The LS category of the product of lens spaces

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We reduce Rudyak’s conjecture that a degree-one map between closed manifolds cannot raise the Lusternik–Schnirelmann category to the computation of the category of the product of two lens spaces $L^n_p \times L^n_q$ with relatively prime $p$ and $q$. We have computed $\text{cat}(L^n_p \times L^n_q)$ for values $p, q > n/2$. It turns out that our computation supports the conjecture.

For spin manifolds $M$ we establish a criterion for the equality $\text{cat} M = \text{dim} M - 1$, which is a K–theoretic refinement of the Katz–Rudyak criterion for $\text{cat} M = \text{dim} M$. We apply it to obtain the inequality $\text{cat}(L^n_p \times L^n_q) \leq 2n - 2$ for all odd $n$ and odd relatively prime $p$ and $q$.

55M30; 55N15

1 Introduction

This paper was motivated by the following conjecture of Rudyak:

**Conjecture 1.1** [19] A degree-one map between closed manifolds cannot raise the Lusternik–Schnirelmann category.

It is known that degree-one maps $f: M \to N$ between manifolds tend to have domain more complex than their image. The Lusternik–Schnirelmann category is a numerical invariant that measures the complexity of a space. Thus, Rudyak’s conjecture that $\text{cat} M \geq \text{cat} N$ for a degree-one map $f: M \to N$ is quite natural. Rudyak (see also the book by Cornea, Lupton, Opera and Tanré [7, page 65]) obtained some partial results supporting the conjecture. In particular, he proved the following:

**Theorem 1.2** [19] Let $f: M \to N$ be a degree-$\pm 1$ map between closed, stably parallelizable $n$–manifolds, $n \geq 4$, such that $2 \text{cat} N \geq n + 4$. Then $\text{cat} M \geq \text{cat} N$.

In this paper we reduce Rudyak’s conjecture to the following question about the LS category of the product of two $n$–dimensional lens spaces ($n = 2k - 1$).
Problem 1.3 Do there exist \( n \) and relatively prime \( p \) and \( q \) such that
\[
\text{cat}(L^n_p \times L^n_q) > n + 1?
\]

We show that an affirmative answer to this problem gives a counterexample to Rudyak’s conjecture.

This paper is devoted to computation of the category of the product \( L^n_p \times L^n_q \) of lens spaces for relatively prime \( p \) and \( q \). Here we use the shorthand notation \( L^n_p = L^n_p(\ell_1, \ldots, \ell_k) \) for a general lens space of dimension \( n = 2k - 1 \), defined for the linear \( \mathbb{Z}_p \)-action on \( S^n \subset \mathbb{C}^k \) determined by the set of natural numbers \( (\ell_1, \ldots, \ell_k) \) with \( (p, \ell_i) = 1 \) for all \( i \).

The obvious inequality \( \text{cat} X \leq \dim X \) and the cup-length lower bound (see Proposition 2.9) give the estimates
\[
(*) \quad n + 1 \leq \text{cat}(L^n_p \times L^n_q) \leq 2n.
\]

In this paper we prove that, for fixed \( n \), the lower bound is almost always sharp.

Theorem 1.4 For every \( n = 2k - 1 \) and primes \( p, q \geq k, p \neq q \), for all lens spaces \( L^n_p \) and \( L^n_q \),
\[
\text{cat}(L^n_p \times L^n_q) = n + 1.
\]

This result still leaves some hope to have \( \text{cat}(L^n_p \times L^n_q) > n + 1 \) for small values of \( p \) (especially for \( p = 2 \)) for some lens spaces.

In the second part of the paper we make an improvement of the upper bound in (*)).

The first improvement comes easily by virtue of the Katz–Rudyak criterion [13]: for a closed \( m \)-manifold \( M \) the inequality \( \text{cat}(M) \leq m - 1 \) holds if and only if \( M \) is inessential. We recall that Gromov calls a \( m \)-manifold \( M \) inessential if a map \( u: M \to B\pi \) that classifies its universal covering can be deformed to the \((m-1)\)-dimensional skeleton \( B\pi^{(m-1)} \). Since for relatively prime \( p \) and \( q \) the product \( L^n_p \times L^n_q \) is inessential, we have \( \text{cat}(L^n_p \times L^n_q) \leq 2n - 1 \). In the paper we improve this inequality to the following:

Theorem 1.5 For all odd \( n \) and odd relatively prime \( p \) and \( q \),
\[
\text{cat}(L^n_p \times L^n_q) \leq 2n - 2.
\]

For that we study a general question: when is the LS category of a closed spin \( m \)-manifold \( M \) less than \( m - 1 \)? We prove in Theorem 6.6 that for a closed \( m \)-manifold \( M \) with \( \pi_2(M) = 0 \), the inequality \( \text{cat} M \leq m - 2 \) holds if and only if the map \( u: M \to B\pi \)
can be deformed to the \((m-2)\)-dimensional skeleton \(B\pi^{(m-2)}\). A deformation of a classifying map of a manifold to the \((m-2)\)-skeleton \(B\pi^{(m-2)}\) is closely related to Gromov’s conjecture on manifolds with positive scalar curvature and it was investigated by Bolotov and Dranishnikov [3]. Combining this with some ideas from [3], we produce a criterion for when a closed spin \(m\)-manifold \(M\) has \(\text{cat } M \leq m-2\). The criterion involves the vanishing of the integral homology and \(ko\)-homology fundamental classes of \(M\) under a map classifying the universal covering of \(M\).

**Theorem 1.6** (Criterion) If \(M\) is a closed, spin, inessential \(m\)-manifold with \(\pi_2(M) = 0\), then

\[\text{cat } M \leq \dim M - 2\]

if and only if \(j_*u_*([M]_{ko}) = 0\), where \(j : B\pi \to B\pi/B\pi^{(m-2)}\) is the quotient map.

Since a closed orientable manifold \(M\) is inessential if and only if \(u_*([M]) = 0\) in \(H_*(B\pi)\) — see Babenko [1] — the Katz–Rudyak criterion for orientable manifolds can be rephrased as follows: \(\text{cat } M \leq m - 1\) if and only if \(u_*([M]) = 0\). Thus, our criterion is a further refinement of the Katz–Rudyak criterion.

It turns out that the vanishing of \(u_*([M])\) in \(H_*(B\pi)\) makes the primary obstruction to a deformation of \(u : M \to B\pi\) to \(B\pi^{(m-2)}\) trivial. It is not difficult to show that the second obstruction lives in the group of coinvariants \(\pi_*(B\pi, B\pi)_\pi\); see [3]. We prove that the group of coinvariants \(\pi_*(B\pi, B\pi^{(m-2)})_\pi\) naturally injects into the homotopy group \(\pi_*(B\pi/B\pi^{(m-2)})\). This closes a gap in the computation of the second obstruction in [3]. Based on that injectivity result we use the real connective K–theory to express the second obstruction in terms of the image of the \(ko\)-fundamental class. The spin condition is needed for the existence of a fundamental class in \(ko\)-theory.

The new upper bound implies that \(\text{cat}(L_p^3 \times L_q^3) = 4\) for all \(p\) and \(q\). Note that for prime \(p\) and \(q\) this fact can be also derived from Theorem 1.4.

We complete the paper with a proof of the upper bound formula for the category of a connected sum of two manifolds:

**Theorem 1.7** \(\text{cat } M \# N \leq \max\{\text{cat } M, \text{cat } N\}\).

Since we use this formula in the paper and its original proof in [16] does not cover all cases, we supply an alternative proof.

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2 Preliminaries

2.1 LS category

The Lusternik–Schnirelmann category, for a topological space \( X \), satisfies \( \text{cat} \ X \leq k \) if there is a cover \( X = U_0 \cup \cdots \cup U_k \) by \( k + 1 \) open subsets each of which is contractible in \( X \). The subsets contractible in \( X \) will be called in this note \( X \)-contractible and The covers of \( X \) by subsets contractible in \( X \) will be called categorical.

Let \( \pi = \pi_1(X) \). We recall that the cup product \( \alpha \smile \beta \) of twisted cohomology classes \( \alpha \in H^i(X; L) \) and \( \beta \in H^j(X; M) \) takes values in \( H^{i+j}(X; L \otimes M) \), where \( L \) and \( M \) are \( \pi \)-modules and \( L \otimes M \) is the tensor product over \( \mathbb{Z} \); see Brown [5]. Then the cup-length of \( X \), denoted as \( \text{cl} \ X \), is defined as the maximal integer \( k \) such that \( \alpha_1 \smile \cdots \smile \alpha_k \neq 0 \) for some \( \alpha_i \in H^{n_i}(X; L_i) \) with \( n_i > 0 \). The following inequalities give estimates on the LS category:

\[
\text{Theorem 2.1} \quad [7] \quad \text{cl}(X) \leq \text{cat} \ X \leq \dim X.
\]

2.2 Ganea–Schwarz approach to the LS category

Given two maps \( f_1: X_1 \to Y \) and \( f_2: X_2 \to Y \), we set

\[
Z = \{(x_1, x_2, t) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}
\]

and define the fiberwise join, or join over \( Y \), of \( f_1 \) and \( f_2 \) as the map

\[
f_1 \ast_Y f_2: Z \to Y, \quad (f_1 \ast_Y f_2)(x_1, x_2, t) = f_1(x_1) = f_2(x_2).
\]

Let \( p_0^X: PX \to X \) be the Serre path fibration. This means that \( PX \) is the space of paths on \( X \) that start at the base point \( x_0 \in X \), and \( p_0^X(\alpha) = \alpha(1) \) for \( \alpha \in PX \). We denote by \( p_n^X: G_n(X) \to X \) the iterated fiberwise join of \( n + 1 \) copies of \( p_0^X \). Thus, the fiber \( F_n = (p_n^X)^{-1}(x_0) \) of the fibration \( p_n^X \) is the join product \( \Omega X \ast \cdots \ast \Omega X \) of \( n + 1 \) copies of the loop space \( \Omega X \) on \( X \). So, \( F_n \) is \((n-1)\)-connected. It is known that \( G_n(X) \) is homotopy equivalent to the mapping cone of the inclusion of the fiber \( F_{n-1} \to G_{n-1}(X) \).

When \( X = K(\pi, 1) \), the loop space \( \Omega X \) is naturally homotopy equivalent to \( \pi \) and the space \( G_n(\pi) = G_n(K(\pi, 1)) \) has the homotopy type of a \( n \)-dimensional complex.

The proof of the following theorem can be found in [7]:

\[
\text{Theorem 2.2} \quad \text{(Ganea, Schwarz)} \quad \text{For a CW space } X, \text{ cat}(X) \leq n \text{ if and only if there exists a section of } p_n^X: G_n(X) \to X.
\]
Theorem 2.3  For a map $f: Y \to X$ to a CW space $X$, $\text{cat}(f) \leq n$ if and only if there exists a lift of $f$ with respect to $p_n^X: G_n(X) \to X$.

We recall that the LS category of a map $f: W \to X$ is the least integer $k$ for which $Y$ can be covered by $k + 1$ open sets $U_0, \ldots, U_k$ such that the restrictions $f|_{U_i}$ are null-homotopic for all $i$.

We use the notation $\pi_*(f) = \pi_*(M_f, X)$, where $M_f$ is the mapping cylinder of $f: X \to Y$. Then $\pi_i(f) = 0$ for $i \leq n$ amounts to saying that $f$ induces isomorphisms $f_*: \pi_i(X) \to \pi_i(Y)$ for $i < n$ and an epimorphism in dimension $n$.

Proposition 2.4 [8] Let $f_j: X_j \to Y_j$, $3 \leq j \leq s$ be a family of maps of CW spaces such that $\pi_i(f_j) = 0$ for $i \leq n_j$. Then the joins satisfy

$$\pi_k(f_1 \ast f_2 \ast \cdots \ast f_s) = 0$$

for $k \leq \min\{n_j\} + s - 1$.

2.3 The Berstein–Schwarz class

Let $\pi$ be a discrete group and $A$ be a $\pi$–module. By $H^*(\pi, A)$ we denote the cohomology of the group $\pi$ with coefficients in $A$ and by $H^*(X; A)$ we denote the cohomology of a space $X$ with the twisted coefficients defined by $A$. The Berstein–Schwarz class of a group $\pi$ is a certain cohomology class $\beta_\pi \in H^1(\pi, I(\pi))$, where $I(\pi)$ is the augmentation ideal of the group ring $\mathbb{Z}\pi$; see Berstein [2] and Schwarz [22]. It is defined as the first obstruction to a lift of $B\pi = K(\pi, 1)$ to the universal covering $E\pi$. The class $\beta_\pi$ is defined by a cocycle $\beta: E\pi^{(1)} \to I(\pi)$. We note that the 1–skeleton of $E\pi$ can be identified with the Cayley graph of $\pi$. For a fixed set $S$ of generators of $\pi$, the Cayley graph $C = C(\pi, S)$ has $V = \pi$ as the set of vertices and $E = \{(\gamma, \gamma s) \mid \gamma \in \pi, s \in S\}$ as the set of edges.

Note that the 1–skeleton of $B\pi$ can be identified with the wedge of circles labeled by $S$. Then the 1–skeleton $E\pi^{(1)}$ of the universal covering equals the Cayley graph $C = C(\pi, S)$. In that case the cocycle $\beta$ takes every edge $[a, b] \subset C$ to $b - a \in I(\pi)$.

Here is a more algebraic definition of $\beta_\pi$. Consider the cohomology long exact sequence generated by the short exact sequence of coefficients

$$0 \to I(\pi) \to \mathbb{Z}\pi \to \mathbb{Z} \to 0,$$
where $\epsilon$ is the augmentation homomorphism. Then $\beta_\pi = \delta(1)$ equals the image of the generator $1 \in H^0(\pi; \mathbb{Z}) = \mathbb{Z}$ under the connecting homomorphism

$$\delta: H^0(\pi; \mathbb{Z}) \to H^1(\pi; I(\pi)).$$

It follows from the definition of the connecting homomorphism $\delta$ (snake lemma) that $\delta(1)$ is defined by the above cocycle $\beta$.

**Theorem 2.5** (Universality [9; 22]) For any cohomology class $\alpha \in H^k(\pi, L)$ there is a homomorphism of $\pi$–modules $I(\pi)^k \to L$ such that the induced homomorphism for cohomology takes $(\beta_\pi)^k \in H^k(\pi; I(\pi)^k)$ to $\alpha$, where $I(\pi)^k = I(\pi) \otimes \cdots \otimes I(\pi)$ and $(\beta_\pi)^k = \beta_\pi \cup \cdots \cup \beta_\pi$.

**Corollary 2.6** [22] The class $(\beta_\pi)^{n+1}$ is the primary obstruction to a section of $p_\pi^B: \pi \to \pi$.

**Corollary 2.7** For any group $\pi$, its cohomological dimension can be expressed as

$$\text{cd}(\pi) = \max\{n \mid (\beta_\pi)^n \neq 0\}.$$

**Corollary 2.8**

$$cl(L^n_p) = n.$$

**Proof** For any lens space $L^n_p$ the inclusion $L^n_p \to B\mathbb{Z}_p$ to the classifying space as the $n$–skeleton takes $(\beta_{\mathbb{Z}_p})^n$ to a nonzero element $\beta^n$. Since $\text{cd}(\mathbb{Z}_p) = \infty$, we obtain $(\beta_{\mathbb{Z}_p})^n \neq 0$. Since the restriction to the $n$–skeleton is injective on $n$–dimensional cohomology groups, the result follows.

**Proposition 2.9**

$$cl(L^n_p \times L^n_q) \geq n + 1.$$

**Proof** Let $\alpha \in H^n(L^n_q) = \mathbb{Z}$ be a generator. Then, in view of the Kunneth formula for local coefficients [4], the cross product

$$\beta^n \times \alpha \in H^{2n+1}(L^n_p \times L^n_q; I(\mathbb{Z}_p)^n)$$

is nontrivial for the above $\beta \in H^1(L^n_p; I(\mathbb{Z}_p))$. 

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3 Some examples of degree-one maps

Let \( M \) be an oriented manifold and \( k \in \mathbb{Z} \setminus \{0\} \); by \( kM \) we denote the connected sum \( M \# \cdots \# M \) of \( |k| \) copies of \( M \), taken with the opposite orientation if \( k \) is negative. For an odd \( n \) and natural \( p > 1 \) we denote by \( L^n_p \) a lens space, i.e. the orbit space \( S^n/\mathbb{Z}_p \) for a free linear action of \( \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \) on the sphere \( S^n \).

**Theorem 3.1** For \( m, n \in 2\mathbb{N} + 1 \) and any relatively prime numbers \( p \) and \( q \) there are \( k, l \in \mathbb{Z} \) such that the manifold

\[
M = k(L^m_p \times S^n) \# l(S^m \times L^n_q)
\]

admits a degree-one map \( \phi: M \to N \) onto \( N = L^m_p \times L^n_q \).

**Proof** Take \( k \) and \( l \) such that \( lp + kq = 1 \). Let \( f: S^m \to L^m_p \) and \( g: S^n \to L^n_q \) be the projections to the orbit space for the \( \mathbb{Z}_p \) and \( \mathbb{Z}_q \) free actions, respectively. We may assume that the above connected sum is obtained by taking the wedge of \( |k| + |l| - 1 \) spheres of dimension \( m+n-1 \) embedded in one of the summands and gluing all other summands along those spheres. Consider the quotient map

\[
\psi: k(L^m_p \times S^n) \# l(S^m \times L^n_q) \to \bigvee_k (L^m_p \times S^n) \vee \bigvee_l (S^m \times L^n_q)
\]

that collapses the wedge of those \((m+n-1)\)-spheres to a point. Let the map

\[
\phi: \bigvee_k (L^m_p \times S^n) \vee \bigvee_l (S^m \times L^n_q) \to L^m_p \times L^n_q
\]

be defined as the union

\[
\phi = \bigcup_k (1 \times g) \cup \bigcup_l (f \times 1).
\]

Note that the degree of \( f \times 1 \) is \( p \), the degree of \( 1 \times g \) is \( q \) and the degree of \( \phi \circ \psi \) is \( lp + kq = 1 \).

**Proposition 3.2** For \( m \leq n \), \( \text{cat}(k(L^m_p \times S^n) \# l(S^m \times L^n_q)) = n + 1 \).

**Proof** It follows from the cup-length estimate that \( \text{cat}(S^m \times L^n_q) \geq n + 1 \) and, generally, \( \text{cat}(k(L^m_p \times S^n) \# l(S^m \times L^n_q)) \geq n + 1 \) when \( l \neq 0 \). By the product formula, \( \text{cat}(S^m \times L^n_p) \leq n + 1 \). Thus, \( \text{cat}(S^m \times L^n_q) = n + 1 \). Then, by the sum formula [16] (see Theorem 7.1),

\[
\text{cat}(k(L^m_p \times S^n) \# l(S^m \times L^n_q)) \leq n + 1.
\]
Now one can see the connection between Rudyak’s conjecture and Problem 1.3. If there exist relatively prime \( p \) and \( q \) and odd \( n \) such that \( \text{cat}(L_p^m \times L_q^n) > n + 1 \), then the map of Theorem 3.1 will be a counter-example to Rudyak’s conjecture.

**Remark** In Theorem 3.1 one can use fake lens spaces. Since every fake lens space is homotopy equivalent to a lens space [23] and the LS category is a homotopy invariant, it suffices to consider only the classical lens spaces.

### 4 On the category of the product of lens spaces

Let \( \tilde{\ell} = (\ell_1, \ldots, \ell_k) \) be a set of mod \( p \) integers relatively prime to \( p \). The lens space \( L_p^{2k-1}(\tilde{\ell}) \) is the orbit space of the action of \( \mathbb{Z}_p = \langle t \rangle \) on the unit sphere \( S^{2k-1} \subset \mathbb{C}^k \) defined by the formula

\[
t(\ell_1, \ldots, \ell_k) = (e^{2\pi i \ell_1/p} z_1, \ldots, e^{2\pi i \ell_k/p}).
\]

We note that for all \( k \) the lens spaces \( L_p^{2k-1}(\tilde{\ell}) \) have a natural CW complex structure with one cell in each dimension up to \( 2k - 1 \) such that \( L_p^{2k-1}(\tilde{\ell}) \) is the \((2k-1)\)–skeleton of \( L_p^{2k+1}(\tilde{\ell}, \ell_{k+1}) \). If \( \alpha: \mathbb{Z}_p \times S^{2k-1} \to S^{2k-1} \) is a free action which is not necessarily linear, its orbit space is called a fake lens space and is denoted by \( L_p^{2k-1}(\alpha) \).

We recall that a closed, oriented \( n \)–manifold \( M \) is called inessential — see Gromov [12] — if a map \( u: M \to B\pi = K(\pi, 1) \) that classifies its universal cover can be deformed to the \((n-1)\)–dimensional skeleton \( B\pi^{(n-1)} \). It is known that a closed, oriented \( n \)–manifold \( M \) is essential if and only if \( u_*([M]) \neq 0 \), where \([M] \in H_n(M; \mathbb{Z})\) denotes the fundamental class [1; 3].

We note that cat \( M = \dim M \) if and only if \( M \) is essential [13]. Clearly, every lens space \( L_p^n \) is essential. In particular, \( \text{cat} L_p^n = n \). Since \( \mathbb{Z}_p \otimes \mathbb{Z}_q = 0 \) for relatively prime \( p \) and \( q \), the product \( L_p^m \times L_q^n \) is inessential. Hence, \( \text{cat}(L_p^m \times L_q^n) \leq m + n - 1 \) for all \( p \) and \( q \).

#### 4.1 Stably parallelizable lens spaces

First we do our computation for stably parallelizable lens spaces.

**Proposition 4.1** For lens spaces \( L_p^m \) and \( L_q^n \) with \( m \leq n \) and \( (p, q) = 1 \) which are homotopy equivalent to stably parallelizable manifolds,

\[
\text{cat}(L_p^m \times L_q^n) = n + 1.
\]
Proof  Let
\[ \phi: M = k(L_p^m \times S^n) \# I(S^m \times L_q^n) \to N = L_p^m \times L_q^n \]
be the map of degree one from Theorem 3.1. Suppose that \( L_p^m \) and \( L_q^n \) are homotopy equivalent to stably parallelizable manifolds \( N_p^m \) and \( N_q^n \), respectively. Then there are homotopy equivalences \( h: M' = k(N_p^m \times S^n) \# I(S^m \times N_q^n) \to M \) and \( h': N = L_p^m \times L_q^n \to N' = N_p^m \times N_q^n \). Since a connected sum and the product of stably parallelizable manifolds are stably parallelizable (see for example [14]), the manifolds \( M' \) and \( N' \) are stably parallelizable. Assume that \( \text{cat}(L_p^m \times L_q^n) \geq n + 2 \). Then
\[
2 \text{cat} N' = 2 \text{cat}(L_p^m \times L_q^n) = 2(n + 2) \geq m + n + 4 = \dim(L_p^m \times L_q^n) + 4.
\]
By Theorem 1.2 applied to the map \( h' \circ \phi \circ h: M' \to N' \) from Theorem 3.1, we obtain a contradiction:
\[
n + 2 = \text{cat} N = \text{cat} N' \leq \text{cat} M' = \text{cat} M = n + 1.
\]
Since all orientable 3-manifolds are stably parallelizable, we obtain:

**Corollary 4.2**  For relatively prime \( p \) and \( q \),
\[
\text{cat}(L_p^3 \times L_q^3) = 4.
\]

There is a characterization of stable parallelizability of lens spaces [10]: the lens space \( L_p^{2k-1}(\ell_1, \ldots, \ell_k) \) is stably parallelizable if and only if \( p \geq k \) and \( \ell_1^j + \cdots + \ell_k^j = 0 \mod p \) for \( j = 1, 2, \ldots, \left[ \frac{1}{2}(k-1) \right] \). We recall that two lens spaces \( L_p^{2k-1}(\ell_1, \ldots, \ell_k) \) and \( L_p^{2k-1}(\ell_1', \ldots, \ell_k') \) are homotopy equivalent [17] if and only if the mod \( p \) equation
\[
\ell_1 \ell_2 \cdots \ell_k = \pm a^k \ell_1' \ell_2' \cdots \ell_k'
\]
has a solution \( a \in \mathbb{Z}_p \). These conditions imply that a lens space is rarely homotopy equivalent to a stably parallelizable one. Nevertheless, Ewing, Moolgavkar, Smith and Stong [10] showed that, for each \( n = 2k - 1 \), for infinitely many primes \( p \) there are stably parallelizable lens spaces \( L_p^n \). Clearly, there are more chances for the existence of stably parallelizable fake lens spaces with given \( n \) and \( p \). Thus, Kwak [15] proved that for every odd \( n = 2k - 1 \) and \( p \geq k \) there is a fake \( n \)-dimensional stably parallelizable lens space. Since every fake lens space is homotopy equivalent to a lens space — see Wall [23] — we obtain that for every \( n = 2k - 1 \) and \( p \geq k \) there is a lens space \( L_p^n \) homotopy equivalent to a stably parallelizable manifold.
4.2 Category of classifying maps

We recall that any map 
\[ u: X \to B\pi = K(\pi, 1) \]
of a CW complex \( X \) that induces an isomorphism of the fundamental group classifies the universal covering \( \tilde{X} \), i.e. \( \tilde{X} \) is obtained as the pull-back of the universal covering \( E\pi \) of \( B\pi \) by means of \( u \). We call such a map a classifying map of \( X \).

Proposition 4.3 Let \( u: X \to B\pi \) be a map classifying the universal covering of a CW complex \( X \). Then the following are equivalent:

1. \( \text{cat}(u) \leq k \).
2. \( u \) admits a lift \( u': X \to G_k(\pi) \) of \( u \) with respect to \( p^n_\pi: G_k(\pi) \to B\pi \).
3. \( u \) is homotopic to a map \( f: X \to B\pi \) with \( f(X) \subset B^{(k)} \).

Proof (1) \( \implies \) (2) is a part of Theorem 2.3.

(2) \( \implies \) (3) Since \( G_k(\pi) \) has the homotopy type of a \( k \)–dimensional complex, the map \( p^n_\pi \) can be deformed to a map \( p' \) with the image in \( B^{(k)} \). Then we can take \( f = p' \circ u' \).

(3) \( \implies \) (1) For a map \( f: X \to B\pi \) with \( f(X) \subset B^{(k)} \) homotopic to \( u \) we obtain \( \text{cat}(u) = \text{cat}(f) \leq \text{cat} B^{(k)} \leq k \).

\[ \square \]

Theorem 4.4 Let \( X \) be an \( n \)–dimensional CW complex with a classifying map \( u: X \to B\pi \) having \( \text{cat} u \leq k \) and with \((n–k)–\)connected universal covering \( \tilde{X} \). Then \( \text{cat} X \leq k \).

Proof Note that the map \( p^X_k \) factors through the pull-back, \( p^X_k = p' \circ q \):

\[ G_k(X) \xrightarrow{q} Z \xrightarrow{f'} G_k(\pi) \]
\[ p^X_k \quad p^n_\pi \]
\[ X \xrightarrow{u} B\pi \]

The condition \( \text{cat} u \leq k \) implies that \( u \) has a lift \( u': X \to G_k(\pi) \), \( u = p^n_\pi u' \). Hence, \( p' \) admits a section \( s: X \to Z \). Since \( X \) is \( n \)–dimensional, to show that \( s \) has a lift with respect to \( q \) it suffices to prove that the homotopy fiber \( F \) of the map \( q \) is \((n–1)–\)connected. Since \( \pi_i(X) = 0 \) for \( 1 < i \leq n–k \), \( B\pi \) is aspherical and \( u \) induces an isomorphism of the fundamental groups, we obtain \( \pi_i(u) = 0 \) for \( i \leq n–k + 1 \). Hence, \( \pi_i(\Omega u) = 0 \) for \( i \leq n–k \). Then, by Proposition 2.4, \( \pi_i(\ast_{k+1} \Omega u) = 0 \) for
\[ i \leq (n-k) + (k+1) - 1 = n. \] The commutative diagram generated by \( q \) and the fibrations \( p_X^k \) and \( p' \),

\[
\begin{array}{cccccccc}
\cdots & \rightarrow & \pi_i(*k+1\Omega(X)) & \rightarrow & \pi_i(G_k(X)) & \rightarrow & \pi_i(X) & \rightarrow & \cdots \\
& \downarrow & *k+1\Omega u & \downarrow & q* & \downarrow & 1 & \downarrow & \\
\cdots & \rightarrow & \pi_i(*k+1\Omega(B\pi)) & \rightarrow & \pi_i(Z) & \rightarrow & \pi_i(X) & \rightarrow & \cdots,
\end{array}
\]

and the five lemma imply that \( \pi_i(q) = 0 \) for \( i \leq n \). Hence, \( \pi_i(F) = 0 \) for \( i \leq n-1 \).

Thus, \( s \) admits a homotopy lift. Therefore, \( p_X^k \) has a homotopy section and, hence, it admits a section. Therefore, by Theorem 2.2, \( \text{cat } X \leq k \).

\section{The main computation}

\textbf{Proposition 4.5} For any two lens spaces \( L_p^n(\tilde{\ell}) \) and \( L_p^n(\tilde{\mu}) \), there is a map

\[ f: L_p^n(\tilde{\ell}) \rightarrow L_p^n(\tilde{\mu}) \]

that induces an isomorphism of the fundamental groups.

\textbf{Proof} Let \( q_1: S^n \rightarrow L_p^n(\tilde{\ell}) \) and \( q_2: S^n \rightarrow L_p^n(\tilde{\mu}) \) be the projections onto the orbit spaces of the corresponding \( \mathbb{Z}_p \)-actions. We note that \( L_p^n(\tilde{\mu}) \) is the \( n \)-skeleton in \( L_p^{n+2}(\tilde{\mu}, 1) \). Let \( \tilde{q}_2: S^{n+2} \rightarrow L_p^n(\tilde{\mu}, 1) \) be the corresponding projection:

\[
\begin{array}{cccccccc}
S^n & \leftarrow & S^n \times S^{n+2} & \rightarrow & S^{n+2} \\
q_1 & \downarrow & q & \downarrow & \tilde{q}_2 \\
L_p^n(\tilde{\ell}) & \leftarrow & S^n \times_{\mathbb{Z}_p} S^{n+2} & \rightarrow & L_p^{n+2}(\tilde{\mu}, 1) & \leftarrow & p_1 & \rightarrow & p_2 \\
\end{array}
\]

Since in the Borel construction for the diagonal \( \mathbb{Z}_p \) action on \( S^n \times S^{n+2} \) the projection \( p_1 \) is \((n+1)\)-connected, it admits a section \( s: L_p^n(\tilde{\ell}) \rightarrow S^n \times_{\mathbb{Z}_p} S^{n+2} \). Then \( f \) is a cellular approximation of \( p_2 \circ s \).

\textbf{Theorem 4.6} For every odd \( n = 2k - 1 \) and distinct primes \( p, q \geq k \),

\[ \text{cat}(L_p^{2k-1} \times L_q^{2k-1}) = n + 1. \]

\textbf{Proof} Let \( L_p^n = L_p^n(\tilde{\ell}) \) and \( L_q^n(\tilde{\ell}') \) for \( \tilde{\ell} = (\ell_1, \ldots, \ell_k) \) and \( \tilde{\ell}' = (\ell'_1, \ldots, \ell'_k) \). By Kwak [15, Theorem 3.1] there are stably parallelizable fake lens spaces \( L_p^n(\alpha) \) and \( L_q^n(\alpha') \). By Wall’s theorem they are homotopy equivalent to lens spaces \( L_p^n(\tilde{\mu}) \) and \( L_q^n(\tilde{\mu}') \) for some \( \tilde{\mu} \) and \( \tilde{\mu}' \). By Proposition 4.1, \( \text{cat}(L_p^n(\tilde{\mu}) \times L_q^n(\tilde{\mu}')) = n + 1 \).
By Proposition 4.3, there is a classifying map \( u: L^n_p(\tilde{\mu}) \times L^n_q(\tilde{\mu}') \rightarrow BW_{pq}^{(n+1)} \). By Proposition 4.5 there are maps \( f_p: L^n_p \rightarrow L^n_p(\tilde{\mu}) \) and \( f_q: L^n_q \rightarrow L^n_q(\tilde{\mu}') \) that induce an isomorphism of the fundamental groups. Therefore,

\[
u' = u \circ (f_p \times f_q): L^n_p \times L^n_q \rightarrow BW_{pq}^{(n+1)}
\]
is a classifying map for \( L^n_p \times L^n_q \). Hence, \( \text{cat}(u') \leq n+1 \). Since the universal covering of the space \( L^n_p \times L^n_q \) is \((n-1)\)-connected, by Theorem 4.4 we obtain \( \text{cat}(L^n_p \times L^n_q) \leq n+1 \). By Proposition 2.9, \( \text{cat}(L^n_p \times L^n_q) = n + 1 \).

**Remark** When \( p \) and \( q \) are relatively prime but not necessarily prime we can prove the equality \( \text{cat}(L^n_p \times L^n_q) = n + 1 \) with a stronger restriction \( p, q \leq n + 3 \). We do not present the proof, since it is more technical. It consists of computation of obstructions for deforming a classifying map \( u: L^n_p \times L^n_q \rightarrow BW_{pq} \) to the \((n+1)\)-skeleton. Vanishing of the first obstruction happens without any restriction on \( p \) and \( q \). Since it is a curious fact on its own it is presented in the next section. The higher obstructions vanish due to the fact that cohomology groups of \( \mathbb{Z}_{pq} \) are \( pq \)-torsions and a theorem of Serre [20] that states that the group \( \pi_{n+k}(S^n) \) has zero \( r \)-torsion component for \( k < 2r - 4 \).

We note that Theorem 4.6 can be stated for all lens spaces \( L^n_p \) with values of \( n \) and \( p \) for which there exists a stably parallelizable fake lens space \( L^n_p(\alpha) \).

**Problem 4.7** For which values of \( n \) and \( p \) is there a stably parallelizable fake lens space \( L^n_k(\alpha) \)?

This does not seem to happen very often when \( p = 2 \). At least, a real \((2k-1)\)-dimensional projective space is stably parallelizable if and only if \( k = 1, 2 \) or \( 4 \).

## 5 The Berstein–Schwarz class for the product of finite cyclic groups

Let \( u: L^n_p \times L^n_q \rightarrow BW_{pq} \) be a classifying map. By Theorem 4.4 and the fact that \( \text{cat}(L^n_p \times L^n_q) \geq n + 1 \), the condition \( \text{cat}(u) \leq n + 1 \) is equivalent to the equality \( \text{cat}(L^n_p \times L^n_q) = n + 1 \). By Proposition 4.3 the inequality \( \text{cat}(u) \leq n + 1 \) is equivalent to the existing of a lift \( u' \) of \( u \) with respect to \( p_n: G_{n+1}(\mathbb{Z}_{pq}) \rightarrow BW_{pq} \). In view of Corollary 2.6 the primary obstruction to such a lift is \( u^*(\beta^{n+2}) \), where \( \beta \) is the Berstein–Schwarz class of \( \mathbb{Z}_{pq} \). We prove that this obstruction is always zero and even more:
Theorem 5.1  For all \( n \) and all relatively prime \( p \) and \( q \),
\[
u^*(\beta^{n+1}) = 0.
\]

Remark  One can show that for sufficiently large \( p \) and \( q \) the higher obstructions are trivial as well, since the homotopy groups of the fiber of \( p_n^X \) do not contain \( r \)-torsions for large \( r \). This would give a result similar to Theorem 4.6, which does not cover small values of \( p \).

We denote by \( \mathbb{Z}(m) \) the group ring \( \mathbb{Z}[\mathbb{Z}/m] \), \( I(m) \) its augmentation ideal, \( \epsilon_m: \mathbb{Z}(m) \to \mathbb{Z} \) its augmentation, and \( \beta_m \) its Bernstein–Schwarz class. Let \( t_m = \sum g \in \mathbb{Z}/m \) \( g \in \mathbb{Z}(m) \). We use the same notation \( t_m \) for a constant map \( t_m: \mathbb{Z}/m \to \mathbb{Z}(m) \) with the value \( t_m \). We note that the group of invariants of \( \mathbb{Z}(m) \) is \( \mathbb{Z} \) generated by \( t_m \). Thus, \( H^0(\mathbb{Z}[\mathbb{Z}/m]; \mathbb{Z}(m)) = \mathbb{Z} \).

Proposition 5.2  Let \( \beta_p \) denote the Bernstein–Schwarz class for the group \( \mathbb{Z}/p = \mathbb{Z}/p \mathbb{Z} \). Then \( \beta_p \) has order \( p \) and hence is \( q \)-divisible for any \( q \) relatively prime to \( p \).

Proof  Let \( t \in \mathbb{Z}/p \) be a generator. We note that
\[
H^0(\mathbb{Z}[\mathbb{Z}/p]; \mathbb{Z}[\mathbb{Z}/p]) = (\mathbb{Z}[\mathbb{Z}/p])^{\mathbb{Z}/p} = \mathbb{Z}\langle t + \cdots + t^{p-1} \rangle
\]
is the group of invariants, which is isomorphic to the subgroup of \( \mathbb{Z}[\mathbb{Z}/p] \) generated by \( 1 + t + \cdots + t^{p-1} \). Then the augmentation homomorphism \( \epsilon: \mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z} \) induces a homomorphism \( \epsilon_*: H^0(\mathbb{Z}[\mathbb{Z}/p]; \mathbb{Z}[\mathbb{Z}/p]) \to H^0(\mathbb{Z}/p; \mathbb{Z}) = \mathbb{Z} \) that takes the generator \( 1 + t + \cdots + t^{p-1} \) to \( p \). Thus, \( p\beta_p = \delta(1) = \delta(p) = 0 \) by exactness of the cohomology long exact sequence associated with the coefficient sequence \( 0 \to I(p) \to \mathbb{Z}/p \to \mathbb{Z} \to 0 \).

Note that \( \beta_p \) generates a subgroup \( G \) of order \( p \) in \( H^1(\pi; I(\mathbb{Z}/p)) \). Therefore it is \( q \)-divisible for \( q \) with \( (p, q) = 1 \).

We recall that the cross product
\[
H^i(X; M) \times H^j(X'; M') \to H^{i+j}(X \times X'; M \otimes \mathbb{Z} M')
\]
is defined for any \( \pi_1(X) \)-module \( M \) and \( \pi_1(X') \)-module \( M' \). Also we note that
\[
H^i(X; M \oplus M') = H^i(X; M) \oplus H^i(X; M').
\]

Proposition 5.3  For relatively prime \( p \) and \( q \) there are \( k, l \in \mathbb{Z} \) such that the Bernstein–Schwarz class \( \beta_{pq} \) is the image of the class
\[
(\beta_p \times 1, k \times \beta_q) \in H^1(\mathbb{Z}/p; I(p) \otimes \mathbb{Z}(q)) \oplus H^1(\mathbb{Z}/q; \mathbb{Z}(p) \otimes I(q))
\]
under the coefficient homomorphism

\[ \phi: I(p) \otimes \mathbb{Z}(q) \oplus \mathbb{Z}(p) \otimes I(q) \to I(pq) \subset \mathbb{Z}(pq) = \mathbb{Z}(p) \otimes \mathbb{Z}(q) \]

defined by the inclusions of the direct summands into \( \mathbb{Z}(p) \otimes \mathbb{Z}(q) \) and the summation.

**Proof** Let \( k \) and \( l \) be such that \( kp + lq = 1 \).

The addition in \( \mathbb{Z}(pq) \) defines the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & I(pq) & \longrightarrow & \mathbb{Z}(pq) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
\phi & & & & & & & & \\
0 & \longrightarrow & I(p) \otimes \mathbb{Z}(q) \oplus \mathbb{Z}(p) \otimes I(q) & \longrightarrow & \mathbb{Z}(pq) \oplus \mathbb{Z}(pq) & \alpha & \longrightarrow & \mathbb{Z}(q) \oplus \mathbb{Z}(p) & \longrightarrow & 0 \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

which defines a commutative square for cohomology:

\[
\begin{array}{ccc}
H^0(\mathbb{Z}_{pq}, \mathbb{Z}) & \longrightarrow & H^1(\mathbb{Z}_{pq}, I(pq)) \\
\delta & & \phi_* \\
H^0(\mathbb{Z}_{pq}, \mathbb{Z}(p) \oplus \mathbb{Z}(q)) & \longrightarrow & H^1(\mathbb{Z}_{pq}, I(p) \otimes \mathbb{Z}(q) \oplus \mathbb{Z}(p) \otimes I(q)) \\
\delta' & & \\
\end{array}
\]

The homomorphism \( \theta: \mathbb{Z}(pq) \to \mathbb{Z}(p) \oplus \mathbb{Z}(q) \) defined on the basis as \( \theta(a \times b) = (lt_q, kt_p) \) is a cochain since it is \( \mathbb{Z}_{pq} \)-equivariant. It is a cocycle, since it is constant. Note that \( (\epsilon_p + \epsilon_q) \circ \theta(a \times b) = kp + lq = 1 \) for any \( a \in \mathbb{Z}_p \) and \( b \in \mathbb{Z}_q \). This means that the cohomology class \([\theta]\) is taken by \( \epsilon_* \) to a generator \( 1 \in H^0(\mathbb{Z}_{pq}, \mathbb{Z}) \). Then \( \beta_{pq} = \delta(1) = \phi \delta'([\theta]) \).

Consider a \( \mathbb{Z}(pq) \)-homomorphism \( \bar{\theta}: \mathbb{Z}(p) \times \mathbb{Z}(q) \to \mathbb{Z}(pq) \oplus \mathbb{Z}(pq) \) defined by the formula \( \bar{\theta}(a \times b) = (a \times lt_q, kt_p \times b) \). Since \( \alpha(\bar{\theta}) = \theta \), by the snake lemma \( \delta'([\theta]) \) is defined by the 1–cocycle \( \delta(\bar{\theta}): C_1 \to I(p) \otimes \mathbb{Z}(q) \oplus \mathbb{Z}(p) \otimes I(q) \). Note that the cellular 1–dimensional chain group \( C_1 \) is defined via the Cayley graph \( C \) of \( \mathbb{Z}_{pq} \).

Note that the Cayley graph \( C(\pi \times \pi', S \times e' \cup e \times S') \) of the product \( \pi \times \pi' \) of two groups with generating sets \( S \) and \( S' \) and units \( e \in \pi \) and \( e' \in \pi' \) equals the 1–skeleton of the product of the Cayley graphs \( C(\pi, S) \times C(\pi', S') \). Thus, \( C = (C^p \times \mathbb{Z}_q) \cup (\mathbb{Z}_p \times C^q) \), where \( C^p \) and \( C^q \) are the Cayley graphs (cycles) for \( \mathbb{Z}_p \) and \( \mathbb{Z}_q \), respectively. Note that

\[
\delta(\bar{\theta})([a_1, a_2] \times b) = \bar{\theta}((a_2 - a_1) \times b) = \bar{\theta}(a_2 \times b) - \bar{\theta}(a_1 \times b) = (a_2 \times lt_q, kt_p \times b) - (a_1 \times lt_q, kt_p \times b) = ((a_2 - a_1) \times lt_q, 0) = (\beta_p \times lt_q)([a_1, a_2] \times b) = (\beta_p \times lt_q, kt_p \times \beta_q)([a_1, a_2] \times b).
\]
Similarly, we have the equality for edges of the type $a \times [b_1, b_2]$. Here $\beta_p$ and $\beta_q$ denote the canonical cochains that define the Berstein–Schwarz classes of $\mathbb{Z}_p$ and $\mathbb{Z}_q$.

Thus, $\delta'([\theta]) = (\beta_p \times l, k \times \beta_q)$ in
\[
H^1(\mathbb{Z}_{pq}, I(p) \otimes \mathbb{Z}(q)) \oplus H^1(\mathbb{Z}_{pq}, \mathbb{Z}(p) \otimes I(q)) = H^1(\mathbb{Z}_{pq}, I(p) \otimes \mathbb{Z}(q) \oplus \mathbb{Z}(p) \otimes I(q)).
\]

\[\square\]

5.1 Proof of Theorem 5.1

We show that $u^*(\beta_{pq}^{n+1}) = 0$, where
\[
u = i_p \times i_q : L^n_p \times L^n_q \to B\mathbb{Z}_p \times B\mathbb{Z}_q = B\mathbb{Z}_{pq}
\]
is the inclusion. Note that $(\beta_p \times lt_q, k t_p \times \beta_q) = \beta_p \times lt_q + k t_p \times \beta_q$. Thus, it suffices to show that $u^*(\beta_p \times lt_q + k t_p \times \beta_q)^{n+1} = 0$. Note that
\[
u^*(\beta_p \times l + k \times \beta_q) = i_p^*(\beta_p) \times l + k \times i_q^*(\beta_q).
\]
Then $(i_p^*(\beta_p) \times l + k \times i_q^*(\beta_q))^{n+1} = (i_p^*(\beta_p) \times l)^{n+1} + (k \times i_q^*(\beta_q))^{n+1} + F$, where $F$ consists of monomials containing both factors.

Claim 1 $(i_p^*(\beta_p) \times l)^{n+1} = 0$ and $(k \times i_q^*(\beta_q))^{n+1} = 0$.

Proof There is an automorphism of the coefficients
\[
(I(p) \otimes \mathbb{Z}(q)) \otimes \cdots \otimes (I(p) \otimes \mathbb{Z}(q)) \to I(p) \otimes \cdots \otimes I(p) \otimes \mathbb{Z}(q) \otimes \cdots \otimes \mathbb{Z}(q)
\]
that takes $(i_p^*(\beta_p) \times l)^{n+1}$ to $i_p^*(\beta_p)^{n+1} \times l^{n+1} = 0$. Similarly, $(k \times i_q^*(\beta_q))^{n+1} = 0$.

Claim 2 $(i_p^*(\beta_p) \times l)A(k \times i_q^*(\beta_q)) = 0$ for any $A$.

Proof Indeed, since $i_p^*(\beta_p)$ is divisible by $q$ (see Proposition 5.2),
\[
(i_p^*(\beta_p) \times l)A(k \times i_q^*(\beta_q)) = \left(\frac{1}{q}(i_p^*(\beta_p) \times l)\right)Aq(k \times i_q^*(\beta_q)) = 0.
\]
Thus, $F = 0$ and the result follows.

6 On the category of $ko$–inessential manifolds

6.1 Deformation into the $(n–2)$–dimensional skeleton

We recall that a classifying map $u : M \to B\pi$ of a closed orientable $n$–manifold $M$ can be deformed into the $(n–1)$–skeleton $B\pi^{(n–1)}$ if and only if $u_*([M]) = 0$, where $[M] \in H_n(M; \mathbb{Z})$ denotes an integral fundamental class; see Babenko [1]. In [3] we
proved the following proposition, which sets the stage for computation of the second obstruction.

**Proposition 6.1** Every inessential \( n \)-manifold \( M \) with a fixed CW structure admits a classifying map \( u: M \to B\pi \) with \( u(M) \subset B\pi^{(n-1)} \) and \( u(M^{(n-1)}) \subset B\pi^{(n-2)} \).

We postpone the proof of the following lemma to the end of the section.

**Lemma 6.2** For any group \( \pi \) and CW complex \( B\pi \), for \( n \geq 5 \) the homomorphism induced by the quotient map

\[
p_*: \pi_n(B\pi, B\pi^{(n-2)}) \to \pi_n(B\pi/B\pi^{(n-2)})
\]

factors through the group of coinvariants as \( p_* = \tilde{p}_* \circ q_* \),

\[
\pi_n(B\pi, B\pi^{(n-2)}) \xrightarrow{q_*} \pi_n(B\pi, B\pi^{(n-2)})_\pi \xrightarrow{\tilde{p}_*} \pi_n(B\pi/B\pi^{(n-2)}),
\]

where \( \tilde{p}_* \) is injective.

We recall that for a \( \pi \)-module \( M \) the group of coinvariants is \( M \otimes_{\mathbb{Z}\pi} \mathbb{Z} \).

**Remark** In the proof of [3, Lemma 4.1] it was stated erroneously that \( \tilde{p}_* \) is bijective. It turns out that the injectivity of \( \tilde{p}_* \) was sufficient for the proof of that lemma to be carried out. Thus, due to Lemma 6.2 the results of [3] that depend on the lemma remain intact.

**Theorem 6.3** Let \( M \) be an \( n \)-manifold with a CW complex structure with one top-dimensional cell. Suppose that a classifying map \( u: M \to B\pi \) satisfies the condition \( u(M^{(n-1)}) \subset B\pi^{(n-2)} \) and let \( \tilde{u}: M/M^{(n-1)} = S^n \to B\pi/B\pi^{(n-2)} \) be the induced map. Then the following are equivalent:

1. There is a deformation of \( u \) in \( B\pi \) to a map \( f: M \to B\pi^{(n-2)} \).
2. \( \tilde{u}_*(1) = 0 \) in \( \pi_n(B\pi/B\pi^{(n-2)}) \), where \( 1 \in \mathbb{Z} = \pi_n(S^n) \).

**Proof** The primary obstruction to deforming \( u \) to \( B\pi^{(n-2)} \) is defined by the cocycle

\[
c_u = u_*: \pi_n(M, M^{(n-1)}) \to \pi_n(B\pi, B\pi^{(n-2)})
\]

with the cohomology class \( o_u = [c_u] \in H^n(M; \pi_n(B\pi, B\pi^{(n-2)})) \). By Poincaré duality, \( o_u \) is dual to the homology class \( PD(o_u) \in H_0(M; \pi_n(B\pi, B\pi^{(n-2)})) = \pi_n(B\pi, B\pi^{(n-2)})_\pi \) represented by \( q_*u_*(1) \), where

\[
q_*: \pi_n(B\pi, B\pi^{(n-2)}) \to \pi_n(B\pi, B\pi^{(n-2)})_\pi
\]
The LS category of the product of lens spaces

is the projection onto the group of coinvariants and

\[ u_*: \pi_n(M, M^{(n-1)}) = \mathbb{Z} \to \pi_n(B\pi, B\pi^{(n-2)}) \]

is induced by \( u \). We note that \( \pi_n(B\pi, B\pi^{(n-2)}) = \pi_n(E\pi, E\pi^{(n-2)}) \). By Lemma 6.2 the homomorphism \( \bar{p}_* \) is injective. Hence, \( \bar{p}_* q_* u_*(1) = 0 \) if and only if \( o_\pi = 0 \). The commutative diagram

\[
\begin{array}{ccc}
\pi_n(M, M^{(n-1)}) & \xrightarrow{u_*} & \pi_n(B\pi, B\pi^{(n-2)}) \\
\downarrow \mathbb{Z} & & \downarrow \mathbb{Z} \\
\pi_n(M/M^{(n-1)}) & \xrightarrow{\bar{u}_*} & \pi_n(B\pi/B\pi^{(n-2)})
\end{array}
\]

implies that \( \bar{u}_*(1) = \bar{p}_* q_* u_*(1) \).

\[ \square \]

### 6.2 ko–inessential manifolds

We recall that an orientable, closed \( n \)-manifold \( M \) is inessential if and only if

\[ u_*([M]) = 0, \text{ where } [M] \in H_n(M; \mathbb{Z}) \text{ is a fundamental class and } u: M \to B\pi \text{ is a classifying map.} \]

We call a closed spin \( n \)-manifold \( M \) ko–inessential if \( u_*([M]_{ko}) = 0 \) in \( ko_n(B\pi) \), where \( ko_* \) denotes the real connective K–theory homology groups.

We recall that for every spectrum \( E \) there is a natural morphism \( S \to E \) of the spherical spectrum. This defines a natural transformation of corresponding (co)homology theories \( \pi_*^S \to E_* \), where \( \pi_*^S \) is the stable homotopy theory. In the case of \( ko_* \) this natural transformation induces an isomorphism \( \pi_*^S(pt) \to ko_i(pt) \) for \( i = 0, 1, 2 \). It allows us in some cases to reduce \( ko_* \) problems to the stable homotopy groups.

We need the following proposition:

**Proposition 6.4** [3] The natural transformation \( \pi_*^S(pt) \to ko_* (pt) \) induces an isomorphism \( \pi_*^S(K/K^{(n-2)}) \to ko_n(K/K^{(n-2)}) \) for any CW complex \( K \).

We recall that spin manifolds are exactly those that admit an orientation in real connective K–theory \( ko_* \).

**Theorem 6.5** A classifying map \( u: M \to B\pi \) of an inessential, closed, spin \( n \)-manifold \( M \), \( n > 3 \), is homotopic to a map \( f: M \to B\pi^{(n-2)} \) if and only if \( j_* u_*([M]_{ko}) = 0 \) in \( ko_n(B\pi, B\pi^{(n-2)}) \), where \( [M]_{ko} \) is a ko–fundamental class.
Proof  By Proposition 6.1 a classifying map $u$ can be chosen to satisfy the condition $u(M^{(n-1)}) \subset B\pi^{(n-2)}$. We show that $\tilde{u}_*(1) = 0$ if and only if $j_*u_*([M]_{ko}) = 0$ and apply Theorem 6.3.

The restriction $n > 3$ implies that $\tilde{u}_*(1)$ survives in the stable homotopy group. In view of Proposition 6.4, the element $\tilde{u}_*(1)$ survives in the composition

$$\pi_n(B\pi / B\pi^{(n-2)}) \rightarrow \pi_n^s(B\pi / B\pi^{(n-2)}) \rightarrow ko_n(B\pi / B\pi^{(n-2)}).$$

The commutative diagram

$$
\begin{array}{ccc}
\pi_n(S^n) & \xrightarrow{\tilde{u}_*} & \pi_n(B\pi / B\pi^{(n-2)}) \\
\cong & & \cong \\
\pi_n^s(S^n) & \xrightarrow{\tilde{u}_*} & \pi_n^s(B\pi / B\pi^{(n-2)}) \\
\cong & & \cong \\
k_{n}(S^n) & \xrightarrow{\tilde{u}_*} & ko_n(B\pi / B\pi^{(n-2)})
\end{array}
$$

implies that $\tilde{u}_*(1) = 0$ for $ko_n$ if and only if $\tilde{u}_*(1) = 0$ for $\pi_n$.

From the diagram with the quotient map $\psi: M \rightarrow M / M^{(n-1)} = S^n$

$$
\begin{array}{ccc}
k_{n}(M) & \xrightarrow{u_*} & ko_n(B\pi) \\
\psi_* & & j_* \\
k_{n}(S^n) & \xrightarrow{\tilde{u}_*} & ko_n(B\pi / B\pi^{(n-2)})
\end{array}
$$

it follows that $j_*u_*([M]_{ko}) = \tilde{u}_*\psi_*([M]_{ko}) = \tilde{u}_*(1)$. Thus, $j_*u_*([M]_{ko}) = 0$ if and only if $\tilde{u}_*(1) = 0$ for $n$–dimensional homotopy groups.

For spin manifolds we prove the following criterion:

**Theorem 6.6**  For a closed spin $n$–manifold $M$ with $\text{cat } M \leq \dim M - 2$,

$$j_*u_*([M]_{ko}) = 0$$

in $ko_n(B\pi, B\pi^{(n-2)})$, where $u: M \rightarrow B\pi$ classifies the universal cover of $M$ and $j: (B\pi, \emptyset) \rightarrow (B\pi, B\pi^{(n-2)})$ is the inclusion.

For a closed, spin, inessential $n$–manifold $M$ with $\pi_2(M) = 0$, $\text{cat } M \leq \dim M - 2$ if and only if $j_*u_*([M]_{ko}) = 0$.
Proof The inequality \( \text{cat } M \leq n - 2 \) implies that the map \( u \) has a lift \( u' \to G_{n-2}(B\pi) \) with \( u = p_{n-2} u' \). Since \( G_{n-2}(B\pi) \) is homotopy equivalent to an \((n-2)\)-dimensional complex, \( p_{n-2} \) can be deformed to \( p' : G_{n-2}(B\pi) \to B\pi^{(n-2)} \). Thus \( u \) can be deformed to \( B\pi^{(n-2)} \). By Theorem 6.5, \( j_* u_*([M]_{ko}) \neq 0 \).

Now let \( \pi_2(M) = 0 \) and \( j_* u_*([M]_{ko}) = 0 \). By Theorem 6.5 the map \( u \) can be deformed to a map \( f : M \to B\pi^{(n-2)} \). By Proposition 4.3, \( \text{cat}(u) \leq n - 2 \). Since \( \pi_2(M) = 0 \), the universal covering of \( M \) is 2–connected. By Theorem 4.4, \( \text{cat } M \leq n - 2 \). \( \square \)

**Proposition 6.7** Let \( M = L^n_p \times L^n_q, m, n > 2 \), be given a \( ko \)–orientation for some relatively prime \( p \) and \( q \) and let \( u : M \to B\mathbb{Z}_{pq} \) be a classifying map of its universal cover. Then \( u_*([M]_{ko}) = 0 \).

**Proof** Note that \( [M]_{ko} = \pm (1 + v)([L^n_p]_{ko} \times [L^n_q]_{ko}) \), where \( v \in \tilde{ko}^0(M) \) is in the reduced \( ko \)–theory and the product is the cap product (see [18, Chapter 5, Proposition 1.9]). Therefore it suffices to show that \( u^p_*([L^n_p]_{ko}) \times u^q_*([L^n_q]_{ko}) = 0 \), where \( u^p : L^n_p \to B\mathbb{Z}_p \) and \( u^q : L^n_q \to B\mathbb{Z}_q \) are classifying maps. This equality follows from the fact that \( ko_m(B\mathbb{Z}_p) \) is \( q \)–divisible and \( ko_n(B\mathbb{Z}_q) \) is a \( q \)–torsion group. \( \square \)

**Corollary 6.8** For \( m, n > 2 \) and odd, relatively prime \( p \) and \( q \), or for \( p \) odd and \( q \) even with \( n = 2k - 1 \) for even \( k \), we have

\[
\text{cat}(L^n_p \times L^n_q) \leq m + n - 2.
\]

**Proof** In this case the lens spaces are spin [11] and we can apply Proposition 6.7. Then Theorem 6.6 and the fact that \( \pi_2(L^n_p \times L^n_q) = 0 \) imply the result. \( \square \)

For \( m = n = 3 \) we obtain a different proof of Corollary 4.2:

**Corollary 6.9** \( \text{cat}(L^3_p \times L^3_q) = 4 \) for all relatively prime \( p \) and \( q \).

### 6.3 Coinvariants

The following lemma can be found in [6, Lemma 3.3]:

**Lemma 6.10** A commutative diagram with exact rows

\[
\begin{array}{c}
A' \longrightarrow A \longrightarrow A'' \longrightarrow 0 \\
\downarrow f' \quad \downarrow f \quad \downarrow f'' \\
0 \longrightarrow C' \longrightarrow C \longrightarrow C''
\end{array}
\]

defines an exact sequence

\[
\ker(f') \to \ker(f) \to \ker(f'') \to \text{coker}(f') \to \text{coker}(f) \to \text{coker}(f'').
\]
Let \( p : E\pi \to B\pi \) be the universal covering. Thus \( p \) is the projection onto the orbit space of a free cellular \( \pi \)-action. Below we use the following abbreviations: \( \pi = \pi_1(B) \), \( B = B\pi \), \( B^k = B^{(k)} \), \( E = E\pi \) and \( E^k = E\pi^{(k)} \).

**Proposition 6.11**  \( p_* : \pi_n(E/E^{n-1}) \to \pi_n(B/B^{n-1}) \) is an epimorphism.

**Proof**  In the commutative diagram

\[
\begin{array}{ccc}
\pi_n(E^n/E^{n-1}) & \xrightarrow{p'_*} & \pi_n(B^n/B^{n-1}) \\
\downarrow & & \downarrow j_* \\
\pi_n(E/E^{n-1}) & \xrightarrow{p_*} & \pi_n(B/B^{n-1})
\end{array}
\]

the homomorphisms \( p'_* \) and \( j_* \) are epimorphisms. The former is surjective since it is induced by a retraction of a wedge of an \( n \)-sphere onto a smaller wedge; the latter is surjective due to the cellular approximation theorem. Therefore, \( p_* \) is an epimorphism. \( \square \)

Recall that \( \pi_*^s \) denotes the stable homotopy groups.

**Corollary 6.12**  For \( n \geq 5 \), the induced homomorphism

\( p'_* : \pi_n^s(E, E^{n-1}) \to \pi_n^s(B, B^{n-1}) \)

is an epimorphism.

**Proof**  This follows from the obvious natural isomorphisms

\[
\pi_n(E/E^{n-1}) = \pi_n^s(E/E^{n-1}) = \pi_n(E, E^{n-1}),
\]

\[
\pi_n(B/B^{n-1}) = \pi_n^s(B/B^{n-1}) = \pi_n(B, B^{n-1}).
\] \( \square \)

### 6.4 Proof of Lemma 6.2

For \( n \geq 5 \), the induced homomorphism

\( p_* : \pi_n(B, B^{n-2}) \to \pi_n(B/B^{n-2}) \)

factors through the group of coinvariants as

\( p_* = \tilde{p}_* \circ q_* \),

\[
\pi_n(B, B^{n-2}) \xrightarrow{q_*} \pi_n(B, B^{n-2}) \xrightarrow{\tilde{p}_*} \pi_n(B/B^{n-2}),
\]

where \( \tilde{p}_* \) is injective.
Note that, for $n \geq 5$,
\[ \pi_n(B, B^{n-2}) = \pi_n(E, E^{n-2}) = \pi^s_n(E, E^{n-2}), \quad \pi_n(B/B^{n-2}) = \pi^s_n(B, B^{n-2}), \]
and the composition
\[ \pi_n(B, B^{n-2}) \xrightarrow{q_*} \pi_n(B, B^{n-2}) \xrightarrow{\bar{p}_*} \pi_n(B/B^{n-2}) \]
coincides with
\[ \pi^s_n(E, E^{n-2}) \xrightarrow{q_*} \pi^s_n(E, E^{n-2}) \xrightarrow{\bar{p}_*} \pi^s_n(B, B^{n-2}), \]
where
\[ \bar{p}_* \circ q_* = p_* : \pi^s_n(E, E^{n-2}) \to \pi^s_n(B, B^{n-2}) \]
is the homomorphism induced by the projection $p$.

Also note that $\pi^s_n(E, E^i)$ inherits a $\pi$–module structure via the $\pi$–action.

We extract from the diagram generated by $p$ and exact $\pi^s_*$–homology sequence of the triple $(E^n, E^{n-1}, E^{n-2})$ the following two diagrams:

\[
\begin{array}{ccccccccc}
\pi^s_{n+1}(E^n, E^{n-1}) & \xrightarrow{j_{n+1}} & \pi^s_n(E^{n-1}, E^{n-2}) & \xrightarrow{\alpha} & \overline{K} & \xrightarrow{} & 0 \\
\downarrow p^1_* & & \downarrow p^2_* & & \downarrow & & \\
\pi^s_{n+1}(B^n, B^{n-1}) & \xrightarrow{j_{n+1}} & \pi^s_n(B^{n-1}, B^{n-2}) & \xrightarrow{} & K & \xrightarrow{} & 0,
\end{array}
\]

where $K$ and $\overline{K}$ are the cokernels of $j_{n+1}$ and $\tilde{j}_{n+1}$, and

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{} & \overline{H} & \xrightarrow{} & \pi^s_n(E^n, E^{n-1}) & \xrightarrow{\tilde{j}_n} & \pi^s_{n-1}(E^{n-1}, E^{n-2}) & \xrightarrow{} & 0 \\
\downarrow \beta & & \downarrow p^3_* & & \downarrow \bar{p}_* & & \downarrow p^4_* & & \downarrow & \\
0 & \xrightarrow{} & H & \xrightarrow{} & \pi^s_n(B^n, B^{n-1}) & \xrightarrow{j_n} & \pi^s_{n-1}(B^{n-1}, B^{n-2}),
\end{array}
\]

where $H$ and $\overline{H}$ are the kernels of $j_n$ and $\tilde{j}_n$. Note that the homomorphisms $p^3_*$ and $p^4_*$ are the direct sums of the augmentation homomorphism
\[ \epsilon: \mathbb{Z} \pi \to \mathbb{Z}. \]

The homomorphisms $p^1_*$ and $p^2_*$ are direct sums of the mod 2 augmentation homomorphisms
\[ \tilde{\epsilon}: \mathbb{Z}_2 \pi \to \mathbb{Z}_2. \]
Also note that $p^i_* \otimes \pi L \mathbb{Z}$ is an isomorphism for $i = 1, 2, 3, 4$. Taking the tensor product of the first diagram with $\mathbb{Z}$ over $\mathbb{Z} \pi$ would give a commutative diagram with the two left vertical arrows isomorphisms. Then, by the five lemma, $\alpha' = \alpha \otimes \pi L \mathbb{Z}$ is an isomorphism.

We argue that $\beta' = \beta \otimes \pi L \mathbb{Z}$ is a monomorphism. Note that $\ker(\beta) \subset \ker(p^3_*) = \bigoplus I(\pi)$, where $I(\pi)$ is the augmentation ideal.

**Claim** \[ \ker(\beta) \otimes \pi L \mathbb{Z} = 0. \]

**Proof** We show that $x \otimes \pi L 1 = 0$ for all $x \in \ker(\beta)$. Let $x = \sum x_i, x_i \in I(\pi)$. It suffices to show that $x_i \otimes \pi L 1 = 0$ for all $x_i$. Note that $x_i = \sum n_j (\gamma_j - e), \gamma_j \in \pi, n_j \in \mathbb{Z}$. Note that $(\gamma - e) \otimes \pi L 1 = 0$ since

\[ (\gamma - e) \otimes \pi L 1 = \gamma \otimes \pi L 1 - e \otimes \pi L 1 = \gamma (e \otimes \pi L 1) - e \otimes \pi L 1 = e \otimes \pi L \gamma(1) - e \otimes \pi L 1 = 0. \]

The tensor product with $\mathbb{Z}$ over $\mathbb{Z} \pi$ of the exact sequence

\[ \ker(\beta) \to H \to \text{im}(\beta) \to 0 \]

implies that

\[ \beta_0 = \beta \otimes \text{id}: H \otimes \pi L \mathbb{Z} = H_\pi \to \text{im}(\beta) \otimes \pi L \mathbb{Z} = \text{im}(\beta) \]

is an isomorphism. The latter equality follows from the fact that both $\text{im}(\beta)$ and $\mathbb{Z}$ are trivial $\pi$–modules. Then $\beta'$ is a monomorphism as the composition of an isomorphism $\beta_0$ and the inclusion $\text{im}(\beta) \to H$.

We consider the diagram of short exact sequences:

\[
\begin{array}{cccc}
0 & \longrightarrow & \bar{K} & \xrightarrow{\phi} \pi^s_n(E^n, E^{n-2}) & \xrightarrow{\psi} \bar{H} & \longrightarrow & 0 \\
& \alpha \downarrow & & p_* \downarrow & & \beta \downarrow & \\
0 & \longrightarrow & K & \xrightarrow{\phi} \pi^s_n(B^n, B^{n-2}) & \xrightarrow{\psi} H & \longrightarrow & 0
\end{array}
\]

Then we apply the tensor product with $\mathbb{Z}$ over $\mathbb{Z} \pi$ to this diagram to obtain the following commutative diagram with exact rows:

\[
\begin{array}{cccc}
\bar{K}_\pi & \xrightarrow{\phi} & \pi^s_n(E^n, E^{n-2})_\pi & \xrightarrow{\psi} \bar{H}_\pi & \longrightarrow & 0 \\
\alpha' \downarrow & & \tilde{p}_* \downarrow & & \beta' \downarrow & \\
0 & \longrightarrow & K & \xrightarrow{\phi} \pi^s_n(B^n, B^{n-2}) & \xrightarrow{\psi} H & \longrightarrow & 0
\end{array}
\]

Lemma 6.10 implies that $\tilde{p}_*$ is a monomorphism.
Next we consider the diagram generated by \((E, E^n, E^{n-2})\) and \((B, B^n, B^{n-2})\),

\[
\begin{array}{c}
\pi^s_{n+1}(E, E^n) \rightarrow \pi^s_n(E^n, E^{n-2}) \rightarrow \pi^s_n(E, E^{n-2}) \rightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
\pi^s_{n+1}(B, B^n) \rightarrow \pi^s_n(B^n, B^{n-2}) \rightarrow \pi^s_n(B, B^{n-2}) \rightarrow 0,
\end{array}
\]

and tensor it with \(\mathbb{Z}\) over \(\mathbb{Z}\pi\) to obtain the following commutative diagram with exact rows:

\[
\begin{array}{c}
\pi^s_{n+1}(E, E^n)_{\pi} \rightarrow \pi^s_n(E^n, E^{n-2})_{\pi} \rightarrow \pi^s_n(E, E^{n-2})_{\pi} \rightarrow 0 \\
p^* \downarrow \quad \bar{p}^* \downarrow \quad \bar{p}^* \\
\pi^s_{n+1}(B, B^n) \rightarrow \pi^s_n(B^n, B^{n-2}) \rightarrow \pi^s_n(B, B^{n-2}) \rightarrow 0.
\end{array}
\]

Since \(p^*_p\) is an epimorphism (see Corollary 6.12) and \(\bar{p}^*_p\) is a monomorphism by the monomorphism version of the five lemma, we obtain that \(\bar{p}^*_p\) is a monomorphism.

### 7 On the category of the sum

The following theorem was proven by R Newton [16] under the assumption that \(\text{cat } M, \text{ cat } N > 2\).

**Theorem 7.1** For closed manifolds \(M\) and \(N\) there is the inequality

\[
\text{cat}(M \# N) \leq \max\{\text{cat } M, \text{ cat } N\}.
\]

His proof is based on obstruction theory. Here we present a proof that works in full generality. Our proof is an application of the following:

**Theorem 7.2** (W Singhof [21, Theorem 4.4]) For any closed \(n\)–manifold \(M\) with \(\text{cat } M = k \geq 2\), there is a categorical partition \(Q_0, \ldots, Q_k\) into manifolds with boundary such that \(Q_i \cap Q_j\) is an \((n-1)\)–manifold with boundary (possibly empty) for all \(i, j\) and each \(Q_i\) admits a deformation retraction onto an \((n-k)\)–dimensional CW complex.

For \(B \subset A \subset X\), a homotopy \(H: A \times I \rightarrow X\) is called a deformation of \(A\) in \(X\) onto \(B\) if \(H|_{A \times \{0\}} = 1_A\), \(H(A \times \{1\}) = B\), and \(H(b, t) = b\) for all \(b \in B\) and \(t \in I = [0, 1]\). The following is well known:
Proposition 7.3 Let \( A \subset M \) be a subset contractible to a point in an \( m \)-manifold \( M \) and let \( B \subset A \) be a closed \( n \)-ball which admits a regular neighborhood. Then there is a deformation of \( A \) in \( M \) onto \( B \).

Proof of Theorem 7.1 Let \( n = \dim M = \dim N \). Suppose that \( \text{cat}(M \# N) \leq k \). We show that \( \text{cat}(M \# N) \leq k \). If \( k = 1 \), the statement obviously follows from the fact that \( M \) and \( N \) are homeomorphic to the sphere. We assume that \( k \geq 2 \). Let \( Q_0, \ldots, Q_k \) be a partition of \( M \) into \( M \)-contractible subsets as in Singhof’s theorem. We may assume that \( Q_0 \cap Q_1 \neq \emptyset \). Moreover, we may assume that there is a closed topological \( n \)-ball \( D \subset Q_0 \cup Q_1 \) with a collar in \( Q_0 \cup Q_1 \) and \( D_0 = D \cap Q_0 \), \( D_1 = D \cap Q_1 \) such that the triad \( (D, D_0, D_1) \) is homeomorphic to the triad \( (B, B_+, B_-) \), where \( B \) is the unit ball in \( \mathbb{R}^n \), \( B_+ = B \cap \mathbb{R}^n_+ \), \( B_- = B \cap \mathbb{R}^n_- \), and \( \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\} \) and \( \mathbb{R}^n_- = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \leq 0\} \) are the half-spaces. Additionally we may assume that the collar of \( D \) intersected with \( Q_0 \cap Q_1 \) defines a collar of \( D \cap Q_0 \cap Q_1 \) in \( Q_0 \cap Q_1 \).

Similarly, we may assume that there is a categorical partition \( V_0, \ldots, V_k \) of \( N \) as in Theorem 7.2 and a closed \( n \)-ball \( D' \) with a collar such that the triad \( (D', D'_0, D'_1) \) is homeomorphic to the triad \( (B, B_+, B_-) \), where \( D'_0 = D' \cap V_0 \), \( D'_1 = D' \cap V_1 \).

We may assume that the connected sum \( M \# N \) is realized as a subset \( M \# N = M \cup N \setminus \text{Int} D \subset M \cup h N \) for some homeomorphism \( h : D' \to D \) that preserves the triad structures.

Let \( W_0 = (Q_0 \setminus \text{Int} D) \cup (V_0 \setminus \text{Int} D') \), \( W_1 = (Q_1 \setminus \text{Int} D) \cup (V_1 \setminus \text{Int} D') \) and \( W_i = Q_i \cup V_i \) for \( i = 2, \ldots, k \). Note that \( Q_i \cap V_i = \emptyset \) for \( i \geq 2 \). By Singhof’s theorem each \( Q_i \) can be deformed to an \( (n-k) \)-dimensional subset \( S_i \) contractible in \( M \). Since \( k \geq 2 \), there is a contraction of \( S_i \) to a point in \( M \) that misses a given point. Hence, there is a contraction of \( S_i \) to a point in \( M \) that misses the ball \( D \). Thus \( Q_i \) for \( i \geq 2 \) can be contracted to a point in \( M \# N \). Similarly, for \( i \geq 2 \) the set \( V_i \) can be contracted to a point in \( M \# N \). Hence the sets \( W_i \) for \( i \geq 2 \) are categorical.

Let \( A_i = Q_i \cap \partial D \) for \( i = 0, 1 \). We show that there is a deformation of \( Q_i \setminus \text{Int} D \) in \( M \# N \) to \( A_i \). The collar of \( Q_i \cap D \) in \( Q_i \) allows us to construct a homeomorphism of \( Q_i \setminus \text{Int} D \) to \( Q_i \) homotopic to the identity. Hence \( Q_i \setminus \text{Int} D \) can be deformed onto an \( (n-k) \)-dimensional subset \( S_i \) contractible in \( M \). A contraction of \( S_i \) to a point can be chosen missing \( c_0 \in \text{Int} D \). By Proposition 7.3 there is a deformation of \( Q_i \setminus \text{Int} D \) in \( M \setminus \{c_0\} \) onto \( A_i \) fixing \( A_i \). Similarly, for \( i = 0, 1 \) there is a deformation of \( V_i \setminus \text{Int} D' \) in \( N \setminus \{c'_0\} \) to \( A_i = V_i \cap \partial D' \) fixing \( A_i \) where \( c'_0 \in \text{Int} D' \). Applying the radial projections from \( c_0 \) and \( c'_0 \) gives us such deformations in \( M \# N \). Pasting these two deformations defines a deformation of \( W_i, i = 0, 1 \), in \( M \# N \) to \( A_i \). Since the sets \( A_i \) are contractible, it follows that the sets \( W_i, i = 0, 1 \) are categorical. \( \square \)
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