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# Gromov width and uniruling for orientable Lagrangian surfaces

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We prove a conjecture of Barraud and Cornea for orientable Lagrangian surfaces. As a corollary, we obtain that displaceable Lagrangian 2–tori have finite Gromov width. In order to do so, we adapt the pearl complex of Biran and Cornea to the nonmonotone situation based on index restrictions for holomorphic disks.

53DXX; 53D12

## 1 Introduction

The present paper is a continuation of results of the author [9], where it was shown that closed *monotone* (see Section 2.1 for the precise definition) Lagrangians in tame symplectic manifolds satisfy a general form of uniruling by holomorphic curves. Here we remove the monotonicity condition in the case of orientable surfaces. Recent examples given by Rizell [22] show that this restriction is not of a technical nature; the results simply do not hold in higher dimensions without other constraints, or even for nonorientable surfaces. In this note we focus our attention mostly on the connection between displacement energy and Gromov width.

Recall that the relative Gromov width of a Lagrangian $L$ is

$$w_G(L) := \sup_{\mathcal{B}(M,L,r)} \pi r^2,$$

where $\mathcal{B}(M,L,r)$ is the set of all symplectic embeddings of $B^{2n}(r)$, the ball of radius $r$ in $\mathbb{C}^n$, such that $B^{2n}(r) \cap \mathbb{R}^n$ is mapped to $L$.

The displacement energy of a Lagrangian is the minimal energy required by a Hamiltonian isotopy to displace it, $E(L) := \inf\{E(\phi) \mid \phi \in \mathrm{Ham}(M,\omega),\ \phi(L) \cap L = \varnothing\}$, where $E(\phi)$ is the energy of a Hamiltonian isotopy; we set $E(L) = \infty$ in case it is not displaceable.

If $L$ is a closed, displaceable and orientable Lagrangian surface, an easy argument shows that it is diffeomorphic to a torus. Our main result is the following:

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**Theorem A** *Let $L$ be a Lagrangian 2–torus in a tame symplectic manifold. Then $w_G(L) \leq 2E(L)$.*

The proof is based on Theorem 3.1, a uniruling result which in turns proves a conjecture of Barraud and Cornea [3] for orientable surfaces.

There are other nonmonotone situations where the Gromov width of a displaceable Lagrangian is known to be finite. Recent results of Borman and McLean [8] show that if $L$ is orientable and admits a metric of nonpositive sectional curvature in a Liouville manifold, then its Gromov width is bounded above by four times its displacement energy.

Uniruling for monotone Lagrangians is established via a mixture of the pearl complex from Biran and Cornea [5; 6; 7] and Lagrangian Floer theory of Floer [11], Oh [20; 21] and Fukaya, Oh, Ohta and Ono [13; 14]. We show how these constructions can be adapted to our case, with an appropriate choice of Novikov ring. The key argument is an elementary index computation of pseudoholomorphic disks given in Lemma 2.2, which itself relies on a technical result of Lazzarini [16]. There is no need to invoke the cluster complex of Cornea and Lalonde [10] or the general Lagrangian Floer theory of [13]; both these theories are much more complicated algebraically and even more so analytically.

We summarize in the next proposition the algebraic structures that we define for orientable surfaces by adapting the techniques of Biran and Cornea, without any monotonicity assumptions. The Novikov ring $\Lambda$ is defined in Section 2.2.2.

**Proposition 1.1** *Given a closed orientable Lagrangian surface $L \subset (M, \omega)$, there is a second-category subset $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}_\omega$ of regular compatible almost complex structures for which the following algebraic structures are defined and depend only on the connected component $[J] \in \pi_0(\mathcal{J}_{\mathrm{reg}})$:*

- *The Lagrangian quantum homology ring* $\mathrm{QH}(L, [J]; \Lambda)$ *of $L$, endowed with the quantum product.*
- *The Lagrangian Floer homology* $\mathrm{HF}(L, H, \{J_t\}; \Lambda)$ *of $L$, where $\boldsymbol{J} := \{J_t\}$ is a generic path of regular almost complex structures and $H$ is a Hamiltonian. It is a left* $\mathrm{QH}(L, [J_0]; \Lambda)$*–module and a right* $\mathrm{QH}(L, [J_1]; \Lambda)$*–module.*
- *The* $\mathrm{QH}(L, [J_0])$*–module isomorphism* $\mathrm{PSS}\colon \mathrm{QH}(L, [J_0]) \to \mathrm{HF}(L, H, \boldsymbol{J})$.

Details of the construction are given mostly for the Lagrangian quantum homology in Section 2.2, as this should emphasize the main ideas; the other structures are only quickly sketched in Section 2.3. The module property of the PSS isomorphism is adapted from [9] and was first proved by Leclercq [17] when there is no bubbling. Finally, a uniruling theorem is given in Section 3, from which we deduce Theorem A.

**Remark 1.2** Assume that $M$ is closed, and denote by $\beta_2^+(M)$ the dimension of the positive definite part of the intersection form on $H_2(M)$. If $L$ is a closed orientable Lagrangian surface of genus $g$ at least 2, then $\beta_2^+(M) \geq 2$. Indeed, $\omega^2 > 0$, and $[L]^2 = -\chi(L) = 2g-2 > 0$. Also, $[L]$ is linearly independent from $\omega$, so that $\beta_2^+(M) \geq 2$.

Much is known about symplectic 4–manifolds and the value of $\beta_2^+(M)$; see, for example, Baldridge [2], Gompf and Stipsicz [15] and McDuff and Salamon [18].

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# 2 Algebraic structures and the proof of Proposition 1.1

## 2.1 Preliminaries

Throughout the paper, $(M,\omega)$ is a tame symplectic four-dimensional manifold. The set of all $\omega$–compatible almost complex structures is denoted by $\mathcal{J}_\omega$, the induced Riemannian metric is $g(v_1,v_2) := \omega(v_1, Jv_2)$, and the associated first Chern class is written as $c_1(M)$ or $c_1$.

The Hamiltonian vector field $X_{H_t}$ of a compactly supported $H\colon S^1 \times M \to \mathbb{R}$ is uniquely defined by $\omega(X_{H_t},\cdot) = -dH_t(\cdot)$. Its time-one flow $\psi_1^H$ defines a Hamiltonian isotopy, and the set of all these isotopies is the group $\mathrm{Ham}(M,\omega)$. The energy of $\psi \in \mathrm{Ham}$ is defined as

$$E(\psi) = \inf_{H \mid \psi_1^H = \psi} \int_{S^1} \Big( \max_M H_t - \min_M H_t \Big)\, dt.$$

Let $L$ be a closed Lagrangian surface in $M$. Then there are two morphisms

$$\omega\colon H_2(M,L) \to \mathbb{R}, \quad \mu\colon H_2(M,L) \to \mathbb{Z}$$

given by the symplectic area and the Maslov index. Recall that this index is defined for every $u\colon (\Sigma, \partial\Sigma) \to (M,L)$, where $\Sigma$ is a surface with boundary, and that it depends only on the homology class of $u$. Finally, we say that $L$ is monotone if two conditions hold:

(1) There exists $\rho > 0$ such that $\omega = \rho\mu$.

(2) The positive generator of the image of $\mu$, denoted by $N_L$, is greater than or equal than two.

In this paper, *no monotonicity conditions* are imposed on Lagrangians.

## 2.2 Lagrangian quantum homology

**2.2.1 Index restriction for $J$–holomorphic disks and spheres; counting disks of Maslov class two** Recall that a disk $u\colon (D^2, S^1) \to (M, L)$ is somewhere injective if there exists $z \in D^2$ such that

$$u^{-1}(u(z)) = z, \quad du(z) \neq 0,$$

and that $u$ is *simple* if the set of injective points is dense.

The space of simple $J$–holomorphic disks in a class $A \in H_2(M, L)$ is denoted by $\mathcal{M}^*(A; J)$. For an $A$–regular $J$, ie one for which the linearization $D_u$ of the $\bar{\partial}_J$ operator at $u$ is onto, this space is a manifold of dimension

$$\dim \mathcal{M}^*(A; J) = n + \mu(A) = 2 + \mu(A). \tag{1}$$

We denote by $\mathcal{J}_{\mathrm{reg}}(A) \subset \mathcal{J}_\omega$ the second-category subset of $A$–regular almost complex structures and define $\mathcal{J}_{\mathrm{reg}} := \bigcap_A \mathcal{J}_{\mathrm{reg}}(A)$. Since $H_2(M, L)$ is countable, $\mathcal{J}_{\mathrm{reg}}$ is also of second category.

It is well known that $J$–holomorphic spheres which are not simple are multiply covered. For nonsimple disks, the same holds when the dimension of the Lagrangian is at least 3 and $J$ is generic, by a theorem of Lazzarini [16]. However, in dimension two the situation is quite different:

**Theorem 2.1** [16, Theorem A] *Let $u\colon (D^2, S^1) \to (M, L)$ be a nonconstant $J$–holomorphic disk. Then there exists a finite family $\{v_i\}$ of simple $J$–holomorphic disks $v_i\colon (D^2, S^1) \to (M, L)$ and positive integers $\{m_i\}$ such that, in $H_2(M, L)$, we have $[u] = \sum_i m_i [v_i]$ and $\bigcup_i v_i(D^2) = u(D^2)$.*

Here $[u]$ denotes the image of the positive generator of $\mathbb{Z} \cong H_2(D^2, S^1)$ by $u_*$.

The following elementary lemma is the key to the construction of all the algebraic structures considered in Proposition 1.1.

**Lemma 2.2** *Let $L$ be an orientable Lagrangian surface, $J \in \mathcal{J}_{\mathrm{reg}}$ a regular almost complex structure and $u\colon (D^2, S^1) \to (M, L)$ a nonconstant $J$–holomorphic disk. Then $\mu(u) \geq 2$. If $u$ is a nonconstant sphere, then $c_1(u) \geq 1$. In both cases $u$ is simple if equality holds.*

**Proof** First, given any orientable Lagrangian, regardless of dimension, the Maslov index of any disk $u$ with boundary on $L$ is even, since the loop $(u|_{S^1})^* TL \in \mathcal{L}(\mathbb{R}^{2n})$ lifts to the double cover $\mathcal{L}^{\mathrm{or}}(\mathbb{R}^{2n})$ of oriented Lagrangian subspaces.

Using the notation of Theorem 2.1, we have $\mu(u)=\sum_i m_i\mu(v_i)$, where all the $v_i$ are nonconstant simple $J$–holomorphic disks. Thus, biholomorphisms of the disk, denoted by $\mathcal{G}:=\mathrm{PSL}(2,\mathbb{R})=\mathrm{Aut}(D^2)\cong S^1\times D^2$, act freely on $\mathcal{M}^*([v_i];J)$, and (1) shows that

$$0\le \dim\mathcal{M}^*([v_i];J)/\mathcal{G}=2+\mu(v_i)-3=\mu(v_i)-1 \quad\Longleftrightarrow\quad 2\le\mu(v_i).$$

Hence, we get that $\mu(u)=\sum_i m_i\mu(v_i)\ge 2$.

As for the last part, $u$ can be written as $u=v\circ d_k$, where $v\colon S^2\to M$ is simple and $d_k\colon S^2\to S^2$ is a degree $k\ge 1$ holomorphic covering map. Then

$$0\le\dim\mathcal{M}^*([v];J)/\mathrm{PSL}(2,\mathbb{C})=4+2c_1(v)-6,$$

and $c_1(u)=kc_1(v)\ge k$. □

Fix $J\in\mathcal{J}_{\mathrm{reg}}$ and $A\in H_2(M,L)$ such that $\mu(A)=2$. Then there is an evaluation map

$$\begin{aligned}\mathrm{ev}\colon \mathcal{M}^*(A;J)\times S^1/\mathcal{G}&\to L,\\ [u,\theta]&\mapsto u(\theta).\end{aligned}$$

The action of $g\in\mathcal{G}$ is given by $g\cdot(u,\theta)=(u\circ g,g^{-1}(\theta))$. The index computations above and Gromov compactness for disks (see Frauenfelder [12]) yield that $\mathcal{M}^*(A;J)\times S^1/\mathcal{G}$ is a closed 2–dimensional manifold, hence ev has a well-defined mod 2 degree, giving the algebraic count of $J$–disks representing $A$ and going through a generic point of $L$, denoted by $d(A,J)$. However, this invariant might depend on the choice of $J$. Indeed, fix a path $\boldsymbol{J}:=\{J_t\}$, $J_t\in\mathcal{J}_\omega$, connecting $J_0$ and $J_1$. Then for a generic choice of such path (see McDuff and Salamon [19, Definition 3.1.6]), the space

$$\mathcal{W}^*(A,\boldsymbol{J}):=\{(t,u)\mid t\in[0,1],\ u\in\mathcal{M}^*(A,J_t)\}$$

provides a cobordism between $\mathcal{M}^*(A,J_0)$ and $\mathcal{M}^*(A,J_1)$. The next lemma is elementary, but it is worth mentioning since it is the possible source of noninvariance for the Lagrangian quantum homology of orientable surfaces; see Section 2.2.2.

**Lemma 2.3** *Fix a generic path $\boldsymbol{J}$. If it is in a connected component of $\mathcal{J}_{\mathrm{reg}}$, called a chamber, then $\mathcal{W}^*(A,\boldsymbol{J})$ is compact, hence $d(A,J_0)=d(A,J_1)$. Otherwise, it admits a compactification which must include nonconstant Maslov zero $J_t$–holomorphic disks for every $J_t\notin\mathcal{J}_{\mathrm{reg}}$.*

**Proof** By Gromov compactness, sequences in $\mathcal{W}^*(A,\boldsymbol{J})$ converge to stable maps $(T,\sum A_k:=A)$, where $T$ is a tree made up of $J_t$–holomorphic bubbles for some fixed $t$; each bubble is either a sphere or a disk $u_k$ representing a class $A_k$. Moreover, their Maslov indices satisfy $\sum \mu(u_k) = \mu(A)$. We use the convention that, for spheres, the Maslov index is twice the Chern class.

Assume first that $\boldsymbol{J} \subset \mathcal{J}_{\mathrm{reg}}$. Then Lemma 2.2 implies that there is only one curve in $T$ and the bubble tree is made of a single simple curve, hence it is already an element of $\mathcal{W}^*(A, \boldsymbol{J})$, which is thus compact. Since degree is invariant under compact cobordisms, we get that $d(A, J_0) = d(A, J_1)$.

Assume now that $\boldsymbol{J}$ is not entirely contained in $\mathcal{J}_{\mathrm{reg}}$. Then, by definition of regularity for the path $\boldsymbol{J}$, we have $\dim \operatorname{coker} D_{u_k} = 1$, since we assume $J_t \notin \mathcal{J}_{\mathrm{reg}}$.

If $u_k$ is a sphere, then, since it is not constant, $\mathrm{PSL}(2, \mathbb{C})$ acts on it and $\dim \ker D_{u_k} \geq 6$. Thus

$$\dim \ker D_{u_k} - \dim \operatorname{coker} D_{u_k} = \operatorname{ind} D_{u_k} = 2n + 2c_1(u_k) = 4 + 2c_1(u_k) \geq 6 - 1 = 5,$$

so $2c_1(u_k) = \mu(u_k) \geq 2$.

If $u_k$ is a nonconstant disk bubble, then $\dim \ker D_{u_k} \geq 3$ and $\dim \operatorname{coker} D_{u_k} = 1$, hence

$$\operatorname{ind} D_{u_k} = n + \mu(u_k) = 2 + \mu(u_k) \geq 3 - 1 = 2 \iff \mu(u_k) \geq 0.$$

Finally, since the total Maslov class of the tree is $\mu(A) = 2$, the bubbles have index at most two, and this concludes the proof. □

**Remark** Examples of Maslov zero nonregular $J$–holomorphic disks in Lagrangian tori can be found in Auroux [1].

**2.2.2 The pearl complex** In this section we adapt the construction of the pearl complex, following the presentation of Biran and Cornea [5] closely. Lemma 2.2 and our choice of Novikov ring guarantee that their original arguments are still valid.

Let $J \in \mathcal{J}_{\mathrm{reg}}$ and $f\colon L \to \mathbb{R}$ be a Morse–Smale function with respect to a generic Riemannian metric $\rho$, and denote by $\phi_t$ the *negative* gradient flow of $(f, \rho)$. The universal Novikov field is

$$\Lambda^{\mathrm{univ}} := \left\{ \sum_k a_k T^{\lambda_k} \;\middle|\; a_k \in \mathbb{Z}_2,\ \lambda_k \in \mathbb{R},\ \lim_{k\to\infty} \lambda_k = \infty \right\}.$$

We set $\Lambda := \Lambda^{\mathrm{univ}}[q, q^{-1}]$ and grade it with $|q| = -1$.

The pearl complex of $f$ is the $\Lambda$–module

$$\mathcal{C}(f, J, \rho) := \operatorname*{span}_{\Lambda} \langle \operatorname{Crit} f \rangle.$$

The differential counts pearly trajectories, which we now describe. Given $x, y \in \operatorname{Crit} f$ and $A \in H_2(M, L)$, the space of pearls from $x$ to $y$ in the class $A$ is denoted by $\mathcal{P}(x, y, A; f, J, \rho)$ and consists of families $(u_1, \dots, u_k)/(\bigoplus_{i=1}^k \mathcal{G}_{-1,1})$ such that:

- $u_i\colon (D^2, S^1) \to (M, L)$ are nonconstant $J$–holomorphic disks.
- $u_1(-1) \in W^u(x)$.
- There exist $0 < t_i < \infty$ such that $\phi_{t_i}(u_i(1)) = u_{i+1}(-1)$, for $i = 1, \dots, k-1$.
- $u_k(1) \in W^s(y)$.
- $\sum_i [u_i] = A$.
- $\mathcal{G}_{-1,1}$ is the subgroup of elements of $\operatorname{Aut}(D^2)$ fixing $\pm 1$ and the direct sum of these groups act on the family $(u_1, \dots, u_k)$.

Whenever $A = 0$, a pearl means a negative gradient flow line from $x$ to $y$, modulo the $\mathbb{R}$–action.

Denote by $\mathcal{P}^{*,d}(x, y, A; f, J, \rho)$ the subspace of pearls for which all the disks are simple and absolutely distinct. A standard transversality argument shows that either $\mathcal{P}^{*,d}(x, y, A; f, J, \rho)$ is empty or is a manifold of dimension $|x| - |y| + \mu(A) - 1$.

In [7, Section 3] Biran and Cornea prove the following crucial result for *monotone* Lagrangians, needed to show that pearls can be used to define a differential, which itself relies on Theorem 2.1 of Lazzarini.

**Proposition 2.4** [7] *Let $L$ be a monotone Lagrangian and $f, \rho$ be as defined above. Then there exists a second-category subset $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}_\omega$ such that, for every $A \in H_2(M, L)$ and every $x, y \in \operatorname{Crit} f$ with $|x| - |y| + \mu(A) - 1 \le 1$, we have:*

(1) $\mathcal{P}^{*,d}(x, y, A; f, J, \rho) = \mathcal{P}(x, y, A; f, J, \rho)$, *ie all pearls are automatically simple and absolutely distinct.*

(2) *If $|x| - |y| + \mu(A) - 1 = 0$, then $\mathcal{P}(x, y, A; f, J, \rho)$ is a compact $0$–dimensional manifold, and hence consists of a finite number of points.*

First, note that Lazzarini's result holds for Lagrangians which are not monotone. Also, a careful inspection of Biran and Cornea's proof shows that, as long as the Maslov index of $J$–holomorphic disks is at least two, then Proposition 2.4 is true, without the monotonicity assumption. By Lemma 2.2, this condition is automatically satisfied for closed orientable surfaces. Hence we can define the pearl differential

$$\partial\colon \mathcal{C}_*(f, J, \rho) \to \mathcal{C}_{*-1}(f, J, \rho),$$
$$x \mapsto \sum_y \Bigg( \sum_{\substack{A \in H_2(M,L) \\ |x|-|y|+\mu(A)-1=0}} \#_2 \mathcal{P}(x, y, A; f, J, \rho) T^{\omega(A)} q^{\mu(A)} \Bigg) y.$$

The coefficient of each $y$ is a well-defined element of the ring $\Lambda$ by standard arguments.

**Remark 2.5** The arguments in this paper work also for Lagrangians whose minimal Maslov number is at least two when restricted to $J$–holomorphic disks, which include the class of monotone Lagrangians and, a fortiori, orientable surfaces. It is, however, not clear to the author how to verify this condition in general.

In order to show that $\partial^2 = 0$, one uses Lemma 2.2 to rule out side bubbling and conclude that a one-dimensional family of pearls can be compactified using only broken flow lines. We denote the resulting Lagrangian quantum homology by $\mathrm{QH}(L, J; \Lambda)$ or $\mathrm{QH}(L, J)$.

The quantum product can also be defined and makes Lagrangian quantum homology a ring with unity, as well as a left module over itself:

$$\circ\colon \mathrm{QH}_p(L, J; \Lambda) \otimes \mathrm{QH}_q(L, J; \Lambda) \to \mathrm{QH}_{p+q-2}(L, J; \Lambda).$$

Comparison morphisms

$$\psi\colon \mathcal{C}(f_1, J_1, \rho_1) \to \mathcal{C}(f_2, J_2, \rho_2)$$

are defined via Morse homotopies and regular paths of almost complex structures. As was pointed out in Lemma 2.3, it is necessary to restrict to a regular path in a connected component of $\mathcal{J}_{\mathrm{reg}}$ to avoid nonregular bubbling of Maslov zero disks. In this case we directly get:

**Proposition 2.6** *For $[J_1] = [J_2] \in \pi_0(\mathcal{J}_{\mathrm{reg}})$, the comparison morphisms are chain maps which induce isomorphisms in homology and are compatible with the quantum product.*

## 2.3 Lagrangian Floer homology and $\mathrm{QH}(L, [J])$–module structure

Fix a Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$ such that $(\phi_1^H)^{-1}(L)$ intersects $L$ transversally, and denote by $\mathcal{O}(H)$ the finite set of contractible time-1 periodic orbits of the Hamiltonian flow $\phi_t^H$.

For each half-disk $u$ contracting such an orbit $\gamma$, recall that there is a Maslov index $\mu(u, \gamma)$ such that $\mu(u, \gamma) + \dim L/2 \in \mathbb{Z}$; see Robbin and Salamon [23]. There is an equivalence relation on tuples $(\gamma, u)$ defined by $(\gamma_1, u) \sim (\gamma_2, v)$ if and only if $\gamma_1 = \gamma_2$ and $\mu(\gamma_1, u) = \mu(\gamma_2, v)$. For each $\gamma$, fix a class $\tilde{\gamma} := [u_\gamma, \gamma]$ and grade it by $|\tilde{\gamma}| = 1 - \mu(\tilde{\gamma})$.

Denote by $\mathcal{J}^I := C^\infty([0, 1], \mathcal{J}_\omega)$ the smooth paths of compatible almost complex structures. Given $\boldsymbol{J} := \{J_t\} \in \mathcal{J}^I$ and $\gamma_-, \gamma_+ \in \mathcal{O}(H)$, the set of Floer strips from $\tilde{\gamma}_-$

to $\tilde{\gamma}_+$ in a homology class $A \in H_2(M, L)$ is the set $\mathcal{M}(\tilde{\gamma}_-, \tilde{\gamma}_+, A; \boldsymbol{J}, H)$ consisting of maps $u\colon \mathbb{R} \times [0,1] \to M$ satisfying these conditions:

- $u(\mathbb{R}, i) \in L$, $i = 0, 1$.
- $\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0$.
- $\lim_{s \to \pm\infty} u(s,t) = \gamma_\pm(t)$.
- $u_{\gamma^-} \# u \# - u_{\gamma^+} = A$.

There exists a second-category subset of regular paths $\mathcal{J}^{\boldsymbol{I}}_{\mathrm{reg}} \subset \mathcal{J}^{\boldsymbol{I}}$ for which the sets of Floer strips are manifolds of dimension $|\tilde{\gamma}_-| - |\tilde{\gamma}_+| + \mu(A)$. Recall that $\mathbb{R}$ acts freely on the space of nonconstant strips, hence the quotient space is also a manifold, of dimension smaller by one.

We further restrict $\mathcal{J}^{\boldsymbol{I}}_{\mathrm{reg}}$ to paths which are included in a chamber of $\mathcal{J}_{\mathrm{reg}}$ and pick such a path $\boldsymbol{J}$. The Floer complex is the graded $\Lambda$–module defined by

$$\mathrm{CF}(L; H, \boldsymbol{J}) := \operatorname*{span}_{\Lambda} \langle \tilde{\gamma} \mid \gamma \in \mathcal{O}(H) \rangle,$$

and the differential is given on generators by

$$\partial\colon \mathrm{CF}_*(L; H, \boldsymbol{J}) \to \mathrm{CF}_{*-1}(L; H, \boldsymbol{J}),$$
$$\tilde{\gamma}_- \mapsto \sum_{\substack{\gamma_+, A \\ |\tilde{\gamma}_-| - |\tilde{\gamma}_+| + \mu(A) = 1}} \#_2(\mathcal{M}(\tilde{\gamma}_-, \tilde{\gamma}_+, A; \boldsymbol{J}, H)/\mathbb{R}) T^{\omega(A)} q^{\mu(A)} \tilde{\gamma}_+.$$

This sum is well-defined by our choice of $\boldsymbol{J}$ and by Gromov compactness. Moreover, $\partial^2 = 0$ follows by standard arguments; one basically mimics the proofs by Oh [20; 21]. Of course, we also need that, for every $\gamma \in \mathcal{O}(H)$ and $A \in \mathbb{R}$, the algebraic numbers of $J_i$–holomorphic disks of Maslov class two and total area at most $A$ going through $\gamma(i)$ are equal, for $i = 0, 1$. This holds by our choice of $\boldsymbol{J} \subset \mathcal{J}_{\mathrm{reg}}$.

As with the pearl complex, if one restricts to a fixed connected component of $\mathcal{J}_{\mathrm{reg}}$, then the Floer homology is canonically independent of all choices.

The module structure is defined as in Charette [9]:

$$\star\colon \mathcal{C}_p(f, J_0) \otimes_\Lambda \mathrm{CF}_q(L, H, \boldsymbol{J}) \to \mathrm{CF}_{p+q-2}(L, H, \boldsymbol{J}),$$
$$x \otimes \tilde{\gamma}_- \mapsto \sum_{\tilde{\gamma}_+,\, A \in H_2(M,L)} \#_2 \mathcal{M}(x, \tilde{\gamma}_-, \tilde{\gamma}_+, A) T^{\omega(A)} q^{\mu(A)} \tilde{\gamma}_+,$$

where $\mathcal{M}(x, \tilde{\gamma}_-, \tilde{\gamma}_+, A)$ is the space of pearls leaving $x$ and entering a Floer strip $u$ (at the point $u(0,0)$) connecting $\tilde{\gamma}_-$ to $\tilde{\gamma}_+$. The homology class is

$$A = \sum_i [v_i] + [u_{\gamma_-} \# u \# - u_{\gamma_+}],$$

where the $v_i$ are the disks appearing in the pearl from $x$ to $u$. This gives $\mathrm{HF}(L, \boldsymbol{J})$ a structure of left $\mathrm{QH}(L, [J_0])$–module.

**2.3.1 The PSS isomorphism** We quickly recall here the remaining structures mentioned in Proposition 1.1; see [5; 9] for more details and proofs.

The PSS–morphism

$$\mathcal{C}_*(f, J) \to \mathrm{CF}_*(L, H, \boldsymbol{J})$$

is defined by counting pearls leaving from a critical point where the last disk in the pearl is a half-disk $u$ satisfying the PSS–equation and converging to an orbit $\tilde{\gamma}_+$. We count these pearls in each homology class $A = \sum v_i + [u\# - u_{\gamma_+}]$. The PSS–equation is

$$\partial_s u + J(s, t, u(s, t))(\partial_t u - \beta(s) X_{H_t}(u)) = 0,$$

where $\beta\colon \mathbb{R} \to [0, 1]$ is a monotone surjective function which is constant close to $\pm\infty$. An easy computation yields the following energy estimate, which is needed to prove Theorem 3.1:

$$E(u) := \int \omega(\partial_s u, J(s, t, u)\partial_s u)\, ds\, dt \le \omega(u) - \int H_t(\gamma_+(t))\, dt + \int \max_M H_t\, dt.$$

This is a chain map, by arguments similar to the ones considered in the previous sections, as long as $J(s, t) \subset \mathcal{J}_{\mathrm{reg}}$ for every $(s, t)$ and the family $J(s, t)$ is generic with $J(-\infty, 0) = J$. It is standard to show that such choices are always possible.

There is a similar $\mathrm{PSS}^{-1}$–morphism defined using pearls starting from an orbit $\tilde{\gamma}_-$ and going into a critical point using a half-disk $u$ satisfying the $\mathrm{PSS}^{-1}$–equation

$$\partial_s u + J(s, t, u(s, t))(\partial_t u - \beta(-s) X_{H_t}(u)) = 0.$$

We count each homology class $A = \sum_i [v_i] + [u_{\gamma_-}\# u]$. The energy of such half-disks verifies

$$E(u) \le \omega(u) + \int H_t(\gamma_-(t))\, dt - \int \min_M H_t\, dt.$$

Recall that a chain homotopy satisfying $d_{\mathrm{pearl}}\psi - \psi d_{\mathrm{pearl}} = \mathrm{id} - \mathrm{PSS}^{-1} \circ \mathrm{PSS}$ is defined by using pearls for which one of the disks $u$ solves the equation

$$\partial_s u + J(s, t, u(s, t))(\partial_t u + \alpha_R(s) X_{H_t}) = 0.$$

Here $R \in (0, \infty)$ depends on $u$, $\alpha_R\colon \mathbb{R} \to [0, 1]$ is smooth and, when $R \ge 1$, we set

$$\alpha_R(s) = \begin{cases} 1 & \text{if } s \in [-R, R], \\ 0 & \text{if } |s| \ge R + 1. \end{cases}$$

We require $|\alpha'_R| \leq 1$ and set $\alpha_R = R\alpha_1$ when $R \leq 1$. The energy bound for such disks is given by

$$E(u) \leq \omega(u) + \int \Big( \max_M H_t - \min_M H_t \Big)\, dt. \tag{2}$$

Finally, there is a chain homotopy

$$\eta\colon (\mathcal{C}(f, J) \otimes_\Lambda \mathcal{C}(g, J))_* \to \mathrm{CF}_{*-1}(L, H, \boldsymbol{J})$$

satisfying

$$\mathrm{PSS}(x \circ y) = x \star \mathrm{PSS}(y) + (\partial\eta - \eta\partial)(x \otimes y),$$

thus the PSS is a $\mathrm{QH}(L)$–module isomorphism.

## 3 A uniruling result and the proof of Theorem A

All the structures defined in Section 2 as well as the energy estimates given there yield a proof of the following uniruling result, which can be found in [9]; the modifications needed using the Novikov ring $\Lambda$ are direct.

**Theorem 3.1** *Let $L$ be an orientable Lagrangian surface and $H$ a nonconstant Hamiltonian. Then, for every $J_0 \in \mathcal{J}_{\mathrm{reg}}$, $\boldsymbol{J} \subset \mathcal{J}_{\mathrm{reg}}$ starting at $J_0$ and every $x_0 \in L \backslash (\phi_1^H)^{-1}(L)$, there exists a nonconstant map $u$ that is either a Floer $\boldsymbol{J}$–strip corresponding to $H$, a $J_t$–holomorphic sphere or a $J_0$–holomorphic disk with boundary on $L$, such that $x_0 \in \Im(u)$ and $0 < E(u) \leq \int(\max H_t - \min H_t)$. If $L$ and $(\phi_1^H)^{-1}(L)$ intersect transversally, then $\mu(u) \leq 2$.*

**Proof of Theorem A** Let $L$ be a closed, orientable and displaceable Lagrangian surface, $E(L)$ its displacement energy, and $H$ a Hamiltonian displacing $L$ chosen such that

$$\int \Big( \max_M H_t - \min_M H_t \Big)\, dt = E(L) + \epsilon.$$

Given a symplectic embedding $e\colon (B^4(r), \mathbb{R}^2) \to (M, L)$, the previous theorem shows that a nonconstant $J$–holomorphic sphere (the constant path $\boldsymbol{J} = J$ is generic since there are no strips) or $J$–holomorphic disk with boundary on $L$, of symplectic area at most $E(L) + \epsilon$ goes through $e(0)$, where $e^*J = J_0$; here $J_0$ denotes the standard complex structure on $\mathbb{C}^2$. Standard arguments (see eg Barraud and Cornea [4, Proof of Corollary 3.10]) then yield $\pi r^2/2 \leq E(L) + \epsilon$ for every $\epsilon$. □

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