Algebraic & Geometric Topology 14 (2014) 3589–3601

Embedded annuli and Jones' conjecture

DOUGLAS J LAFOUNTAIN
WILLIAM W MENASCO

We show that after stabilizations of opposite parity and braid isotopy, any two braids in the same topological link type cobound embedded annuli. We use this to prove the generalized Jones' conjecture relating the braid index and algebraic length of closed braids within a link type, following a reformulation of the problem by Kawamuro.

57M25; 57R17, 20F36

1 Introduction

Consider an oriented unknotted braid axis A in S^3 whose complement fibers over S^1 with oriented disc fibers $\{A(\theta)\}_{\theta \in S^1}$. For a given oriented topological link type $\mathcal{L}$ with m components, we study closed braid representatives $\beta \in \mathcal{L}$ which are embedded in $S^3 \setminus A$ and which transversely intersect the disc fibers $A(\theta)$ positively.

Given two such representatives $\beta_1, \beta_2 \in \mathcal{L}$ braided about a common axis A, a topological isotopy of β_1 to β_2 sweeps out m immersed annuli cobounded by β_1 and β_2, one annulus per component of $\mathcal{L}$. Our first result shows that by negatively stabilizing β_1 and positively stabilizing β_2 in a judicious manner, we may in fact obtain m disjointly embedded annuli cobounded by these respective stabilizations.

Proposition 1.1 *Let $\beta_1, \beta_2 \in \mathcal{L}$ be braided about a common unknotted braid axis in S^3. Then there exist two braids $\widehat{\beta}_1, \widehat{\beta}_2 \in \mathcal{L}$ whose m components pairwise cobound m disjointly embedded annuli, such that $\widehat{\beta}_1$ is obtained by negatively stabilizing β_1 and $\widehat{\beta}_2$ is obtained by positively stabilizing β_2.*

For any closed braid β, we denote its braid index by $n(\beta)$ and its algebraic length by $\ell(\beta)$. We use the above proposition to prove the following theorem, namely the generalized Jones' conjecture, which relates the braid index and algebraic length of closed braids within a topological link type; see Jones [8] and Kawamuro [9].

Theorem 1.2 (Jones' conjecture) *Let $\beta, \beta_0 \in \mathcal{L}$ be two closed braids such that $n(\beta_0)$ is minimal for its link type. Then*

$$|\ell(\beta) - \ell(\beta_0)| \leq n(\beta) - n(\beta_0).$$

Published: 15 January 2015 DOI: 10.2140/agt.2014.14.3589

As noted by others, the veracity of Jones' conjecture yields immediate applications to the study of transverse links in the contact 3–sphere, quasipositive and strongly quasipositive braids, representations of braid groups and polynomial invariants for links; see Dynnikov and Prasolov [6], Etnyre and Van Horn-Morris [7], Kawamuro [9; 10] and Stoimenow [11]. Recently, Dynnikov and Prasolov provided a proof of Jones' conjecture by studying bypasses for rectangular diagrams representing Legendrian links [6]. We present here an independent proof using an alternative approach.

The outline of the paper is as follows: in Section 2 we briefly review some background material, and then prove Proposition 1.1 using braid foliation techniques of Birman and Menasco; in particular, our argument is inspired by the constructions in [4] and [5, Section 2]. In Section 3 we then establish Theorem 1.2 by proving an equivalent statement proposed by Kawamuro [9; 10].

Acknowledgement The authors would like to thank Joan Birman and Keiko Kawamuro for their helpful comments on a preliminary version of this paper.

2 Stabilizing to embedded annuli

Let β be braided about an unknotted braid axis A with braid fibration $\{A(\theta)\}_{\theta \in S^1}$. The *braid index* of β, denoted by $n(\beta)$, is the number of intersections of β with any disc $A(\theta)$. The *algebraic length* of β, denoted by $\ell(\beta)$, is the sum of the signed crossings in any regular braid projection of β. Given β, we may alter it through standard moves, namely *braid isotopy* in the complement of A, which does not change n nor ℓ; *exchange moves*, which change neither n nor ℓ (see the left side of Figure 1); and *stabilization (destabilization)*, which increases (decreases) n and either increases or decreases ℓ depending on whether the stabilization or destabilization is positive or negative (see the right side of Figure 1).

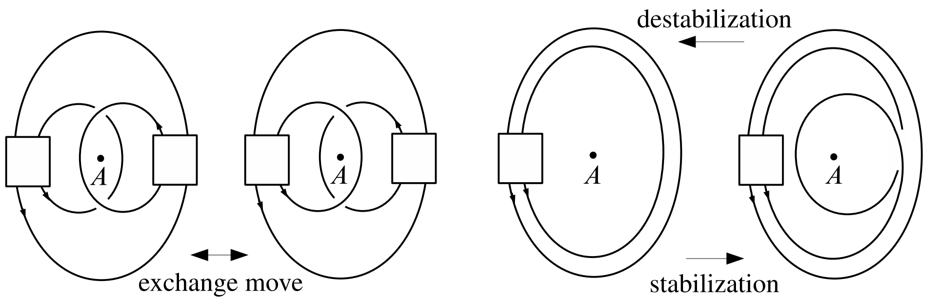

Figure 1: On the left, an exchange move; on the right, stabilization (destabilization)

Throughout this paper we will be studying braids using various embedded oriented annuli and bigon discs. By general position, we may assume that A intersects our surface of interest S transversely in finitely many points, called *vertices*, of either positive or negative parity depending on whether the orientation of A agrees or disagrees with the orientation of S at that vertex. Provided the boundary components, or boundary arcs, of our surface S are transverse to discs $A(\theta)$ in the braid fibration, by standard braid foliation arguments [2] we may also assume that there are finitely many points of tangency between S and the disc fibers $A(\theta)$, all of which yield simple saddle singularities in the foliation of S induced by the braid fibration. We will generically refer to such points as *singularities*, and again these will be of positive or negative parity depending on whether the orientation of S agrees or disagrees with the orientation of $A(\theta)$ at those points.

Also by standard arguments [2], the nonsingular leaves in the $A(\theta)$–foliation of our discs and annuli will be either s*–arcs*, whose endpoints are on two different boundary components of an annulus (or different boundary arcs of a bigon disc); a*–arcs*, with one endpoint on a vertex and one endpoint on a boundary component or arc; or b*–arcs*, whose endpoints are on two different vertices of opposite parity. Furthermore, singular leaves can then be classified by the types of nonsingular leaves that interact to form the saddle singularity; specifically, in general we will have *aa*–, *bb*–, *ab*–, *as*– and *abs*–singularities. These are depicted in Figure 2, where the oriented transverse boundary is given by bold black arrows; we will refer to the gray-shaded regions as *aa*–, *bb*–, *ab*–, *as*– and *abs*–tiles, according to the singularity which they contain.

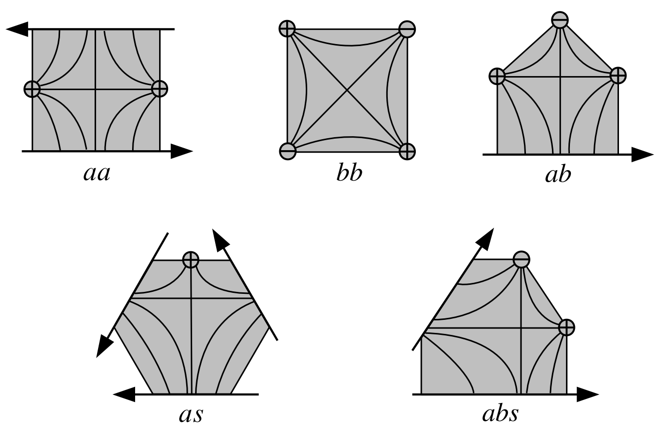

Figure 2: The five types of saddle singularities, classified by the types of nonsingular arcs interacting to form the singularity. The parities of the vertices may be reversed provided the orientations of the transverse boundary arcs are compatibly reversed.

The *valence* of a vertex is the number of singular leaves for which it serves as an endpoint; alternatively, the valence is the number of (one-parameter families of) a–arcs or b–arcs to which the vertex is adjacent. Any vertex may therefore be labeled as type (α, β), where α and β are the number of a–arcs and b–arcs to which the vertex is adjacent, with the valence of the vertex being $\alpha + \beta$.

As discussed in detail in [2] and shown in Figure 3, valence-1 vertices of type $(1, 0)$ indicate a destabilization of the braided boundary of a surface which results in the elimination of that vertex, and valence-2 vertices of type $(1, 1)$ or $(0, 2)$ indicate an exchange move with a corresponding elimination of two vertices. Also, any two consecutive singularities of like parity at the two ends of a family of b–arcs adjacent to a vertex may have their $A(\theta)$–order interchanged using braid isotopy, thus reducing the valence of that vertex by one; see the bottom right picture in Figure 3. This reordering of consecutive singularities of like parity is referred to as a *standard change of fibration* and always occurs for odd-valence vertices of type $(0, 2k+1)$ for $k > 0$; in particular, a standard change of fibration always exists for vertices of type $(0, 3)$. As a result, using destabilizations, exchange moves and braid isotopy we can reduce the total number of vertices provided there are such vertices of valence 1, 2 or 3.

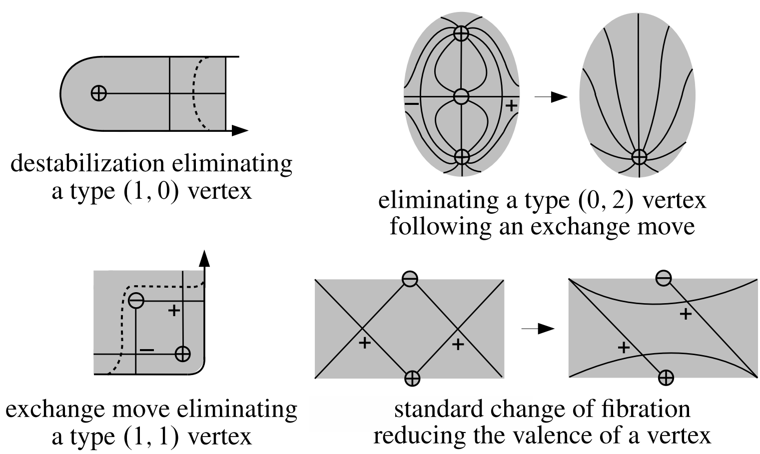

Figure 3: Eliminating, and reducing the valence of, vertices using destabilization, exchange moves and braid isotopy. In the figures on the left, the braided boundary is indicated by the bold black arrow, and the subdisc cobounded by it and the dashed line is eliminated following destabilization and an exchange move, respectively.

With this background, we can now proceed with the proof of Proposition 1.1, namely that given braids $\beta_1, \beta_2 \in \mathcal{L}$, then after negative stabilizations of β_1 and positive stabilizations of β_2 we obtain two braids $\widehat{\beta}_1, \widehat{\beta}_2$ which cobound m embedded annuli, one for each component of the link type $\mathcal{L}$.

Proof We consider β_1 and β_2 braided about a common axis A, and furthermore think of A as the z–axis in $\mathbb{R}^3$. Using braid isotopy, we may assume that

$$\beta_1 \subset \mathbb{R}^3_- = \{(x,y,z) \in \mathbb{R}^3 \mid z < 0\} \quad \text{and} \quad \beta_2 \subset \mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3 \mid z > 0\},$$

so that β_1 is unlinked from β_2. Throughout most of the proof β_1 and β_2 will remain fixed, and we will be considering other braids and links which are smoothly isotopic to β_1 and β_2; it is only at the end of the proof that we will stabilize both β_1 and β_2 to the desired $\widehat{\beta}_1$ and $\widehat{\beta}_2$, and include a final braid isotopy which links $\widehat{\beta}_1$ and $\widehat{\beta}_2$ to obtain the embedded annuli.

We begin with β_1 and take a braided push-off β_1' of β_1 such that $\beta_1', \beta_1 \subset \mathbb{R}^3_-$ and $\beta_1' \sqcup \beta_1$ cobound m embedded annuli, one for each component of $\mathcal{L}$. Now consider a regular braid projection of $\beta_1' \sqcup \beta_1$ onto the $z = -1$ plane in $\mathbb{R}^3_-$, and wherever there is a double point at which β_1' passes under β_1, we imagine isotoping β_1' locally upwards in the z–direction through β_1. The result is β_2', which is in fact braid isotopic to β_1', but which is now unlinked from β_1, yet still in $\mathbb{R}^3_-$. Furthermore, for each of these r crossing changes, we have a bigon disc D_i, $1 \le i \le r$, cobounded by arcs of β_1' and β_2' and intersected once by β_1; see Figure 4. We will therefore think of β_1' and β_2' as being almost identical as sets in $\mathbb{R}^3$, except where they differ by the bigon discs D_i.

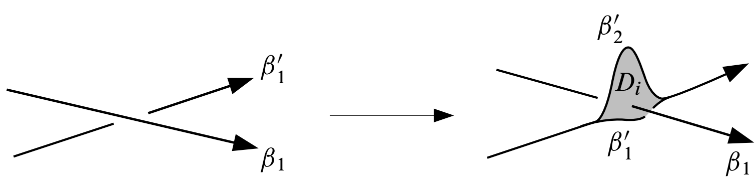

Figure 4: An under-crossing of β_1' passes through β_1 to produce β_2', and produces a bigon disc D_i which is intersected once by β_1.

We may now vertically braid isotope β_2' in the positive z–direction so that $\beta_2' \subset \mathbb{R}^3_+$, and the points along β_1' which are shared by β_2' will experience an induced isotopy into $\mathbb{R}^3_+$, yet with β_1' still differing from β_2' by the bigon discs D_i, which remain punctured once by β_1 in $\mathbb{R}^3_-$. Then, since $\beta_2', \beta_2 \in \mathcal{L}$, there is an ambient isotopy of $\mathbb{R}^3_+$, relative to the xy–plane, which takes β_2' to β_2. The braid β_1 remains fixed, but the points along β_1' which are shared with β_2' will experience an induced isotopy, and hence β_1' will be isotoped to a link L_1' which continues to cobound with β_1 a total of m embedded annuli. Similarly, the bigon discs D_i will experience an induced isotopy. The boundary of each new D_i consists of two arcs, namely a boundary arc $\partial_i^+ \subset \beta_2$

and a boundary arc $\partial_i^- \subset L_1'$; see Figure 5. Observe that the braid β_1 intersects each disc D_i once in $\mathbb{R}^3_-$, close to the ∂_i^- boundary arc. Since $\partial_i^+ \subset \beta_2$, it is transverse to the fibration $\{A(\theta)\}$. However, the ∂_i^- boundary arc will not necessarily be transverse to the fibration $\{A(\theta)\}$ in $\mathbb{R}^3_+$, although it will still be transverse to the portions of discs $A(\theta)$ in $\mathbb{R}^3_-$.

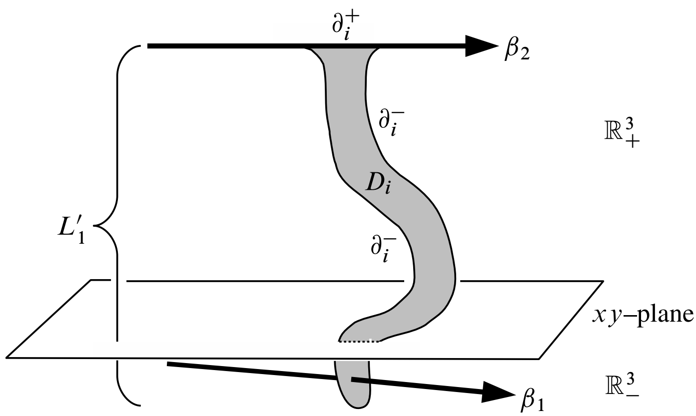

Figure 5: The link L_1' and discs D_i after isotoping β_2' to β_2 in $\mathbb{R}^3_+$

We now apply Alexander's theorem for links in $\mathbb{R}^3_+$ so as to make the ∂_i^- boundary arcs for the discs D_i transverse to the fibration $\{A(\theta)\}$, and we can do so without changing the conjugacy class of β_2; see [1] and also [5, §2]. This takes the link L_1' to a braid β_1'' which continues to cobound with β_1 a total of m embedded annuli. Moreover, we can now realize bigon discs D_i whose braid foliations consist of bands of s–arcs alternating with regions tiled by *aa*–, *ab*– and *bb*–tiles, with either a single *as*–tile serving as a transition between the s–band and the tiled region, or two *abs*–tiles serving as the transition on either end of the tiled region. We orient each D_i so as to agree with the orientation of the ∂_i^- boundary arc, so that a–arcs along ∂_i^- connect to positive vertices and a–arcs along ∂_i^+ connect to negative vertices; we then work to simplify the foliation of each D_i.

First, consider all singular leaves which intersect the ∂_i^- boundary arc; by slightly perturbing D_i, if necessary, we may assume β_1 does not intersect any such singular leaves. If any of these singular leaves has its other endpoint on a vertex v, we may stabilize the ∂_i^- arc along that singular leaf so as to remove that vertex v from the foliation of D_i (see Figure 6).

Since an *abs*–tile implies the existence of such a stabilization of ∂_i^-, we may assume that all *abs*–tiles are eliminated by this process of stabilization, as well as any *as*–tiles containing negative vertices. As a result, the foliation of each D_i intersects the ∂_i^+

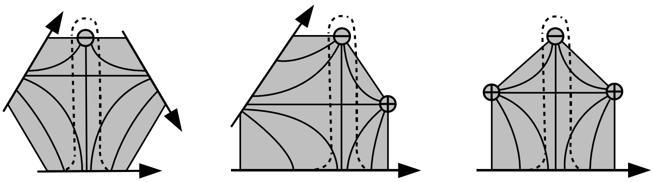

Figure 6: We can stabilize ∂_i^- along singular leaves if any of these three configurations occur.

boundary arc along a single *as*–tile having a positive vertex, and any *ab*–tiles in D_i must have boundary along ∂_i^- and therefore can be eliminated by stabilization of ∂_i^-. This eliminates all *bb*–tiles from D_i, as any collection of *bb*–tiles must eventually be glued to either *ab*– or *abs*–tiles. Thus, the foliation of each D_i consists of a band of s–arcs with trees of *aa*–tiles extending off of it along the ∂_i^- boundary arc, with a single *as*–tile connecting each tree of the forest to the band of s–arcs; see the left-hand side of Figure 7.

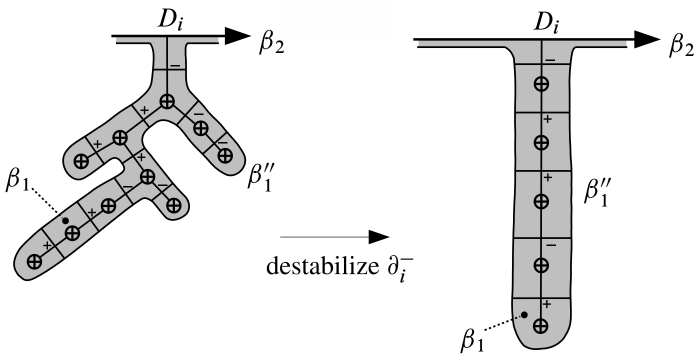

Figure 7: On the left, D_i, whose vertex-singularity graph is a tree after stabilizing ∂_i^-; on the right, D_i, whose graph is a linear tree after destabilizing ∂_i^-

We can now destabilize the ∂_i^- boundary arc along each outermost *aa*–tile containing a valence-1 vertex, as long as it does not contain the point of intersection with β_1; the result is that each D_i may be assumed to be a linear string of *aa*–tiles as in the right-hand side of Figure 7, where the outermost tile farthest from β_2 contains the intersection with β_1. Observe that throughout this simplification of the foliation, both β_2 and β_1 are fixed, and the resulting braid β_1'' continues to cobound with β_1 a total of m embedded annuli.

We now examine the resulting linear foliation on a single D_i; it consists of positive vertices, along with a sequence of singularities. The result is a twisted band, with the parity of the half-twists given by the signs of the singularities. It is then evident that if anytime a negative singularity is consecutive with a positive singularity, we may perform an exchange move so as to reorder those two singularities; see Figures 8 and 9. Moreover, this exchange move involves an isotopy of D_i which is performed in a regular neighborhood of the subdisc of D_i cobounded by ∂_i^- and the singular leaves associated with the two singularities; thus the isotopy fixes both β_1, β_2 and the other D_i.

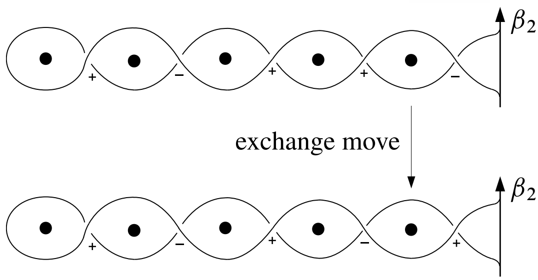

Figure 8: An exchange move involving the two consecutive singularities closest to the β_2–end of the D_i linear tree. The black dots represent intersections of the braid axis A with D_i.

In this way, we may arrange that all positive singularities are stacked at the β_2–end of the foliation of D_i and all negative singularities are stacked at the β_1–end of the foliation of D_i. We then can negatively destabilize ∂_i^- through those singularities, but in doing so we will induce negative stabilizations of β_1 so as to avoid β_1'' passing through β_1, as depicted in Figures 9 and 10. (See "microflypes" in [5, Section 2.3].) The result of negatively stabilizing β_1 will be our desired $\widehat{\beta}_1$, which cobounds with our new β_1'' a total of m embedded annuli. We then observe in Figure 9 that stabilizing β_2 along the remaining positive singularities and allowing it to pass through $\widehat{\beta}_1$ will yield our desired $\widehat{\beta}_2$, where in fact $\widehat{\beta}_2 = \beta_1''$ and as a result $\widehat{\beta}_2$ cobounds with $\widehat{\beta}_1$ a total of m embedded annuli. □

3 Jones' conjecture

Before proving Theorem 1.2, we recall a reformulation of the generalized Jones' conjecture as described by Kawamuro [9; 10]. To do so, observe that for any braid $\beta \in \mathcal{L}$ there is an ordered pair $(\ell(\beta), n(\beta))$; we then have the following definition:

Definition 3.1 Let $\beta \in \mathcal{L}$. The *cone* of β is defined to be $(\ell(\beta), n(\beta))$ along with all pairs (ℓ, n) that can be achieved by stabilizing β.

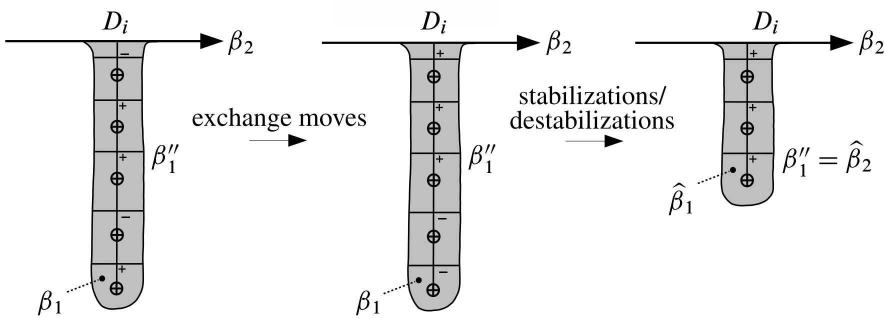

Figure 9: On the left, reordering of singularities using exchange moves so that all positive singularities are stacked near the β_2–end of D_i and all negative singularities are stacked near the β_1–end. On the right, negative destabilizations of ∂_i^-, along with the corresponding negative stabilizations of β_1 which yield $\widehat{\beta}_1$, and the subsequent positive stabilizations of β_2 which yield $\widehat{\beta}_2$.

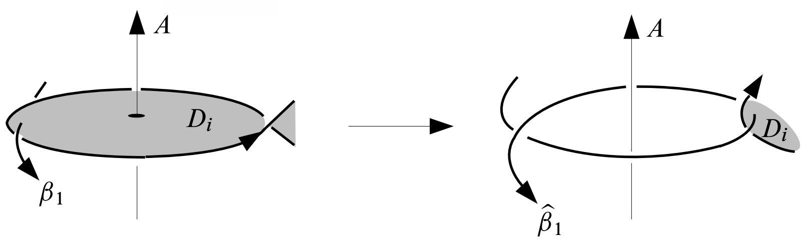

Figure 10: A destabilization of the braided boundary of D_i induces a stabilization of β_1.

In the (ℓ, n)–plane, the cone of β indeed has the shape of a cone; it consists of a lattice of points at or above $(\ell(\beta), n(\beta))$ which are contained within the region bounded by lines of slope +1 and -1 passing through the point $(\ell(\beta), n(\beta))$; see Figure 11.

An equivalent reformulation of Theorem 1.2 due to Kawamuro is then the following proposition, which we prove.

Proposition 3.2 *Let $\beta_0 \in \mathcal{L}$ be at minimum braid index for the link type $\mathcal{L}$. If $\beta \in \mathcal{L}$, then the cone of β is contained in the cone of β_0.*

Proof Suppose for contradiction that there is a $\beta_1 \in \mathcal{L}$ whose cone contains points outside the cone of β_0. Then in fact $(\ell(\beta_1), n(\beta_1))$ must be outside the cone of β_0. We assume for the moment that $(\ell(\beta_1), n(\beta_1))$ is such that $\ell(\beta_1) < \ell(\beta)$ for any β in the cone of β_0 with $n(\beta_1) = n(\beta)$. In other words, as we look at the cone of β_0, we see β_1 *to the left* of the cone of β_0; see Figure 12.

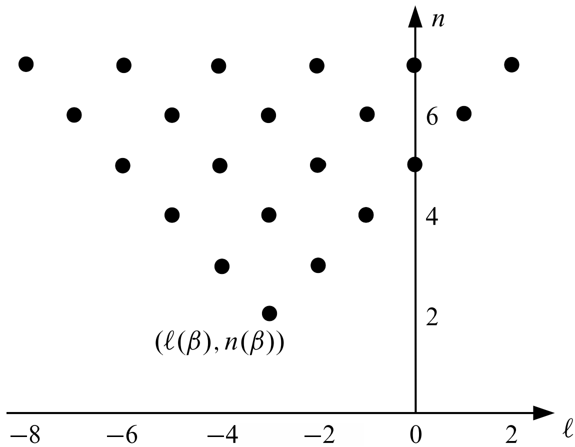

Figure 11: The cone of a braid β in the (ℓ, n)–plane

By Proposition 1.1 there exist $\widehat{\beta}_0, \widehat{\beta}_1 \in \mathcal{L}$ such that the m components of each braid pairwise cobound m embedded annuli, where $\widehat{\beta}_0$ is obtained from β_0 by positively stabilizing, and $\widehat{\beta}_1$ is obtained from β_1 by negatively stabilizing. As a result, we observe that $\widehat{\beta}_1$ lies *to the left and outside of* the cone of β_0, and $\widehat{\beta}_0$ lies *to the right and outside of* the cone for β_1; see Figure 12. Our goal in the proof of the current proposition is to use the embedded annuli to find a braid $\widehat{\beta}_0^*$ which is obtained from $\widehat{\beta}_0$ by destabilizations, braid isotopy and exchange moves, and a braid $\widehat{\beta}_1^*$ which is obtained from $\widehat{\beta}_1$ by destabilizations, braid isotopy and exchange moves, such that $(\ell(\widehat{\beta}_0^*), n(\widehat{\beta}_0^*)) = (\ell(\widehat{\beta}_1^*), n(\widehat{\beta}_1^*))$. Since neither braid isotopy nor exchange moves change the algebraic length or braid index of a braid, the conclusion is that $n(\widehat{\beta}_0^*) = n(\widehat{\beta}_1^*) < n(\beta_0)$, which is our desired contradiction since β_0 is at minimum braid index. It is then evident that if we begin with β_1 *to the right* of the cone of β_0, we can reverse the roles of β_0, β_1 in Proposition 1.1 to achieve a similar contradiction.

So it remains to consider a representative annulus $\mathcal{A}$ cobounded by a component of $\widehat{\beta}_0$ and a component of $\widehat{\beta}_1$. First, we observe that any *aa*– or *as*–tile must separate from $\mathcal{A}$ a subdisc Δ which is cobounded by the associated singular leaf and a subarc of a single component of $\partial\mathcal{A}$. By an Euler characteristic calculation [3, Lemma 7], we are guaranteed that

$$V(1,1) + 2V(0,2) + V(0,3) \geq 4,$$

where $V(\alpha, \beta)$ is the number of vertices of type (α, β) in Δ. Therefore, the tiling of Δ contains either valence-1 vertices which we can remove by destabilizing, or valence-2 or valence-3 vertices which can then be eliminated after braid isotopy and exchange

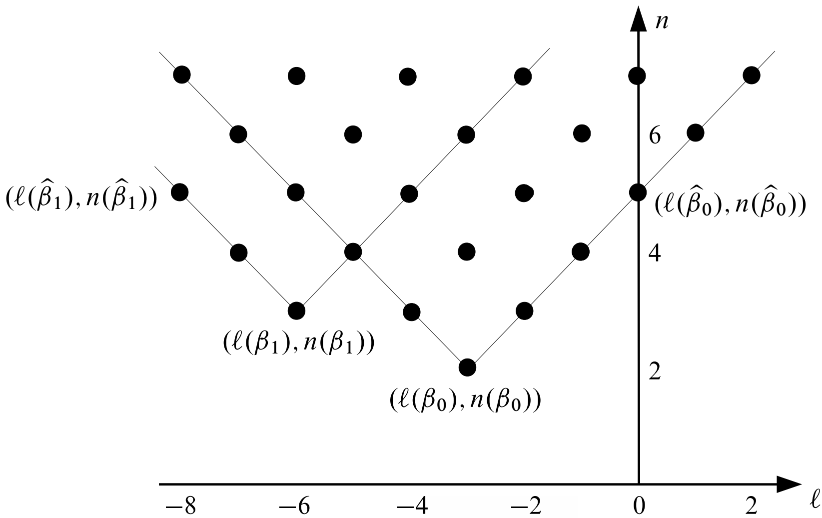

Figure 12: Two cones with neither one contained in the other

moves. We may therefore assume that we can always eliminate *aa*– or *as*–tiles. The result then is an annulus $\mathcal{A}$ whose foliation consists of s–bands alternating with bigon discs Δ_i such that two *abs*–tiles serve as the transition on either end of each bigon disc. Then, by a related Euler characteristic calculation [5, Lemma 6.3.1] for our annulus $\mathcal{A}$, we know that

$$V(1,1) + 2V(0,2) + V(0,3) = 2E(s) + V(2,1) + 2V(3,0) + \sum_{v=4}^{\infty}\sum_{\alpha=0}^{v}(v+\alpha-4)V(\alpha, v-\alpha),$$

where $E(s)$ is twice the number of s–bands; furthermore, if both sides equal zero we know that only vertices of type $(1,2)$ or $(0,4)$ appear.

If $E(s) > 0$ we may therefore eliminate all vertices in the foliation of $\mathcal{A}$ using braid isotopy, exchange moves and destabilizations, and we obtain an annulus whose foliation consists entirely of s–arcs. If this is the case for all of our m embedded annuli, then we are done, since the resulting $\widehat{\beta}_0^*$ and $\widehat{\beta}_1^*$ will have the same braid index and algebraic length.

Otherwise, if $E(s) = 0$ for some annulus, then again using braid isotopy, exchange moves and destabilizations we obtain either an annulus entirely foliated by s–arcs (in which case we are done), or an annulus whose foliation is tiled entirely by *ab*– or *bb*–tiles in which every vertex is either of type $(1,2)$ or $(0,4)$. If any of the remaining vertices of type $(1,2)$ or $(0,4)$ are adjacent to consecutive singularities of like parity on either end of a one-parameter family of b–arcs, then we may perform a standard change of fibration to obtain a vertex of type either $(1,1)$ or $(0,3)$, which we may then remove following braid isotopy and exchange moves. We may therefore assume we obtain a tiling in which consecutive singularities around any vertex alternate parity (see Figure 13). The tiling of $\mathcal{A}$ will then be composed of a subannulus of some number r of ab–tiles along $\widehat{\beta}_0^*$, and a subannulus of the same number r of ab–tiles along $\widehat{\beta}_1^*$, along with k subannuli containing a number $2r$ of bb–tiles which interpolate between the subannuli of ab–tiles; in Figure 13, $r = 3$ and $k = 1$. However, in this case the resulting braids $\widehat{\beta}_0^*$ and $\widehat{\beta}_1^*$ will then have the same braid index and algebraic length, and we achieve the desired contradiction. To justify this last statement, observe that if the annulus $\mathcal{A}$ is oriented so as to agree with $\widehat{\beta}_1^*$, we may stabilize $\widehat{\beta}_1^*$ along singular leaves in $\mathcal{A}$ some number $r(k+1)$ of times to remove all negative vertices in $\mathcal{A}$, and then destabilize along valence-1 vertices the same number $r(k+1)$ of times to remove all positive vertices from $\mathcal{A}$; moreover, the parities of all of these stabilizations and destabilizations are identical, and thus $(\ell(\widehat{\beta}_0^*), n(\widehat{\beta}_0^*)) = (\ell(\widehat{\beta}_1^*), n(\widehat{\beta}_1^*))$. □

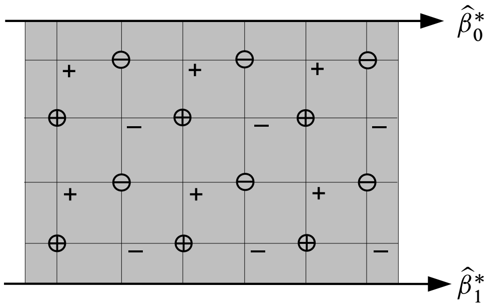

Figure 13: A tiling of $\mathcal{A}$ such that all vertices are either valence-4 or valence-3, and consecutive singularities around any vertex alternate sign

References

[1] **J W Alexander**, *A lemma on systems of knotted curves*, Proc. Nat. Acad. Sci. U.S.A. 9 (1923) 93–95

[2] **J S Birman**, **E Finkelstein**, *Studying surfaces via closed braids*, J. Knot Theory Ramifications 7 (1998) 267–334 MR1625362

[3] **J S Birman**, **W W Menasco**, *Studying links via closed braids, V: The unlink*, Trans. Amer. Math. Soc. 329 (1992) 585–606 MR1030509

[4] **J S Birman**, **W W Menasco**, *On Markov's theorem*, J. Knot Theory Ramifications 11 (2002) 295–310 MR1905686

[5] **J S Birman**, **W W Menasco**, *Stabilization in the braid groups, I: MTWS*, Geom. Topol. 10 (2006) 413–540 MR2224463

[6] **I A Dynnikov**, **M V Prasolov**, *Bypasses for rectangular diagrams. A proof of the Jones conjecture and related questions*, Tr. Mosk. Mat. Obs. 74 (2013) 115–173 In Russian; translated in Trans. Moscow Math. Soc. 2013, 97–144

[7] **J B Etnyre**, **J Van Horn-Morris**, *Fibered transverse knots and the Bennequin bound*, Int. Math. Res. Not. 2011 (2011) 1483–1509 MR2806512

[8] **V F R Jones**, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. 126 (1987) 335–388 MR908150

[9] **K Kawamuro**, *The algebraic crossing number and the braid index of knots and links*, Algebr. Geom. Topol. 6 (2006) 2313–2350 MR2286028

[10] **K Kawamuro**, *Conjectures on the braid index and the algebraic crossing number*, from: "Intelligence of low dimensional topology 2006", (J S Carter, S Kamada, L H Kauffman, A Kawauchi, T Kohno, editors), Ser. Knots Everything 40, World Sci. Publ., Hackensack, NJ (2007) 151–155 MR2371720

[11] **A Stoimenow**, *Properties of closed* 3*–braids* (2007) arXiv:math/0606435

Department of Mathematics, Western Illinois University
Macomb, IL 61455, USA

Department of Mathematics, University at Buffalo
Buffalo, NY 14260, USA

d-lafountain@wiu.edu, menasco@buffalo.edu

http://www.wiu.edu/users/dl127/, http://www.math.buffalo.edu/~menasco/

Received: 17 October 2013 Revised: 2 January 2014