

Systolic volume of homology classes

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Given an integer homology class of a finitely presentable group, the systolic volume quantifies how tight a geometric realization of this class could be. In this paper, we study various aspects of this numerical invariant showing that it is a complex and powerful tool for investigating topological properties of homology classes of finitely presentable groups.

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1 Introduction

Let G be a finitely presentable group and fix a nontrivial homology class $\mathbf{a} \in H_m(G, \mathbb{Z})$ of dimension $m \geq 1$. In order to study topological and algebraic properties of the class \mathbf{a} , we consider the various ways this class can be realized by a pseudomanifold endowed with a polyhedral metric. For such realizations, the two main geometrical quantities are the volume of the pseudomanifold and the lengths of loops representing nontrivial elements of G . The systolic volume turns out to be the simplest way to compare these geometric quantities in order to form a topological invariant and is defined as follows.

A *geometric cycle* (X, f) representing a class \mathbf{a} is a pair (X, f) consisting of an orientable pseudomanifold X of dimension m and a continuous map $f: X \rightarrow K(G, 1)$ such that $f_*[X] = \mathbf{a}$, where $[X]$ denotes the fundamental class of X and $K(G, 1)$ the Eilenberg–Mac Lane space.¹ The representation is said to be *normal* if in addition the induced map $f_{\#}: \pi_1(X) \rightarrow G$ is an epimorphism. Given a geometric cycle (X, f) and a polyhedral metric g on X (see Babenko [3] for a definition), the *relative homotopic systole* $\text{sys}_f(X, g)$ is defined as the least length of a loop γ in X such that $f \circ \gamma$ is not contractible. The *systolic volume* of (X, f) is then the value

$$\mathfrak{S}_f(X) := \inf_g \frac{\text{vol}(X, g)}{g \text{ sys}_f(X, g)^m},$$

¹This definition slightly differs from the original one by Gromov in [17], where in addition a geometric cycle is provided with a polyhedral metric on X .

where the infimum is taken over all polyhedral metrics g on X and $\text{vol}(X, g)$ denotes the m -dimensional volume of X . When $f: X \rightarrow K(\pi_1(X), 1)$ is the classifying map (induced by an isomorphism between fundamental groups), we simply denote by $\mathfrak{S}(X)$ the systolic volume of the pair (X, f) . From [17, Section 6], we have that for any dimension $m \geq 1$,

$$\sigma_m := \inf_{(X, f)} \mathfrak{S}_f(X) > 0,$$

the infimum being taken over all geometric cycles (X, f) representing a nontrivial homology class of dimension m . The following notion was introduced by Gromov (see [17, Section 6]):

Definition *The systolic volume of the pair (G, \mathbf{a}) is defined as the number*

$$\mathfrak{S}(G, \mathbf{a}) := \inf_{(X, f)} \mathfrak{S}_f(X),$$

where the infimum is taken over all geometric cycles (X, f) representing the class \mathbf{a} .

Any integer class admits a representation by a geometric cycle; see Theorem I below. The systolic volume of (G, \mathbf{a}) is thus well-defined and satisfies $\mathfrak{S}(G, \mathbf{a}) \geq \sigma_m$. But it is not clear whether the infimum value $\mathfrak{S}(G, \mathbf{a})$ is actually a minimum and what the structure of a minimal geometric cycle would be. When the class \mathbf{a} is representable by a manifold, the systolic volume coincides with the systolic volume of any normal representation of \mathbf{a} by a manifold; see Babenko [3; 4] and Brunnbauer [10]. A manifold is an example of *admissible* pseudomanifold, that is a special type of pseudomanifolds for which any element of the fundamental group can be represented by a curve not going through the singular locus of X . In this article, we first prove the following result; see Section 2.

Theorem I *Let G be a finitely presentable group. Any homology class $\mathbf{a} \in H_m(G, \mathbb{Z})$ admits a normal representation by an admissible geometric cycle. Furthermore, any such representation by an admissible geometric cycle (X, f) is minimal in the sense that*

$$\mathfrak{S}_f(X) = \mathfrak{S}(G, \mathbf{a}).$$

Here an admissible geometric cycle (X, f) stands for a geometric cycle whose pseudomanifold X is admissible. Thus the infimum in the definition of systolic volume of a homology class is a minimum. Furthermore the systolic volume of an admissible orientable pseudomanifold X depends only on the image of its fundamental class $f_*[X] \in H_m(\pi_1(X), \mathbb{Z})$. In Section 6 we exhibit an example showing that the condition of normalization (that is, $f_{\#}$ is an epimorphism between fundamental groups) cannot be

relaxed in our theorem. Finally observe that working with the class of pseudomanifolds is necessary and not merely a formal generalization: there exist classes that are not representable by manifolds; see Thom [27].

In order to understand the systolic volume invariant, we study its distribution along the real line. Here two phenomena appear. First, the systolic volume function does not avoid arbitrarily large intervals.

Proposition I *Let $m \geq 3$. For any interval $I \subset \mathbb{R}^+$ of length at least σ_m , there exists a pair (G, \mathbf{a}) consisting of a finitely presentable group and a homology class of dimension m such that $\mathfrak{S}(G, \mathbf{a}) \in I$.*

Secondly, there is no finiteness result for systolic volume in dimension $m \geq 3$. In order to avoid irrelevant nonfiniteness results, we introduce the following definition. A class $\mathbf{a} \in H_m(G, \mathbb{Z})$ is said to be *reducible* if there exists a subgroup $H \subsetneq G$ and a class $\mathbf{b} \in H_m(H, \mathbb{Z})$ such that $i_*(\mathbf{b}) = \mathbf{a}$, where i denotes the canonical inclusion. Otherwise the class will be called *irreducible*. If the class is reducible then $\mathfrak{S}(G, \mathbf{a}) \leq \mathfrak{S}(H, \mathbf{b})$ by definition. Now observe that it is trivial to form infinite sequences of classes with bounded systolic volume which reduce to the same class. Indeed consider a homology class $\mathbf{b} \in H_m(H, \mathbb{Z})$ of dimension $m \geq 2$. Then for any positive integer n ,

$$\mathfrak{S}(H * \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_n, \mathbf{a}_n) \leq \mathfrak{S}(H, \mathbf{b}),$$

where $\mathbf{a}_n = (i_n)_*(\mathbf{b})$ denotes the image of the class \mathbf{b} by the natural inclusion $i_n: H \hookrightarrow H * \mathbb{Z} * \dots * \mathbb{Z}$. The existence of infinite sequences of pairwise distinct irreducible classes with bounded systolic volume is much more meaningful and will be proven in Section 4.

Theorem II *For any dimension $m \geq 3$ there exists an infinite sequence of finitely presentable pairwise distinct groups $\{G_i\}$ for which at least one irreducible class $\mathbf{a}_i \in H_m(G_i, \mathbb{Z})$ satisfies $\mathfrak{S}(G_i, \mathbf{a}_i) \leq 1$.*

So we have to introduce some topological or algebraic restrictions in order to get finiteness results. For instance, given a finitely presentable group G , a homology class $\mathbf{a} \in H_m(G, \mathbb{Z})$ and a positive number T , the number of integer multiple classes $k\mathbf{a}$ whose systolic volume is less than T is at least $T \cdot \ln T$ (up to some multiplicative constant). More precisely, the following theorem will be proven in Section 5.

Theorem III *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$, where $m \geq 3$. There exists a positive number $C(G, \mathbf{a})$ depending only on the pair (G, \mathbf{a}) such that*

$$\mathfrak{S}(G, k\mathbf{a}) \leq C(G, \mathbf{a}) \cdot \frac{k}{\ln(1+k)}$$

for any positive integer k . In particular,

$$\lim_{k \rightarrow \infty} \frac{\mathfrak{S}(G, k\mathbf{a})}{k} = 0.$$

This result shows that the systolic volume of multiples of a class is a sublinear function, which is remarkable. For some special classes \mathbf{a} (for which simplicial volume is not zero; see Section 5.3), we know after Gromov [17] that there exists a positive number $C'(G, \mathbf{a})$ depending only on the pair (G, \mathbf{a}) such that

$$(1-1) \quad \mathfrak{S}(G, k\mathbf{a}) \geq C'(G, \mathbf{a}) \cdot \frac{k}{(\ln(1+k))^m}.$$

Moreover, for fundamental groups π_l of orientable surfaces Σ_l of genus $l \geq 1$ and for the corresponding fundamental classes $[\Sigma_l]$, we know by [17] and Buser and Sarnak [11] that

$$\mathfrak{S}(\pi_l, k[\Sigma_l]) \sim \frac{k}{(\ln(1+k))^2},$$

where $f \sim g$ means that there exists some positive constants c and C such that $c \cdot f \leq g \leq C \cdot f$. This naturally leads to the following conjecture.

Conjecture *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ a class of nonzero simplicial volume where $m \geq 3$. Then*

$$\mathfrak{S}(G, k\mathbf{a}) \sim \frac{k}{(\ln(1+k))^m}.$$

In Section 6, we explore the case of nilmanifolds, and more specifically the case of the Heisenberg group of dimension 3. We obtain a new illustration of the possible behavior of the systolic volume of cyclic coverings. The study of the systolic volume of cyclic coverings in terms of the number of sheets has been suggested by Gromov [18], and the first result in this direction can be found in Babenko and Balacheff [5].

The Heisenberg group \mathcal{H} of dimension 3 is the group of triangular matrices

$$\left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \mid x, y, z \in \mathbb{R} \right\}.$$

The subset $\mathcal{H}(\mathbb{Z})$ of \mathcal{H} composed of matrices with integer coefficients (ie matrices for which $x, y, z \in \mathbb{Z}$) is a lattice, and we will denote by $M_{\mathcal{H}} = \mathcal{H}/\mathcal{H}(\mathbb{Z})$ the corresponding nilmanifold. First of all, we obtain the following explicit upper bound for the systolic volume of multiples of the fundamental class $M_{\mathcal{H}}$.

Theorem IV Let $\mathbf{a} = [M_{\mathcal{H}}] \in H_3(\mathcal{H}(\mathbb{Z}), \mathbb{Z})$ be the fundamental class of $M_{\mathcal{H}}$. Then

$$\mathfrak{S}(\mathcal{H}(\mathbb{Z}), k\mathbf{a}) \leq 19 \cdot \mathfrak{S}(\mathcal{H}(\mathbb{Z}), \mathbf{a})$$

for any integer $k \geq 1$.

The constant appearing here is the one involved in the resolution of the classical Waring problem (see Balasubramanian, Deshouillers and Dress [7; 8]): any integer number decomposes into a sum of at most 19 fourth powers. The idea of using the solution of the Waring problem in order to bound from above the function $\mathfrak{S}(G, k\mathbf{a})$ when $(G, \mathbf{a}) = (\mathcal{H}(\mathbb{Z}), [M_{\mathcal{H}}])$ carries over to any pair (G, \mathbf{a}) , where G is a nilpotent graded group without torsion and \mathbf{a} denotes the fundamental class of the corresponding nilmanifold; see Theorem 6.2.

Now consider the sequence of lattices $\{\mathcal{H}_n(\mathbb{Z})\}_{n=1}^{\infty}$ of \mathcal{H} , where $\mathcal{H}_n(\mathbb{Z})$ denotes the subset of matrices whose integer coefficients satisfy $x \in n\mathbb{Z}$ and $y, z \in \mathbb{Z}$. Denote by $M_{\mathcal{H}_n} = \mathcal{H}/\mathcal{H}_n(\mathbb{Z})$ the corresponding nilmanifolds. The manifold $M_{\mathcal{H}_n}$ is a cyclic covering with n sheets of $M_{\mathcal{H}}$, and the proof of Theorem III implies that

$$\mathfrak{S}(M_{\mathcal{H}_n}) \leq C \cdot \frac{n}{\ln(1+n)};$$

see Remark 5.5. The fact that the function $\mathfrak{S}(M_{\mathcal{H}_n})$ goes to infinity is not obvious, and its proof relies on a new invariant for groups called *simplicial complexity* and defined in Babenko, Balacheff and Bulteau [6]. More precisely, we have the following.

Proposition [6] *The function $\mathfrak{S}(M_{\mathcal{H}_n})$ satisfies the inequality*

$$\mathfrak{S}(M_{\mathcal{H}_n}) \geq a \frac{\ln n}{\exp(b\sqrt{\ln \ln n})},$$

where a and b are two positive constants.

In particular,

$$\lim_{n \rightarrow +\infty} \mathfrak{S}(M_{\mathcal{H}_n}) = +\infty.$$

Note that in this case $\|M_{\mathcal{H}_n}\|_{\Delta} = 0$ and the lower bound (1-1) does not apply. For any integer n the manifold $M_{\mathcal{H}_n}$ gives a nonnormal realization of the class $n[M_{\mathcal{H}}]$. So the normalization condition in Theorem I cannot be relaxed.

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2 Manifolds with singularities as extrema of the systolic volume

The systolic volume of a homology class is defined by an infimum. We may wonder if this infimum is reached and what would be the structure of a minimizing pseudomanifold. If the homology class \mathbf{a} is realized by a manifold, we know that its systolic volume is a minimum which coincides with the systolic volume of any normal representation of \mathbf{a} by a manifold; see [3; 4; 10]. But there exist classes which do not admit representations by manifolds (see [27]), and for such a class \mathbf{a} it is not even clear whether $\mathfrak{S}(G, \mathbf{a})$ is actually a minimum.

Let X be a pseudomanifold of dimension m . The *singular locus* of X is by definition the set $\Sigma(X)$ of points of X which do not have a neighborhood homeomorphic to an m -dimensional ball. By definition of a pseudomanifold, $\Sigma(X)$ is a simplicial subcomplex of codimension at least 2.

Definition 2.1 A pseudomanifold X is said to be *admissible* if the natural inclusion $X \setminus \Sigma(X) \subset X$ induces an epimorphism on fundamental groups.

That is, a pseudomanifold is admissible if any element of the fundamental group can be represented by a loop of $X \setminus \Sigma(X)$. A geometric cycle (X, f) representing some homology class \mathbf{a} will be called *admissible* if the pseudomanifold X is admissible.

Example 2.2 Let M be a triangulated manifold and $N \subset M$ be a simplicial subcomplex of codimension greater than or equal to 2. Denote by $\sharp N$ the set of connected components of N . The simplicial complex $M/\sharp N$ obtained from M by contracting the connected components of N into distinct points is an admissible pseudomanifold. The singular locus $\Sigma(M/\sharp N)$ consists of the points corresponding to the connected components of N .

Example 2.3 Let M be a manifold with boundary $\partial M = A \times P$, where A is a manifold and P is a connected manifold. The result of the fibred contraction of P is an admissible pseudomanifold \bar{M} homeomorphic to the space

$$M \cup_{\partial M} A \times CP,$$

where CP stands for the cone over P and the singular locus $\Sigma(\bar{M})$ is homeomorphic to A . Remark that if P is simply connected then $\pi_1(\bar{M}) = \pi_1(M)$. The pseudomanifold \bar{M} obtained that way is a particular example of a *singular P -manifold*; see Baas [1] and Section 2.3 for the general construction.

Remark that an admissible pseudomanifold of dimension 2 is a surface. In particular it does not possess any singularity.

Theorem I is a direct consequence of the following two propositions.

Proposition 2.4 *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ be a homology class of dimension $m \geq 3$. Suppose that there exists a normal representation of the class \mathbf{a} by an admissible geometric cycle (X, f) . Then*

$$\mathfrak{S}(G, \mathbf{a}) = \mathfrak{S}_f(X).$$

The condition of normalization saying that $f_{\#}$ is an epimorphism cannot be dropped; see Section 6 and the example of the Heisenberg group.

The following proposition, together with Proposition 2.4, shows that for any pair (G, \mathbf{a}) the systolic volume $\mathfrak{S}(G, \mathbf{a})$ is actually a minimum.

Proposition 2.5 *Let K be a CW-complex whose fundamental group is finitely generated. Any homology class $\mathbf{a} \in H_*(K, \mathbb{Z})$ admits a normal representation by an admissible geometric cycle (X, f) .*

Before proving Propositions 2.4 and 2.5, we need some technical results.

2.1 Technical lemmas

Hopf's trick perfectly adapts to the setting of admissible pseudomanifolds. Consider a map

$$f: (X, X_1) \rightarrow (Y, Y_1)$$

between two relative manifolds of the same dimension $m \geq 3$. Suppose that f is transversal at $y \in Y \setminus Y_1$, ie there exists an embedded m -disk D such that:

- $y \in D \subset Y \setminus Y_1$.
- $f^{-1}(D) = \bigcup_{i=1}^n D_i$ is a disjoint union of m -disks embedded in $X \setminus X_1$.
- The restriction of f to $f^{-1}(D)$ is a covering map with base space D and n sheets.

Set $x_i = D_i \cap f^{-1}(y)$. The technical trick by Hopf is essentially contained in the following lemma.

Lemma 2.6 *Suppose that there exists an m -disk embedded in $D' \subset X \setminus X_1$ with the following properties:*

- (1) $D_1, D_2 \subset D'$ and $D' \cap D_i = \emptyset$ if $i > 2$.
- (2) For any path γ from x_1 to x_2 in D' , the loop $f(\gamma)$ is contractible in Y .
- (3) For any orientation of D , the orientations on D_1 and D_2 induced by f are not compatible in D' .

Then there exists a homotopy $\{f_t\}_{0 \leq t \leq 1}$ of $f_0 = f$ which is constant on $X \setminus D'$ and such that

$$f_1^{-1}(D) = \bigcup_{i>2} D_i,$$

the last union being empty if $n = 2$.

We refer the reader to Epstein [14, pages 378–380] for a proof of Lemma 2.6, which will work in our setting, as the construction of the homotopy f_t occurs in D' and therefore carries over to our context. We will now state the corresponding version of Hopf's theorem in the orientable case.

Lemma 2.7 *Let X be an admissible orientable connected pseudomanifold and (Y, Y_1) an orientable relative manifold of the same dimension $m \geq 3$. Suppose that $f: X \rightarrow Y$ is a map of degree k inducing an epimorphism between fundamental groups.*

Then there exists a homotopy $\{f_t\}_{0 \leq t \leq 1}$ of $f_0 = f$ such that $f_1^{-1}(Y \setminus Y_1)$ is homeomorphic to the disjoint union of k disks and the restriction of f_1 to the union of these disks is a covering map with base space $Y \setminus Y_1$ and k sheets.

The degree of f stands here for the absolute value of the multiple defined by the induced map $f_*: H_m(X; \mathbb{Z}) \rightarrow H_m(Y, Y_1; \mathbb{Z})$. A corresponding version of Lemma 2.7 also holds in the nonorientable context with the notion of absolute degree.

Proof Consider a point $y \in Y \setminus Y_1$. We can assume that $y \notin f(\Sigma(X))$ and the map f is transversal at y . Let $D \subset Y \setminus Y_1$ be a disk containing the point y such that $f^{-1}(D) = \bigcup_{i=1}^n D_i$ is a disjoint union of m -disks embedded in $X \setminus \Sigma(X)$ and such that the restriction of f to $\bigcup_{i=1}^n D_i$ is a covering map with n sheets and base space D . We have $n \geq k$ and suppose that $n > k$. We can choose generators of $H_m(X; \mathbb{Z})$ and $H_m(Y, Y_1; \mathbb{Z})$ such that the map f_* induced on m -dimensional homology is simply the multiplication by k . This induces an orientation both on X and $Y \setminus Y_1$, and also on disks $\{D_i\}_{i=1}^n$ and D . As $n > k$, there exist two disks, D_1 and D_2 , such that $f|_{D_1}$ reverses the orientation and $f|_{D_2}$ preserves it. We now follow step by step the proof

of [14, Theorem 4.1]. Join the two points $x_i = f^{-1}(y) \cap D_i$, $i = 1, 2$ by a simple curve $\gamma \subset X \setminus \Sigma(X)$. Because f induces an epimorphism between fundamental groups, there exists a loop α based at x_1 such that $f(\alpha)$ and $f(\gamma)$ are homotopic as loops based at y . As X is admissible, we can furthermore choose α in $X \setminus \Sigma(X)$. The concatenation $\alpha^{-1} \star \gamma$ and its evident modification in a neighborhood of the concatenation define a simple curve $\beta \subset X \setminus \Sigma(X)$ joining x_1 and x_2 . The loop $f(\beta)$ is contractible in Y relatively to y . We then define the disk C as a small enough neighborhood of β , and apply Lemma 2.6. Remark that the choice of C implies a possible diminution of the size of the disks $\{D_i\}_{i=1}^n$ and D . The end of the proof is straightforward; see [14] for the missing details. \square

2.2 Admissible geometric cycles are minima of the systolic volume

In this subsection, we prove Proposition 2.4. For this we use in a decisive way the comparison and extension techniques elaborated in [3], as well as the ideas contained in this article. For the reader's convenience, let first recall these two techniques.

We first present the comparison principle which originally appeared as [3, Proposition 3.2]. Given two pseudomanifolds X_1 and X_2 recall that a simplicial map from X_1 to X_2 is called m -monotone if the preimage of any open m -simplex is either an open m -simplex or empty.

Proposition 2.8 (Comparison principle) *Let (X_1, f_1) and (X_2, f_2) be two geometric cycles representing an homology class $\mathbf{a} \in H_m(G, \mathbb{Z})$ and suppose that there exists an m -monotone simplicial map $p: X_1 \rightarrow X_2$ such that $f_1 = f_2 \circ p$. Then*

$$\mathfrak{S}_{f_1}(X_1) \leq \mathfrak{S}_{f_2}(X_2).$$

Now we state the extension principle; see [3, Proposition 3.6].

Proposition 2.9 (Extension principle) *Let (X_1, f_1) be a geometric cycle representing a homology class of dimension m . Suppose that X_2 is a CW-complex obtained from X_1 by adding a finite number of cells with dimension from 3 to $m - 1$. Then*

$$\mathfrak{S}_{f_1}(X_1) = \mathfrak{S}_{f_2}(X_2),$$

where $f_2: X_2 \rightarrow K(G, 1)$ is the unique (up to homotopy) extension of f_1 .

We now present the proof of Proposition 2.4.

Let (X_1, f_1) be a geometric cycle representing the class \mathbf{a} . The pseudomanifold X_1 admits a cell decomposition with only one m -cell (see for example Sabourau [24]).

This allows us to describe X_1 as a relative m -manifold (Y, Y') , where Y denotes the m -cell and Y' lies in the $(m-1)$ -skeleton. Following [3] we construct an extension of X_1 as follows.

We start by adding a finite number of 1- and 2-cells to X_1 such that the resulting CW-complex $X_1(2)$ has fundamental group G and the inclusion map $i: X_1 \hookrightarrow X_1(2)$ satisfies $i_{\#} = (f_1)_{\#}$. More precisely, given a finitely generated presentation of G we first construct a 2-dimensional finite cellular complex $K^{(2)}$ by considering the wedge sum of 1-cells corresponding to the generators of the presentation of G , and then gluing 2-cells according to its relations. The finite cellular complex $X(2)$ is then obtained as the wedge sum of X_1 and $K^{(2)}$, with additional 2-cells glued along the image of $(f_1)_{\#}(\gamma_i) * \gamma_i^{-1}$ for a finite set $\{\gamma_i\}_{i \in I}$ of generators of $\pi_1(X_1)$.

Lemma 2.10 *We have*

$$\mathfrak{S}_{f_1}(X_1) = \mathfrak{S}(X(2)).$$

Proof In fact, using the comparison principle, we first observe that

$$\mathfrak{S}_{f_1}(X_1) \leq \mathfrak{S}(X_1(2)).$$

Now fix a positive ε and a metric g_1 on X_1 such that

$$\frac{\text{vol}(X_1, g_1)}{\text{sys}_{f_1}(X_1, g_1)^m} \leq \mathfrak{S}_{f_1}(X_1) + \varepsilon.$$

Now define the metric g on $X_1(2)$ which coincides with g_1 on X_1 , for which the length of each additional 1-cell is exactly $\text{sys}_{f_1}(X_1, g_1)$ and for which each additional 2-cell is a round hemisphere whose curvature is chosen accordingly to the length of its boundary. It is straightforward to see that any closed curve of $X_1(2)$ can be homotoped to a curve lying in the union of X_1 and the 1-skeleton of $K^{(2)}$ without increasing the length. In particular $\text{sys}(X_1(2), g) = \text{sys}_{f_1}(X_1, g_1)$. But $\text{vol}(X_1(2), g) = \text{vol}(X_1, g_1)$ for dimensional reasons as $m \geq 3$ so we get

$$\mathfrak{S}(X_1(2)) \leq \frac{\text{vol}(X_1(2), g)}{\text{sys}(X_1(2), g)^m} \leq \mathfrak{S}_{f_1}(X_1) + \varepsilon,$$

which concludes the proof. □

Then, for each dimension k going from 3 to m , we add k -cells to $X_1(k-1)$ such that the new CW-complex $X_1(k)$ thus obtained satisfies $\pi_i(X_1(k)) = 0$ for $1 < i < k-1$. At the end we get a CW-complex denoted by $X_1(m)$ which is m -aspherical and whose fundamental group is G . Remark that $Y \setminus Y'$ is an m -cell of $X_1(m)$. By adding cells of dimension higher than m , we can realize the Eilenberg-Mac Lane space $K(G, 1)$ as an extension of $X_1(m)$.

By assumption there exists an admissible pseudomanifold X and a map $f: X \rightarrow K(G, 1)$ giving a realization of the class $a \in H_m(G, \mathbb{Z})$ such that $f_{\#}: \pi_1(X) \rightarrow G$ is an epimorphism. By the cellular approximation theorem and according to [3, Lemma 3.10] which applies to this situation, we can find a map

$$g: X \rightarrow X_1(m) \subset K(G, 1)$$

homotopic to f in $K(G, 1)$ such that

$$(2-1) \quad g_*[X] = i_*[X_1]$$

in $H_m(X_1(m), \mathbb{Z})$. Since g is homotopic to f , the induced map $g_{\#}$ is still an epimorphism and

$$\mathfrak{S}_f(X) = \mathfrak{S}_g(X).$$

Let $\{Y\} \cup \{Y_i\}_{i \in I}$ denote the m -cells of $X_1(m)$ (this list can be finite or infinite). To each m -cell Y_i or Y is associated the relative manifold (Y_i, \hat{Y}_i) or (Y, \hat{Y}) , where \hat{Y}_* denotes the closure of the union of all the other cells of Y_* . The map g induces maps

$$\tilde{g}: X \rightarrow (Y, \hat{Y}) \quad \text{and} \quad \tilde{g}_i: X \rightarrow (Y_i, \hat{Y}_i) \quad \text{for all } i \in I,$$

which still induce epimorphisms between fundamental groups. From (2-1), we deduce that the degree of \tilde{g} is equal to 1, and that the degree of each \tilde{g}_i is 0. As X is compact, $g(X)$ intersects only a finite number of m -cells. We then apply Lemma 2.7 to each relative manifold (Y_i, \hat{Y}_i) such that $Y_i \cap g(X) \neq \emptyset$. In this way we obtain a map

$$g_1: X \rightarrow Z \subset X_1(m-1)$$

homotopic to g where Z denotes the subcomplex of $X_1(m-1)$ obtained from $X_1(2)$ by adding the cells with dimension from 3 to $m-1$ which intersect $g_1(X)$. Observe that these cells are in finite number so

$$\mathfrak{S}(Z) = \mathfrak{S}(X_1(2))$$

according to the extension principle. The map can be chosen to be m -monotone and simplicial; see [3, Proposition 3.13]. Thus

$$\mathfrak{S}_g(X) = \mathfrak{S}_{g_1}(X) \leq \mathfrak{S}(Z)$$

according to the comparison principle.

To summarize, we have shown that

$$\mathfrak{S}_f(X) \leq \mathfrak{S}_{f_1}(X_1)$$

for any representation (X_1, f_1) of \mathbf{a} which proves that

$$\mathfrak{S}_f(X) = \mathfrak{S}(G, \mathbf{a}).$$

2.3 Manifolds with singularities of prescribed type according to Baas

In order to prove Proposition 2.5, we briefly recall a type of singular manifolds introduced by N Baas. We follow the presentation of [1].

First recall the following definition due to J Cerf; see [12].

Definition 2.11 A manifold with general corners of dimension m is a Hausdorff space locally modeled on the m -cube $[0, 1]^m$.

In other words, manifolds with general corners are manifolds whose boundary looks like the boundary of the cube. A manifold with general corners can be smoothed through a process which has been well studied (see [12] for instance).

To define manifolds with reasonable singularities, N Baas — following D Sullivan’s idea — considers manifolds with general corners whose boundary decomposes as products and contracts some terms of these products, like in Example 2.3. To describe his construction let us recall the following definition.

Definition 2.12 A manifold M^m with general corners is said to be *decomposed* if its boundary decomposes into a union

$$\partial M = \partial_0 M \cup \partial_1 M \cup \dots \cup \partial_n M$$

of codimension-1 submanifolds with general corners. Here the union means the identification along a common part of the boundaries of the $\partial_i M$.

If M is a decomposed manifold, each of its boundary terms turns out to be a decomposed manifold too. More precisely, by setting

$$(2-2) \quad \begin{cases} \partial_j(\partial_i M) = \partial_j M \cap \partial_i M & \text{if } j \neq i, \\ \partial_i(\partial_i M) = \emptyset, \end{cases}$$

we get

$$\partial(\partial_i M) = \bigcup_{j=0}^n \partial_j(\partial_i M)$$

and each $\partial_i M$ is again a decomposed manifold.

Example 2.13 If M denotes the m -dimensional cube, its boundary is naturally decomposed into $(m - 1)$ -faces:

$$\partial M = \partial_0 M \cup \partial_1 M \cup \dots \cup \partial_{2m-1} M.$$

We consider a finite sequence of closed manifolds $S = \{P_0 = *, P_1, \dots, P_n\}$ ordered by increasing dimension. These manifolds will play the role of “singularities” after their contraction. Before the contraction process, the following definition describes how the manifolds should look in order to prescribe the type of singularities we will obtain.

Definition 2.14 A manifold M with general corners is said to be an S -manifold if:

- (1) For any subset $\omega \subset \{0, 1, \dots, n\}$, there exists a decomposed manifold $M(\omega)$ such that:
 - (a) $M(\emptyset) = M$.
 - (b) $\partial M(\omega) = \bigcup_{i \notin \omega} \partial_i M(\omega)$.
 - (c) For every $i \in \{1, \dots, n\} \setminus \omega$, there exists a diffeomorphism

$$(2-3) \quad \beta(\omega, i): \partial_i M(\omega) \simeq M(\omega, i) \times P_i,$$

where $(\omega, i) = \omega \cup \{i\}$.

- (2) For any subset $\omega \subset \{0, 1, \dots, n\}$, and for all $i, j \in \{1, \dots, n\} \setminus \omega$, the diagram

$$\begin{array}{ccccc} \partial_j \partial_i M(\omega) & \xrightarrow{\beta(\omega, i)} & \partial_j M(\omega, i) \times P_i & \xrightarrow{\beta(\omega, i, j) \times \text{id}} & M(\omega, i, j) \times P_j \times P_i \\ \parallel & & & & \downarrow \text{id} \times T \\ \partial_j M(\omega) \cap \partial_i M(\omega) & & & & \\ \parallel & & & & \\ \partial_i \partial_j M(\omega) & \xrightarrow{\beta(\omega, j)} & \partial_j M(\omega, i) \times P_j & \xrightarrow{\beta(\omega, j, i) \times \text{id}} & M(\omega, i, j) \times P_i \times P_j \end{array}$$

is commutative, where T denotes the transposition.

The first part of the definition describes the local structure of the product on the boundary of the decomposed manifold M . The diagram describes how the boundary components are glued together. We now define the particular class of singular manifolds pointed out by N Baas.

Definition 2.15 To any S -manifold M we associate the singular manifold M_S defined as the quotient space $M(\emptyset) / \sim$, where

$$a \sim b \quad \text{if} \quad a, b \in \partial_{i_1} \dots \partial_{i_k} M, \quad i_j \geq 1, \quad k \geq 1,$$

and

$$(2-4) \quad \begin{aligned} \text{pr} \circ \beta(i_1, \dots, i_k) \circ \dots \circ \beta(i_1, i_2) \circ \beta(\emptyset, i_1)(a) \\ = \text{pr} \circ \beta(i_1, \dots, i_k) \circ \dots \circ \beta(i_1, i_2) \circ \beta(\emptyset, i_1)(b). \end{aligned}$$

Here

$$(2-5) \quad \text{pr}: M(i_1, \dots, i_k) \times P_{i_1} \times \dots \times P_{i_k} \rightarrow M(i_1, \dots, i_k)$$

denotes projection on the first factor.

If the elements of S are connected manifolds, then every manifold M with singularities of type S is an admissible pseudomanifold. If not, the following remark will be of fundamental importance in the next section.

Remark 2.16 For each $i = 1, \dots, n$, we decompose the manifold P_i into connected components Q_{ij} and set

$$T = \{Q_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq k_i\}.$$

Given a manifold M_S with singularities of type S modeled on M , the local S -structure (2-3) on ∂M defines a local T -structure. The commutative diagram of Definition 2.14 and the equivalence (2-4) allow us to define an equivalence relation on M denoted by \sim_T and such that the projections (2-5) only occur along the factors of type Q_{ij} . This gives rise to a T -singular manifold M_T defined as the quotient $M(\emptyset)/\sim_T$. A class for the relation \sim_T being a subclass of the relation \sim_S , we get a canonical map of degree 1:

$$(2-6) \quad q: M_T \rightarrow M_S.$$

2.4 Realization of homology classes by admissible geometric cycles

We now prove Proposition 2.5. Following Milnor [21] and Novikov [22], the complex cobordism ring Ω_*^U is isomorphic to the ring of integer polynomials $\Omega_*^U = \mathbb{Z}[x_1, x_2, \dots]$, where each generator x_k is of degree $2k$ and can be represented by a manifold P_k . Each representant P_k can be chosen as a complex algebraic manifold; see Stong [26] for instance. But the connectivity of this complex manifold is not clear in general (if $k = p^s - 1$, where p is a prime number, P_k can be chosen as $\mathbb{C}P^k$). Define the sequence of singularities

$$(2-7) \quad S = \{P_1, P_2, \dots\}.$$

According to Baas' theorem [1, Corollaire 5.1], given a homology class $a \in H_m(X, \mathbb{Z})$, there exists a manifold M_S with singularities of type S of dimension m and a map

$f: M_S \rightarrow X$ such that $f_*[M_S] = \mathbf{a}$. The elements in S are not necessarily connected manifolds. So we proceed as in Remark 2.16, and obtain in this way a new manifold with singularities M' representing \mathbf{a} which is now an admissible pseudomanifold. Finally, we add if necessary 1–handles to M' and extend the map $f' = f \circ q$ (where q denotes the canonical map from M' to M_S of degree 1) in such a way that $f'_\#$ becomes an epimorphism between fundamental groups. This concludes the proof.

Remark 2.17 The admissible pseudomanifold M' which realizes \mathbf{a} can be chosen as a manifold with singularities whose singularities are more specific. In $\mathbb{C}P^m \times \mathbb{C}P^n$ with $m \leq n$, we consider the hypersurface of degree (1, 1) given by

$$H_{m,n} = \{z_0w_0 + z_1w_1 + \dots + z_mw_m = 0\},$$

where (z_0, z_1, \dots, z_m) and (w_0, w_1, \dots, w_n) denote the homogeneous coordinates in $\mathbb{C}P^m$ and $\mathbb{C}P^n$ respectively. The manifolds $H_{m,n}$ are known as *Milnor’s manifolds*. The cobordism classes of the $\{H_{m,n}\}_{m \leq n}$ together with the family of classes $\{\mathbb{C}P^s\}_{s \geq 1}$ give rise to a spanning family of Ω_*^U (see Hirzebruch [19] and [22]). The classes in Ω_*^U are thus linear combinations (with integer coefficients) of cobordism classes $\{H_{m,n}\}_{m \leq n}$ and $\{\mathbb{C}P^s\}_{s \geq 1}$. So we can choose the $\{P_k\}_{k \geq 1}$ as a disjoint union of some of these manifolds endowed with an adequate orientation. Taking into account Remark 2.16, an admissible pseudomanifold which represents the class $\mathbf{a} \in H_*(K, \mathbb{Z})$ can be chosen as a manifold with singularities of type $\{\mathbb{C}P^s, H_{m,n}\}$.

3 Relative density of the values of systolic volume

The aim of this section is to show that the set of values of systolic volume over the set of homology classes (resp. over the set of orientable manifolds) of fixed dimension is a relatively dense set in the following sense.

Definition 3.1 Given a subset $A \subset \mathbb{R}^+$ and a positive number d , A is said to be *d–dense* in \mathbb{R}^+ if for any interval $I \subset \mathbb{R}^+$ of length $|I| > d$, the intersection $I \cap A$ is not empty.

For a fixed dimension $m \geq 3$, define

$$\Sigma_m := \{\mathfrak{S}(G, \mathbf{a}) \mid G \text{ finitely presentable group and } \mathbf{a} \neq 0 \in H_m(G, \mathbb{Z})\},$$

$$\sigma_m := \inf \Sigma_m.$$

Similarly, set

$$\Omega_m^+ := \{\mathfrak{S}(M) \mid M \text{ orientable essential manifold of dimension } m\},$$

$$\omega_m^+ := \inf \Omega_m^+.$$

Recall that $\sigma_m > 0$ by [17] and that an orientable manifold M is said to be *essential* if the image of its fundamental class under its classifying map is not zero. In particular, $\omega_m^+ \geq \sigma_m$. The main result of this section is the following proposition.

Proposition 3.2 *For any dimension $m \geq 3$, the set Σ_m (resp. Ω_m^+) is σ_m -dense (resp. ω_m^+ -dense) in \mathbb{R}^+ .*

Proposition I of the introduction is a direct consequence of this statement. In order to prove Proposition 3.2, we will study the behavior of systolic volume under the operation of connected sum. Part of these results will also be useful in the next section.

Remark 3.3 Remark that the set Ω_m of values taken by the systolic volume over all manifolds of the same dimension m (not necessarily orientable) contains the subset Ω_m^+ : it is also a relatively dense set of \mathbb{R}^+ with density ω_m^+ . It is not clear if this density can be decreased and the answer may depend on the parity of the dimension.

Fix a morphism of groups $\pi: G \rightarrow G'$ and a homology class $\mathbf{a} \in H_m(G, \mathbb{Z})$.

Proposition 3.4 *If $m \geq 3$,*

$$\mathfrak{S}(G', \pi_*\mathbf{a}) \leq \mathfrak{S}(G, \mathbf{a}).$$

Proof According to Proposition 2.5, fix an admissible geometric cycle (X, f) representing normally the class \mathbf{a} . The admissible geometric cycle $(X, \pi \circ f)$ is a representation of the class $\pi_*(\mathbf{a})$, so

$$\mathfrak{S}(G', \pi_*(\mathbf{a})) \leq \mathfrak{S}_{\pi \circ f}(X) \leq \mathfrak{S}_f(X) = \mathfrak{S}(G, \mathbf{a})$$

by Proposition 2.4. □

Corollary 3.5 *Let X_1 and X_2 be two orientable admissible pseudomanifolds of dimension $m \geq 3$. Then*

$$\max\{\mathfrak{S}(X_1), \mathfrak{S}(X_2)\} \leq \mathfrak{S}(X_1 \# X_2),$$

where $X_1 \# X_2$ denotes the connected sum of X_1 and X_2 .

Proof Denote by $f_j: X_j \rightarrow K(\pi_1(X_j), 1)$ the classifying map for $j = 1, 2$. Observe that $\pi_1(X_1 \# X_2) = \pi_1(X_1) * \pi_1(X_2)$. We have a natural monomorphism $i_j: \pi_1(X_j) \hookrightarrow$

$\pi_1(X_1) * \pi_1(X_2)$ and a natural epimorphism $s_j: \pi_1(X_1) * \pi_1(X_2) \rightarrow \pi_1(X_j)$ such that $s_j \circ i_j = \text{id}_{\pi_1(X_j)}$, $s_2 \circ i_1 = 0$ and $s_1 \circ i_2 = 0$. By Proposition 3.4,

$$\begin{aligned} \mathfrak{S}(X_1 \# X_2) &= \mathfrak{S}(\pi_1(X_1) * \pi_1(X_2), (i_1 \circ f_1)_*[X_1] + (i_2 \circ f_2)_*[X_2]) \\ &\geq \mathfrak{S}(s_j(\pi_1(X_1) * \pi_1(X_2)), (s_j)_* \circ ((i_1 \circ f_1)_*[X_1] + (i_2 \circ f_2)_*[X_2])) \\ &\geq \mathfrak{S}(\pi_1(X_j), (f_j)_*[X_j]) = \mathfrak{S}(X_j) \end{aligned}$$

for $j = 1, 2$. □

Furthermore, we have the following comparison result:

Proposition 3.6 *Let X_1 and X_2 be two orientable pseudomanifolds of dimension $m \geq 3$. Then*

$$\mathfrak{S}(X_1 \# X_2) \leq \mathfrak{S}(X_1) + \mathfrak{S}(X_2).$$

Proof The contraction of the gluing sphere into a point gives rise to a natural projection map

$$p: X_1 \# X_2 \rightarrow X_1 \vee X_2$$

which induces an isomorphism between fundamental groups if $m \geq 3$. Applying the comparison principle (see [3, Proposition 3.2]), we get

$$\mathfrak{S}(X_1 \# X_2) \leq \mathfrak{S}(X_1 \vee X_2) = \mathfrak{S}(X_1) + \mathfrak{S}(X_2). \quad \square$$

With these two comparison results, we can now prove Proposition 3.2.

We first prove the σ_m -density of the set Σ_m . Consider any interval $I \subset \mathbb{R}^+$ such that $|I| > \sigma_m$. Choose a pair (G, \mathbf{a}) composed of a finitely presentable group and an m -class of its homology such that $\sigma_m \leq \mathfrak{S}(G, \mathbf{a}) < |I|$. According to Proposition 2.5, we can choose an admissible geometric cycle (X, f) representing normally the class $\mathbf{a} \in H_m(G, \mathbb{Z})$ which satisfies $\mathfrak{S}_f(X) = \mathfrak{S}(G, \mathbf{a})$ by Proposition 2.4. If the map $f_\#$ is not a monomorphism, we can directly contract some loops in X and get a new admissible geometric cycle (X', f') representing \mathbf{a} such that $f'_\#$ is an isomorphism. So

$$\mathfrak{S}_f(X) = \mathfrak{S}(X').$$

The sequence $\{a_n = \mathfrak{S}(\#_n X')\}_{n=1}^\infty$ is increasing by Corollary 3.5 and satisfies $a_{n+1} - a_n \leq \mathfrak{S}(X') < |I|$ by Proposition 3.6. By Sabourau [25], we have

$$\mathfrak{S}\left(\#_n X\right) \geq C_m \frac{n}{\exp(C'_m \sqrt{\log n})},$$

where C_m and C'_m are two positive numbers depending only on the dimension m (strictly speaking, this inequality is proved for manifolds, but carries over perfectly to pseudomanifolds). So the sequence $\{a_n\}$ is not bounded and does not intersect I .

The proof of the ω_m^+ -density of Ω_m^+ is similar: for any interval $I \subset \mathbb{R}^+$ with $|I| > \omega_m^+$, we can argue as previously starting with the sequence $\{a_n = \mathfrak{S}(\#_n M)\}_{n=1}^\infty$, where M is an orientable essential manifold of dimension m with $\omega_m^+ \leq \mathfrak{S}(M) < |I|$.

4 Nonfiniteness of irreducible homology classes

This section deals with the following natural question:

Question 4.1 Given a positive number T , how many finitely presentable groups G are there, such that any essential (orientable) manifold M of dimension m with fundamental group G satisfies $\mathfrak{S}(M) \leq T$?

In dimension 2, this number is bounded from below by $c \cdot T(\ln T)^2$ and from above by $C \cdot T(\ln T)^2$ for some positive constants c and C ; see [11; 18]. In fact the situation is quite rigid in dimension 2. Even the finiteness of the systolic volume over the set of finite simplicial complexes of dimension 2 holds. More precisely, recall that:

- The *systolic area* of a finitely presentable group G is defined as

$$\mathfrak{S}(G) = \inf_{\pi_1(P)=G} \mathfrak{S}(P),$$

where the infimum is taken over all finite simplicial complexes P of dimension 2 with fundamental group G .

- A finitely presentable group G is said to be *of zero free index* if G cannot be written as a free product $H * F_n$ for some $n > 0$; compare Rudyak and Sabourau [23].

Then the number of finitely presentable groups G of zero free index such that $\mathfrak{S}(G) \leq T$ does not exceed $K T^3$ where $K > 1$ is an explicit constant; see [23].

The situation is rather different in higher dimensions. Let M be an essential manifold of dimension $m \geq 4$ whose fundamental group is of zero free index and N a nonessential manifold of the same dimension with fundamental group of zero free index. The fundamental group $\pi_1(M) * \pi_1(N)$ of the connected sum $M \# N$ is still of zero free index. By Proposition 3.6 we get

$$\mathfrak{S}(M \# N) \leq \mathfrak{S}(M).$$

(Recall that nonessential orientable manifolds have systolic volume equal to zero; see Babenko [2] for instance). That is, while staying in the class of groups of zero free index we can considerably modify the fundamental group of a manifold without increasing the systolic volume. So there is no hope to obtain finiteness results in this context. This is why we introduce the following.

Definition 4.2 Let G be a finitely presentable group. A class $\mathbf{a} \in H_m(G, \mathbb{Z})$ is said to be reducible if there exists a proper subgroup $H < G$ and a class $\mathbf{b} \in H_m(H, \mathbb{Z})$ such that $i_*(\mathbf{b}) = \mathbf{a}$, where i denotes the canonical inclusion. In the contrary case, the class \mathbf{a} is said to be irreducible.

Furthermore, we will say that a manifold M is reducible (resp. irreducible) if the image of its fundamental class $[M]$ (under the classifying map) in $H_m(\pi_1(M), \mathbb{Z})$ is a reducible (resp. irreducible) class.

Example 4.3 Let M be an aspherical manifold of dimension m (that is $\pi_k(M) = 0$ for $k > 1$). Then M is irreducible.

On the other hand if M and N are nonsimply connected manifolds and N is not essential then the connected sum $M \# N$ is always reducible.

Example 4.4 Let G be a finite group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ be a class of order $|G|$. Then \mathbf{a} is irreducible.

This last example shows that the fundamental class of a lens manifold is irreducible.

Remark that it is possible that:

- Any multiple of an irreducible class is irreducible as in the case $G = \mathbb{Z}_p$ for p a prime number.
- Each multiple of an irreducible class is reducible as in the case of tori \mathbb{T}^m .

On the other hand, there exist groups G and classes $\mathbf{a} \in H_m(G, \mathbb{Z})$ which are *completely reducible* in the following sense: \mathbf{a} is reducible, and any class $\mathbf{b} \in H_m(H, \mathbb{Z})$, where H is a proper subgroup $H < G$ and such that $i_*(\mathbf{b}) = \mathbf{a}$ is also reducible.

Given a positive number T and a positive integer m , we denote by $\mathcal{F}(m, T)$ the set of finitely presentable groups G such that there exists an irreducible class $\mathbf{a} \in H_m(G, \mathbb{Z})$ with $\mathfrak{S}(\mathbf{a}) \leq T$. Theorem II of the introduction can be now restated as follows.

Theorem 4.5 *The set $\mathcal{F}(m, 1)$ is infinite for any dimension $m \geq 3$.*

Proof Let p be a prime number and set

$$G(p, m) := \underbrace{\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_m.$$

Denote by $\phi_p: \pi_1(\mathbb{T}^m) \rightarrow G(p, m)$ the natural projection and set

$$\mathbf{a}(p, m) := (\phi_p)_*[\mathbb{T}^m] \in H_m(G(p, m), \mathbb{Z}).$$

In order to prove that $\mathbf{a}(p, m) \neq 0$ in $H_m(G(p, m), \mathbb{Z})$, we will show that the reduction modulo p of $\mathbf{a}(p, m)$ is not null in $H_m(G(p, m), \mathbb{Z}_p)$. Consider the generators v_1, \dots, v_m of $H^1(G(p, m), \mathbb{Z}_p)$ corresponding to the natural projections of $G(p, m)$ on each factor. The elements $u_i := (\phi_p)^*(v_i)$, $i = 1, 2, \dots, m$ generate the group $H^1(\mathbb{T}^m, \mathbb{Z}_p)$, and $u := u_1 \cup \cdots \cup u_m$ generates the group $H^m(\mathbb{T}^m, \mathbb{Z}_p)$. So $u \cap [\mathbb{T}^m]_p = 1$, where $[\mathbb{T}^m]_p$ denotes the reduction modulo p of the fundamental class $[\mathbb{T}^m]$. This implies

$$(v_1 \cup \cdots \cup v_m) \cap (\phi_p)_*[\mathbb{T}^m]_p = (\phi_p)^*(v_1) \cup \cdots \cup (\phi_p)^*(v_m) \cap [\mathbb{T}^m]_p = 1.$$

This proves the nontriviality of $(\phi_p)_*[\mathbb{T}^m]_p$, and thus of $(\phi_p)_*[\mathbb{T}^m]$.

We now prove the irreducibility of $\mathbf{a}(p, m)$. Let us suppose the contrary. Any proper subgroup H of $G(p, m)$ is also a \mathbb{Z}_p -vector subspace of $G(p, m)$ of dimension $k < m$. Associated to some complement of H in $G(p, m)$, we construct a projection map

$$\pi: G(p, m) \rightarrow H$$

which is the identity on H . Fix a basis of the free \mathbb{Z} -module $\pi_1(\mathbb{T}^m)$ such that the composition

$$\pi \circ \phi_p: \pi_1(\mathbb{T}^m) \rightarrow H$$

decomposes as

$$\pi \circ \phi_p = \psi \circ \rho_{\#},$$

where $\rho_{\#}$ is induced by some projection $\rho: \mathbb{T}^m \rightarrow \mathbb{T}^k$ and $\psi: \pi_1(\mathbb{T}^k) \rightarrow H$ corresponds to the reduction modulo p . Now assume that $\mathbf{a}(p, m) = i_*(\mathbf{b})$, where $\mathbf{b} \in H_m(H, \mathbb{Z})$ and $i: H \rightarrow G(p, m)$ denotes the inclusion. Then

$$\mathbf{b} = \pi_*(\mathbf{a}(p, m)) = \pi_* \circ (\phi_p)_*[\mathbb{T}^m] = \psi_* \circ \rho_*[\mathbb{T}^m] = \psi_*(0) = 0$$

as $\mathbf{b} = \pi_* \circ i_*(\mathbf{b})$. This gives a contradiction the class $\mathbf{a}(p, m)$ being nontrivial.

In order to conclude the proof, remark that

$$\mathfrak{S}(\mathbf{a}(p, m)) = \mathfrak{S}_{\phi_p}(\mathbb{T}^m) \leq \mathfrak{S}(\mathbb{T}^m) \leq 1,$$

for any $m \geq 3$ and any prime p . □

This theorem implies the following unexpected result in dimensions $m \geq 4$.

Corollary 4.6 *For any dimension $m \geq 4$, there exists an infinite number of irreducible orientable manifolds M of dimension m with pairwise nonisomorphic fundamental groups such that $\mathfrak{S}(M) \leq 1$.*

Proof By construction, every class $\mathbf{a}(p, m)$ is representable by a manifold. If $m \geq 4$, such a manifold can be modified by surgery in order to get a new manifold denoted by $M(p, m)$ such that $\pi_1(M(p, m)) = G(p, m)$ and $\Phi_*[M(p, m)] = \mathbf{a}(p, m)$, where $\Phi: M(p, m) \rightarrow K(G(p, m), 1)$ denotes the classifying map; see [3]. The infinite sequence of manifolds $\{M(p, m)\}$ where p runs over all prime numbers gives an infinite sequence of irreducible orientable manifolds M of dimension m with pairwise nonisomorphic fundamental groups such that $\mathfrak{S}(M) \leq 1$. \square

The following natural question remains open:

Question 4.7 Consider the systolic volume $\mathfrak{S}(\cdot)$ as a function over all irreducible orientable manifolds of dimension m . Does there exist a positive number C such that the number of distinct values of the function $\mathfrak{S}(\cdot)$ less than C is infinite?

5 Systolic volume of multiple classes

Given a finitely presentable group G and a homology class $\mathbf{a} \in H_*(G, \mathbb{Z})$, how does the function $\mathfrak{S}(G, k\mathbf{a})$ behave in terms of the integer variable k ? In this section, we prove Theorem III which we restate here for the reader's convenience.

Theorem 5.1 *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$, where $m \geq 3$. Then there exists a positive number $C(G, \mathbf{a})$ depending only on the pair (G, \mathbf{a}) so that*

$$(5-1) \quad \mathfrak{S}(G, k\mathbf{a}) \leq C(G, \mathbf{a}) \cdot \frac{k}{\ln(1+k)}$$

for any positive integer k .

Before proving this result, we put the question in a more general context.

5.1 Systolic volume of the sum of homology classes

Let G_1 and G_2 be two finitely presentable groups and denote by $i_j: G_j \hookrightarrow G_1 * G_2$ for $j = 1, 2$ the natural monomorphisms. Fix two integer homology classes $\mathbf{a}_1 \in H_m(G_1, \mathbb{Z})$ and $\mathbf{a}_2 \in H_m(G_2, \mathbb{Z})$. The natural isomorphism

$$H_m(G_1 * G_2, \mathbb{Z}) \simeq H_m(G_1, \mathbb{Z}) \oplus H_m(G_2, \mathbb{Z})$$

for positive m allows us to denote by $\mathbf{a}_1 + \mathbf{a}_2$ the class $(i_1)_*(\mathbf{a}_1) + (i_2)_*(\mathbf{a}_2)$.

Proposition 5.2 *If $m \geq 3$,*

$$\mathfrak{S}(G_1 * G_2, \mathbf{a}_1 + \mathbf{a}_2) \leq \mathfrak{S}(G_1, \mathbf{a}_1) + \mathfrak{S}(G_2, \mathbf{a}_2).$$

Proof For any $\varepsilon > 0$ and for $j = 1, 2$ we choose a geometric cycle (X_j, f_j) of dimension m representing \mathbf{a}_j and satisfying

$$\mathfrak{S}_{f_j}(X_j) \leq \mathfrak{S}(G_j, \mathbf{a}_j) + \frac{1}{2}\varepsilon.$$

The geometric cycle $(X_1 \# X_2, f_1 \# f_2)$ obtained as the connected sum of (X_1, f_1) and (X_2, f_2) represents the class $\mathbf{a}_1 + \mathbf{a}_2$. By the comparison principle (see [3]),

$$\mathfrak{S}_{f_1 \# f_2}(X_1 \# X_2) \leq \mathfrak{S}_{f_1 \vee f_2}(X_1 \vee X_2) = \mathfrak{S}_{f_1}(X_1) + \mathfrak{S}_{f_2}(X_2),$$

where $(X_1 \vee X_2, f_1 \vee f_2)$ denotes the wedge sum of (X_1, f_1) and (X_2, f_2) . As ε can be chosen arbitrarily small, we get the result. \square

If \mathbf{a}_1 and \mathbf{a}_2 are two homology classes of dimension m of the same group G , we deduce the following subadditivity property of the systolic volume.

Corollary 5.3 *Let \mathbf{a}_1 and \mathbf{a}_2 be two classes of $H_m(G, \mathbb{Z})$ with $m \geq 3$. Then*

$$\mathfrak{S}(G, \mathbf{a}_1 + \mathbf{a}_2) \leq \mathfrak{S}(G, \mathbf{a}_1) + \mathfrak{S}(G, \mathbf{a}_2).$$

Proof Indeed, if we denote by $\pi: G * G \rightarrow G$ the epimorphism defined by $\pi \circ i_j = \text{id}_G$, then $\pi_*(\mathbf{a}_1 + \mathbf{a}_2) = \mathbf{a}_1 + \mathbf{a}_2$. By Proposition 3.4, we get the result. \square

As a direct consequence of this corollary, we derive that

$$\mathfrak{S}(G, k\mathbf{a}) \leq k \cdot \mathfrak{S}(G, \mathbf{a})$$

for any $\mathbf{a} \in H_m(G, \mathbb{Z})$ with $m \geq 3$ and any integer k . In particular the limit

$$\lim_{k \rightarrow \infty} \frac{\mathfrak{S}(G, k\mathbf{a})}{k}$$

exists. Theorem 5.1 permits us to conclude that this limit is always zero.

5.2 Sublinear upper bound for the systolic volume of multiple classes

Theorem 5.1 is related to the behavior of systolic volume for connected sum and is a direct consequence of the following result.

Theorem 5.4 Let X be a connected pseudomanifold of dimension $m \geq 3$ and denote by $\#_k X$ the connected sum of k copies of X . There exists a positive number $C(X)$ depending only on the topology of X such that

$$(5-2) \quad \mathfrak{S}\left(\#_k X\right) \leq C(X) \cdot \frac{k}{\ln(1+k)}$$

for any positive integer k . If k is large enough, this last inequality (5-2) is satisfied for

$$C(X) = m \cdot c(X) \cdot \ln c(X),$$

where $c(X)$ stands for the minimal number of m -cubes in a cubical decomposition of X .

Recall that a cubical decomposition of X is a family of m -dimensional embedded topological cubes covering X and such that any nonempty intersection between two cubes occurs along a lower-dimensional face. In particular, any $(m - 1)$ -face belongs to exactly two m -cubes. The existence of such decompositions can be proved by considering a triangulation of X and decomposing each m -simplex of this triangulation into $m + 1$ m -cubes.

Remark 5.5 Theorem 5.4 substantially improves [5, Theorem A], and the upper bound (5-2) still holds for a sequence of cyclic covering, the details being similar to those considered in the sequel (compare with [5]).

Proof The proof relies on ideas and techniques appearing in [5]. We consider a minimal cubical decomposition Θ of X with $c(X)$ elements. From each cube $C \in \Theta$ we remove an open cube $C' \simeq]1/4, 3/4[^m \subset [0, 1]^m \simeq C$ and denote by X' the polyhedron thus obtained. Each set (called a *sleeve* of X')

$$\bar{C} = C \setminus C' \simeq \partial C \times [0, 1/2]$$

is endowed with the product metric $g_{\partial C} \times \frac{1}{N} dt$, where $g_{\partial C}$ denotes the metric on ∂C induced by a realization of $C \simeq [0, 1]^m \subset \mathbb{R}^m$ in the standard Euclidean vector space and N is a positive integer to be chosen later. Choose the metrics $\{g_{\partial C}\}_{C \in \Theta}$ to coincide on each nonempty intersection between two m -cubes, and denote by g' the Riemannian polyhedral metric thus obtained on X' . By construction:

- The complex X' is homeomorphic to X minus $c(X)$ disjoint open m -disks, and in particular $\pi_1(X') = \pi_1(X)$.
- $\text{vol}(X', g') = m \cdot \frac{1}{N} \cdot c(X)$.
- The obvious retraction of X' onto $\Theta^{(m-1)}$ (where $\Theta^{(m-1)}$ denotes the $(m - 1)$ -skeleton of Θ) is distance nonincreasing.

Fix an m -cube $\mathcal{C} \in \Theta$. We define the star $\overline{\text{st}}(\mathcal{C})$ as the union of all cubes with nonempty intersection with \mathcal{C} (including \mathcal{C} itself). The open star $\text{st}(\mathcal{C})$ is then defined as the union of the interiors of all cubes having a nonempty intersection with \mathcal{C} , so the closure of $\text{st}(\mathcal{C})$ is exactly $\overline{\text{st}}(\mathcal{C})$. The following lemma will be important in what follows.

Lemma 5.6 *If a relative curve γ of $(X', \overline{\mathcal{C}})$ is not entirely included in $X' \cap \text{st}(\mathcal{C})$, then $l_{g'}(\gamma) \geq 2$.*

Proof Let p belongs to the boundary $\partial(\overline{\text{st}}(\mathcal{C})) = \overline{\text{st}}(\mathcal{C}) \setminus \text{st}(\mathcal{C})$. Then

$$d_{g'}(p, \partial\mathcal{C}) \geq 1$$

which proves the lemma.

Indeed, suppose that this is not the case. The point p lies in the $(m - 1)$ -face F of an m -cube $\mathcal{C}_1 \subset \overline{\text{st}}(\mathcal{C})$. But F belongs to exactly two m -cubes \mathcal{C}_1 and \mathcal{C}_2 . As $d_{g'}(F, \partial\mathcal{C}) \leq d_{g'}(p, \partial\mathcal{C}) < 1$, we conclude that $F \cap \mathcal{C} \neq \emptyset$ which implies that $\mathcal{C}_2 \subset \overline{\text{st}}(\mathcal{C})$. This together with the condition $d_{g'}(p, \partial\mathcal{C}) < 1$ implies that $p \in \text{st}(\mathcal{C})$ which is a contradiction. \square

According to the construction of graphs with large girth by Erdős and Sachs [15], there exists for any positive integers n and N such that

$$n \geq 2 \sum_{t=1}^{N-2} (c(X) - 1)^t$$

a $c(X)$ -regular graph Γ with $2n$ vertices whose girth—defined as the least number of edges composing a cycle—is at least N . The graph Γ can be thought as a 1-dimensional polyhedron on which we define the PL-metric h for which all edges have length $1/N$. The girth's bound thus says that $\text{sys}(\Gamma, h) \geq 1$.

Denote by v_1, \dots, v_{2n} the vertices of Γ and let $(X'_1, g'_1), \dots, (X'_{2n}, g'_{2n})$ be $2n$ copies of the Riemannian polyhedron (X', g') . We define a map $F: X'_1 \cup \dots \cup X'_{2n} \rightarrow \Gamma$ as follows. For each $i = 1, \dots, 2n$:

- Set $F(p) = v_i$ if $p \in \Theta_i^{m-1}$, where Θ_i^{m-1} denotes the $(m - 1)$ -skeleton of the cubical decomposition Θ_i corresponding to X'_i .
- Denote by $e^i_1, \dots, e^i_{c(X)}$ the edges adjacent to v_i , and by $\overline{\mathcal{C}}^i_1, \dots, \overline{\mathcal{C}}^i_{c(X)}$ the sleeves of X'_i corresponding to the cubical decomposition Θ_i . For each $j = 1, \dots, c(X)$, each edge e^i_j is isometric to $([0, 1], \frac{1}{N} dt)$ with 0 being identified with the vertex v_i . Then for each point $p = (x, t)$ belonging to the sleeve $\overline{\mathcal{C}}^i_j \simeq \partial\mathcal{C}^i_j \times [0, \frac{1}{2}]$ set $F_i(p) = t$.

Observe that each restriction $F|_{X'_i}: (X'_i, g'_i) \rightarrow (\Gamma, h)$ is distance nonincreasing.

Now to each edge $e = (v_i, v_j) \in E(\Gamma)$ correspond exactly two sleeves \bar{C} and \bar{C}' belonging respectively to the two distinct copies X'_i and X'_j of X' and such that their images under F cover the edge e . We choose a PL-isometry denoted by ϕ_e between $(\partial C \times \{\frac{1}{2}\}, g'_i)$ and $(\partial C' \times \{\frac{1}{2}\}, g'_j)$. The quotient space

$$X(n) = (X'_1 \cup \dots \cup X'_{2n}) / \{\phi_e\}_{e \in E(\Gamma)}$$

is a polyhedron homeomorphic to

$$\left(\#_{2n} X\right) \# \left(\#_{n(c-2)+1} S^1 \times S^{m-1}\right).$$

The metrics $\{g'_i\}_{i=1, \dots, n}$ are compatible with the quotient map $X'_1 \cup \dots \cup X'_n \twoheadrightarrow X(n)$ and give rise to a Riemannian polyhedral metric on $X(n)$ that we denote by $g(n)$. The map F induces a new map still denoted by F from $X(n)$ onto Γ . By construction this map $F: (X(n), g(n)) \rightarrow (\Gamma, h)$ is distance nonincreasing.

According to [5, Proposition 1] the addition of 1-handles does not change the value of the systolic volume, and therefore

$$\mathfrak{S}\left(\#_{2n} X\right) = \mathfrak{S}(X(n)) \leq \frac{\text{vol}(X(n), g(n))}{(\text{sys}(X(n), g(n)))^m}.$$

We have

$$\text{vol}(X(n), g(n)) = 2n \cdot \text{vol}(X', g') = 2n \cdot m \cdot \frac{1}{N} \cdot c(X).$$

Lemma 5.7 *We have*

$$(5-3) \quad \text{sys}(X(n), g(n)) \geq 1.$$

Proof Let $\gamma: [0, 1] \rightarrow X(n)$ be a closed curve which is not contractible.

If its image $F(\gamma)$ is not contractible in Γ , then

$$l_{g(n)}(\gamma) \geq l_h(F(\gamma)) \geq \text{sys}(\Gamma, h) \geq 1$$

as F is distance nonincreasing.

Now suppose that $F(\gamma)$ is contractible in Γ . We will show that $l_{g(n)}(\gamma) \geq 2$ using Lemma 5.6.

First of all, we can assume that γ is minimizing in its own homotopy class. The contractibility of $F(\gamma)$ implies that we can find an edge $[v_i, v_j]$ of Γ joining two vertices v_i and v_j , a point $v \in]v_i, v_j]$ and a triplet $0 \leq t_1 < t_* < t_2 \leq 1$ such that:

- $v = F(\gamma(t_*))$.
- $F(\gamma([t_1, t_2])) \subset [v_i, v]$.
- $F(\gamma(t_1)) = F(\gamma(t_2)) = v_i$.

Denote by m_{ij} the midpoint of the edge $[v_i, v_j]$, and by

$$\bar{C}_i = \overline{F^{-1}([v_i, m_{ij}])} \quad \text{and} \quad \bar{C}_j = \overline{F^{-1}([m_{ij}, v_j])}$$

the two corresponding sleeves.

If $v \in]v_i, m_{ij}]$, we contract the portion of the curve $\{\gamma(t)\}_{t \in [t_1, t_2]}$ using the projection $\pi_i: \bar{C}_i \simeq \partial C_i \times [0, \frac{1}{2}] \rightarrow \partial C_i \times \{0\}$ given by the formula $\pi_i(p, t) = (p, 0)$. This contraction strictly decreases the length of γ which is in contradiction with its minimality. If $v \in]m_{ij}, v_j[$, we contract the portion of the curve $\{\gamma(t)\}_{t \in [t_1, t_2]}$ lying in \bar{C}_j using the projection $\pi_j: \bar{C}_j \simeq \partial C_j \times [0, \frac{1}{2}] \rightarrow \partial C_j \times \{\frac{1}{2}\}$ given by the formula $\pi_j(p, t) = (p, \frac{1}{2})$. This contraction also strictly decreases the length of γ which is again in contradiction with its minimality. So the point v must coincide with v_j .

The restriction of γ to the interval $[t_1, t_2]$ is thus the concatenation $\gamma_1 \star \gamma_2 \star \gamma_3$ of three curves with $\gamma_1, \gamma_3 \subset F^{-1}([v_i, v_j])$ and $\gamma_2 \subset F^{-1}(v_j) = \Theta_j^{m-1}$. We will show that γ_2 is not entirely contained in the open star $\text{st}(C_j)$ corresponding to the sleeve \bar{C}_j , and thus

$$l_{g(n)}(\gamma) \geq l_{g'}(\gamma_2) \geq 2$$

according to Lemma 5.6 which concludes the proof of the lemma.

Indeed suppose that γ_2 is entirely contained in the open star $\text{st}(C_j)$. Observe first that $\gamma_2 \not\subset \bar{C}_j$. Otherwise we contract the portion of the curve $\{\gamma(t)\}_{t \in [t_1, t_2]}$ lying in \bar{C}_j using the projection $\pi_j: \bar{C}_j \simeq \partial C_j \times [0, \frac{1}{2}] \rightarrow \partial C_j \times \{\frac{1}{2}\}$ given by the formula $\pi_j(p, t) = (p, \frac{1}{2})$. This contraction strictly decreases the length of γ which is in contradiction with its minimality. Now the orthogonal projections of $\text{st}(\bar{C}_j) \cap \Theta_j^{m-1}$ on each face of $\bar{C}_j \cap \Theta_j^{m-1}$ are correctly defined and coherent. So by projecting γ_2 orthogonally on $\bar{C}_j \cap \Theta_j^{m-1}$ we do not change the homotopy class of γ , and since $\gamma_2 \not\subset \bar{C}_j$, the length of γ strictly decreases. This is again a contradiction with its minimality. □

So

$$(5-4) \quad \mathfrak{S}\left(\frac{\# X}{2n}\right) \leq \frac{\text{vol}(X(n), g(n))}{(\text{sys}(X(n), g(n)))^m} \leq m \cdot \frac{2n}{N} \cdot c(X)$$

for any positive integers n and N such that

$$n \geq \frac{2}{c(X)-2} [(c(X)-1)^{N-1} - (c(X)-1)].$$

Fix a positive integer n . As $c(X) \geq 2m + 1 \geq 7$, we choose N such that

$$n \in [(c(X)-1)^{N-1}, (c(X)-1)^N],$$

and get that

$$\mathfrak{S}\left(\#_{2n} X\right) \leq m \cdot c(X) \cdot \ln(c(X) - 1) \cdot \frac{2n}{\ln(2n+1)}$$

according to (5-4). Furthermore

$$\begin{aligned} \mathfrak{S}\left(\#_{2n+1} X\right) &\leq \mathfrak{S}\left(\#_{2n} X\right) + \mathfrak{S}(X) \\ &\leq m \cdot c(X) \cdot \ln(c(X) - 1) \cdot \frac{2n}{\ln(2n+1)} + \mathfrak{S}(X) \\ &\leq m \cdot c(X) \cdot \ln(c(X) - 1) \cdot \frac{2n+1}{\ln(2n+2)} + \mathfrak{S}(X). \end{aligned}$$

For n large enough, we thus get the universal upper bound (5-2) which concludes the proof. □

5.3 Homology classes with positive simplicial volume

Recall the following definition (see Gromov [16]).

Definition 5.8 Let X be a pseudomanifold of dimension m . Its *simplicial volume* is the quantity

$$\|X\|_{\Delta} = \inf \left\{ \sum_i |r_i| \mid [X] = \sum_i r_i \sigma_i^m \right\},$$

where the infimum is taken over the set of representations of the fundamental class $[X]$ by singular simplicial chains with real coefficients.

If G denotes a finitely presentable group and \mathbf{a} a homology class of dimension m , the simplicial volume of \mathbf{a} is then the number

$$\|\mathbf{a}\|_{\Delta} = \inf \{ \|X\| \mid X \text{ representing } \mathbf{a} \}.$$

For homology classes whose simplicial volume is positive, the function $\mathfrak{S}(G, k\mathbf{a})$ goes to infinity and the following result provides a better information about its asymptotic behavior.

Corollary 5.9 Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ be a homology class of dimension $m \geq 3$ such that $\|\mathbf{a}\|_{\Delta} > 0$. Then there exist two positive numbers $C(G, \mathbf{a})$ and $C'(G, \mathbf{a})$ depending only on the pair (G, \mathbf{a}) such that

$$C'(G, \mathbf{a}) \cdot \frac{k}{(\ln(1+k))^m} \leq \mathfrak{S}(G, k\mathbf{a}) \leq C(G, \mathbf{a}) \cdot \frac{k}{\ln(1+k)}$$

for any positive integer k .

Proof The lower bound is a direct consequence of the following inequality of Gromov (see [17, Theorem 6.4.D']): any pseudomanifold X of dimension m satisfies the inequality

$$C'_m \frac{\|X\|_\Delta}{(\ln(2 + \|X\|_\Delta))^m} \leq \mathfrak{S}(X),$$

where C'_m is a positive number depending only on the dimension m . It remains to remark that, if X represents the class ka , then $\|X\|_\Delta = \|ka\|_\Delta = k\|a\|_\Delta$. The upper bound then follows by Theorem 5.1. \square

5.4 Large oscillations of systolic volume

The following example shows that the function $k \mapsto \mathfrak{S}(G, ka)$ may have arbitrarily large oscillations.

Let $m = 2l + 1 \geq 3$ be an odd integer and $q \geq 2$ an integer. Let X be an essential manifold of dimension m (for example aspherical) and $f: X \rightarrow K(\pi_1(X), 1)$ be its classifying map. If X is not aspherical, we assume that the image of its fundamental class $f_*[X]$ is an element of infinite order in $H_m(\pi_1(X), \mathbb{Z})$. Set $a = f_*[X]$. Fix a generator $b \in H_m(\mathbb{Z}_q, \mathbb{Z}) = \mathbb{Z}_q$. For each $1 \leq l \leq q - 1$, we choose a normal representation of lb by a manifold Y_l with $\pi_1(Y_l) = \mathbb{Z}_q$. For relatively prime l and q , the corresponding lens space can be chosen to be Y_l . Set

$$D = \max_{1 \leq k \leq q} \mathfrak{S}(\pi_1(X), ka)$$

and fix any positive number C . Consider the free product

$$G_n = \pi_1(X) * \underbrace{\mathbb{Z}_q * \dots * \mathbb{Z}_q}_n,$$

and pick in

$$H_m(G_n, \mathbb{Z}) = H_m(\pi_1(X), \mathbb{Z}) \oplus H_m(\mathbb{Z}_q, \mathbb{Z}) \oplus H_m(\mathbb{Z}_q, \mathbb{Z}) \oplus \dots \oplus H_m(\mathbb{Z}_q, \mathbb{Z})$$

the class $c = a \oplus \underbrace{b \oplus \dots \oplus b}_n$. If X_k is a manifold representing the class ka , then

$$X_k \# \underbrace{Y_k \# \dots \# Y_k}_n$$

represents the class kc . By Corollary 3.5,

$$\mathfrak{S}(G_n, kc) \geq \mathfrak{S}(\underbrace{Y_k \# \dots \# Y_k}_n).$$

Now [25, Theorem A] implies that if n is chosen large enough, we have $\mathfrak{S}(G_n, kc) > C$ for any $1 \leq k \leq q - 1$. Besides $qc = qa$ in $H_m(G_n, \mathbb{Z})$, and so $\mathfrak{S}(G_n, qc) \leq D$.

5.5 Systolic generating function

If G is a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ is a homology class of infinite order, the study of the sequence $\{\mathfrak{S}(G, k\mathbf{a})\}_{k=1}^\infty$ is equivalent to the study of analytic properties of the following generating function:

$$\sigma_{G,\mathbf{a}}(z) = \sum_{k=1}^\infty \mathfrak{S}(G, k\mathbf{a}) \cdot z^k.$$

If the group G is clearly identified by the context, we simplify the notation into $\sigma_{\mathbf{a}}(z)$. The upper bound (5-1) implies that $\sigma_{\mathbf{a}}(z)$ is an analytic function on the disk $|z| < 1$. Furthermore the complex point $z = 1$ is a singular point of this function as $\mathfrak{S}(G, k\mathbf{a}) \geq \sigma_m > 0$.

In general there is no hope for $\sigma_{G,\mathbf{a}}(z)$ to be a rational function. Indeed, if \mathbf{a} is a class with positive simplicial volume, Corollary 5.9 teaches us that $z = 1$ is not a pole of the corresponding systolic generating function, so $\sigma_{\mathbf{a}}(z)$ is not rational. It is well known that the rationality of the generating function of a numerical sequence is equivalent to the recurrence of this sequence. We deduce the following

Proposition 5.10 *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ a homology class of infinite order. If the simplicial volume of \mathbf{a} is positive, the sequence of systolic volumes $\{\mathfrak{S}(G, k\mathbf{a})\}_{k=1}^\infty$ does not satisfy any linear recurrence equation.*

Nevertheless the rationality of $\sigma_{\mathbf{a}}(z)$ seems plausible for classes \mathbf{a} with a bounded sequence of systolic volume $\{\mathfrak{S}(G, k\mathbf{a})\}_{k=1}^\infty$. Tori give a model of this type of behavior for multiple classes.

Proposition 5.11 *Let $3 \leq m \leq n$ be two integers. Any class $\mathbf{a} \in H_m(\mathbb{Z}^n, \mathbb{Z})$ satisfies the inequality*

$$\mathfrak{S}(\mathbb{Z}^n, \mathbf{a}) \leq \binom{n}{m} \cdot \mathfrak{S}(\mathbb{T}^m),$$

where $\binom{n}{m}$ denotes the binomial coefficient.

Proof Fix a basis of $H_m(\mathbb{Z}^n, \mathbb{Z})$ composed of embedded m -tori, and write the class \mathbf{a} in this basis as

$$\mathbf{a} = \sum_{i=1}^{\binom{n}{m}} k_i [\mathbb{T}_i^m],$$

where $k_i \in \mathbb{Z}$ for each i . By Corollary 5.3,

$$(5-5) \quad \mathfrak{S}(G, \mathbf{a}) \leq \sum_{i=1}^{\binom{n}{m}} \mathfrak{S}(G, k_i[\mathbb{T}_i^m]).$$

Observe that

$$(5-6) \quad \mathfrak{S}(\mathbb{Z}^n, k[\mathbb{T}_i^m]) \leq \mathfrak{S}(\mathbb{T}^m)$$

for any integer k . In fact, if $f: \mathbb{T}^m \rightarrow \mathbb{T}_i^m$ denotes a map of degree k , then the geometric cycle (\mathbb{T}^m, f) represents the class $k[\mathbb{T}_i^m]$. By adding 1–handles, we can normalize this representation into a geometric cycle $(\mathbb{T}^m \# (S^1 \times S^{m-1}) \# \dots \# (S^1 \times S^{m-1}), \tilde{f})$. We get

$$\begin{aligned} \mathfrak{S}(\mathbb{Z}^n, k[\mathbb{T}_i^m]) &= \mathfrak{S}_{\tilde{f}}(\mathbb{T}^m \# (S^1 \times S^{m-1}) \# \dots \# (S^1 \times S^{m-1})) \\ &\leq \mathfrak{S}(\mathbb{T}^m \# (S^1 \times S^{m-1}) \# \dots \# (S^1 \times S^{m-1})) = \mathfrak{S}(\mathbb{T}^m). \end{aligned}$$

Now we deduce the result by combining inequalities (5-5) and (5-6). \square

We close this chapter with the following.

Conjecture 5.12 *If $\mathbf{a} = [\mathbb{T}^m] \in H_m(\mathbb{Z}^m, \mathbb{Z})$, then the associated systolic generating function is*

$$\sigma_{\mathbf{a}}(z) = \mathfrak{S}(\mathbb{T}^m) \cdot \frac{z}{1-z}.$$

6 The Heisenberg group, nilmanifolds and the Waring problem

The nilpotent groups give particularly interesting examples: the systolic volume of multiples of certain homology classes are bounded, albeit certain of these multiples admit (nonnormalized) representations by manifolds whose systolic volume is not bounded. This phenomena already appears in the simplest case of nilpotent nonabelian groups, that is the Heisenberg group.

6.1 Nilmanifolds and the Waring problem

Consider a nilpotent group G of finite type without torsion. The classical result of Mal'cev [20] implies that there exists a simply connected nilpotent Lie group $\mathcal{G}(G)$ such that G embeds in $\mathcal{G}(G)$ as a lattice, that is as a cocompact discrete subgroup.

Denote by $\mathcal{L}(G)$ the Lie algebra of $\mathcal{G}(G)$ and suppose that $\mathcal{L}(G)$ is graded in the following way:

$$(6-1) \quad \mathcal{L}(G) = \bigoplus_{k=1}^s \mathcal{L}_k, \quad [\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j},$$

where $\mathcal{L}_{i+j} = 0$ if $i + j > s$. We do not suppose here that

$$(6-2) \quad \left\{ \mathcal{L}_{(i)} = \bigoplus_{k=i}^s \mathcal{L}_k \right\}_{i=1}^s$$

is a lower central series, s being not in general the nilpotency class of $\mathcal{L}(G)$. For any $t \in \mathbb{R}$, a natural homothety δ_t is associated to the decomposition (6-1) by the formula

$$\delta_t(v) = t^k v \quad \text{if } v \in \mathcal{L}_k.$$

This homothety δ_t is an endomorphism of $\mathcal{L}(G)$ for any real parameter t . By the Baker–Campbell–Hausdorff formula, and taking into account the structure (6-1), the homothety δ_t generates a homothety Δ_t of \mathcal{G} .

Definition 6.1 The nilpotent group G is said to be graded if there exists a graduation (6-1) of the corresponding Lie algebra $\mathcal{L}(G)$ such that for integer parameters the corresponding homotheties of $\mathcal{G}(G)$ preserve the lattice $G \subset \mathcal{G}(G)$, that is

$$\Delta_n(G) \subset G \quad \text{for all } n \in \mathbb{Z}.$$

If G is a nilpotent graded group, let

$$d(G) = \sum_{k=1}^s k \dim \mathcal{L}_k$$

be the weighted dimension of its corresponding Lie algebra $\mathcal{L}(G)$. Remark that if the sequence of subalgebras (6-2) is the lower central series, $d(G)$ coincides with the degree of polynomial growth of G ; see Wolf [28] and Bass [9].

Denote by $M = \mathcal{G}(G)/G$ the nilmanifold corresponding to the nilpotent group G . If G is graded, the homothety $\Delta_n: \mathcal{G}(G) \rightarrow \mathcal{G}(G)$ defined using the graduation induces for every positive integer n a map

$$\tilde{\Delta}_n: M \rightarrow M$$

of degree n^d with $d = d(G)$. Let $\mathbf{a} = [M] \in H_m(G, \mathbb{Z})$ be the fundamental class of M and k be a positive integer. We can represent the class $k\mathbf{a}$ by the connected

sum of a uniformly bounded number of copies of M as follows. By a result of Hilbert (see Ellison [13]), there exists an integer $\mathcal{K}(d)$ such that

$$(6-3) \quad k = \sum_{i=1}^s a_i^d,$$

where each coefficient a_i is a positive integer and $s \leq \mathcal{K}(d)$. Now the class $k\mathbf{a}$ is represented by the geometric cycle $(\#_{i=1}^s M_i, f)$, where $M_i \simeq M$ for $i = 1, \dots, s$ and

$$f: \#_{i=1}^s M_i \rightarrow \bigvee_{i=1}^s M_i \xrightarrow{\bigvee_{i=1}^s \tilde{\Delta}_{a_i}} M,$$

the first map being the contraction of the connected sum into a wedge. We easily compute that $\deg f = \sum_{i=1}^s a_i^d = k$ and so

$$\mathfrak{S}(G, k\mathbf{a}) \leq \mathfrak{S}\left(\#_{i=1}^s M\right).$$

Then we apply Proposition 3.6 in order to derive the following result.

Theorem 6.2 *Let G be a graded nilpotent group. If $\mathbf{a} = [\mathcal{G}(G)/G]$ denotes the fundamental class of the corresponding nilmanifold, then*

$$\mathfrak{S}(G, k\mathbf{a}) \leq \mathcal{K}(d(G)) \cdot \mathfrak{S}(G, \mathbf{a})$$

for any positive integer k .

6.2 Family of lattices in the Heisenberg group

Consider the Heisenberg group of dimension 3 composed of the following set of upper triangular matrices:

$$\mathcal{H} = \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \mid x, y, z \in \mathbb{R} \right\}.$$

The subset $\mathcal{H}(\mathbb{Z})$ of matrices of \mathcal{H} with integer coefficients (ie for which $x, y, z \in \mathbb{Z}$) is a lattice, and we denote by $M_{\mathcal{H}} = \mathcal{H}/\mathcal{H}(\mathbb{Z})$ the corresponding nilmanifold. The fundamental group $\mathcal{H}(\mathbb{Z})$ of $M_{\mathcal{H}}$ satisfies the assumptions of Theorem 6.2. In fact the homotheties $\{\Delta_t\}_{t>0}$ are given by the formula

$$\Delta_t \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) = \left(\begin{array}{ccc} 1 & tx & t^2z \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{array} \right),$$

so $\Delta_n(\mathcal{H}(\mathbb{Z})) \subset \mathcal{H}(\mathbb{Z})$ for any integer $n \geq 1$. The map Δ_n factorizes through a map

$$\tilde{\Delta}_n: M_{\mathcal{H}} \rightarrow M_{\mathcal{H}}$$

for which $\deg(\tilde{\Delta}_n) = n^4$. The resolution of the Waring problem for the sum of fourth powers (see [7; 8]) gives that any integer number decomposes into a sum of at most 19 fourth powers. That is, with the notation of Theorem 6.2, we have $d(\mathcal{H}(\mathbb{Z})) = 4$ and $\mathcal{K}(4) = 19$. Theorem IV of the introduction now easily follows.

Corollary 6.3 *Let $a = [M_{\mathcal{H}}] \in H_3(\mathcal{H}(\mathbb{Z}), \mathbb{Z})$ be the fundamental class of $M_{\mathcal{H}}$. Then*

$$\mathfrak{S}(\mathcal{H}(\mathbb{Z}), ka) \leq 19 \cdot \mathfrak{S}(\mathcal{H}(\mathbb{Z}), a)$$

for any positive integer k .

The different lattices of \mathcal{H} give rise to nilmanifolds whose systolic behavior is particularly interesting. Consider the sequence of lattices $\{\mathcal{H}_n(\mathbb{Z})\}_{n=1}^{\infty}$ of \mathcal{H} , where $\mathcal{H}_n(\mathbb{Z})$ is the subset of matrices of \mathcal{H} such that $x \in n\mathbb{Z}$ and $y, z \in \mathbb{Z}$. Denote by $M_n = M_{\mathcal{H}_n} = \mathcal{H}/\mathcal{H}_n(\mathbb{Z})$ the corresponding nilmanifolds. The manifold M_n is a cyclic covering of $M_{\mathcal{H}}$ with n sheets, so

$$(6-4) \quad \mathfrak{S}(M_n) \leq C \frac{n}{\ln n},$$

according to the version of Theorem 5.4 for cyclic coverings. The fact that the function $\mathfrak{S}(M_n)$ goes to infinity is not obvious. For instance the simplicial volume of these manifolds is zero, and thus the corresponding lower bound (see Corollary 5.9) does not apply. We are nevertheless able to prove the following proposition; see [6].

Proposition 6.4 [6] *The function $\mathfrak{S}(M_{\mathcal{H}_n})$ satisfies the inequality*

$$\mathfrak{S}(M_{\mathcal{H}_n}) \geq a \frac{\ln n}{\exp(b\sqrt{\ln \ln n})},$$

where a and b are two positive constants. In particular,

$$\lim_{n \rightarrow +\infty} \mathfrak{S}(M_{\mathcal{H}_n}) = +\infty.$$

As the cover $M_n \rightarrow M_{\mathcal{H}}$ has n sheets, the manifold M_n represents the class $n[M_{\mathcal{H}}] \in H_3(\mathcal{H}(\mathbb{Z}), \mathbb{Z})$ for any positive integer n . This representation is not normalized, and Corollary 6.3 together with Proposition 6.4 shows that the assumption of normalization cannot be dropped in Proposition 2.4.

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