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Quilted strips, graph associahedra, and A_∞ n–modules

SIKIMETI MA'U

We consider moduli spaces of quilted strips with markings. By identifying each compactified moduli space with the nonnegative real part of a projective toric variety, we conclude that it is homeomorphic under the moment map to the moment polytope. The moment polytopes in these cases belong to a certain class of graph associahedra, which include the associahedra and permutahedra as special cases. In fact, these graph associahedra are precisely the polytopes whose facet combinatorics encode the A_∞ equations of A_∞ n–modules.

14H10; 14M25

1 Introduction

Let $\boldsymbol{r} = (r_0, r_1, \dots, r_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ be a tuple of nonnegative integers, not all zero. In this note we will define an associated moduli space $Q(\boldsymbol{r})$ of quilted strips with $\boldsymbol{r}$ markings, and consider its Grothendieck–Knudsen compactification $\overline{Q}(\boldsymbol{r})$. For the same tuple $\boldsymbol{r}$, we will define a graph $G(\boldsymbol{r})$ and show the following.

Theorem 1.1 *The compactified moduli space $\overline{Q}(\boldsymbol{r})$ is homeomorphic as a manifold-with-corners to (a convex hull realization of) the graph associahedron of the graph $G(\boldsymbol{r})$.*

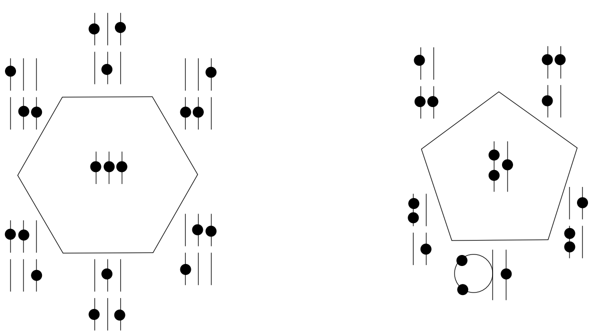

Figure 1: Two moduli spaces of quilted strips with 3 markings

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Figure 1 illustrates two cases for $r = 3$. The hexagon is the graph associahedron of a triangle, which is the permutahedron P_3. The pentagon is the graph associahedron of a path on three vertices, the associahedron K_4.

Graph associahedra are a class of convex polytopes introduced independently by Carr and Devadoss [2], Davis, Januszkiewicz, and Scott [3], Postnikov [7], and Toledano-Laredo [9]. The family of graphs that appear as dual graphs to these moduli spaces includes paths and complete graphs, which produce associahedra and permutahedra respectively. The associahedra are parameter spaces for homotopy associativity, while the permutahedra are parameter spaces for homotopy commutativity. The graph associahedra that appear in this note are the parameter spaces for a mixture of homotopy associativity and homotopy commutativity. In fact this particular class of graph associahedra also arises in the work of Bloom on a spectral sequence for links in monopole Floer homology [1].

The combinatorics of moduli spaces of Riemann surfaces with markings are intimately connected to higher algebraic structures. This is manifest, for instance, in constructions such as Gromov–Witten invariants and Fukaya's A_∞ categories which are based on pseudoholomorphic curve theory. Quilted strips with markings belong to a class of generalized Riemann surfaces called *quilts*, which are Riemann surfaces decorated with some real-analytic submanifolds called *seams*. The moduli spaces of quilted strips with markings govern the combinatorics of algebraic structures that we call A_∞ n–modules, which are A_∞ analogues of modules over a tensor product of n algebras. Since quilts are domains for a generalized pseudoholomorphic curve theory developed by Wehrheim and Woodward [10] which employ Lagrangian correspondences as boundary conditions for the seams, this A_∞ algebraic structure arises naturally in Lagrangian Floer theory.

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2 Quilted strips

Let $\mathbb{C}$ denote the complex plane with complex coordinate $z = x + iy$, and let $\Sigma = \{z \in \mathbb{C} \,|\, 0 < \operatorname{Re} z = x < 1\}$ denote the infinite strip of unit width in $\mathbb{C}$. Let $n \geq 1$, and fix an increasing sequence of real values $0 = a_0 < a_1 < \cdots < a_n = 1$. Each a_k

determines a vertical line $C_k = \{z = x + iy | x = a_k\}$ in Σ, which we orient in the direction of increasing imaginary component (ie upwards).

Definition 2.1 The lines $C_0, \dots, C_k$ are called *seams* and $Q = (\Sigma, C_0, \dots, C_n)$ is called a *quilted strip*.

We denote by $\mathrm{Conf}_m^+(C_k)$ the set of configurations of m distinct points in C_k whose order is compatible with the orientation of C_k.

Definition 2.2 Let $\boldsymbol{r} = (r_0, r_1, \dots, r_n) \in \mathbb{Z}_{\geq 0}^{n+1}$. A *quilted strip with* $\boldsymbol{r}$ *marked points* is a tuple $(\boldsymbol{z}_0, \dots, \boldsymbol{z}_n) \in \mathrm{Conf}_{r_0}^+(C_0) \times \cdots \times \mathrm{Conf}_{r_n}^+(C_n)$.

We write $|\boldsymbol{r}| := r_0 + \cdots + r_n$ for the total number of marked points on the quilted strip. When $|\boldsymbol{r}| \geq 1$, there is a free and proper $\mathbb{R}$ action on $\mathrm{Conf}_{r_0}^+(C_0) \times \cdots \times \mathrm{Conf}_{r_n}^+(C_n)$, given by simultaneous vertical translation. Explicitly, let

$$(\boldsymbol{z}_0, \dots, \boldsymbol{z}_n) \in \mathrm{Conf}_{r_0}^+(C_0) \times \cdots \times \mathrm{Conf}_{r_n}^+(C_n),$$

with the entries of $\boldsymbol{z}_i \in \mathrm{Conf}_{r_i}^+(C_i)$ denoted by $(z_{i,1}, \dots, z_{i,r_i})$. For each $a \in \mathbb{R}$, the action $a \cdot (\boldsymbol{z}_0, \dots, \boldsymbol{z}_n)$ is the map $z_{k,l} \mapsto z_{k,l} + ia$, $k = 0, \dots, n, l = 1, \dots, r_k$.

Definition 2.3 Let $\boldsymbol{r} = (r_0, r_1, \dots, r_n) \in \mathbb{Z}_{\geq 0}^{n+1}$, with $|\boldsymbol{r}| \geq 1$. The *moduli space of quilted strips with* $\boldsymbol{r}$ *marked points* is the quotient space

$$Q(\boldsymbol{r}) := \mathrm{Conf}_{r_0}^+(C_0) \times \cdots \times \mathrm{Conf}_{r_n}^+(C_n)/\mathbb{R},$$

which is a smooth, noncompact $(|\boldsymbol{r}| - 1)$–dimensional manifold. We denote the equivalence class of $(\boldsymbol{z}_0, \dots, \boldsymbol{z}_n)$ by $[\boldsymbol{z}_0, \dots, \boldsymbol{z}_n]$.

Alternative definition

An alternative definition of $Q(\boldsymbol{r})$ is in terms of the upper half-plane $H = \{z \in \mathbb{C} \,\big|\, \mathrm{Im}\, z = y > 0\}$. The strip Σ is biholomorphic to $H \setminus \{0\}$, for instance by the map $z \mapsto \exp(-i\pi z)$. Under this map the oriented lines $C_k \subset \Sigma$ become oriented rays emanating from 0 in H, the fixed distances between the vertical lines in the strip Σ determine fixed angles between the rays in the half-plane H, and vertical translations of Σ become dilations of H, ie multiplication by a positive real scalar. Thus, an equivalent definition of $Q(\boldsymbol{r})$ is as the moduli space of configurations of marked points on the rays, modulo dilations. See Figure 2 for an illustration.

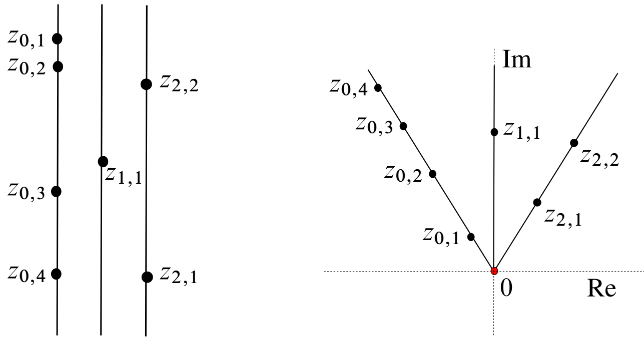

Figure 2: Two views of a quilted strip with $\boldsymbol{r} = (4, 1, 2)$ markings

2.1 Compactification

Let $|\boldsymbol{r}| \geq 1$, and consider the space $Q(\boldsymbol{r})$ defined in terms of marked points on rays from the origin, modulo dilations. Then there is a canonical embedding

$$\iota\colon Q(\boldsymbol{r}) \hookrightarrow \overline{\mathcal{M}}_{0,|\boldsymbol{r}|+2},$$
$$[z_0, \dots, z_n] \mapsto [z_0, \dots, z_n, 0, \infty],$$

where $\overline{\mathcal{M}}_{0,|\boldsymbol{r}|+2}$ is the Grothendieck–Knudsen compactification of the moduli space of genus zero curves with $|\boldsymbol{r}| + 2$ marked points.

Definition 2.4 The compactification $\overline{Q}(\boldsymbol{r})$ is the closure of $Q(\boldsymbol{r})$ in $\overline{\mathcal{M}}_{0,|\boldsymbol{r}|+2}$.

3 Explicit description of $\overline{Q}(\boldsymbol{r})$

Elements of $\overline{\mathcal{M}}_{0,|\boldsymbol{r}|+2}$ are called *stable nodal curves with markings*. An explicit description of the compactified moduli space $\overline{\mathcal{M}}_{0,|\boldsymbol{r}|+2}$ is given in [6, Appendix D], whose terminology and notation we will use to give an explicit description of $\overline{Q}(\boldsymbol{r})$.

3.1 Combinatorial types in $\overline{Q}(\boldsymbol{r})$

The combinatorial types of elements in the compactification $\overline{Q}(\boldsymbol{r})$ contain structure reflecting the prescribed order of marked points along the seams C_i. The prescribed order of marked points can be represented by *ribbon structures* in the combinatorial types. Recall that if $T = (V, E)$ is a tree, then a *ribbon structure* Φ on T is a cyclic ordering of the edges at each vertex,

$$\Phi = \{f_v\colon \{1, 2, \dots, |E_v|\} \to E_v \mid f_v \text{ is a bijection}\}.$$

By embedding each vertex of the tree and its adjacent edges in the plane in the cyclic order $f_v(1), \dots, f_v(|E_v|)$, the ribbon structure determines, up to planar isotopy, a planar embedding of T.

Definition 3.1 Let $\boldsymbol{r} = (r_0, \dots, r_n) \in \mathbb{Z}_{\geq 0}^{n+1}$. An *$\boldsymbol{r}$–labeled ribbon tree* consists of:

- A tree $T = (V, E)$ with vertices V and finite edges E.
- A partition of the vertices $V = V_{\mathrm{spine}} \sqcup V_0 \sqcup \cdots \sqcup V_n$ with $|V_{\mathrm{spine}}| \geq 1$.
- A partition of the edges $E = E_{\mathrm{spine}} \sqcup E_0 \sqcup \cdots \sqcup E_n$.
- A set of semi-infinite edges $E^\infty = \{e^+, e^-, \{e_{i,j}\}_{i=0,\dots,n, j=1,\dots,r_i}\}$ and a map $\Lambda\colon E^\infty \to V$ such that $\Lambda(e^\pm) \in V_{\mathrm{spine}}$ and $\Lambda(e_{i,j}) \in V_i \cup V_{\mathrm{spine}}$.
- A ribbon structure Φ_i for each subgraph

$$T_i = (V_{\mathrm{spine}} \cup V_i, E_{\mathrm{spine}} \cup E_i \cup \{e^+, e^-, \{e_{i,j}\}_{j=1,\dots,r_i}\}) \quad \text{for } i = 0 \text{ to } n.$$

These data satisfy the following conditions:

- Each T_i is a tree; we call it the i^{th} *page*.
- The subgraph $(V_{\mathrm{spine}}, E_{\mathrm{spine}} \cup \{e^+, e^-\})$ is a path between the semi-infinite edges e^+ and e^-; we call it the *spine* of T.
- The planar embedding of each T_i determined by its ribbon structure induces the ordering $e^-, e_{i,1}, \dots, e_{i,r_i}, e^+$ of its semi-infinite edges.
- When $i \neq j$ there are no edges between V_i and V_j.

An $\boldsymbol{r}$–labeled ribbon tree $(T, E^\infty, \Lambda, \{\Phi_k\}_{k=0,\dots,n})$ is *stable* if every vertex is adjacent to at least 3 edges (semi-infinite or finite).

3.2 Stable, nodal quilted strips with markings

Consider a stable $\boldsymbol{r}$–labeled ribbon tree $\Gamma = (T, E^\infty, \Lambda, \{\Phi_k\}_{k=0,\dots,n})$. Γ is a combinatorial type for $\overline{Q}(\boldsymbol{r})$, as follows. Forgetting the extra structure at first, consider only the underlying stable $(|\boldsymbol{r}| + 2)$–labeled tree $\widehat{\Gamma} := (T, \Lambda)$ as a combinatorial type for $\overline{\mathcal{M}}_{0,|\boldsymbol{r}|+2}$. An element of $\overline{\mathcal{M}}_{0,|\boldsymbol{r}|+2}$ with combinatorial type $\widehat{\Gamma}$ is a stable, nodal, marked genus-zero curve

$$(\Gamma, \{z_{v,e}\}_{v \in e, e \in E}, \{z_x\}_{x \in E^\infty})$$

up to equivalence. It is in the closure of $\overline{Q}(\boldsymbol{r})$ if and only if the marked points satisfy the following additional constraints:

(1) For each $v \in V_{\text{spine}}$, let e_v^+ (resp. e_v^-) $\in E_{\text{spine}} \cup \{e^+, e^-\}$ be the edge containing v that is closest to the semi-infinite edge e^+ (resp. e^-). There is a Möbius transformation $\phi\colon S^2 \to S^2$ for which $\phi(z_{v,e_v^+}) = \infty$, $\phi(z_{v,e_v^-}) = 0$, $\phi(z_{v,e}) \in C_i$ if $e \in E_i$, and $\phi(z_{i,j}) \in C_i$ if $\Lambda(i,j) = v$. That is, Z_v is equivalent to a quilted strip with marked points.

(2) For each $v \in V_k$, $k = 0, \dots, n$, there is a Möbius transformation $\phi\colon S^2 \to S^2$ such that $\phi(z_{v,e}) \in \partial D$ for all $z_{v,e} \in Z_v$, and the cyclic order of the marked points $\{\phi(z_{v,e}) | e \in E_v\}$ is given by the cyclic order of the edges E_v in the ribbon structure Φ_k at v.

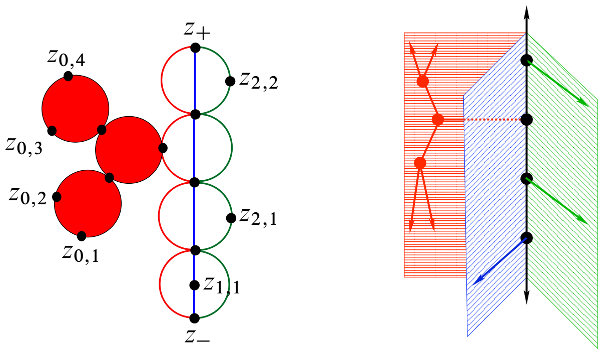

Figure 3: An element in $\overline{Q}(4,1,2)$ shown alongside its combinatorial type

Example 3.2 Figure 3 depicts an element in the compactification of $Q(4,1,2)$ with its combinatorial type, a stable $(4,1,2)$–labeled tree whose three pages are in different colors, and spine is indicated in black.

Proposition 3.3 *Let $|\boldsymbol{r}| \geq 1$. Then $\overline{Q}(\boldsymbol{r})$ is a smooth manifold-with-corners of real dimension $|\boldsymbol{r}| - 1$, and has a stratification*

$$\overline{Q}(\boldsymbol{r}) := \bigcup_{\Gamma} Q_\Gamma,$$

where Γ ranges over all $(n+1)$–paged ribbon trees with $\boldsymbol{r}$ leaves, and each stratum Q_Γ is a smooth submanifold-with-corners of real codimension $|E|$.

Proof This follows directly from the construction of local charts using cross-ratios, which in turn is a direct application of the proof of [6, Appendix D, Theorem D.4.5] which constructs local cross-ratio charts for $\overline{\mathcal{M}}_{0,|\boldsymbol{r}|+2}$. □

4 Charts using simple ratios

We now describe an explicit set of coordinate charts on $\overline{Q}(\boldsymbol{r})$ which determine the same topology on $\overline{Q}(\boldsymbol{r})$ as cross-ratio coordinate charts. The reason for introducing these new charts is that the coordinates satisfy relations that are algebraically very simple and in fact toric. This will lead to the direct connection with toric varieties later.

From now on, we view elements of $Q(\boldsymbol{r})$ as marked points on fixed rays in $H\setminus\{0\}$, modulo dilations. Let $x_{i,j}$ denote the distance between marked points $z_{i,j}$ and $z_{i,j-1}$ on the i^{th} ray. For each i we understand $z_{i,0}=0$, which is the point where the rays meet (see Figure 4 for an illustration). The distances $x_{i,j}$ are only determined on elements of $Q(\boldsymbol{r})$ up to multiplication by positive real scalars, so the $x_{i,j}$ are in fact real, positive homogeneous coordinates $\boldsymbol{x} := (x_{0,1}:\ldots:x_{n,r_n})$. This identifies $Q(\boldsymbol{r})$ with the open subset of $\mathbb{RP}^{|\boldsymbol{r}|-1}$ consisting of $\boldsymbol{x}$ for which all the $x_{i,j}$ are nonzero with the same sign, however the compactification $\overline{Q}(\boldsymbol{r})$ does *not* come from its closure in the compact space $\mathbb{RP}^{|\boldsymbol{r}|-1}$.

Each ratio $x_{i,j}/x_{k,l}$ is a well-defined function on $Q(\boldsymbol{r})$ and we call it a *simple ratio* coordinate. We next describe how to extend simple ratio coordinates to the compactified moduli space $\overline{Q}(\boldsymbol{r})$.

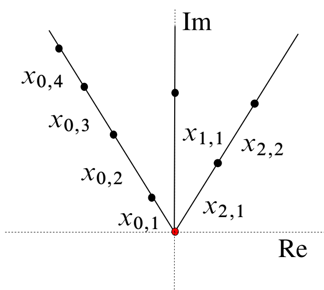

Figure 4: Real homogeneous coordinates on $Q(4,1,2)$

Let Γ be a combinatorial type for an element in the compactification $\overline{Q}(\boldsymbol{r})$, and let $(\Gamma,\{z_{v,e}\}_{v\in e,e\in E},\{z_x\}_{x\in E^\infty})\in Q_\Gamma$.

For each $v\in V$, take any Möbius transformation that sends the marked point $z_{v,+}$ to ∞. Note that such transformations differ only by rotations, translations, or dilations in $\mathbb{C}$. After such a transformation, the remaining marked points in $Z_v\setminus\{z_{v,+}\}$ either lie on rays corresponding to the images of $C_0,\ldots,C_n$, or on a straight line corresponding to the image of ∂D. If $z,z'\in Z_v\setminus\{z_{v,+}\}$ are a pair of markings that lie next to each other on the same ray or straight line, there is a unique pair of labels (k,j) and $(k,j+1)$ for which $z = z_{v,(k,j)}$ and $z' = z_{v,(k,j+1)}$. We will label the distance between z and z' by $x_{k,j+1}$, noting as before that this distance is only well-defined up to dilations. Let $X_v \subset \{x_{i,j} \mid i = 0, \dots, n,\ j = 1, \dots, r_i\}$ denote the subset of homogeneous coordinates associated to v in this way.

Example 4.1 In Figure 3, let v be the vertex in the spine that has the two semi-infinite edges labeled by $+$ and $(2,2)$ incident to it. In this case a Möbius transformation sending $z_{v,+}$ to ∞ maps the three arcs to three rays C_0, C_1, C_2 with one marked point on C_2 and another marked point where the three rays meet. In this case $X_v = \{x_{2,2}\}$. As another example from Figure 3, let v' be the vertex on the red page (V_0, E_0) with no semi-infinite edges. Once a Möbius transformation maps $z_{v',+}$ to ∞, the remaining two marked points lie on a straight line, and $X_{v'} = \{x_{0,2}\}$.

Lemma 4.2 *Let Γ be a combinatorial type for $\overline{Q}(\boldsymbol{r})$, and consider the open set $U_\Gamma = \bigcup_{\Gamma' \le \Gamma} Q_{\Gamma'}$.*

(1) *Let $v \in V_\Gamma$. Any pair $x_{i,j}$ and $x_{i',j'}$ of distinct elements in X_v determines a continuous function $\xi\colon U_\Gamma \to (0,\infty)$ given by $\xi := x_{i',j'}/x_{i,j}$.*

(2) *Let $e = (v_e^+, v_e^-)$ be an edge in Γ, where v_e^+ is the vertex closest to the vertex $v_+ = \Lambda(+)$. Any pair $(x_{i,j}, x_{i',j'}) \in X_{v+} \times X_{v-}$ determines a continuous function $\xi\colon U_\Gamma \to [0,\infty)$ given by $\xi := x_{i',j'}/x_{i,j}$, and $\xi = 0$ if and only if the combinatorial type $\widetilde{\Gamma} \le \Gamma$ has the edge e.*

Proof The proof is immediate once we express each ratio function ξ as a function of the cross-ratios in a local cross-ratios chart. Since it is straightforward to do this explicitly, we omit the details. □

We will call the functions defined in Lemma 4.2 *simple ratio coordinates*. Observe that relations between simple ratios are generated by relations of the form $(x_{i,j}/x_{k,l}) \cdot (x_{k,l}/x_{m,n}) = x_{i,j}/x_{m,n}$ and $(x_{i,j}/x_{k,l}) \cdot (x_{k,l}/x_{i,j}) = 1$. As we shall see later, these algebraically simple relations are what allow the moduli spaces to be identified with the nonnegative real part of a projective toric variety. Recursion relations between cross-ratio coordinates, by contrast, are *not* toric.

Definition 4.3 Let Γ be a combinatorial type for $\overline{Q}(\boldsymbol{r})$. A *simple ratio chart* for the open set $U_\Gamma \subset \overline{Q}(\boldsymbol{r})$ is a map

$$\Phi_\Gamma\colon U_\Gamma \longrightarrow [0,\infty)^{|E|} \times (0,\infty)^{|\boldsymbol{r}|-1-|E|}$$

determined by a collection of $|\boldsymbol{r}| - 1$ simple ratios, as follows.

(1) For each vertex v, fix one element $x_{i,j} \in X_v$ to use as denominator, and take the $|X_v|-1$ functions $\{x_{i',j'}/x_{i,j} \,|\, x_{i',j'} \in X_v, x_{i',j'} \neq x_{i,j}\}$;

(2) For each edge e, with $e = (v^+, v^-)$, take the simple ratio $x_{k,l}/x_{i,j}$, where $x_{i,j} \in X_{v^+}$ is the element used as the denominator for v^+, and $x_{k,l} \in X_{v^-}$ is the element used as the denominator for v^-.

Example 4.4 For the combinatorial type of Figure 3, the chart consists of the ratios $x_{0,4}/x_{0,3}, x_{0,2}/x_{0,3}, x_{0,3}/x_{0,1}, x_{2,2}/x_{0,1}, x_{0,1}/x_{2,1}, x_{2,1}/x_{1,1}$.

From the relations between simple ratios we see that transition functions between charts correspond to multiplication by nonzero simple ratios. We also observe that the simple ratio charts associated to the open sets $\{U_\Gamma \,|\, \Gamma \text{ maximal}\}$ (ie the subset of simple ratio charts associated to *maximal* combinatorial types) cover $\overline{Q}(\boldsymbol{r})$ and so form an atlas.

Lemma 4.5 *The topology defined on $\overline{Q}(\boldsymbol{r})$ by the simple ratio charts is identical to the Grothendieck–Knudsen topology.*

Proof It suffices to express each simple ratio in a simple ratio chart for U_Γ as a continuous function of the cross-ratios in a cross-ratio chart for U_Γ, and vice-versa. Again it is extremely straightforward to write concrete expressions down explicitly, so we omit the details. □

5 Connection with graph associahedra

Next we recall the definition of graph associahedra given by Carr and Devadoss [2]. Let $G = (V, E)$ be a finite graph with vertices V and edges E, with no multiple edges and no loops.

Definition 5.1 A *tube* is a proper connected subset of vertices of G. A *tubing* of G is a collection of tubes, such that each pair of tubes in the tubing satisfies the following admissibility conditions:

(1) a pair of tubes may be *nested* provided that the inner tube is a proper subset of the outer tube.

(2) a pair of tubes may be *disjoint* provided that there is no edge connecting a vertex in one tube with a vertex in the other tube.

Definition 5.2 The *graph associahedron* K_G is a simple polytope whose facets of codimension k are indexed by tubings containing k tubes. A facet of codimension k is contained in a facet of codimension k' if and only if the set of tubes in the k'–tubing contains the set of tubes in the k–tubing.

5.1 Partial order

The set of tubings of G is partially ordered by inclusion, ie $\mathcal{T} \leq \mathcal{T}'$ if and only if all the tubes in the tubing $\mathcal{T}$ are also tubes in the tubing $\mathcal{T}'$. The *maximal tubings* are those which are maximal with respect to this partial order. They correspond to zero-dimensional facets, and have exactly $|V|-1$ tubes.

Definition 5.3 We introduce some terminology.

- Two maximal tubings are called *neighbors* if they differ by one tube.
- For a given tubing $\mathcal{T}$, a tube $T \in \mathcal{T}$ is called *the minimal tube for vertex* v if $v \in T$ and v is not in any tube contained in T. In this case we also call v a *maximal vertex of* T.

In a maximal tubing of G, each tube T determines a unique vertex v_T of V – namely, the unique maximal vertex of T – and there is exactly one vertex which is not contained in any tube.

Devadoss [4] produced the following algorithm for assigning an integral vector to each maximal tubing, such that the convex hull of the vectors realizes the graph associahedron as a polytope.

Definition 5.4 To each maximal tubing of G assign a *weight vector*, which is a function $\boldsymbol{w}\colon V \to \mathbb{Z}$, by induction on the number of vertices in a tube.

(1) If v is the only vertex in a tube, $\boldsymbol{w}(v) = 0$.

(2) Otherwise, let T be the minimal tube containing v. By maximality, all other vertices in the tube T are contained in tubes of smaller size than $|T|$, so by induction the function $\boldsymbol{w}$ is defined on them already. Then set $\boldsymbol{w}(v) = 3^{|T|-2} - \sum_{v' \in T, v' \neq v} \boldsymbol{w}(v')$.

In other words, if a tube T has two or more vertices in it, the sum of all weights of its vertices should be $3^{|T|-2}$. For the purposes of the algorithm, think of the last vertex in the process (which is not contained in any tube) as belonging to a tube of size $|V|$.

Returning to our moduli spaces, we will associate a graph $G(\boldsymbol{r})$ to each moduli space $Q(\boldsymbol{r})$ as follows.

Definition 5.5 Fix an $\boldsymbol{r} \in \mathbb{Z}^n_{\geq 0}$ with $|\boldsymbol{r}| \geq 1$. View the elements of $Q(\boldsymbol{r})$ as configurations of rays in $H \setminus \{0\}$ with markings. Define a graph $G(\boldsymbol{r})$ as follows. The vertices of $G(\boldsymbol{r})$ are indexed by the *finite* open line segments in $C_0 \cup \cdots \cup C_n \setminus \{z_{i,j}\}_{i=0,\dots,n;\, j=1,\dots,r_i}$. Two vertices in $G(\boldsymbol{r})$ are connected by an edge if and only if they correspond to finite open line segments whose closures intersect either at a marked point $z_{i,j}$ or at 0. In particular, $G(\boldsymbol{r})$ is a complete graph with paths adjoined to some of its vertices.

Example 5.6 The graph $G(4,1,3,1,3)$ is depicted in Figure 5.

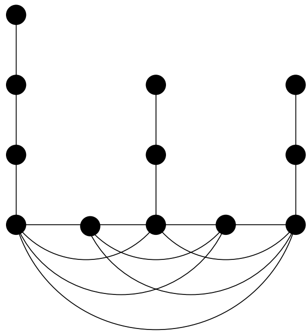

Figure 5: The graph $G(4,1,3,1,3)$

The following combinatorial dictionary between tubings of $G(\boldsymbol{r})$ and the combinatorial types of $\overline{Q}(\boldsymbol{r})$ is immediate, so we omit its proof:

Lemma 5.7 *There is a canonical bijection between tubings of $G(\boldsymbol{r})$ and combinatorial types of $\overline{Q}(\boldsymbol{r})$ which respects their respective poset structures. Under this bijection, each tube in a tubing determines an edge in the combinatorial type.*

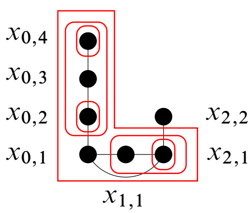

Figure 6: The tubing of the graph $G(4,1,2)$ that corresponds to the combinatorial type depicted in Figure 3

Example 5.8 Figure 6 shows the tubing of $G(4,1,2)$ that corresponds to the combinatorial type of Figure 3.

5.2 Simple ratio charts from tubings

The combinatorial dictionary of Lemma 5.7 allows a simple ratios chart to be read off from a tubing $\mathcal{T}$.

- Fixing a simple ratio for each edge in the combinatorial type $\Gamma_{\mathcal{T}}$ is equivalent to picking a simple ratio for each tube in $\mathcal{T}$. So let T be a tube in $\mathcal{T}$. Pick a vertex which is maximal in T and let $x_{\mathrm{in}}(T)$ denote the variable indexing it, and pick a vertex that is immediately outside T and let $x_{\mathrm{out}}(T)$ denote the variable indexing it. Assign the simple ratio $x_{\mathrm{in}}(T)/x_{\mathrm{out}}(T)$ to the tube T.
- Next we pick the simple ratios in the chart that are associated to the vertices of the combinatorial type $\Gamma_{\mathcal{T}}$. Let $\mathcal{T}=\{T_1,\dots,T_k\}$ be the tubing of $G(\boldsymbol{r})$. For each $i\in\{1,\dots,k\}$ let W_i denote the subset of vertices of $G(\boldsymbol{r})$ for which the tube T_i is minimal, and let W_* denote the subset of vertices of $G(\boldsymbol{r})$ which are contained in none of the tubes of $\mathcal{T}$. Hence $W_1,\dots,W_k,W_*$ is a partition of the vertices of the graph $G(\boldsymbol{r})$. For each W_i, fix a variable $x_{m,n}$ indexing a vertex in W_i to serve as denominator, and for every other vertex in W_i, indexed by the variable $x_{p,q}$, say, we associate the simple ratio $x_{p,q}/x_{m,n}$.

Example 5.9 For the tubing in Figure 6, the simple ratios chart is made up of the ratios $x_{0,4}/x_{0,3}, x_{0,2}/x_{0,3}, x_{0,3}/x_{0,1}, x_{2,2}/x_{0,1}, x_{0,1}/x_{2,1}, x_{2,1}/x_{1,1}$, which is the same chart as in Example 4.4.

6 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on two lemmas about weights.

Lemma 6.1 *Let $G=(V,E)$ be a connected graph with no loops and no multiple edges. Let $\mathcal{T}$ be a maximal tubing of G and $v\in V$ a vertex.*

(1) *If v is the maximal vertex for a tube $T\in\mathcal{T}$ with $|T|\geq 3$, then $w(v)>3^{|T|-3}$.*

(2) *If v is not contained in any tube of $\mathcal{T}$ and $|V|\geq 3$, then $w(v)>3^{|V|-3}$.*

Proof First observe that any connected subgraph of G is again a connected graph, and a tube is a connected subgraph of G, so it is enough to prove (b). If we remove the vertex v from G, as well as all edges incident to v, the resulting graph can have several connected components. Suppose it has $k\geq 1$ connected components. By maximality of the tubing $\mathcal{T}$, each connected component must be a tube. Call these tubes $T_1,\dots,T_k$, each containing $n_1,\dots,n_k$ vertices respectively. We first claim that $w(T_i)\leq 3^{|V|-k-2}$ for each i. To see this, note that $1\leq n_i$ for each i, and $n_1+\cdots+n_k=|V|-1$, so $n_i\leq |V|-k$ for each i. If $n_i\geq 2$, then by the algorithm for weights $w(T_i) = 3^{n_i-2} \le 3^{|V|-k-2}$. If $n_i = 1$, then by the base step of the weights algorithm $w(T_i) = 0 < 3^{|V|-k-2}$. Therefore, we can write

$$w(v) = w(V) - \sum_{i=1}^{k} w(T_i) \ge 3^{|V|-2} - k3^{|V|-k-2} = 3^{|V|-3}(3 - k3^{1-k}) > 3^{|V|-3}$$

since $0 < k3^{1-k} \le 1$ for $k \ge 1$. $\square$

Lemma 6.2 *Let $\mathcal{T} = \{T_1, \dots, T_{|\boldsymbol{r}|-1}\}$ and $\widetilde{\mathcal{T}}$ be two maximal tubings of $G(\boldsymbol{r})$, with weight vectors $\boldsymbol{w}$ and $\widetilde{\boldsymbol{w}}$ respectively. Then there are integers $m_1, \dots, m_{|\boldsymbol{r}|-1} \ge 0$ such that*

$$\boldsymbol{x}^{\widetilde{\boldsymbol{w}}-\boldsymbol{w}} = \xi_1^{m_1} \xi_2^{m_2} \cdots \xi_{|\boldsymbol{r}|-1}^{m_{|\boldsymbol{r}|-1}},$$

where each simple ratio ξ_i is the simple ratio chart coordinate corresponding to the tube T_i. Moreover if $\widetilde{\mathcal{T}}$ and $\mathcal{T}$ are neighboring maximal tubings, then $\boldsymbol{x}^{\widetilde{\boldsymbol{w}}-\boldsymbol{w}} = \xi^m$ for some $m \ge 1$, where ξ is the simple ratio chart coordinate corresponding to the one tube T in $\mathcal{T}$ that is not also a tube in $\widetilde{\mathcal{T}}$.

Proof We will use the tubing $\mathcal{T} = \{T_1, T_2, \dots, T_{|\boldsymbol{r}|-1}\}$ as a reference tubing. Each tube T_i determines a pair of vertices in $G(\boldsymbol{r})$, the vertex immediately inside T_i and the vertex immediately outside T_i, so we will write $x_{\text{in}}(T_i)$ and $x_{\text{out}}(T_i)$ respectively for the corresponding homogeneous variables of type $x_{i,j}$. The simple ratio chart coordinate for the tube T_i is $\xi = x_{\text{in}}(T_i)/x_{\text{out}}(T_i)$. Let x_* denote the variable corresponding to the one vertex not contained in any tube; thus the $|\boldsymbol{r}|$ vertices of $G(\boldsymbol{r})$ are indexed by

$$V = \{x_{\text{in}}(T_1), \dots, x_{\text{in}}(T_{|\boldsymbol{r}|-1}), x_*\}. \tag{1}$$

Let $\boldsymbol{w} = (w_1, \dots, w_{|\boldsymbol{r}|-1}, w_*)$ be the weight vector for the tubing $\mathcal{T}$ with respect to the indexing (1). Let $w(T_i)$ denote the *sum* of all the weights in the tube T_i, and let $w(V)$ denote the sum of all weights in the graph. Then

$$\begin{aligned}
\boldsymbol{x}^{\boldsymbol{w}} &:= x_{\text{in}}(T_1)^{w_1} x_{\text{in}}(T_2)^{w_2} \cdots x_{\text{in}}(T_{|\boldsymbol{r}|-1})^{w_{|\boldsymbol{r}|-1}} x_*^{w_*} \\
&= \left(\frac{x_{\text{in}}(T_1)}{x_{\text{out}}(T_1)}\right)^{w(T_1)} \left(\frac{x_{\text{in}}(T_2)}{x_{\text{out}}(T_2)}\right)^{w(T_2)} \cdots \left(\frac{x_{\text{in}}(T_{|\boldsymbol{r}|-1})}{x_{\text{out}}(T_{|\boldsymbol{r}|-1})}\right)^{w(T_{|\boldsymbol{r}|-1})} x_*^{w(V)} \\
&=: \xi_1^{w(T_1)} \cdots \xi_{|\boldsymbol{r}|-1}^{w(T_{|\boldsymbol{r}|-1})} x_*^{w(V)}.
\end{aligned}$$

Now consider the other maximal tubing $\widetilde{\mathcal{T}}$. Let us denote its weight vector by $\widetilde{\boldsymbol{w}} = (\widetilde{w}_1, \dots, \widetilde{w}_{|\boldsymbol{r}|-1}, \widetilde{w}_*)$, where the entries in this vector *are still with respect to the indexing of the vertices in* (1). Let $\widetilde{w}(T_i)$ denote the sum of the $\widetilde{w}$ weights of the vertices in T_i, noting that T_i is *not* necessarily a tube in $\widetilde{\mathcal{T}}$. Then again we can write

$$\begin{aligned} \boldsymbol{x}^{\widetilde{\boldsymbol{w}}} &:= x_{\mathrm{in}}(T_1)^{\widetilde{w}_1} x_{\mathrm{in}}(T_2)^{\widetilde{w}_2} \cdots x_{\mathrm{in}}(T_{|\boldsymbol{r}|-1})^{\widetilde{w}_{|\boldsymbol{r}|-1}} x_*^{\widetilde{w}_*} \\ &= \left(\frac{x_{\mathrm{in}}(T_1)}{x_{\mathrm{out}}(T_1)}\right)^{\widetilde{w}(T_i)} \left(\frac{x_{\mathrm{in}}(T_2)}{x_{\mathrm{out}}(T_2)}\right)^{\widetilde{w}(T_2)} \cdots \left(\frac{x_{\mathrm{in}}(T_{|\boldsymbol{r}|-1})}{x_{\mathrm{out}}(T_{|\boldsymbol{r}|-1})}\right)^{\widetilde{w}(T_{|\boldsymbol{r}|-1})} x_*^{\widetilde{w}(V)} \\ &=: \xi_1^{\widetilde{w}(T_1)} \cdots \xi_{|\boldsymbol{r}|-1}^{\widetilde{w}(T_{|\boldsymbol{r}|-1})} x_*^{\widetilde{w}(V)} \end{aligned}$$

and therefore, since $w(V) = \widetilde{w}(V)$, we get

$$\boldsymbol{x}^{\widetilde{\boldsymbol{w}}-\boldsymbol{w}} = \xi_1^{\widetilde{w}(T_1)-w(T_1)} \cdots \xi_k^{\widetilde{w}(T_k)-w(T_k)}.$$

We will show that $\widetilde{w}(T_i) - w(T_i) \geq 0$ for all $i = 1, \dots, k$. If T_i is a tube in the tubing $\widetilde{\mathcal{T}}$ as well, then $\widetilde{w}(T_i) - w(T_i) = 0$. So suppose that T_i is *not* a tube in $\widetilde{\mathcal{T}}$. In this case, there is at least one vertex v in T_i that, in the tubing $\widetilde{\mathcal{T}}$, is the maximal vertex for a tube T which contains all of T_i (or, if v is not contained in any tube, simply replace T with V in the next calculation).

Case (i) $|T_i| \geq 2$. Then $|T| \geq 3$, and by Lemma 6.1,

$$\widetilde{w}(T_i) \geq \widetilde{w}(v) > 3^{|T|-3} \geq 3^{|T_i|-2} = w(T_i).$$

Case (ii) $|T_i| = 1$, so $w(T_i) = 0$ but $\widetilde{w}(T_i) > 0$ as T_i is not a tube in $\widetilde{\mathcal{T}}$. □

Example 6.3 Figure 7 shows the maximal tubing $\mathcal{T}$ of Figure 6 on the left, with its weights $\boldsymbol{w}$, and to its right a neighboring maximal tubing $\widetilde{\mathcal{T}}$, with weights $\widetilde{\boldsymbol{w}}$. Then $\boldsymbol{x}^{\widetilde{\boldsymbol{w}}-\boldsymbol{w}} = x_{1,1}^{71} x_{0,1}^{-71}$, and $x_{1,1}/x_{0,1}$ is the simple ratio in the chart for $\mathcal{T}$ associated to the tube in $\mathcal{T}$ that is not in $\mathcal{T}'$.

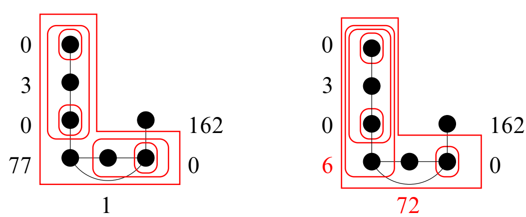

Figure 7: The weights for neighboring maximal tubings, exactly two of which are different (in red).

Proof of Theorem 1.1 Let $\boldsymbol{x} = (x_{0,1} : \dots : x_{0,r_0} : \dots : x_{n,1} : \dots : x_{n,r_n})$ denote the real homogeneous coordinates on $Q(\boldsymbol{r})$, and write $\boldsymbol{z}$ for the complexification of these homogeneous coordinates. Let $\mathcal{T}_1, \dots, \mathcal{T}_N$ be all the maximal tubings of $G(\boldsymbol{r})$, with weight vectors $\boldsymbol{w}_1, \dots, \boldsymbol{w}_N$ respectively. Define a projective toric variety $X \subset \mathbb{CP}^{N-1}$ by taking the closure in $\mathbb{CP}^{N-1}$ of the image of the map

$$z \mapsto (z^{\boldsymbol{w}_1} : \dots : z^{\boldsymbol{w}_N}). \tag{2}$$

Let us denote the nonnegative real part of X by $X_{\geq}$. Let $\mathbb{A}_1, \dots, \mathbb{A}_N$ denote the affine charts on $\mathbb{CP}^{N-1}$, ie $\mathbb{A}_i$ consists of all elements of $\mathbb{CP}^{N-1}$ whose i^{th} coordinate is nonzero. The sets $\{X_{\geq} \cap \mathbb{A}_i\}_{i=1,\dots,N}$ cover $X_{\geq}$, and we will show that $X_{\geq}$ has the structure of a manifold-with-corners, with charts

$$\psi_i \colon X_{\geq} \cap \mathbb{A}_i \xrightarrow{\cong} [0,\infty)^{|\boldsymbol{r}|-1}.$$

Indeed, $X_{\geq} \cap \mathbb{A}_i$ is the closure in $\mathbb{A}_i$ of the set of all points

$$(\boldsymbol{x}^{\boldsymbol{w}_1 - \boldsymbol{w}_i} : \dots : \underbrace{1}_{i} : \dots : \boldsymbol{x}^{\boldsymbol{w}_N - \boldsymbol{w}_i}),$$

where the entries of $\boldsymbol{x}$ are real and positive. By Lemma 6.2, each $\boldsymbol{x}^{\boldsymbol{w}_j - \boldsymbol{w}_i}$ is a product of the simple ratios in the simple ratios chart for the maximal tubing $\mathcal{T}_i$, and if a maximal tubing $\mathcal{T}_j$ is a neighbor of $\mathcal{T}_i$, then $\boldsymbol{x}^{\boldsymbol{w}_j - \boldsymbol{w}_i} = \xi^m$ for some $m \geq 1$, where ξ is the simple ratio corresponding to the tube in $\mathcal{T}_i$ that is no longer a tube in $\mathcal{T}_j$. There are $|\boldsymbol{r}| - 1$ maximal tubings which are neighbors of $\mathcal{T}_i$, so if we write $\xi_1, \dots, \xi_{|\boldsymbol{r}|-1}$ for the simple ratios in the chart for $\mathcal{T}_i$, then (up to rearranging the order of the entries) we can write

$$X_{\geq} \cap \mathbb{A}_i = \{(\xi_1^{m_1} : \xi_2^{m_2} : \dots : \xi_{|\boldsymbol{r}|-1}^{m_{|\boldsymbol{r}|-1}} : * : \dots : * : 1) \in \mathbb{CP}^{N-1} \mid \xi_i \in [0,\infty)\}$$

where $m_j \geq 1$, and each $*$ is some product of the form $\xi_1^{n_1} \xi_2^{n_2} \cdots \xi_{|\boldsymbol{r}|-1}^{n_{|\boldsymbol{r}|-1}}$ for $n_k \geq 0$. The map $\xi \mapsto \xi^m$ for $m \geq 1$ is an automorphism of the domain $[0,\infty)$, giving a direct identification

$$X_{\geq} \cap \mathbb{A}_i \cong (\xi_1, \dots, \xi_{|\boldsymbol{r}|-1}) \cong [0,\infty)^{|\boldsymbol{r}|-1}$$

of the affine chart on $X_{\geq} \cap \mathbb{A}_i$ with the simple ratio chart on U_{Γ_i}, showing that $X_{\geq}$ and $\overline{Q}(\boldsymbol{r})$ are homeomorphic as manifolds-with-corners. By the theory of toric varieties (see [5; 8]) we also know that $X_{\geq}$ and the moment polytope for X are homeomorphic as manifolds-with-corners. In this case the moment polytope is the convex hull of the weight vectors, which by [4] realizes the graph associahedron for $G(\boldsymbol{r})$. □

Remark As a closing remark, we mention that the combinatorics of these polytopes govern the algebraic structure of nonunital left A_∞ n–modules, whose counterparts in ordinary algebra are left modules over a tensor product of algebras. A left A_∞ n–module over an n–tuple of A_∞ algebras $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ consists of a graded vector space M, and a collection of multilinear maps $\mu^{\boldsymbol{r}|1} \colon \mathcal{A}_1^{\otimes r_1} \otimes \dots \otimes \mathcal{A}_n^{\otimes r_n} \otimes M \to M$

of degree $1-|\boldsymbol{r}|$ for each $\boldsymbol{r}=(r_1,\dots,r_n)\in\mathbb{Z}^n_{\geq 0}$, where $\mathcal{A}_i^{\otimes 0}:=\mathbb{F}$, the underlying field. Like any A_∞ algebraic structure, the maps $\mu^{\boldsymbol{r}|1}$ are then required to satisfy a collection of quadratic A_∞ relations.

In terms of the moduli spaces, the maps $\mu^{\boldsymbol{r}|1}$ can be represented by quilted strips with $\boldsymbol{r}$ markings; see Figure 8.

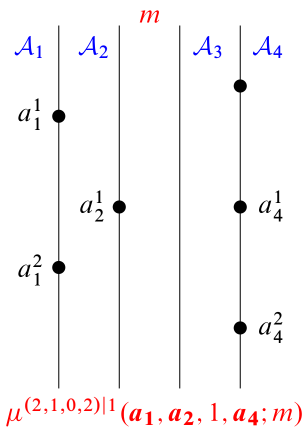

Figure 8: A labeled quilted strip in $Q(2,1,0,3)$ represents an operation $\mu^{(2,1,0,3)|1}$

Each seam of the quilted strip is labeled by an A_∞ algebra, and the marked points on that seam by elements of that A_∞ algebra. The two infinite ends of the strip are labeled by the module M, one end represents the input $m\in M$, and the other by the output $\mu^{\boldsymbol{r}|1}(\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n;m)$. In the A_∞ n–module equations, each quadratic term either contains $\mu^{\boldsymbol{0}|1}$ or $\mu^1_{\mathcal{A}_j}$, or it corresponds to a codimension-one facet of $\overline{Q}(\boldsymbol{r})$.

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Department of Computer Science, Università degli Studi di Verona
Strada le Grazie 15, 37134 Verona Italy

sikimeti.mau@univr.it

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