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Index theory of the de Rham complex on manifolds with periodic ends

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We study the de Rham complex on a smooth manifold with a periodic end modeled on an infinite cyclic cover $\tilde{X} \to X$. The completion of this complex in exponentially weighted L^2 norms is Fredholm for all but finitely many exceptional weights determined by the eigenvalues of the covering translation map $H_*(\tilde{X}) \to H_*(\tilde{X})$. We calculate the index of this weighted de Rham complex for all weights away from the exceptional ones.

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1 Introduction

Let M be a smooth closed orientable manifold of dimension n. The de Rham complex of complex-valued differential forms on M,

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \longrightarrow \cdots \longrightarrow \Omega^n(M) \longrightarrow 0,$$

is known to be Fredholm in a suitable L^2 completion. This means as usual that the images of d_k are closed and the vector spaces $\ker d_k / \operatorname{im} d_{k-1}$ are finite-dimensional. The alternating sum of the dimensions of these spaces is called the index of the de Rham complex. Since $\ker d_k / \operatorname{im} d_{k-1}$ is isomorphic to the singular cohomology $H^k(M;\mathbb{C})$ by the de Rham theorem, the above index equals $\chi(M)$, the Euler characteristic of M.

This paper extends these classical results to certain noncompact manifolds; those with periodic ends. It builds on the earlier work of Miller [10] and Taubes [16] and can be viewed as a continuation of our research in [12] and [13] on the index theory of elliptic operators on such manifolds.

By a manifold with a periodic end we mean an open Riemannian manifold M whose end, in the sense of Hughes and Ranicki [7], is modeled on an infinite cyclic cover $\tilde{X}$ of a compact manifold X associated with a primitive cohomology class $\gamma \in H^1(X;\mathbb{Z})$;

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the case of several ends can be treated similarly. To be precise, the manifold M is of the form

$$Z_\infty = Z \cup W_0 \cup W_1 \cup W_2 \cup \cdots ,$$

where W_k are isometric copies of the fundamental segment W obtained by cutting X open along an oriented connected submanifold Y Poincaré dual to γ, and Z is a smooth compact manifold with boundary Y.

The de Rham complex of M can be completed in the L^2 norm using (over the end) a Riemannian measure dx lifted from that on X. This completion is, however, not Fredholm; see Remark 2.3. To rectify this problem, we will use L^2_δ norms, which are the L^2 norms on M with respect to the measure $e^{\delta f(x)}\,dx$ over the end. Here δ is a real number and $f\colon \widetilde{X} \to \mathbb{R}$ is a smooth function such that $f(\tau(x)) = f(x)+1$ with respect to the covering translation $\tau\colon \widetilde{X} \to \widetilde{X}$. The L^2_δ completion of the de Rham complex on M will be denoted by $\Omega^*_\delta(M)$.

Theorem 1.1 *Let M be a smooth Riemannian manifold with a periodic end modeled on $\widetilde{X}$, and suppose that $H_*(M;\mathbb{C})$ is finite-dimensional. Then $\Omega^*_\delta(M)$ is Fredholm for all but finitely many δ of the form $\delta = \ln|\lambda|$, where λ is a root of the characteristic polynomial of $\tau_*\colon H_*(\widetilde{X};\mathbb{C}) \to H_*(\widetilde{X};\mathbb{C})$.*

Conditions on X that guarantee that $H_*(M;\mathbb{C})$ is finite-dimensional can be found in Section 2, together with a proof of Theorem 1.1.

Given a manifold M as in the above theorem, the complex $\Omega^*_\delta(M)$ has a well-defined index $\operatorname{ind}_\delta(M)$. Miller [10] showed that $\operatorname{ind}_\delta(M)$ is an even or odd function of δ according to whether $\dim M = n$ is even or odd, and that $\operatorname{ind}_\delta(M) = (-1)^n\chi(M)$ for sufficiently large $\delta > 0$. We add to this knowledge the following result.

Theorem 1.2 *Let M be as in Theorem 1.1. Then $\operatorname{ind}_\delta(M)$ is a piecewise constant function of δ whose only jumps occur at $\delta = \ln|\lambda|$, where λ is a root of the characteristic polynomial $A_k(t)$ of $\tau_*\colon H_k(\widetilde{X};\mathbb{C}) \to H_k(\widetilde{X};\mathbb{C})$ for some $k = 0,\dots,n-1$. Every such λ contributes $(-1)^{k+1}$ times its multiplicity as a root of $A_k(t)$ to the jump.*

Together with the results of [10] this completes the calculation of the function $\operatorname{ind}_\delta(M)$. Theorem 1.2 is proved in Section 3. The last section of the paper contains discussion as well as calculations of $\operatorname{ind}_\delta(M)$ for two important classes of examples. The first class consists of manifolds with infinite cylindrical ends studied earlier by Atiyah, Patodi and Singer [1], and the second of manifolds arising in the study of knotted 2–spheres in S^4.

Finally, we will remark that the de Rham complex is a special case of the more general concept of an elliptic complex. The index theory for elliptic complexes on closed manifolds was developed by Atiyah and Singer, whose famous index theorem [2] expresses the index of such a complex in purely topological terms. More generally, Atiyah, Patodi and Singer [1] computed the index for certain elliptic operators (that is, elliptic complexes of length two) on manifolds with cylindrical ends. In our paper [12] we extended their result to general manifolds with periodic ends; our index formula involves a new periodic η–invariant, generalizing the η–invariant of Atiyah, Patodi and Singer from the cylindrical setting.

It would be interesting to compare the above formula for $\mathrm{ind}_\delta(M)$ with the formula in [12] for the index of the operator $d + d^*$ obtained by wrapping up the de Rham complex. The issue is that both the de Rham complex and the operator $d + d^*$ are Fredholm only when completed in L^2_δ norms with $\delta \neq 0$. Since the dual space of $L^2_\delta(M)$ equals $L^2_{-\delta}(M)$ rather than $L^2_\delta(M)$, this completion does not commute with the wrap-up procedure, hence the index of the operator $d+d^*$ completed in L^2_δ norms need not match the index $\mathrm{ind}_\delta(M)$ of the current paper. Atiyah, Patodi and Singer dealt with a similar issue in [1, Proposition 4.9] using a rather precise relation between L^2 harmonic forms and cohomology on manifolds with cylindrical ends; comparing the above indices would likely entail developing such a relation for general manifolds with periodic ends.

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2 The Fredholm property

In this section, we will prove Theorem 1.1 by reducing it to a statement about twisted cohomology of X.

2.1 The Fourier–Laplace transform

The de Rham complex of M is an elliptic complex on the end-periodic manifold M, hence we can use the general theory of such complexes due to Taubes [16]. According to that theory, it is sufficient to check the Fredholm property of the de Rham complex of $\tilde{X}$ completed in the L^2 norm on $\tilde{X}$ with respect to the measure $e^{\delta f(x)}\,dx$. The latter complex can be studied using the Fourier–Laplace transform, defined by

$$\hat{\omega}_z = \sum z^k \cdot (\tau^*)^k \omega, \quad z \in \mathbb{C}^*,$$

on compactly supported forms ω on $\widetilde{X}$, and extended by continuity to L^2 forms. The summation in the above formula extends over all integers k, which makes the form $\widehat{\omega}_z$ invariant with respect to τ^*. The form $\widehat{\omega}_z$ then defines a form on X which is denoted by the same symbol. An application of the Fourier–Laplace transform to the de Rham complex on $\widetilde{X}$ results in a family of twisted de Rham complexes

$$\cdots \longrightarrow \Omega^k(X) \xrightarrow{d-\ln z\, df} \Omega^{k+1}(X) \longrightarrow \cdots \tag{1}$$

parameterized by $z \in \mathbb{C}^*$. The following result is proved in Taubes [16, Lemma 4.3].

Proposition 2.1 *Let M be a smooth Riemannian manifold with a periodic end modeled on $\widetilde{X}$. For any given δ, the complex $\Omega^*_\delta(M)$ is Fredholm if and only if the complexes (1) are exact for all z such that $|z| = e^\delta$.*

The cohomology of complex (1) is of course the twisted de Rham cohomology $H^*_z(X;\mathbb{C})$ with coefficients in the complex line bundle with flat connection $-\ln z\, df$.

Proposition 2.2 *Let M be a smooth Riemannian manifold with a periodic end modeled on $\widetilde{X}$. Then the following three conditions are equivalent:*

(i) *$\Omega^*_\delta(M)$ is Fredholm for all $\delta \in \mathbb{R}$ away from a discrete set.*

(ii) *$H^*_z(X;\mathbb{C})$ vanishes for all $z \in \mathbb{C}^*$ away from a discrete set.*

(iii) *$H^*_z(X;\mathbb{C})$ vanishes for at least one $z \in \mathbb{C}^*$.*

Proof Observe that the complexes (1) form a holomorphic family of elliptic complexes on $\mathbb{C}^*$; therefore, exactness of (1) at one point is equivalent to exactness away from a discrete set by the analytic Fredholm theorem. The rest of the statement follows from the preceding discussion. □

Remark 2.3 The usual L^2 completion of the de Rham complex on M, that is, the complex $\Omega^*_0(M)$ is not Fredholm because $H^0_z(X;\mathbb{C})$ is not zero when $z = 1$.

2.2 Finite-dimensionality

Fix a finite cell complex structure on X, lift it to $\widetilde{X}$, and consider the chain complex $C_*(\widetilde{X},\mathbb{C})$ and its homology $H_*(\widetilde{X};\mathbb{C})$. The group of integers acts on both by covering translations making them into finitely generated modules over $\mathbb{C}[t,t^{-1}]$. The twisted de Rham theorem tells us that the cohomology $H^*_z(X;\mathbb{C})$ of complex (1) is isomorphic to the cohomology of the complex $\operatorname{Hom}_{\mathbb{C}[t,t^{-1}]}(C_*(\widetilde{X},\mathbb{C}),\mathbb{C}_z)$, where $\mathbb{C}_z$ denotes a copy of $\mathbb{C}$ viewed as a $\mathbb{C}[t,t^{-1}]$ module with $p(t)$ acting via multiplication by $p(z)$.

Proposition 2.4 *Let M be a smooth Riemannian manifold with a periodic end modeled on $\widetilde{X}$. Then the following two conditions are equivalent:*

(i) $H_*(M;\mathbb{C})$ *is a finite-dimensional vector space.*

(ii) $H^*_z(X;\mathbb{C})$ *vanishes for at least one $z \in \mathbb{C}^*$.*

Proof It is immediate from the Mayer–Vietoris principle that $H^*(M;\mathbb{C})$ is finite-dimensional if and only if $H^*(\widetilde{X};\mathbb{C})$ is finite-dimensional. Since $H_*(\widetilde{X};\mathbb{C})$ is a finitely generated module over the principal ideal domain $\mathbb{C}[t,t^{-1}]$, it admits a primary decomposition

$$H_*(\widetilde{X};\mathbb{C}) = \mathbb{C}[t,t^{-1}]^{\ell} \oplus \mathbb{C}[t,t^{-1}]/(p_1) \oplus \cdots \oplus \mathbb{C}[t,t^{-1}]/(p_m); \tag{2}$$

therefore, $H^*(\widetilde{X};\mathbb{C})$ is a finite-dimensional vector space if and only if $\ell = 0$ in this decomposition. According to the universal coefficient theorem,

$$H^*_z(X;\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(H_*(\widetilde{X};\mathbb{C}), \mathbb{C}_z) \oplus \mathrm{Ext}_{\mathbb{C}[t,t^{-1}]}(H_*(\widetilde{X};\mathbb{C}), \mathbb{C}_z),$$

hence vanishing of $H^*_z(X;\mathbb{C})$ for at least one z implies that $\ell = 0$. On the other hand, an easy calculation shows that

$$\mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(V, \mathbb{C}_z) = \mathrm{Ext}_{\mathbb{C}[t,t^{-1}]}(V, \mathbb{C}_z) = 0$$

for any module $V = \mathbb{C}[t,t^{-1}]/(p)$ such that $p(z) \neq 0$. Therefore, $\ell = 0$ implies that $H^*_z(X;\mathbb{C})$ must vanish for all z away from the roots of the polynomials $p_1, \ldots, p_m$. □

2.3 Proof of Theorem 1.1

It follows from Propositions 2.2 and 2.4 that if $H^*(\widetilde{X};\mathbb{C})$ is finite-dimensional, the complex $\Omega^*_\delta(M)$ is Fredholm for all δ away from a discrete set. To finish the proof of Theorem 1.1 we just need to identify this discrete set. According to Proposition 2.1, it consists of $\delta = \ln|z|$, where $z \in \mathbb{C}^*$ are the complex numbers for which $H^*_z(X;\mathbb{C})$ fails to be zero. To find them, note that the free part in the prime decomposition (2) vanishes, making $H_*(\widetilde{X};\mathbb{C})$ into a torsion module,

$$H_*(\widetilde{X};\mathbb{C}) = \mathbb{C}[t,t^{-1}]/(p_1) \oplus \cdots \oplus \mathbb{C}[t,t^{-1}]/(p_m). \tag{3}$$

According to Milnor [11, Assertion 4], the order ideal $(p_1 \cdots p_m)$ of this module is spanned by the characteristic polynomial of $\tau_*\colon H_*(\widetilde{X};\mathbb{C}) \to H_*(\widetilde{X};\mathbb{C})$. The calculation with the universal coefficient theorem as in the proof of Proposition 2.4 now completes the proof. □

2.4 A sufficient condition

Let M be a smooth orientable manifold with a periodic end modeled on $\widetilde{X}$. Vanishing of $\chi(X)$ is obviously a necessary condition for the vector space $H_*(M;\mathbb{C})$ to be finite-dimensional. To come up with a sufficient condition, observe that the derivative df defines a closed 1–form on X, and let $\xi = [df] \in H^1(X;\mathbb{C})$ be its cohomology class. The cup product with ξ gives rise to the chain complex

$$H^0(X;\mathbb{C}) \xrightarrow{\cup\xi} H^1(X;\mathbb{C}) \xrightarrow{\cup\xi} \cdots \xrightarrow{\cup\xi} H^n(X;\mathbb{C}). \tag{4}$$

Proposition 2.5 *Suppose the chain complex (4) is exact. Then $H_*(M;\mathbb{C})$ is a finite-dimensional vector space for any smooth orientable manifold with periodic end modeled on $\widetilde{X}$.*

This proposition can be derived as a special case of Taubes [16, Theorem 3.1]. That proof is analytic in nature; here is another proof which is purely topological.

According to Proposition 2.4, it is sufficient to prove that the twisted cohomology $H^*_z(X;\mathbb{C})$ vanishes for at least one $z \in \mathbb{C}^*$. We will show that it does so for all $z \neq 1$ in a sufficiently small neighborhood of 1. For such z, there is a spectral sequence (E^*_r, d_r) which starts at $E^*_1 = H^*(X;\mathbb{C})$, converges to $H^*_z(X;\mathbb{C})$ and whose differentials are given by the Massey products with the class ξ; see Farber [5, Section 10.9] and Pajitnov [14]. The convergence of this spectral sequence to zero is therefore a necessary and sufficient condition for the finite-dimensionality of $H_*(M;\mathbb{C})$. The chain complex (4) is the term (E^*_1, d_1) of that spectral sequence, hence its exactness is sufficient for the finite-dimensionality of $H^*(X;\mathbb{C})$.

Note that the vanishing of the Euler characteristic of X is not a sufficient condition for $H_*(M;\mathbb{C})$ to be finite-dimensional. An example is provided by the connected sum of $S^1 \times S^{n-1}$ with any manifold that is not a rational homology sphere but has Euler characteristic 2.

3 The index calculation

Let M be a smooth Riemannian manifold with periodic end modeled on the infinite cyclic cover $\widetilde{X}$, and assume that $H_*(M;\mathbb{C})$ is finite-dimensional. Let $\tau\colon \widetilde{X} \to \widetilde{X}$ be a covering translation, and denote by $A_k(t)$ the characteristic polynomial of $\tau_*\colon H_k(\widetilde{X};\mathbb{C}) \to H_k(\widetilde{X};\mathbb{C})$. (The polynomial $A_1(t)$ is known as the Alexander polynomial of the fundamental group of X.) Denote by Δ the set of all δ of the form $\delta = \ln|\lambda|$, where λ is a root of the product polynomial $A_0(t)\cdots A_{n-1}(t)$.

According to Theorem 1.1, the complex $\Omega^*_\delta(M)$ is Fredholm for all δ away from Δ. Its index $\mathrm{ind}_\delta(M)$ is a piecewise constant function away from Δ, where it may jump. We wish to calculate the size of these jumps.

3.1 Excision principle

Let δ_1 and δ_2 be two weights in $\mathbb{R} - \Delta$, and complete the de Rham complex of $\tilde{X}$ in the L^2 norm with respect to the measures $e^{\delta_1 f(x)}\,dx$ on the negative end of $\tilde{X}$ and $e^{\delta_2 f(x)}\,dx$ on the positive end. This complex will be denoted by $\Omega^*_{\delta_1\delta_2}(\tilde{X})$. This is a Fredholm complex, whose index will be denoted by $\mathrm{ind}_{\delta_1\delta_2}(\tilde{X})$.

Proposition 3.1 $$\mathrm{ind}_{\delta_2}(M) - \mathrm{ind}_{\delta_1}(M) = \mathrm{ind}_{\delta_1\delta_2}(\tilde{X}).$$

Proof Let $c \in \mathbb{R}$ be a regular value of $f\colon \tilde{X} \to \mathbb{R}$; then $Y = f^{-1}(c)$ is a submanifold of $\tilde{X}$ separating it as $\tilde{X} = \tilde{X}_- \cup \tilde{X}_+$. Write $M = Z \cup \tilde{X}_+$ for some smooth compact manifold Z with boundary Y. An application of the excision principle to these two splittings yields

$$\mathrm{ind}_{\delta_1}(M) + \mathrm{ind}_{\delta_1\delta_2}(\tilde{X}) = \mathrm{ind}_{\delta_2}(M) + \mathrm{ind}_{\delta_1\delta_1}(\tilde{X}).$$

Note that the complex $\Omega^*_{\delta_1\delta_1}(\tilde{X})$ is exact because the complexes (1) obtained from it by the Fourier–Laplace transform are exact for all z with $|z| = e^{\delta_1}$. Therefore, $\mathrm{ind}_{\delta_1\delta_1}(\tilde{X}) = 0$ and the proof is complete. □

3.2 Computing the cohomology of $\Omega^*_{\delta_1\delta_2}(\tilde{X})$

We will proceed by several reductions, the first being from weighted forms to weighted cellular cochains. To be precise, fix a finite cell complex structure on X and lift it to $\tilde{X}$. Also, introduce the Hilbert space $\ell^2_{\delta_1\delta_2}$ of the sequences $\{x_k \mid k \in \mathbb{Z}\}$ of complex numbers such that

$$\sum_{k<0} e^{2\delta_1 k}|x_k|^2 < \infty \quad \text{and} \quad \sum_{k>0} e^{2\delta_2 k}|x_k|^2 < \infty.$$

Theorem 2.17 of Miller [10], which is a weighted version of the L^2 de Rham theorem [4; 9] establishes an isomorphism between the cohomology of $\Omega^*_{\delta_1\delta_2}(\tilde{X})$ and the cellular cohomology of $\tilde{X}$ with $\ell^2_{\delta_1\delta_2}$ coefficients. Miller actually uses weighted *simplicial* cohomology, but this is readily seen to be isomorphic to the more standard and convenient cellular version.

Proposition 3.2 *View $\ell^2_{\delta_1\delta_2}$ as a $\mathbb{C}[t,t^{-1}]$ module with t acting as the right shift operator, $t(x_k)=x_{k+1}$. Then for all but finitely many δ_1 and δ_2, the cohomology of $\widetilde{X}$ with $\ell^2_{\delta_1\delta_2}$ coefficients equals the homology of the complex*

$$\mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}\big(C_*(\widetilde{X},\mathbb{C}),\ell^2_{\delta_1\delta_2}\big). \tag{5}$$

Proof This will follow as soon as we show that the images of the boundary operators ∂ in complex (5) are closed. These boundary operators

$$\partial\colon \big(\ell^2_{\delta_1\delta_2}\big)^k \to \big(\ell^2_{\delta_1\delta_2}\big)^\ell$$

are matrices whose entries are Laurent polynomials in t. Since $\mathbb{C}[t,t^{-1}]$ is a principal ideal domain, each ∂ will have a diagonal matrix in properly chosen bases. The statement now follows from the fact that the operator $t-\lambda\colon \ell^2_{\delta_1\delta_2}\to\ell^2_{\delta_1\delta_2}$ is Fredholm for all λ with $|\lambda|$ different from e^{δ_1} and e^{δ_2}; see for instance Conway [3, Proposition 27.7 (c)]. □

The universal coefficient theorem now tells us that the $\ell^2_{\delta_1\delta_2}$ cohomology of $\widetilde{X}$ is isomorphic to

$$\mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}\big(H_*(\widetilde{X};\mathbb{C}),\ell^2_{\delta_1\delta_2}\big)\oplus \mathrm{Ext}_{\mathbb{C}[t,t^{-1}]}\big(H_*(\widetilde{X};\mathbb{C}),\ell^2_{\delta_1\delta_2}\big). \tag{6}$$

Recall from the proof of Theorem 1.1 that $H_*(\widetilde{X};\mathbb{C})$ is a torsion module (3) whose order ideal is spanned by the characteristic polynomial of $\tau_*\colon H_*(\widetilde{X};\mathbb{C})\to H_*(\widetilde{X};\mathbb{C})$. Therefore, our next step will be to compute (6), one cyclic module at a time.

Lemma 3.3 *Let λ be a complex number such that $|\lambda|$ is different from e^{δ_1} and e^{δ_2}. Then, for any cyclic module $V=\mathbb{C}[t,t^{-1}]/(t-\lambda)^m$, we have*

$$\mathrm{Ext}_{\mathbb{C}[t,t^{-1}]}\big(V,\ell^2_{\delta_1\delta_2}\big)=0.$$

Proof We know from the proof of Proposition 3.2 that the operator $t-\lambda\colon \ell^2_{\delta_1\delta_2}\to\ell^2_{\delta_1\delta_2}$ is Fredholm. In addition, one can easily check that all finite sequences belong to its image. Since such sequences are dense in $\ell^2_{\delta_1\delta_2}$, the operator $t-\lambda$ is surjective, and so are the operators $(t-\lambda)^m$ for all m. The result is now immediate from the definition of Ext. □

Lemma 3.4 *Let λ be a complex number such that $|\lambda|$ is different from e^{δ_1} and e^{δ_2}. Assume that $\delta_2<\delta_1$. Then, for any cyclic module $V=\mathbb{C}[t,t^{-1}]/(t-\lambda)^m$, the dimension of* $\mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(V,\ell^2_{\delta_1\delta_2})$ *is m if $e^{\delta_2}<|\lambda|<e^{\delta_1}$ and zero otherwise.*

Proof For such a module, $\mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(V,\ell^2_{\delta_1\delta_2})$ equals the kernel of the operator $(t-\lambda)^m\colon \ell^2_{\delta_1\delta_2}\to\ell^2_{\delta_1\delta_2}$. And computing this kernel is a straightforward exercise with infinite series. □

3.3 Proof of Theorem 1.2

Let λ be a root of the product polynomial $A_0(t)\cdots A_{n-1}(t)$ of multiplicity $m = m_0+\cdots+m_{n-1}$, where m_k is the multiplicity of λ as a root of $A_k(t)$. Choose generic δ_1 and δ_2 so that $e^{\delta_2} < |\lambda| < e^{\delta_1}$ and there are no other roots of $A_0(t)\cdots A_{n-1}(t)$ whose absolute values fit in this interval. It follows from Proposition 3.1 and the cohomology calculation in the previous section that

$$\operatorname{ind}_{\delta_1}(M) = \operatorname{ind}_{\delta_2}(M) - \sum(-1)^k m_k,$$

which is exactly the formula claimed in Theorem 1.2. □

4 Discussion and examples

Let M be a smooth Riemannian manifold of dimension n with a periodic end modeled on $\widetilde{X}$, and suppose that $H^*(\widetilde{X};\mathbb{C})$ is finite-dimensional. Then, for any $\delta \in \mathbb{R} - \Delta$, the de Rham complex $\Omega^*_\delta(M)$ is Fredholm and its index is

$$\operatorname{ind}_\delta(M) = (-1)^n \chi(M) + \sum(-1)^k \,\#\{\lambda \mid A_k(\lambda) = 0, |\lambda| > e^\delta\},$$

where the roots λ of $A_k(t)$ are counted with their multiplicities. This formula is obtained by combining Theorem 1.2 with a theorem by Miller [10] according to which $\operatorname{ind}_\delta(M) = (-1)^n\chi(M)$ for sufficiently large $\delta > 0$. In that paper Miller also shows that the function $\operatorname{ind}_\delta(M)$ is even or odd depending on whether n is even or odd. This is consistent with the above formula because of Blanchfield duality, which says that $A_k(\lambda) = 0$ if and only if $A_{n-k-1}(1/\lambda) = 0$ with matching multiplicities.

Example 4.1 A manifold with product end is a smooth Riemannian manifold whose end is modeled on $\widetilde{X} = \mathbb{R} \times Y$, where Y is a closed Riemannian manifold. The metric on $\mathbb{R} \times Y$ is presumed to be the product metric. The index theory on such manifolds has been studied by Atiyah, Patodi and Singer [1]. The covering translation induces an identity map τ_* on the homology of $\mathbb{R} \times Y$. Since $\lambda = 1$ is the only root of the characteristic polynomial of τ_*, the complex $\Omega^*_\delta(M)$ is Fredholm for all $\delta \neq 0$. Its index $\operatorname{ind}_\delta(M)$ equals $\chi(M)$ if the dimension of M is even, and $\operatorname{sign}(-\delta)\cdot\chi(M)$ if the dimension of M is odd. Note that the same is true for any manifold whose periodic end is modeled on $\widetilde{X}$ such that the characteristic polynomial of $\tau_*\colon H_*(\widetilde{X};\mathbb{C}) \to H_*(\widetilde{X};\mathbb{C})$ only has unitary roots.

Example 4.2 This example originates in Fox's "Quick trip" [6, Example 11]. Fox constructs a 2–knot in the 4–sphere with the property that the infinite cyclic cover of its exterior has first homology isomorphic to the additive group of dyadic rationals. A nice plumbing construction of this knot described in Rolfsen's book [15, Section 7.F] shows that it has a Seifert surface diffeomorphic to $S^1 \times S^2 - D^3$. This is shown in Figure 1, where the red circle indicates a standardly embedded 2–sphere in the 4–sphere.

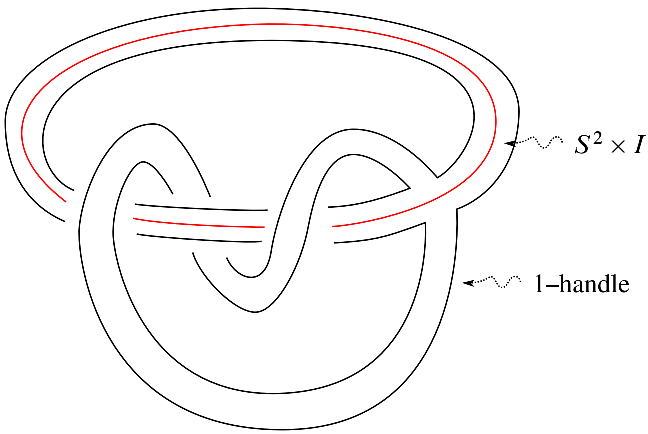

Figure 1: Plumbed 3–manifold bounding a knot

The Seifert surface is obtained from $S^2 \times I$ by adjoining a 3–dimensional 1–handle that links the 2–sphere twice, as indicated. Perform a surgery on the knot so that the Seifert surface is capped off by the core 3–disk of the surgery. The resulting manifold X has the integral homology of $S^1 \times S^3$. It follows from a calculation in [15, Section 7.F] that the characteristic polynomials $A_k(t)$ of the covering translation $\tau_*\colon H_k(\tilde{X};\mathbb{C}) \to H_k(\tilde{X};\mathbb{C})$ (which are the same as the Alexander polynomials) are as follows: $A_0(t) = t-1$, $A_1(t) = t-2$, $A_2(t) = t^{-1}-2$ and $A_3(t) = t-1$. Cut $\tilde{X}$ along a copy of $S^1 \times S^2$ and fill it in by $D^2 \times S^2$ to obtain an end-periodic manifold M. A straightforward calculation shows that $\chi(M) = 2$. The complex $\Omega^*_\delta(M)$ is Fredholm away from $\delta = 0$ and $\delta = \pm \ln 2$. Its index is equal to 1 if $0 < |\delta| < \ln 2$, and is equal to 2 otherwise.

One can construct many more such examples; for instance it is known [8] that any integer polynomial $A(t)$ satisfying $A(1) = \pm 1$ is the first Alexander polynomial of a knot in the 4–sphere, with $A(t^{-1})$ the second polynomial (describing H_2 of the infinite cyclic cover). As in Example 4.2, such knots can be constructed (see [15, Section 7.F, Exercise 6]) as the boundary of a once-punctured connected sum of copies of $S^2 \times S^1$.

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