Norm minima in certain Siegel leaves

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In this paper we shall illustrate that each polytopal moment-angle complex can be understood as the intersection of the minima of corresponding Siegel leaves and the unit sphere, with respect to the maximum norm. Consequently, an alternative proof of a rigidity theorem of Bosio and Meersseman is obtained; as piecewise linear manifolds, polytopal real moment-angle complexes can be smoothed in a natural way.

57R30; 57R70, 05E45

1 Introduction

An admissible configuration of \( m \) complex vectors in \( \mathbb{C}^{d/2} \) (\( m > d \) with \( d \) even) satisfying so called Siegel and weak hyperbolicity conditions (see Meersseman [15, page 82], and Section 2 for a real analogue), gives rise to a free action on \( \mathbb{C}^m \) via exponential functions. There are two types of leaves in the holomorphic foliation given by this action: a leaf is of Siegel type if the origin is not in its closure; otherwise it is said to be of Poincaré type.

These objects originated in the work of C Camacho, N Kuiper and J Palls [6] on the complex analogue of a dynamical system for which the real version appeared in an earlier work of Poincaré, and were later developed and generalized by S López de Medrano and A Verjovsky [14] and L Meersseman [15]. From their works, the projectivization of the minima of all Siegel leaves, with respect to the Euclidean norm, can be endowed with the structure of a compact, complex manifold \( C^{\infty} \)–embedded in \( \mathbb{C}P^{m-1} \), which is not symplectic except in the trivial case. This class of complex manifolds is now named LVM manifolds.

On the other hand, by a direct calculation, the space of minima of all Siegel leaves can be described by \( d \) real quadrics arising from the given configuration in \( \mathbb{R}^d \), whose intersection with the unit Euclidean sphere in \( \mathbb{C}^m \) is transverse, hence it is a smooth manifold of real dimension \( 2m - d - 1 \). F Bosio and L Meersseman [3] observed that this method also works for odd \( d \), and call these manifolds embedded in spheres links.

This special class of links is a model for polytopal moment-angle manifolds. In general their topology is known to be complicated (see [3] and Gitler and López de
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Medrano [10]), for instance, arbitrary torsion can appear in the cohomology, as well as nonvanishing triple Massey products (see Baskakov [2] and Denham and Suciu [9]); in the case $d = 2$, the classification work [13] by S López de Medrano shows that they are diffeomorphic to a triple product of spheres or to the connected sum of sphere products. An important way to understand them is that they inherit the natural $(S^1)^m$–action on $\mathbb{C}^m$, with each quotient space homeomorphic (as manifolds with corners) to a simple convex polytope. Via the basic construction originating from reflection group theory and then generalized by M W Davis and T Januszkiewicz in their influential work [8], each link discussed above is homeomorphic to a moment-angle complex (named in Buchstaber and Panov [5]), ie a polyhedral product with pairs $(D^2, S^1)$ corresponding to the boundary complex of a simplicial polytope.

The polyhedral product model was studied in detail and generalized by V Buchstaber and T Panov in [5]. Later a more categorical treatment by A Bahri, M Bendersky, F R Cohen and S Gitler [1] provided a penetrating viewpoint from homotopy theory.

These spaces have spawned a large body of work; see most notably that by Davis and Januszkiewicz [8] on quasitoric varieties, Buchstaber and Panov [5] on moment-angle complexes, Goresky and MacPherson [11] on complements of complex arrangements, S López de Medrano [13] on the topology of these varieties, as well as many others. The interconnections between these subjects is developed in the beautiful book [4] by Buchstaber and Panov.

The objective of this paper is to show that, for an admissible configuration of $m$ real vectors in $\mathbb{R}^d$ whose centroid is located at the origin, the corresponding foliation provides a direct relation between the model of links and the model of polyhedral products: there are continuous paths in the space of the union of all Siegel leaves (which is the complement of a coordinate subspace arrangement in $\mathbb{C}^m$) such that each point of the link is connected by a path to a unique point in the respective moment-angle complex, yielding a homeomorphism between them. Every path is parameterized by real numbers $p \in [1, \infty)$, with each $p$ associated to the intersection of the $L^p$–norm minima in the Siegel leaves and the $L^p$–norm unit sphere in $\mathbb{C}^m$, which is a topological manifold homeomorphic to the link. In this way, we can understand each polytopal moment-angle complex as the intersection of the unit sphere and the minima of all Siegel leaves, with respect to the $L^\infty$–norm.

This paper develops a more analytic approach to these spaces in the spirit of the work [3] by Bosio and Meersseman.

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2 Notation and main results

Let $A = (A_1, A_2, \ldots, A_m)$ be an $m$–tuple of vectors in $\mathbb{R}^d$, with $m > d \geq 0$ ($A_i \equiv 0$ when $d = 0$); occasionally we treat such a tuple as a $(d \times m)$–matrix. Denote by $[m]$ the set $\{1, 2, \ldots, m\}$, and for $I \subset [m]$, let $A(I)$ be the subtuple $(A_i)_{i \in I}$ and $\text{conv} A$ (resp. $\text{conv} A(I)$) the convex hull of vectors from $A$ (resp. from $A(I)$).

We say $A$ is admissible if it satisfies the following two conditions (cf [3, Lemma 0.3]):

*1 (Siegel condition) $0 \in \text{conv} A$.
*2 (Weak hyperbolicity condition) If $0 \in \text{conv} A(I)$, then we have $\text{card}(I) > d$ (where card refers to the cardinality).

Up to Section 5, we always assume that $A$ is admissible.

Let $\mathbb{R}_{>0}$ be the set of positive real numbers, in which $p \geq 1$ is a real number. For each $z = (z_i)_{i=1}^m \in \mathbb{C}^m$, denote by $\|z\|_p$ its $L^p$–norm, namely $\|z\|_p = \left(\sum_{i=1}^m |z_i|^p\right)^{1/p}$, where $|z_i| = \sqrt{z_i \bar{z}_i}$.

With respect to an $m$–tuple $A$, there is a smooth foliation $\mathcal{F}$ of $\mathbb{C}^m$ given by the orbits of the action

$$F: \mathbb{C}^m \times \mathbb{R}^d \to \mathbb{C}^m,$$

$$(z, T) \mapsto (z_i e^{(A_i, T)})_{i=1}^m.$$

For each $z \in \mathbb{C}^m$, let $L_z$ be the leaf passing through $z$. We call $L_z$ a Siegel leaf if $0$ is not in its closure, otherwise we say the leaf $L_z$ is of Poincaré type. It follows that the union of all Siegel leaves can be described by the set (see [6; 3] and Meersseman and Verjovsky [16])

$$S_A = \{z \in \mathbb{C}^m | 0 \in \text{conv} A(I_z)\},$$

where $I_z$ is the set of nonzero entries for $z = (z_i)_{i=1}^m$, i.e $I_z = \{i \in [m] | |z_i| \neq 0\}$. With an argument involving foliations, complex analysis and the convexity, the following fact is a combination of the works mentioned above, which is our starting point:

**Theorem 1** (cf [3, Lemma 0.8, pages 61–62]) For each $z \in S_A$, there is a unique point $f_2(z)$ in the leaf $L_z$ such that its $L^2$–norm $\|f_2(z)\|_2$ is minimal and positive. The foliation $\mathcal{F}$ is trivial when restricted to $S_A$, and

$$\Phi_A(2): X_A(2) \times \mathbb{R}^d \times \mathbb{R}_{>0} \to S_A,$$

$$(z, T, r) \mapsto r(z_i e^{(A_i, T)})_{i=1}^m.$$
is a global diffeomorphism, where $X_A(2)$ is given by the transverse intersection

$$
\begin{cases}
\sum_{i=1}^{m} A_i |z_i|^2 = 0, \\
\|z\|_2 = 1,
\end{cases}
$$

and is thus a smooth manifold.

It follows that there is a smooth function

$$
T_2: S_A \rightarrow \mathbb{R}^d
$$

such that $f_2(z) = F(z, T_2(z))$, and after differentiating $F(z, T)$ with respect to $T \in \mathbb{R}^d$, one easily checks that the critical point corresponding to the minimum satisfies

$$
\sum_{i=1}^{m} A_i |z_i|^2 e^{2(A_i, T)} = 0,
$$

in which $T_2(z)$ is the unique solution. Moreover, $f_2/\|f_2\|_2: S_A \rightarrow X_A(2)$ is a smooth retraction.

Following their approach, we consider the space of $L^p$–norm minima of those Siegel leaves. Our first main theorem is the following, whose proof is based on some real analysis and will be given in Section 3.

**Theorem 2** Let $X_A(p)$ be the intersection

$$
\begin{cases}
\sum_{i=1}^{m} A_i |z_i|^p = 0, \\
\|z\|_p = 1.
\end{cases}
$$

There is a unique point $f_p(z)$ in the leaf $L_z$ for each element $z \in S_A$, whose $L^p$–norm $\|f_p(z)\|_p$ is minimal and positive, and the restriction of the smooth function $f_2/\|f_2\|_2: S_A \rightarrow X_A(2)$ to $X_A(p)$ induces a homeomorphism onto $X_A(2)$ for all $p \geq 1$. Moreover,

$$
\Phi_A(p): X_A(p) \times \mathbb{R}^d \times \mathbb{R}_{>0} \rightarrow S_A,
$$

$$(z, T, r) \mapsto r(z e^{(A_i, T)})_{i=1}^{m},$$

is a homeomorphism.

Similar to (4), for each $p$ we can define a continuous function $T_p: S_A \rightarrow \mathbb{R}^d$ such that $f_p/\|f_p\|_p: S_A \rightarrow X_A(p)$ is a retraction, where $f_p(z) = F(z, T_p(z))$ is the function of $L^p$–norm minima in the leaf $L_z$. 

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It is interesting to imagine what will happen when \( p \) tends to infinity, and we will discuss this in Section 4. First note that the set

\[
K_A = \{ \sigma \subset [m] \mid \emptyset \in \text{conv} A([m] \setminus \sigma) \}
\]

is an abstract simplicial complex (see [3, Lemma 0.12]), ie all subsets of \( \sigma \) will be in \( K_A \) if \( \sigma \) is. It turns out that with each \( z \in S_A \) fixed, \( T_p(z) \) and \( f_p(z)/\|f_p(z)\|_p \) are continuous in \( p \in [1, \infty) \) (see Proposition 4.2); when \( p \) goes to infinity, \( f_p(z)/\|f_p(z)\|_p \) approaches the moment-angle complex \((D^2, S^1)^K_A\) (see Section 4.1 and Proposition 4.4 for details), which is a subset of the intersection of \( S_A \) with the \( L^\infty \)–norm unit sphere in \( \mathbb{C}^m \) (\( \|z\|_\infty = \max\{\|z_i\|_{i=1}^m\} \)).

We say that the tuple \( A \) is centered at the origin if the centroid of all vectors in \( A \) are located at the origin:

\[
\sum_{i=1}^m A_i = 0.
\]

Under this additional assumption, \( K_A \) is isomorphic to the boundary of a convex polytope arising from the Gale transform of \( A \) (see Proposition 5.3); based on a result of Panov and Ustinovsky [18], in Section 5 we will show that \( f_p(z)/\|f_p(z)\|_p \) converges to a unique point in \((D^2, S^1)^K_A \) as \( p \) tends to infinity. With a similar treatment as the one for Theorem 2, the following theorem holds:

**Theorem 3** Assume that \( A \) is an admissible tuple centered at the origin. Then the restriction \( f_2/\|f_2\|_2\rangle_{(D^2, S^1)^K_A} : (D^2, S^1)^K_A \to X_A(2) \) is a homeomorphism. Moreover,

\[
\Phi_A(\infty) : (D^2, S^1)^K_A \times \mathbb{R}^d \times \mathbb{R}_{>0} \to S_A,
\]

\[
(z, T, r) \mapsto r(zi e^{A_i T})_{i=1}^m,
\]

is a homeomorphism.

Therefore, we can understand such a moment-angle complex \((D^2, S^1)^K_A \) as “\( X_A(\infty) \)”, namely the intersection of the \( L^\infty \)–norm minima in the Siegel leaves with the \( L^\infty \)–norm unit sphere in \( \mathbb{C}^m \) (the reader is encouraged to imagine the deformation from \( X_A(1) \) to \( X_A(\infty) \) in the case \( d = 0 \)).

As an application, in Section 6 we give an alternative proof for a rigidity theorem of Bosio and Meersseman [3, Theorem 4.1]: if two admissible \( m \)–tuples \( A \) and \( A' \) are both centered at the origin such that \( K_A \) and \( K_{A'} \) are isomorphic simplicially, then there is a diffeomorphism between associated links \( X_A(2) \) and \( X_{A'}(2) \) (see Proposition 6.1 for more details).
From its definition (1), notice that each leaf $L_z$ is contained in $S_A \cap \mathbb{R}^m$ if and only if $z \in S_A \cap \mathbb{R}^m$. Hence the theorems and properties above are also true when restricted to the subspace $\mathbb{R}^m$ in $\mathbb{C}^m$.

At last in Section 6, we shall illustrate that the restriction of $f_2/\|f_2\|_2$ to the real moment-angle complex $(D^1, S^0)^{K_A} = (D^2, S^1)^{K_A} \cap \mathbb{R}^m$ is a piecewise differentiable homeomorphism onto $X_A(2) \cap \mathbb{R}^m$, provided that $A$ is admissible and centered at the origin (see Definition 6.2, Lemma 6.3 and Proposition 6.4 for more details). In this way these real moment-angle complexes can be smoothed as piecewise linear manifolds.

3 Proof of Theorem 2

We start with a well-known lemma due to Meersseman and Verjovsky, whose proof is omitted here:

**Lemma 3.1** [16, Lemma 1.1; 3, Lemma 0.3] For an admissible tuple $A = (A_i)_{i=1}^m$, let $\tilde{A} = (\tilde{A}_i)_{i=1}^m$ be the augmentation with $\tilde{A}_i = (A_i^T, 1)^T \in \mathbb{R}^{d+1}$, $i = 1, 2, \ldots, m$. Then for any $I \subset [m]$ such that $0 \in \text{conv} A(I)$, the rank of the subtuple $\tilde{A}(I)$ is $d + 1$.

**Proposition 3.2** For each $z \in S_A$ given, there is a unique point $f_p(z)$ in the leaf $L_z$ such that $\|f_p(z)\|_p$ is minimal and positive.

**Proof** Uniqueness (cf [6; 15; 16]) Assume $F_z$ has two local minima, ie $T_1$ and $T_2$ in $\mathbb{R}^d$ that are both critical points of $(\|F(z, T)\|_p)^p = \sum_{i=1}^m |z_i|^p e^{p(A_i^T T)}$, which means

$$
\sum_{i=1}^m A_i |z_i|^p e^{p(A_i^T T_i)} = 0, \quad j = 1, 2.
$$

We define a function $h: [0, 1] \to \mathbb{R}$ such that $h(t) = (\|F(z, (1-t)T_1 + tT_2)\|_p)^p$; clearly

$$
\frac{dh}{dt} = p \sum_{i=1}^m \langle A_i, T_2 - T_1 \rangle |z_i|^p e^{p(A_i^T (1-t)T_1 + tT_2)}.
$$

From Lemma 3.1, the subtuple $A(I_2)$ has rank $d$ ($I_2 \subset [m]$ consists of entries $i$ such that $z_i \neq 0$), which is independent of $z \in S_A$, thus there exists $i \in I_2$ such that $\langle A_i, T_2 - T_1 \rangle$ does not vanish; it follows that the second derivative of $h$ is strictly positive, hence its first derivative (9) is strictly increasing, which is a contradiction.

**Existence** First from the Cauchy–Schwarz inequality

$$
\|F(z, T)\|_2 \leq \|F(z, T)\|_1 \leq \sqrt{m} \|F(z, T)\|_2,
$$

together with Theorem 1 and Lemma 3.3 below, we conclude that \( \| F(z, T) \|_1 \) bounds away from zero, and stays large whenever \( \| T \|_2 \) is large. Thus the minimum of \( \| F(z, T) \|_1 \) is positive, and it appears only when \( T \) is in the interior of a ball of finite radius. So the case \( p = 1 \) is clear. For general cases when \( p \neq 1, 2 \), Hölder’s inequality implies

\[
(11) \quad \| F(z, T) \|_p \leq \| F(z, T) \|_1 \leq \sqrt[p]{m} \| F(z, T) \|_p;
\]

here \( q > 1 \) such that \( 1/p + 1/q = 1 \). We can repeat the previous argument and then the proof is completed.

Lemma 3.3 With \( z \in \mathcal{S}_A \) given, for any \( N > 0 \), there exists \( R > 0 \) such that \( \| F(z, T) \|_2 > N \) whenever \( \| T \|_2 > R \).

Proof Let \( T_2(z) \) be the point in \( \mathbb{R}^d \) such that \( \| F(z, T_2(z)) \|_2 \) is minimal (see (4) for details). Denote by \( u(t; T_1, T_2) \) the derivative of \( (\| F(z, (1-t)T_1 + tT_2) \|_2)^2 \) with respect to \( t \in [0, 1] \), for \( T_1, T_2 \in \mathbb{R}^d \), and let \( B(r, T_2(z)) \) be the ball with radius \( r \) centered at \( T_2(z) \). Since \( T_2(z) \) is the unique minimum, for all \( y \in \partial B(1, T_2(z)) \) on the boundary, there is a positive \( \varepsilon \) such that

\[
(\| F(z, y) \|_2)^2 - (\| F(z, T_2(z)) \|_2)^2 = \int_0^1 u(t; T_2(z), y) \, dt > \varepsilon;
\]

therefore we can choose \( t(y) \in (0, 1) \) such that

\[
u(t(y); T_2(z), y) > \varepsilon,
\]

by the mean value theorem. For \( r > 1 \), assume \( y_r \in \partial B(r, T_2(z)) \) with \( y \in \partial B(1, T_2(z)) \) on the ray from \( T_2(z) \) to \( y_r \); by the monotonicity of \( u(t; T_2, y_r) \) (see the uniqueness part in the proof of Proposition 3.2), we have

\[
u(t; y, y_r) > u(t(y); T_2, y),
\]

thus

\[
(\| F(z, y_r) \|_2)^2 - (\| F(z, T_2(z)) \|_2)^2
= \int_0^1 u(t; T_2, y) \, dt + \int_0^1 u(t; y, y_r) \, dt > \varepsilon + (r - 1)\varepsilon,
\]

from which the conclusion follows.

The function of minima \( f_p: \mathcal{S}_A \to \mathcal{S}_A \) is well-defined by Proposition 3.2; but except for the case \( p = 2 \), it remains to prove its continuity. In what follows we shall illustrate this by showing the continuity of the restriction \( f_p/\| f_p \|_p \big|_{X_A(2)} \) first, and then it will follow from the global diffeomorphism \( \Phi_A(2) \) defined in Theorem 1.

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**Proposition 3.4** The restriction \(f_2/\|f_2\|_2 |_{X_A(p)}\) of the smooth function

\[
f_2/\|f_2\|_2: S_A \to X_A(2)
\]

induces a homeomorphism onto \(X_A(2)\), whose inverse is

\[
f_p/\|f_p\|_p |_{X_A(2)}: X_A(2) \to X_A(p).
\]

**Proof** Consider the function

\[
\Phi_A: S_A \times \mathbb{R}^d \times \mathbb{R}_{>0} \to S_A,
\]

\[
(z, T, r) \mapsto r(\sum_{i=1}^m z_i e^{(A_i,T)})^m,
\]

from Theorem 1 and Proposition 3.2. Given \(z \in S_A\), its image under \(\Phi_A\) intersects both \(X_A(p)\) and \(X_A(2)\) exactly once, respectively, hence \(f_2/\|f_2\|_2 |_{X_A(p)}\) is a bijection. Moreover, it is easy to see that \(X_A(p)\) is compact and \(X_A(2)\) is Hausdorff; since a closed subspace of a compact space is compact, and a compact subspace of a Hausdorff space is closed, it follows that \(f_2/\|f_2\|_2 |_{X_A(p)}\) is closed and hence a homeomorphism by the bijectiveness. As a conclusion, its inverse \(f_p/\|f_p\|_p |_{X_A(2)}\) is continuous. \(\square\)

**Theorem 3.5** The continuous function

\[
\Phi_A(p): X_A(p) \times \mathbb{R}^d \times \mathbb{R}_{>0} \to S_A,
\]

\[
(z, T, r) \mapsto r(\sum_{i=1}^m z_i e^{(A_i,T)})^m,
\]

is a homeomorphism for all \(p \geq 1\).

**Proof** It suffices to find a continuous inverse for \(\Phi_A(p)\). Suppose \(f_p(z)/\|f_p(z)\|_p = (x_i(z))_{i=1}^m\). For \((z, T, r) \in X_A(2) \times \mathbb{R}^d \times \mathbb{R}_{>0}\), we can rewrite

\[
z = \rho^{-1}(z) F(f_p(z)/\|f_p(z)\|_p, T_2(f_p(z)/\|f_p(z)\|_p))
\]

\[
= \rho^{-1}(z) \left( \sum_{i=1}^m x_i(z)e^{(A_i,T_2(f_p(z)/\|f_p(z)\|_p))} \right)^m
\]

where \(\rho(z) = \|(x_i(z)e^{(A_i,T_2(f_p(z)/\|f_p(z)\|_p))})_{i=1}^m\|_2\). The continuity of \(\rho^{-1}(z)\), \(x_i(z)\) and \(e^{(A_i,T_2(f_p(z)/\|f_p(z)\|_p))}\) follows from Proposition 3.4 (by Theorem 1, \(T_2\) is smooth). Observe that

\[
\Phi_A(2)(z, T, r) = r(\sum_{i=1}^m z_i e^{(A_i,T)})^m = r\rho^{-1}(z) \left( \sum_{i=1}^m x_i(z)e^{(A_i,T+T_2(f_p(z)/\|f_p(z)\|_p))} \right)^m
\]

\[
= \Phi_A(p)\left( f_p(z)/\|f_p(z)\|_p, T + T_2(f_p(z)/\|f_p(z)\|_p), r\rho^{-1}(z) \right).
\]
hence we have a coordinate transition function
\[
\varphi: X_A(2) \times \mathbb{R}^d \times \mathbb{R}_{>0} \to X_A(p) \times \mathbb{R}^d \times \mathbb{R}_{>0},
\]
\[
(z, T, r) \mapsto \left( f_p(z)/\|f_p(z)\|_p, T + T_2(f_p(z)/\|f_p(z)\|_p), r\rho^{-1}(z) \right).
\]
It is straightforward to check the continuity of \(\varphi\), thus \(\varphi \circ (\Phi_A(2))^{-1}\) is the inverse of \(\Phi_A(p)\). \(\square\)

**Corollary 3.6** The function
\[
T_p: S_A \to \mathbb{R}^d \quad \text{such that} \quad f_p(z) = F(z, T_p(z))
\]
is well-defined and continuous. That is to say, for each \(z \in S_A\), \(T_p(z)\) is the unique solution of the equation
\[
\sum_{i=1}^{m} A_i |z|^P e^{p(A_i, T)} = 0,
\]
which depends continuously on \(z\).

## 4 When \(p\) tends to infinity

In this section we treat \(T_p(z)\) and \(f_p(z)\) (defined in Corollary 3.6 and Proposition 3.2 respectively) as functions of \(p \in [1, \infty)\), with \(z \in S_A\) fixed.

**Lemma 4.1** There exists a bound \(N(z)\) such that \(\|T_p(z)\|_2 < N(z)\) for all \(p \in [1, \infty)\).

**Proof** By definition, \(\|F(z, T_p(z))\|_p\) is the unique minimum in the leaf \(L_z\). Suppose that on the contrary, there exists a sequence \(\{p_k\}_{k=1}^\infty\) tending to infinity such that \(\|T_{p_k}(z)\|_2 > k\) for each \(k\). First by Lemma 3.3 and the Cauchy–Schwarz inequality \((10)\), \(\|F(z, T)\|_1\) becomes arbitrarily large whenever \(\|T\|_2\) is large enough, thus there exists \(N > 0\) such that for all \(k > N\), \(m \|F(z, T_1(z))\|_1 < \|F(z, T_{p_k}(z))\|_1\).

Then by Hölder’s inequality \((11)\), we have
\[
\sqrt[q_k]{m} \|F(z, T_1(z))\|_{p_k} \leq \sqrt[q_k]{m} \|F(z, T_1(z))\|_1 < \|F(z, T_{p_k}(z))\|_1 \leq \sqrt[q_k]{m} \|F(z, T_{p_k}(z))\|_{p_k},
\]
where \(1/p_k + 1/q_k = 1\). It follows that \(\|F(z, T_{p_k}(z))\|_{p_k}\) is strictly greater than \(\|F(z, T_1(z))\|_{p_k}\), yielding a contradiction. \(\square\)

**Proposition 4.2** The function \(T_p(z)\) is continuous for all \(p \in [1, \infty)\).
Proof Suppose again on the contrary there is a sequence \( \{p_k\}_{k=1}^{\infty} \) with \( \lim_k p_k = p_0 \), but \( \|T_{p_k}(z) - T_{p_0}(z)\|_2 \geq \delta \), for some \( \delta > 0 \). Without loss of generality we may assume that \( \lim_k T_{p_k} = T_0 \neq T_{p_0}(z) \), or we can choose a subsequence satisfying the property, by the lemma above. Consider the smooth function
\[
\mu: [1, \infty) \times \mathbb{R}^d \to \mathbb{R}^d,
(p, T) \mapsto \sum_{i=1}^{m} A_i |z_i|^p e^{p(A_i, T)};
\]
we have \( 0 = \lim_k \mu(p_k, T_k(z)) = \mu(p_0, T_0) \) by continuity, contradicting the uniqueness (see Corollary 3.6).

Corollary 4.3 As a function of \( p \in [1, \infty) \), \( f_p(z)/\|f_p(z)\|_p \) is continuous with its image in the \( L^p \)-link \( X_A(p) \) (defined by (6)), and we have
\[
(13) \quad \lim_{p \to \infty} \|f_p(z)/\|f_p(z)\|_p\|_\infty = 1.
\]

Proof Denote \( f_p(z)/\|f_p(z)\|_p \) by \( y(p) = (y_i(p))_{i=1}^{m} \). Observe that
\[
1 = \|y(p)\|_p = \|y(p)\|_\infty \left( \sum_{i=1}^{m} \frac{|y_i(p)|}{\|y(p)\|_\infty} \right)^{1/p},
\]
where the last term in the bracket does not exceed \( m \), thus (13) holds as desired.

4.1 Moment-angle complexes

Let \( K_A \) be the simplicial complex defined by (7). The associated moment-angle complex \( (D^2, S^1)^{K_A} \) is defined as the polyhedral product
\[
(D^2, S^1)^{K_A} = \bigcup_{\sigma \in K_A} D(\sigma), \quad D(\sigma) = \prod_{i=1}^{m} Y_i,
\]
\[
Y_i = \begin{cases} D^2 = \{|z| \leq 1 \quad |z| \in \mathbb{C} \} & \text{if } i \in \sigma, \\ S^1 = \{|z| = 1 \quad |z| \in \mathbb{C} \} & \text{otherwise.} \end{cases}
\]
The proposition below implies that \( f_p(z)/\|f_p(z)\|_p \in X_A(p) \) approaches \( (D^2, S^1)^{K_A} \) as \( p \) tends to infinity.

Proposition 4.4 Let \( S_\infty \) be the unit sphere of \( \mathbb{C}^m \) with respect to the \( L^\infty \)-norm, and let \( z \in S_A \) be a given point. Then for every point \( z' = (z_i')_{i=1}^{m} \in S_\infty \cap S_A \setminus (D^2, S^1)^{K_A} \), \( f_p(z)/\|f_p(z)\|_p \) will go outside of the set
\[
C(z') = \{(z_i)_{i=1}^{m} \in \mathbb{C}^m \mid |z_i| \leq |z_i'| \quad \text{for all } i = 1, 2, \ldots, m\},
\]
whenever \( p \) is sufficiently large.
We claim that the vector we use here is combinatorial, in which Gale transforms play an essential role.

To see this, first note that because \( x \) and \( n \) vector (14) shall lie in \( U \) by those in these works. Already proven in \([16; 3]\), where Gale transforms have been intensely used. The approach here is motivated by the admissibility of \( A \), which means \( S_\infty = (D^2, S^1)^{K_A} \) and we have nothing to prove; otherwise \( B \) is compact thus there is an open neighborhood \( U_B \) such that \( 0 \notin U_B \).

Suppose the contrary, namely there is a sequence \( \{p_k\}_{k=1}^\infty \) tending to infinity such that \( x_k = (x_{ki})_{i=1}^m = f_{p_k}(z)/\|f_{p_k}(z)\|_p \in C(z') \). Since \( C(z') \) is compact, we may assume that \( \{x_k\}_{k=1}^\infty \) converges to a point \( x_0 = (x_{0i})_{i=1}^m \in C(z') \), without loss of generality.

We claim that the vector

\[
\sum_{i=1}^m A_i |x_{ki}|^{p_k}
\]

lies in \( U_B \) whenever \( k \) is large enough. Notice that this will be a contradiction since \( x_k \in X_A(p) \), whose definition implies that the vector above should always be zero.

To see this, first note that because \( x_0 \notin (D^2, S^1)^{K_A} \), there exists \( \tau \notin K_A \), such that \( |x_{0i}| < \delta < 1 \) for all \( i \in \tau \). This means for those \( i \in \tau \), there exists an \( N > 0 \) such that \( |x_{ki}| < \delta < 1 \) holds when \( k > N \); thus for any given \( \varepsilon > 0 \), we can find \( N_\varepsilon > N \) such that \( |x_{ki}|^{p_k} < \varepsilon \) for all \( k > N_\varepsilon \). It is not difficult to see that, if \( \varepsilon \) is small enough, vector (14) shall lie in \( U_B \), as claimed.

\[\square\]

5 The convergence

In this section we shall prove that the function \( f_p(z)/\|f_p(z)\| : S_A \to X_A(p) \) indeed converges to a point in \( (D^2, S^1)^{K_A} \), as one may expect from Proposition 4.4, with an additional assumption that \( A \) is centered at the origin (see (8)). The main technique we use here is combinatorial, in which Gale transforms play an essential role.\footnote{I would like to thank the referee for pointing out that analogues of Lemma 5.2 and Proposition 5.3 are already proven in [16; 3], where Gale transforms have been intensely used. The approach here is motivated by those in these works.}

Suppose \( V = (V_1, V_2, \ldots, V_m) \) is a tuple of vectors in \( \mathbb{R}^{m-d-1} \) such that the affine dimension of \( V \) is \( m-d-1 \), ie the matrix with columns \( (V_i^T, 1)^T \) \( (i = 1, 2, \ldots, m) \) has rank \( m - d \).

Denote by \( A_V = (A_1, A_2, \ldots, A_m) \) the Gale transform of \( V \) (see Grünbaum [12, Chapter 5.4, pages 85–86]), which is the transpose of a basis of solutions of the following
linear system:

\[
\begin{aligned}
\sum_{i=1}^{m} V_i x_i &= 0, \\
\sum_{i=1}^{m} x_i &= 0.
\end{aligned}
\] (15)

It is clear that each \( A_i \) is a vector in \( \mathbb{R}^d \) and different choices of \( A_V \) are linearly equivalent.

Recall that for any \( J \subset [m] \), the subtuple \( V(J) = (V_i)_{i \in J} \) is a face of \( V \) if the intersection of \( \text{conv} V([m] \setminus J) \) with the affine space spanned by vectors in \( V(J) \) is empty (see [12, Chapter 5.4, page 88]). For instance, if \( V \) consists of the vertices of a convex polytope \( P \), then \( V(J) \) is a face of \( V \) when and only when \( \text{conv} V(J) \) is a face of \( P \). Now we need two facts about Gale transforms:

**Proposition 5.1** [12, Chapter 5.4, page 88] Let \( V = (V_i)_{i=1}^{m} \) be a tuple of vectors in \( \mathbb{R}^{m-d-1} \), whose affine dimension is \( m - d - 1 \), and let \( A_V = (A_i)_{i=1}^{m} \) be its Gale transform.

Then for any \( I \subset [m] \), \( \text{conv} V([m] \setminus I) \) is a face of \( V \) if and only if either \( I \) is empty or \( 0 \) is in the relative interior of \( \text{conv} A_V(I) \).

Moreover, \( V \) coincides with the vertex set of a convex polytope \( P \) if and only if either

1. \( d = 0 \) (thus \( P \) is a simplex) or
2. for every open halfspace \( H^+ \) of \( \mathbb{R}^d \) containing \( 0 \) in its closure, we have that \( \text{card}(\{i \mid A_i \in H^+\}) \geq 2 \).

It follows that if \( V \) is centered at the origin, then the double Gale transform of \( V \) gives the same configuration in \( \mathbb{R}^{m-d-1} \). However, this is not true in general (see Remark 5.5).

Based on the facts above, we have the following lemma (in which we use the same notation as in Proposition 5.1).

**Lemma 5.2** Suppose that every vector of \( V \) is a face, and every face of \( V \) has at most \( m - d - 1 \) vectors. Then \( V \) coincides with the vertex set of a convex polytope \( P \), whose boundary is simplicial.
**Proof** Let $A_V = (A_i)_{i=1}^m$ be the Gale transform of $V$. It suffices to show either (i) or (ii) in Proposition 5.1 holds. Note that the case $d = 0$ is trivial: this happens if and only if $V$ spans an $(m - 1)$--simplex in $\mathbb{R}^{m-1}$.

Now suppose that $d > 0$. Notice that by Proposition 5.1, the Siegel and weak hyperbolicity conditions hold for $A_V$, with the assumption above. Moreover, since every vector in $V$ is a face, we have $0 \in \text{conv} A_V(J)$, for all $J$ with $\text{card}(J) = m-1$.

Let $H^+$ be an open halfspace of $\mathbb{R}^d$ with $0$ on the boundary. From its admissibility, $A_V$ has rank $d$, with a neighborhood of $0 \in \mathbb{R}^d$ contained in $\text{conv} A_V$ (see Lemma 3.1); hence there exists $A_i \in A_V$ such that $A_i \in H^+$. Observe that now $0 \in \text{conv} A_V([m] \setminus \{i\})$ with $A_V([m] \setminus \{i\})$ again being admissible, by the same argument, there exists another $A_j \in A_V$ with $A_j \in H^+$, which means (ii) holds hence the statement follows. □

**Proposition 5.3** Let $K_A$ be the simplicial complex induced from an admissible $m$--tuple $A = (A_i)_{i=1}^m$ centered at the origin, with vectors in $\mathbb{R}^d$. Let the tuple $V = (V_i)_{i=1}^m$ be the transpose of a basis of the system

$$
\begin{align*}
\sum_{i=1}^m A_i x_i &= 0, \\
\sum_{i=1}^m x_i &= 0.
\end{align*}
$$

Then $\{V_i\}_{(i) \in K_A}$ is the vertex set of a convex polytope $P_A$ of affine dimension $m - d - 1$, with each $V_j$ in its interior, where $\{j\} \not\in K_A$. Moreover, the boundary of $P_A$ is isomorphic to $K_A$ and we can assume that $P_A$ contains $0$ in its interior.

**Proof** First from Lemma 3.1, the affine dimension of $V$ is $m - d - 1$.

Since the centroid of $A$ is $0$, now $A$ is the Gale transform of $V$, with the subtuple $(V_i)_{(i) \in K_A}$ satisfying the assumptions in Lemma 5.2; thus it coincides with the vertex set of a convex polytope $P_A$ whose boundary is simplicial. For those $\{j\} \not\in K_A$, if $V_j$ lies outside, or on the boundary of $P_A$, it must be in a face of $V$ that is contained in a supporting hyperplane of $P_A$; by Proposition 5.1, this is impossible.

The last statement is also a consequence of Proposition 5.1, together with the observation that any translation of the form $V + v_0 = (V_i + v_0)_{i=1}^m$ also satisfies (16). □

**Example 5.4** Let $A$ be the 5--tuple given by the matrix

$$
\begin{pmatrix}
0 & 0 & 1 & 1 & -2 \\
1 & \frac{1}{2} & 0 & 0 & -\frac{3}{2}
\end{pmatrix}
$$
which is admissible and centered at the origin. By solving (16) we can choose \( V \) that is given by
\[
\begin{pmatrix}
0 & 0 & -1 & 1 & 0 \\
6 & -9 & 2 & 0 & 1
\end{pmatrix}.
\]
Observe that the last point \((0, 1)^T\) is in the interior of the square spanned by the other four vertices.

**Remark 5.5** Note that Proposition 5.3 is independent of the choice of \( V \). If the centroid of \( A \) is not at the origin, Proposition 5.3 may not hold. Consider the case that \( A \) is given by the matrix (one can check its admissibility)
\[
\begin{pmatrix}
1 & 1 & 4 & -2 \\
4 & -2 & 1 & 1
\end{pmatrix},
\]
then we choose \( V = (-1, -1, 1, 1) \) by (16), but now points \( V_2 = (-1) \) and \( V_4 = (1) \) are not contained in the interior of \( P_A = \text{conv}(V_1, V_3) \). This is because the Gale transform of \( V \) can be
\[
\begin{pmatrix}
0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0
\end{pmatrix},
\]
which is no longer admissible.

The following proposition is essentially due to Panov and Ustinovsky [18].

**Proposition 5.6** Let \( A = (A_i)_{i=1}^m \) be an admissible \( m \)-tuple centered at the origin. Then for each \( z \in S_A \) given, there is a unique pair \((r, T) \in \mathbb{R}_{>0} \times \mathbb{R}^d \) such that \( \Phi_A(z, T, r) = r(z e^{(A_i, T)})_{i=1}^m \in (D^2, S^1)^{K_A} \) (see Section 4.1 for the definition).

**Proof** The proof which is presented here is adapted from Panov [17, Theorem 9.2, pages 37–40]. Observe that in the trivial case when \( d = 0 \), ie \( K_A \) is a simplex, we can simply take \( r = \|z\|^{-1}_\infty \). In what follows suppose \( d > 0 \).

Let \( \mathbb{R}_{\geq 0} \) (resp. \( \mathbb{R}_{\leq 0} \)) be the set of nonnegative (resp. nonpositive) real numbers. Note that it suffices to prove the cases when \( z \in (\mathbb{R}_{\geq 0})^m \), since for each \( z = (z_i)_{i=1}^m \in \mathbb{C}^m \), there is a rotation \( e^{\sqrt{-1}\theta} = (e^{\sqrt{-1}\theta} i)_{i=1}^m \in (S^1)^m \) such that
\[
e^{\sqrt{-1}\theta} z = (e^{\sqrt{-1}\theta} i)_{i=1}^m \in (\mathbb{R}_{\geq 0})^m,
\]
and we have
\[
e^{\sqrt{-1}\theta} \Phi_A(z, T, r) = \Phi_A(e^{\sqrt{-1}\theta} z, T, r).
\]
For the tuple \( A = (A_i)_{i=1}^m \), let \( V = (V_i)_{i=1}^m \) be the tuple defined in Proposition 5.3, which satisfies (16). Since \( A \) is centered at the origin, the row vectors of \( \tilde{V} = (\tilde{V}_i)_{i=1}^m \)
with $\tilde{V}_i = (V_i^T, 1)^T$ are a basis of the orthogonal complement of the space spanned by the row vectors of $A$.

Let $\alpha$ be the linear morphism
\[
\alpha: \mathbb{R}^m \to \mathbb{R}^{m-d},
\]
\[
(x_i)_{i=1}^m \mapsto \sum_{i=1}^m \tilde{V}_i x_i.
\]

For $x = (x_i)_{i=1}^m \in (\mathbb{R}_{\geq 0})^m$, we shall abbreviate $(\ln(x_i))_{i=1}^m$ as $\ln(x)$ in what follows.

First we consider the case $z \in S_A \cap (\mathbb{R}_{> 0})^m$. Observe that there exists a pair $(r, T) \in \mathbb{R}_{> 0} \times \mathbb{R}^d$ such that $y = (y_i)_{i=1}^m = \Phi_A(z, T, r)$ when and only when $\ln(y) - w - \ln(z) = ((A_i, T_i))_{i=1}^m$, where $w = (w_i)_{i=1}^m$ with $w_i \equiv \ln(r)$, and this happens if and only if the vector $\ln(y) - w - \ln(z)$ belongs to $\text{Ker}(\alpha)$.

Let $(\mathbb{R}_{\leq 0}, 0)^K_A$ be the polyhedral product
\[
(\mathbb{R}_{\leq 0}, 0)^K_A = \bigcup_{\sigma \in K_A} D(\sigma), \quad D(\sigma) = \prod_{i=1}^m Y_i, \quad Y_i = \begin{cases} \mathbb{R}_{\leq 0} & \text{if } i \in \sigma, \\ \{0\} & \text{otherwise}, \end{cases}
\]
and it is clear that $y \in (D^2, S^1)^K_A \cap (\mathbb{R}_{> 0})^m$ if and only if $\ln(y) \in (\mathbb{R}_{\leq 0}, 0)^K_A$, hence now it suffices to find a unique pair $(u, c) \in ((\mathbb{R}_{\leq 0}, 0)^K_A, \mathbb{R})$, such that
\[
\sum_{i=1}^m (V_i^T, 1)^T (u_i + c) = \sum_{i=1}^m (V_i^T, 1)^T \ln(z_i)
\]
holds, where $u = (u_i)_{i=1}^m$. Let $\tilde{P}_A$ be the convex polytope spanned by $\{-V_i\}_{i \in K_A}$. By Proposition 5.3, $\tilde{P}_A$ contains a neighborhood of 0 in its interior, and the boundary of $\tilde{P}_A$ is the union $-\bigcup_{\sigma \in K_A} \text{conv} V(\sigma)$, which is simplicially isomorphic to $K_A$. Therefore every vector $v$ in $\mathbb{R}^{m-d-1}$ has a unique expression $\rho v_0$, where $\rho \in \mathbb{R}_{\geq 0}$ and $v_0$ lies in the relative interior of the corresponding face. Together with the observation $\sum_{i=1}^m V_i = 0$ (see (16)), we conclude that there exists a pair $(u, c) \in ((\mathbb{R}_{\leq 0}, 0)^K_A, \mathbb{R})$ such that
\[
\sum_{i=1}^m V_i u_i = \sum_{i \in K_A} V_i u_i = \sum_{i=1}^m V_i \ln(z_i), \quad \sum_{i=1}^m (\ln(z_i) - u_i) = \sum_{i=1}^m c = mc,
\]
namely (18) holds, which is unique by the construction.

Next we consider general case when $z \in S_A \cap \mathbb{R}_{\geq 0}^m$ with $\tilde{I}_z = \{i \mid z_i = 0\}$ not empty. First note that by definition, $\tilde{I}_z$ is a simplex of $K_A$. Let $\pi_z: \mathbb{R}^{m-d-1} \to \mathbb{R}^{m-d-1-\text{card}(\tilde{I}_z)}$
be the orthogonal projection onto the linear subspace

$$\bigcap_{i \in I_z} \{ v \in \mathbb{R}^{m-d} \ | \ \langle v, V_i \rangle = 0 \},$$

and denote by $\text{Link}(I_z, K_A)$ the union

$$\{ \sigma \in K_A \ | \ (\sigma \cup I_z) \in K_A, \ \sigma \cap I_z = \emptyset \},$$

which is a subcomplex of $\text{Star}(I_z, K_A) = \{ \sigma \in K_A \ | \ I_z \subseteq \sigma \}$. It is not difficult to see that in the image of $\pi_z$, $\pi_z(\text{conv} \mathcal{V}(\text{Star}(I_z, K_A)))$ is a convex polytope bounded by $\pi_z(\text{conv} \text{Link}(I_z, K_A))$ (for example, by induction on $\text{card}(I_z)$). Then by a similar argument as in the previous case, we deduce that there exists a unique $u = (u_i)_{i=1}^m$ in the polyhedral product $(\mathbb{R} \leq 0, 0)_{\text{Link}(I_z, K_A)}$ (defined by replacing $K_A$ with $\text{Link}(I_z, K_A)$ in (17)), such that

$$\pi_z \left( \sum_{i=1}^m V_i u_i \right) = \pi_z \left( \sum_{i=1}^m V_i \chi(z_i) \ln(z_i) \right), \quad \chi(z_i) \ln(z_i) = \begin{cases} \ln(z_i) & \text{if } |z_i| > 0, \\ 0 & \text{otherwise}; \end{cases}$$

note that vectors of $\{V_i\}_{i \in I_z}$ are linearly independent, hence we have a unique $x = (x_i)_{i=1}^m \in \mathbb{R}^m$ with $I_x \subseteq I_z$, such that

$$\sum_{i=1}^m V_i (u_i + x_i) = \sum_{i=1}^m V_i \chi(z_i) \ln(z_i)$$

holds. With $c$ obtained from

$$\sum_{i=1}^m (\chi(z_i) \ln(z_i) - u_i - x_i) = mc,$$

we have

$$\sum_{i=1}^m (V_i^T, 1)^T (u_i + x_i + c) = \sum_{i=1}^m (V_i^T, 1)^T \chi(z_i) \ln(z_i).$$

At last, by solving $T \in \mathbb{R}^d$ from

$$\langle A_i, T \rangle = \chi(z_i) \ln(z_i) - u_i - x_i - c$$

for $i = 1, 2, \ldots, m$, and setting $r = e^c$, we have $\Phi_A(z, T, r) \in (D^2, S^1)^K_A$ as desired; the uniqueness follows from the arguments above and the observation that the rank of $A$ is $d$. \qed

From Proposition 5.6, we can define a map $f_\infty: S_A \to S_A$, with $f_\infty(z)$ the point in the leaf $L_z$ such that $f_\infty(z)/\|f_\infty(z)\|_\infty \in (D^2, S^1)^K_A$. 

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The proofs of Proposition 5.7 and Theorem 5.8 are similar to the ones for Proposition 3.4 and Theorem 3.5, respectively, which we shall omit here.

**Proposition 5.7** With the assumption that $A$ is admissible and centered at the origin, the restriction $f_2/\|f_2\|_2|_{(D^2,S^1)^K_A}: (D^2,S^1)^K_A \rightarrow X_A(2)$ is a homeomorphism, whose inverse is the restriction $f_\infty/\|f_\infty\|_\infty|_{X_A(2)}: X_A(2) \rightarrow (D^2,S^1)^K_A$.

**Theorem 5.8** The continuous function
\[
\Phi_A(\infty): (D^2,S^1)^K_A \times \mathbb{R}^d \times \mathbb{R}_{>0} \rightarrow S_A.
\]
\[
(z,T,r) \mapsto r (z_i e^{(A_i,T)})^m_i,
\]
is a homeomorphism, provided that $A$ is admissible and centered at the origin.

Recall that for each $p \in [1,\infty)$, we have defined $T_p: S_A \rightarrow \mathbb{R}^d$ such that $f_p(z) = F(z,T_p(z))$ has the minimal $L^p$–norm in each leaf $F_z$. By Theorem 3.5, $T_p$ is the composition of $\Phi_A^{-1}(p)$ and the projection onto $\mathbb{R}^d$, and $f_p(z)/\|f_p(z)\|_p$ is the composition of $\Phi_A^{-1}(p)$ and the projection onto $X_A(p)$.

**Corollary 5.9** Let $T_\infty: S_A \rightarrow \mathbb{R}^d$ be the composition of $\Phi_A^{-1}(\infty)$ and the projection onto $\mathbb{R}^d$, with $A$ admissible and centered at the origin. Then we have
\[
\lim_{p \rightarrow \infty} T_p(z) = T_\infty(z),
\]
which means
\[
\lim_{p \rightarrow \infty} f_p(z)/\|f_p(z)\|_p = f_\infty(z)/\|f_\infty(z)\|_\infty,
\]
with any $z \in S_A$ given.

**Proof** By Lemma 4.1, Corollary 4.3 and Proposition 4.4, there exists a sequence $\{p_k\}_{k=1}^\infty$ such that $\{T_{p_k}(z)\}_{k=1}^\infty$ converges to a point $T_0 \in \mathbb{R}^d$, and $\{f_{p_k}(z)\}_{k=1}^\infty$ converges to some $y_0$ such that $y_0/\|y_0\|_\infty \in (D^2,S^1)^K_A$. We claim that
\[
\lim_{p \rightarrow \infty} T_p(z) = T_0 = T_\infty(z)
\]
with
\[
\lim_{p \rightarrow \infty} f_p(z)/\|f_p(z)\|_p = y_0/\|y_0\|_\infty = f_\infty(z)/\|f_\infty(z)\|_\infty.
\]
Note that
\[
y_0/\|y_0\|_\infty = \lim_{k \rightarrow \infty} \Phi_A(z, T_k(z), ||f_{p_k}||^{-1}_{p_k}) = \Phi_A(z, T_0, ||y_0||^{-1}_{\infty}) \in (D^2,S^1)^K_A,
\]
which is uniquely determined by \( z \) (see Proposition 5.6), therefore \( y_0/\|y_0\|_\infty \) must be \( f_\infty(z)/\|f_\infty(z)\|_\infty \) and \( T_0 \) must be \( T_\infty(z) \). It is not difficult to see that the argument above is independent of the choice of the sequence \( \{p_k\}_k^\infty \), hence the claim holds and the proof is completed.

\[ \square \]

6 Some applications

In this section we shall revisit several known results from another perspective. First notice that by Proposition 5.3, a simplicial complex \( K_A \) induced from an admissible tuple that is centered at the origin can be realized as the boundary of a convex polytope dual to a simple one; the converse is also true: for a convex polytope with simplicial boundary, the Gale transform of its vertices will be a tuple with the property above.

Our first application is an alternative proof of a rigidity theorem on polytopal moment-angle manifolds, due to Bosio and Meersseman:

**Proposition 6.1** [3, Theorem 4.1] Let \( K_A \) and \( K_{A'} \) be the simplicial complexes induced from two admissible \( m \)–tuples \( A \) and \( A' \) that are centered at the origin, respectively. If there is a simplicial isomorphism \( \phi: K_A \to K_{A'} \), then there is a diffeomorphism between \( X_A(2) \) and \( X_{A'}(2) \).

**Proof** Observe that under the assumption, \( \phi \) can be extended as a bijection from \([m]\) to itself (possibly not unique), and let \( \bar{\phi}: S_A \to S_{A'} \) be the diffeomorphism via permuting coordinates with respect to \( \phi \). Clearly \( \bar{\phi} \) gives a homeomorphism between associated moment-angle complexes \( (D^2, S^1)^{K_A} \) and \( (D^2, S^1)^{K_{A'}} \). On the other hand, we have a smooth map \( (f'_2/\|f'_2\|_2) \circ \bar{\phi}: X_A(2) \to X_{A'}(2) \) given in the diagram

\[
\begin{array}{ccc}
S_A & \xrightarrow{\phi} & S_{A'} \\
\downarrow \text{diffeo.} & & \downarrow \text{homeo.} \\
X_A(2) & \xrightarrow{(f'_2/\|f'_2\|_2) \circ \bar{\phi}} & X_{A'}(2) \\
\downarrow \text{homeo.} & & \downarrow \text{homeo.} \\
(D^2, S^1)^{K_A} & \xrightarrow{\phi} & (D^2, S^1)^{K_{A'}}
\end{array}
\]

where \( f'_2: S_{A'} \to S_{A'} \) is the function of \( L^2 \)–norm minima of Siegel leaves. By commutativity, it follows that \( (f'_2/\|f'_2\|_2) \circ \bar{\phi} \) is a homeomorphism (see Theorem 1 and Proposition 5.7), whose inverse can be constructed by interchanging the roles of \( A \) and \( A' \), which is also smooth.

\[ \square \]
In what follows we shall discuss everything with $\mathbb{C}^m$ replaced by its subspace $\mathbb{R}^m$. In the foliation $\mathcal{F}$ given by the action (1), a leaf $L_z$ lies in $S_A \cap \mathbb{R}^m$ if and only if $z \in S_A \cap \mathbb{R}^m$. Therefore all properties hold true when restricted to the real case.

We will still use the same notation as in the previous sections, with the exception that the notation $D^1, S^0/K_A$ is used for the associated real moment-angle complex, i.e. the intersection of $(D^2, S^1)K_A$ with $\mathbb{R}^m$ (see Section 4.1 for details).

Notice that the real version of Proposition 6.1 holds, namely the $\mathbb{Z}_2$–equivariant (where $\mathbb{Z}_2^m$ acts on $X_A(2)$ by changing the signs of coordinates) smooth structures on $X_A(2)$ are determined by combinatorial types of $K_A$. This can be deduced from a result of Wiemeler in [21, Corollary 5.2] (see also Davis [7, Corollary 1.3]).

A polyhedron $X$ is a piecewise linear (abbreviated PL) $n$–manifold if given a certain triangulation, the link of each vertex is PL homeomorphic to the boundary of an $n$–simplex or to an $(n−1)$–simplex (i.e. these homeomorphisms become simplicial after suitable subdivisions on both sides). Note that this property is independent of the triangulation chosen for $X$ (see e.g. [19, pages 20–22]).

**Definition 6.2** (Whitehead triangulation) Let $X$ be a polyhedron and $M$ a smooth manifold. A map $\eta: X \to M$ is a piecewise differentiable (abbreviated PD) homeomorphism if there exists a triangulation of $X$ such that the restriction of $\eta$ to each simplex is smooth with the Jacobian matrix nondegenerate. Such a PD homeomorphism $\eta$ is called a Whitehead triangulation of $M$, and also a smoothing of $X$.

Note that by Propositions 3.4 and 5.7, the smooth function $f_2/\|f_2\|_2: S_A \to X_A(2)$ induces a homeomorphism when restricted to either $(D^1, S^0)K_A$ or $X_A(1)$. Moreover, the following lemma holds:

**Lemma 6.3** Let $A = (A_i)_{i=1}^m$ be an admissible tuple centered at the origin. If a space $Y \subset \mathbb{R}^m$ is either

(a) the intersection of the $L^p$–link $X_A(p)$ (defined by (6)) with the first orthant of $\mathbb{R}^m$ (i.e. points with nonnegative coordinates), for any $p \geq 1$, or

(b) a component of the polyhedral product $D(\sigma) = (D^1, S^0)^{\sigma}$ (see Section 4.1 for definition, with the pair replaced), for any $\sigma \in K_A$ with maximal dimension,

then $Y$ is a smooth manifold with corners, and the differential of $f_2/\|f_2\|_2$ at any point of $Y$ induces a linear injection between corresponding tangent spaces.
As a corollary, we find that with given triangulations, the restriction of \( f_2/\| f_2 \|_2 \) to either \((D^1, S^0) / X_A^m\) or \( X_A(2) \) will be a Whitehead triangulation of \( X_A(2) \). By a theorem of Whitehead [20], if there is a PD homeomorphism from a polyhedron \( X \) to a smooth manifold \( M \), then \( X \) is a PL manifold, and the PL structure on \( X \) is

**Proof** First we show that each \( Y \) is indeed a smooth manifold with corners, in both cases. For (b) this is obvious since \( Y \) is a cube of dimension \( m - d - 1 \). As for (a), observe that for each \( \sigma \in K_A \) with \( \text{card}(\sigma) = k \), the augmented subtuple \( \tilde{A}([m] \setminus \sigma) \) has rank \( d + 1 \), where \( \tilde{A} = (\tilde{A}_i)_{i=1}^m \) with \( \tilde{A}_i = (A_i^T, 1)^T \) (see Lemma 3.1), therefore the row vectors of \( \tilde{A} \), together with canonical basis vectors \( e_i \in \mathbb{R}^m \) (the vector with only \( i \textsuperscript{th} \) coordinate nonzero, which is one) for all \( i \in \sigma \), form a matrix of rank \( d + k + 1 \). This means that the intersection

\[
Y \bigcap_{i \in \sigma} F_i
\]

is transverse, where \( F_i = \{(x_i)_{i=1}^m \in \mathbb{R}^m \mid x_i = 0\} \).

Recall that \( \Phi_A(2) : X_A(2) \times \mathbb{R}^d \times \mathbb{R}_{>0} \rightarrow S_A \) is a diffeomorphism such that \( f_2/\| f_2 \|_2 \circ \Phi_A(2) \) is the identity on \( X_A(2) \) (see Theorem 1). Let

\[
d\Phi_A(2)_x : \mathbb{R}^{m-d-1} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^m \in T\Phi_A(2)_x|_\xi S_A.
\]

be the differential of \( \Phi_A(2) \) at the point \( x \), and let \( \xi \) be the linear subspace \( \{0\} \times \mathbb{R}^d \times \mathbb{R} \subset \mathbb{R}^m \) of dimension \( d + 1 \). It suffices to show that for all \( y = (y_i)_{i=1}^m \in Y \) with \( x = (x_i)_{i=1}^m = f_2(y)/\| f_2(y) \|_2 \), the intersection of the image of \( d\Phi_A(2)_x|_\xi \) with the tangent space \( T_y Y \) is trivial.

For (a), note that from its definition (6), the tangent space \( T_y Y \) is the orthogonal complement of the \((d+1)\)–space spanned by the row vectors of the \(((d+1) \times m)\)–matrix

\[
\tilde{A}_{y_{p-1}} = ((A_i^T, 1)^T y_i^{p-1})_{i=1}^m
\]

and the image of \( d\Phi_A(2)_x|_\xi \) is spanned by the row vectors of \( \tilde{A}_y = ((A_i^T, 1)^T y_i)_{i=1}^m \). From the previous argument, the subtuple \( \tilde{A}_y_{p-1}(I_y) \) has rank \( d + 1 \) (\( I_y \subset [m] \) is the set of nonzero entries of \( y \)), hence any row vector of \( \tilde{A}_y(I_y) \) cannot be orthogonal to the corresponding one in \( \tilde{A}_y_{p-1}(I_y) \), otherwise itself must be zero (since we can write each \( y_i^p \) as a square).

As for (b), the tangent space at \( y \in (D^1, S^0)^\sigma \) is spanned by \( \{e_i \mid i \in \sigma \} \), where \( \text{card}(\sigma) = m - d - 1 \). But we have shown that the row vectors of \( \tilde{A}_y(I_y) \) and the basis of \( T_y Y \) has a full rank \( m \), therefore the intersection of the image of \( d\Phi_A(2)_x|_\xi \) with \( T_y Y \) must be trivial. \( \square \)

As a corollary, we find that with given triangulations, the restriction of \( f_2/\| f_2 \|_2 \) to either \((D^1, S^0) / X_A^m\) or \( X_A(1) \) will be a Whitehead triangulation of \( X_A(2) \). By a theorem of Whitehead [20], if there is a PD homeomorphism from a polyhedron \( X \) to a smooth manifold \( M \), then \( X \) is a PL manifold, and the PL structure on \( X \) is.
uniquely determined by the smooth structure given on \( M \). Consequently, it follows that \((D^1, S^0)^{K_A}\) and \(X_A(1)\) are homeomorphic as PL manifolds.

At last, we make a conclusion to end this section.

**Proposition 6.4** For each simplicial complex \( K_A \) induced from an admissible \( m \)-tuple \( A \) centered at the origin, there is a PD homeomorphism from \((D^1, S^0)^{K_A}\) onto the smooth manifold \( X_A(2) \), thus \((D^1, S^0)^{K_A}\) is a PL manifold of dimension \( m - d - 1 \). If \((D^2, S^1)^{K_A}\) has an exotic PL structure, then either it is not smoothable, or \( X_A(2) \) must have different smooth structures.

**References**


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