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The LS category of the product of lens spaces

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We reduce Rudyak's conjecture that a degree-one map between closed manifolds cannot raise the Lusternik–Schnirelmann category to the computation of the category of the product of two lens spaces $L_p^n \times L_q^n$ with relatively prime p and q. We have computed $\operatorname{cat}(L_p^n \times L_q^n)$ for values p, $q > n/2$. It turns out that our computation supports the conjecture.

For spin manifolds M we establish a criterion for the equality $\operatorname{cat} M = \dim M - 1$, which is a K–theoretic refinement of the Katz–Rudyak criterion for $\operatorname{cat} M = \dim M$. We apply it to obtain the inequality $\operatorname{cat}(L_p^n \times L_q^n) \leq 2n - 2$ for all odd n and odd relatively prime p and q.

55M30; 55N15

1 Introduction

This paper was motivated by the following conjecture of Rudyak:

Conjecture 1.1 [19] *A degree-one map between closed manifolds cannot raise the Lusternik–Schnirelmann category.*

It is known that degree-one maps $f\colon M \to N$ between manifolds tend to have domain more complex than their image. The Lusternik–Schnirelmann category is a numerical invariant that measures the complexity of a space. Thus, Rudyak's conjecture that $\operatorname{cat} M \geq \operatorname{cat} N$ for a degree-one map $f\colon M \to N$ is quite natural. Rudyak (see also the book by Cornea, Lupton, Opera and Tanré [7, page 65]) obtained some partial results supporting the conjecture. In particular, he proved the following:

Theorem 1.2 [19] *Let $f\colon M \to N$ be a degree-± 1 map between closed, stably parallelizable n–manifolds, $n \geq 4$, such that $2 \operatorname{cat} N \geq n + 4$. Then $\operatorname{cat} M \geq \operatorname{cat} N$.*

In this paper we reduce Rudyak's conjecture to the following question about the LS category of the product of two n–dimensional lens spaces ($n = 2k - 1$).

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Problem 1.3 *Do there exist n and relatively prime p and q such that*

$$\operatorname{cat}(L_p^n \times L_q^n) > n+1\,?$$

We show that an affirmative answer to this problem gives a counterexample to Rudyak's conjecture.

This paper is devoted to computation of the category of the product $L_p^n \times L_q^n$ of lens spaces for relatively prime p and q. Here we use the shorthand notation $L_p^n = L_p^n(\ell_1, \dots, \ell_k)$ for a general lens space of dimension $n = 2k-1$, defined for the linear $\mathbb{Z}_p$–action on $S^n \subset \mathbb{C}^k$ determined by the set of natural numbers $(\ell_1, \dots, \ell_k)$ with $(p, \ell_i) = 1$ for all i.

The obvious inequality $\operatorname{cat} X \le \dim X$ and the cup-length lower bound (see Proposition 2.9) give the estimates

$$(*) \qquad n+1 \le \operatorname{cat}(L_p^n \times L_q^n) \le 2n.$$

In this paper we prove that, for fixed n, the lower bound is almost always sharp.

Theorem 1.4 *For every $n = 2k-1$ and primes p, $q \ge k$, $p \ne q$, for all lens spaces L_p^n and L_q^n,*

$$\operatorname{cat}(L_p^n \times L_q^n) = n+1.$$

This result still leaves some hope to have $\operatorname{cat}(L_p^n \times L_q^n) > n+1$ for small values of p (especially for $p = 2$) for some lens spaces.

In the second part of the paper we make an improvement of the upper bound in $(*)$. The first improvement comes easily by virtue of the Katz–Rudyak criterion [13]: *for a closed m–manifold M the inequality* $\operatorname{cat}(M) \le m-1$ *holds if and only if M is inessential.* We recall that Gromov calls a m–manifold M *inessential* if a map $u\colon M \to B\pi$ that classifies its universal covering can be deformed to the $(m-1)$–dimensional skeleton $B\pi^{(m-1)}$. Since for relatively prime p and q the product $L_p^n \times L_q^n$ is inessential, we have $\operatorname{cat}(L_p^n \times L_q^n) \le 2n-1$. In the paper we improve this inequality to the following:

Theorem 1.5 *For all odd n and odd relatively prime p and q,*

$$\operatorname{cat}(L_p^n \times L_q^n) \le 2n-2.$$

For that we study a general question: when is the LS category of a closed spin m–manifold M less than $m-1$? We prove in Theorem 6.6 that for a closed m–manifold M with $\pi_2(M) = 0$, the inequality $\operatorname{cat} M \le m-2$ holds if and only if the map $u\colon M \to B\pi$

can be deformed to the $(m-2)$–dimensional skeleton $B\pi^{(m-2)}$. A deformation of a classifying map of a manifold to the $(m-2)$–skeleton $B\pi^{(m-2)}$ is closely related to Gromov's conjecture on manifolds with positive scalar curvature and it was investigated by Bolotov and Dranishnikov [3]. Combining this with some ideas from [3], we produce a criterion for when a closed spin m–manifold M has $\operatorname{cat} M \le m-2$. The criterion involves the vanishing of the integral homology and ko–homology fundamental classes of M under a map classifying the universal covering of M.

Theorem 1.6 (Criterion) *If M is a a closed, spin, inessential m–manifold with $\pi_2(M)=0$, then*

$$\operatorname{cat} M \le \dim M - 2$$

if and only if $j_*u_*([M]_{ko})=0$, where $j\colon B\pi \to B\pi/B\pi^{(m-2)}$ is the quotient map.

Since a closed orientable manifold M is inessential if and only if $u_*([M])=0$ in $H_*(B\pi)$ — see Babenko [1] — the Katz–Rudyak criterion for orientable manifolds can be rephrased as follows: $\operatorname{cat} M \le m-1$ if and only if $u_*([M])=0$. Thus, our criterion is a further refinement of the Katz–Rudyak criterion.

It turns out that the vanishing of $u_*([M])$ in $H_*(B\pi)$ makes the primary obstruction to a deformation of $u\colon M \to B\pi$ to $B\pi^{(m-2)}$ trivial. It is not difficult to show that the second obstruction lives in the group of coinvariants $\pi_m(B\pi, B\pi)_\pi$; see [3]. We prove that the group of coinvariants $\pi_m(B\pi, B\pi^{(m-2)})_\pi$ naturally injects into the homotopy group $\pi_m(B\pi/B\pi^{(m-2)})$. This closes a gap in the computation of the second obstruction in [3]. Based on that injectivity result we use the real connective K–theory to express the second obstruction in terms of the image of the ko–fundamental class. The spin condition is needed for the existence of a fundamental class in ko-theory.

The new upper bound implies that $\operatorname{cat}(L_p^3 \times L_q^3)=4$ for all p and q. Note that for prime p and q this fact can be also derived from Theorem 1.4.

We complete the paper with a proof of the upper bound formula for the category of a connected sum of two manifolds:

Theorem 1.7 $\operatorname{cat} M \# N \le \max\{\operatorname{cat} M, \operatorname{cat} N\}.$

Since we use this formula in the paper and its original proof in [16] does not cover all cases, we supply an alternative proof.

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2 Preliminaries

2.1 LS category

The Lusternik–Schnirelmann category, for a topological space X, satisfies $\operatorname{cat} X \le k$ if there is a cover $X = U_0 \cup \cdots \cup U_k$ by $k+1$ open subsets each of which is contractible in X. The subsets contractible in X will be called in this note *X–contractible* and The covers of X by subsets contractible in X will be called *categorical.*

Let $\pi = \pi_1(X)$. We recall that the cup product $\alpha \smile \beta$ of twisted cohomology classes $\alpha \in H^i(X; L)$ and $\beta \in H^j(X; M)$ takes values in $H^{i+j}(X; L \otimes M)$, where L and M are π–modules and $L \otimes M$ is the tensor product over $\mathbb{Z}$; see Brown [5]. Then the cup-length of X, denoted as $cl(X)$, is defined as the maximal integer k such that $\alpha_1 \smile \cdots \smile \alpha_k \neq 0$ for some $\alpha_i \in H^{n_i}(X; L_i)$ with $n_i > 0$. The following inequalities give estimates on the LS category:

Theorem 2.1 [7] $$cl(X) \le \operatorname{cat} X \le \dim X.$$

2.2 Ganea–Schwarz approach to the LS category

Given two maps $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$, we set

$$Z = \{(x_1, x_2, t) \in X_1 * X_2 \mid f_1(x_1) = f_2(x_2)\}$$

and define the *fiberwise join*, or *join over* Y, of f_1 and f_2 as the map

$$f_1 *_Y f_2\colon Z \to Y, \quad (f_1 *_Y f_2)(x_1, x_2, t) = f_1(x_1) = f_2(x_2).$$

Let $p_0^X\colon PX \to X$ be the Serre path fibration. This means that PX is the space of paths on X that start at the base point $x_0 \in X$, and $p_0^X(\alpha) = \alpha(1)$ for $\alpha \in PX$. We denote by $p_n^X\colon G_n(X) \to X$ the iterated fiberwise join of $n+1$ copies of p_0^X. Thus, the fiber $F_n = (p_n^X)^{-1}(x_0)$ of the fibration p_n^X is the join product $\Omega X * \cdots * \Omega X$ of $n+1$ copies of the loop space ΩX on X. So, F_n is $(n-1)$–connected. It is known that $G_n(X)$ is homotopy equivalent to the mapping cone of the inclusion of the fiber $F_{n-1} \to G_{n-1}(X)$.

When $X = K(\pi, 1)$, the loop space ΩX is naturally homotopy equivalent to π and the space $G_n(\pi) = G_n(K(\pi, 1))$ has the homotopy type of a n–dimensional complex.

The proof of the following theorem can be found in [7]:

Theorem 2.2 (Ganea, Schwarz) *For a CW space X, $\operatorname{cat}(X) \le n$ if and only if there exists a section of $p_n^X\colon G_n(X) \to X$.*

This theorem can be extended to maps:

Theorem 2.3 *For a map $f\colon Y \to X$ to a CW space X, $\mathrm{cat}(f) \le n$ if and only if there exists a lift of f with respect to $p_n^X\colon G_n(X) \to X$.*

We recall that the LS category of a map $f\colon Y \to X$ is the least integer k for which Y can be covered by $k+1$ open sets $U_0, \dots, U_k$ such that the restrictions $f|_{U_i}$ are null-homotopic for all i.

We use the notation $\pi_*(f) = \pi_*(M_f, X)$, where M_f is the mapping cylinder of $f\colon X \to Y$. Then $\pi_i(f) = 0$ for $i \le n$ amounts to saying that f induces isomorphisms $f_*\colon \pi_i(X) \to \pi_i(Y)$ for $i < n$ and an epimorphism in dimension n.

Proposition 2.4 [8] *Let $f_j\colon X_j \to Y_j$, $3 \le j \le s$ be a family of maps of CW spaces such that $\pi_i(f_j) = 0$ for $i \le n_j$. Then the joins satisfy*

$$\pi_k(f_1 * f_2 * \cdots * f_s) = 0$$

for $k \le \min\{n_j\} + s - 1$.

2.3 The Berstein–Schwarz class

Let π be a discrete group and A be a π–module. By $H^*(\pi, A)$ we denote the cohomology of the group π with coefficients in A and by $H^*(X; A)$ we denote the cohomology of a space X with the twisted coefficients defined by A. The Berstein–Schwarz class of a group π is a certain cohomology class $\beta_\pi \in H^1(\pi, I(\pi))$, where $I(\pi)$ is the augmentation ideal of the group ring $\mathbb{Z}\pi$; see Berstein [2] and Schwarz [22]. It is defined as the first obstruction to a lift of $B\pi = K(\pi, 1)$ to the universal covering $E\pi$. The class β_π is defined by a cocycle $\beta\colon E\pi^{(1)} \to I(\pi)$. We note that the 1–skeleton of $E\pi$ can be identified with the Cayley graph of π. For a fixed set S of generators of π, the Cayley graph $C = C(\pi, S)$ has $V = \pi$ as the set of vertices and $E = \{[\gamma, \gamma s] \mid \gamma \in \pi,\ s \in S\}$ as the set of edges.

Note that the 1–skeleton of $B\pi$ can be identified with the wedge of circles labeled by S. Then the 1–skeleton $E\pi^{(1)}$ of the universal covering equals the Cayley graph $C = C(\pi, S)$. In that case the cocycle β takes every edge $[a, b] \subset C$ to $b - a \in I(\pi)$.

Here is a more algebraic definition of β_π. Consider the cohomology long exact sequence generated by the short exact sequence of coefficients

$$0 \longrightarrow I(\pi) \longrightarrow \mathbb{Z}\pi \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where ϵ is the augmentation homomorphism. Then $\beta_\pi = \delta(1)$ equals the image of the generator $1 \in H^0(\pi;\mathbb{Z}) = \mathbb{Z}$ under the connecting homomorphism

$$\delta\colon H^0(\pi;\mathbb{Z}) \to H^1(\pi; I(\pi)).$$

It follows from the definition of the connecting homomorphism δ (snake lemma) that $\delta(1)$ is defined by the above cocycle β.

Theorem 2.5 (Universality [9; 22]) *For any cohomology class $\alpha \in H^k(\pi, L)$ there is a homomorphism of π–modules $I(\pi)^k \to L$ such that the induced homomorphism for cohomology takes $(\beta_\pi)^k \in H^k(\pi; I(\pi)^k)$ to α, where $I(\pi)^k = I(\pi)\otimes\cdots\otimes I(\pi)$ and $(\beta_\pi)^k = \beta_\pi \smile \cdots \smile \beta_\pi$.*

Corollary 2.6 [22] *The class $(\beta_\pi)^{n+1}$ is the primary obstruction to a section of $p_n^{B\pi}\colon G_n(\pi) \to B\pi$.*

Corollary 2.7 *For any group π, its cohomological dimension can be expressed as*

$$\mathrm{cd}(\pi) = \max\{n \mid (\beta_\pi)^n \neq 0\}.$$

Corollary 2.8 $$cl(L_p^n) = n.$$

Proof For any lens space L_p^n the inclusion $L_p^n \to B\mathbb{Z}_p$ to the classifying space as the n–skeleton takes $(\beta_{\mathbb{Z}_p})^n$ to a nonzero element β^n. Since $\mathrm{cd}(\mathbb{Z}_p) = \infty$, we obtain $(\beta_{\mathbb{Z}_p})^n \neq 0$. Since the restriction to the n–skeleton is injective on n–dimensional cohomology groups, the result follows. □

Proposition 2.9 $$cl(L_p^n \times L_q^n) \geq n+1.$$

Proof Let $\alpha \in H^n(L_q^n) = \mathbb{Z}$ be a generator. Then, in view of the Kunneth formula for local coefficients [4], the cross product

$$\beta^n \times \alpha \in H^{2n+1}(L_p^n \times L_q^n; I(\mathbb{Z}_p)^n)$$

is nontrivial for the above $\beta \in H^1(L_p^n; I(\mathbb{Z}_p))$. □

3 Some examples of degree-one maps

Let M be an oriented manifold and $k \in \mathbb{Z} \setminus \{0\}$; by kM we denote the connected sum $M \# \cdots \# M$ of $|k|$ copies of M, taken with the opposite orientation if k is negative. For an odd n and natural $p > 1$ we denote by L^n_p a lens space, ie the orbit space $S^n/\mathbb{Z}_p$ for a free linear action of $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ on the sphere S^n.

Theorem 3.1 *For m, $n \in 2\mathbb{N} + 1$ and any relatively prime numbers p and q there are k, $l \in \mathbb{Z}$ such that the manifold*

$$M = k(L^m_p \times S^n) \# l(S^m \times L^n_q)$$

admits a degree-one map $\phi\colon M \to N$ onto $N = L^m_p \times L^n_q$.

Proof Take k and l such that $lp + kq = 1$. Let $f\colon S^m \to L^m_p$ and $g\colon S^n \to L^n_q$ be the projections to the orbit space for the $\mathbb{Z}_p$ and $\mathbb{Z}_q$ free actions, respectively. We may assume that the above connected sum is obtained by taking the wedge of $|k| + |l| - 1$ spheres of dimension $m + n - 1$ embedded in one of the summands and gluing all other summands along those spheres. Consider the quotient map

$$\psi\colon k(L^m_p \times S^n) \# l(S^m \times L^n_q) \to \bigvee_k (L^m_p \times S^n) \vee \bigvee_l (S^m \times L^n_q)$$

that collapses the wedge of those $(m+n-1)$–spheres to a point. Let the map

$$\phi\colon \bigvee_k (L^m_p \times S^n) \vee \bigvee_l (S^m \times L^n_q) \to L^m_p \times L^n_q$$

be defined as the union

$$\phi = \bigcup_k (1 \times g) \cup \bigcup_l (f \times 1).$$

Note that the degree of $f \times 1$ is p, the degree of $1 \times g$ is q and the degree of $\phi \circ \psi$ is $lp + kq = 1$. □

Proposition 3.2 *For $m \le n$,* $\operatorname{cat}(k(L^m_p \times S^n) \# l(S^m \times L^n_q)) = n + 1$.

Proof It follows from the cup-length estimate that $\operatorname{cat}(S^m \times L^n_q) \ge n + 1$ and, generally, $\operatorname{cat}(k(L^m_p \times S^n) \# l(S^m \times L^n_q)) \ge n + 1$ when $l \ne 0$. By the product formula, $\operatorname{cat}(S^m \times L^n_r) \le n + 1$. Thus, $\operatorname{cat}(S^m \times L^n_r) = n + 1$. Then, by the sum formula [16] (see Theorem 7.1),

$$\operatorname{cat}(k(L^m_p \times S^n) \# l(S^m \times L^n_q)) \le n + 1.$$ □

Now one can see the connection between Rudyak's conjecture and Problem 1.3. If there exist relatively prime p and q and odd n such that $\operatorname{cat}(L^n_p \times L^n_q) > n+1$, then the map of Theorem 3.1 will be a counter-example to Rudyak's conjecture.

Remark In Theorem 3.1 one can use fake lens spaces. Since every fake lens space is homotopy equivalent to a lens space [23] and the LS category is a homotopy invariant, it suffices to consider only the classical lens spaces.

4 On the category of the product of lens spaces

Let $\bar{\ell} = (\ell_1, \dots, \ell_k)$ be a set of mod p integers relatively prime to p. The lens space $L^{2k-1}_p(\bar{\ell})$ is the orbit space of the action of $\mathbb{Z}_p = \langle t \rangle$ on the unit sphere $S^{2k-1} \subset \mathbb{C}^k$ defined by the formula

$$t(z_1, \dots, z_k) = (e^{2\pi i \ell_1/p} z_1, \dots, e^{2pii\ell_k/p}).$$

We note that for all k the lens spaces $L^{2k-1}_p(\bar{\ell})$ have a natural CW complex structure with one cell in each dimension up to $2k-1$ such that $L^{2k-1}_p(\bar{\ell})$ is the $(2k-1)$–skeleton of $L^{2k+1}_p(\bar{\ell}, \ell_{k+1})$. If $\alpha\colon \mathbb{Z}_p \times S^{2k-1} \to S^{2k-1}$ is a free action which is not necessarily linear, its orbit space is called a *fake lens space* and is denoted by $L^{2k-1}_p(\alpha)$.

We recall that a closed, oriented n–manifold M is called *inessential* — see Gromov [12] — if a map $u\colon M \to B\pi = K(\pi, 1)$ that classifies its universal cover can be deformed to the $(n-1)$–dimensional skeleton $B\pi^{(n-1)}$. It is known that a closed, oriented n–manifold M is essential if and only if $u_*([M]) \neq 0$, where $[M] \in H_n(M; \mathbb{Z})$ denotes the fundamental class [1; 3].

We note that $\operatorname{cat} M = \dim M$ if and only if M is essential [13]. Clearly, every lens space L^n_p is essential. In particular, $\operatorname{cat} L^n_p = n$. Since $\mathbb{Z}_p \otimes \mathbb{Z}_q = 0$ for relatively prime p and q, the product $L^m_p \times L^n_q$ is inessential. Hence, $\operatorname{cat}(L^m_p \times L^n_q) \le m+n-1$ for all p and q.

4.1 Stably parallelizable lens spaces

First we do our computation for stably parallelizable lens spaces.

Proposition 4.1 *For lens spaces L^m_p and L^n_q with $m \le n$ and $(p,q) = 1$ which are homotopy equivalent to stably parallelizable manifolds,*

$$\operatorname{cat}(L^m_p \times L^n_q) = n+1.$$

Proof Let

$$\phi\colon M = k(L_p^m \times S^n) \# l(S^m \times L_q^n) \to N = L_p^m \times L_q^n$$

be the map of degree one from Theorem 3.1. Suppose that L_p^m and L_q^n are homotopy equivalent to stably parallelizable manifolds N_p^m and N_q^n, respectively. Then there are homotopy equivalences $h\colon M' = k(N_p^m \times S^n) \# l(S^m \times N_q^n) \to M$ and $h'\colon N = L_p^m \times L_q^n \to N' = N_p^m \times N_q^n$. Since a connected sum and the product of stably parallelizable manifolds are stably parallelizable (see for example [14]), the manifolds M' and N' are stably parallelizable. Assume that $\operatorname{cat}(L_p^m \times L_q^n) \geq n+2$. Then

$$2 \operatorname{cat} N' = 2 \operatorname{cat}(L_p^m \times L_q^n) = 2(n+2) \geq m+n+4 = \dim(L_p^m \times L_q^n) + 4.$$

By Theorem 1.2 applied to the map $h' \circ \phi \circ h\colon M' \to N'$ from Theorem 3.1, we obtain a contradiction:

$$n+2 = \operatorname{cat} N = \operatorname{cat} N' \leq \operatorname{cat} M' = \operatorname{cat} M = n+1. \qquad \square$$

Since all orientable 3-manifolds are stably parallelizable, we obtain:

Corollary 4.2 *For relatively prime p and q,*

$$\operatorname{cat}(L_p^3 \times L_q^3) = 4.$$

There is a characterization of stable parallelizability of lens spaces [10]: the lens space $L_p^{2k-1}(\ell_1, \dots, \ell_k)$ is stably parallelizable if and only if $p \geq k$ and $\ell_1^{2j} + \cdots + \ell_k^{2j} = 0 \bmod p$ for $j = 1, 2, \dots, \left[\frac{1}{2}(k-1)\right]$. We recall that two lens spaces $L_p^{2k-1}(\ell_1, \dots, \ell_k)$ and $L_p^{2k-1}(\ell_1', \dots, \ell_k')$ are homotopy equivalent [17] if and only if the mod p equation

$$\ell_1 \ell_2 \cdots \ell_k = \pm a^k \ell_1' \ell_2' \cdots \ell_k'$$

has a solution $a \in \mathbb{Z}_p$. These conditions imply that a lens space is rarely homotopy equivalent to a stably parallelizable one. Nevertheless, Ewing, Moolgavkar, Smith and Stong [10] showed that, for each $n = 2k-1$, for infinitely many primes p there are stably parallelizable lens spaces L_p^n. Clearly, there are more chances for the existence of stably parallelizable fake lens spaces with given n and p. Thus, Kwak [15] proved that for every odd $n = 2k-1$ and $p \geq k$ there is a fake n–dimensional stably parallelizable lens space. Since every fake lens space is homotopy equivalent to a lens space — see Wall [23] — we obtain that for every $n = 2k-1$ and $p \geq k$ there is a lens space L_p^n homotopy equivalent to a stably parallelizable manifold.

4.2 Category of classifying maps

We recall that any map $u\colon X \to B\pi = K(\pi, 1)$ of a CW complex X that induces an isomorphism of the fundamental group classifies the universal covering $\widetilde{X}$, ie $\widetilde{X}$ is obtained as the pull-back of the universal covering $E\pi$ of $B\pi$ by means of u. We call such a map a *classifying map* of X.

Proposition 4.3 *Let $u\colon X \to B\pi$ be a map classifying the universal covering of a CW complex X. Then the following are equivalent:*

(1) $\mathrm{cat}(u) \le k$.

(2) *u admits a lift $u'\colon X \to G_k(\pi)$ of u with respect to $p^\pi_n\colon G_k(\pi) \to B\pi$.*

(3) *u is homotopic to a map $f\colon X \to B\pi$ with $f(X) \subset B^{(k)}$.*

Proof (1) $\Longrightarrow$ (2) is a part of Theorem 2.3.

(2) $\Longrightarrow$ (3) Since $G_k(\pi)$ has the homotopy type of a k–dimensional complex, the map p^π_k can be deformed to a map p' with the image in $B\pi^{(k)}$. Then we can take $f = p' \circ u'$.

(3) $\Longrightarrow$ (1) For a map $f\colon X \to B\pi$ with $f(X) \subset B^{(k)}$ homotopic to u we obtain $\mathrm{cat}(u) = \mathrm{cat}(f) \le \mathrm{cat}\, B\pi^{(k)} \le k$. □

Theorem 4.4 *Let X be an n–dimensional CW complex with a classifying map $u\colon X \to B\pi$ having $\mathrm{cat}\, u \le k$ and with $(n-k)$–connected universal covering $\widetilde{X}$. Then $\mathrm{cat}\, X \le k$.*

Proof Note that the map p^X_k factors through the pull-back, $p^X_k = p' \circ q$:

$$\begin{array}{ccccc} G_k(X) & \xrightarrow{q} & Z & \xrightarrow{f'} & G_k(\pi) \\ & \searrow^{p^X_k} & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p^\pi_k} \\ & & X & \xrightarrow{u} & B\pi \end{array}$$

The condition $\mathrm{cat}\, u \le k$ implies that u has a lift $u'\colon X \to G_k(\pi)$, $u = p^\pi_k u'$. Hence, p' admits a section $s\colon X \to Z$. Since X is n–dimensional, to show that s has a lift with respect to q it suffices to prove that the homotopy fiber F of the map q is $(n-1)$–connected. Since $\pi_i(X) = 0$ for $1 < i \le n-k$, $B\pi$ is aspherical and u induces an isomorphism of the fundamental groups, we obtain $\pi_i(u) = 0$ for $i \le n-k+1$. Hence, $\pi_i(\Omega u) = 0$ for $i \le n-k$. Then, by Proposition 2.4, $\pi_i(*_{k+1}\Omega u) = 0$ for

$i \le (n-k)+(k+1)-1 = n$. The commutative diagram generated by q and the fibrations p^k_X and p',

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_i(*_{k+1}\Omega(X)) & \longrightarrow & \pi_i(G_k(X)) & \longrightarrow & \pi_i(X) & \longrightarrow \cdots \\ & \downarrow {\scriptstyle *_{k+1}\Omega u} & & \downarrow {\scriptstyle q_*} & & \downarrow {\scriptstyle 1} & \\ \cdots \longrightarrow & \pi_i(*_{k+1}\Omega(B\pi)) & \longrightarrow & \pi_i(Z) & \longrightarrow & \pi_i(X) & \longrightarrow \cdots, \end{array}$$

and the five lemma imply that $\pi_i(q) = 0$ for $i \le n$. Hence, $\pi_i(F) = 0$ for $i \le n-1$. Thus, s admits a homotopy lift. Therefore, p^X_k has a homotopy section and, hence, it admits a section. Therefore, by Theorem 2.2, $\operatorname{cat} X \le k$. □

4.3 The main computation

Proposition 4.5 *For any two lens spaces $L^n_p(\bar\ell)$ and $L^n_p(\bar\mu)$, there is a map*

$$f\colon L^n_p(\bar\ell) \to L^n_p(\bar\mu)$$

that induces an isomorphism of the fundamental groups.

Proof Let $q_1\colon S^n \to L^n_p(\bar\ell)$ and $q_2\colon S^n \to L^n_p(\bar\mu)$ be the projections onto the orbit spaces of the corresponding $\mathbb{Z}_p$–actions. We note that $L^n_p(\bar\mu)$ is the n–skeleton in $L^{n+2}_p(\bar\mu, 1)$. Let $\bar q_2\colon S^{n+2} \to L^n_p(\mu, 1)$ be the corresponding projection:

$$\begin{array}{ccccc} S^n & \longleftarrow & S^n \times S^{n+2} & \longrightarrow & S^{n+2} \\ \downarrow {\scriptstyle q_1} & & \downarrow {\scriptstyle q} & & \downarrow {\scriptstyle \bar q_2} \\ L^n_p(\bar\ell) & \xleftarrow{p_1} & S^n \times_{\mathbb{Z}_p} S^{n+2} & \xrightarrow{p_2} & L^{n+2}(\bar\mu, 1) \end{array}$$

Since in the Borel construction for the diagonal $\mathbb{Z}_p$ action on $S^n \times S^{n+2}$ the projection p_1 is $(n+1)$–connected, it admits a section $s\colon L^n_p(\bar\ell) \to S^n \times_{\mathbb{Z}_p} S^{n+2}$. Then f is a cellular approximation of $p_2 \circ s$. □

Theorem 4.6 *For every odd $n = 2k-1$ and distinct primes $p, q \ge k$,*

$$\operatorname{cat}(L^{2k-1}_p \times L^{2k-1}_q) = n+1.$$

Proof Let $L^n_p = L^n_p(\bar\ell)$ and $L^n_q(\bar\ell')$ for $\bar\ell = (\ell_1, \dots, \ell_k)$ and $\bar\ell' = (\ell'_1, \dots, \ell'_k)$. By Kwak [15, Theorem 3.1] there are stably parallelizable fake lens spaces $L^n_p(\alpha)$ and $L^n_q(\alpha')$. By Wall's theorem they are homotopy equivalent to lens spaces $L^n_p(\bar\mu)$ and $L^n_q(\bar\mu')$ for some $\bar\mu$ and $\bar\mu'$. By Proposition 4.1, $\operatorname{cat}(L^n_p(\bar\mu) \times L^n_q(\bar\mu')) = n+1$.

By Proposition 4.3, there is a classifying map $u\colon L^n_p(\bar\mu)\times L^n_q(\bar\mu')\to B\mathbb{Z}^{(n+1)}_{pq}$. By Proposition 4.5 there are maps $f_p\colon L^n_p\to L^n_p(\bar\mu)$ and $f_q\colon L^n_q\to L^n_q(\bar\mu')$ that induce an isomorphism of the fundamental groups. Therefore,

$$u'=u\circ(f_p\times f_q)\colon L^n_p\times L^n_q\to B\mathbb{Z}^{(n+1)}_{pq}$$

is a classifying map for $L^n_p\times L^n_q$. Hence, $\mathrm{cat}(u')\le n+1$. Since the universal covering of the space $L^n_p\times L^n_q$ is $(n-1)$–connected, by Theorem 4.4 we obtain $\mathrm{cat}(L^n_p\times L^n_q)\le n+1$. By Proposition 2.9, $\mathrm{cat}(L^n_p\times L^n_q)=n+1$. □

Remark When p and q are relatively prime but not necessarily prime we can prove the equality $\mathrm{cat}(L^n_p\times L^n_q)=n+1$ with a stronger restriction $p,\ q\ge n+3$. We do not present the proof, since it is more technical. It consists of computation of obstructions for deforming a classifying map $u\colon L^n_p\times L^n_q\to B\mathbb{Z}_{pq}$ to the $(n+1)$–skeleton. Vanishing of the first obstruction happens without any restriction on p and q. Since it is a curious fact on its own it is presented in the next section. The higher obstructions vanish due to the fact that cohomology groups of $\mathbb{Z}_{pq}$ are pq–torsions and a theorem of Serre [20] that states that the group $\pi_{n+k}(S^n)$ has zero r–torsion component for $k<2r-4$.

We note that Theorem 4.6 can be stated for all lens spaces L^n_p with values of n and p for which there exists a stably parallelizable fake lens space $L^n_p(\alpha)$.

Problem 4.7 *For which values of n and p is there a stably parallelizable fake lens space $L^n_k(\alpha)$?*

This does not seem to happen very often when $p=2$. At least, a real $(2k-1)$–dimensional projective space is stably parallelizable if and only if $k=1$, 2 or 4.

5 The Berstein–Schwarz class for the product of finite cyclic groups

Let $u\colon L^n_p\times L^n_q\to B\mathbb{Z}_{pq}$ be a classifying map. By Theorem 4.4 and the fact that $\mathrm{cat}(L^n_p\times L^n_q)\ge n+1$, the condition $\mathrm{cat}(u)\le n+1$ is equivalent to the equality $\mathrm{cat}(L^n_p\times L^n_q)=n+1$. By Proposition 4.3 the inequality $\mathrm{cat}(u)\le n+1$ is equivalent to the existing of a lift u' of u with respect to $p_n\colon G_{n+1}(\mathbb{Z}_{pq})\to B\mathbb{Z}_{pq}$. In view of Corollary 2.6 the primary obstruction to such a lift is $u^*(\beta^{n+2})$, where β is the Berstein–Schwarz class of $\mathbb{Z}_{pq}$. We prove that this obstruction is always zero and even more:

Theorem 5.1 *For all n and all relatively prime p and q,*

$$u^*(\beta^{n+1}) = 0.$$

Remark One can show that for sufficiently large p and q the higher obstructions are trivial as well, since the homotopy groups of the fiber of p_n^π do not contain r–torsions for large r. This would give a result similar to Theorem 4.6, which does not cover small values of p.

We denote by $\mathbb{Z}(m)$ the group ring $\mathbb{Z}\mathbb{Z}_m$ of $\mathbb{Z}_m$, $I(m)$ its augmentation ideal, $\epsilon_m\colon \mathbb{Z}(m) \to \mathbb{Z}$ its augmentation, and β_m its Berstein–Schwarz class. Let $t_m = \sum_{g\in\mathbb{Z}_m} g \in \mathbb{Z}(m)$. We use the same notation t_m for a constant map $t_m\colon \mathbb{Z}_m \to \mathbb{Z}(m)$ with the value t_m. We note that the group of invariants of $\mathbb{Z}(m)$ is $\mathbb{Z}$ generated by t_m. Thus, $H^0(\mathbb{Z}_m; \mathbb{Z}(m)) = \mathbb{Z}$.

Proposition 5.2 *Let β_p denote the Berstein–Schwarz class for the group $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. Then β_p has order p and hence is q–divisible for any q relatively prime to p.*

Proof Let $t \in \mathbb{Z}_p$ be a generator. We note that

$$H^0(\mathbb{Z}_p; \mathbb{Z}\mathbb{Z}_p) = (\mathbb{Z}\mathbb{Z}_p)^{\mathbb{Z}_p} = \mathbb{Z}\langle 1 + t + \cdots + t^{p-1}\rangle$$

is the group of invariants, which is isomorphic to the subgroup of $\mathbb{Z}\mathbb{Z}_p$ generated by $1 + t + \cdots + t^{p-1}$. Then the augmentation homomorphism $\epsilon\colon \mathbb{Z}\mathbb{Z}_p \to \mathbb{Z}$ induces a homomorphism $\epsilon_*\colon H^0(\mathbb{Z}_p; \mathbb{Z}\mathbb{Z}_p) \to H^0(\mathbb{Z}_p; \mathbb{Z}) = \mathbb{Z}$ that takes the generator $1 + t + \cdots + t^{p-1}$ to p. Thus, $p\beta_p = p\delta(1) = \delta(p) = 0$ by exactness of the cohomology long exact sequence associated with the coefficient sequence $0 \to I(p) \to \mathbb{Z}(p) \to \mathbb{Z} \to 0$.

Note that β_p generates a subgroup G of order p in $H^1(\pi; I(\mathbb{Z}\pi))$. Therefore it is q–divisible for q with $(p,q) = 1$. □

We recall that the cross product

$$H^i(X; M) \times H^j(X'; M') \to H^{i+j}(X \times X'; M \otimes_{\mathbb{Z}} M')$$

is defined for any $\pi_1(X)$–module M and $\pi_1(X')$–module M'. Also we note that

$$H^i(X; M \oplus M') = H^i(X; M) \oplus H^i(X; M').$$

Proposition 5.3 *For relatively prime p and q there are $k, l \in \mathbb{Z}$ such that the Berstein–Schwarz class β_{pq} is the image of the class*

$$(\beta_p \times l, k \times \beta_q) \in H^1(\mathbb{Z}_{pq}; I(p) \otimes \mathbb{Z}(q)) \oplus H^1(\mathbb{Z}_{pq}; \mathbb{Z}(p) \otimes I(q))$$

under the coefficient homomorphism

$$\phi\colon I(p)\otimes\mathbb{Z}(q)\oplus\mathbb{Z}(p)\otimes I(q)\to I(pq)\subset\mathbb{Z}(pq)=\mathbb{Z}(p)\otimes\mathbb{Z}(q)$$

defined by the inclusions of the direct summands into $\mathbb{Z}(p)\otimes\mathbb{Z}(q)$ *and the summation.*

Proof Let k and l be such that $kp+lq=1$.

The addition in $\mathbb{Z}(pq)$ defines the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I(pq) & \longrightarrow & \mathbb{Z}(pq) & \xrightarrow{\epsilon_{pq}} & \mathbb{Z} & \longrightarrow & 0\\
 & & \phi\uparrow & & +\uparrow & & \uparrow\epsilon_p+\epsilon_q & & \\
0 & \longrightarrow & I(p)\otimes\mathbb{Z}(q)\oplus\mathbb{Z}(p)\otimes I(q) & \longrightarrow & \mathbb{Z}(pq)\oplus\mathbb{Z}(pq) & \xrightarrow{\alpha} & \mathbb{Z}(q)\oplus\mathbb{Z}_(p) & \longrightarrow & 0
\end{array}$$

which defines a commutative square for cohomology:

$$\begin{array}{ccc}
H^0(\mathbb{Z}_{pq},\mathbb{Z}) & \xrightarrow{\delta} & H^1(\mathbb{Z}_{pq},I(pq))\\
\epsilon_*\uparrow & & \uparrow\phi_*\\
H^0(\mathbb{Z}_{pq},\mathbb{Z}(p)\oplus\mathbb{Z}(q)) & \xrightarrow{\delta'} & H^1(\mathbb{Z}_{pq},I(p)\otimes\mathbb{Z}(q)\oplus\mathbb{Z}(p)\otimes I(q))
\end{array}$$

The homomorphism $\theta\colon\mathbb{Z}(pq)\to\mathbb{Z}(p)\oplus\mathbb{Z}(q)$ defined on the basis as $\theta(a\times b)=(lt_q,kt_p)$ is a cochain since it is $\mathbb{Z}_{pq}$–equivariant. It is a cocycle, since it is constant. Note that $(\epsilon_p+\epsilon_q)\circ\theta(a\times b)=kp+lq=1$ for any $a\in\mathbb{Z}_p$ and $b\in\mathbb{Z}_q$. This means that the cohomology class $[\theta]$ is taken by ϵ_* to a generator $1\in H^0(\mathbb{Z}_{pq};\mathbb{Z})$. Then $\beta_{pq}=\delta(1)=\phi\delta'([\theta])$.

Consider a $\mathbb{Z}(pq)$–homomorphism $\bar\theta\colon\mathbb{Z}(p)\times\mathbb{Z}(q)\to\mathbb{Z}(pq)\oplus\mathbb{Z}(pq)$ defined by the formula $\bar\theta(a\times b)=(a\times lt_q,kt_p\times b)$. Since $\alpha(\bar\theta)=\theta$, by the snake lemma $\delta'([\theta])$ is defined by the 1–cocycle $\delta(\bar\theta)\colon C_1\to I(p)\otimes\mathbb{Z}(q)\oplus\mathbb{Z}(p)\otimes I(q)$. Note that the cellular 1–dimensional chain group C_1 is defined via the Cayley graph C of $\mathbb{Z}_{pq}$.

Note that the Cayley graph $C(\pi\times\pi',S\times e'\cup e\times S')$ of the product $\pi\times\pi'$ of two groups with generating sets S and S' and units $e\in\pi$ and $e'\in\pi'$ equals the 1–skeleton of the product of the Cayley graphs $C(\pi,S)\times C(\pi',S')$. Thus, $C=(C^p\times\mathbb{Z}_q)\cup(\mathbb{Z}_p\times C^q)$, where C^p and C^q are the Cayley graphs (cycles) for $\mathbb{Z}_p$ and $\mathbb{Z}_q$, respectively. Note that

$$\begin{aligned}
\delta(\bar\theta)([a_1,a_2]\times b)&=\bar\theta((a_2-a_1)\times b)=\bar\theta(a_2\times b)-\bar\theta(a_1\times b)\\
&=(a_2\times lt_q,kt_p\times b)-(a_1\times lt_q,kt_p\times b)=((a_2-a_1)\times lt_q,0)\\
&=(\beta_p\times lt_q)([a_1,a_2]\times b)=(\beta_p\times lt_q,kt_p\times\beta_q)([a_1,a_2]\times b).
\end{aligned}$$

Similarly, we have the equality for edges of the type $a\times[b_1,b_2]$. Here β_p and β_q denote the canonical cochains that define the Berstein–Schwarz classes of $\mathbb{Z}_p$ and $\mathbb{Z}_q$.

Thus, $\delta'([\theta])=(\beta_p\times l, k\times\beta_q)$ in

$$H^1(\mathbb{Z}_{pq}, I(p)\otimes\mathbb{Z}(q))\oplus H^1(\mathbb{Z}_{pq},\mathbb{Z}(p)\otimes I(q)) \\ = H^1(\mathbb{Z}_{pq}, I(p)\otimes\mathbb{Z}(q)\oplus\mathbb{Z}(p)\otimes I(q)). \quad \square$$

5.1 Proof of Theorem 5.1

We show that $u^*(\beta_{pq}^{n+1})=0$, where

$$u=i_p\times i_q\colon L_p^n\times L_q^n\to B\mathbb{Z}_p\times B\mathbb{Z}_q=B\mathbb{Z}_{pq}$$

is the inclusion. Note that $(\beta_p\times lt_q, kt_p\times\beta_q)=\beta_p\times lt_q+kt_p\times\beta_q$. Thus, it suffices to show that $u^*(\beta_p\times lt_q+kt_p\times\beta_q)^{n+1}=0$. Note that

$$u^*(\beta_p\times l+k\times\beta_q)=i_p^*(\beta_p)\times l+k\times i_q^*(\beta_q).$$

Then $(i_p^*(\beta_p)\times l+k\times i_q^*(\beta_q))^{n+1}=(i_p^*(\beta_p)\times l)^{n+1}+(k\times i_q^*(\beta_q))^{n+1}+F$, where F consists of monomials containing both factors.

Claim 1 $(i_p^*(\beta_p)\times l)^{n+1}=0$ and $(k\times i_q^*(\beta_q))^{n+1}=0$.

Proof There is an automorphism of the coefficients

$$(I(p)\otimes\mathbb{Z}(q))\otimes\cdots\otimes(I(p)\otimes\mathbb{Z}(q))\to I(p)\otimes\cdots\otimes I(p)\otimes\mathbb{Z}(q)\otimes\cdots\otimes\mathbb{Z}(q)$$

that takes $(i_p^*(\beta_p)\times l)^{n+1}$ to $i_p^*(\beta_p)^{n+1}\times l^{n+1}=0$. Similarly, $(k\times i_q^*(\beta_q))^{n+1}=0$.

Claim 2 $(i_p^*(\beta_p)\times l)A(k\times i_q^*(\beta_q))=0$ for any A.

Proof Indeed, since $i_p^*(\beta_p)$ is divisible by q (see Proposition 5.2),

$$(i_p^*(\beta_p)\times l)A(k\times i_q^*(\beta_q))=\Big(\frac{1}{q}(i_p^*(\beta_p)\times l)\Big)Aq(k\times i_q^*(\beta_q))=0.$$

Thus, $F=0$ and the result follows.

6 On the category of *ko*–inessential manifolds

6.1 Deformation into the $(n-2)$–dimensional skeleton

We recall that a classifying map $u\colon M\to B\pi$ of a closed orientable n–manifold M can be deformed into the $(n-1)$–skeleton $B\pi^{(n-1)}$ if and only if $u_*([M])=0$, where $[M]\in H_n(M;\mathbb{Z})$ denotes an integral fundamental class; see Babenko [1]. In [3] we proved the following proposition, which sets the stage for computation of the second obstruction.

Proposition 6.1 *Every inessential n–manifold M with a fixed CW structure admits a classifying map $u\colon M \to B\pi$ with $u(M) \subset B\pi^{(n-1)}$ and $u(M^{(n-1)}) \subset B\pi^{(n-2)}$.*

We postpone the proof of the following lemma to the end of the section.

Lemma 6.2 *For any group π and CW complex $B\pi$, for $n \geq 5$ the homomorphism induced by the quotient map*

$$p_*\colon \pi_n(B\pi, B\pi^{(n-2)}) \to \pi_n(B\pi/B\pi^{(n-2)})$$

factors through the group of coinvariants as $p_* = \bar{p}_* \circ q_*$,

$$\pi_n(B\pi, B\pi^{(n-2)}) \xrightarrow{q_*} \pi_n(B\pi, B\pi^{(n-2)})_\pi \xrightarrow{\bar{p}_*} \pi_n(B\pi/B\pi^{(n-2)}),$$

where $\bar{p}_*$ is injective.

We recall that for a π–module M the group of coinvariants is $M \otimes_{\mathbb{Z}\pi} \mathbb{Z}$.

Remark In the proof of [3, Lemma 4.1] it was stated erroneously that $\bar{p}_*$ is bijective. It turns out that the injectivity of $\bar{p}_*$ was sufficient for the proof of that lemma to be carried out. Thus, due to Lemma 6.2 the results of [3] that depend on the lemma remain intact.

Theorem 6.3 *Let M be an n–manifold with a CW complex structure with one top-dimensional cell. Suppose that a classifying map $u\colon M \to B\pi$ satisfies the condition $u(M^{(n-1)}) \subset B\pi^{(n-2)}$ and let $\bar{u}\colon M/M^{(n-1)} = S^n \to B\pi/B\pi^{(n-2)}$ be the induced map. Then the following are equivalent:*

(1) *There is a deformation of u in $B\pi$ to a map $f\colon M \to B\pi^{(n-2)}$.*
(2) *$\bar{u}_*(1) = 0$ in $\pi_n(B\pi/B\pi^{(n-2)})$, where $1 \in \mathbb{Z} = \pi_n(S^n)$.*

Proof The primary obstruction to deforming u to $B\pi^{(n-2)}$ is defined by the cocycle

$$c_u = u_*\colon \pi_n(M, M^{(n-1)}) \to \pi_n(B\pi, B\pi^{(n-2)})$$

with the cohomology class $o_u = [c_u] \in H^n(M; \pi_n(B\pi, B\pi^{(n-2)}))$. By Poincaré duality, o_u is dual to the homology class $PD(o_u) \in H_0(M; \pi_n(B\pi, B\pi^{(n-2)})) = \pi_n(B\pi, B\pi^{(n-2)})_\pi$ represented by $q_*u_*(1)$, where

$$q_*\colon \pi_n(B\pi, B\pi^{(n-2)}) \to \pi_n(B\pi, B\pi^{(n-2)})_\pi$$

is the projection onto the group of coinvariants and

$$u_*\colon \pi_n(M, M^{(n-1)}) = \mathbb{Z} \to \pi_n(B\pi, B\pi^{(n-2)})$$

is induced by u. We note that $\pi_n(B\pi, B\pi^{(n-2)}) = \pi_n(E\pi, E\pi^{(n-2)})$. By Lemma 6.2 the homomorphism $\bar{p}_*$ is injective. Hence, $\bar{p}_* q_* u_*(1) = 0$ if and only if $o_u = 0$. The commutative diagram

$$\begin{array}{ccccc} \pi_n(M, M^{(n-1)}) & \xrightarrow{u_*} & \pi_n(B\pi, B\pi^{(n-2)}) & \xrightarrow{q_*} & \pi_n(B\pi, B\pi^{(n-2)}) \\ {\scriptstyle =}\downarrow & & & & \downarrow{\scriptstyle \bar{p}_*} \\ \mathbb{Z} & \xrightarrow{\quad = \quad} & \pi_n(M/M^{(n-1)}) & \xrightarrow{\bar{u}_*} & \pi_n(B\pi/B\pi^{(n-2)}) \end{array}$$

implies that $\bar{u}_*(1) = \bar{p}_* q_* u_*(1)$. □

6.2 *ko*–inessential manifolds

We recall that an orientable, closed n–manifold M is inessential if and only if $u_*([M]) = 0$, where $[M] \in H_n(M;\mathbb{Z})$ is a fundamental class and $u\colon M \to B\pi$ is a classifying map. We call a closed spin n–manifold M *ko–inessential* if $u_*([M]_{ko}) = 0$ in $ko_n(B\pi)$, where ko_* denotes the real connective K–theory homology groups.

We recall that for every spectrum E there is a natural morphism $S \to E$ of the spherical spectrum. This defines a natural transformation of corresponding (co)homology theories $\pi_*^s \to E_*$, where π_*^s is the stable homotopy theory. In the case of ko_* this natural transformation induces an isomorphism $\pi_i^s(\mathrm{pt}) \to ko_i(\mathrm{pt})$ for $i = 0, 1, 2$. It allows us in some cases to reduce ko_* problems to the stable homotopy groups.

We need the following proposition:

Proposition 6.4 [3] *The natural transformation $\pi_*^s(\mathrm{pt}) \to ko_*(\mathrm{pt})$ induces an isomorphism $\pi_n^s(K/K^{(n-2)}) \to ko_n(K/K^{(n-2)})$ for any CW complex K.*

We recall that spin manifolds are exactly those that admit an orientation in real connective K–theory ko_*.

Theorem 6.5 *A classifying map $u\colon M \to B\pi$ of an inessential, closed, spin n–manifold M, $n > 3$, is homotopic to a map $f\colon M \to B\pi^{(n-2)}$ if and only if $j_* u_*([M]_{ko}) = 0$ in $ko_n(B\pi, B\pi^{(n-2)})$, where $[M]_{ko}$ is a ko–fundamental class.*

Proof By Proposition 6.1 a classifying map u can be chosen to satisfy the condition $u(M^{(n-1)}) \subset B\pi^{(n-2)}$. We show that $\bar{u}_*(1) = 0$ if and only if $j_*u_*([M]_{ko}) = 0$ and apply Theorem 6.3.

The restriction $n > 3$ implies that $\bar{u}_*(1)$ survives in the stable homotopy group. In view of Proposition 6.4, the element $\bar{u}_*(1)$ survives in the composition

$$\pi_n(B\pi/B\pi^{(n-2)}) \to \pi_n^s(B\pi/B\pi^{(n-2)}) \to ko_n(B\pi/B\pi^{(n-2)}).$$

The commutative diagram

$$\begin{array}{ccc}
\pi_n(S^n) & \xrightarrow{\bar{u}_*} & \pi_n(B\pi/B\pi^{(n-2)}) \\
\cong \downarrow & & \cong \downarrow \\
\pi_n^s(S^n) & \xrightarrow{\bar{u}_*} & \pi_n^s(B\pi/B\pi^{(n-2)}) \\
\cong \downarrow & & \cong \downarrow \\
ko_n(S^n) & \xrightarrow{\bar{u}_*} & ko_n(B\pi/B\pi^{(n-2)})
\end{array}$$

implies that $\bar{u}_*(1) = 0$ for ko_n if and only if $\bar{u}_*(1) = 0$ for π_n.

From the diagram with the quotient map $\psi\colon M \to M/M^{(n-1)} = S^n$

$$\begin{array}{ccc}
ko_n(M) & \xrightarrow{u_*} & ko_n(B\pi) \\
\psi_* \downarrow & & j_* \downarrow \\
ko_n(S^n) & \xrightarrow{\bar{u}_*} & ko_n(B\pi/B\pi^{(n-2)}),
\end{array}$$

it follows that $j_*u_*([M]_{ko}) = \bar{u}_*\psi_*([M]_{ko}) = \bar{u}_*(1)$. Thus, $j_*u_*([M]_{ko}) = 0$ if and only if $\bar{u}_*(1) = 0$ for n–dimensional homotopy groups. □

For spin manifolds we prove the following criterion:

Theorem 6.6 *For a closed spin n–manifold M with* cat $M \le$ dim $M - 2$,

$$j_*u_*([M]_{ko}) = 0$$

in $ko_n(B\pi, B\pi^{(n-2)})$, where $u\colon M \to B\pi$ classifies the universal cover of M and $j\colon (B\pi, \varnothing) \to (B\pi, B\pi^{(n-2)})$ is the inclusion.

For a closed, spin, inessential n–manifold M with $\pi_2(M) = 0$, cat $M \le$ dim $M - 2$ *if and only if $j_*u_*([M]_{ko}) = 0$.*

Proof The inequality $\operatorname{cat} M \le n-2$ implies that the map u has a lift $u' \to G_{n-2}(B\pi)$ with $u = p^{\pi}_{n-2}u'$. Since $G_{n-2}(B\pi)$ is homotopy equivalent to an $(n-2)$–dimensional complex, p^{π}_{n-2} can be deformed to $p'\colon G_{n-2}(B\pi) \to B\pi^{(n-2)}$. Thus u can be deformed to $B\pi^{(n-2)}$. By Theorem 6.5, $j_*u_*([M]_{ko}) \neq 0$.

Now let $\pi_2(M)=0$ and $j_*u_*([M]_{ko})=0$. By Theorem 6.5 the map u can be deformed to a map $f\colon M \to B\pi^{(n-2)}$. By Proposition 4.3, $\operatorname{cat}(u) \le n-2$. Since $\pi_2(M)=0$, the universal covering of M is 2–connected. By Theorem 4.4, $\operatorname{cat} M \le n-2$. □

Proposition 6.7 *Let $M = L^m_p \times L^n_q$, $m, n > 2$, be given a ko–orientation for some relatively prime p and q and let $u\colon M \to B\mathbb{Z}_{pq}$ be a classifying map of its universal cover. Then $u_*([M]_{ko}) = 0$.*

Proof Note that $[M]_{ko} = \pm(1+v)([L^m_p]_{ko} \times [L^n_q]_{ko})$, where $v \in \widetilde{ko}^0(M)$ is in the reduced ko–theory and the product is the cap product (see [18, Chapter 5, Proposition 1.9]). Therefore it suffices to show that $u^p_*([L^m_p]_{ko}) \times u^q_*([L^n_q]_{ko}) = 0$, where $u^p\colon L^m_p \to B\mathbb{Z}_p$ and $u^q\colon L^n_q \to B\mathbb{Z}_q$ are classifying maps. This equality follows from the fact that $ko_m(B\mathbb{Z}_p)$ is q–divisible and $ko_n(B\mathbb{Z}_q)$ is a q–torsion group. □

Corollary 6.8 *For $m, n > 2$ and odd, relatively prime p and q, or for p odd and q even with $n = 2k-1$ for even k, we have*

$$\operatorname{cat}(L^m_p \times L^n_q) \le m+n-2.$$

Proof In this case the lens spaces are spin [11] and we can apply Proposition 6.7. Then Theorem 6.6 and the fact that $\pi_2(L^m_p \times L^n_q) = 0$ imply the result. □

For $m = n = 3$ we obtain a different proof of Corollary 4.2:

Corollary 6.9 $\operatorname{cat}(L^3_p \times L^3_q) = 4$ *for all relatively prime p and q.*

6.3 Coinvariants

The following lemma can be found in [6, Lemma 3.3]:

Lemma 6.10 *A commutative diagram with exact rows*

$$\begin{array}{ccccccc} & & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & & \end{array}$$

defines an exact sequence

$$\ker(f') \to \ker(f) \to \ker(f'') \to \operatorname{coker}(f') \to \operatorname{coker}(f) \to \operatorname{coker}(f'').$$

Let $p\colon E\pi \to B\pi$ be the universal covering. Thus p is the projection onto the orbit space of a free cellular π–action. Below we use the following abbreviations: $\pi = \pi_1(B)$, $B = B\pi$, $B^k = B^{(k)}$, $E = E\pi$ and $E^k = E\pi^{(k)}$.

Proposition 6.11 *$p_*\colon \pi_n(E/E^{n-1}) \to \pi_n(B/B^{n-1})$ is an epimorphism.*

Proof In the commutative diagram

$$\begin{array}{ccc} \pi_n(E^n/E^{n-1}) & \xrightarrow{p'_*} & \pi_n(B^n/B^{n-1}) \\ \big\downarrow & & \big\downarrow{\scriptstyle j_*} \\ \pi_n(E/E^{n-1}) & \xrightarrow{p_*} & \pi_n(B/B^{n-1}) \end{array}$$

the homomorphisms p'_* and j_* are epimorphisms. The former is surjective since it is induced by a retraction of a wedge of an n–sphere onto a smaller wedge; the latter is surjective due to the cellular approximation theorem. Therefore, p_* is an epimorphism. □

Recall that π_*^s denotes the stable homotopy groups.

Corollary 6.12 *For $n \geq 5$, the induced homomorphism*

$$p'_*\colon \pi_n^s(E, E^{n-1}) \to \pi_n^s(B, B^{n-1})$$

is an epimorphism.

Proof This follows from the obvious natural isomorphisms

$$\pi_n(E/E^{n-1}) = \pi_n^s(E/E^{n-1}) = \pi_n(E, E^{n-1}),$$
$$\pi_n(B/B^{n-1}) = \pi_n^s(B/B^{n-1}) = \pi_n(B, B^{n-1}).$$ □

6.4 Proof of Lemma 6.2

For $n \geq 5$, the induced homomorphism

$$p_*\colon \pi_n(B, B^{n-2}) \to \pi_n(B/B^{n-2})$$

factors through the group of coinvariants as $p_* = \bar{p}_* \circ q_*$,

$$\pi_n(B, B^{n-2}) \xrightarrow{q_*} \pi_n(B, B^{n-2})_\pi \xrightarrow{\bar{p}_*} \pi_n(B/B^{n-2}),$$

where $\bar{p}_*$ is injective.

Note that, for $n \geq 5$,

$$\pi_n(B, B^{n-2}) = \pi_n(E, E^{n-2}) = \pi_n^s(E, E^{n-2}), \quad \pi_n(B/B^{n-2}) = \pi_n^s(B, B^{n-2}),$$

and the composition

$$\pi_n(B, B^{n-2}) \xrightarrow{q_*} \pi_n(B, B^{n-2})_\pi \xrightarrow{\bar{p}_*} \pi_n(B/B^{n-2})$$

coincides with

$$\pi_n^s(E, E^{n-2}) \xrightarrow{q_*} \pi_n^s(E, E^{n-2})_\pi \xrightarrow{\bar{p}_*} \pi_n^s(B, B^{n-2}),$$

where

$$\bar{p}_* \circ q_* = p_* \colon \pi_n^s(E, E^{n-2}) \to \pi_n^s(B, B^{n-2})$$

is the homomorphism induced by the projection p.

Also note that $\pi_*^s(E, E^i)$ inherits a π–module structure via the π–action.

We extract from the diagram generated by p and exact π_*^s–homology sequence of the triple (E^n, E^{n-1}, E^{n-2}) the following two diagrams:

$$\begin{array}{ccccccc}
\pi_{n+1}^s(E^n, E^{n-1}) & \xrightarrow{\bar{j}_{n+1}} & \pi_n^s(E^{n-1}, E^{n-2}) & \longrightarrow & \overline{K} & \longrightarrow & 0 \\
\downarrow{\scriptstyle p_*^1} & & \downarrow{\scriptstyle p_*^2} & & \downarrow{\scriptstyle \alpha} & & \\
\pi_{n+1}^s(B^n, B^{n-1}) & \xrightarrow{j_{n+1}} & \pi_n^s(B^{n-1}, B^{n-2}) & \longrightarrow & K & \longrightarrow & 0,
\end{array}$$

where K and $\overline{K}$ are the cokernels of j_{n+1} and $\bar{j}_{n+1}$, and

$$\begin{array}{ccccccc}
0 & \longrightarrow & \overline{H} & \longrightarrow & \pi_n^s(E^n, E^{n-1}) & \xrightarrow{\bar{j}_n} & \pi_{n-1}^s(E^{n-1}, E^{n-2}) \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle p_*^3} & & \downarrow{\scriptstyle p_*^4} \\
0 & \longrightarrow & H & \longrightarrow & \pi_n^s(B^n, B^{n-1}) & \xrightarrow{j_n} & \pi_{n-1}^s(B^{n-1}, B^{n-2}),
\end{array}$$

where H and $\overline{H}$ are the kernels of j_n and $\bar{j}_n$. Note that the homomorphisms p_*^3 and p_*^4 are the direct sums of the augmentation homomorphism

$$\epsilon \colon \mathbb{Z}\pi \to \mathbb{Z}.$$

The homomorphisms p_*^1 and p_*^2 are direct sums of the $\operatorname{mod} 2$ augmentation homomorphisms

$$\bar{\epsilon} \colon \mathbb{Z}_2\pi \to \mathbb{Z}_2.$$

Also note that $p_*^i \otimes_\pi 1_{\mathbb{Z}}$ is an isomorphism for $i = 1, 2, 3, 4$. Taking the tensor product of the first diagram with $\mathbb{Z}$ over $\mathbb{Z}\pi$ would give a commutative diagram with the two left vertical arrows isomorphisms. Then, by the five lemma, $\alpha' = \alpha \otimes_\pi 1_{\mathbb{Z}}$ is an isomorphism.

We argue that $\beta' = \beta \otimes_\pi 1_{\mathbb{Z}}$ is a monomorphism. Note that $\ker(\beta) \subset \ker(p_*^3) = \bigoplus I(\pi)$, where $I(\pi)$ is the augmentation ideal.

Claim $\ker(\beta) \otimes_\pi \mathbb{Z} = 0.$

Proof We show that $x \otimes_\pi 1 = 0$ for all $x \in \ker(\beta)$. Let $x = \sum x_i$, $x_i \in I(\pi)$. It suffices to show that $x_i \otimes_\pi 1 = 0$ for all x_i. Note that $x_i = \sum n_j(\gamma_j - e)$, $\gamma_j \in \pi$, $n_J \in \mathbb{Z}$. Note that $(\gamma - e) \otimes_\pi 1 = 0$ since

$$(\gamma - e) \otimes_\pi 1 = \gamma \otimes_\pi 1 - e \otimes_\pi 1 = \gamma(e \otimes_\pi 1) - e \otimes_\pi 1 = e \otimes_\pi \gamma(1) - e \otimes_\pi 1 = 0.$$

The tensor product with $\mathbb{Z}$ over $\mathbb{Z}\pi$ of the exact sequence

$$\ker(\beta) \to \overline{H} \to \operatorname{im}(\beta) \to 0$$

implies that

$$\beta_0 = \beta \otimes \mathrm{id}\colon \overline{H} \otimes_\pi \mathbb{Z} = \overline{H}_\pi \to \operatorname{im}(\beta) \otimes_\pi \mathbb{Z} = \operatorname{im}(\beta)$$

is an isomorphism. The latter equality follows from the fact that both $\operatorname{im}(\beta)$ and $\mathbb{Z}$ are trivial π–modules. Then β' is a monomorphism as the composition of an isomorphism β_0 and the inclusion $\operatorname{im}(\beta) \to H$.

We consider the diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{K} & \xrightarrow{\bar\phi} & \pi_n^s(E^n, E^{n-2}) & \xrightarrow{\bar\psi} & \overline{H} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & K & \xrightarrow{\phi} & \pi_n^s(B^n, B^{n-2}) & \xrightarrow{\psi} & H & \longrightarrow & 0
\end{array}$$

Then we apply the tensor product with $\mathbb{Z}$ over $\mathbb{Z}\pi$ to this diagram to obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & \overline{K}_\pi & \xrightarrow{\bar\phi} & \pi_n^s(E^n, E^{n-2})_\pi & \xrightarrow{\bar\psi} & \overline{H}_\pi & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha'} & & \downarrow{\scriptstyle\tilde p_*} & & \downarrow{\scriptstyle\beta'} & & \\
0 & \longrightarrow & K & \xrightarrow{\phi} & \pi_n^s(B^n, B^{n-2}) & \xrightarrow{\psi} & H & &
\end{array}$$

Lemma 6.10 implies that $\tilde p_*$ is a monomorphism.

Next we consider the diagram generated by (E, E^n, E^{n-2}) and (B, B^n, B^{n-2}),

$$\begin{array}{ccccccc}
\pi^s_{n+1}(E, E^n) & \longrightarrow & \pi^s_n(E^n, E^{n-2}) & \longrightarrow & \pi^s_n(E, E^{n-2}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi^s_{n+1}(B, B^n) & \longrightarrow & \pi^s_n(B^n, B^{n-2}) & \longrightarrow & \pi^s_n(B, B^{n-2}) & \longrightarrow & 0,
\end{array}$$

and tensor it with $\mathbb{Z}$ over $\mathbb{Z}\pi$ to obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\pi^s_{n+1}(E, E^n)_\pi & \longrightarrow & \pi^s_n(E^n, E^{n-2})_\pi & \longrightarrow & \pi^s_n(E, E^{n-2})_\pi & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle p'_*} & & \Big\downarrow{\scriptstyle \tilde{p}_*} & & \Big\downarrow{\scriptstyle \bar{p}_*} & & \\
\pi^s_{n+1}(B, B^n) & \longrightarrow & \pi^s_n(B^n, B^{n-2}) & \longrightarrow & \pi^s_n(B, B^{n-2}) & \longrightarrow & 0
\end{array}$$

Since p'_* is an epimorphism (see Corollary 6.12) and $\tilde{p}_*$ is a monomorphism by the monomorphism version of the five lemma, we obtain that $\bar{p}_*$ is a monomorphism.

7 On the category of the sum

The following theorem was proven by R Newton [16] under the assumption that $\operatorname{cat} M$, $\operatorname{cat} N > 2$.

Theorem 7.1 *For closed manifolds M and N there is the inequality*

$$\operatorname{cat}(M \# N) \le \max\{\operatorname{cat} M, \operatorname{cat} N\}.$$

His proof is based on obstruction theory. Here we present a proof that works in full generality. Our proof is an application of the following:

Theorem 7.2 (W Singhof [21, Theorem 4.4]) *For any closed n–manifold M with $\operatorname{cat} M = k \ge 2$, there is a categorical partition $Q_0, \dots, Q_k$ into manifolds with boundary such that $Q_i \cap Q_j$ is an $(n-1)$–manifold with boundary (possibly empty) for all i, j and each Q_i admits a deformation retraction onto an $(n-k)$–dimensional CW complex.*

For $B \subset A \subset X$, a homotopy $H\colon A \times I \to X$ is called a *deformation* of A in X onto B if $H_{A\times\{0\}} = 1_A$, $H(A \times \{1\}) = B$, and $H(b, t) = b$ for all $b \in B$ and $t \in I = [0, 1]$. The following is well known:

Proposition 7.3 *Let $A \subset M$ be a subset contractible to a point in an m–manifold M and let $B \subset A$ be a closed n–ball which admits a regular neighborhood. Then there is a deformation of A in M onto B.*

Proof of Theorem 7.1 Let $n = \dim M = \dim N$. Suppose that $\operatorname{cat} M$, $\operatorname{cat} N \le k$. We show that $\operatorname{cat}(M \# N) \le k$. If $k = 1$, the statement obviously follows from the fact that M and N are homeomorphic to the sphere. We assume that $k \ge 2$. Let $Q_0, \dots Q_k$ be a partition of M into M–contractible subsets as in Singhof's theorem. We may assume that $Q_0 \cap Q_1 \ne \varnothing$. Moreover, we may assume that there is a closed topological n–ball $D \subset Q_0 \cup Q_1$ with a collar in $Q_0 \cup Q_1$ and $D_0 = D \cap Q_0$, $D_1 = D \cap Q_1$ such that the triad (D, D_0, D_1) is homeomorphic to the triad (B, B_+, B_-), where B is the unit ball in $\mathbb{R}^n$, $B_+ = B \cap \mathbb{R}^n_+$, $B_- = B \cap \mathbb{R}^n_-$, and $\mathbb{R}^n_+ = \{(x_1, \dots x_n) \in \mathbb{R}^n \mid x_n \ge 0\}$ and $\mathbb{R}^n_- = \{(x_1, \dots x_n) \in \mathbb{R}^n \mid x_n \le 0\}$ are the half-spaces. Additionally we may assume that the collar of D intersected with $Q_0 \cap Q_1$ defines a collar of $D \cap Q_0 \cap Q_1$ in $Q_0 \cap Q_1$.

Similarly, we may assume that there is a categorical partition $V_0, \dots, V_k$ of N as in Theorem 7.2 and a closed n–ball D' with a collar such that the triad (D', D'_0, D'_1) is homeomorphic to the triad (B, B_+, B_-), where $D'_0 = D' \cap V_0$, $D'_1 = D' \cap V_1$.

We may assume that the connected sum $M \# N$ is realized as a subset $M \# N = M \cup N \setminus \operatorname{Int} D \subset M \cup_h N$ for some homeomorphism $h\colon D' \to D$ that preserves the triad structures.

Let $W_0 = (Q_0 \setminus \operatorname{Int} D) \cup (V_0 \setminus \operatorname{Int} D')$, $W_1 = (Q_1 \setminus \operatorname{Int} D) \cup (V_1 \setminus \operatorname{Int} D')$ and $W_i = Q_i \cup V_i$ for $i = 2, \dots, k$. Note that $Q_i \cap V_i = \varnothing$ for $i \ge 2$. By Singhof's theorem each Q_i can be deformed to an $(n-k)$–dimensional subset S_i contractible in M. Since $k \ge 2$, there is a contraction of S_i to a point in M that misses a given point. Hence, there is a contraction of S_i to a point in M that misses the ball D. Thus Q_i for $i \ge 2$ can be contracted to a point in $M \# N$. Similarly, for $i \ge 2$ the set V_i can be contracted to a point in $M \# N$. Hence the sets W_i for $i \ge 2$ are categorical.

Let $A_i = Q_i \cap \partial D$ for $i = 0, 1$. We show that there is a deformation of $Q_i \setminus \operatorname{Int} D$ in $M \# N$ to A_i. The collar of $Q_i \cap D$ in Q_i allows us to construct a homeomorphism of $Q_i \setminus \operatorname{Int} D$ to Q_i homotopic to the identity. Hence $Q_i \setminus \operatorname{Int} D$ can be deformed onto an $(n-k)$–dimensional subset S_i contractible in M. A contraction of S_i to a point can be chosen missing $c_0 \in \operatorname{Int} D$. By Proposition 7.3 there is a deformation of $Q_i \setminus \operatorname{Int} D$ in $M \setminus \{c_0\}$ onto A_i fixing A_i. Similarly, for $i = 0, 1$ there is a deformation of $V_i \setminus \operatorname{Int} D'$ in $N \setminus \{c'_0\}$ to $A_i = V_i \cap \partial D'$ fixing A_i where $c'_0 \in \operatorname{Int} D'$. Applying the radial projections from c_0 and c'_0 gives us such deformations in $M \# N$. Pasting these two deformations defines a deformation of W_i, $i = 0, 1$, in $M \# N$ to A_i. Since the sets A_i are contractible, it follows that the sets W_i, $i = 0, 1$. are categorical. □

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