Uniformly finite homology and amenable groups

MATTHIAS BLANK

FRANCESCA DIANA

Uniformly finite homology is a coarse invariant for metric spaces; in particular, it is a quasi-isometry invariant for finitely generated groups. In this article, we study uniformly finite homology of finitely generated amenable groups and prove that it is infinite-dimensional in many cases. The main idea is to use different transfer maps to distinguish between classes in uniformly finite homology. Furthermore we show that there are infinitely many classes in degree zero that cannot be detected by means.

1 Introduction

Uniformly finite homology was introduced by Block and Weinberger [5] to study the large-scale structure of metric spaces having bounded geometric complexity. It is a coarse invariant in the sense that two quasi-isometric metric spaces have isomorphic uniformly finite homology. The chains considered are the ones introduced by Roe [19] to define coarse homology, but with an additional boundedness condition on the coefficients (Definition 2.1).

The uniformly finite homology groups of a metric space X are denoted by $H_*^{\mathrm{uf}}(X; \mathbb{R})$. One of the main properties of $H_0^{\mathrm{uf}}(X; \mathbb{R})$ is the relation with the notion of amenability for spaces having coarse bounded geometry (in particular for finitely generated groups). The following result is due to Block and Weinberger [5, Theorem 3.1]:

Theorem 1.1 *If X is a metric space with coarse bounded geometry, then X is amenable if and only if $H_0^{\mathrm{uf}}(X; \mathbb{R}) \neq 0$.*

On the other hand, it is not so clear how $H_0^{\mathrm{uf}}(X; \mathbb{R})$ looks in the amenable case. Moreover, higher-degree uniformly finite homology groups are not yet well understood; see Block and Weinberger [6] and Dranishnikov [12].

In this paper, we show that uniformly finite homology groups of amenable groups are infinite-dimensional in many cases. Our main result is:

Theorem 3.8 *Let $n \in \mathbb{N}$ and let G be a finitely generated amenable group. Let $H \leq G$ be a subgroup such that $[G : H] = \infty$ and that the map*

$$H_n(i)\colon H_n(H; \mathbb{R}) \longrightarrow H_n(G; \mathbb{R})$$

induced by the inclusion $i\colon H \hookrightarrow G$ is nontrivial. Then $\dim_{\mathbb{R}}(H_n^{\mathrm{uf}}(G; \mathbb{R})) = \infty$.

Therefore, for many amenable groups we can use regular group homology to deduce that uniformly finite homology is infinite-dimensional.

The zero-degree uniformly finite homology group has been used for a number of different applications. For instance, rigidity problems for uniformly discrete metric spaces with bounded geometry have been investigated with the help of uniformly finite homology; see Dymarz [15] and Whyte [23]. Vanishing classes in $H_0^{\mathrm{uf}}(X; \mathbb{R})$ have also been used to construct aperiodic tilings for nonamenable manifolds [5].

Higher-degree uniformly finite homology groups prove to be a powerful tool to investigate large-scale notions of dimensions. Dranishnikov [12; 13; 14] uses the "lifting" map $H_*(G; \mathbb{R}) \to H_*^{\mathrm{uf}}(G; \mathbb{R})$ to study the macroscopic dimensions of some manifolds.

We also present some examples that follow directly from our main theorem. For instance, from the calculation of group homology of nilpotent groups we can deduce:

Example 4.4 (Nilpotent groups) *Let G be a finitely generated virtually nilpotent group of Hirsch rank $h \in \mathbb{N}$. Then*

$$H_k^{\mathrm{uf}}(G; \mathbb{R}) = \begin{cases} \text{infinite-dimensional} & \text{if } k \in \{0, \ldots, h-1\}, \\ \mathbb{R} & \text{if } k = h, \\ 0 & \text{else.} \end{cases}$$

Our main technique for proving Theorem 3.8 is to use invariant means on G and the corresponding transfer maps to distinguish between different homology classes. This is straightforward in the degree-zero case and doing so we give a very simple proof that any infinite amenable group G has infinite-dimensional homology group $H_0^{\mathrm{uf}}(G; \mathbb{R})$. In order to extend this to the higher-degree case, we construct functions in $\ell^\infty(G)$ that are invariant with respect to a subgroup $H \leq G$ and can be distinguished by means.

We also define a subspace $\hat{H}_0^{\mathrm{uf}}(G; \mathbb{R}) \subset H_0^{\mathrm{uf}}(G; \mathbb{R})$ of classes that can *not* be detected by means and hence are different from the classes constructed before. In the last part of this article, we give a geometric condition for classes in uniformly finite homology to be in $\hat{H}_0^{\mathrm{uf}}(G; \mathbb{R})$. We then present a method of differentiating between such classes, following Whyte. Hence we can show:

Theorem 5.1 *Let G be a finitely generated infinite amenable group. Then*

$$\dim_{\mathbb{R}} \hat{H}_0^{\mathrm{uf}}(G; \mathbb{R}) = \infty.$$

The article is structured in the following way. In Section 2 we introduce uniformly finite homology both following Block and Weinberger and in terms of group homology with ℓ^∞ coefficients and discuss its properties most relevant for this article. Section 3 contains our main results and proofs. In Section 4 we present some examples and immediate applications of our main theorem. Finally, in Section 5 we introduce and study *sparse classes* in $H_0^{\mathrm{uf}}(G; \mathbb{R})$ and prove Theorem 5.1.

2 Basic facts and notation

In this section we fix some notation and present the main object we will investigate, namely uniformly finite homology. This was introduced by Block and Weinberger as a coarse homology invariant. The basic references for uniformly finite homology are [5; 23] and Nowak and Yu [17]. We will also define group homology with ℓ^∞ coefficients. For finitely generated groups there is a canonical isomorphism between the corresponding chain complexes (Remark 2.6). Hence, to study uniformly finite homology we will often use the description in terms of ℓ^∞ coefficients.

Definition 2.1 Let (X, d) be a metric space.

(i)   For each $n \in \mathbb{N}$ let $C_n^{\mathrm{uf}}(X; \mathbb{R})$ be the vector space of functions $c \colon X^{n+1} \to \mathbb{R}$ satisfying:

 (a)   The map c is bounded.

 (b)   For all $r \in \mathbb{R}_{>0}$ there exists a $K_r \in \mathbb{R}_{>0}$ (depending on c) such that for all $y \in X^{n+1}$
$$|\{x \in B_r(y) \mid c(x) \neq 0\}| \leq K_r.$$
   Here we consider X^{n+1} endowed with the metric
$$d_n(x, y) := \max_{i \in \{0,\dots,n\}} d(x_i, y_i).$$

 (c)   There exists an $R_c \in \mathbb{R}_{>0}$ (depending on c) such that for all $x \in X^{n+1}$ for which $\sup_{i,j \in \{0,\dots,n\}} d(x_i, x_j) > R_c$, we have $c(x) = 0$.

We will write such a function also as a formal sum $\sum_{x \in X^{n+1}} c(x) \cdot x$.

(ii) Define for each $n \in \mathbb{N}_{>0}$ a boundary operator

$$\partial_n \colon C_n^{\mathrm{uf}}(X;\mathbb{R}) \longrightarrow C_{n-1}^{\mathrm{uf}}(X;\mathbb{R})$$

by setting for each $x \in X$

$$\partial_n(x) = \sum_{j=0}^{n}(-1)^j (x_0,\ldots,\widehat{x}_j,\ldots,x_n)$$

and extending to $C_n^{\mathrm{uf}}(X;\mathbb{R})$ in the obvious way. In this fashion, we get indeed a chain complex.

(iii) The homology of $(C_n^{\mathrm{uf}}(X;\mathbb{R}), \partial_n)_{n\in\mathbb{N}}$ is called the *uniformly finite homology of X* and denoted by $H_*^{\mathrm{uf}}(X;\mathbb{R})$.

An important fact about uniformly finite homology is that it is a coarse invariant; see [5, Proposition 2.1].

Proposition 2.2 Let X and Y be metric spaces and $f \colon X \to Y$ a quasi-isometry. Then f induces a chain map

$$C_n^{\mathrm{uf}}(X;\mathbb{R}) \longrightarrow C_n^{\mathrm{uf}}(Y;\mathbb{R}),$$

$$\sum_{x\in X^{n+1}} c(x)\cdot x \longmapsto \sum_{x\in X^{n+1}} c(x)\cdot (f(x_0),\ldots,f(x_n)).$$

The induced map in homology

$$H_*^{\mathrm{uf}}(X;\mathbb{R}) \longrightarrow H_*^{\mathrm{uf}}(Y;\mathbb{R})$$

is an isomorphism in every degree.

In particular, we can define the uniformly finite homology of a group G with a fixed set of generators S, by endowing G with the word metric with respect to S. We recall the definition here.

Definition 2.3 Let G be a group with generating set S. The *word metric on G with respect to S* is the metric defined as

$$d_S(g,h) := \min\{n \in \mathbb{N} \mid \text{there exist } s_1,\ldots,s_n \in S \cup S^{-1} \text{ such that } g^{-1}\cdot h = s_1\cdots s_n\}$$

for any $g, h \in G$.

By Proposition 2.2 the uniformly finite homology of a finitely generated group does not depend on the finite generating set up to canonical isomorphism. Hence, this defines a quasi-isometry invariant for finitely generated groups. Notice that in this case condition (ib) in Definition 2.1 can be dropped.

Most important for us and many applications is the following result of Block and Weinberger [5, Theorem 3.1]:

Theorem 2.4 *Let G be a finitely generated group. Then G is amenable if and only if $H_0^{\mathrm{uf}}(G;\mathbb{R}) \neq 0$.*

We define now group homology with ℓ^∞ coefficients. Let G be a (discrete) group. Consider the $\mathbb{R}[G]$–chain complex $(C_*(G;\mathbb{R}), \partial_*)$ defined as follows:

(i) For any $n \in \mathbb{N}$, set

$$C_n(G;\mathbb{R}) := \bigoplus_{(g_0,\ldots,g_n)\in G^{n+1}} \mathbb{R}\cdot(g_0,\ldots,g_n)$$

with the G–action given by $g\cdot(g_0,\ldots,g_n) = (gg_0,\ldots,gg_n)$.

(ii) For any $n \in \mathbb{N}_{>0}$, let ∂_n be the boundary map

$$\partial_n\colon C_n(G;\mathbb{R}) \longrightarrow C_{n-1}(G;\mathbb{R}),$$

$$(g_0,\ldots,g_n) \longmapsto \sum_{j=0}^{n}(-1)^j(g_0,\ldots,\widehat{g}_j,\ldots,g_n).$$

Consider the space of real-valued bounded functions on G, $\ell^\infty(G)$; this is a left $\mathbb{R}[G]$–module with respect to the action

$$G \times \ell^\infty(G) \longrightarrow \ell^\infty(G),$$

$$(g,\varphi) \longmapsto g\cdot\varphi = (g' \longmapsto \varphi(g^{-1}g')).$$

Let $C_*(G;\ell^\infty(G))$ be the $\mathbb{R}$–chain complex given by

$$C_*(G;\ell^\infty(G)) := \bar{C}_*(G;\mathbb{R}) \otimes_{\mathbb{R}[G]} \ell^\infty(G),$$

where $\bar{C}_*(G;\mathbb{R})$ denotes the right $\mathbb{R}[G]$–module obtained by $C_*(G;\mathbb{R})$ via the canonical involution $g \mapsto g^{-1}$.

Definition 2.5 For a group G we define the *group homology of G with coefficients in $\ell^\infty(G)$* by $H_*(G;\ell^\infty(G)) := H_*(C_*(G;\ell^\infty(G)))$.

The following remark was observed by Brodzki, Niblo and Wright [7]:

Remark 2.6 For a finitely generated group G, the uniformly finite chain complex $(C_*^{\mathrm{uf}}(G;\mathbb{R}), \partial_*)$ is canonically chain isomorphic to $(C_*(G;\ell^\infty(G)), \partial_*)$.

Indeed, for any $n \in \mathbb{N}$, the simplices in a fixed uniformly finite n–chain c are tuples $(g_0, \dots, g_n) \in G^{n+1}$ having diameter less than a uniform constant $R_c \in \mathbb{R}_{>0}$. These simplices are contained in the G–orbits of the finitely many simplices of the form $(e, t_1, \dots, t_n)$ of diameter less than R_c. Hence, the following chain isomorphism is well defined:

$$\rho_n \colon C_n^{\mathrm{uf}}(G;\mathbb{R}) \longrightarrow C_n(G;\ell^\infty(G)),$$

$$\sum_{(g_0,\dots,g_n)\in G^{n+1}} c(g_0, \dots, g_n) \cdot (g_0, \dots, g_n) \longmapsto \sum_{(t_1,\dots,t_n)\in G^n} (e, t_1, \dots, t_n) \otimes \varphi_{(t_1,\dots,t_n)}.$$

Here for each $(t_1, \dots, t_n) \in G^n$ the map $\varphi_{(t_1,\dots,t_n)} \in \ell^\infty(G)$ is given by

$$\varphi_{(t_1,\dots,t_n)}(g) = c(g^{-1}, g^{-1} \cdot t_1, \dots, g^{-1} \cdot t_n).$$

In particular, $H_*^{\mathrm{uf}}(G;\mathbb{R}) \cong H_*(G;\ell^\infty(G))$. Therefore, certain aspects of uniformly finite homology are accessible through methods of ordinary group homology.

We recall the definition of amenable groups that will be the main class of groups we will study in this article.

Definition 2.7 Let G be a group. A *mean* is a linear map $m : \ell^\infty(G) \to \mathbb{R}$ satisfying the following properties:

  (i) We have $m(\chi_G) = 1$ for the characteristic function $\chi_G \in \ell^\infty(G)$ of the group.

  (ii) For any $\varphi \in \ell^\infty(G)$ such that $\varphi \geq 0$ (ie, $\varphi(g) \geq 0$ for all $g \in G$) we have $m(\varphi) \geq 0$.

A mean m is said to be *left G–invariant* if:

  (iii) For any $g \in G$ and any $\varphi \in \ell^\infty(G)$, we have $m(g \cdot \varphi) = m(\varphi)$.

Similarly one could define right G–invariant means.

Definition 2.8 A group G is *amenable* if it admits a left G–invariant mean.

Remark 2.9 The class of amenable groups contains for example all solvable, all finite and all subexponential groups and is closed under taking extensions, subgroups, quotients, etc; see Ceccherini-Silberstein and Coornaert [10] and Paterson [18].

For an amenable group G, let $M(G)$ be the set of left G–invariant means and let $LM(G)$ be its linear span in $\mathrm{Hom}_{\mathbb{R}}(\ell^\infty(G), \mathbb{R})$. We have the following result due to Chou [11]:

Theorem 2.10 *If G is an infinite amenable group, then G has exactly $2^{2^{|G|}}$ left invariant means, where $|G|$ denotes the cardinality of G. Thus $LM(G)$ is infinite-dimensional.*

3 Main results

In this section, we compute uniformly finite homology of finitely generated (discrete) amenable groups in many cases: in Theorem 3.7 we show that in degree zero this is always infinite-dimensional, while in Theorem 3.8 we calculate higher-degree uniformly finite homology.

3.1 Transfer via means

Let G be an amenable group. Every left G–invariant mean $m \in M(G)$ induces a transfer map $m_*\colon C_*(G; \ell^\infty(G)) \to C_*(G; \mathbb{R})$, which averages the coefficients. Our next proposition is similar to a result of Attie [2, Proposition 2.15]:

Proposition 3.1 *Let G be an amenable group and let $\chi_G \in \ell^\infty(G)$ be the characteristic function of G. Then every mean $m \in M(G)$ induces a transfer map*

$$m_*\colon H_*(G; \ell^\infty(G)) \longrightarrow H_*(G; \mathbb{R}),$$
$$[c \otimes \varphi] \longmapsto [m(\varphi) \cdot c],$$

which is a left inverse to the map

$$i_*\colon H_*(G; \mathbb{R}) \longrightarrow H_*(G; \ell^\infty(G)),$$
$$[c] \longmapsto [c \otimes \chi_G],$$

induced by the canonical inclusion $\mathbb{R} \hookrightarrow \ell^\infty(G)$ as constant functions.

Proof Consider $\mathbb{R}$ as G–module with the trivial action; then every $m \in M(G)$ is a G–equivariant map $m\colon \ell^\infty(G) \to \mathbb{R}$ and thus induces a change-of-coefficients chain map

$$m_*\colon C_*(G; \ell^\infty(G)) \longrightarrow C_*(G; \mathbb{R}),$$
$$c \otimes \varphi \longmapsto m(\varphi) \cdot c.$$

The inclusion $i\colon \mathbb{R} \to \ell^\infty(G)$ is also G–equivariant and induces a chain map

$$i_*\colon C_*(G;\mathbb{R}) \longrightarrow C_*(G;\ell^\infty(G)),$$

$$c \longmapsto c \otimes \chi_G.$$

Obviously, for every mean $m \in M(G)$ the map m_* is a left inverse for i_*. Applying homology proves the claim. $\square$

Remark 3.2 Under the canonical identification between uniformly finite homology and homology with ℓ^∞ coefficients for finitely generated groups, the maps m_* defined in Proposition 3.1 translate (by precomposition with ρ_*) to transfer maps for uniformly finite homology that we denote by $\bar{m}_*$.

Hence, for any finitely generated amenable group there is an inclusion

$$H_*(G;\mathbb{R}) \lhook\joinrel\longrightarrow H_*^{\mathrm{uf}}(G;\mathbb{R}).$$

Remark 3.3 Given $m \in M(G)$, in degree zero we have $C_0(G;\ell^\infty(G)) \cong \ell^\infty(G)$ and the transfer map m_0 coincides with the mean m. Also $C_0^{\mathrm{uf}}(G;\mathbb{R}) \cong \ell^\infty(G)$ and ρ_0 is just the canonical inversion

$$\ell^\infty(G) \longrightarrow \ell^\infty(G),$$

$$\varphi \longmapsto (g \longmapsto \varphi(g^{-1})).$$

Therefore the transfer map $\bar{m}_0$ on $C_0^{\mathrm{uf}}(G;\mathbb{R})$ coincides with the corresponding right-invariant mean.

Definition 3.4 The *uniformly finite homological dimension* of a group G is defined by

$$\mathrm{hd}_{\mathrm{uf}}\,G = \sup\{n \in \mathbb{N} \mid H_n^{\mathrm{uf}}(G;\mathbb{R}) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

Because uniformly finite homology is invariant under quasi-isometry, we obtain the following fact as an immediate corollary of Proposition 3.1.

Corollary 3.5 *The Hirsch rank of a finitely generated virtually nilpotent group G equals $\mathrm{hd}_{\mathrm{uf}}(G)$ and is therefore a quasi-isometry invariant.*

Proof We can assume G to be nilpotent; indeed, by definition, the Hirsch rank of a virtually nilpotent group is the Hirsch rank of any nilpotent subgroup $G' \leq G$ of finite index, and since G is finitely generated it is quasi-isometric to G'. Clearly $\mathrm{hd}_{\mathrm{uf}}\,G \leq \mathrm{hd}_\mathbb{R}\,G$, where $\mathrm{hd}_\mathbb{R}$ denotes the homological dimension of G.

Conversely, for a finitely generated nilpotent group the homological dimension coincides with the largest integer n for which $H_n(G; \mathbb{R}) \neq 0$ and with its Hirsch rank; see Stammbach [21]. In view of the inclusion $H_*(G; \mathbb{R}) \hookrightarrow H_*^{\mathrm{uf}}(G; \mathbb{R})$ obtained in Proposition 3.1, this integer must be smaller than or equal to $\mathrm{hd}_{\mathrm{uf}}\, G$. □

3.2 Degree zero

Definition 3.6 We call the subspace

$$\widehat{H}_0^{\mathrm{uf}}(G; \mathbb{R}) := \{c \in H_0^{\mathrm{uf}}(G; \mathbb{R}) \mid \text{for all } m \in M(G),\ \bar{m}_0(c) = 0\}$$

the *mean-invisible part of* $H_0^{\mathrm{uf}}(G; \mathbb{R})$.

Theorem 3.7 *Let* G *be a finitely generated infinite amenable group. Then*

$$\dim_{\mathbb{R}} H_0^{\mathrm{uf}}(G; \mathbb{R}) / \widehat{H}_0^{\mathrm{uf}}(G; \mathbb{R}) = \infty.$$

In particular, $\dim_{\mathbb{R}} H_0^{\mathrm{uf}}(G; \mathbb{R}) = \infty$.

Proof By definition we have the canonical inclusion

$$LM(G) \subseteq \ell^{\infty}(G)^* = C_0(G; \ell^{\infty}(G))^*.$$

Since this inclusion corresponds to the construction of the chain maps in Proposition 3.1 in degree 0, this induces a well-defined injection

$$LM(G) \hookrightarrow H_0(G; \ell^{\infty}(G))^*,$$
$$m \longmapsto ([g \otimes \varphi] \longmapsto m(\varphi)).$$

In view of Remarks 2.6 and 3.2, this translates into an inclusion

$$LM(G) \hookrightarrow H_0^{\mathrm{uf}}(G; \mathbb{R})^*,$$
$$m \longmapsto ([c] \longmapsto \bar{m}_0(c)).$$

By the definition of $\widehat{H}_0^{\mathrm{uf}}(G; \mathbb{R})$ this also induces an injection

$$LM(G) \hookrightarrow (H_0^{\mathrm{uf}}(G; \mathbb{R}) / \widehat{H}_0^{\mathrm{uf}}(G; \mathbb{R}))^*.$$

Hence, by Theorem 2.10 we have $\dim_{\mathbb{R}} H_0^{\mathrm{uf}}(G; \mathbb{R}) / \widehat{H}_0^{\mathrm{uf}}(G; \mathbb{R}) = \infty$. □

3.3 Higher degrees

Now we consider uniformly finite homology in higher degrees. We construct infinitely many different classes coming from elements in $\ell^\infty(G)$ that are invariant with respect to the action of an infinite index subgroup.

Theorem 3.8 *Let $n \in \mathbb{N}$ and let G be a finitely generated amenable group. Let $H \leq G$ be a subgroup such that $[G : H] = \infty$ and that the map*

$$H_n(i) \colon H_n(H; \mathbb{R}) \longrightarrow H_n(G; \mathbb{R})$$

induced by the inclusion $i \colon H \hookrightarrow G$ is nontrivial. Then $\dim_\mathbb{R}(H_n^{\mathrm{uf}}(G; \mathbb{R})) = \infty$.

The idea behind the proof of Theorem 3.8 is to construct a family of infinitely many means on G that can be distinguished by a family of H–invariant functions (Theorem 3.11). Following Mitchell [16], in Lemma 3.10 we give a condition for a subset $\Gamma \subset G$ to support a left G–invariant mean m such that $m(\chi_\Gamma) = 1$. We construct infinitely many such subsets and we separate them using Lemma 3.9.

We will also give an alternative proof of Theorem 3.8 in the case that H is a normal subgroup of G.

Lemma 3.9 *Let G be an amenable group and $H \leq G$ such that $[G : H] = \infty$. Let $\pi \colon G \to H\backslash G$ be the canonical projection. Then for any pair of finite subsets $T, T' \subseteq G$ there exists $g \in G$ such that $\pi(T \cdot g) \cap \pi(T') = \varnothing$.*

Proof For each $g \in G$ such that $\pi(T \cdot g) \cap \pi(T') \neq \varnothing$ there exist $t \in T$ and $t' \in T'$ such that $tg \in Ht'$, hence $g \in t^{-1}Ht'$. Thus, if $\pi(T \cdot g) \cap \pi(T') \neq \varnothing$ for all $g \in G$, then $G = T^{-1}HT'$. Since G is amenable, there exists a bi-G–invariant mean m, inducing a finitely additive probability measure μ on the power set of G, given by $\mu(A) = m(\chi_A)$, [10, Proposition 4.4.4]. Moreover, since H has infinite index in G it follows that $\mu(H) = 0$. Then we have

$$1 = \mu(G) = \mu(T^{-1}HT') \leq \sum_{t \in T, t' \in T'} \mu(t^{-1}Ht') = \sum_{t \in T, t' \in T'} \mu(H) = 0,$$

hence a contradiction. $\square$

Lemma 3.10 *Let G be a finitely generated amenable group and $\Gamma \leq G$ a subset. Assume that for each finite set $F \subset G$ there exists $g \in G$ such that $F \cdot g \subset \Gamma$. Then there exists a left G–invariant mean m on G such that $m(\chi_\Gamma) = 1$.*

Proof Let $(F_n)_{n\in\mathbb{N}}$ be a (left) Følner sequence in G (Definition 5.5). Choose for each $n \in \mathbb{N}$ a $g_n \in G$, such that $F_n \cdot g_n \subset \Gamma$. Then $(F_n \cdot g_n)_{n\in\mathbb{N}}$ is also a left Følner sequence, since for all $h \in G$ and all $n \in \mathbb{N}$

$$\frac{|h \cdot F_n \cdot g_n \setminus F_n \cdot g_n|}{|F_n \cdot g_n|} = \frac{|h \cdot F_n \setminus F_n|}{|F_n|}.$$

For each $n \in \mathbb{N}$ consider the function

$$m_n \colon \ell^\infty(G) \longrightarrow \mathbb{R},$$

$$x \longmapsto \frac{1}{|F_n|} \sum_{h \in F_n} x(h \cdot g_n).$$

By [10, Theorem 4.9.2] there is a subsequence of $(m_n)_{n\in\mathbb{N}}$ that converges in the weak-$*$–topology to a left G–invariant mean m. Since $m_n(\chi_\Gamma) = 1$ for all $n \in \mathbb{N}$, we also have $m(\chi_\Gamma) = 1$. $\qquad\square$

Theorem 3.11 *Let G be a finitely generated amenable group and $H \leq G$ a subgroup such that $[G : H] = \infty$. Then there exists an infinite family $(m_j)_{j \in J}$ of left G–invariant means and an infinite family $(f_j)_{j \in J}$ of (left) H–invariant functions in $\ell^\infty(G)$ such that $m_k(f_j) = \delta_{k,j}$ for any $k, j \in J$.*

Proof Let $n \in \mathbb{N}$ and let $\pi \colon G \to H\backslash G$ be the canonical projection. Consider G equipped with the word metric with respect to a (finite) set of generators. We inductively construct finite sets $A_l^k \subseteq G$ for all $k \in \{1, \dots, n\}$ and for all $l \in \mathbb{N}$ such that:

- The family $(\pi(A_l^k))_{k \in \{1,\dots,n\}, l \in \mathbb{N}}$ is pairwise disjoint.

- For all $k \in \{1, \dots, n\}$ and $l \in \mathbb{N}$ there exists $g \in G$ such that $A_l^k = B_l(e) \cdot g$, where $B_l(e)$ denotes the ball of radius l centered at the identity element.

Let $A_1^1 := B_1(e)$; assume that the sets have been constructed for all indices smaller than or equal to (l, k) (using lexicographic order). Then:

- If $k < n$, by Lemma 3.9 there exists $g \in G$ such that

$$\pi\left(\bigcup_{(l',k')\leq(l,k)} A_{l'}^{k'}\right) \cap \pi(B_l(e) \cdot g) = \varnothing.$$

  Set $A_l^{k+1} := B_l(e) \cdot g$.

- If $k = n$, by Lemma 3.9 there exists $g \in G$ such that

$$\pi \left(\bigcup_{(l',k') \leq (l,k)} A_{l'}^{k'} \right) \cap \pi(B_{l+1}(e) \cdot g) = \varnothing.$$

Set $A_{l+1}^1 := B_{l+1}(e) \cdot g$.

Now set for all $k \in \{1, \dots, n\}$

$$T^k := \bigcup_{l \in \mathbb{N}} H A_l^k.$$

Notice that $T^1, \dots, T^n$ are pairwise disjoint by construction.

For any finite subset $F \subseteq G$, there exists $r \in \mathbb{N}$ such that $F \subseteq B_r(e)$, hence for all $k \in \{1, \dots, n\}$ there exists $g \in G$ such that $F \cdot g \subseteq B_r(e) \cdot g \subseteq T^k$. So by Lemma 3.10, for each $k \in \{1, \dots, n\}$ there exists a left G–invariant mean m_k such that $m_k(\chi_{T^k}) = 1$. Moreover, for each $k \in \{1, \dots, n\}$ and for each $j \in \{1, \dots, n\} \setminus \{k\}$ we have that $m_k(\chi_{T^j}) = 0$ since the sets $T^1, \dots, T^n$ are pairwise disjoint and $m_k(G) = 1$. By definition of T^k, the functions χ_{T^k} are H–invariant for any $k \in \{1, \dots, n\}$.

We have constructed a family of left G–invariant means $\{m_k\}_{k \in \{1,\dots,n\}}$ and a family of bounded H–invariant functions $\{\chi_{T^k}\}_{k \in \{1,\dots,n\}}$ on G. We can repeat the same construction for any $n \in \mathbb{N}$, so we can have an arbitrarily large finite family of means and of functions satisfying the theorem. Using a slightly different induction step, it is possible to construct a family of finite sets $A_l^k \subseteq G$ for all $k \in \mathbb{N}$ and for all $l \in \mathbb{N}$ as above. In this way we are able to construct directly an infinite family of means and of functions satisfying the theorem.   □

We are now ready to prove our main theorem.

First proof of Theorem 3.8 If $\dim_{\mathbb{R}} H_n(G; \mathbb{R}) = \infty$ then by Proposition 3.1 we immediately conclude that $H_n^{\mathrm{uf}}(G; \mathbb{R})$ must be infinite-dimensional.

So we assume $\dim_{\mathbb{R}} H_n(G; \mathbb{R}) < \infty$. We prove the statement for $H_n(G; \ell^\infty(G))$ and in view of Remark 2.6 the theorem will also follow for $H_n^{\mathrm{uf}}(G; \mathbb{R})$.

Let $c \in C_n(H; \mathbb{R})$ be a cycle such that $[i(c)] \in H_n(G; \mathbb{R})$ is nontrivial. Consider the space $\ell^\infty(G)^H := \{\varphi \in \ell^\infty(G) \mid \text{for all } h \in H, \ h \cdot \varphi = \varphi\}$. Then

$$S_c := \{i(c) \otimes \varphi \in C_n(G; \ell^\infty(G)) \mid \varphi \in \ell^\infty(G)^H\}$$

is a subspace of cycles in $C_n(G; \ell^\infty(G))$. Indeed, for every $\varphi \in \ell^\infty(G)^H$ one can define the H–equivariant map $f_\varphi \colon \mathbb{R} \to \ell^\infty(G)$, $1 \mapsto \varphi$, which induces a chain map $C_n(i, f_\varphi) \colon C_n(H; \mathbb{R}) \to C_n(G; \ell^\infty(G))$ that maps c to $i(c) \otimes \varphi$.

Let $(m_j)_{j \in J}$ and $(f_j)_{j \in J}$ be as in Theorem 3.11; since for all $j \in J$, the map f_j is H–left invariant, the elements $(i(c) \otimes f_j)_{j \in J} \in C_n(G; \ell^\infty(G))$ belong to S_c, so they are all cycles in $C_n(G; \ell^\infty(G))$. Now consider the family of induced transfer maps $(m_{j*})_{j \in J} \in \mathrm{Hom}_\mathbb{R}(H_n(G; \ell^\infty(G)), H_n(G; \mathbb{R}))$. For any $k, j \in J$, by Theorem 3.11 $m_{k*}([i(c) \otimes f_j]) = \delta_{i,j} \cdot [i(c)]$, hence the family $(m_{j*})_{j \in J}$ is linearly independent in $\mathrm{Hom}_\mathbb{R}(H_n(G; \ell^\infty(G)), H_n(G; \mathbb{R}))$. Thus $H_n(G; \ell^\infty(G))$ is infinite-dimensional. $\square$

Now we give a more direct proof of the main result under the additional assumption that H is normal in G.

Second proof of Theorem 3.8 Let H now be normal in G. Since H is a subgroup of the amenable group G, it is also amenable. Let m_0 be a left H–invariant mean. Consider the transfer map

$$\tau \colon \ell^\infty(G) \longrightarrow \ell^\infty(G/H),$$

$$\varphi \longmapsto \big(gH \longmapsto m_0((g^{-1} \cdot \varphi)|_H)\big).$$

For any $\varphi \in \ell^\infty(G)$, the map $\tau(\varphi)$ is well defined. Indeed, let g_1, g_2 be elements of G such that $g_1 H = g_2 H$; then $g_2 = g_1 h$ for some $h \in H$, hence

$$\tau(\varphi)(g_2 H) = m_0((g_2^{-1} \cdot \varphi)|_H) = m_0((h^{-1} g_1^{-1} \cdot \varphi)|_H) = m_0((g_1^{-1} \cdot \varphi)|_H) = \tau(\varphi)(g_1 H)$$

by the H–invariance of m_0. It is clear that $\tau(\varphi)$ is bounded for any $\varphi \in \ell^\infty(G)$.

We also have a map induced by the canonical projection $\pi \colon G \to G/H$,

$$\pi^* \colon \ell^\infty(G/H) \longrightarrow \ell^\infty(G),$$

$$\psi \longmapsto \psi \circ \pi.$$

It is easy to see that $\mathrm{Im}(\pi^*) \subseteq \ell^\infty(G)^H$; indeed, for any $\psi \in \ell^\infty(G/H)$, any $h \in H$ and any $g \in G$,

$$h \cdot \pi^*(\psi)(g) = \pi^*(\psi)(h^{-1} g) = \psi(h^{-1} g H) = \psi(g H).$$

Notice that the equality $h^{-1} g H = g H$ holds since we have assumed H to be normal in G. The composition

$$\ell^\infty(G/H) \xrightarrow{\pi^*} \ell^\infty(G)^H \xrightarrow{\tau|_{\ell^\infty(G)^H}} \ell^\infty(G/H)$$

is the identity on $\ell^\infty(G/H)$. Indeed, for any $\psi \in \ell^\infty(G/H)$ and any class $gH \in G/H$ we have

$$\tau(\psi \circ \pi)(gH) = m_0(g^{-1} \cdot (\psi \circ \pi))|_H) = m_0(h \mapsto \psi(gH)) = \psi(gH),$$

since the function $h \mapsto \psi(gH)$ is constant. In particular, $\tau|_{\ell^\infty(G)^H}$ is surjective. It is easy to see that $\tau(\chi_G) = 1$ and that $\tau(\varphi) \geq 0$ for any $\varphi \geq 0$ in $\ell^\infty(G)$. Moreover, τ is G–equivariant: indeed, for any $g, g' \in G$ and any $\varphi \in \ell^\infty(G)$, we have

$$\tau(g' \cdot \varphi)(gH) = m_0((g^{-1} \cdot g' \cdot \varphi)|_H) = m_0(((g'^{-1}g)^{-1} \cdot \varphi)|_H)$$
$$= \tau(\varphi)(g'^{-1}gH) = g' \cdot \tau(\varphi)(gH).$$

By dualising τ we get a map

$$\tau^* \colon \ell^\infty(G/H)^* \longrightarrow \ell^\infty(G)^*,$$
$$F \longmapsto F \circ \tau.$$

Since H is a normal subgroup of G with infinite index, G/H is also an infinite amenable group, so restricting τ^* to $LM(G/H)$ we have a well-defined map

$$\tau^* \colon LM(G/H) \longrightarrow LM(G).$$

Indeed, let $m \in M(G/H)$ be a mean on G/H. Then, since $\tau(\chi_G) = 1$ and $\tau(\varphi) \geq 0$ for any $\varphi \geq 0$ in $\ell^\infty(G)$, it immediately follows that $\tau^*(m)$ satisfies conditions (i) and (ii) of Definition 2.7. Condition (iii) follows by the G–equivariance of τ, which implies that $\tau^*(m)$ is a left G–invariant mean.

Let $[S_c]$ be the space of classes represented by elements of S_c as defined in the first proof of Theorem 3.8. Now consider the diagram

$$
\begin{array}{ccc}
LM(G) & \xrightarrow{\ \ \Phi\ \ } & \mathrm{Hom}_\mathbb{R}(H_n(G;\ell^\infty(G)), H_n(G;\mathbb{R})) \\
{\scriptstyle \tau^*}\Big\uparrow & & \Big\downarrow{\scriptstyle \mathrm{res}} \\
LM(G/H) & \xrightarrow{\ \ \overline{\Phi}\ \ } & \mathrm{Hom}_\mathbb{R}([S_c], H_n(G;\mathbb{R}))
\end{array}
$$

where the upper horizontal map Φ takes every mean $m \in M(G)$ to the induced homomorphism m_*. Here res denotes the restriction map and the lower horizontal map is given by composition.

Since $H_n(G;\mathbb{R})$ can be assumed to be finite-dimensional, to prove the theorem it suffices to show that $\overline{\Phi}$ is injective; indeed, by Theorem 2.10 of Chou, since G/H is infinite amenable, $LM(G/H)$ is infinite-dimensional and from the injectivity of $\overline{\Phi}$ it would follow that $[S_c] \subseteq H_n(G;\ell^\infty(G))$ must be infinite-dimensional.

We now show that $\overline{\Phi}$ is injective. So, consider two means $m_1 \neq m_2 \in LM(G/H)$; then there exists $\psi \in \ell^\infty(G/H)$ such that $m_1(\psi) \neq m_2(\psi)$. By surjectivity of $\tau|_{\ell^\infty(G)^H}$ we know that there exists $\varphi \in \ell^\infty(G)^H$ such that $\tau(\varphi) = \psi$. Consider now the image of $[i(c) \otimes \varphi] \in [S_c]$ by the homomorphisms $\overline{\Phi}(m_1), \overline{\Phi}(m_2) \in \mathrm{Hom}([S_c], H_n(G; \mathbb{R}))$. We have for $k \in \{1, 2\}$

$$\overline{\Phi}(m_k)([i(c) \otimes \varphi]) = \Phi(\tau^*(m_k))([i(c) \otimes \varphi]) = [m_k(\tau(\varphi)) \cdot i(c)] = m_k(\psi) \cdot [i(c)].$$

Since $m_1(\psi) \neq m_2(\psi)$ and $[i(c)] \neq 0$, the two classes $m_1(\psi) \cdot [i(c)]$ and $m_2(\psi) \cdot [i(c)]$ are also different in $H_n(G; \mathbb{R})$, which implies that $\overline{\Phi}(m_1) \neq \overline{\Phi}(m_2)$; in particular, $\overline{\Phi}$ is injective. $\qquad \square$

4 Examples

Corollary 4.1 *Let G be a finitely generated amenable group. Assume that $H_1(G; \mathbb{R})$ is nontrivial, ie that the abelianization of G is not a torsion group. Then*

$$H_1^{\mathrm{uf}}(G; \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } G \text{ is virtually } \mathbb{Z}, \\ \text{infinite-dimensional} & \text{otherwise.} \end{cases}$$

Proof Let $g \in G$ be an element such that $(1, g) \in H_1(G; \mathbb{R})$ is a nontrivial cycle. Such a g exists by assumption and the isomorphism $H_1(G; \mathbb{R}) \cong G_{\mathrm{ab}} \otimes \mathbb{R}$. If G is not virtually $\mathbb{Z}$, the claim follows from Theorem 3.8, since then $[G : \langle g \rangle] = \infty$. And for the virtually $\mathbb{Z}$ case see the following example. $\qquad \square$

Example 4.2 Let G be a finitely generated infinite amenable group. Consider semidirect products of the form $G \rtimes \mathbb{Z}^l$ for $l \in \mathbb{N}$. Then for all $k \in \{0, \ldots, l\}$,

$$\dim_{\mathbb{R}} H_k^{\mathrm{uf}}(G \rtimes \mathbb{Z}^l; \mathbb{R}) = \infty.$$

Similar results hold if one replaces $\mathbb{Z}^l$ by an amenable group with nonvanishing homology in the correct degrees. In particular, for all $l \in \mathbb{N}$

$$H_k^{\mathrm{uf}}(\mathbb{Z}^l; \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } k = l, \\ \text{infinite-dimensional} & \text{if } k \in \{0, \ldots, l-1\}, \\ 0 & \text{else.} \end{cases}$$

Proof The splitting map $\mathbb{Z}^l \hookrightarrow G \rtimes \mathbb{Z}^l$ induces a nontrivial map in degree $0, \ldots, l$ in homology and hence the first part follows from Theorem 3.8. The product $\mathbb{Z}^{l-1} \times \mathbb{Z}$ is a special case. The degrees $k \geq l$ follow because the group is a Poincaré duality group and $H^0(G; \ell^\infty(G)) \cong \ell^\infty(G)^G \cong \mathbb{R}$ for all groups G. $\qquad \square$

Example 4.3 For the integral three-dimensional Heisenberg group Heis_3 we get

$$H_k^{\text{uf}}(\text{Heis}_3;\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } k = 3, \\ \text{infinite-dimensional} & \text{if } k \in \{0, 1, 2\}, \\ 0 & \text{else.} \end{cases}$$

Proof We only have to consider $k = 2$, since the cases $k = 0, 1$ follow directly from Theorem 3.7 and Corollary 4.1. The higher degrees are a consequence of Poincaré duality. Consider the presentation $\text{Heis}_3 \cong \langle x, y, z \,|\, [x, y], [y, z], y^{-1}[x, y]\rangle$. By Hopf's theorem (see Brown [8, II.5, Theorem 3]), to compute the second homology it suffices to look at the Schur multiplier which, in this case, is generated by the symbols $[x, y]$, $[y, z]$. Hence the inclusion of the subgroup generated by x and y is nontrivial in homology in degree 2, so we can apply Theorem 3.8 again. $\square$

Actually, the last examples are simple special cases of a more general result:

Example 4.4 Let G be a finitely generated virtually nilpotent group of Hirsch rank $h \in \mathbb{N}$. Then

$$H_k^{\text{uf}}(G;\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } k = h, \\ \text{infinite-dimensional} & \text{if } k \in \{0, \dots, h-1\}, \\ 0 & \text{else.} \end{cases}$$

Proof We closely follow the calculation of the homology groups of nilpotent groups of Baumslag, Miller and Short [4] to see that our condition in Theorem 3.8 is satisfied in degrees $0, \dots, h-1$. After passing to a finite-index subgroup, we may assume G is torsion-free and nilpotent. Then we can write G as a split extension of the form

$$1 \longrightarrow N \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 1$$

for $N \subset G$ a normal subgroup of Hirsch rank $h-1$. The Hochschild–Serre spectral sequence for this split extension induces a short exact sequence

$$0 \longrightarrow H_0(\mathbb{Z}; H_i(N;\mathbb{R})) \longrightarrow H_i(G;\mathbb{R}) \longrightarrow H_1(\mathbb{Z}; H_{i-1}(N;\mathbb{R})) \longrightarrow 0$$

for all $i \in \mathbb{N}_{>0}$. The map on the left-hand side is one edge map of the spectral sequence and is induced by the canonical map $H_i(N;\mathbb{R}) \to H_i(G;\mathbb{R})$ under the identification $H_i(N;\mathbb{R})_{\mathbb{Z}} \cong H_0(\mathbb{Z}, H_i(N;\mathbb{R}))$; see Weibel [22]. In particular, the canonical map $H_i(N;\mathbb{R}) \to H_i(G;\mathbb{R})$ is nontrivial if $H_i(N;\mathbb{R})_{\mathbb{Z}}$ is nontrivial. But these homology groups are nontrivial [4, proof of Theorem 16]. The degrees $k \geq h$ follow since finitely generated nilpotent groups are Poincaré duality groups [8, VIII.10, Example 1]. $\square$

Example 4.5 Consider $A \in \mathrm{Sl}(2, \mathbb{Z})$. Then for the semidirect product $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ given by the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ induced by A, we have

$$H_k^{\mathrm{uf}}(\mathbb{Z}^2 \rtimes_A \mathbb{Z}; \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } k = 3, \\ \text{infinite-dimensional} & \text{if } k \in \{0, 1, 2\}, \\ 0 & \text{else.} \end{cases}$$

In particular, this example includes cocompact lattices in Sol [20].

Proof This is similar to Example 4.4. We only have to consider $k = 2$; the other cases follow as in the other examples. By the Hochschild–Serre spectral sequence we get a short exact sequence

$$0 \longrightarrow H_0(\mathbb{Z}; H_2(\mathbb{Z}^2; \mathbb{R})) \longrightarrow H_2(\mathbb{Z}^2 \rtimes_A \mathbb{Z}; \mathbb{R}) \longrightarrow H_1(\mathbb{Z}; H_1(\mathbb{Z}^2; \mathbb{R})) \longrightarrow 0.$$

By Poincaré duality and the universal coefficient theorem, the dimension of the middle term is equal to $\dim_{\mathbb{R}} H_1(\mathbb{Z} \rtimes_A \mathbb{Z}; \mathbb{R}) = 3$. Since the dimension of the term at the right-hand-side is at most 2, the map $H_0(\mathbb{Z}; H_2(\mathbb{Z}^2; \mathbb{R})) \to H_2(\mathbb{Z}^2 \rtimes_A \mathbb{Z}; \mathbb{R})$ is nontrivial. In particular the canonical map $H_2(\mathbb{Z}^2; \mathbb{R}) \to H_2(\mathbb{Z}^2 \rtimes_A \mathbb{Z}; \mathbb{R})$ is also nontrivial. Hence $H_2^{\mathrm{uf}}(\mathbb{Z}^2 \rtimes_A \mathbb{Z}; \mathbb{R})$ is infinite-dimensional by Theorem 3.8. $\square$

There are many well-known examples of finitely generated amenable groups having nontrivial real homology in each degree. If G is such a group, the group $G \times \mathbb{Z}$ satisfies $\dim_{\mathbb{R}} H_n^{\mathrm{uf}}(G \times \mathbb{Z}; \mathbb{R}) = \infty$ for all $n \in \mathbb{N}$ by Theorem 3.8. For instance:

Example 4.6 There exists a finitely presented metabelian group G such that for all $n \in \mathbb{N}$,

$$\dim_{\mathbb{R}} H_n^{\mathrm{uf}}(G; \mathbb{R}) = \infty.$$

Proof For example, Baumslag and Dyer have shown [3] that Baumslag's metabelian group $B := \langle a, s, t \mid a^s = a \cdot a^t, [a, a^t] = 1 = [s, t] \rangle$ has nontrivial homology in degree $n \geq 3$. Set $G := B \times \mathbb{Z}^3$. $\square$

Example 4.7 Consider Thompson's group $F := \langle x_0, x_1, \ldots \mid x_n^{x_i} = x_{n+1} \text{ for } i < n \rangle$. We have

$$F \text{ amenable} \quad \Longrightarrow \quad \text{for all } n \in \mathbb{N}, \ \dim_{\mathbb{R}} H_n^{\mathrm{uf}}(F; \mathbb{R}) = \infty.$$

Proof We follow Kenneth Brown's calculation of the homology of F [9]. Consider F as the group of dyadic piecewise linear homeomorphisms $[0, 1] \to [0, 1]$.

Brown observes that in this description the commutator subgroup of F can be seen as $F' = \{f \in F \mid f'(0) = 1 = f'(1)\}$. There is a product map

$$*\colon F \times F \longrightarrow F,$$

$$(f, g) \longmapsto \left(t \longmapsto \begin{cases} f(2 \cdot t)/2 & 0 \leq t \leq \frac{1}{2} \\ g(2 \cdot t - 1)/2 & \frac{1}{2} \leq t \leq 1 \end{cases}\right)$$

Define the subgroup $G := \{f \in F \mid f'(0) = 1\} \leq F$. The product map restricts both to F' and G and induces a natural product in homology on $H_*(F'; \mathbb{R})$, $H_*(G; \mathbb{R})$ and $H_*(F; \mathbb{R})$.

Brown [9, Theorem 4.1] shows that $H_*(F; \mathbb{R})$ is generated as a ring by elements $\varepsilon, \alpha, \beta$, that alternating products $\alpha \cdot \beta \cdot \alpha \cdots$ and $\beta \cdot \alpha \cdot \beta \cdots$ form a basis of $H_*(F; \mathbb{R})$ in positive degree and that the image of $H_*(F'; \mathbb{R}) \to H_*(F; \mathbb{R})$ contains $\alpha \cdot \beta \in H_2(F; \mathbb{R})$ and hence is nontrivial in even degrees.

Since F' is strictly contained in G, the image of the map $H_*(G; \mathbb{R}) \to H_*(F; \mathbb{R})$ contains a nontrivial linear combination of α and β (since they form a basis of $H_1(F; \mathbb{R})$), hence the map $H_*(G; \mathbb{R}) \to H_*(F; \mathbb{R})$ is nontrivial in each degree and we may apply Theorem 3.8. □

5 Sparse classes in $H_0^{\mathrm{uf}}(G; \mathbb{R})$

In this section, we give a geometric condition for classes in $H_0^{\mathrm{uf}}(G; \mathbb{R})$ to be mean-invisible and show that in an infinite amenable group, there are infinitely many linear independent classes of this type. In particular, we show the following.

Theorem 5.1 *Let G be a finitely generated infinite amenable group. Then*

$$\dim_{\mathbb{R}} \hat{H}_0^{\mathrm{uf}}(G; \mathbb{R}) = \infty.$$

In this section, we always consider finitely generated groups equipped with a word metric with respect to some finite generating set (Definition 2.3).

5.1 Distinguishing classes by asymptotic behavior

Let G be a finitely generated discrete group with a word metric d. For any $S \subseteq G$ let

$$\partial_r(S) := \{g \in G \mid 0 < d(g, S) \leq r\}.$$

The following theorem due to Whyte [23, Theorem 7.6] gives a characterization of trivial classes in uniformly finite homology in degree 0.

Theorem 5.2 *Let G be a finitely generated group. Let c be a cycle in $C_0^{\mathrm{uf}}(G;\mathbb{R})$. Then $[c] = 0 \in H_0^{\mathrm{uf}}(G;\mathbb{R})$ if and only if there exist $C, r \in \mathbb{N}_{>0}$ such that for all $S \subseteq G$ finite,*

$$\left| \sum_{s \in S} c(s) \right| \leq C \cdot |\partial_r S|.$$

Remark 5.3 Whyte states Theorem 5.2 for uniformly finite homology with $\mathbb{Z}$ coefficients. However, for infinite finitely generated groups in degree zero, there is an isomorphisms of uniformly finite homology with $\mathbb{Z}$ coefficients and uniformly finite homology with $\mathbb{R}$ coefficients induced by the canonical inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ [23, Lemma 7.7]. It is easy to see that this isomorphism translates the original statement of Whyte to Theorem 5.2.

Whyte's criterion can be reformulated in terms of the asymptotic behavior of a degree zero chain in comparison to the behavior of a Følner sequence. First, we introduce a notion to compare the asymptotic behavior of functions:

Definition 5.4 Let $\alpha, \beta \colon \mathbb{N} \to \mathbb{R}_{>0}$ be two functions.

(i) We write $\alpha \prec \beta$ if

$$\lim_{n \to \infty} \frac{\alpha(n)}{\beta(n)} = 0.$$

(ii) We write $\alpha \sim \beta$ if

$$\lim_{n \to \infty} \frac{\alpha(n)}{\beta(n)} \in \mathbb{R}_{>0}.$$

(iii) We also write $\alpha \preceq \beta$ if $\alpha \sim \beta$ or $\alpha \prec \beta$.

We recall the definition of a Følner sequence for finitely generated groups. Also recall that a finitely generated group is amenable if and only if it admits a Følner sequence [10, Theorem 4.9.2].

Definition 5.5 Let G be a finitely generated group. A *Følner sequence* in G is a sequence $(S_j)_{j \in \mathbb{N}}$ of nonempty finite subsets of G such that for each $r \in \mathbb{R}_{>0}$

$$\lim_{j \to \infty} \frac{|\partial_r(S_j)|}{|S_j|} = 0.$$

Now fix a Følner sequence $S := (S_j)_{j \in \mathbb{N}}$ of a finitely generated amenable group G.

For all $c \in \ell^\infty(G)$ we define a function

$$\beta_c^S \colon \mathbb{N} \longrightarrow \mathbb{R},$$

$$n \longmapsto \frac{\left|\sum_{s \in S_n} c(s)\right|}{|S_n|}.$$

Finally, we also consider the behavior of the boundaries

$$\sigma_S \colon \mathbb{N} \longrightarrow \mathbb{R}_{>0},$$

$$n \longmapsto \frac{|\partial_1 S_n|}{|S_n|}.$$

By comparing the behavior of these functions, we can distinguish between classes in $H_0^{\mathrm{uf}}(G; \mathbb{R})$:

Lemma 5.6 *Let G be a finitely generated amenable group and S a Følner sequence in G. Let $c \in \ell^\infty(G)$.*

(i)   *We have $0 \preceq \beta_c^S \preceq 1$.*

(ii)   *If $\beta_c^S \succ \sigma_S$, then $[c] \neq 0 \in H_0^{\mathrm{uf}}(G; \mathbb{R})$.*

(iii)   *Conversely, if $[c] \neq 0 \in H_0^{\mathrm{uf}}(G; \mathbb{R})$ then there exists a Følner sequence S' such that $\sigma_{S'} \prec \beta_c^{S'}$.*

(iv)   *More generally, if $(c_n)_{n \in \mathbb{N}}$ is a sequence in $\ell^\infty(G)$ such that $\sigma_S \prec \beta_{c_0}^S$ and for all $n \in \mathbb{N}$,*

$$\beta_{c_n}^S \prec \beta_{c_{n+1}}^S,$$

then the family $([c_n])_{n \in \mathbb{N}}$ of classes in $H_0^{\mathrm{uf}}(G; \mathbb{R})$ is linearly independent.

Proof   (i)   This is obvious by definition.

(ii)   Assume $[c] = 0$. By Whyte's criterion there exist $C, r \in \mathbb{N}_{>0}$ such that for all $n \in \mathbb{N}$

$$0 < \frac{1}{C} \leq \frac{|\partial_r S_n|}{|\sum_{s \in S} c(s)|} \leq |B_r(e)| \cdot \frac{|\partial_1 S_n|}{|\sum_{s \in S} c(s)|}.$$

Hence $\beta_c^S \not\succ \sigma_S$.

(iii)   By Whyte's criterion, if $[c] \neq 0$, for each $n \in \mathbb{N}$ there exists a finite subset $S_n' \subseteq G$ such that

$$\left|\sum_{s \in S_n'} c(s)\right| > n \cdot |\partial_1 S_n'|;$$

hence

$$\frac{1}{\|c\|_\infty} \cdot \frac{|\partial_1 S'_n|}{|S'_n|} \leq \frac{|\partial_1 S'_n|}{|\sum_{s \in S'_n} c(s)|} < \frac{1}{n}.$$

In particular $S' := (S'_n)_{n \in \mathbb{N}}$ is a Følner sequence and $\sigma_{S'} \prec \beta_c^{S'}$.

(iv) For each $c \in \ell^\infty(G)$ define the subspace

$$C_0^{\mathrm{uf}}(G; \mathbb{R})^{\preceq c} := \left\{ c' \in C_0^{\mathrm{uf}}(G; \mathbb{R}) \;\middle|\; \lim_{n \to \infty} \frac{\sum_{s \in S_n} c'(s)}{\sum_{s \in S_n} c(s)} \text{ exists} \right\}.$$

Choose a splitting of $\mathbb{R}$–vector spaces $C_0^{\mathrm{uf}}(G; \mathbb{R}) = C_0^{\mathrm{uf}}(G; \mathbb{R})^{\preceq c} \oplus V$ and define a linear map

$$\gamma_c^S \colon C_0^{\mathrm{uf}}(G; \mathbb{R}) = C_0^{\mathrm{uf}}(G; \mathbb{R})^{\preceq c} \oplus V \longrightarrow \mathbb{R},$$

$$(c', c'') \longmapsto \lim_{n \to \infty} \frac{\sum_{s \in S_n} c'(s)}{\sum_{s \in S_n} c(s)}.$$

If $a \in C_0^{\mathrm{uf}}(G; \mathbb{R})$ is a boundary, the function β_a^S / σ_S is bounded by Whyte's criterion. Therefore, if $\sigma_S \prec \beta_c^S$ then $\beta_a^S \prec \beta_c^S$ and hence $\gamma_c^S(a) = 0$. Thus γ_c^S induces a map

$$\overline{\gamma_c^S} \colon H_0^{\mathrm{uf}}(G; \mathbb{R}) \longrightarrow \mathbb{R}.$$

In our situation we have $\gamma_{c_i}^S(c_j) = \delta_{ij}$ for all $j \leq i$ in $\mathbb{N}$. Hence, $([c_n])_{n \in \mathbb{N}}$ is linearly independent. $\qquad\square$

Example 5.7 The sequence $(\{n^k \mid n \in \mathbb{N}\})_{k \in \mathbb{N}_{>0}}$ of subsets of $\mathbb{Z}$ satisfies the conditions of the last part of Lemma 5.6, and hence induces a sequence of linearly independent classes in $H_0^{\mathrm{uf}}(\mathbb{Z}; \mathbb{R})$.

5.2 Sparse classes

We now introduce a geometric condition that will ensure that a subset is mean-invisible:

Definition 5.8 Let G be a finitely generated group. We call a subset $\Gamma \subseteq G$ *(asymptotically) sparse* if there exists $C \in \mathbb{N}$ such that for all $r \in \mathbb{N}_{>0}$, there exists $R \in \mathbb{N}_{>0}$ such that for all $g \in G \setminus B_R(e)$,

$$|\Gamma \cap B_r(g)| \leq C.$$

Example 5.9 For each $k \in \mathbb{N}_{>1}$ the subsets $\{n^k \mid n \in \mathbb{N}\} \subseteq \mathbb{Z}$ are sparse.

Lemma 5.10 *If $\Gamma \subseteq G$ is sparse and G a finitely generated infinite amenable group we have*

$$[\chi_\Gamma] \in \hat{H}_0^{\mathrm{uf}}(G; \mathbb{R}).$$

Proof Let m be a left-invariant mean on G and $C \in \mathbb{N}$ as in the definition of sparse. Let $r \in \mathbb{N}_{>0}$. Here we write $\chi(S) := \chi_S$ for the characteristic function of any subset $S \subseteq G$. Since G is infinite we have for all $R \in \mathbb{N}_{>0}$ that $\bar{m}(\chi(\Gamma)) = \bar{m}(\chi(\Gamma \setminus B_R(e)))$. Hence we may assume that for all $g \in G$,

$$|\{\gamma \in \Gamma \mid \text{there exists } h \in B_r(e) \text{ such that } \gamma \cdot h = g\}| = |\Gamma \cap B_r(g)| \leq C.$$

Therefore the coefficients of $\sum_{h \in B_r(e)} \chi(\Gamma \cdot h)$ are bounded by C. So we see that

$$|B_r(e)| \cdot \bar{m}(\chi(\Gamma)) = \sum_{h \in B_r(e)} \bar{m}(\chi(\Gamma \cdot h)) = \bar{m}\left(\sum_{h \in B_r(e)} \chi(\Gamma \cdot h) \right) \leq C.$$
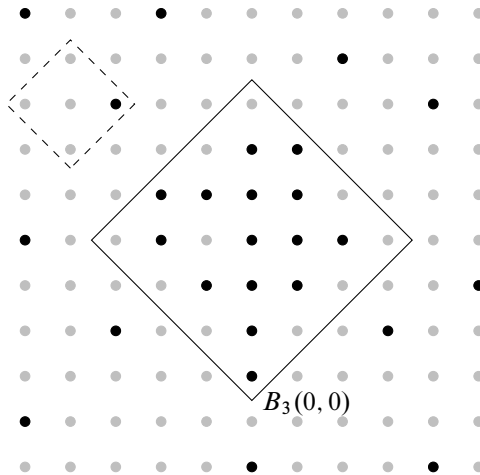
So $\bar{m}(\chi(\Gamma)) = 0$.   □

Figure 1: The black dots form a sparse subset in $\mathbb{Z}^2$ with the standard word metric

5.3 Constructing sparse classes

We recall the notion of tilings, which will be the building blocks for our sparse classes.

Definition 5.11 Let G be a finitely generated group with the word metric. For $r \in \mathbb{N}_{>0}$ we call a subset $T \subseteq G$ an *r–tiling for G* if:

 (i) For all $g_1, g_2 \in T$ such that $B_r(g_1) \cap B_r(g_2) \neq \varnothing$, we have $g_1 = g_2$.

 (ii) $G = \bigcup_{g \in T} B_{2 \cdot r}(g)$.

By Zorn's lemma, there exists an r–tiling [10, Proposition 5.6.3] for all $r \in \mathbb{N}_{>0}$.

Lemma 5.12 [10, Proposition 5.6.4] *Let G be a finitely generated amenable group and $(S_j)_{j \in \mathbb{N}}$ a Følner sequence. Let $r \in \mathbb{N}_{>0}$ and let T be an r–tiling for G. Set*

$$T_j := \{g \in T \mid B_r(g) \subseteq S_j\} \subseteq S_j.$$

Then there exists an $l(T) \in \mathbb{N}$ such that for all $j \geq l(T)$

$$\frac{1}{|2 \cdot B_{2 \cdot r}(e)|} \leq \frac{|T_j|}{|S_j|} \leq \frac{1}{|B_r(e)|}.$$

Remark 5.13 The idea of the following proof of Theorem 5.1 is the following. Let $(S_j)_{j \in \mathbb{N}}$ be a Følner sequence. For each "ring" $S_j \setminus S_{j-1}$ we choose a radius $r(j) \in \mathbb{N}_{>0}$ and put part of an $r(j)$–tiling on this ring; see Figure 2. By letting the sequence $(r(j))_{j \in \mathbb{N}}$ go to infinity, we make sure to get a sparse subset. On the other hand, we let $(r(j))_{j \in \mathbb{N}}$ grow just slowly enough to be sure to get a nontrivial class in $H_0^{\mathrm{uf}}(G; \mathbb{R})$. We then vary the growth of the sequence $(r(j))_{j \in \mathbb{N}}$ and use Lemma 5.6 to get an infinite family of linearly independent classes in $H_0^{\mathrm{uf}}(G; \mathbb{R})$.

Proof of Theorem 5.1 By a result of T Adachi [1] there exists a monotonous and exhausting Følner sequence of G. Hence there is a Følner sequence $(S_j)_{j \in \mathbb{N}}$ such that for each $j \in \mathbb{N}$ there exists an $R(j) \in \mathbb{N}_{>0}$ such that

$$(*) \qquad\qquad S_{j-1} \subseteq B_{R(j)}(e) \subseteq B_{3 \cdot R(j)}(e) \subseteq S_j.$$

Furthermore we can assume $\lim_{j \to \infty}(|S_{j-1}|/|S_j|) = 0$.

Choose for each $r \in \mathbb{N}_{>0}$ an r–tiling T^r of G. For each $j \in \mathbb{N}_{>0}$ and each sequence $c = (c(l))_{l \in \mathbb{N}}$ of positive real numbers converging to 0 define $r(j, c)$ to be the maximal number $r \in \{1, \ldots, R(j)\}$ satisfying

$$\frac{|B_r(e)|}{4 \cdot |B_{2 \cdot r}(e)|} \geq \frac{|S_{j-1}|}{|S_j|},$$

$$\frac{1}{4 \cdot |B_{2 \cdot r}(e)|} \geq \sqrt{c(j)},$$

$$l(T^r) \leq j.$$

If no such r exists, we set $r(j, c) = 1$.

The conditions on $r(j, c)$ will force $(r(j, c))_{j \in \mathbb{N}}$ to grow slowly enough so that the class Γ_c we are about to construct will satisfy $c \prec \beta_{\Gamma_c}^S$.
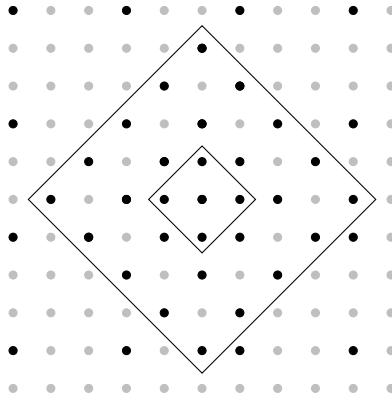
Figure 2: Construction of a sparse subset in $\mathbb{Z}^2$ (Remark 5.13)

By our assumption on $(S_j)_{j \in \mathbb{N}}$ we have $\lim_{j \to \infty} r(j, c) = \infty$. Define the subset

$$\Gamma_c = T_0^{r(0,c)} \cup \bigcup_{j \in \mathbb{N}_{>0}} \left(T_j^{r(j,c)} \setminus T_{j-1}^{r(j,c)} \right) \subseteq G.$$

For each $r \in \mathbb{N}$ consider $j \in \mathbb{N}$ such that $r(j, c) > 2 \cdot r$. By condition $(*)$ we know that for each $g \in G \setminus S_j$ the ball $B_r(g)$ intersects at most two sets $S_l \setminus S_{l-1}$ and $S_{l+1} \setminus S_l$. Because of the tiling condition and since $r(j, c) > 2 \cdot r$ the ball also contains at most one point each from

$$T_{l+1}^{r(l+1,c)} \setminus T_l^{r(l+1,c)} \quad \text{and} \quad T_l^{r(l,c)} \setminus T_{l-1}^{r(l,c)},$$

hence $|B_r(g) \cap \Gamma_c| \leq 2$, so Γ_c is sparse. We also have for all $j \in \mathbb{N}_{>0}$, by definition of Γ,

$$\begin{aligned}
\frac{|\Gamma_c \cap S_j|}{|S_j|} &\geq \frac{|T_j^{r(j,c)}|}{|S_j|} - \frac{|T_{j-1}^{r(j,c)}|}{|S_j|} \\
&\geq \frac{1}{2 \cdot |B_{2 \cdot r(j,c)}(e)|} - \frac{|S_{j-1}|}{|B_{r(j,c)}(e)| \cdot |S_j|} \quad \text{(by Lemma 5.12)} \\
&\geq \frac{1}{4 \cdot |B_{2 \cdot r(j,c)}(e)|} \geq \sqrt{c(j)},
\end{aligned}$$

where the last inequality holds by definition of $r(j, c)$. Therefore, $\beta_{\Gamma_c} \succ c$.

Since Γ_c is sparse, $\lim_{j \to \infty} \beta_{\Gamma_c}(j) = \lim_{j \to \infty} |\Gamma_c \cap S_j| / |S_j| = 0$. Now inductively define a sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of sparse subsets by setting $\Gamma_0 = \Gamma_{\sigma_S}$ and $\Gamma_{n+1} = \Gamma_{\beta_{\Gamma_n}^S}$ for all $n \in \mathbb{N}$. Then $\sigma_S \prec \beta_{\Gamma_0} \prec \beta_{\Gamma_1} \prec \cdots$. Hence by Lemma 5.6 we have found a sequence of linearly independent classes in $\widehat{H}_0^{\mathrm{uf}}(G; \mathbb{R})$. □

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Fakultät für Mathematik, *Universität Regensburg*
93040 Regensburg, Germany

matthias.blank@mathematik.uni-regensburg.de,
francesca.diana@mathematik.uni-regensburg.de

http://www.mathematik.uni-r.de/blank/,
http://homepages-nw.uni-regensburg.de/~dif13273/