow[u, shift right] & A \arrow[l, shift left] \arrow[u, shift right]
\end{tikzcd}
$$

Figure 3: A homotopically discrete double groupoid

Note that

$$(A{\times_B}A){\times_{(g,g)}}(A{\times_B}A) \cong (A{\times_C}A){\times_{(f,f)}}(A{\times_C}A)$$

via the map $(a,b,c,d) \mapsto (a,c,b,d)$, and more generally

$$(A{\times_B}A){\times_{(g,g)}} \overset{k}{\cdots} {\times_{(g,g)}}(A{\times_B}A) \cong (A{\times_C}A){\times_{(f,f)}} \overset{k}{\cdots} {\times_{(f,f)}}(A{\times_C}A)$$

for each $k \geq 2$. It follows that

$$
(N^{(1)}G)_{k-1} = \begin{cases} A^f & \text{if } k = 1, \\ (A{\times_C} \overset{k}{\cdots} {\times_C}A)^{(f,\dots,f)} & \text{if } k \geq 2, \end{cases}
\tag{3-8}
$$

$$
(N^{(2)}G)_{k-1} = \begin{cases} A^g & \text{if } k = 1, \\ (A{\times_B} \overset{k}{\cdots} {\times_B}A)^{(g,\dots,g)} & \text{if } k \geq 2. \end{cases}
\tag{3-9}
$$

Therefore $(N^{(1)}G)_k$ and $(N^{(2)}G)_k$ are homotopically discrete groupoids for all $k \geq 0$. Moreover, applying π_0 vertically to each column in Figure 3 yields the groupoid B^h, that is,

$$
\begin{tikzcd}
B{\times_D}B{\times_D}B \arrow[r] & B{\times_D}B \arrow[r, shift left] \arrow[r, shift left=2] & B. \arrow[l, shift left]
\end{tikzcd}
\tag{3.10}
$$

Similarly, applying π_0 horizontally in each row yields C^ℓ.

3.11 Remark The construction of Example 3.6 makes sense in any category with enough limits. Conversely, any map $v\colon A^f \to C^\ell$ with a section v' has a map of objects $g\colon A \to C$ and induces a map $h\colon B \to D$ on π_0, which fits into a commuting square as in (3-7).

3.12 Definition Recall from Section 2.6(g) that if $X \in [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$ is an $(n-1)$–fold simplicial object in groupoids, $\bar{\pi}_0^{(n)} X$ is the $(n-1)$–fold simplicial set obtained by applying π_0 (the coequalizer of the source and target maps of the groupoid) in each $(n-1)$–fold simplicial dimension of X. If $\mathrm{c}X$ denotes the discrete groupoid on a set X (see Definition 2.23), $c\colon \mathsf{Set} \to \mathsf{Gpd}$ is right adjoint to π_0, and the unit of the adjunction $\gamma\colon \mathrm{Id} \to c\pi_0$ induces a natural transformation of $(n-1)$–simplicial groupoids

$$\bar{\gamma}\colon X \to \bar{c}^{(n)}\, \bar{\pi}_0^{(n)} X.$$

3.13 Remark Let $G \in \mathsf{Gpd}^n$ be an n–fold groupoid and

$$X = N^{(n-1)} \cdots N^{(1)} G \in [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]\,.$$

Let us suppose that $\bar{\pi}_0^{(n)} X$ is the multinerve of an $(n-1)$–fold groupoid, denoted by $\Pi_0^{(n)} G$, so that

$$\bar{\pi}_0^{(n)} X = N_{(n-1)} \Pi_0^{(n)} G\ .$$

Then $\bar{c}^{(n)}\, \bar{\pi}_0^{(n)} X$ is the multinerve of an n–fold groupoid $\mathrm{c}^{(n)} \Pi_0^{(n)} G$ (discrete in the new n^{th} direction) and

$$\bar{\gamma} = N^{(n-1)} \cdots N^{(1)} \gamma^{(n)}$$

for a map of n–fold groupoids $\gamma^{(n)}\colon G \to \mathrm{c}^{(n)} \Pi_0^{(n)} G$.

3.14 Remark Since $\pi_0\colon \mathsf{Gpd} \to \mathsf{Set}$ preserves products and coproducts, it preserves fiber products over discrete groupoids. Therefore, the same is true of $\bar{\pi}_0^{(n)}$.

3.15 Lemma *Let $G \in \mathsf{Gpd}^n_{\mathrm{hd}}$ be a homotopically discrete n–fold groupoid. Then:*

(a) *If $N^{(i)}\colon \mathsf{Gpd}^n \to [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-1}]$ for some $1 \le i \le n$ is as in Notation 2.18, then $(N^{(i)} G)_k$ is homotopically discrete for all $k \ge 0$.*

(b) *The $(n-1)$–simplicial set $\overline{\pi}_0^{(n)} N^{(n-1)} \cdots N^{(1)} G$ is the multinerve of a homotopically discrete $(n-1)$–fold groupoid $\Pi_0^{(n)} G$, and there is a commutative diagram:*

$$\begin{tikzcd}
\mathsf{Gpd}^n_{\mathrm{hd}} \arrow[r, "N^{(n-1)} \dots N^{(1)}"] \arrow[d, "\Pi_0^{(n)}"'] & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]} \arrow[d, "\overline{\pi}_0^{(n)}"] \\
\mathsf{Gpd}^{n-1}_{\mathrm{hd}} \arrow[r, "N_{(n-1)}"] & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Set}]}
\end{tikzcd}$$

(c) *The map of n–fold groupoids $\gamma^{(n)}\colon G \to \mathrm{c}^{(n)} \Pi_0^{(n)} G$ of Remark 3.13 is a geometric weak equivalence (Definition 2.21).*

(d) *The set $\Pi_0^{(1)} \cdots \Pi_0^{(n)} G$ is isomorphic to $\pi_0 BG$ (see Definition 2.21).*

(e) *If we let $\gamma_{(n)}$ denote the composite*

$$G \xrightarrow{\gamma^{(n)}} \mathrm{c}^{(n)} \Pi_0^{(n)} G \xrightarrow{\mathrm{c}^{(n)} \gamma^{(n-1)}} \mathrm{c}^{(n-1)}\, \mathrm{c}^{(n)} \Pi_0^{(n-1)} \Pi_0^{(n)} G \longrightarrow \cdots \longrightarrow \mathrm{c}^{(1)} \cdots \mathrm{c}^{(n)} \Pi_0^{(1)} \cdots \Pi_0^{(n)} G,$$

it induces a geometric weak equivalence

$$\widehat{\mu}_k\colon G_1 \times_{G_0} \overset{k}{\cdots} \times_{G_0} G_1 \to G_1 \times_{G_0^d} \overset{k}{\cdots} \times_{G_0^d} G_1 \quad \textit{for all } k \geq 2 \tag{3-16}$$

(where G_0^d is as in Definition 2.23).

Proof By Definition 3.3, G (as an object of $\mathsf{Gpd}^2(\mathsf{Gpd}^{n-2})$) has the form of Figure 3 for some commuting square

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[d, "g"] & B \arrow[l, bend right, "f'"'] \arrow[d, "h"'] \\
C \arrow[r, "\ell"] \arrow[u, bend left, "g'"] & D \arrow[l, bend left, "\ell'"] \arrow[u, bend right, "h'"']
\end{tikzcd} \tag{3-17}$$

of $(n-2)$–fold groupoids, as in (3-7), by Remark 3.11.

(a) By (3-8) and (3-9), the statement holds for $n = 2$. Suppose by induction that it holds for $n-1$: then $(N^{(1)} G)_0 = A^f$ is in $\mathsf{Gpd}^{n-1}_{\mathrm{hd}}$. Also $(N^{(1)} G)_{k-1} = (A \times_C \overset{k}{\cdots} \times_C A)^{(f,\dots,f)}$ for $k \geq 2$. By definition and the induction hypothesis,

$$(f, \dots, f)\colon A \times_C \overset{k}{\cdots} \times_C A \longrightarrow B \times_D \overset{k}{\cdots} \times_D B$$

is a morphism with a section in $\mathsf{Gpd}^{n-1}_{\mathrm{hd}}$, Hence, by definition, $(N^{(1)}G)_{k-1} \in \mathsf{Gpd}^{n-1}_{\mathrm{hd}}$. Similarly for any $N^{(i)}G$.

(b) By (3-17) and part (a), $N^{(1)}\overline{\pi}_0^{(n)}G$ is the nerve of the $(n{-}1)$– fold homotopically discrete groupoid $\Pi_0^{(n)}G := B^h$, and the map $\overline{\gamma}^{(n)}$ lifts to a map of n–fold groupoids.

(c)–(e) By induction on $n \geq 2$. For $n = 2$, we saw that $\Pi_0^{(2)}G = B^h$, and since each column in Figure 3 is homotopically discrete, we see from (3.10) that the rightmost column is equivalent to B, the next to $B \times_D B$, and so on. Thus $N^{(1)}\gamma^{(2)}\colon N^{(1)}G \to N^{(1)}c\Pi_0^{(2)}G$ induces dimensionwise weak equivalences of simplicial spaces, so a weak equivalence of classifying spaces. Since B^h is a homotopically discrete groupoid, it is weakly equivalent to $\mathrm{c}D$ (in the notation of (3-17)), which is $(\pi_0 BG)^d$.

By (3-8), for each $k \geq 2$,

$$G_1 \times_{G_0} \overset{k}{\cdots} \times_{G_0} G_1 = (N^{(1)}G)_{k-1} = (A \times_C \overset{k}{\cdots} \times_C A)^{(f,\dots,f)} \cong B \times_D \overset{k}{\cdots} \times_D B,$$

while since G_1 is homotopically discrete and G_0^d is discrete,

$$G_1 \times_{G_0^d} \overset{k}{\cdots} \times_{G_0^d} G_1$$

is homotopically discrete (see Remark 3.4), and thus it is also weakly equivalent to $B \times_D \overset{k}{\cdots} \times_D B$. Hence (3-16) holds for $n = 2$.

In the induction step, $N^{(1)}G$ is a simplicial $(n{-}1)$–fold homotopically discrete groupoid (by (3-8) again), and thus by the induction hypothesis for $n-1$ we have a weak equivalence

$$\big(N^{(1)}\gamma_{(n-1)}\big)_r\colon \big(N^{(1)}G\big)_r \to \big(\mathrm{c}^{(2)}\cdots\mathrm{c}^{(n)}N^{(1)}\Pi_0^{(2)}\cdots\Pi_0^{(n)}G\big)_r =: P_r$$

in each simplicial dimension $r \geq 0$. Applying the $(n{-}1)$–fold nerve $N_{(n-1)}$ to both sides, we obtain a map of n–fold simplicial sets $N_{(n)}G \to P_\bullet$ which is a weak equivalence in each simplicial dimension, so induces a weak equivalence

$$\mathrm{Diag}_{(n)}\, N_{(n)}G \to \mathrm{Diag}_{(n)}\, P_\bullet.$$

However, $P_\bullet$ is discrete in all but the first simplicial direction, where it is (the nerve of) a homotopically discrete groupoid

$$H := \Pi_0^{(2)}\cdots\Pi_0^{(n)}G.$$

In fact, $H = (B^d)^{h^d}$, in the notation of Definition 3.1, where $h^d\colon B^d \to D^d$ is the discretization of the map $h\colon B \to D$ in (3-17).

Therefore, $\mathrm{Diag}_{(n)} P_\bullet = BH$ has $\pi_0 BH = \pi_0 H^d = \pi_0 BG$ while $\pi_i BH = 0$ for $i \geq 1$, and the map $\gamma_{(n)} = \gamma^{(1)} \circ \gamma_{(n-1)}$ induces the requisite weak equivalence. Since also $\gamma_{(n)} = \mathrm{c}^{(n)}\gamma_{(n-1)} \circ \gamma^{(n)}$, we deduce by induction that $\gamma^{(n)}$ is a geometric weak equivalence, too.

To show (3-16), note that by (3.10) we have

$$(\Pi_0^{(n)} G)_2 = \Pi_0^{(n-1)}(G_1 \times_{G_0} G_1) = B \times_D B \times_D B = (B \times_D B) \times_B (B \times_D B),$$

which by the induction hypothesis (3-16) and Remark 3.14 equals

$$\begin{aligned}\Pi_0^{(n-1)} G_1 \times_{\Pi_0^{(n-1)} G_0} \Pi_0^{(n-1)} G_1 &\simeq \Pi_0^{(n-1)} G_1 \times_{(\Pi_0^{(n-1)} G_0)^d} \Pi_0^{(n-1)} G_1 \\ &= \Pi_0^{(n-1)}(G_1 \times_{G_0^d} G_1).\end{aligned}$$

That is, we have a commuting square

$$\begin{tikzcd}
G_1 \times_{G_0} G_1 \arrow[r, "\gamma^{(n-1)}"] \arrow[d, "\widehat{\mu}_2"'] & \Pi_0^{(n-1)}(G_1 \times_{G_0} G_1) \arrow[d, "\simeq"] \\
G_1 \times_{G_0^d} G_1 \arrow[r, "\gamma^{(n-1)}"'] & \Pi_0^{(n-1)}(G_1 \times_{G_0^d} G_1)
\end{tikzcd}$$

in which three of the maps are geometric weak equivalences, so $\widehat{\mu}_2$ is, too.

Similarly for all $k > 2$. □

From part (d) of the lemma we see the following.

3.18 Corollary *If G is a homotopically discrete n–fold groupoid, the map $\gamma_{(n)}\colon G \to \mathrm{c}^{(1)} \cdots \mathrm{c}^{(n)} \Pi_0^{(1)} \cdots \Pi_0^{(n)} G$ is a geometric weak equivalence, so BG is homotopically trivial (that is, $\pi_i BG = 0$ for all $i \geq 1$).*

3.B Weakly globular n–fold groupoids

We are now in a position to define the main notion of this section. At first glance, it does not appear to be fully algebraic, since it uses the concept of a geometric weak equivalence. However, as we shall show in Corollary 4.8 below, this concept has an equivalent purely algebraic description.

3.19 Definition For each $n \geq 1$, the full subcategory $\mathsf{Gpd}^n_{\mathrm{wg}}$ of Gpd^n, whose objects are called *weakly globular n–fold groupoids*, is defined by induction on n, as follows:

For $n=1$, any groupoid is weakly globular; suppose we have defined $\mathsf{Gpd}_{\mathrm{wg}}^{n-1}$. We say that an n–fold groupoid

$$G = \big(G_1^{(1)} \rightrightarrows G_0^{(1)} \big)$$

is *weakly globular* if:

(i) $G_0 := G_0^{(1)}$ is in $\mathsf{Gpd}_{\mathrm{hd}}^{n-1}$.

(ii) $G_1 := G_1^{(1)}$ is in $\mathsf{Gpd}_{\mathrm{wg}}^{n-1}$, and for each $k \geq 2$, $G_1 \times_{G_0} \overset{k}{\cdots} \times_{G_0} G_1$ is in $\mathsf{Gpd}_{\mathrm{wg}}^{n-1}$.

(iii) The $(n-1)$–simplicial set $\bar{\pi}_0^{(n)} N^{(n-1)} \cdots N^{(1)} G$ is the nerve of a weakly globular $(n-1)$–fold groupoid $\Pi_0^{(n)} G$ such that

$$N_{(n-1)} \Pi_0^{(n)} G = \bar{\pi}_0^{(n)} N^{(n-1)} \cdots N^{(1)} G\,.$$

(iv) The map of $(n-1)$–fold groupoids

$$G_1 \times_{G_0} \overset{k}{\cdots} \times_{G_0} G_1 \xrightarrow{\widehat{\mu}_k} G_1 \times_{G_0^d} \overset{k}{\cdots} \times_{G_0^d} G_1$$

induced by $\gamma_{(n)}\colon G_0 \to G_0^d$ is a geometric weak equivalence for all $k \geq 2$.

Note the special role played by the *first* of the n–directions in this definition. Also, note that we have a functor $\Pi_0^{(n)}$ making the diagram

$$\begin{tikzcd} \mathsf{Gpd}_{\mathrm{wg}}^{n} \arrow[r, "N^{(n-1)}\dots N^{(1)}"] \arrow[d, "\Pi_0^{(n)}"'] & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]} \arrow[d, "\bar{\pi}_0^{(n)}"] \\ \mathsf{Gpd}_{\mathrm{wg}}^{n-1} \arrow[r, "N_{(n-1)}"] & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Set}].} \end{tikzcd}$$

commute.

3.20 Remark For $n=2$, the above definition is slightly more general than [11, Definition 2.21]. In fact, in [11], G is required to be symmetric, and both maps in $G_1 \rightrightarrows G_0$ are required to be fibrations of groupoids; the latter implies conditions (iii) and (iv).

Note that if $G \in \mathsf{Gpd}_{\mathrm{wg}}^{n}$, not only is $G_1 \times_{G_0} \overset{k}{\cdots} \times_{G_0} G_1 \in \mathsf{Gpd}_{\mathrm{wg}}^{n-1}$ (by Definition 3.19), but also $G_1 \times_{G_0^d} \overset{k}{\cdots} \times_{G_0^d} G_1 \in \mathsf{Gpd}_{\mathrm{wg}}^{n-1}$. We show this for $k=2$, the general case being similar. In fact we observe more generally that the pullback P of $G \to H \leftarrow G'$ with G, G' in $\mathsf{Gpd}_{\mathrm{wg}}^{n}$ and H discrete is an object of $\mathsf{Gpd}_{\mathrm{wg}}^{n}$.

We proceed by induction on n: For $n=1$ the statement is clear, since $\mathsf{Gpd}_{\mathrm{wg}}^{1} = \mathsf{Gpd}$. Suppose it is true for $n-1$. We have $P_0 = G_0 \times_{H_0} G_0' \in \mathsf{Gpd}_{\mathrm{hd}}^{n-1}$ since $G_0, G_0' \in \mathsf{Gpd}_{\mathrm{hd}}^{n-1}$, and H_0 is discrete (by Remark 3.4). Furthermore, $P_1 = G_1 \times_{H_1} G'_1 \in \mathsf{Gpd}^{n-1}_{\mathrm{hd}}$ by the induction hypothesis.

Likewise, since H is discrete,

$$P_1 \times_{P_0} P_1 \cong (G_1 \times_{G_0} G_1) \times_{(H_1 \times_{H_0} H_1)} (G'_1 \times_{G'_0} G'_1) \\ = (G_1 \times_{G_0} G_1) \times_{H_0} (G'_1 \times_{G'_0} G'_1). \tag{3-21}$$

Therefore $P_1 \times_{P_0} P_1 \in \mathsf{Gpd}^{n-1}_{\mathrm{wg}}$ by the induction hypothesis. For the same reason, $P_1 \times_{P_0} \overset{k}{\cdots} \times_{P_0} P_1 \in \mathsf{Gpd}^{n-1}_{\mathrm{wg}}$. Since π_0 commutes with fiber products over discrete objects, we have

$$\overline{\pi}^{(n)}_0 P = \Pi^{(n)}_0 G \times_H \Pi^{(n)}_0 G,$$

and this is in $\mathsf{Gpd}^{n-1}_{\mathrm{wg}}$ by the induction hypothesis.

Finally,

$$P_1 \times_{P^d_0} P_1 = (G_1 \times_{G^d_0} G_1) \times_{H_0} (G'_1 \times_{G'^d_0} G'_1)\ . \tag{3-22}$$

Since there are geometric weak equivalences

$$G_1 \times_{G_0} G_1 \to G_1 \times_{G^d_0} G_1 \quad \text{and} \quad G'_1 \times_{G'_0} G'_1 \to G'_1 \times_{G'^d_0} G'_1,$$

by (3-21) and (3-22) this induces a geometric weak equivalence

$$P_1 \times_{P_0} P_1 \to P_1 \times_{P^d_0} P_1\ .$$

Similarly, one shows that for each $k \geq 2$, there is a geometric weak equivalence

$$P_1 \times_{P_0} \overset{k}{\cdots} \times_{P_0} P_1 \to P_1 \times_{P^d_0} \overset{k}{\cdots} \times_{P^d_0} P_1\ .$$

This completes the proof that $P \in \mathsf{Gpd}^n_{\mathrm{wg}}$.

3.23 Definition For any n–fold groupoid G and $1 \leq k \leq n$, we define its *k–fold object of arrows* to be the $(n-k)$–fold groupoid

$$\mathcal{W}_{(n,k)} G := G^{(1\cdots k)}_{\underbrace{1\cdots 1}_{k}},$$

using the indexing conventions of Section 2.6(b).

3.24 Remark By Definition 3.19(ii), if G is weakly globular, so is $\mathcal{W}_{(n,1)} G$, so by induction we have a functor $\mathcal{W}_{(n,k)} \colon \mathsf{Gpd}^n_{\mathrm{wg}} \to \mathsf{Gpd}^{n-k}_{\mathrm{wg}}$, since

$$\mathcal{W}_{(n,k)} = \mathcal{W}_{(n-k+1,1)} \mathcal{W}_{(n-k+2,1)} \cdots \mathcal{W}_{(n-1,1)} \mathcal{W}_{(n,1)}. \tag{3-25}$$

3.26 Algebraic homotopy groups and algebraic weak equivalences For a weakly globular n–fold groupoid G, we define the k^{th} *algebraic homotopy group* of G at $x_0 \in G_{0\overset{n}{\cdots}0}$ to be

$$\omega_k(G;x_0) \cong \begin{cases} \mathcal{W}_{(n,n)}G(x_0,x_0) & \text{if } k=n, \\ \mathcal{W}_{(n-k,n-k)}\big(\Pi_0^{(k+1)}\cdots\Pi_0^{(n)}G\big)(x_0,x_0) & \text{if } 0<k<n, \end{cases} \tag{3-27}$$

with the 0^{th} *algebraic homotopy set* of G defined as

$$\omega_0(G) := \Pi_0^{(1)}\cdots\Pi_0^{(n)}G.$$

Here $\mathcal{W}_{(n,n)}G(a,b)$ (see Definition 3.23) is the set of morphisms from a to b in the groupoid $\mathcal{W}_{(n,n-1)}G$ (in the n^{th} direction), so in particular $\mathcal{W}_{(n,n)}G(a,a)$ is the group of automorphisms of a (which is abelian for $n \geq 2$).

A map $f\colon G \to G'$ of weakly globular n–fold groupoids is called an *algebraic weak equivalence* if it induces bijections on the k^{th} algebraic homotopy groups (set) for all $x_0 \in G_{0\overset{n}{\cdots}0}$ and $0 \leq k \leq n$.

3.28 Definition For each $n \geq 0$, let $\mathsf{P}^n\mathsf{Top}$ denote the full subcategory of Top consisting of spaces X for which the natural map $X \to P^nX$ is a weak equivalence (that is, $\pi_i(X,x)=0$ for all $x \in X$ and $i>n$). An *n–type* is an object in $\mathsf{P}^n\mathsf{Top}$ (or in the corresponding full subcategory $\mathrm{ho}(\mathsf{P}^n\mathsf{Top})$ of $\mathrm{ho}\mathsf{Top}$).

We use similar notation for n–Postnikov simplicial sets (where for a Kan complex X (see Goerss and Jardine [31, Section I.3]), we can use $\mathrm{csk}_{n+1}X$ as a model for the n^{th} Postnikov section P^nX).

For any $n \geq 0$, a map $f\colon X \to Y$ in $[\Delta^{\mathrm{op}}, \mathsf{Set}]$ (or in Top)is called an *n–equivalence* if it induces isomorphisms $f_*\colon \pi_0X \to \pi_0Y$ (of sets), and $f_\#\colon \pi_i(X,x) \to \pi_i(Y,f(x))$ for every $1 \leq i \leq n$ and $x \in X_0$.

We recall the following notion and fact from our earlier paper [11]:

3.29 Definition A map $f\colon W \to V$ of bisimplicial sets is called a *diagonal n–equivalence* if $f_k^h\colon W_k^h \to V_k^h$ is an $(n-k)$–equivalence for each $k \leq n$.

3.30 Proposition [11, Proposition 3.9] *If $f\colon W \to V$ is a diagonal n–equivalence, then the induced map* $\mathrm{Diag}\, f\colon \mathrm{Diag}\, W \to \mathrm{Diag}\, V$ *is an n–equivalence.*

3.31 Lemma *For any $G \in \mathsf{Gpd}^n_{\mathrm{wg}}$, the map $\overline{\gamma}$ of Definition 3.12 corresponds to a map of n–fold groupoids $\gamma^{(n)}\colon G \to \mathrm{c}^{(n)}\Pi_0^{(n)}G$ with $\overline{\gamma} = N^{(n-1)}\dots N^{(1)}\gamma^{(n)}$, which induces an $(n-1)$–equivalence $B\gamma^{(n)}\colon BG \to B\,\mathrm{c}^{(n)}\Pi_0^{(n)}G$ on classifying spaces.*

Proof By Definition 3.19 and Remark 3.13 the map $\overline{\gamma}$ corresponds to a map of n–fold groupoids as stated. We show that this is an $(n-1)$–equivalence by induction on n. It is clear for $n=1$. Suppose, inductively, it holds for $n-1$.

By construction we have

$$\big(\Pi_0^{(n)}G\big)_r := \big(N^{(n)}\Pi_0^{(n)}G\big)_r^{(n)} = \Pi_0^{(n-1)}\big(N^{(n)}G\big)_r^{(n)},$$

and therefore, for each $r \geq 0$ there is a map

$$\big(N^{(n)}\gamma^{(n-1)}\big)_r\colon \big(\Pi_0^{(n)}G\big)_r \to \big(\mathrm{c}^{(n)}\Pi_0^{(n-1)}G\big)_r.$$

By taking realizations, we obtain a map of simplicial spaces $B\gamma^{(n-1)}$. We claim that the corresponding map of bisimplicial sets is a diagonal $(n-1)$–equivalence (see Definition 3.29). In fact, since $G_0 = (N^{(n)}G)_0^{(n)}$ is homotopically discrete, by Lemma 3.15, $(B\gamma^{(n-1)})_0$ is a weak equivalence, hence in particular an $(n-1)$–equivalence. By the induction hypothesis $(B\gamma^{(n-1)})_r$ is a $(n-2)$–equivalence for all $r \geq 1$. Hence $B\gamma^{(n-1)}$ is an $(n-1)$–equivalence by Proposition 3.30. □

3.32 Remark From Lemmas 3.15 and 3.31 we see that a homotopically discrete n–fold groupoid is weakly globular.

4 n–types

In this section we prove one of the main result of this paper, Theorem 4.32, which asserts that all n–types are modeled by weakly globular n–fold groupoids.

4.A The homotopy type of a weakly globular n–fold groupoid

We start by showing that if $G \in \mathsf{Gpd}^n_{\mathrm{wg}}$, its classifying space BG (see Definition 2.21) is an n–type; that is, $\pi_i(BG,x)=0$ for all $x \in BG$ and $i > n$. We prove this using a spectral sequence computation of $\pi_i(BG,x)$. In Section 5, we give an alternative proof using a comparison with Tamsamani's weak n–groupoids.

In [44], Quillen constructed a spectral sequence for a bisimplicial group, which was generalized by Bousfield and Friedlander in [14, Appendix B] to define the *Bousfield–Friedlander spectral sequence* of a bisimplicial set $X_{\bullet\bullet} \in [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]$, with

$$E^2_{s,t} = \pi^h_s \pi^v_t X_{\bullet\bullet} \Rightarrow \pi_{s+t}\,\mathrm{Diag}\,X_{\bullet\bullet}. \tag{4-1}$$

See Dwyer, Kan and Stover [26, Section 8.4] for an alternative construction when $X_{\bullet\bullet}$ is connected in each simplicial dimension. The spectral sequence need not converge otherwise; however, we have the following *sufficient* condition for convergence (see [14, Section B.3]):

4.2 Definition Think of a bisimplicial set $X_{\bullet\bullet} \in [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]$ as a (horizontal) simplicial object in $[\Delta^{\mathrm{op}}, \mathsf{Set}]$ (with the simplicial direction inside $[\Delta^{\mathrm{op}}, \mathsf{Set}]$ thought of as being vertical). In this notation, a *k–π_t-matching collection* at $a \in X_{n,0}$ (for $0 \le k \le n$) is a set of elements $x_i \in \pi_t(X_{n-1\bullet}, d_i^h a)$ $(0 \le i \le n, i \ne k)$, such that

$$(d_i^h)_* x_j = (d_{j-1}^h)_* x_i \tag{4-3}$$

for every $0 \le i < j \le n$ $(i, j \ne k)$.

We say that $X_{\bullet\bullet}$ satisfies the *π_*–Kan condition* if for every $n, t \ge 1$, $0 \le k \le n$, $a \in X_{n,0}$, and k–π_t–matching collection $(x_i)_{i\ne k}^n$ at a, there is a fill-in $w \in \pi_t^v(X_{n\bullet}, a)$ such that $(d_i^h)_* w = x_i$ for all $0 \le i \le n$ $(i \ne k)$.

By [14, Theorem B.5], if $X_{\bullet\bullet}$ satisfies the π_*–Kan condition – for example, if each $X_{n\bullet}$ is connected – then the spectral sequence (4-1) converges.

4.4 Notation For any simplicial set Y and $t \ge 1$, the t^{th} homotopy group $\pi_t(Y, y)$, as $y \in Y$ varies, constitutes a *semi-discrete groupoid*, in the sense of [11, Section 1], that is, a disjoint union of groups (abelian, if $t \ge 2$). We denote it by $\hat{\pi}_t Y$.

4.5 Lemma *Let $G_\bullet \in \mathsf{Gpd}([\Delta^{\mathrm{op}}, \mathsf{Set}])$ be a groupoid in $[\Delta^{\mathrm{op}}, \mathsf{Set}]$, such that*

$$G_1 \times_{G_0} \overset{k}{\cdots} \times_{G_0} G_1 \to G_1 \times_{c\pi_0 G_0} \overset{k}{\cdots} \times_{c\pi_0 G_0} G_1$$

is a weak equivalence of simplicial sets for all $k \ge 2$, with G_0 a homotopically trivial simplicial set. Then the bisimplicial set $X_{\bullet\bullet} := NG_\bullet$ satisfies the π_*–Kan condition, and for each $t \ge 1$, $\hat{\pi}_t X_{\bullet\bullet}$ is a groupoid object in semi-discrete groupoids, so is 2–coskeletal.

Proof We think of the simplicial direction as vertical. Let $X_k = (NG_\bullet)_k$. Since $X_0 = G_0$ is homotopically trivial (that is, a disjoint union of contractible spaces), the groupoid $\hat{\pi}_t X_0$ is discrete on $\pi_0 G_0$, so any k–π_t–matching collection for $n = 1$ is trivial.

For $n = 2$, note that $X_2 = X_1 \times_{X_0} X_1$, so any $a \in X_{2,0}$ is of the form $a = (a', a'')$, where $d_1 a' = d_0 a'' =: b$ Moreover, $d_0 a = a'$, $d_1 a = a' \star a''$ (where $\star$ denotes the groupoid composition), and $d_2 a = a''$.

So if $t \ge 1$ there are three cases for a k–π_t–matching collection $(x_i \in \pi_t^v(X_1, d_i a))_{i \ne k}$ at a:

(i) When $k = 1$, the fill-in $w \in \pi_t^v(X_2, a)$ for x_0 and x_2 is the pull-back pair (x_0, x_2) in

$$\pi_t^v(X_2, a) = \pi_t^v(X_1, a') \times_{\pi_t^v(X_0, b)} \pi_t^v(X_1, a'').$$

(ii) When $k=0$, the fill-in $w=(y,x_2)$ for x_1 and x_2 should satisfy $x_1=d_1w=y\star x_2$, so $y=x_1\star(x_2)^{-1}$, using the groupoid structure on $\hat{\pi}^v_t X_1$.

(iii) The case $k=2$ is similar.

For $n>2$ the proof of the π_*–Kan condition is analogous; however, because $\hat{\pi}^v_t X_{\bullet\bullet}$ is 2–coskeletal, we do not even need to verify it, since the spectral sequence (4-1) from the E^2–term on then depends only on the 2–truncation of $X_{\bullet\bullet}$ in the horizontal direction. □

In order to study the homotopy groups of the n–fold diagonal dNG of an n–fold groupoid, we think of it as an iterative construction in which we take diagonals in successive bisimplicial bidirections. The weak globularity allows us to iteratively apply Lemma 4.5, and thus the Bousfield–Friedlander spectral sequence.

4.6 Theorem *For any weakly globular n–fold groupoid $G\in\mathsf{Gpd}^n_{\mathrm{wg}}$, BG is an n–type, and for each base point $x_0\in G_{0\overset{n}{\cdots}0}$ we have natural isomorphisms*

(4-7) $$\pi_k(BG;x_0)\cong\omega_k(G;x_0)\quad\textit{for}\quad 0<k\le n,\quad\textit{and}\quad \pi_0BG\cong\omega_0(G)$$

(see (3-27)).

Proof Since BG is the geometric realization of dNG, we prove the theorem simplicially, for dNG, by induction on n.

Using the convention of Remark 2.27, for each $a\in\Delta^{n-2}$ we have a double groupoid $G^{(1,2)}(a)\in\mathsf{Gpd}^2$ (in the notation of Section 2.6(b)). Assuming that the first of the n directions of G is not among those of Δ^{n-2}, $N^{(1)}G^{(1,2)}(a)\in[\Delta^{\mathrm{op}},\ \mathsf{Gpd}]$ satisfies the hypotheses of Lemma 4.5, by Definition 3.19. Therefore, the Bousfield–Friedlander spectral sequence for the bisimplicial set

$$X(a):=N^{(1,2)}G^{(1,2)}(a)$$

converges to $\pi_*\operatorname{Diag}X(a)$. Moreover, $\pi^v_tX(a)$ is 2–coskeletal for each $t\ge1$, by the lemma, as is $\pi^v_0X(a)$ (by Definition 3.19 again). Thus in the E^2–term of the spectral sequence only the two right columns of two bottom rows can be non-zero, so that $\operatorname{Diag}X(a)$ is a 2–type. In fact, the rightmost column is zero (except at the bottom), so we can read off the homotopy groups of $\operatorname{Diag}X(a)$ from those of $X(a)$.

Since Diag is functorial in $a\in{\Delta^{n-2}}^{\mathrm{op}}$, the resulting object $Y:=\operatorname{Diag}^{(1,2)}N^{(1,2)}G$ is in $[\Delta^{\mathrm{op}},\ \mathsf{Gpd}^{n-2}]$, with each $Y(a)\in[\Delta^{\mathrm{op}},\ \mathsf{Set}]$ a simplicial 2–type. Since G_0 was a homotopically discrete $(n-1)$–fold groupoid, the object Y^v_0 (in dimension 0 in the first (simplicial) direction) is a homotopically discrete $(n-2)$–fold groupoid. Moreover, for any choice of a third (groupoid) direction i, and each $b \in \Delta^{n-3}$, by Definition 3.19, we have a bisimplicial groupoid

$$Z_{\bullet\bullet *} := N^{(1,2)} G^{(1,2,i)}(b)$$

(where the third index is the groupoid direction). This has a weak equivalence of bisimplicial sets

$$Z_{\bullet\bullet k} = Z_{\bullet\bullet 1} \times_{Z_{\bullet\bullet 0}} \overset{k}{\cdots} \times_{Z_{\bullet\bullet 0}} Z_{\bullet\bullet 1} \xrightarrow{\simeq} Z_{\bullet\bullet 1} \times_{G_0^d} \overset{k}{\cdots} \times_{G_0^d} Z_{\bullet\bullet 1}$$

for each $k \geq 2$, natural in b (note that G^d is independent of b). This map therefore induces a weak equivalence in the bisimplicial direction (see Remark 2.22). Thus each simplicial groupoid $Y(b) = \operatorname{Diag} Z_{\bullet\bullet *}$ satisfies the hypotheses of Lemma 4.5.

Now assume by descending induction on $2 \leq k < n$ that we have $Y \in [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-k}]$, with $Y(a) \in [\Delta^{\mathrm{op}}, \mathsf{Set}]$ a k–type for each $a \in \Delta^{n-k}$, with Y_0^v a homotopically discrete $(n-k)$–fold groupoid. Here the first (vertical) direction is simplicial.

For any choice of a second (groupoid) direction, and each $b \in \Delta^{n-k-1}$, the simplicial groupoid $Y^{(1,2)}(b) \in [\Delta^{\mathrm{op}}, \mathsf{Gpd}]$ satisfies the hypotheses of Lemma 4.5. Therefore, (4-1) converges, with only the two right columns of the bottom k rows non-zero, and $\operatorname{Diag} Y(a)$ is thus a $(k+1)$–type. When $k = n-1$, Y is a simplicial groupoid which is an $(n-1)$–type in the simplicial direction, with BG appearing as the realization of $\operatorname{Diag} Y$.

For any weakly globular double groupoid G, the E^2–term of the Bousfield–Friedlander spectral sequence for the bisimplicial set $X_{\bullet\bullet} = N^h N^v G$ survives to E^∞. Moreover, because G_0 is homotopically trivial, $E^2_{1,0} = \pi_1 \pi_0 X_{\bullet\bullet} = 0$, so in fact by Lemma 4.5,

$$\pi_i(\operatorname{Diag} X_{\bullet\bullet}, x_0) = \begin{cases} E^2_{0,0} = \pi_0\pi_0(X_{\bullet\bullet}, x_0) & \text{if } i = 0, \\ E^2_{0,1} = \pi_0\pi_1(X_{\bullet\bullet}, x_0) & \text{if } i = 1, \\ E^2_{1,1} = \pi_1\pi_1(X_{\bullet\bullet}, x_0) & \text{if } i = 2, \end{cases}$$

for each choice of a base-point x_0 in G_{00}. Actually, $\pi_1\pi_1(X_{\bullet\bullet}, x_0)$ is just the automorphism group of G_1, that is, $\mathcal{W}_{(2,2)} G(x_0, x_0)$

Therefore, given a weakly globular n–fold groupoid G, by what we have shown above we see that

$$\pi_n(BG; x_0) \cong \omega_n(G; x_0)$$

for each $x_0 \in G_{0,\ldots,0}$. Moreover, by Lemma 3.31 we have

$$\pi_i(BG, x_0) \cong \pi_i\big(B\Pi_0^{(n-k+1)} \cdots \Pi_0^{(n)} G, x_0\big)$$

for all $0 \leq i \leq n-k$, and $\Pi_0^{(n-k+1)} \cdots \Pi_0^{(n)} G$ is an $(n{-}k)$–weakly globular $(n{-}k)$–fold groupoid, so in particular (4-7) holds for each $0 \leq k \leq n$. □

Observe that Theorem 4.6 provides an intrinsic algebraic definition of the notion of geometric weak equivalences among weakly globular n–fold groupoids, since we have the following corollary:

4.8 Corollary (a) *A map of weakly globular n–fold groupoids is a geometric weak equivalence (Definition 2.21) if and only if it is an algebraic weak equivalence (Section 3.26).*

(b) *The notion of a weakly globular n–fold groupoid G is purely algebraic.*

4.9 Remark It follows from above that the functor $\Pi_0^{(n)}\colon \mathsf{Gpd}^n_{\mathrm{wg}} \to \mathsf{Gpd}^{n-1}_{\mathrm{wg}}$ preserves geometric weak equivalences and serves as an algebraic $(n{-}1)$–Postnikov section functor.

4.B An iterative description of $Q_{(n)}$

We now use the notions of the previous section to provide a more transparent iterative description of the fundamental n–fold groupoid functor $Q_{(n)}X$ (Definition 2.30) for a Kan complex X.

4.10 Definition For any simplicial set X, let

$$L_k X := \begin{cases} \mathrm{Dec}\, X & \text{if } k = 0 \\ \mathrm{Dec}\, X \times_X \overset{k+1}{\cdots} \times_X \mathrm{Dec}\, X & \text{if } k \geq 1. \end{cases} \tag{4-11}$$

4.12 Remark If X is a Kan complex, we have a natural fibration of simplicial sets $u\colon \mathrm{Dec}\, X \to X$ (see Section 2.5), yielding the internal groupoid $(\mathrm{Dec}\, X)^u \in \mathsf{Gpd}[\Delta^{\mathrm{op}}, \mathsf{Set}]$ of Definition 3.1. We see that

$$(N(\mathrm{Dec}\, X)^u)_k = L_k X = L_1 X \times_{\mathrm{Dec}\, X} \overset{k}{\cdots} \times_{\mathrm{Dec}\, X} L_1 X \tag{4-13}$$

for all $k \geq 1$, so we may denote the bisimplicial set $N(\mathrm{Dec}\, X)^u$ by $L_\bullet X$. This is depicted in Figure 4, where the vertical maps are induced by those indicated in the rightmost column, and the horizontal maps are structure maps for the pullbacks, as in (3-2).

If X is reduced, $\mathrm{Dec}\, X$ is contractible, so $L_1 X$ models the loop space ΩX. In general, $L_1 X$ is homotopy equivalent to the "path object" $\mathbb{P}X$ of Duskin [25, Section 2.2].

$$
\begin{tikzcd}
\cdots \arrow[r] & X_3 \times_{X_2} X_3 \times_{X_2} X_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & X_3 \times_{X_2} X_3 \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & X_3 \arrow[d, shift right=2, "d_0"'] \arrow[d, "d_1" description] \arrow[d, shift left=2, "d_2"] \\
\cdots \arrow[r] & X_2 \times_{X_1} X_2 \times_{X_1} X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left] \arrow[d, shift right] & X_2 \times_{X_1} X_2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_1"'] \arrow[d, shift left] \arrow[d, shift right] & X_2 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
\cdots \arrow[r] & X_1 \times_{X_0} X_1 \times_{X_0} X_1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \times_{X_0} X_1 \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] & X_1
\end{tikzcd}
$$

Figure 4: Corner of $L_\bullet X$

4.14 Lemma *Let X be a Kan complex, and $\mathrm{c}X$ the corresponding bisimplicial set, constant in the horizontal direction.*

(a) *There is a natural map of bisimplicial sets $\phi\colon \mathrm{c}X \to L_\bullet X$, which is a dimensionwise weak equivalence (as horizontal simplicial sets, in each vertical dimension; see Figure 4), so induces a weak equivalence $\mathrm{Diag}\,\phi\colon X \to \mathrm{Diag}\, L_\bullet X$.*

(b) *We have $N^{(n)} Q_{(n)} X = \bar{Q}^{(n)}_{(n-1)} L_\bullet X$, that is, for each $k \geq 0$,*

$$(N^{(n)} Q_{(n)} X)_k = Q_{(n-1)} L_k X. \tag{4-15}$$

Thus, for each $k \geq 1$,

$$Q_{(n-1)} L_k X \cong Q_{(n-1)} L_1 X \times_{Q_{(n-1)}\,\mathrm{Dec}\, X} \overset{k}{\cdots} \times_{Q_{(n-1)}\,\mathrm{Dec}\, X} Q_{(n-1)} L_1 X. \tag{4-16}$$

(c) *If X is homotopically trivial then, for $k \geq 1$,*

$$Q_{(n)} L_k X \cong Q_{(n)}\, \mathrm{Dec}\, X \times_{Q_{(n)} X} \overset{k+1}{\cdots} \times_{Q_{(n)} X} Q_{(n)}\, \mathrm{Dec}\, X. \tag{4-17}$$

Proof (a) The section $\sigma\colon X \to \mathrm{Dec}\, X$ to the augmentation $\varepsilon = \bar{d}_*\colon \mathrm{Dec}\, X \to X$, given in dimension i by the degeneracy $s_i\colon X_i \to X_{i+1}$ (see Section 2.7), fits into a diagram

$$
\begin{tikzcd}
X \arrow[r, "\sigma"] \arrow[d, equal] & \mathrm{Dec}\, X \arrow[r, "\varepsilon"] \arrow[d, "\varepsilon"] & X \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}
\tag{4-18}
$$

of vertical arrows in $[\Delta^{\mathrm{op}}, \mathrm{Set}]$, where the horizontal composite is the identity. Applying the construction of Definition 3.1 to each vertical arrow we obtain

$$\mathrm{c}X \xrightarrow{\phi} L_\bullet X \xrightarrow{\overline{\varepsilon}} \mathrm{c}X.$$

Here the map of simplicial sets $\phi_i\colon \mathrm{c}(X_i) \to (L_\bullet X)_i$ in each internal simplicial dimension i is given by the vertical maps in

$$\begin{tikzcd}
\cdots \arrow[r] & X_i \arrow[r, shift left=2, "="] \arrow[r, "="] \arrow[r, shift right=2, "="'] \arrow[d, "{(s_i,s_i,s_i)}"'] & X_i \arrow[r, shift left, "="] \arrow[r, shift right, "="'] \arrow[d, "{(s_i,s_i)}"] & X_i \arrow[d, "s_i"] \\
\cdots \arrow[r] & X_{i+1} \times_{X_i} X_{i+1} \times_{X_i} X_{i+1} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_{i+1} \times_{X_i} X_{i+1} \arrow[r, shift left] \arrow[r, shift right] & X_{i+1}.
\end{tikzcd} \tag{4-19}$$

Since the lower row in (4-19) is the nerve of a homotopically discrete groupoid, the vertical map is a weak equivalence (with inverse induced by the right square in (4-18)).

(b) We will show that, for $n \geq 2$,

$$N^{(n)} Q_{(n)} X = \overline{Q}^{(2)}_{(n-1)} N^{(2)} \widehat{\pi}^{(2)}_1 \operatorname{Or}_{(2)} X, \tag{4-20}$$

where $\overline{Q}^{(2)}_{(n-1)}$ is obtained by applying $Q_{(n-1)}$ in each simplicial dimension in the second direction to the bisimplicial object $N^{(2)} \widehat{\pi}^{(2)}_1 \operatorname{Or}_{(2)}$.

By Lemma 2.28, we have

$$\overline{\operatorname{Or}}^{(2)}_{(n-1)} N^{(2)} \widehat{\pi}^{(2)}_1 \operatorname{Or}_{(2)} X = N^{(n)} \widehat{\pi}^{(n)}_1 \operatorname{Or}_{(n)} X, \tag{4-21}$$

and since, by the definition of $Q_{(n-1)}$,

$$\overline{Q}^{(2)}_{(n-1)} N^{(2)} \widehat{\pi}^{(2)}_1 \operatorname{Or}_{(2)} X = \widehat{\pi}^{(1)}_1 \cdots \widehat{\pi}^{(n-1)}_1 \overline{\operatorname{Or}}^{(2)}_{(n-1)} N^{(2)} \widehat{\pi}^{(2)}_1 \operatorname{Or}_{(2)} X$$

we deduce that

$$\overline{Q}^{(2)}_{(n-1)} N^{(2)} \widehat{\pi}^{(2)}_1 \operatorname{Or}_{(2)} X = \widehat{\pi}^{(1)}_1 \cdots \widehat{\pi}^{(n-1)}_1 N^{(n)} \widehat{\pi}^{(n)}_1 \operatorname{Or}_{(n)} X. \tag{4-22}$$

Since $Q_{(n)} X := \widehat{\pi}^{(1)}_1 \cdots \widehat{\pi}^{(n)}_1 \operatorname{Or}_{(n)} X$ and $\operatorname{Or}_{(n)} X$ is $(n,2)$–fibrant, in order to show (4-20) it suffices to show by induction on $n \geq 2$ that

$$N^{(n)} \widehat{\pi}^{(1)}_1 \cdots \widehat{\pi}^{(n)}_1 Y = \widehat{\pi}^{(1)}_1 \cdots \widehat{\pi}^{(n-1)}_1 N^{(n)} \widehat{\pi}^{(n)}_1 Y \tag{4-23}$$

for any $(n,2)$–fibrant n–fold simplicial set Y. For $n = 2$,

$$N^{(2)} \widehat{\pi}^{(1)}_1 \widehat{\pi}^{(2)}_1 Y = \widehat{\pi}^{(1)}_1 N^{(2)} \widehat{\pi}^{(2)}_1 Y$$

by Lemma 2.35 and Proposition 2.34.

In the induction step, let $G_\bullet$ be the simplicial $(n-1)$–fold groupoid $\hat{\pi}_1^{(2)}\cdots\hat{\pi}_1^{(n)}Y$. By Lemma 2.37, $G_\bullet$ is $(n-1,2)$–fibrant, so for each $a\in\Delta^{n-2}$, the simplicial groupoid $G_\bullet(a)$ (in the first groupoid direction of $G_\bullet$) is $(2,2)$–fibrant. Thus by Lemma 2.35 we have $N^{(n)}\hat{\pi}_1^{(1)}G_\bullet(a)=\hat{\pi}_1^{(1)}N^{(n)}G_\bullet(a)$, so

$$N^{(n)}\hat{\pi}_1^{(1)}\cdots\hat{\pi}_1^{(n)}Y=N^{(n)}\hat{\pi}_1^{(1)}G_\bullet=\hat{\pi}_1^{(1)}N^{(n)}G_\bullet=\hat{\pi}_1^{(1)}N^{(n)}\hat{\pi}_1^{(2)}\cdots\hat{\pi}_1^{(n)}Y.$$

If we think of Y as a simplicial $(n-1,2)$–fibrant $(n-1)$–fold simplicial set $Y_\bullet^{(1)}$ (in the first direction), by the induction hypotheses

$$N^{(n)}\hat{\pi}_1^{(2)}\cdots\hat{\pi}_1^{(n)}Y_m^{(1)}=\hat{\pi}_1^{(2)}\cdots\hat{\pi}_1^{(n-1)}N^{(n)}\hat{\pi}_1^{(n)}Y_m^{(1)}$$

for each $m\geq 0$, so (4-23) holds for Y, too. This concludes the proof of (4-20).

Observe that

$$N^{(2)}\hat{\pi}_1^{(2)}\operatorname{Or}_{(2)}X=NA^f \tag{4-24}$$

with $A^u\in\mathsf{Gpd}([\Delta^{\mathrm{op}},\mathsf{Set}])$ as in (3-2), for the map $u\colon\operatorname{Dec}X\to X$ of simplicial sets. In fact, $\hat{\pi}_1^{(2)}\operatorname{Or}_{(2)}X$, thought of as a simplicial object in Gpd, has $(\hat{\pi}_1^{(2)}\operatorname{Or}_{(2)}X)_k=\hat{\pi}_1\operatorname{Dec}^kX$ in simplicial dimension k. This is isomorphic to the homotopically discrete groupoid $(X_k)^{u_k}$ (where $u_k\colon X_k\to X_{k-1}$ is a map of sets). Hence from (4-20) and (4-24) we conclude that

$$N^{(n)}Q_{(n)}X=\bar{Q}_{(n-1)}NA^u.$$

Since $(NA^u)_k=L_kX$ for each $k\geq 0$, (4-15) follows.

In particular, since $Q_{(n)}X\in\mathsf{Gpd}^n_{\mathrm{wg}}$, we have, for $k\geq 2$,

$$\begin{aligned}Q_{(n-1)}L_kX&=(N^{(n)}Q_{(n)}X)_k^{(n)}\\&\cong(N^{(n)}Q_{(n)}X)_1^{(n)}\times_{(N^{(n)}Q_{(n)}X)_0^{(n)}}\overset{k}{\cdots}\times_{(N^{(n)}Q_{(n)}X)_0^{(n)}}(N^{(n)}Q_{(n)}X)_1^{(n)}\end{aligned}$$

so by (4-15) we have

$$Q_{(n-1)}L_kX\cong(Q_{(n-1)}L_1X)\times_{(Q_{(n-1)}\operatorname{Dec}X)}\overset{k}{\cdots}\times_{(Q_{(n-1)}\operatorname{Dec}X)}(Q_{(n-1)}L_1X).$$

(c) By induction on n. For $n=1$, $Q_{(1)}=\hat{\pi}_1$. Since by hypothesis X is homotopically trivial and $u\colon\operatorname{Dec}X\to X$ is a fibration, $L_1X=\operatorname{Dec}X\times_X\operatorname{Dec}X$ is also homotopically trivial; hence $\hat{\pi}_1L_1X$ is a homotopically discrete groupoid, and is therefore isomorphic to A^f where $f\colon A\to B$ is the obvious map

$$X_1\times_{X_0}X_1\longrightarrow X_0\times_{\pi_0X}X_0.$$

On the other hand, $\hat{\pi}_1 \operatorname{Dec} X \cong (X_1)^{d_0}$ and $\hat{\pi}_1 X = (X_0)^{\gamma}$ (for $\gamma\colon X_0 \to \pi_0 X$), so

$$\hat{\pi}_1 L_1 X \cong \hat{\pi}_1 \operatorname{Dec} X \times_{\hat{\pi}_1 X} \hat{\pi}_1 \operatorname{Dec} X.$$

In the induction step, applying $N^{(n)}$ to both sides of (4-17), we must show that for each $k \geq 1$ and $i \geq 1$ we have

$$\begin{aligned}&\big(N^{(n)} Q_{(n)} L_k X\big)^{(n)}_{i-1}\\ &\quad\cong \big(N^{(n)} Q_{(n)} \operatorname{Dec} X\big)^{(n)}_{i-1} \times_{(N^{(n)} Q_{(n)} X)^{(n)}_{i-1}} \overset{k-1}{\cdots} \times_{(N^{(n)} Q_{(n)} X)^{(n)}_{i-1}} \big(N^{(n)} Q_{(n)} \operatorname{Dec} X\big)^{(n)}_{i-1}\end{aligned}$$

or equivalently (after applying (b)), that

$$\begin{aligned}Q_{(n-1)} L_i (L_k X)\\ \cong Q_{(n-1)} L_i \operatorname{Dec} X \times_{Q_{(n-1)} L_i X} \overset{k+1}{\cdots} \times_{Q_{(n-1)} L_i X} Q_{(n-1)} L_i \operatorname{Dec} X.\end{aligned} \tag{4-25}$$

Since X is homotopically trivial, so are $\operatorname{Dec} X$ and $L_k X$ (since $u\colon \operatorname{Dec} X \to X$ is a fibration), so we can apply induction hypothesis (c) for $(n-1)$ to replace the left-hand side of (4-25) by

$$Q_{(n-1)} \operatorname{Dec}(L_k X) \times_{Q_{(n-1)} L_k X} \overset{i+1}{\cdots} \times_{Q_{(n-1)} L_k X} Q_{(n-1)} \operatorname{Dec}(L_k X),$$

and since Dec commutes with fiber products, and thus with L_k, this equals

$$\begin{aligned}&\big(Q_{(n-1)}\big(\operatorname{Dec}^2 X \times_{\operatorname{Dec} X} \overset{k+1}{\cdots} \times_{\operatorname{Dec} X} \operatorname{Dec}^2 X\big)\big)\\ &\qquad \times_{(Q_{(n-1)}(\operatorname{Dec} X \times_X \overset{k+1}{\cdots} \times_X \operatorname{Dec} X))} \overset{i+1}{\cdots} \times_{(Q_{(n-1)}(\operatorname{Dec} X \times_X \overset{k+1}{\cdots} \times_X \operatorname{Dec} X))}\\ &\qquad\qquad \big(Q_{(n-1)}\big(\operatorname{Dec}^2 X \times_{\operatorname{Dec} X} \overset{k+1}{\cdots} \times_{\operatorname{Dec} X} \operatorname{Dec}^2 X\big)\big)\end{aligned}$$

If we write $A := Q_{(n-1)} \operatorname{Dec}^2 X$, $B := Q_{(n-1)} \operatorname{Dec} X$, and $C := Q_{(n-1)} X$, applying (b) for $(n-1)$ to this last expression yields

$$\big(A \times_B \overset{k+1}{\cdots} \times_B A\big) \times_{(B \times_C \overset{k-1}{\cdots} \times_C B))} \overset{i+1}{\cdots} \times_{(B \times_C \overset{k-1}{\cdots} \times_C B))} \big(A \times_B \overset{k+1}{\cdots} \times_B A\big). \tag{4-26}$$

Similarly, (c) applied to the right-hand side of (4-25) yields

$$\big(A \times_B \overset{i+1}{\cdots} \times_B A)\big) \times_{(B \times_C \overset{i-1}{\cdots} \times_C B))} \overset{k+1}{\cdots} \times_{(B \times_C \overset{i-1}{\cdots} \times_C B))} \big(A \times_B \overset{i+1}{\cdots} \times_B A)\big), \tag{4-27}$$

and the two limits (4-26) and (4-27) are evidently equal, proving (4-25). □

4.C Modeling n–types

In the last part of this section we finally show that weakly globular n–fold groupoids indeed model n–types.

4.28 Proposition *Let X be a Kan complex. Then:*

(a) *There is a natural n–equivalence $\psi^X_{(n)}\colon X \to dNQ_{(n)}X$.*

(b) *$Q_{(n)}$ preserves weak equivalences of Kan complexes.*

(c) *If X is homotopically trivial (that is, all higher homotopy groups vanish), then $Q_{(n)}X$ is a homotopically trivial n–fold groupoid.*

(d) *$Q_{(n)}X$ is a weakly globular n–fold groupoid, and $\Pi^{(n)}_0 Q_{(n)}X$ is isomorphic to $Q_{(n-1)}X$.*

Proof By induction on n. The claim is immediate for $n=1$ (with $Q_{(0)}X := \pi_0 X$ and $\psi^X_{(1)}\colon X \to N\widehat{\pi}_1 X \simeq P^1X$ the Postnikov structure map).

(a) We assume that we have a map

$$\psi^X_{(n-1)}\colon X \to \mathrm{Diag}_{(n-1)} N_{(n-1)} Q_{(n-1)} X,$$

natural in X. Applying this to the simplicial object $L_\bullet X \in [\Delta^{\mathrm{op}}, [\Delta^{\mathrm{op}}, \mathsf{Set}]]$ (which is fibrant in each simplicial dimension, by Remark 2.8), we obtain a map of bisimplicial sets

$$\psi^{L_\bullet X}_{(n-1)}\colon L_\bullet X \to \overline{\mathrm{Diag}}^{(n)}_{(n-1)} \bar{N}_{(n-1)} \bar{Q}^{(n)}_{(n-1)} L_\bullet X,$$

which is an $(n-1)$–equivalence in each simplicial dimension.

However, in simplicial dimension 0 we have $L_0X = \mathrm{Dec}\, X$, which is homotopically trivial, while $Q_{(n-1)}\mathrm{Dec}\, X$ is a homotopically discrete $(n-1)$–fold groupoid by induction assumption (c) for $n-1$, so $dNQ_{(n-1)}\mathrm{Dec}\, X$ is homotopically trivial by Corollary 3.18. Thus $\psi^{L_0X}_{(n-1)}$ is actually a geometric weak equivalence, so $\psi^{L_\bullet X}_{(n-1)}$ is a diagonal n–equivalence (see Definition 3.29), which implies that

$$\mathrm{Diag}\,\psi^{L_\bullet X}_{(n-1)}\colon \mathrm{Diag}\, L_\bullet X \to \mathrm{Diag}_{(n)} \bar{N}_{(n-1)} \bar{Q}^{(n)}_{(n-1)} L_\bullet X \tag{4-29}$$

is an n–equivalence by Proposition 3.30.

Now by (4-15), $N^{(n)}Q_{(n)}X = \bar{Q}^{(n)}_{(n-1)} L_\bullet X$, so together with the map $\phi\colon \mathrm{c}X \to L_\bullet X$ of Lemma 4.14(a) we have maps of bisimplicial sets

$$\mathrm{c}X \xrightarrow{\phi} L_\bullet X \xrightarrow{\psi^{L_\bullet X}_{(n-1)}} \overline{\mathrm{Diag}}^{(n)}_{(n-1)} \bar{N}_{(n-1)} \bar{Q}^{(n)}_{(n-1)} L_\bullet X = \overline{\mathrm{Diag}}^{(n)}_{(n-1)} N^{(n)} Q_{(n)} X.$$

Applying Diag to both maps we see that the first is a weak equivalence, while the second is an n–equivalence, because (4-29) is such. We define the composite to be $\psi^X_{(n)}\colon X \to dNQ_{(n)}X$, which is therefore an n–equivalence.

(b) Let $f\colon X \to Y$ be a weak equivalence of Kan complexes. Since by part (a), $X \to \mathrm{Diag}_{(n)} Q_{(n)}X$ and $Y \to \mathrm{Diag}_{(n)} Q_{(n)}Y$ are n–equivalences, it follows that

$\mathrm{Diag}_{(n)}\, Q_{(n)} f$ is an n–equivalence. Furthermore, by Theorem 4.6, $\mathrm{Diag}_{(n)}\, Q_{(n)} X$ and $\mathrm{Diag}_{(n)}\, Q_{(n)} Y$ are n–types. Hence $\mathrm{Diag}_{(n)}\, Q_{(n)} f$ is a weak equivalence.

(c) Since X is homotopically trivial, by Lemma 4.14 for each $k \geq 1$ we have

$$\begin{aligned}(N^{(n)} Q_{(n)} X)_k &= Q_{(n-1)} L_k X \\ &= Q_{(n-1)} \operatorname{Dec} X \times_{Q_{(n-1)} X} \overset{k+1}{\cdots} \times_{Q_{(n-1)} X} Q_{(n-1)} \operatorname{Dec} X.\end{aligned}$$

Therefore $Q_{(n)} X = A^f$, where $A = Q_{(n-1)} \operatorname{Dec} X$ and by induction

$$f := Q_{(n-1)}\varepsilon\colon Q_{(n-1)} \operatorname{Dec} X \to Q_{(n-1)} X$$

is a map of homotopically discrete $(n-1)$–fold groupoids with a section $Q_{(n-1)}\sigma$ (see Section 2.7). Hence, $Q_{(n)} X$ is homotopically discrete, by definition.

(d) To show that $Q_{(n)} X$ is weakly globular (Definition 3.19), we think of it as a groupoid in Gpd^{n-1}, with $(n-1)$–fold groupoid of objects $(Q_{(n)} X)_0$ and $(n-1)$–fold groupoid of arrows $(Q_{(n)} X)_1$. Note that

$$(Q_{(n)} X)_0 = Q_{(n-1)} \operatorname{Dec} X$$

by (4-15) with $k = 0$, and since $\operatorname{Dec} X$ is homotopically discrete, $(Q_{(n)} X)_0^{(n)}$ is homotopically discrete, by (c).

Similarly,

$$(N^{(n)} Q_{(n)} X)_1^{(n)} = Q_{(n-1)} L_1 X \in \mathsf{Gpd}_{\mathrm{wg}}^{n-1},$$

and by (4-16)

$$\begin{aligned}(N^{(n)} Q_{(n)} X)_k &= (Q_{(n)} X)_1 \times_{(Q_{(n)} X)_0} \overset{k}{\cdots} \times_{(Q_{(n)} X)_0} (Q_{(n)} X)_1 \\ &= Q_{(n-1)} (L_1 X \times_{\operatorname{Dec} X} \overset{k}{\cdots} \times_{\operatorname{Dec} X} L_1 X),\end{aligned}$$

so $(N^{(n)} Q_{(n)} X)_k$ is weakly globular for each $k \geq 0$.

If we apply $\Pi_0^{(n-1)}$ in each simplicial dimension in the n^{th} direction, by Lemma 4.14(b) and the induction hypothesis, then

$$(\overline{\Pi}_0^{(n-1)} N^{(n)} Q_{(n)} X)_k^{(n)} = \Pi_0^{(n-1)} Q_{(n-1)} L_k X = Q_{(n-2)} L_k X = (Q_{(n-1)} X)_k,$$

where $(N^{(n-1)} Q_{(n-1)} X)_k^{(n-1)}$ is abbreviated to $(Q_{(n-1)} X)_k$.

This shows that $\Pi_0^{(n)} G$ lands in weakly globular $(n-1)$–fold groupoids, and that

$$\Pi_0^{(n)} Q_{(n)} X \cong Q_{(n-1)} X.$$

To prove that $Q_{(n)}X \in \mathsf{Gpd}^n_{\mathrm{wg}}$, it remains to show that in each simplicial dimension $k \geq 2$ (in the n^{th} direction), the map

$$(Q_{(n)}X)_1 \times_{(Q_{(n)}X)_0} \overset{k}{\cdots} \times_{(Q_{(n)}X)_0} (Q_{(n)}X)_1 \longrightarrow (Q_{(n)}X)_1 \times_{(Q_{(n)}X)_0^d} \overset{k}{\cdots} \times_{(Q_{(n)}X)_0^d} (Q_{(n)}X)_1 \tag{4-30}$$

is a geometric weak equivalence. By Lemma 4.14(b) we have

$$\begin{aligned}(Q_{(n)}X)_1 \times_{(Q_{(n)}X)_0} \overset{k}{\cdots} \times_{(Q_{(n)}X)_0} (Q_{(n)}X)_1 &= (N^{(n)}Q_{(n)}X)_k \\ &= Q_{(n-1)}L_k X \\ &\cong Q_{(n-1)}(L_1 X \times_{\operatorname{Dec} X} \overset{k}{\cdots} \times_{\operatorname{Dec} X} L_1 X),\end{aligned}$$

where the second equality is (4-15) and the third is (4-16).

Since $\operatorname{Dec} X$ is homotopically trivial, $Q_{(n-1)} \operatorname{Dec} X$ is homotopically discrete by (c), and so

$$(Q_{(n)}X)_0^d = (Q_{(n-1)} \operatorname{Dec} X)^d = Q_{(n-1)}c(\pi_0 \operatorname{Dec} X) = Q_{(n-1)}c(X_0)$$

by (a) and Lemma 3.15(d), where $c(X_0)$ is the constant simplicial set on X_0.

Since X is a Kan complex and $Q_{(n-1)}$ commutes with fiber products over discrete objects, by Remark 2.41, we have

$$\begin{aligned}(Q_{(n)}X)_1 &\times_{(Q_{(n)}X)_0^d} \overset{k}{\cdots} \times_{(Q_{(n)}X)_0^d} (Q_{(n)}X)_1 \\ &= Q_{(n-1)}L_1 X \times_{Q_{(n-1)}c(X_0)} \overset{k}{\cdots} \times_{Q_{(n-1)}c(X_0)} Q_{(n-1)}L_1 X \\ &\cong Q_{(n-1)}(L_1 X \times_{c(X_0)} \overset{k}{\cdots} \times_{c(X_0)} L_1 X).\end{aligned}$$

Since $\operatorname{Dec} X \to X$ is a fibration, so is $L_1 X \to \operatorname{Dec} X$, and $\operatorname{Dec} X \to c(X_0)$ is a weak equivalence; thus the map

$$L_1 X \times_{\operatorname{Dec} X} \overset{k}{\cdots} \times_{\operatorname{Dec} X} L_1 X \to L_1 X \times_{c(X_0)} \overset{k}{\cdots} \times_{c(X_0)} L_1 X$$

is a weak equivalence of Kan complexes. Therefore, by (b), (4-30) is a weak equivalence, as required. $\square$

Recall from Definition 3.28 that $\mathsf{P}^n\mathsf{Top}$ denotes the full subcategory of Top consisting of spaces T for which the natural map $T \to P^n T$ is a weak equivalence, and $\mathrm{ho}\mathsf{P}^n\mathsf{Top}$ is the corresponding full subcategory of the homotopy category $\mathrm{ho}\mathsf{Top}$ of topological spaces.

4.31 Definition Let $\mathrm{ho}\,\mathsf{Gpd}^n_{\mathrm{wg}}$ denote the localization of the category $\mathsf{Gpd}^n_{\mathrm{wg}}$ with respect to the (algebraic) weak equivalences (see Corollary 4.8 and compare with Gabriel and Zisman [29]).

4.32 Theorem *The functors $\widehat{Q}_{(n)}\colon \mathsf{Top} \to \mathsf{Gpd}^n_{\mathrm{wg}}$ and $B\colon \mathsf{Gpd}^n_{\mathrm{wg}} \to \mathsf{Top}$ induce functors*

$$\mathrm{ho}\mathsf{P}^n\mathsf{Top} \underset{\widehat{Q}_{(n)}}{\overset{B}{\rightleftarrows}} \mathrm{ho}\,\mathsf{Gpd}^n_{\mathrm{wg}}\ , \tag{4-33}$$

with $B \circ \widehat{Q}_{(n)} \cong \mathrm{Id}_{\mathrm{ho}\mathsf{P}^n\mathsf{Top}}$, so $\widehat{Q}_{(n)}\colon \mathrm{ho}\mathsf{P}^n\mathsf{Top} \to \mathrm{ho}\,\mathsf{Gpd}^n_{\mathrm{wg}}$ is a faithful embedding.

Proof By Theorem 4.6 and Proposition 4.28 both functors $\widehat{Q}_n = Q_{(n)}S$ and B preserve weak equivalences, and therefore induce corresponding functors on the homotopy categories. Also, for any $T \in \mathsf{P}^n\mathsf{Top}$, by Theorem 4.6 and Proposition 4.28, there is a span

$$B\widehat{Q}_{(n)}T \longleftarrow |ST| \longrightarrow T, \tag{4-34}$$

where the map on the left is a homotopy equivalence and the map on the right is a weak homotopy equivalence. It follows that T and $B\widehat{Q}_{(n)}T$ are weakly equivalent in $\mathsf{P}^n\mathsf{Top}$; that is, $B \circ \widehat{Q}_{(n)} \cong \mathrm{Id}_{\mathrm{ho}\mathsf{P}^n\mathsf{Top}}$. □

4.35 Weakly globular double groupoids We can strengthen Theorem 4.32 for $n=2$ to obtain an equivalence

$$\mathrm{ho}\mathsf{P}^2\mathsf{Top} \approx \mathrm{ho}\,\mathsf{Gpd}^2_{\mathrm{wg}},$$

where on the right-hand side we use the (internally defined) algebraic weak equivalences of $\mathsf{Gpd}^2_{\mathrm{wg}}$ itself.

As in Bullejos, Cegarra and Duskin [19, Theorem 2.5], for any double groupoid G one can construct a map $\varepsilon_\bullet\colon \mathrm{Or}_{(2)}\, dNG \to N_{(2)}G$. By Cegarra, Heredia and Remedios [22, Theorem 8], if G is weakly globular (and therefore $(2,2)$–fibrant), dNG is a Kan complex. Therefore, $P_{(2)}\,\mathrm{Or}_{(2)}\, dNG$ and $P_{(2)}N_{(2)}G = G$ are weakly globular double groupoids. Since we have a homotopy equivalence of Kan complexes $\xi\colon dNG \to \mathcal{S}\|dNG\| = \mathcal{S}BG$, we also have a geometric weak equivalence of weakly globular double groupoids

$$Q_{(2)}dNG \xrightarrow{Q_{(2)}\xi} Q_{(2)}\mathcal{S}BG = \widehat{Q}_{(2)}BG. \tag{4-36}$$

Therefore, the algebraic homotopy groups $\omega_*Q_{(2)}dNG$ are isomorphic by (3-27) to

$$\pi_*BQ_{(2)}dNG \cong \pi_*B\widehat{Q}_{(2)}BG \cong \pi_*BG$$

(using (4-34) for $T := BG$). By Theorem 4.6, $\omega_* G \cong \pi_* BG$, and since $\omega_* G \cong \pi_* dNG$, also by (3-27), we conclude that $\omega_* Q_{(2)} dNG \cong \omega_* G$. One can verify that this isomorphism is induced by the map

$$P_{(2)}\varepsilon_\bullet \colon P_{(2)} \operatorname{Or}_{(2)} dNG = Q_{(2)} dNG \to P_{(2)} N_{(2)} G = G,$$

which is therefore a geometric weak equivalence of double groupoids.

Together with a map of double groupoids induced by (4-36), we obtain a zig-zag of geometric weak equivalences

$$\widehat{Q}_{(2)} BG \xleftarrow{Q_{(2)}\xi} Q_{(2)} dNG \xrightarrow{P_{(2)}\varepsilon_\bullet} G.$$

This implies that (4-33) is an equivalence of localized categories when $n = 2$.

5 Tamsamani's model and weakly globular *n*–fold groupoids

In this section we construct a comparison functor from weakly globular n–fold groupoids to Tamsamani's weak n–groupoids, which preserves homotopy types.

5.A Tamsamani's weak *n*–groupoids

We begin with a brief recapitulation of the notion of a Tamsamani weak n–groupoid, starting with a modified definition. This differs somewhat from the original definition in Tamsamani [48, Section 5] (compare Simpson [47, Section 15.2] and Paoli [42, Section 8]), which was motivated by the goal of modeling higher *categories*, rather than groupoids.

5.1 Definition The category Tam^n of *Tamsamani weak n–groupoids* is a full subcategory of $[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$, defined inductively as follows:

(a) $\mathsf{Tam}^1 := \mathsf{Gpd}$ is the category of groupoids.

(b) Each $X \in \mathsf{Tam}^n$ is a simplicial object in Tam^{n-1} (in the first simplicial direction). We therefore have an inclusion functor $J_n \colon \mathsf{Tam}^n \to [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$.

(c) The 0^{th} Tamsamani weak $(n-1)$–groupoid X_0 is discrete (that is, a constant $(n-1)$–fold simplicial set).

(d) The Segal maps $\mu_k \colon X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ (Definition 2.3) are geometric weak equivalences of Tamsamani weak $(n-1)$–groupoids for each $k \geq 2$: that is, $B\mu_k \colon BX_k \to B(X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1)$ is a weak equivalence of topological spaces, where $B \colon \mathsf{Tam}^n \to \mathsf{Top}$ is the realization functor of (1-3).

(e) The $(n-1)$–simplicial set $\overline{\pi}_0^{(n)} J_n X$ is the nerve of a Tamsamani weak $(n-1)$ groupoid $\Pi_0^{(n)} X$, and we have a commutative diagram:

$$\begin{tikzcd}
\mathsf{Tam}^n \arrow[rr, "J_n"] \arrow[d, "\Pi_0^{(n)}"'] & & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]} \arrow[d, "\overline{\pi}_0^{(n)}"] \\
\mathsf{Tam}^{n-1} \arrow[r, "J_{n-1}"'] & {[\Delta^{n-2^{\mathrm{op}}}, \mathsf{Gpd}]} \arrow[r, "\bar{N}"'] & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Set}]}
\end{tikzcd}$$

Furthermore, $\Pi_0^{(n)}$ preserves geometric weak equivalences.

5.2 Tamsamani's original definition Tamsamani's original approach (as described in [42, Section 8]) gave an inductive definition of the category $\underline{\mathsf{Tam}}^n \subseteq [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$ equipped with a class of maps called *n–equivalences* for each $n \geq 1$, The following assumptions must be satisfied:

(a) $\underline{\mathsf{Tam}}^1 := \mathsf{Gpd}$ (with 1–equivalences being equivalences of groupoids).

(b) Each $X \in \underline{\mathsf{Tam}}^n$ is a simplicial object in $\underline{\mathsf{Tam}}^{n-1}$.

(c) X_0 is discrete.

(d) The Segal maps $\mu_k\colon X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ are $(n-1)$–equivalences in the category $\underline{\mathsf{Tam}}^{n-1}$ for each $k \geq 2$.

(e) The functor $\overline{\pi}_0^{(1)} \overline{\pi}_0^{(2)} \cdots \overline{\pi}_0^{(n)}\colon \underline{\mathsf{Tam}}^n \to \mathsf{Set}$, (see Definition 3.12) takes n–equivalences to bijections and preserves fiber products over discrete objects.

Note that (d) and (e) together imply that the Tamsamani *fundamental groupoid* functor

$$T_{(n)}^{\mathrm{Tm}} := \overline{\pi}_0^{(2)} \cdots \overline{\pi}_0^{(n-1)} \overline{\pi}_0^{(n)},$$

when applied to $X \in \underline{\mathsf{Tam}}^n$, lands in groupoids.

(f) For every a and b in the set X_0, the fiber of $X_{(a,b)}$ of $(d_0, d_1)\colon X_1 \to X_0 \times X_0$ is a Tamsamani weak $(n-1)$–groupoid.

(g) A map $f\colon X \to Y$ in $\underline{\mathsf{Tam}}^n$ is an n–equivalence if and only if:

(i) The map $T_{(n)}^{\mathrm{Tm}} f\colon T_{(n)}^{\mathrm{Tm}} X \to T_{(n)}^{\mathrm{Tm}} Y$ is an equivalence of groupoids.
(ii) $f_{(a,b)}\colon X_{(a,b)} \to Y_{(a,b)}$ is an $(n-1)$–equivalence for every $(a,b) \in X_0 \times X_0$.

5.3 Remark Note that if $g\colon X \to Y$ is a morphism in $\underline{\mathsf{Tam}}^n$ with Y discrete, then X is isomorphic to $\coprod_{y\in Y} g^{-1}\{y\}$, where the coproduct is taken in $\underline{\mathsf{Tam}}^n$ (compare Lemma 6.7 below).

This implies that if $X \in \underline{\mathsf{Tam}}^n$, then X_1 is isomorphic to the coproduct over all $a, b \in X_0$ of $X_1(a,b) \in \underline{\mathsf{Tam}}^{n-1}$ (where $X_1(a,b)$ is the fiber of $(d_0, d_1)\colon X_1 \to X_0 \times X_0$).

From this and from (e) we deduce that if $X \in \underline{\mathsf{Tam}}^n$, the $(n-1)$–simplicial set $\bar{\pi}_0^{(n)} J_n X$ is the nerve of an object $\Pi_0^{(n)} X$ of $\underline{\mathsf{Tam}}^{n-1}$ and we have the commutative diagram:

$$\begin{tikzcd}
\underline{\mathsf{Tam}}^n \arrow[rr, "J_n"] \arrow[d, "\Pi_0^{(n)}"'] & & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]} \arrow[d, "\bar{\pi}_0^{(n)}"] \\
\underline{\mathsf{Tam}}^{n-1} \arrow[r, "J_{n-1}"'] & {[\Delta^{n-2^{\mathrm{op}}}, \mathsf{Gpd}]} \arrow[r, "\bar{N}"'] & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Set}]}
\end{tikzcd}$$

Furthermore, $\Pi_0^{(n)}$ takes n–equivalences to $(n-1)$–equivalences, and one can therefore replace (g) in the definition above by the following:

(i) The map $\Pi_0^{(n)} f\colon \Pi_0^{(n)} X \to \Pi_0^{(n)} Y$ is an $(n-1)$– equivalence in $\underline{\mathsf{Tam}}^{n-1}$.

(ii) $f_{(a,b)}\colon X_{(a,b)} \to Y_{(a,b)}$ is an $(n-1)$–equivalence for every $(a,b) \in X_0 \times X_0$.

We recall the following fact from [42, Lemma 10.1]:

5.4 Lemma *A map $f\colon X \to Y$ in $\underline{\mathsf{Tam}}^n$ is an n–equivalence if and only if it is a geometric weak equivalence.*

5.5 Proposition *The categories Tam^n and $\underline{\mathsf{Tam}}^n$ are identical.*

Proof By induction on $n \geq 1$, starting with $\mathsf{Tam}^1 = \mathsf{Gpd} = \underline{\mathsf{Tam}}^1$. The fact that $\underline{\mathsf{Tam}}^n$ is contained in Tam^n is immediate (by the induction hypothesis and Lemma 5.4), while the other direction follows from Remark 5.3 and Lemma 5.4 again. □

5.6 Definition Let $\mathrm{ho}\,\mathsf{Tam}^n$ denote the localization of the category Tam^n with respect to the n–equivalences.

5.7 Theorem (Tamsamani [48, Theorem 8.0]) *There is a* Poincaré n–groupoid *functor $F^n_{\mathrm{Tm}}\colon \mathsf{Top} \to \mathsf{Tam}^n$ which, together with $B\colon \mathsf{Tam}^n \to \mathsf{Top}$, induces equivalences of categories*

$$\mathrm{ho}\mathsf{P}^n\mathsf{Top} \underset{F^n_{\mathrm{Tm}}}{\overset{B}{\rightleftarrows}} \mathrm{ho}\,\mathsf{Tam}^n . \tag{5-8}$$

For every $T \in \mathsf{Top}$, there is a zig-zag of weak equivalences in $\mathsf{P}^n\mathsf{Top}$ between $BF^n_{\mathrm{Tm}}T$ and P^nT, and for every $X \in \mathsf{Tam}^n$, there is a natural weak equivalence $X \to F^n_{\mathrm{Tm}}BX$.

5.B Comparison with weakly globular n–fold groupoids

We construct iteratively a discretization functor $D_n\colon \mathsf{Gpd}^n_{\mathrm{wg}} \to \mathsf{Tam}^n$, which preserves the homotopy type.

5.9 Two simplicial constructions Let $\mathcal{C}$ be a (co)complete category, $X \in [\Delta^{\mathrm{op}}, \mathcal{C}]$ a simplicial object, and $\gamma\colon X_0 \to W$ a map in $\mathcal{C}$. In this context we mimic the construction of a new simplicial object $Y \in [\Delta^{\mathrm{op}}, \mathcal{C}]$ described in our paper [11, Section 3], as follows:

Consider the pushout in $\mathcal{C}$

(5-10)
$$\begin{tikzcd} X_0 \arrow[r, "s_{(n)}"] \arrow[d, "\gamma"'] & X_n \arrow[d, "f_n"] \\ W \arrow[r, "\sigma_{(n)}"'] & Y_n \end{tikzcd}$$

where $s_{(n)}$ is induced by the unique morphism $[0] \to [n]$ in Δ^{op}. For any morphism $\phi\colon [n] \to [m]$ in Δ^{op}, $\phi s_{(n)} = s_{(m)}$ by the uniqueness, so that

$$f_m \phi s_{(n)} = f_m s_{(m)} = \sigma_{(m)} f_0\colon X_0 \to Y_m.$$

By the universal property of pushouts there exists a unique $\widehat{\phi}\colon Y_n \to Y_m$ with $\widehat{\phi} f_n = f_m \widehat{\phi}$ and $\widehat{\phi}\sigma_{(n)} = \sigma_{(m)}$. In particular, we have maps $\widehat{d}_i\colon Y_n \to Y_{n-1}$ for $0 \le i \le n$, and $\widehat{\sigma}_i\colon Y_{n-1} \to Y_n$ for $0 \le i < n$. The maps $\widehat{d}_i$ and $\widehat{\sigma}_i$ satisfy the simplicial identities, so that Y is a simplicial object in $\mathcal{C}$. In fact, if

$$[n] \xrightarrow{\phi} [m] \xrightarrow{\psi} [k]$$

are morphisms in Δ^{op} and $\xi = \psi \circ \phi$, then

$$\widehat{\xi}\sigma_{(n)} = \sigma_{(k)} = \widehat{\psi}\sigma_{(m)} = \widehat{\psi}\widehat{\phi}\sigma_{(n)},$$
$$\widehat{\xi} f_n = f_k \widehat{\xi} = f_k \widehat{\psi}\widehat{\phi} = \widehat{\psi} f_m \widehat{\phi} = \widehat{\psi}\widehat{\phi} f_n.$$

It follows by universal property of pushouts that $\widehat{\xi} = \widehat{\psi}\widehat{\phi}$. In particular, since the simplicial identities are satisfied by d_i and σ_i, they are satisfied by $\widehat{d}_i$ and $\widehat{\sigma}_i$. So we have a map of simplicial objects $f\colon X \to Y$.

Note that if $\gamma'\colon W \to X_0$ is a section for γ (with $\gamma\gamma' = \mathrm{Id}$), we may construct a new simplicial object $X^\gamma \in [\Delta^{\mathrm{op}}, \mathcal{C}]$ by setting

$$X^\gamma_n = \begin{cases} W & \text{if } n = 0, \\ X_n & \text{if } n > 0. \end{cases}$$

Let $d_0, d_1\colon X_1 \to X_0$ and $\sigma_0 = s_{(1)}\colon X_0 \to X_1$ be the face and degeneracy maps of X, and let $d'_0,\, d'_1\colon X_1 \to W$, and $\sigma'_0\colon W \to X_1$, respectively, denote $d'_i = \gamma d_i$ $(i = 0, 1)$, and $\sigma'_0 = \sigma_0\gamma'$. All other face and degeneracy operators of X^γ are the same as those of X.

Finally, we define a map $h\colon X^\gamma \to Y$ in $[\Delta^{\mathrm{op}}, \mathscr{C}]$ by setting $h_0 := \mathrm{Id}$ and $h_n := f_n$ for $n > 0$. In fact, $d'_i = \widehat{d}_i f_1$; also, $f_1\sigma_0 = \widehat{\sigma}_0\gamma$, which implies $f_1\sigma_0\gamma' = \widehat{\sigma}_0\gamma\gamma' = \widehat{\sigma}_0$. All other identities are the same as for f.

5.11 The functor *D* Let $[\Delta^{\mathrm{op}}, \mathsf{Set}]^2_h$ be the full subcategory of bisimplicial sets X such that the simplicial set X_0 is homotopically trivial, through a weak equivalence $\gamma\colon X_0 \to X_0^d$ with a section $\gamma'\colon X_0^d \to X_0$ with $\gamma\gamma' = \mathrm{Id}$, where X_0^d is the constant simplicial set on $\pi_0 X_0$. Let $[\Delta^{\mathrm{op}}, \mathsf{Set}]^2_d$ denote the full subcategory of bisimplicial sets X such that the simplicial set X_0 is constant. We construct a functor

$$D\colon [\Delta^{\mathrm{op}}, \mathsf{Set}]^2_h \to [\Delta^{\mathrm{op}}, \mathsf{Set}]^2_d$$

by setting $DX := X^\gamma$ (in the notation of Section 5.9).

5.12 Lemma *Let $D\colon [\Delta^{\mathrm{op}}, \mathsf{Set}]^2_h \to [\Delta^{\mathrm{op}}, \mathsf{Set}]^2_d$ be as above. Then for each $X \in [\Delta^{\mathrm{op}}, \mathsf{Set}]^2_h$, DX and X have the same homotopy type.*

Proof We construct a bisimplicial set Y and weak equivalences

$$X \xrightarrow{f} Y \xleftarrow{h} DX$$

using the construction of Section 5.9, for $\mathscr{C} = [\Delta^{\mathrm{op}}, \mathsf{Set}]$, $W := X_0^d$ and $\gamma\colon X_0 \to X_0^d$ as above. Since γ is a weak equivalence and $s_{(n)}$ is a cofibration of simplicial sets, the right vertical map f_n in (5-10) is a weak equivalence for each $n \geq 0$, that is, we have a map of bisimplicial sets $f\colon X \to Y$ which is a levelwise weak equivalence. Thus Bf is also a weak equivalence.

Since the map $h\colon DX \to Y$ of Section 5.9 is a levelwise weak equivalence, Bh is a weak equivalence. In conclusion, f and h are weak equivalences, so that

$$\mathrm{Diag}\, X \simeq \mathrm{Diag}\, DX. \qquad \square$$

5.13 Definition We define the 0–*discretization* functor

$$\mathrm{Disc}_0\colon \mathsf{Gpd}^n_{\mathrm{wg}} \to [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-1}_{\mathrm{wg}}]$$

on any weakly globular n–fold groupoid G as follows: set (see Notation 2.18)

$$(\mathrm{Disc}_0\, G)_k := \begin{cases} G_0^d & \text{if } k = 0, \\ (N^{(1)}G)_k & \text{if } k > 0. \end{cases}$$

If $d_0, d_1\colon G_1 \to G_0$ are the source and target maps, and $\sigma_0\colon G_0 \to G_1$ is the degeneracy operator (all in Gpd^{n-1}), we define $d'_0, d'_1\colon (\mathrm{Disc}_0\, G)_1 \to (\mathrm{Disc}_0\, G)_0$ and $\sigma'_0\colon (\mathrm{Disc}_0\, G)_0 \to (\mathrm{Disc}_0\, G)_1$ by $d'_i = \gamma d_i$ $(i = 0, 1)$ and $\sigma'_0 = \sigma_0 \gamma'$. All other face and degeneracy operators of $\mathrm{Disc}_0\, G$ are those of G. Since $\gamma\gamma' = \mathrm{Id}$, all simplicial identities hold for $\mathrm{Disc}_0\, G$.

5.14 Lemma *For any weakly globular n–fold groupoid $G \in \mathsf{Gpd}^n_{\mathrm{wg}}$, $\mathrm{Diag}_{(n)}\, G$ and $\mathrm{Diag}_{(n)}\, \mathrm{Disc}_0\, G$ are weakly equivalent.*

Proof $\mathrm{Diag}_{(n)}\, G$ is the diagonal of the bisimplicial set X with

$$X_k := \mathrm{Diag}_{(n-1)}(N^{(n)} G)^{(n)}_k$$

for all $k \geq 0$, while $\mathrm{Diag}_{(n)}\, \mathrm{Disc}_0\, G$ is the diagonal of the bisimplicial set Y with $Y_0 := G^d_0$ and $Y_k := \mathrm{Diag}_{(n-1)}\, (N^{(n)} G)^{(n)}_k$ for $k \geq 1$. By construction, $X \in [\Delta^{\mathrm{op}}, \mathsf{Set}]^2_h$ and $Y = DX$. Hence, by Lemma 5.12, $\mathrm{Diag}_{(n)}\, G = \mathrm{Diag}\, X \simeq \mathrm{Diag}\, Y = \mathrm{Diag}_{(n)}\, \mathrm{Disc}_0\, G$. □

5.15 Notation Let $T^{\mathrm{wg}}_{(n)}\colon \mathsf{Gpd}^n_{\mathrm{wg}} \to \mathsf{Gpd}$ denote the weakly globular *fundamental groupoid* functor, that is, the composite

$$T^{\mathrm{wg}}_{(n)} := \Pi^{(2)}_0 \cdots \Pi^{(n-1)}_0 \Pi^{(n)}_0 \tag{5-16}$$

(see Definitions 3.12 and 3.19).

By construction, for all $i \geq 0$,

$$(T^{\mathrm{wg}}_{(n)} G)_i = \pi_0 T^{\mathrm{wg}}_{(n-1)} G_i. \tag{5-17}$$

5.18 Definition For each $n \geq 1$, we define discretization functors

$$D_n\colon \mathsf{Gpd}^n_{\mathrm{wg}} \to [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$$

by induction on n, starting with $D_1 := \mathrm{Id}\colon \mathsf{Gpd} \to \mathsf{Gpd}$. For $n \geq 2$, we let D_n be the composite

$$\mathsf{Gpd}^n_{\mathrm{wg}} \xrightarrow{N^{(n)}} [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-1}_{\mathrm{wg}}] \xrightarrow{\mathrm{Disc}_0} [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-1}_{\mathrm{wg}}] \xrightarrow{\bar{D}_{n-1}} [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}],$$

where $\bar{D}_{n-1}$ is obtained by applying D_{n-1} in each simplicial dimension.

5.19 Theorem *The functor D_n lands in Tam^n. Furthermore, $T^{\mathrm{Tm}}_{(n)} D_n = T^{\mathrm{wg}}_{(n)}$, and for each $G \in \mathsf{Gpd}^n_{\mathrm{wg}}$, we have a natural weak equivalence*

$$\mathrm{Diag}_{(n)}\, G \simeq \mathrm{Diag}_{(n)}\, D_n G.$$

Proof By induction on $n \geq 2$. For $n = 2$, note that $D_2 G = \mathrm{Disc}_0\, N^{(2)} G$ is in Tam^2 for any weakly globular double groupoid G, since for each $k \geq 2$ by Definition 3.19(iv) we have

$$\begin{aligned} (D_2 G)_k &= G_1 \times_{G_0} \overset{k}{\cdots} \times_{G_0} G_1 \\ &\simeq G_1 \times_{G_0^d} \overset{k}{\cdots} \times_{G_0^d} G_1 \\ &\simeq (D_2 G)_1 \times_{(D_2 G)_0} \overset{k}{\cdots} \times_{(D_2 G)_0} (D_2 G)_1 . \end{aligned}$$

Furthermore, $T^{\mathrm{Tm}}_{(2)} D_2 G = T^{\mathrm{wg}}_{(2)} G = \Pi^{(2)}_0 G$ is a groupoid. Hence by definition, $D_2 G \in \mathsf{Tam}^2$. By Lemma 5.14, $B D_2 G \simeq dNG$ since $G \in \mathsf{Gpd}^2_{\mathrm{wg}}$.

In the induction step, note that $(D_n G)_0 = G^d_0$ is discrete. So to prove that $D_n G$ is in Tam^n, it remains to show the following:

(a) The Segal maps

$$\mu_k \colon (D_n G)_k \to (D_n G)_1 \times_{(D_n G)_0} \overset{k}{\cdots} \times_{(D_n G)_0} (D_n G)_1$$

are geometric weak equivalences.

(b) $T^{\mathrm{Tm}}_{(n)} D_n G$ is a groupoid.

Note that by Definition 5.18 and by the inductive hypothesis, for $k \geq 2$ we have

$$\begin{aligned} \mathrm{Diag}_{(n-1)} (D_n G)_k &= \mathrm{Diag}_{(n-1)}\, D_{n-1} \big(G_1 \times_{G_0} \overset{k}{\cdots} \times_{G_0} G_1 \big) \\ &\simeq dN \big(G_1 \times_{G_0} \overset{k}{\cdots} \times_{G_0} G_1 \big), \end{aligned}$$

and by Definition 3.19(iv) and the inductive hypothesis this is weakly equivalent to

$$dN \big(G_1 \times_{G_0^d} \overset{k}{\cdots} \times_{G_0^d} G_1 \big) \simeq \mathrm{Diag}_{(n-1)}\, D_{n-1} G_1 \times_{dNG_0^d} \overset{k}{\cdots} \times_{dNG_0^d} \mathrm{Diag}_{(n-1)}\, D_{n-1} G_1$$

which is $\mathrm{Diag}_{(n)} ((D_n G)_1 \times_{(D_n G)_0} \overset{k}{\cdots} \times_{(D_n G)_0} (D_n G)_1)$ by Definition 5.18. Thus each Segal map μ_k is a geometric weak equivalence. This proves (a).

To prove (b), note that by definition of $T^{\mathrm{Tm}}_{(n)}$, (5-16) and (5-17), we have

$$\begin{aligned} \big(T^{\mathrm{Tm}}_{(n)} (D_n G) \big)_0 = \pi_0 T^{\mathrm{Tm}}_{(n-1)} (D_n G)_0 &= \pi_0 T^{\mathrm{Tm}}_{(n-1)} G^d_0 \\ &= G^d_0 = \pi_0 N T^{\mathrm{wg}}_{(n-1)} G_0 = \big(T^{\mathrm{wg}}_{(n)} G \big)_0, \end{aligned}$$

where $\pi_0 T^{\mathrm{Tm}}_{(n)} X = \Pi^{(1)}_0 \Pi^{(2)}_0 \cdots \Pi^{(n)}_0 X$.

Furthermore,

$$\big(T^{\mathrm{Tm}}_{(n)} D_n G\big)_k = \pi_0 T^{\mathrm{Tm}}_{(n-1)}(D_n G)_k = \pi_0 T^{\mathrm{Tm}}_{(n-1)} D_{n-1}(N^{(n)}G)_k$$

for $k \geq 1$. By induction we therefore have

$$\pi_0 T^{\mathrm{Tm}}_{(n-1)} D_{n-1}(N^{(n)}G)_k = \pi_0 T^{\mathrm{wg}}_{(n-1)}(N^{(n)}G)_k = \big(T^{\mathrm{wg}}_{(n)} G\big)_k,$$

It follows that $T^{\mathrm{Tm}}_{(n)} D_n G = T^{\mathrm{wg}}_{(n)} G$, as claimed. Since $T^{\mathrm{wg}}_{(n)} G$ is a groupoid, so is $T^{\mathrm{Tm}}_{(n)} D_n G$. This concludes the proof that $D_n G \in \mathsf{Tam}^n$.

Finally, we show that $\mathrm{Diag}_{(n)} D_n G \simeq dNG$. Let $Y = \mathrm{Disc}_0\, N^{(n)} G \in [\Delta^{\mathrm{op}}, \mathsf{Gpd}^{n-1}_{\mathrm{wg}}]$. By Lemma 5.14, $dNY \simeq dNG$. Furthermore, $\mathrm{Diag}_{(n)} D_n G$ is the realization of the bisimplicial set Z with $Z_k := \mathrm{Diag}_{(n-1)} D_{n-1} Y_k$. By induction, $Z_k \simeq \mathrm{Diag}_{(n-1)} Y_k$, so that $\mathrm{Diag}\, Z \simeq dNY \simeq dNG$, as required. □

5.20 Remark Since by Tamsamani [48, Section 8], $BD_n G$ is an n–type, it follows from Theorem 5.19 that the realization of a weakly globular n–fold groupoid is an n–type. This provides an alternative proof of the first statement in Theorem 4.6. Moreover, [48, Section 5] provides a formula for the homotopy groups

$$\pi_n(BD_n G, x) = \mathrm{Aut}_{\mathscr{C}_n(D_n G)}(\mathrm{Id}_x),$$

where $\mathscr{C}_n(D_n G)$ is the groupoid $\mathcal{W}_{(n,n-1)} G$. This matches (4-7).

6 Weakly globular pseudo n–fold groupoids

We now introduce the category $\mathsf{PsGpd}^n_{\mathrm{wg}}$ of weakly globular pseudo n–fold groupoids, and prove Theorem 6.23, stating that there is a zig-zag of weak equivalences between any $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$ and $\widehat{Q}_{(n)} BX$. This implies our second main result (Theorem 6.28), stating that $\widehat{Q}_{(n)}$ induces an equivalence $\mathrm{ho}\mathsf{P}^n\mathsf{Top} \simeq \mathrm{ho}\,\mathsf{PsGpd}^n_{\mathrm{wg}}$.

6.A Types of pseudo n–fold groupoids

The notion of a weakly globular pseudo n–fold groupoid is a further relaxation of $\mathsf{Gpd}^n_{\mathrm{wg}}$, similarly defined using a subcategory of homotopically discrete objects.

6.1 Definition For each n, we introduce a full subcategory $\mathsf{PsGpd}^n_{\mathrm{hd}}$ of $[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$, whose objects are called *homotopically discrete pseudo n–fold groupoids*. These categories are defined by induction on $n \geq 1$, as follows:

(a) The category $\mathsf{PsGpd}^1_{\mathrm{hd}} = \mathsf{Gpd}^1_{\mathrm{hd}}$ consists of the homotopically discrete groupoids.

(b) If $X \in \mathsf{PsGpd}^n_{\mathrm{hd}}$, then $X_k \in \mathsf{PsGpd}^{n-1}_{\mathrm{hd}}$ for all $k \geq 0$, where k is the simplicial dimension in the first direction (see Definition 3.19).

(c) If $X \in \mathsf{PsGpd}^n_{\mathrm{hd}}$, the $(n-1)$–simplicial set $\bar{\pi}^{(n)}_0 J_n X$ is the nerve of an object $\Pi^{(n)}_0 X$ of $\mathsf{PsGpd}^{n-1}_{\mathrm{hd}}$ and the following diagram commutes (where J_n denotes the inclusion):

$$\begin{tikzcd}
\mathsf{PsGpd}^n_{\mathrm{hd}} \arrow[rr, "J_n"] \arrow[d, "\Pi^{(n)}_0"'] & & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]} \arrow[d, "\bar{\pi}^{(n)}_0"] \\
\mathsf{PsGpd}^{n-1}_{\mathrm{hd}} \arrow[r, "J_{n-1}"'] & {[\Delta^{n-2^{\mathrm{op}}}, \mathsf{Gpd}]} \arrow[r, "\bar{N}"'] & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Set}]}
\end{tikzcd}$$

Furthermore, the map $\bar{\gamma}$ of Definition 3.12 induces a map $\gamma^{(n)}\colon X \to \mathrm{c}^{(n)}\Pi^{(n)}_0 X$ in $\mathsf{PsGpd}^n_{\mathrm{hd}}$ which is a weak equivalence of groupoids in each multi-simplicial dimension (and thus a geometric weak equivalence by Remark 2.22).

(d) For each $k \geq 2$, the induced Segal map

$$X_k \xrightarrow{\hat{\mu}_k} X_1 \times_{X^d_0} \overset{k}{\cdots} \times_{X^d_0} X_1 \tag{6-2}$$

of (1-4) is a geometric weak equivalence.

Note that condition (c) implies that the composite $\gamma_{(n)}$ of

$$X \xrightarrow{\gamma^{(n)}} \mathrm{c}^{(n)}\Pi^{(n)}_0 X \xrightarrow{\mathrm{c}^{(n)}\gamma^{(n-1)}} \cdots \longrightarrow \mathrm{c}^{(1)}\cdots\mathrm{c}^{(n)}\Pi^{(1)}_0\cdots\Pi^{(n)}_0 X, \tag{6-3}$$

is a geometric weak equivalence, so that BX is a homotopically trivial simplicial set (that is, a 0–type).

6.4 Definition We now use Definition 6.1 to specify, for each $n \geq 1$, another full subcategory $\mathsf{PsGpd}^n_{\mathrm{wg}}$ of $[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$, whose objects are called *weakly globular pseudo n–fold groupoids*, defined by induction on $n \geq 1$.

(a) $\mathsf{PsGpd}^1_{\mathrm{wg}} := \mathsf{Gpd}$.

(b) If $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$, then $X_0 \in \mathsf{PsGpd}^{n-1}_{\mathrm{hd}}$ and $X_k \in \mathsf{PsGpd}^{n-1}_{\mathrm{wg}}$ for all $k \geq 1$.

(c) If $X \in \mathsf{PsGpd}^n_{\mathrm{hd}}$, the $(n-1)$–simplicial set $\bar{\pi}^{(n)}_0 J_n X$ is the nerve of an object $\Pi^{(n)}_0 X$ of $\mathsf{PsGpd}^{n-1}_{\mathrm{hd}}$ and the following diagram commutes (where J_n denotes the inclusion):

$$\begin{tikzcd}
\mathsf{PsGpd}^n_{\mathrm{wg}} \arrow[rr, "J_n"] \arrow[d, "\Pi_0^{(n)}"'] & & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]} \arrow[d, "\overline{\pi}_0^{(n)}"] \\
\mathsf{PsGpd}^{n-1}_{\mathrm{wg}} \arrow[r, "J_{n-1}"'] & {[\Delta^{n-2^{\mathrm{op}}}, \mathsf{Gpd}]} \arrow[r, "\bar{N}"'] & {[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Set}]}
\end{tikzcd}$$

Furthermore, $\Pi_0^{(n)}$ preserves geometric weak equivalences.

(d) For each $k \geq 2$, the induced Segal map

$$X_k \xrightarrow{\widehat{\mu}_k} X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$$

is a geometric weak equivalence.

6.5 Remark Both Tam^n and $\mathsf{Gpd}^n_{\mathrm{wg}}$ are full subcategories of $\mathsf{PsGpd}^n_{\mathrm{wg}}$, and $\mathsf{Gpd}^n_{\mathrm{hd}}$ is a full subcategory of $\mathsf{PsGpd}^n_{\mathrm{hd}}$.

6.6 Example When $n = 2$, a weakly globular pseudo double groupoid is just a simplicial object in groupoids $X \in [\Delta^{\mathrm{op}}, \mathsf{Gpd}]$ such that X_0 is a homotopically discrete groupoid, the simplicial set $\overline{\pi}_0^{(2)} X$ is the nerve of a groupoid, and for each $k \geq 2$, the induced Segal map

$$X_k \xrightarrow{\widehat{\mu}_k} X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$$

is an equivalence of groupoids.

6.7 Lemma *If $f\colon X \to Y$ is a map in $\mathsf{PsGpd}^n_{\mathrm{wg}}$, and Y is discrete (that is, the constant $(n-1)$–fold simplicial object on a discrete groupoid), then X is the coproduct in $\mathsf{PsGpd}^n_{\mathrm{wg}}$ of the fibers $X^{-1}(a)$, taken over all $a \in Y$.*

Proof By induction on $n \geq 1$, where for $n = 1$, X is a groupoid, which is a coproduct of its connected components. The n^{th} step follows from the $(n-1)^{\mathrm{st}}$, since coproducts in $\mathsf{PsGpd}^n_{\mathrm{wg}}$ are those of $[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$, namely, disjoint unions, which are therefore taken dimensionwise. □

6.8 Corollary *If $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$, then X_1 is isomorphic to the coproduct in $\mathsf{PsGpd}^{n-1}_{\mathrm{wg}}$ of $X_1(a,b)$ (the fiber of $(\gamma_{(n)} d_0, \gamma_{(n)} d_1)\colon X_1 \to X_0^d \times X_0^d$, taken over all $(a,b) \in X_0^d \times X_0^d$).*

6.9 Definition We now define the notion of *n–equivalence* for maps of weakly globular pseudo n–fold groupoids by induction on $n \geq 1$, where a 1–equivalence is simply an equivalence of groupoids.

A map $f\colon X \to Y$ in $\mathsf{PsGpd}^n_{\mathrm{wg}}$ is an n–equivalence if:

(a) $\Pi_0^{(n)} f\colon \Pi_0^{(n)} X \to \Pi_0^{(n)} Y$ is an $(n-1)$– equivalence in $\mathsf{PsGpd}^{n-1}_{\mathrm{wg}}$.

(b) For every $a, b \in X_0^d$, the map $f(a,b)\colon X_1(a,b) \to Y_1(f(a), f(b))$ is also an $(n-1)$– equivalence in $\mathsf{PsGpd}^{n-1}_{\mathrm{wg}}$.

6.B Comparison with Tamsamani's weak *n*–groupoids

We describe a procedure for transforming a weakly globular pseudo n–fold groupoid X into a Tamsamani weak n–groupoid, without altering the homotopy type. The construction is done in two stages:

In the first, we use the general construction of Section 5.9 to produce $\mathrm{Disc}_0 X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$, in which only X_0 is discretized (as in Section 5.B). This time we must proceed by induction on the n simplicial directions in order to obtain a zig-zag of intermediate objects (in Lemma 6.21), all weakly equivalent in $\mathsf{PsGpd}^n_{\mathrm{wg}}$ (which was not possible in $\mathsf{Gpd}^n_{\mathrm{wg}}$).

In the second stage, we define the full discretization functors $D_n\colon \mathsf{PsGpd}^n_{\mathrm{wg}} \to \mathsf{Tam}^n$ by induction on $n \geq 2$, with $D_2 := \mathrm{Disc}_0$, so as to make each X_k a Tamsamani weak $(n-1)$–groupoid.

First, we need some technical facts about weakly globular pseudo n–fold groupoids:

6.10 Lemma *If $f\colon X \to Y$ is a map in $[\Delta^{\mathrm{op}}, \mathsf{PsGpd}^{n-1}_{\mathrm{wg}}]$ which is a weak equivalence in each simplicial dimension, with $Y_0 \in \mathsf{PsGpd}^{n-1}_{\mathrm{hd}}$ and $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$, then for each $k \geq 2$ the induced Segal maps of (6-2) for Y are geometric weak equivalences.*

Proof First note that f induces an isomorphism $X_0^d \cong Y_0^d$, so by Corollary 6.8 it follows that

$$f_1\colon X_1 \to Y_1$$

is the coproduct over $(a,b) \in X_0^d \times X_0^d$ of its restrictions $f_1(a,b)\colon X_1(a,b) \to Y_1(a,b)$. Since the classifying space functor $B\colon \mathsf{PsGpd}^{n-1}_{\mathrm{wg}} \to \mathsf{Top}$ commutes with disjoint unions, the fact that f_1 is a weak equivalence implies that each $f_1(a,b)$ is a geometric weak equivalence in $\mathsf{PsGpd}^{n-1}_{\mathrm{wg}}$.

Moreover, since X_0^d is discrete,

$$X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \cong \coprod_{a_0,\dots,a_k \in X_0^d} X_1(a_0,a_1) \times X_1(a_1,a_2) \times \cdots \times X_1(a_{k-1},a_k) \tag{6-11}$$

and similarly for Y.

Now consider the commutative diagram

$$\begin{tikzcd} X_k \arrow[r, "\widehat{\mu}_k^X", "\simeq"'] \arrow[d, "f_k"', "\simeq"] & X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \arrow[d, "(f_1 \times \cdots \times f_1)"] \\ Y_k \arrow[r, "\widehat{\mu}_k^Y"'] & Y_1 \times_{Y_0^d} \overset{k}{\cdots} \times_{Y_0^d} Y_1 \end{tikzcd} \tag{6-12}$$

of induced Segal maps, where the left vertical map is a geometric weak equivalence by assumption, as is the top horizontal induced Segal map (since $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$), while the right horizontal map is a geometric weak equivalence because of (6-11). Therefore, μ_k^Y is a geometric weak equivalence, too. □

6.13 Lemma *Consider a pushout diagram*

$$\begin{tikzcd} A \arrow[r, hook, "j"] \arrow[d, "\gamma^{(n)}"', "\simeq"] & B \arrow[d, "g"] \\ \mathrm{c}^{(n)} \Pi_0^{(n)} A \arrow[r, "h"'] & C \end{tikzcd} \tag{6-14}$$

in $[\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$, *with* $A \in \mathsf{PsGpd}^n_{\mathrm{hd}}$ *and* j *monic. Then:*

(a) *If* $B \in \mathsf{PsGpd}^n_{\mathrm{wg}}$, *so is* C.

(b) *If* $B \in \mathsf{PsGpd}^n_{\mathrm{hd}}$, *so is* C.

Proof By induction on $n \geq 1$:

First note that for any $n \geq 1$, g is a geometric weak equivalence, since f is, because $\mathrm{Diag}_{(n)}$ preserves pushouts, A is in $\mathsf{PsGpd}^n_{\mathrm{hd}}$, and $\mathrm{Diag}_{(n)}\, j$ is a cofibration of simplicial sets.

When $n = 1$, (6-14) is a diagram of groupoids, so (a) is clear, and (b) follows from Joyal and Street [35, Corollary 3].

In general, since the pushout is taken in a diagram category, C_0 is the pushout of the objects in simplicial dimension 0, which is therefore in $\mathsf{PsGpd}^{n-1}_{\mathrm{hd}}$ by (b) for $n-1$, while for $k \geq 1$, C_k is in $\mathsf{PsGpd}^{n-1}_{\mathrm{wg}}$ by (a) for $n-1$.

Since the functor $\Pi_0^{(n)}$ is defined by applying π_0 to each groupoid, π_0 commutes with pushouts of groupoids, and $\pi_0\gamma$ is an isomorphism, we see that $\Pi_0^{(n)} C = \Pi_0^{(n)} B$ is in $\mathsf{PsGpd}^{n-1}_{\mathrm{wg}}$ by (a) for $n-1$.

Finally, the Segal condition follows from Lemma 6.10 for g, since g_k is a weak equivalence for each $k \geq 0$, $B \in \mathsf{PsGpd}^{n}_{\mathrm{wg}}$, and $C_0 \in \mathsf{PsGpd}^{n-1}_{\mathrm{hd}}$.

This shows (a). Part (b) is immediate. □

6.15 Proposition *Assume given a weakly globular pseudo n–fold groupoid X, and let $Y \in [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$ be the result of applying the construction of Section 5.9 to the map $\gamma\colon X_0 \to W$ for $W := (\mathrm{c}^{(n)}\Pi_0^{(n)} X)_0$ and $\mathscr{C} = [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$; then Y is actually in $\mathsf{PsGpd}^{n}_{\mathrm{wg}}$. Moreover, the maps*

$$X \xrightarrow{f} Y \xleftarrow{h} X^{\gamma} \tag{6-16}$$

are geometric weak equivalences in $\mathsf{PsGpd}^{n}_{\mathrm{wg}}$, where X^{γ} is as in Section 5.9.

Proof First, note that $Y_0 := W$ is in $\mathsf{PsGpd}^{n-1}_{\mathrm{hd}}$, by Definition 6.4. Furthermore, for any $k \geq 1$, Y_k is defined by the pushout square of (5-10)

$$\begin{tikzcd} X_0 \arrow[r, "s_{(k)}"] \arrow[d, "\gamma"'] & X_k \arrow[d, "f_k"] \\ (\mathrm{c}^{(n)}\Pi_0^{(n)} X)_0 \arrow[r, "\sigma_{(k)}"'] & Y_k \end{tikzcd} \tag{6-17}$$

where γ is a geometric weak equivalence since X_0 is in $\mathsf{PsGpd}^{n-1}_{\mathrm{hd}}$, and the iterated degeneracy map $s_{(k)}$ is one-to-one since it has a left inverse $d_{(k)}$. Thus by Lemma 6.13, $Y_k \in \mathsf{PsGpd}^{n-1}_{\mathrm{wg}}$.

The maps f_k in (6-17) are geometric weak equivalences, since after applying $\mathrm{Diag}_{(n)}$ we obtain a pushout of a weak equivalence along a cofibration in $[\Delta^{\mathrm{op}}, \mathsf{Set}]$. Therefore, by Lemma 6.10 applied to f, the induced Segal maps for Y are weak equivalences.

Finally, $\Pi_0^{(n)} Y$ is obtained by applying π_0 to each groupoid of $Y \in [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$, and since this commutes with pushouts and $\pi_0\gamma$ is an isomorphism, we see that $\Pi_0^{(n)} Y \cong \Pi_0^{(n)} X$, so in particular it is in $\mathsf{PsGpd}^{n-1}_{\mathrm{wg}}$. This shows that $Y \in \mathsf{PsGpd}^{n}_{\mathrm{wg}}$.

Since each f_k is a geometric weak equivalence, as is $h_0 = \gamma$ and $h_k = \mathrm{Id}$ for $k \geq 1$, the two maps f and h are geometric weak equivalences in $\mathsf{PsGpd}^{n}_{\mathrm{wg}}$. □

6.18 Notation Let $T^{\mathrm{ps}}_{(n)}\colon \mathsf{PsGpd}^n_{\mathrm{wg}} \to \mathsf{Gpd}$ denote the *fundamental groupoid* functor for $\mathsf{PsGpd}^n_{\mathrm{wg}}$, that is, the composite

$$T^{\mathrm{ps}}_{(n)} := \Pi^{(2)}_0 \cdots \Pi^{(n-1)}_0 \Pi^{(n)}_0. \tag{6-19}$$

6.20 Definition For each $n \geq 2$ we define a sequence of functors $\mathrm{Disc}^{(k)}_0\colon \mathsf{PsGpd}^n_{\mathrm{wg}} \to \mathsf{PsGpd}^n_{\mathrm{wg}}$ (where $1 \leq k \leq n$) by setting

$$\mathrm{Disc}^{(k)}_0 X := X^{\gamma^{(k)}_{(n)}}$$

(in the notation of Section 5.9), where

$$\gamma^{(k)}_{(n)}\colon X_0 \to \big(\mathrm{c}^{(k)} \cdots \mathrm{c}^{(n)} \Pi^{(k)}_0 \cdots \Pi^{(n)}_0 X\big)_0$$

is the composite of the first k maps of (6-3) in dimension 0. We write Disc_0 for $\mathrm{Disc}^{(1)}_0$.

6.21 Lemma *For each $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$ we have a sequence of natural geometric weak equivalences*

$$\begin{tikzcd}[column sep=small]
X \arrow[dr, "f^{(n)}"] & & \mathrm{Disc}^{(n)}_0 X \arrow[dl, "h^{(n)}"'] \arrow[dr, "f^{(n-1)}"] & & & \mathrm{Disc}_0 X \arrow[dl, "h^{(1)}"'] \\
& Y^{(n)} & & Y^{(n-1)} \quad \cdots & Y^{(1)} &
\end{tikzcd}$$

Proof Each $Y^{(k)}$ is obtained by applying Proposition 6.15 to $\mathrm{Disc}^{(k+1)}_0 X$, where $\mathrm{Disc}^{(n+1)}_0 X := X$, and using (6-16). □

6.22 Definition We now define *discretization functors*

$$D_n\colon \mathsf{PsGpd}^n_{\mathrm{wg}} \to [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}]$$

for each $n \geq 1$ by induction on n, starting with $D_1 := \mathrm{Id}\colon \mathsf{Gpd} \to \mathsf{Gpd}$. For $n \geq 2$, we define D_n inductively to be the composite

$$\mathsf{PsGpd}^n_{\mathrm{wg}} \hookrightarrow [\Delta^{\mathrm{op}}, \mathsf{PsGpd}^{n-1}_{\mathrm{wg}}] \xrightarrow{\mathrm{Disc}_0} [\Delta^{\mathrm{op}}, \mathsf{PsGpd}^{n-1}_{\mathrm{wg}}] \xrightarrow{\overline{D}_{n-1}} [\Delta^{n-1^{\mathrm{op}}}, \mathsf{Gpd}],$$

where $\overline{D}_{n-1}$ is obtained by applying D_{n-1} in each simplicial dimension.

Note that D_2 is simply $\mathrm{Disc}_0\colon \mathsf{PsGpd}^2_{\mathrm{wg}} \to \mathsf{Tam}^2$.

6.23 Theorem *The functor D_n lands in Tam^n and preserves geometric weak equivalences and fiber products over discrete objects. Furthermore, for every weakly globular pseudo n–fold groupoid $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$, the groupoid $T^{\mathrm{Tm}}_{(n)} D_n X$ is isomorphic to $T^{\mathrm{ps}}_{(n)} X$, and there is a zig-zag of weak equivalences between $D_n X$ and X in the category $\mathsf{PsGpd}^n_{\mathrm{wg}}$.*

Proof By induction on $n \geq 2$. For $n = 2$, $D_2 X = \mathrm{Disc}_0 X$ is clearly in Tam^2 for any $X \in \mathsf{PsGpd}^2_{\mathrm{wg}}$.

In the induction step, note that $(D_n X)_0 = X_0^d$ is discrete and $(D_n X)_k = D_{n-1} X_k$ is in Tam^{n-1}, by induction. So to prove that $D_n X$ is in Tam^n, it remains to show the following:

(a) The Segal maps

$$\mu_k \colon (D_n X)_k \to (D_n X)_1 \times_{(D_n X)_0} \overset{k}{\cdots} \times_{(D_n X)_0} (D_n X)_1 \tag{6-24}$$

are $(n-1)$–equivalences.

(b) $\Pi_0^{(n)} D_n X$ is in Tam^{n-1}.

To show (a), note that since $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$, the induced Segal maps

$$X_k \xrightarrow{\hat{\mu}_k} X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$$

are geometric weak equivalences for all $k \geq 2$. Since by induction D_{n-1} preserves geometric weak equivalences, we have weak equivalences

$$D_{n-1} X_k \xrightarrow{\simeq} D_{n-1}\big(X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1\big).$$

Moreover, $(D_n X)_1 = D_{n-1} X_1$ and $(D_n X)_0 = X_0^d$ is discrete, so the right-hand side is an iterated fiber product over discrete objects, and thus (again by induction)

$$D_{n-1}\big(X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1\big) = (D_n X)_1 \times_{(D_n X)_0} \overset{k}{\cdots} \times_{(D_n X)_0} (D_n X)_1,$$

which proves (a) for n.

To show (b), by Tamsamani's original definition 5.2 and Proposition 5.5 it suffices to show that $T^{\mathrm{Tm}}_{(n)} D_n X$ is a groupoid, which we do by showing that it is isomorphic to $T^{\mathrm{ps}}_{(n)} X$. We have

$$\big(T^{\mathrm{Tm}}_{(n)} D_n X\big)_0 = \pi_0 T^{\mathrm{Tm}}_{(n-1)} (D_n X)_0 = \pi_0 T^{\mathrm{Tm}}_{(n-1)} X_0^d = X_0^d.$$

and

$$\big(T^{\mathrm{Tm}}_{(n)} D_n X\big)_k = \pi_0 T^{\mathrm{Tm}}_{(n-1)} (D_n X)_k = \pi_0 T^{\mathrm{Tm}}_{(n-1)} D_{n-1} X_k = \pi_0 T^{\mathrm{ps}}_{(n-1)} X_k = \big(T^{\mathrm{ps}}_{(n)} X\big)_k.$$

for $k \geq 1$, where we use the induction hypothesis for the equality before last.

It follows that

$$T_{(n)}^{\mathrm{Tm}} D_n X = T_{(n)}^{\mathrm{ps}} X,$$

and since the latter is a groupoid, so is $T_{(n)}^{\mathrm{Tm}} D_n X$. This concludes the proof that $D_n X$ is in Tam^n.

Finally, we obtain the required natural zig-zag of geometric weak equivalences

$$D_n X \to \cdots \leftarrow \mathrm{Disc}_0\, X \to \cdots \leftarrow X,$$

by induction on $n \geq 1$, where the right-hand zig-zag is provided by Lemma 6.21.

For $n = 1$, we have $D_1 X = X$, while for $n \geq 2$ we use Definition 6.22 to identify $(D_n X)_k$ with $(\bar{D}_{n-1} X)_k$ for $k \geq 1$:

$$\begin{tikzcd}
D_n X \arrow[d] & \cdots \arrow[r] & D_{n-1} X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & D_{n-1} X_1 \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d] & X_0^d \arrow[d, equal] \\
\vdots & & \vdots & \vdots & X_0^d \arrow[d, equal] \\
\mathrm{Disc}_0\, X \arrow[u] & \cdots \arrow[r] & D_{n-1} X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[u] & X_1 \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[u] & X_0^d
\end{tikzcd}$$

using induction to obtain the right-hand vertical zig-zag in each simplicial dimension.

□

6.25 Remark The functor $D_n \colon \mathsf{PsGpd}^n_{\mathrm{wg}} \to \mathsf{Tam}^n$ extends the functor $D_n \colon \mathsf{Gpd}^n_{\mathrm{wg}} \to \mathsf{Tam}^n$ of Definition 5.18.

6.26 Remark If $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$, it follows from Theorem 6.23 that BX is an n–type.

6.27 Definition Let $\mathrm{ho}\,\mathsf{PsGpd}^n_{\mathrm{wg}}$ denote the localization of the category $\mathsf{PsGpd}^n_{\mathrm{wg}}$ with respect to the geometric weak equivalences.

6.28 Theorem *The functors $\widehat{Q}_{(n)} \colon \mathsf{Top} \to \mathsf{Gpd}^n_{\mathrm{wg}}$ and $B \colon \mathsf{PsGpd}^n_{\mathrm{wg}} \to \mathsf{Top}$, together with the inclusion $J \colon \mathsf{Gpd}^n_{\mathrm{wg}} \hookrightarrow \mathsf{PsGpd}^n_{\mathrm{wg}}$, induce equivalences of categories*

$$\mathrm{ho}\mathsf{P}^n\mathsf{Top} \underset{J \circ \widehat{Q}_{(n)}}{\overset{B}{\rightleftarrows}} \mathrm{ho}\,\mathsf{PsGpd}^n_{\mathrm{wg}}. \tag{6-29}$$

Moreover, for every $T \in \mathsf{Top}$, there is a zig-zag of weak equivalences in $\mathsf{P}^n\mathsf{Top}$ between $P^n T$ and $B\widehat{Q}_{(n)} T$, and for $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$ there is a zig-zag of geometric weak equivalences between X and $\widehat{Q}_{(n)} BX$ in $\mathsf{PsGpd}^n_{\mathrm{wg}}$.

Proof All three functors preserve weak equivalences, so we have induced functors as in (6-29). For any n–type T, we have an isomorphism in $\mathrm{hoP}^n\mathsf{Top}$ between T and $B\hat{Q}_{(n)}T$ by Theorem 4.32, which also implies (see Remark 6.26) that for any $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$ we have a homotopy equivalence (of CW complexes) in Top

$$BX \xrightarrow{\simeq} B\hat{Q}_{(n)}BX. \tag{6-30}$$

By Theorem 6.23 we also have zig-zags of geometric weak equivalences in $\mathsf{PsGpd}^n_{\mathrm{wg}}$

$$D_n X \to \cdots \leftarrow X \qquad \text{and} \qquad D_n\hat{Q}_{(n)}BX \to \cdots \leftarrow \hat{Q}_{(n)}BX. \tag{6-31}$$

Therefore, after applying B to (6-31) we have homotopy equivalences of CW complexes

$$BD_nX \xrightarrow{\simeq} BX \qquad \text{and} \qquad B\hat{Q}_{(n)}BX \xrightarrow{\simeq} BD_n\hat{Q}_{(n)}BX.$$

Combining these with (6-30) yields a weak equivalence

$$BD_nX \to BD_n\hat{Q}_{(n)}BX$$

in Top, which by Theorem 5.7 implies that D_nX and $D_n\hat{Q}_{(n)}BX$ are isomorphic in $\mathrm{ho}\,\mathsf{Tam}^n$, and thus in $\mathrm{ho}\,\mathsf{PsGpd}^n_{\mathrm{wg}}$. By (6-31) we see that X and $J\hat{Q}_{(n)}BX$ are weakly equivalent through a zig-zag in $\mathsf{PsGpd}^n_{\mathrm{wg}}$. □

6.32 Remark Note that Theorem 6.23 implies that the functor D_n induces an equivalence of categories

$$\mathrm{ho}\,\mathsf{PsGpd}^n_{\mathrm{wg}} \simeq \mathrm{ho}\,\mathsf{Tam}^n.$$

Together with Theorem 4.32 and Theorem 5.7 this implies the equivalence of categories (6-29). In the course of the proof of Theorem 6.28 we have further shown that any weakly globular pseudo n–fold groupoid $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$ has two different functorial partial strictifications: the Tamsamani weak n–groupoid D_nX, and the weakly globular n–fold groupoid $\hat{Q}_{(n)}BX \in \mathsf{Gpd}^n_{\mathrm{wg}}$, each equipped with zig-zags of weak equivalences in $\mathsf{PsGpd}^n_{\mathrm{wg}}$ from X

$$D_nX \to \cdots \leftarrow X \to \cdots \leftarrow \hat{Q}_{(n)}BX. \tag{6-33}$$

6.34 Definition As in Definition 3.23, for any weakly globular pseudo n–fold groupoid X and $1 \le k \le n$, we define its *k–fold object of arrows* to be the pseudo $(n-k)$–fold groupoid $\mathcal{W}_{(n,k)}X := X^{(1\cdots k)}_{\underbrace{1\cdots 1}_{k}}$.

6.35 Algebraic homotopy groups and algebraic weak equivalences In analogy to Section 3.26, for any weakly globular pseudo n–fold groupoid $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$, we define the k^{th} *algebraic homotopy group* of X at $x_0 \in X_{\underset{n}{0 \dots 0}}$ to be

$$\omega_k(X; x_0) \cong \begin{cases} \mathcal{W}_{(n,n)} X(x_0, x_0) & \text{if } k = n, \\ \mathcal{W}_{(n-k,n-k)}(\Pi_0^{(k+1)} \cdots \Pi_0^{(n)} X)(x_0, x_0) & \text{if } 0 < k < n, \end{cases} \tag{6-36}$$

with the 0^{th} *algebraic homotopy set* of X defined as

$$\omega_0(X) := \Pi_0^{(1)} \cdots \Pi_0^{(n)} X.$$

A map $f\colon X \to Y$ of weakly globular pseudo n–fold groupoids is called an *algebraic weak equivalence* if it induces bijections on the k^{th} algebraic homotopy groups (set) for all $x_0 \in X_{\underset{n}{0 \dots 0}}$ and $0 \le k \le n$.

6.37 Remark As for weakly globular n–fold groupoids (see Remark 5.20), our definition of algebraic homotopy groups for $\mathsf{PsGpd}^n_{\mathrm{wg}}$ generalizes that of Tamsamani [48, Section 5], and since $D_n X$ and X by Remark 6.32 have the same algebraic homotopy groups, by construction, both provide an algebraic way of calculating the homotopy groups of BX, as in Theorem 4.6.

Using this fact, one can show that a map $f\colon X \to Y$ in $\mathsf{PsGpd}^n_{\mathrm{wg}}$ is an n–equivalence (Definition 6.9) if and only if it is a geometric weak equivalence.

7 Applications and further directions

In this section we provide an application for our model of n–types, and indicate some directions for future work.

7.A Modeling $(k-1)$–connected n–types

We now provide an algebraic model of $(k-1)$–connected n–types, and relate it to the homotopy types of iterated loop spaces. This was mentioned by Baez and Dolan in [2] as a desirable feature for models of n–types (see also Berger [10]).

Recall that a space X is $(k-1)$*–connected* if $\pi_0 X = 0$ and $\pi_i(X, x) = 0$ for $1 \le i \le k-1$, and all $x \in X$. We denote the category of $(k-1)$–connected pointed n–types by $\mathsf{P}^n_k\mathsf{Top}_*$.

7.1 Lemma *If X is a k–connected pointed Kan complex, X is naturally weakly equivalent to a $(k-1)$–reduced Kan complex $\widehat{X}$, that is, $\widehat{X}_i = \{*\}$ for $1 \le i \le k-1$.*

Proof See Goerss and Jardine [31, Section III.3]. □

7.2 Definition For any k–connected pointed topological space $T \in \mathsf{Top}_*$, let $\mathcal{S}^{\mathrm{red}}T$ denote the canonical k–reduced version $\widehat{\mathcal{S}T}$ of the singular set $\mathcal{S}T$.

7.3 Definition A homotopically discrete pseudo n–fold groupoid $X \in \mathsf{PsGpd}^n_{\mathrm{hd}}$ is *contractible* if $\pi_0 BX$ is trivial (so that BX is contractible).

More generally, a weakly globular pseudo n–fold groupoid $X \in \mathsf{PsGpd}^n_{\mathrm{wg}}$ is called (n,k)*–weakly globular* if for each $0 \le r < k$, the homotopically discrete pseudo $(n-r-1)$–fold groupoid

$$X^{(1\cdots r+1)}_{\underbrace{1\cdots1}_{r}0} = (\mathcal{W}_{(n,r)}X)^{(r+1)}_0$$

is contractible. This is the pseudo $(n-r-1)$–fold groupoid of objects of the pseudo $(n-r)$–fold groupoid $\mathcal{W}_{(n,r)}X \in \mathsf{PsGpd}^{n-r}_{\mathrm{wg}}$ (see Definition 6.34).

In particular, when $r=0$, this just means that the pseudo $(n-1)$–fold groupoid of objects $X^{(n)}_0$ of X in the n^{th} direction (which is a homotopically discrete pseudo $(n-1)$–fold groupoid) is in fact contractible.

We let $\mathsf{PsGpd}^{(n,k)}_{\mathrm{wg}}$ denote the full subcategory of (n,k)–weakly globular pseudo n–fold groupoids in $\mathsf{PsGpd}^n_{\mathrm{wg}}$. Similarly, $\mathsf{Gpd}^{(n,k)}_{\mathrm{wg}}$ is the full subcategory of (n,k)–weakly globular pseudo n–fold groupoids in $\mathsf{Gpd}^n_{\mathrm{wg}}$.

We now want to show that $\mathsf{PsGpd}^{(n,k)}_{\mathrm{wg}}$ is an algebraic model of $(k-1)$–connected n–types. For this, we need the following:

7.4 Lemma *If X is a $(k-1)$–reduced Kan complex, then $Q_{(n)}X$ is (n,k)–weakly globular.*

Proof By Lemma 4.14(b), $(Q_{(n)}X)^{(n)}_0 = Q_{(n-1)}\operatorname{Dec} X$. Since $\operatorname{Dec} X \simeq c(X_0) = c(*)$ and $Q_{(n-1)}$ preserves weak equivalences of Kan complexes by Proposition 4.28(b), we have $Q_{(n-1)}\operatorname{Dec} X \simeq Q_{(n-1)}(*) = *$. Therefore, $dN(Q_{(n)}X)^{(n)}_0$ is contractible.

We now show by induction on $1 \le r < k$ that

(7-5) $$\mathcal{W}_{(n,r)}Q_{(n-1)}X := (N^{(n-r+1,\dots,n)}Q_{(n)}X)^{(n-r+1,\dots,n)}_{\underbrace{1\cdots1}_{r}} = Q_{(n-r)}L^r_1X$$

(in the notation of (4-11), where $L^r_1X := L^{r-1}_1(L_1X)$ for $r \ge 2$, and $L^1_1X = L_1X$)). The case $r=1$ is Lemma 4.14(b) for $k=1$, which implies that we have an isomorphism

(7-6) $$\mathcal{W}_{(n,1)}Q_{(n-1)}X := (N^{(n)}Q_{(n)}X)^{(n)}_1 \cong Q_{(n-1)}L_1X$$

of $(n-1)$–fold groupoids. In the induction step, since L_1X is still a Kan complex, by (3-25) we can apply the induction hypothesis to the right-hand side of (7-6) (using the fact that $\mathcal{W}_{(n,r)} = \mathcal{W}_{(n-1,r-1)}\mathcal{W}_{(n,1)}$, by Equation (3-25)), to deduce that

$$\mathcal{W}_{(n,r)}Q_{(n-1)}L_1X \cong Q_{(n-r)}L_1^{r-1}(L_1X),$$

which yields (7-5). From this and Lemma 4.14(a) (for $k=0$) we have

$$(\mathcal{W}_{(n,r)}Q_{(n)}X)_0 = Q_{(n-r-1)}\operatorname{Dec} L_1^rX,$$

and since $\operatorname{Dec} L_1^rX \simeq c(L_1^rX)_0$, we have $dN(\mathcal{W}_{(n,r)}Q_{(n)}X)_0 \simeq c(L_1^rX)_0$.

Note that since X is $(k-1)$–reduced, $\operatorname{Dec} X$, and thus L_1X, are $(k-2)$–reduced, so by induction L_1^rX is $(k-r-1)$–reduced. Thus as long as $r<k$, L_1^rX is 0–reduced, so $dN(\mathcal{W}_{(n,r)}Q_{(n)}X)_0$ is contractible. □

7.7 Proposition *The functors $Q_{(n)}$ and B induce equivalences of categories*

$$\operatorname{ho}\mathsf{P}^n_k\mathsf{Top}_* \underset{J\circ Q_{(n)}\circ\mathcal{S}^{\mathrm{red}}}{\overset{B}{\leftrightarrows}} \operatorname{ho}\mathsf{PsGpd}^{(n,k)}_{\mathrm{wg}}.$$

Proof If $T\in\mathsf{P}^n_k\mathsf{Top}_*$, then $\mathcal{S}^{\mathrm{red}}T$ is $(k-1)$–reduced, so by Lemma 7.4, $Q_{(n)}X\in\mathsf{PsGpd}^{(n,k)}_{\mathrm{wg}}$. The result follows immediately from Theorem 6.28. □

7.8 Remark By Theorem 4.32, the composition of $\mathcal{W}_{(n,k)}$ of Definition 6.34 with the classifying space functor B lands in $\mathsf{P}^n\mathsf{Top}$ so its restriction to $\mathsf{PsGpd}^{(n,k)}_{\mathrm{wg}}$ lands in the category $\mathsf{P}^{n-k}\mathsf{Top}$ of $(n-k)$–types.

Moreover, if $T=\Omega^kY$ is an $(n-k)$–type k–fold loop space, applying the k–fold delooping functor

$$E_{(k)}\colon P^{n-k}_{\Omega^k} \to \mathsf{P}^n_k\mathsf{Top}_*$$

of May [40, Theorem 13.1] yields the $(k-1)$–connected n–type $Y=E_{(k)}T\in\mathsf{P}^n_k\mathsf{Top}_*$. In fact:

7.9 Proposition *For any $(k-1)$–connected n–type $Y\in\mathsf{P}^n_k\mathsf{Top}_*$, we have a zig-zag of weak equivalences in $\mathsf{P}^{n-k}\mathsf{Top}$ between $B\mathcal{W}_{(n,k)}\hat{Q}_{(n)}Y$ and Ω^kY, so the weakly globular $(n-k)$–fold groupoid $\mathcal{W}_{(n,k)}\hat{Q}_{(n)}Y$ is an algebraic model for Ω^kY.*

Proof By induction on k. Let $G:=\hat{Q}_{(n)}Y\in\mathsf{Gpd}^{(n,k)}_{\mathrm{wg}}$, so $BG\cong Y$ in $\operatorname{ho}\mathsf{P}^n_k\mathsf{Top}_*$.

For $k=1$, consider the simplicial $(n-1)$–fold groupoid $N^{(n)}G$. Applying the classifying space functor $B\colon\mathsf{Gpd}^{n-1}\to\mathsf{Top}$ in each simplicial dimension yields a simplicial space $Y_\bullet = (\overline{B}^{(n)} N^{(n)} G)_\bullet$. Thus $Y_0 = B(N^{(n)}G)^{(n)}_0$ is contractible, and the Segal maps for $Y_\bullet$ are isomorphisms (since $N^{(n)}G^{(n)}_\bullet$ is the nerve of an internal groupoid), hence in particular geometric weak equivalences.

As G is weakly globular, applying the functor $T^{\mathrm{wg}}_{(n)}$ of Definition 5.18 yields a groupoid, and $\pi_0 Y_\bullet = N T^{\mathrm{wg}}_{(n)} G$. Since Y_0 is contractible, $\pi_0 Y_\bullet$ is the nerve of a group. Thus Y_1 has a homotopy inverse (see Dold [24, Theorems 6.3 and 6.4]), so it follows from Segal [45, Proposition 1.5] that $Y_1 \simeq \Omega \| Y_\bullet \|$. That is,

$$BG^{(n)}_1 = B\mathcal{W}_{(n,1)} G \simeq \Omega BG.$$

Since $\Omega BG \cong \Omega Y$ in $\mathrm{ho}\,\mathsf{P}^{n-1}_{k-1}\mathsf{Top}_*$, it follows that $B\mathcal{W}_{(n,1)} G \cong \Omega Y$ in $\mathrm{ho}\,\mathsf{P}^{n-1}_{k-1}\mathsf{Top}_*$. In the induction step, let

$$H := \mathcal{W}_{(n,1)} G = (N^{(n)} G)^{(n)}_1$$

in $\mathsf{Gpd}^{(n-1,k-1)}_{\mathrm{wg}}$, where by the inductive hypothesis $B\mathcal{W}_{(n-1,k-1)} H \cong \Omega^{k-1} BH$ in $\mathrm{ho}\,\mathsf{P}^{n-k}\mathsf{Top}_*$. By what we have shown above for $k = 1$ we have

$$BH = B(N^{(n)} G)^{(n)}_1 \simeq \Omega BG.$$

It follows that there are isomorphisms

$$B\mathcal{W}_{(n,k)} G = B\mathcal{W}_{(n-1,k-1)} H \cong \Omega^{k-1} BH \cong \Omega^{k-1}(\Omega BG) = \Omega^k BG.$$

in $\mathrm{ho}\,\mathsf{P}^{n-k}\mathsf{Top}_*$. □

7.B Further directions

One motivation in constructing our model for n–types was to obtain useful algebraic approximations of homotopy theories, that is, of simplicially enriched categories.

Recall that if $\langle \mathcal{V}, \otimes, I \rangle$ is any monoidal category, we denote by $\mathcal{V}$–Cat the collection of all (not necessarily small) $\mathcal{V}$–categories, that is, categories enriched in $\mathcal{V}$ (see Borceux [13, Section 6.2]). We obtain further variants by applying any (strictly) monoidal functor $P \colon \langle \mathcal{V}, \otimes \rangle \to \langle \mathcal{V}', \otimes' \rangle$ to a $\mathcal{V}$–category $\mathcal{C}$. For example, given a simplicially enriched category $X_\bullet$, for each $n \geq 1$ we have a $\mathsf{P}^n[\Delta^{\mathrm{op}}, \mathsf{Set}]$–category $Y_\bullet := P^n X_\bullet$, in which each mapping space $Y_\bullet(a,b)$ is the n^{th} Postnikov section $P^n X_\bullet(a,b)$.

7.10 *n*–track categories For $n \geq 2$, an *n–track category* is a category enriched in weakly globular n–fold groupoids $(\mathsf{Gpd}^n_{\mathrm{wg}}, \times)$, with respect to the cartesian monoidal structure. The category of n–track categories is denoted by Track_n.

Since $Q_{(n)}\colon [\Delta^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{Gpd}^n_{\mathrm{wg}}$ preserves products (see Remark 2.41), it induces a functor

$$S_{(n)}\colon [\Delta^{\mathrm{op}}, \mathsf{Set}]\text{–}\mathsf{Cat} \longrightarrow \mathsf{Track}_n$$

from simplicial categories to n–track categories. Furthermore, the functors

$$\Pi_0^{(n)}\colon \mathsf{Gpd}^n_{\mathrm{wg}} \to \mathsf{Gpd}^{n-1}_{\mathrm{wg}}$$

giving the Postnikov decomposition of $\mathsf{Gpd}^n_{\mathrm{wg}}$ induce functors

$$P^{n-1}\colon \mathsf{Track}_n \to \mathsf{Track}_{n-1}$$

providing the Postnikov decomposition of simplicially enriched categories.

For $n=1$, the corresponding k–invariant was described by Baues and Wirsching in [9] in terms of the Baues–Wirsching cohomology of categories, and a similar result was obtained in our paper [11] for $n=2$, using an algebraically defined cohomology of track categories. The extension of this to general n via an appropriate cohomology of $(n-1)$–track categories will be investigated in the future.

7.11 Spectral sequences In [7], Baues and Blanc introduced the notion of the *Postnikov n–stem* $\mathcal{P}[n]X$ of a topological space X, that is, the system of $(k-1)$–connected $(n+k)$–Postnikov sections $P^{n+k}X\langle k-1\rangle$ $(k=0,1,\ldots)$, with the natural maps between them.

They then show that the E^{n+2}–term of the homotopy spectral sequence of a (co)simplicial space $W_\bullet$ (respectively, $W^\bullet$) depends only on the simplicial n–stems $\mathcal{P}[n]W_\bullet$ or $\mathcal{P}[n]W^\bullet$. Thus we can in principle use the $(n+k)$–fold groupoid models of each W_m or W^m, as in Definition 7.3 to extract information about the d^{n+1}–differentials.

However, in many cases of interest – including the (stable or unstable) Adams spectral sequence, the Eilenberg–Moore spectral sequence and others – a more “algebraic” approach can be used, using the notion of n^{th} *order derived functors* introduced by Baues and Blanc in [6].

For example, the (unstable) $\mathbb{F}_p$–Adams spectral sequence for a (simply connected) space X constructed by Bousfield and Kan in [15] is the homotopy spectral sequence of a cosimplicial space $W^\bullet$ obtained as a $\mathbb{F}_p$–resolution of X. It can be shown that the E^{n+2}–term of this spectral sequence depends only on the n–Postnikov sections of the mapping spaces $\mathrm{map}(X,E)$ and $\mathrm{map}(E,E')$ for various products of $\mathbb{F}_p$–Eilenberg–Mac Lane space E and E'. Thus we do not need a full algebraic model for the $\mathsf{P}^n[\Delta^{\mathrm{op}}, \mathsf{Set}]$–category Top, but only for the small subcategory with objects X and E as above. Since all mapping spaces in this category are themselves simplicial $\mathbb{F}_p$–vector spaces, the associated n–track category is correspondingly simplified. The case $n=1$ was treated in great detail by Baues in [5], and some progress on the case $n=2$ has been made by Baues and Frankland in work that is still under way. However, it is clear from Baues and Blanc [8] that a better conceptual framework, such as an algebraic model for such "linear" n–track categories, will be needed before any further progress can be made for $n \geq 2$.

Appendix A: Fibrancy conditions on n–fold simplicial sets

In this appendix we prove some technical facts about $\mathrm{Or}_{(n)}$:

A.1 Remark Given a map of simplicial sets $f\colon A \to B$ and $m \geq 2$, let $P := \mathrm{Or}_{(m)} A$, $Q := \mathrm{Or}_{(m)} B$, and $F = \mathrm{Or}_{(m)} f\colon P \to Q$. From the description in Section 2.9 we see by induction on m (using (2-11)) that for every multi-index $(p_1 \cdots p_m)$ the map of sets $F_{(p_1 \cdots p_m)}\colon P_{(p_1 \cdots p_m)} \to Q_{(p_1 \cdots p_m)}$ is simply $f_\ell\colon A_\ell \to B_\ell$, for $\ell := m-1+p_1+\cdots+p_m$ (see (2-10)).

A.2 Lemma *If $Y = \mathrm{Or}_{(n)} X \in [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$ for some $X \in [\Delta^{\mathrm{op}}, \mathsf{Set}]$ and $n \geq 2$, then for any two of its n directions $1 \leq p \neq q \leq n$, the lower right corner of the bisimplicial set $Z = Y^{(p,q)} \in [\Delta^{2^{\mathrm{op}}}, \mathsf{Set}]$ (see Section 2.6(b)) has the form*

$$\text{(A-3)} \qquad \begin{tikzcd}[column sep=huge, row sep=huge]
 & X_{s+2} \arrow[r, shift left, "d_{k+3}"] \arrow[r, shift right, "d_{k+2}"'] \arrow[d, shift right=2, "d_i"'] \arrow[d, "d_{i+1}" description] \arrow[d, shift left=2, "d_{i+2}"] & X_{s+1} \arrow[d, shift right=2, "d_i"'] \arrow[d, "d_{i+1}" description] \arrow[d, shift left=2, "d_{i+2}"] \\
X_{s+2} \arrow[r, shift left=2, "d_{k+3}"] \arrow[r, "d_{k+2}" description] \arrow[r, shift right=2, "d_{k+1}"'] \arrow[d, shift right, "d_i"'] \arrow[d, shift left, "d_{i+1}"] & X_{s+1} \arrow[r, shift left, "d_{k+2}"] \arrow[r, shift right, "d_{k+1}"'] \arrow[d, shift right, "d_i"'] \arrow[d, shift left, "d_{i+1}"] & X_s \arrow[d, shift right, "d_i"'] \arrow[d, shift left, "d_{i+1}"] \\
X_{s+1} \arrow[r, shift left=2, "d_{k+2}"] \arrow[r, "d_{k+1}" description] \arrow[r, shift right=2, "d_k"'] & X_s \arrow[r, shift left, "d_{k+1}"] \arrow[r, shift right, "d_k"'] & X_{s-1}
\end{tikzcd}$$

for some $s \geq n$ and $0 \leq i < k < s$.

Proof By induction on $n \geq 2$, where the case $n=2$ is depicted in Figure 1 of Section 2. Using (2-11), we see that

$$Y = \overline{\mathrm{Or}}^{(2)}_{(n-1)} \mathrm{Or}_{(2)} X,$$

so if we number the n directions of Y so as to the start with the horizontal direction of $\mathrm{Or}_{(2)}\, X$, then for any $1 < p \neq q \leq n$ the bisimplicial set $Z = Y^{(p,q)} \in [\Delta^{n^{\mathrm{op}}}, \mathsf{Set}]$ is contained in the $(n-1)$–fold simplicial set $\mathrm{Or}_{(n-1)}\, Q_{t\bullet}$, for one of the vertical simplicial sets of $Q := \mathrm{Or}_{(2)}\, X$. Thus the claim for such a Z follows by the induction hypothesis.

Thus it suffices to treat the case $1 = p < q$. Since the corresponding *vertical* maps in each of the vertical simplicial sets $Q_{t\bullet}$, for various t, have the same labels (in terms of the original face maps of X), the same will be true after applying the functor $\mathrm{Or}_{(n-1)}$ to each of them. This implies that the vertical maps in (A-3) are indeed both labeled d_i, d_{i+1}, for some $i < k+1$. However, since each of the simplicial sets $Q_{t\bullet}$ is obtained by repeated applications of Dec to X (see (2-12)), we must have omitted at least the maximally labeled face map $d_{k+1}\colon X_{k+1} \to X_k$, by definition of Dec. Therefore, among the various face maps of X appearing in $\mathrm{Or}_{(n-1)}\, Q_{t\bullet}$, the map $d_{k+1}\colon X_{k+1} \to X_k$ cannot appear. Thus, we actually have $i < k$.

From Figure 1 (or from the fact that the bisimplicial set Q, as a (vertical) simplicial object over $[\Delta^{\mathrm{op}}, \mathsf{Set}]$, is the resolution of X produced by the comonad Dec'), we see that the horizontal maps in Q are always the face maps of *maximal* consecutive indices for any given $Q_{i,j} = X_{i+j+1}$: for example, the bottom left horizontal maps in Figure 1 are $d_1, d_2\colon X_2 \to X_1$.

On the other hand, by Remark A.1 (for $m = n-1$), the two pairs of *horizontal* maps in (A-3) are just those that appear in the rightmost sequence of horizontal maps in Figure 1: namely, $d_k, d_{k+1}\colon X_{k+1} \to X_k$ and $d_{k+1}, d_{k+2}\colon X_{k+2} \to X_{k+1}$. Thus when $p = 1$, in fact $k = s-1$ in (A-3) (as for the front and back squares in Figure 2). $\square$

Proposition (Proposition 2.39) *If $X \in [\Delta^{\mathrm{op}}, \mathsf{Set}]$ is a Kan complex, then $Y := \mathrm{Or}_{(n)}\, X$ is $(n,2)$–fibrant.*

Proof For every $1 \leq p \leq n$, the simplicial set $Y^{(p)}$ is obtained from X by repeated applications of Dec and Dec', so it is still a Kan complex, and the same is true of $\mathrm{csk}_2\, Y^{(p)}$.

For each bisimplicial set of the form (A-3), denote by W and Z the middle and right vertical simplicial sets, respectively, with $\phi\colon W \to Z$ the horizontal map in $[\Delta^{\mathrm{op}}, \mathsf{Set}]$ given by $d_k\colon W_0 = X_s \to Z_0 = X_{s-1}$, $d_{k+1}\colon W_1 = X_{s+1} \to Z_1 = X_s$, and so on. Similarly, denote by U and V the middle and bottom horizontal simplicial sets, respectively, with $\psi\colon U \to V$ the vertical map in $[\Delta^{\mathrm{op}}, \mathsf{Set}]$ given by $d_i\colon U_j = X_{s+j} \to V_j = X_{s+j-1}$ for all $j \geq 0$.

By Definition 2.31 and Lemma A.2, in order to verify that Y is $(n,2)$–fibrant, we must check that $\mathrm{csk}_2\, \phi$ and $\mathrm{csk}_2\, \psi$ are fibrations for any choice of (A-3) with $i < k$. This means that we must show that a lifting $\widehat{g}$ exists for every solid commuting square of one of the two following forms

(A-4)
$$\begin{tikzcd} \Lambda^j[m] \arrow[r, "f"] \arrow[d, "i_j"'] & U \arrow[d, "\psi"] \\ \Delta[m] \arrow[r, "g"'] \arrow[ur, dotted, "\widehat{g}"] & V \end{tikzcd} \qquad \begin{tikzcd} \Lambda^j[m] \arrow[r, "f"] \arrow[d, "i_j"'] & W \arrow[d, "\phi"] \\ \Delta[m] \arrow[r, "g"'] \arrow[ur, dotted, "\widehat{g}"] & Z \end{tikzcd}$$

(a) (b)

for $m=1,2$ and $0 \le j \le m$ (where $\Lambda^j[m] \subseteq \partial\Delta[m]$ consists of all but the j^{th} face of $\Delta[m]$, and $i_j\colon \Lambda^j[m] \hookrightarrow \Delta[m]$ is the inclusion).

Case 1 When $m=1$ in (A-4) (a), the map $f\colon \Lambda^j[1] \to U$ $(j=0,1)$ corresponds to a 0–simplex $\widetilde{\sigma} \in U_0$, that is, an s–simplex $\sigma \in X_s$ (since $U_0 = X_s$ and $\Lambda^j[1] \cong \Delta[0]$), and the map $g\colon \Delta[1] \to V$ corresponds to a 1–simplex $\widetilde{\tau} \in V_1$, that is, an s–simplex $\tau \in X_s$.

Commutativity of the solid square in (A-4)(a) – that is, $\psi \circ f = g \circ i_j$ – means that $d_j^{V_1}(\widetilde{\tau}) = \psi(\widetilde{\sigma})$, that is,

(A-5)
$$d_{k+j}^{X_s}\tau = d_i^{X_s}\sigma.$$

A lift $\widehat{g}\colon \Delta[1] \to U$ corresponds to a 1–simplex $\widetilde{\omega} \in U_1$, that is, an $(s+1)$–simplex $\omega \in X_{s+1}$, and commutativity of the two triangles in (A-4) (a) translates into the two conditions $d_j^{U_1}(\widetilde{\omega}) = \widetilde{\sigma}$ and $\psi(\widetilde{\omega}) = \widetilde{\tau}$, that is,

(A-6)
$$d_{k+1+j}^{X_{s+1}}\,\omega = \sigma \quad \text{and} \quad d_i^{X_s}\,\omega = \tau.$$

Combining (A-5) and (A-6) yields the simplicial identity

(A-7)
$$d_i d_{k+1+j}\omega = d_{k+j} d_i \omega,$$

since $i < k$.

The two s–simplices σ and τ satisfying (A-5) define a map from the following pushout P in $[\Delta^{\mathrm{op}}, \mathsf{Set}]$:

$$\begin{tikzcd} \Delta[s-1] \arrow[r, "\eta_i"] \arrow[d, "\eta_{k+j}"'] & \Delta[s] \arrow[d] \arrow[ddr, bend left, "\sigma"] & \\ \Delta[s] \arrow[r] \arrow[drr, bend right, "\tau"'] & P \arrow[ul, phantom, "\boxed{\text{PO}}"] \arrow[dr, dotted, "(\sigma,\tau)"] & \\ & & X \end{tikzcd}$$

Since P is a union of two s–simplices along a common face, it is a contractible subspace of $\Delta[s+1]$, so $P \hookrightarrow \Delta[s+1]$ is an acyclic cofibration in $[\Delta^{\mathrm{op}}, \mathsf{Set}]$. Because X is fibrant, a lift $\omega\colon \Delta[s+1] \to X$ for (σ,τ) – and thus $\widehat{g}\colon \Delta[m] \to U$ – always exists.

Case 2 When $m = 2$ in (A-4)(a), the map $f\colon \Lambda^j[2] \to U$ $(j = 0,1,2)$ corresponds to a pair of 1–simplices $\widetilde{\alpha}, \widetilde{\beta} \in U_1$ with $d_p\widetilde{\alpha} = d_q\widetilde{\beta}$, where

(A-8) $$(p,q) = \begin{cases} (1,1) & \text{if } j = 0, \\ (0,1) & \text{if } j = 1, \\ (0,0) & \text{if } j = 2. \end{cases}$$

This means that we have $\alpha, \beta \in X_{s+1} = U_1$ with

(A-9) $$d^{X_{s+1}}_{k+1+p}\alpha = d^{X_s}_{k+1+q}\beta.$$

The map $g\colon \Delta[2] \to V$ corresponds to $\sigma \in X_{s+1} = V_2$, and the map $\widehat{g}\colon \Delta[2] \to U$ corresponds to $\omega \in X_{s+2}$.

Commutativity of the solid square in (A-4)(a) means that

(A-10) $$d^{X_{s+1}}_{k+p}\sigma = d^{X_s}_i\alpha \qquad \text{and} \qquad d^{X_{s+1}}_{k+q}\sigma = d^{X_s}_i\beta.$$

Commutativity of the upper triangle in (A-4)(a) means

(A-11) $$d^{X_{s+2}}_{k+1+p}\omega = \alpha \qquad \text{and} \qquad d^{X_{s+2}}_{k+1+q}\omega = \beta,$$

and commutativity of the lower triangle in (A-4)(a) means

(A-12) $$d^{X_{s+1}}_i\omega = \sigma.$$

Combining (A-10), (A-11) and (A-12) yields the two simplicial identities

(A-13) $$d_i d_{k+1+p}\omega = d_{k+p}d_i\omega \qquad \text{and} \qquad d_i d_{k+1+q}\omega = d_{k+q}d_i\omega.$$

since $i < k$. The existence of ω follows as above.

The analogous cases for (A-4)(b) are obtained from these by applying the inversion I^* of Remark 2.2. □

References

[1] **M Artin**, **B Mazur**, *On the van Kampen theorem*, Topology 5 (1966) 179–189 MR0192495

[2] **J C Baez**, **J Dolan**, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. 36 (1995) 6073–6105 MR1355899

[3] **M A Batanin**, *Monoidal globular categories as a natural environment for the theory of weak n–categories*, Adv. Math. 136 (1998) 39–103 MR1623672

[4] **H J Baues**, *Combinatorial homotopy and 4–dimensional complexes*, de Gruyter Expositions in Mathematics 2, de Gruyter, Berlin (1991) MR1096295

[5] **H-J Baues**, *The algebra of secondary cohomology operations*, Progress in Mathematics 247, Birkhäuser, Basel (2006) MR2220189

[6] **H-J Baues**, **D Blanc**, *Higher order derived functors* arXiv:1210.7437

[7] **H J Baues**, **D Blanc**, *Stems and spectral sequences*, Algebr. Geom. Topol. 10 (2010) 2061–2078 MR2728484

[8] **H-J Baues**, **D Blanc**, *Higher order derived functors and the Adams spectral sequence*, J. Pure Appl. Algebr 219 (2015) 199–239 MR3250522

[9] **H J Baues**, **G Wirsching**, *Cohomology of small categories*, J. Pure Appl. Algebra 38 (1985) 187–211 MR814176

[10] **C Berger**, *Double loop spaces, braided monoidal categories and algebraic 3–type of space*, from: "Higher homotopy structures in topology and mathematical physics", Contemp. Math. 227, Amer. Math. Soc. (1999) 49–66 MR1665460

[11] **D Blanc**, **S Paoli**, *Two-track categories*, J. K–Theory 8 (2011) 59–106 MR2826280

[12] **F Borceux**, *Handbook of categorical algebra 1: Basic category theory*, Encycl. Math. Appl. 50, Cambridge Univ. Press (1994) MR1291599

[13] **F Borceux**, *Handbook of categorical algebra 2: Categories and structures*, Encycl. Math. Appl. 51, Cambridge Univ. Press (1994) MR1313497

[14] **A K Bousfield**, **E M Friedlander**, *Homotopy theory of Γ–spaces, spectra, and bisimplicial sets*, from: "Geometric applications of homotopy theory, II", Lecture Notes in Math. 658, Springer, Berlin (1978) 80–130 MR513569

[15] **A K Bousfield**, **D M Kan**, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics 304, Springer, Berlin (1972) MR0365573

[16] **R Brown**, **K A Hardie**, **K H Kamps**, **T Porter**, *A homotopy double groupoid of a Hausdorff space*, Theory Appl. Categ. 10 (2002) 71–93 MR1883479

[17] **R Brown**, **P J Higgins**, **R Sivera**, *Nonabelian algebraic topology*, EMS Tracts in Mathematics 15, Eur. Math. Soc., Zürich (2011) MR2841564

[18] **R Brown**, **C B Spencer**, *Double groupoids and crossed modules*, Cahiers Topologie Géom. Différentielle 17 (1976) 343–362 MR0440553

[19] **M Bullejos**, **A M Cegarra**, **J Duskin**, *On cat^n–groups and homotopy types*, J. Pure Appl. Algebra 86 (1993) 135–154 MR1215642

[20] **J G Cabello**, **A R Garzón**, *Closed model structures for algebraic models of n–types*, J. Pure Appl. Algebra 103 (1995) 287–302 MR1357790

[21] **P Carrasco**, **A M Cegarra**, *Group-theoretic algebraic models for homotopy types*, J. Pure Appl. Algebra 75 (1991) 195–235 MR1137837

[22] **A M Cegarra**, **B A Heredia**, **J Remedios**, *Double groupoids and homotopy 2–types*, Appl. Categ. Structures 20 (2012) 323–378 MR2943635

[23] **D-C Cisinski**, *Batanin higher groupoids and homotopy types*, from: "Categories in algebra, geometry and mathematical physics", Contemp. Math. 431, Amer. Math. Soc. (2007) 171–186 MR2342828

[24] **A Dold**, *Partitions of unity in the theory of fibrations*, Ann. of Math. 78 (1963) 223–255 MR0155330

[25] **J W Duskin**, *Simplicial matrices and the nerves of weak n–categories, I: Nerves of bicategories*, Theory Appl. Categ. 9 (2001/02) 198–308 MR1897816

[26] **W G Dwyer**, **D M Kan**, **C R Stover**, *The bigraded homotopy groups $\pi_{i,j}X$ of a pointed simplicial space X*, J. Pure Appl. Algebra 103 (1995) 167–188 MR1358761

[27] **P J Ehlers**, **T Porter**, *Joins for (augmented) simplicial sets*, J. Pure Appl. Algebra 145 (2000) 37–44 MR1732286

[28] **G Ellis**, **R Steiner**, *Higher-dimensional crossed modules and the homotopy groups of $(n+1)$–ads*, J. Pure Appl. Algebra 46 (1987) 117–136 MR897011

[29] **P Gabriel**, **M Zisman**, *Calculus of fractions and homotopy theory*, Ergeb. Math. Grenzgeb. 35, Springer, New York (1967) MR0210125

[30] **P G Glenn**, *Realization of cohomology classes in arbitrary exact categories*, J. Pure Appl. Algebra 25 (1982) 33–105 MR660389

[31] **P G Goerss**, **J F Jardine**, *Simplicial homotopy theory*, Progress in Mathematics 174, Birkhäuser, Basel (1999) MR1711612

[32] **A Grothendieck**, *Pursuing stacks* Available at `http://www.math.jussieu.fr/~maltsin/ps.html`

[33] **K A Hardie**, **K H Kamps**, **R W Kieboom**, *A homotopy bigroupoid of a topological space*, Appl. Categ. Structures 9 (2001) 311–327 MR1836257

[34] **L Illusie**, *Complexe cotangent et déformations, II*, Lecture Notes in Mathematics 283, Springer, Berlin (1972) MR0491681

[35] **A Joyal**, **R Street**, *Pullbacks equivalent to pseudopullbacks*, Cahiers Topologie Géom. Différentielle Catég. 34 (1993) 153–156 MR1223657

[36] **A Joyal**, **M Tierney**, *Algebraic homotopy types*, handwritten lecture notes (1984)

[37] **O Leroy**, *Sur une notion de 3–catégorie adapté à l'homotopie*, preprint number 94-10, Dept. Sciences & Math., Univ. Montpellier (1994)

[38] **J-L Loday**, *Spaces with finitely many nontrivial homotopy groups*, J. Pure Appl. Algebra 24 (1982) 179–202 MR651845

[39] **S MacLane**, **J H C Whitehead**, *On the 3–type of a complex*, Proc. Nat. Acad. Sci. U. S. A. 36 (1950) 41–48 MR0033519

[40] **J P May**, *The geometry of iterated loop spaces*, Lectures Notes in Mathematics 271, Springer, Berlin (1972) MR0420610

[41] **I Moerdijk**, **J-A Svensson**, *Algebraic classification of equivariant homotopy 2–types, I*, J. Pure Appl. Algebra 89 (1993) 187–216 MR1239560

[42] **S Paoli**, *Weakly globular* cat^n*–groups and Tamsamani's model*, Adv. Math. 222 (2009) 621–727 MR2538023

[43] **T Porter**, *n–types of simplicial groups and crossed n–cubes*, Topology 32 (1993) 5–24 MR1204402

[44] **D G Quillen**, *Spectral sequences of a double semi-simplicial group*, Topology 5 (1966) 155–157 MR0195097

[45] **G Segal**, *Categories and cohomology theories*, Topol. 13 (1974) 293–312 MR0353298

[46] **C Simpson**, *Homotopy types of strict 3–groupoids* arXiv:math.CT/9810059

[47] **C Simpson**, *Homotopy theory of higher categories*, New Mathematical Monographs 19, Cambridge Univ. Press (2012) MR2883823

[48] **Z Tamsamani**, *Sur des notions de n–catégorie et n–groupoïde non strictes via des ensembles multi-simpliciaux*, K–Theory 16 (1999) 51–99 MR1673923

Department of Mathematics, University of Haifa
31905 Haifa, Israel

Department of Mathematics, University of Leicester
Leicester LE1 7RH, UK

blanc@math.haifa.ac.il, sp424@le.ac.uk

Received: 6 September 2013 Revised: 20 February 2014