Index theory of the de Rham complex on manifolds with periodic ends

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We study the de Rham complex on a smooth manifold with a periodic end modeled on an infinite cyclic cover \( \mathcal{X} \to X \). The completion of this complex in exponentially weighted \( L^2 \) norms is Fredholm for all but finitely many exceptional weights determined by the eigenvalues of the covering translation map \( H_*(\mathcal{X}) \to H_*(\mathcal{X}) \). We calculate the index of this weighted de Rham complex for all weights away from the exceptional ones.

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1 Introduction

Let \( M \) be a smooth closed orientable manifold of dimension \( n \). The de Rham complex of complex-valued differential forms on \( M \),

\[
0 \longrightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \longrightarrow \cdots \longrightarrow \Omega^n(M) \longrightarrow 0,
\]
is known to be Fredholm in a suitable \( L^2 \) completion. This means as usual that the images of \( d_k \) are closed and the vector spaces \( \ker d_k / \text{im} d_{k-1} \) are finite-dimensional. The alternating sum of the dimensions of these spaces is called the index of the de Rham complex. Since \( \ker d_k / \text{im} d_{k-1} \) is isomorphic to the singular cohomology \( H^k(M; \mathbb{C}) \) by the de Rham theorem, the above index equals \( \chi(M) \), the Euler characteristic of \( M \).

This paper extends these classical results to certain noncompact manifolds; those with periodic ends. It builds on the earlier work of Miller [10] and Taubes [16] and can be viewed as a continuation of our research in [12] and [13] on the index theory of elliptic operators on such manifolds.

By a manifold with a periodic end we mean an open Riemannian manifold \( M \) whose end, in the sense of Hughes and Ranicki [7], is modeled on an infinite cyclic cover \( \mathcal{X} \) of a compact manifold \( X \) associated with a primitive cohomology class \( \gamma \in H^1(X; \mathbb{Z}) \);
the case of several ends can be treated similarly. To be precise, the manifold \( M \) is of the form
\[
Z_{\infty} = Z \cup W_0 \cup W_1 \cup W_2 \cup \cdots,
\]
where \( W_k \) are isometric copies of the fundamental segment \( W \) obtained by cutting \( X \) open along an oriented connected submanifold \( Y \) Poincaré dual to \( \gamma \), and \( Z \) is a smooth compact manifold with boundary \( Y \).

The de Rham complex of \( M \) can be completed in the \( L^2 \) norm using (over the end) a Riemannian measure \( dx \) lifted from that on \( X \). This completion is, however, not Fredholm; see Remark 2.3. To rectify this problem, we will use \( L^2 \) norms, which are the \( L^2 \) norms on \( M \) with respect to the measure \( e^{\delta f(x)} dx \) over the end. Here \( \delta \) is a real number and \( f: \tilde{X} \to \mathbb{R} \) is a smooth function such that \( f(\tau(x)) = f(x) + 1 \) with respect to the covering translation \( \tau: \tilde{X} \to \tilde{X} \). The \( L^2_{\delta} \) completion of the de Rham complex on \( M \) will be denoted by \( \Omega^*_\delta(M) \).

**Theorem 1.1** \( M \) be a smooth Riemannian manifold with a periodic end modeled on \( \tilde{X} \), and suppose that \( H_\bullet(M; \mathbb{C}) \) is finite-dimensional. Then \( \Omega^*_\delta(M) \) is Fredholm for all but finitely many \( \delta \) of the form \( \delta = \ln |\lambda| \), where \( \lambda \) is a root of the characteristic polynomial of \( \tau_*: H_\bullet(\tilde{X}; \mathbb{C}) \to H_\bullet(\tilde{X}; \mathbb{C}) \).

Conditions on \( X \) that guarantee that \( H_\bullet(M; \mathbb{C}) \) is finite-dimensional can be found in Section 2, together with a proof of Theorem 1.1.

Given a manifold \( M \) as in the above theorem, the complex \( \Omega^*_\delta(M) \) has a well-defined index \( \text{ind}_\delta(M) \). Miller [10] showed that \( \text{ind}_\delta(M) \) is an even or odd function of \( \delta \) according to whether \( \dim M = n \) is even or odd, and that \( \text{ind}_\delta(M) = (-1)^n \chi(M) \) for sufficiently large \( \delta > 0 \). We add to this knowledge the following result.

**Theorem 1.2** \( M \) be as in Theorem 1.1. Then \( \text{ind}_\delta(M) \) is a piecewise constant function of \( \delta \) whose only jumps occur at \( \delta = \ln |\lambda| \), where \( \lambda \) is a root of the characteristic polynomial \( A_k(t) \) of \( \tau_*: H_k(\tilde{X}; \mathbb{C}) \to H_k(\tilde{X}; \mathbb{C}) \) for some \( k = 0, \ldots, n-1 \). Every such \( \lambda \) contributes \( (-1)^{k+1} \) times its multiplicity as a root of \( A_k(t) \) to the jump.

Together with the results of [10] this completes the calculation of the function \( \text{ind}_\delta(M) \). Theorem 1.2 is proved in Section 3. The last section of the paper contains discussion as well as calculations of \( \text{ind}_\delta(M) \) for two important classes of examples. The first class consists of manifolds with infinite cylindrical ends studied earlier by Atiyah, Patodi and Singer [1], and the second of manifolds arising in the study of knotted 2–spheres in \( S^4 \).

Finally, we will remark that the de Rham complex is a special case of the more general concept of an elliptic complex. The index theory for elliptic complexes on closed
manifolds was developed by Atiyah and Singer, whose famous index theorem [2] expresses the index of such a complex in purely topological terms. More generally, Atiyah, Patodi and Singer [1] computed the index for certain elliptic operators (that is, elliptic complexes of length two) on manifolds with cylindrical ends. In our paper [12] we extended their result to general manifolds with periodic ends; our index formula involves a new periodic \( \eta \)-invariant, generalizing the \( \eta \)-invariant of Atiyah, Patodi and Singer from the cylindrical setting.

It would be interesting to compare the above formula for \( \text{ind}_\delta(M) \) with the formula in [12] for the index of the operator \( d + d^* \) obtained by wrapping up the de Rham complex. The issue is that both the de Rham complex and the operator \( d + d^* \) are Fredholm only when completed in \( L^2_\delta \) norms with \( \delta \neq 0 \). Since the dual space of \( L^2_\delta(M) \) equals \( L^2_{-\delta}(M) \) rather than \( L^2_\delta(M) \), this completion does not commute with the wrap-up procedure, hence the index of the operator \( d + d^* \) completed in \( L^2_\delta \) norms need not match the index \( \text{ind}_\delta(M) \) of the current paper. Atiyah, Patodi and Singer dealt with a similar issue in [1, Proposition 4.9] using a rather precise relation between \( L^2 \) harmonic forms and cohomology on manifolds with cylindrical ends; comparing the above indices would likely entail developing such a relation for general manifolds with periodic ends.

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## 2 The Fredholm property

In this section, we will prove Theorem 1.1 by reducing it to a statement about twisted cohomology of \( X \).

### 2.1 The Fourier–Laplace transform

The de Rham complex of \( M \) is an elliptic complex on the end-periodic manifold \( M \), hence we can use the general theory of such complexes due to Taubes [16]. According to that theory, it is sufficient to check the Fredholm property of the de Rham complex of \( \tilde{X} \) completed in the \( L^2 \) norm on \( \tilde{X} \) with respect to the measure \( e^{\delta f(x)} \, dx \). The latter complex can be studied using the Fourier–Laplace transform, defined by

\[
\hat{\omega}_z = \sum z^k \cdot (\tau^*)^k \omega, \quad z \in \mathbb{C}^*.
\]
on compactly supported forms $\omega$ on $\tilde{X}$, and extended by continuity to $L^2$ forms. The summation in the above formula extends over all integers $k$, which makes the form $\tilde{\omega}_z$ invariant with respect to $\tau^*$. The form $\tilde{\omega}_z$ then defines a form on $X$ which is denoted by the same symbol. An application of the Fourier–Laplace transform to the de Rham complex on $\tilde{X}$ results in a family of twisted de Rham complexes

$$
\cdots \longrightarrow \Omega^k(X) \xrightarrow{d - \ln z \, df} \Omega^{k+1}(X) \longrightarrow \cdots
$$

parameterized by $z \in \mathbb{C}^*$. The following result is proved in Taubes [16, Lemma 4.3].

**Proposition 2.1** Let $M$ be a smooth Riemannian manifold with a periodic end modeled on $\tilde{X}$. For any given $\delta$, the complex $\Omega^*_\delta(M)$ is Fredholm if and only if the complexes (1) are exact for all $z$ such that $|z| = e^\delta$.

The cohomology of complex (1) is of course the twisted de Rham cohomology $H^*_z(X; \mathbb{C})$ with coefficients in the complex line bundle with flat connection $- \ln z \, df$.

**Proposition 2.2** Let $M$ be a smooth Riemannian manifold with a periodic end modeled on $\tilde{X}$. Then the following three conditions are equivalent:

(i) $\Omega^*_\delta(M)$ is Fredholm for all $\delta \in \mathbb{R}$ away from a discrete set.

(ii) $H^*_z(X; \mathbb{C})$ vanishes for all $z \in \mathbb{C}^*$ away from a discrete set.

(iii) $H^*_z(X; \mathbb{C})$ vanishes for at least one $z \in \mathbb{C}^*$.

**Proof** Observe that the complexes (1) form a holomorphic family of elliptic complexes on $\mathbb{C}^*$; therefore, exactness of (1) at one point is equivalent to exactness away from a discrete set by the analytic Fredholm theorem. The rest of the statement follows from the preceding discussion.

**Remark 2.3** The usual $L^2$ completion of the de Rham complex on $M$, that is, the complex $\Omega^*_0(M)$ is not Fredholm because $H^0_z(X; \mathbb{C})$ is not zero when $z = 1$.

### 2.2 Finite-dimensionality

Fix a finite cell complex structure on $X$, lift it to $\tilde{X}$, and consider the chain complex $C_*(\tilde{X}; \mathbb{C})$ and its homology $H_*(\tilde{X}; \mathbb{C})$. The group of integers acts on both by covering translations making them into finitely generated modules over $\mathbb{C}[t, t^{-1}]$. The twisted de Rham theorem tells us that the cohomology $H^*_z(X; \mathbb{C})$ of complex (1) is isomorphic to the cohomology of the complex Hom$_{\mathbb{C}[t, t^{-1}]}(C_*(\tilde{X}; \mathbb{C}), \mathbb{C}_z)$, where $\mathbb{C}_z$ denotes a copy of $\mathbb{C}$ viewed as a $\mathbb{C}[t, t^{-1}]$ module with $p(t)$ acting via multiplication by $p(z)$.  

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Proposition 2.4  Let \( M \) be a smooth Riemannian manifold with a periodic end modeled on \( \widetilde{X} \). Then the following two conditions are equivalent:

(i)  \( H_*(M; \mathbb{C}) \) is a finite-dimensional vector space.

(ii) \( H^*_z(X; \mathbb{C}) \) vanishes for at least one \( z \in \mathbb{C}^* \).

Proof  It is immediate from the Mayer–Vietoris principle that \( H^*(M; \mathbb{C}) \) is finite-dimensional if and only if \( H^*(\widetilde{X}; \mathbb{C}) \) is finite-dimensional. Since \( H_*^*(\widetilde{X}; \mathbb{C}) \) is a finitely generated module over the principal ideal domain \( \mathbb{C}[t, t^{-1}] \), it admits a primary decomposition

\[
H^*_z(X; \mathbb{C}) = \mathbb{C}[t, t^{-1}]^\ell \oplus \mathbb{C}[t, t^{-1}]/(p_1) \oplus \cdots \oplus \mathbb{C}[t, t^{-1}]/(p_m);
\]

therefore, \( H^*_z(X; \mathbb{C}) \) is a finite-dimensional vector space if and only if \( \ell = 0 \) in this decomposition. According to the universal coefficient theorem,

\[
H^*_z(X; \mathbb{C}) = \text{Hom}_{\mathbb{C}[t, t^{-1}]}(H^*_z(\widetilde{X}; \mathbb{C}), \mathbb{C}_2) \oplus \text{Ext}_{\mathbb{C}[t, t^{-1}]}(H^*_z(\widetilde{X}; \mathbb{C}), \mathbb{C}_2),
\]

hence vanishing of \( H^*_z(X; \mathbb{C}) \) for at least one \( z \) implies that \( \ell = 0 \). On the other hand, an easy calculation shows that

\[
\text{Hom}_{\mathbb{C}[t, t^{-1}]}(V, \mathbb{C}_2) = \text{Ext}_{\mathbb{C}[t, t^{-1}]}(V, \mathbb{C}_2) = 0
\]

for any module \( V = \mathbb{C}[t, t^{-1}]/(p) \) such that \( p(z) \neq 0 \). Therefore, \( \ell = 0 \) implies that \( H^*_z(X; \mathbb{C}) \) must vanish for all \( z \) away from the roots of the polynomials \( p_1, \ldots, p_m \).

\[\square\]

2.3 Proof of Theorem 1.1

It follows from Propositions 2.2 and 2.4 that if \( H^*(\widetilde{X}; \mathbb{C}) \) is finite-dimensional, the complex \( \Omega^*_\delta(M) \) is Fredholm for all \( \delta \) away from a discrete set. To finish the proof of Theorem 1.1 we just need to identify this discrete set. According to Proposition 2.1, it consists of \( \delta = \ln |z| \), where \( z \in \mathbb{C}^* \) are the complex numbers for which \( H^*_z(X; \mathbb{C}) \) fails to be zero. To find them, note that the free part in the prime decomposition (2) vanishes, making \( H^*_z(\widetilde{X}; \mathbb{C}) \) into a torsion module,

\[
H_*(\widetilde{X}; \mathbb{C}) = \mathbb{C}[t, t^{-1}]/(p_1) \oplus \cdots \oplus \mathbb{C}[t, t^{-1}]/(p_m).
\]

According to Milnor [11, Assertion 4], the order ideal \( (p_1 \cdots p_m) \) of this module is spanned by the characteristic polynomial of \( \tau_*: H_*(\widetilde{X}; \mathbb{C}) \to H_*(\widetilde{X}; \mathbb{C}) \). The calculation with the universal coefficient theorem as in the proof of Proposition 2.4 now completes the proof.

\[\square\]
2.4 A sufficient condition

Let $M$ be a smooth orientable manifold with a periodic end modeled on $\widetilde{X}$. Vanishing of $\chi(X)$ is obviously a necessary condition for the vector space $H_\ast(M; \mathbb{C})$ to be finite-dimensional. To come up with a sufficient condition, observe that the derivative $df$ defines a closed 1–form on $X$, and let $\xi = [df] \in H^1(X; \mathbb{C})$ be its cohomology class. The cup product with $\xi$ gives rise to the chain complex

$$
H^0(X; \mathbb{C}) \xrightarrow{\cup \xi} H^1(X; \mathbb{C}) \xrightarrow{\cup \xi} \cdots \xrightarrow{\cup \xi} H^n(X; \mathbb{C}).
$$

Proposition 2.5 Suppose the chain complex (4) is exact. Then $H_\ast(M; \mathbb{C})$ is a finite-dimensional vector space for any smooth orientable manifold with periodic end modeled on $\widetilde{X}$.

This proposition can be derived as a special case of Taubes [16, Theorem 3.1]. That proof is analytic in nature; here is another proof which is purely topological.

According to Proposition 2.4, it is sufficient to prove that the twisted cohomology $H^\ast_\xi(X; \mathbb{C})$ vanishes for at least one $z \in \mathbb{C}^\ast$. We will show that it does so for all $z \neq 1$ in a sufficiently small neighborhood of 1. For such $z$, there is a spectral sequence $(E^r_\ast, d_r)$ which starts at $E^1_\ast = H^\ast(X; \mathbb{C})$, converges to $H^\ast_\xi(X; \mathbb{C})$ and whose differentials are given by the Massey products with the class $\xi$; see Farber [5, Section 10.9] and Pajitnov [14]. The convergence of this spectral sequence to zero is therefore a necessary and sufficient condition for the finite-dimensionality of $H_\ast(M; \mathbb{C})$. The chain complex (4) is the term $(E^1_\ast, d_1)$ of that spectral sequence, hence its exactness is sufficient for the finite-dimensionality of $H^\ast(X; \mathbb{C})$.

Note that the vanishing of the Euler characteristic of $X$ is not a sufficient condition for $H_\ast(M; \mathbb{C})$ to be finite-dimensional. An example is provided by the connected sum of $S^1 \times S^{n-1}$ with any manifold that is not a rational homology sphere but has Euler characteristic 2.

3 The index calculation

Let $M$ be a smooth Riemannian manifold with periodic end modeled on the infinite cyclic cover $\widetilde{X}$, and assume that $H_\ast(M; \mathbb{C})$ is finite-dimensional. Let $\tau: \widetilde{X} \to \widetilde{X}$ be a covering translation, and denote by $A_k(t)$ the characteristic polynomial of $\tau_\ast: H_k(\widetilde{X}; \mathbb{C}) \to H_k(\widetilde{X}; \mathbb{C})$. (The polynomial $A_1(t)$ is known as the Alexander polynomial of the fundamental group of $X$.) Denote by $\Delta$ the set of all $\delta$ of the form $\delta = \ln |\lambda|$, where $\lambda$ is a root of the product polynomial $A_0(t) \cdots A_{n-1}(t)$. 

According to Theorem 1.1, the complex $\Omega^*_{\delta}(M)$ is Fredholm for all $\delta$ away from $\Delta$. Its index $\text{ind}_\delta(M)$ is a piecewise constant function away from $\Delta$, where it may jump. We wish to calculate the size of these jumps.

### 3.1 Excision principle

Let $\delta_1$ and $\delta_2$ be two weights in $\mathbb{R} - \Delta$, and complete the de Rham complex of $\tilde{X}$ in the $L^2$ norm with respect to the measures $e^{\delta_1 f(x)} dx$ on the negative end of $\tilde{X}$ and $e^{\delta_2 f(x)} dx$ on the positive end. This complex will be denoted by $\Omega^*_{\delta_1 \delta_2}(\tilde{X})$. This is a Fredholm complex, whose index will be denoted by $\text{ind}_{\delta_1 \delta_2}(\tilde{X})$.

**Proposition 3.1**

$$\text{ind}_{\delta_2}(M) = \text{ind}_{\delta_1}(M) = \text{ind}_{\delta_1 \delta_2}(\tilde{X}).$$

**Proof** Let $c \in \mathbb{R}$ be a regular value of $f$: $\tilde{X} \to \mathbb{R}$; then $Y = f^{-1}(c)$ is a submanifold of $\tilde{X}$ separating it as $\tilde{X} = \tilde{X}_- \cup \tilde{X}_+$. Write $M = Z \cup \tilde{X}_+$ for some smooth compact manifold $Z$ with boundary $Y$. An application of the excision principle to these two splittings yields

$$\text{ind}_{\delta_1}(M) + \text{ind}_{\delta_1 \delta_2}(\tilde{X}) = \text{ind}_{\delta_2}(M) + \text{ind}_{\delta_1}(\tilde{X}).$$

Note that the complex $\Omega^*_{\delta_1 \delta_2}(\tilde{X})$ is exact because the complexes (1) obtained from it by the Fourier–Laplace transform are exact for all $z$ with $|z| = e^{\delta_1}$. Therefore, $\text{ind}_{\delta_1}(\tilde{X}) = 0$ and the proof is complete. $\square$

### 3.2 Computing the cohomology of $\Omega^*_{\delta_1 \delta_2}(\tilde{X})$

We will proceed by several reductions, the first being from weighted forms to weighted cellular cochains. To be precise, fix a finite cell complex structure on $X$ and lift it to $\tilde{X}$. Also, introduce the Hilbert space $\ell^2_{\delta_1 \delta_2}$ of the sequences $\{x_k \mid k \in \mathbb{Z}\}$ of complex numbers such that

$$\sum_{k < 0} e^{2\delta_1 k} |x_k|^2 < \infty \quad \text{and} \quad \sum_{k > 0} e^{2\delta_2 k} |x_k|^2 < \infty.$$

Theorem 2.17 of Miller [10], which is a weighted version of the $L^2$ de Rham theorem [4; 9] establishes an isomorphism between the cohomology of $\Omega^*_{\delta_1 \delta_2}(\tilde{X})$ and the cellular cohomology of $\tilde{X}$ with $\ell^2_{\delta_1 \delta_2}$ coefficients. Miller actually uses weighted simplicial cohomology, but this is readily seen to be isomorphic to the more standard and convenient cellular version.
Proposition 3.2 View $\ell^2_{\delta_1 \delta_2}$ as a $\mathbb{C}[t, t^{-1}]$ module with $t$ acting as the right shift operator, $t(x_k) = x_{k+1}$. Then for all but finitely many $\delta_1$ and $\delta_2$, the cohomology of $\tilde{X}$ with $\ell^2_{\delta_1 \delta_2}$ coefficients equals the homology of the complex

$$\hom_{\mathbb{C}[t, t^{-1}]}(C_\ast(\tilde{X}, \mathbb{C}), \ell^2_{\delta_1 \delta_2}).$$

Proof This will follow as soon as we show that the images of the boundary operators $\partial$ in complex (5) are closed. These boundary operators

$$\partial: \left(\ell^2_{\delta_1 \delta_2}\right)^k \to \left(\ell^2_{\delta_1 \delta_2}\right)^\ell$$

are matrices whose entries are Laurent polynomials in $t$. Since $\mathbb{C}[t, t^{-1}]$ is a principal ideal domain, each $\partial$ will have a diagonal matrix in properly chosen bases. The statement now follows from the fact that the operator $t - \lambda: \ell^2_{\delta_1 \delta_2} \to \ell^2_{\delta_1 \delta_2}$ is Fredholm for all $\lambda$ with $|\lambda|$ different from $e^{\delta_1}$ and $e^{\delta_2}$; see for instance Conway [3, Proposition 27.7 (c)].

The universal coefficient theorem now tells us that the $\ell^2_{\delta_1 \delta_2}$ cohomology of $\tilde{X}$ is isomorphic to

$$\hom_{\mathbb{C}[t, t^{-1}]}(H_\ast(\tilde{X}; \mathbb{C}), \ell^2_{\delta_1 \delta_2}) \oplus \ext_{\mathbb{C}[t, t^{-1}]}(H_\ast(\tilde{X}; \mathbb{C}), \ell^2_{\delta_1 \delta_2}).$$

Recall from the proof of Theorem 1.1 that $H_\ast(\tilde{X}; \mathbb{C})$ is a torsion module (3) whose order ideal is spanned by the characteristic polynomial of $\tau_\ast: H_\ast(\tilde{X}; \mathbb{C}) \to H_\ast(\tilde{X}; \mathbb{C})$. Therefore, our next step will be to compute (6), one cyclic module at a time.

Lemma 3.3 Let $\lambda$ be a complex number such that $|\lambda|$ is different from $e^{\delta_1}$ and $e^{\delta_2}$. Then, for any cyclic module $V = \mathbb{C}[t, t^{-1}]/(t - \lambda)^m$, we have

$$\ext_{\mathbb{C}[t, t^{-1}]}(V, \ell^2_{\delta_1 \delta_2}) = 0.$$ 

Proof We know from the proof of Proposition 3.2 that the operator $t - \lambda: \ell^2_{\delta_1 \delta_2} \to \ell^2_{\delta_1 \delta_2}$ is Fredholm. In addition, one can easily check that all finite sequences belong to its image. Since such sequences are dense in $\ell^2_{\delta_1 \delta_2}$, the operator $t - \lambda$ is surjective, and so are the operators $(t - \lambda)^m$ for all $m$. The result is now immediate from the definition of $\ext$.

Lemma 3.4 Let $\lambda$ be a complex number such that $|\lambda|$ is different from $e^{\delta_1}$ and $e^{\delta_2}$. Assume that $\delta_2 < \delta_1$. Then, for any cyclic module $V = \mathbb{C}[t, t^{-1}]/(t - \lambda)^m$, the dimension of $\hom_{\mathbb{C}[t, t^{-1}]}(V, \ell^2_{\delta_1 \delta_2})$ is $m$ if $e^{\delta_2} < |\lambda| < e^{\delta_1}$ and zero otherwise.

Proof For such a module, $\hom_{\mathbb{C}[t, t^{-1}]}(V, \ell^2_{\delta_1 \delta_2})$ equals the kernel of the operator $(t - \lambda)^m: \ell^2_{\delta_1 \delta_2} \to \ell^2_{\delta_1 \delta_2}$. And computing this kernel is a straightforward exercise with infinite series. 

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3.3 Proof of Theorem 1.2

Let $\lambda$ be a root of the product polynomial $A_0(t) \cdots A_{n-1}(t)$ of multiplicity $m = m_0 + \cdots + m_{n-1}$, where $m_k$ is the multiplicity of $\lambda$ as a root of $A_k(t)$. Choose generic $\delta_1$ and $\delta_2$ so that $e^{\delta_2} < |\lambda| < e^{\delta_1}$ and there are no other roots of $A_0(t) \cdots A_{n-1}(t)$ whose absolute values fit in this interval. It follows from Proposition 3.1 and the cohomology calculation in the previous section that

$$\text{ind}_{\delta_1}(M) = \text{ind}_{\delta_2}(M) - \sum (-1)^k m_k,$$

which is exactly the formula claimed in Theorem 1.2.

\[ \square \]

4 Discussion and examples

Let $M$ be a smooth Riemannian manifold of dimension $n$ with a periodic end modeled on $\tilde{X}$, and suppose that $H^*(\tilde{X}; \mathbb{C})$ is finite-dimensional. Then, for any $\delta \in \mathbb{R} - \Delta$, the de Rham complex $\Omega^*_\delta(M)$ is Fredholm and its index is

$$\text{ind}_\delta(M) = (-1)^n \chi(M) + \sum (-1)^k \# \{ \lambda \mid A_k(\lambda) = 0, |\lambda| > e^\delta \},$$

where the roots $\lambda$ of $A_k(t)$ are counted with their multiplicities. This formula is obtained by combining Theorem 1.2 with a theorem by Miller [10] according to which $\text{ind}_\delta(M) = (-1)^n \chi(M)$ for sufficiently large $\delta > 0$. In that paper Miller also shows that the function $\text{ind}_\delta(M)$ is even or odd depending on whether $n$ is even or odd. This is consistent with the above formula because of Blanchfield duality, which says that $A_k(\lambda) = 0$ if and only if $A_{n-k-1}(1/\lambda) = 0$ with matching multiplicities.

**Example 4.1** A manifold with product end is a smooth Riemannian manifold whose end is modeled on $\tilde{X} = \mathbb{R} \times Y$, where $Y$ is a closed Riemannian manifold. The metric on $\mathbb{R} \times Y$ is presumed to be the product metric. The index theory on such manifolds has been studied by Atiyah, Patodi and Singer [1]. The covering translation induces an identity map $\tau_\ast$ on the homology of $\mathbb{R} \times Y$. Since $\lambda = 1$ is the only root of the characteristic polynomial of $\tau_\ast$, the complex $\Omega^*_\delta(M)$ is Fredholm for all $\delta \neq 0$. Its index $\text{ind}_\delta(M)$ equals $\chi(M)$ if the dimension of $M$ is even, and $\text{sign}(-\delta) \cdot \chi(M)$ if the dimension of $M$ is odd. Note that the same is true for any manifold whose periodic end is modeled on $\tilde{X}$ such that the characteristic polynomial of $\tau_\ast$: $H_\ast(\tilde{X}; \mathbb{C}) \to H_\ast(\tilde{X}; \mathbb{C})$ only has unitary roots.
Example 4.2  This example originates in Fox’s “Quick trip” [6, Example 11]. Fox constructs a 2–knot in the 4–sphere with the property that the infinite cyclic cover of its exterior has first homology isomorphic to the additive group of dyadic rationals. A nice plumbing construction of this knot described in Rolfsen’s book [15, Section 7.F] shows that it has a Seifert surface diffeomorphic to $S^1 \times S^2 - D^3$. This is shown in Figure 1, where the red circle indicates a standardly embedded 2–sphere in the 4–sphere.

![Figure 1: Plumbed 3–manifold bounding a knot](image)

The Seifert surface is obtained from $S^2 \times I$ by adjoining a 3–dimensional 1–handle that links the 2–sphere twice, as indicated. Perform a surgery on the knot so that the Seifert surface is capped off by the core 3–disk of the surgery. The resulting manifold $X$ has the integral homology of $S^1 \times S^3$. It follows from a calculation in [15, Section 7.F] that the characteristic polynomials $A_k(t)$ of the covering translation $\tau_*: H_k(\tilde{X}; \mathbb{C}) \to H_k(\tilde{X}; \mathbb{C})$ (which are the same as the Alexander polynomials) are as follows: $A_0(t) = t - 1$, $A_1(t) = t - 2$, $A_2(t) = t^{-1} - 2$ and $A_3(t) = t - 1$. Cut $\tilde{X}$ along a copy of $S^1 \times S^2$ and fill it in by $D^2 \times S^2$ to obtain an end-periodic manifold $M$. A straightforward calculation shows that $\chi(M) = 2$. The complex $\Omega_\delta^+(M)$ is Fredholm away from $\delta = 0$ and $\tilde{\delta} = \pm \ln 2$. Its index is equal to 1 if $0 < |\delta| < \ln 2$, and is equal to 2 otherwise.

One can construct many more such examples; for instance it is known [8] that any integer polynomial $A(t)$ satisfying $A(1) = \pm 1$ is the first Alexander polynomial of a knot in the 4–sphere, with $A(t^{-1})$ the second polynomial (describing $H_2$ of the infinite cyclic cover). As in Example 4.2, such knots can be constructed (see [15, Section 7.F, Exercise 6]) as the boundary of a once-punctured connected sum of copies of $S^2 \times S^1$. 

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References


