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Higher Hochschild cohomology of the Lubin–Tate ring spectrum

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We construct a spectral sequence computing factorization homology of an $\mathcal{E}_d$–algebra in spectra using as an input an algebraic version of higher Hochschild homology due to Pirashvili. This induces a full computation of higher Hochschild cohomology when the algebra is étale. As an application, we compute higher Hochschild cohomology of the Lubin–Tate ring spectrum.

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This paper is devoted to higher Hochschild cohomology. Given E an $\mathcal{E}_\infty$–ring spectrum, the Hochschild cohomology of an associative algebra A in $\mathbf{Mod}_E$ with coefficients in a bimodule M is the derived homomorphisms object in the category of A–A–bimodules with source A and target M. Higher Hochschild cohomology is the generalization of this construction when A is an $\mathcal{E}_d$–algebra instead of an associative algebra. In this case, we need to replace the notion of bimodule by the notion of operadic $\mathcal{E}_d$–module and the definition becomes

$$\mathrm{HH}_{\mathcal{E}_d}(A|E,M)=\mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{Mod}_A^{\mathcal{E}_d}}(A,M),$$

where $\underline{\mathrm{Hom}}_{\mathbf{Mod}_A^{\mathcal{E}_d}}$ denotes the homomorphism object in the category of operadic $\mathcal{E}_d$–modules over A.

For practical reasons, we use a different but equivalent definition of higher Hochschild cohomology inspired by factorization homology. For A an $\mathcal{E}_d$–algebra in $\mathbf{Mod}_E$ and V a d–dimensional framed manifold, there is a spectrum $\int_V A$ called the factorization homology of A over V. This construction is functorial with respect to maps of $\mathcal{E}_d$–algebras and with respect to embeddings of framed d–manifolds. Moreover, $V\mapsto\int_V A$ is a symmetric monoidal functor. This implies that $\int_{S^{d-1}\times\mathbb{R}} A$ is an $\mathcal{E}_1$–algebra in spectra. This $\mathcal{E}_1$–algebra serves as a universal enveloping algebra for the category of operadic $\mathcal{E}_d$–modules over A. More precisely, we prove in Proposition 3.19 the identity

$$\mathrm{HH}_{\mathcal{E}_d}(A|E,M)\simeq\mathbb{R}\underline{\mathrm{Hom}}_A^{S^{d-1}\times[0,1]}(A,M),$$

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where the right-hand side is an explicit construction given by a homotopy limit of a certain functor over the poset of disks on the manifold $S^{d-1}\times[0,1]$. In Corollary 3.15, we prove an equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{S^{d-1}\times[0,1]}(A,M)\simeq\mathbb{R}\underline{\mathrm{Hom}}^{[0,1]}_{\int_{S^{d-1}\times(0,1)}A}(A,M),$$

where the right-hand side is a suitable generalization of the homomorphisms between left modules over an $\mathcal{E}_1$– (as opposed to associative) algebra. Thus, we reduce the computation of higher Hochschild cohomology to the computation of the derived homomorphisms between two left modules over an $\mathcal{E}_1$–algebra.

With this last description, we see that, in order to make explicit computations of higher Hochschild cohomology, the first step is to compute $\int_{S^{d-1}\times\mathbb{R}}A$ with its $\mathcal{E}_1$–structure. In Section 5, we construct a spectral sequence that computes the factorization homology of an $\mathcal{E}_d$–algebra over any framed manifold:

Proposition 5.4 *Let A be an $\mathcal{E}_d$–algebra in $\mathbf{Mod}_E$, let M be a framed d–manifold and let K be a homology theory with a $\mathbb{Z}/2$–equivariant Künneth isomorphism. There is a spectral sequence*

$$\mathrm{E}^2_{s,t}=\mathrm{HH}^M_{s,t}(K_*A)\Longrightarrow K_{s+t}\left(\int_M A\right).$$

Let us say a few words about the E^2–page. Given a commutative ring k, Pirashvili defines a functor $(X,A)\mapsto\mathrm{HH}^X(A)$, where X is a simplicial set, A is a commutative algebra in k–modules and $\mathrm{HH}^X(A)$ is a chain complex of k–modules. When $X=S^1$, this object is quasi-isomorphic to ordinary Hochschild homology. Our spectral sequence computing factorization homology is given by Pirashvili's higher Hochschild homology on the E^2–page.

In Section 6, we make an explicit computation in the case of the Lubin–Tate spectrum (also known as Morava E–theory) E_n. Using the étaleness of the algebra $(K_n)_*E_n$, we can prove that for any $\mathcal{E}_d$–structure on E_n that induces the correct multiplication on K_n–homology, the unit map

$$E_n\to\int_{S^{d-1}\times\mathbb{R}}E_n$$

is a K_n–homology equivalence. Using the fact that E_n is K_n–local, this implies the following theorem:

Proposition 6.4 *The map $\mathrm{HH}_{\mathcal{E}_d}(E_n)\to E_n$ is a weak equivalence.*

In Section 7, we prove an étale base-change theorem for étale algebras:

Theorem 7.9 *Let T be a commutative algebra in $\mathbf{Mod}_E$ that is (K–locally) étale as an $\mathcal{E}_d$–algebra. That is to say that the $\mathcal{E}_d$–version of the cotangent complex of E defined in Definition 2.7 of Francis* [6] *is (K–locally) contractible. Then, for any (K–local) $\mathcal{E}_d$–algebra A over T, the base-change map*

$$\mathrm{HH}_{\mathcal{E}_d}(A|E) \xrightarrow{\sim} \mathrm{HH}_{\mathcal{E}_d}(A|T)$$

is an equivalence.

In particular, this result combined with our computation implies that for any K_n–local $\mathcal{E}_d$–algebra A over E_n, the base-change map

$$\mathrm{HH}_{\mathcal{E}_d}(A|E_n) \to \mathrm{HH}_{\mathcal{E}_d}(A|\mathbb{S})$$

is a weak equivalence.

The full strength of the results proved in this paper is unnecessary in the case of E_n since it is known to be a commutative ring spectrum. However, we think that the method presented here could be used in other contexts, where one has to deal with $\mathcal{E}_d$–algebras that are not commutative.

Conventions

We denote by $\boldsymbol{S}$ the category of simplicial sets with its usual model structure. We use boldface letters to denote categories. We use calligraphic letters like $\mathcal{A}$ to denote operads. All our categories and operads are enriched in $\boldsymbol{S}$. Note that given a topological operad or category, we can turn it into a simplicially enriched operad or category by applying the functor Sing to each mapping space. We allow ourselves to do this operation implicitly.

We denote by $\mathbf{Mod}_E$ the simplicial category of modules over a commutative symmetric ring spectrum E. This category is symmetric monoidal for the relative tensor product over E. Moreover, it has two model structures: the positive model structure, denoted by $\mathbf{Mod}_E^+$, and the absolute model structure, denoted by $\mathbf{Mod}_E$. We refer the reader to Section 1 for more details. We often write $\boldsymbol{C}$ instead of $\mathbf{Mod}_E$ in the sections where the results do not depend a lot on the symmetric monoidal model category.

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1 Review of operads and factorization homology

We recall a few notations. We denote by **Fin** the category whose objects are the nonnegative integers and with

$$\mathbf{Fin}(m,n) = \mathbf{Set}(\{1,\dots,m\},\{1,\dots,n\}).$$

We abuse notation and write n for the finite set $\{1,\dots,n\}$.

To an operad $\mathcal{M}$ with one color, we can assign its PROP $\boldsymbol{M}$. This is a category whose set of objects coincides with the set of objects of **Fin** and with

$$\boldsymbol{M}(m,n) = \bigsqcup_{f\in\mathbf{Fin}(m,n)} \prod_{i\in n} \mathcal{M}(f^{-1}(i)).$$

Note that **Fin** is the PROP associated to the commutative operad. The construction of the associated PROP is a functor from operads to categories. In particular, the unique map $\mathcal{M}\to\mathcal{C}om$ induces a map $\boldsymbol{M}\to\mathbf{Fin}$.

An $\mathcal{M}$–algebra A in a simplicially enriched symmetric monoidal category $\boldsymbol{C}$ induces a symmetric monoidal simplicial functor $\boldsymbol{M}\to\boldsymbol{C}$ that we also denote by A.

Let E be a commutative ring in symmetric spectra. We denote by $\mathbf{Mod}_E^+$ the category of modules over E equipped with the positive model structure (constructed in Schwede [17, Theorem III.3.2] under the name projective positive stable model structure). The category $\mathbf{Mod}_E^+$ is a closed symmetric monoidal model category for the smash product over E (denoted by $-\otimes_E -$). It is also a simplicial model category. Moreover, the two structures are compatible in the sense that the tensor of simplicial sets and E–modules

$$-\otimes -\colon \boldsymbol{S}\times\mathbf{Mod}_E^+\to\mathbf{Mod}_E^+$$

sending (X,M) to $(E\wedge\Sigma_+^\infty X)\otimes_E M$ is a Quillen left bifunctor.

There is another model structure on $\mathbf{Mod}_E$ called the absolute model structure and that we denote by $\mathbf{Mod}_E$ (its construction can also be found in [17, Thorem III.3.2]). Its weak equivalences are the same as in the positive model structure but there are more cofibrations. In particular, the important fact for us is that the unit E is cofibrant in the absolute model structure but not in the positive model structure. The model category $\mathbf{Mod}_E$ is also a closed symmetric monoidal simplicial model category. The advantage of the positive model structure is that the smash product is much better behaved. In particular, the following theorem would be false for the absolute model structure:

Theorem 1.1 *The category $\mathbf{Mod}_E^+$ is a closed symmetric monoidal cofibrantly generated simplicial model category satisfying the following properties:*

- *For any operad $\mathcal{M}$ in $\boldsymbol{S}$, the category $\mathbf{Mod}_E^+[\mathcal{M}]$ of $\mathcal{M}$–algebras in $\mathbf{Mod}_E^+$ has a model category structure where weak equivalences and fibrations are created by the forgetful functor $\mathbf{Mod}_E^+[\mathcal{M}] \to (\mathbf{Mod}_E^+)^{\mathrm{Col}(\mathcal{M})}$.*
- *If $\alpha\colon \mathcal{M} \to \mathcal{N}$ is a is a map of operads, the adjunction*
$$\alpha_!\colon \mathbf{Mod}_E^+[\mathcal{M}] \leftrightarrows \mathbf{Mod}_E^+[\mathcal{N}] :\alpha^*$$
is a Quillen adjunction. It is, moreover, a Quillen equivalence if α is a weak equivalence.
- *The forgetful functor $\mathbf{Mod}_E^+[\mathcal{M}] \to (\mathbf{Mod}_E)^{\mathrm{Col}(\mathcal{M})}$ sends cofibrant objects to cofibrant objects.*

Proof See Theorems 3.4.1 and 3.4.3 of Pavlov and Scholbach [14]. □

Remark 1.2 All the operads that we consider in this work have a finite number of colors. The only kind of weak equivalences we will have to consider are maps that induce a bijection on the set of colors and induce weak equivalences on each space of operations.

The little disk operad

There is a topological category whose objects are d–manifolds without boundary and with space of maps between M and N given by $\mathrm{Emb}(M,N)$, the topological space of smooth embeddings with the weak C^1 topology.

Definition 1.3 A *framed d–manifold* is a pair (M,σ_M) where M is a d–manifold and σ_M is a smooth section of the $\mathrm{GL}(d)$–principal bundle $\mathrm{Fr}(TM)$.

If M and N are two framed d–manifolds, we define a space of framed embeddings, denoted by $\mathrm{Emb}_f(M,N)$ as in Definition V.8.3 of Andrade [1]. We now recall this construction. First, given a diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle v} \\ X & \xrightarrow[u]{} & Z \end{array}$$

in the category of topological spaces over a fixed topological space W, we define its homotopy pullback as in [1, Chapter V.9] to be the space of triples $(y,p,z) \in X \times Z^{[0,1]} \times Y$ such that $p(0) = u(x)$, $p(1) = v(y)$ and such that the image of p in $W^{[0,1]}$ is a constant path. It can be shown that this is indeed a model for the homotopy pullback in the model category $\mathbf{Top}_{/W}$.

Definition 1.4 Let M and N be two framed d–dimensional manifolds. The *topological space of framed embeddings from M to N*, denoted by $\mathrm{Emb}_f(M,N)$, is given by the following homotopy pullback in the category of topological spaces over $\mathrm{Map}(M,N)$:

$$\begin{tikzcd}
\mathrm{Emb}_f(M,N) \arrow[r] \arrow[d] & \mathrm{Map}(M,N) \arrow[d] \\
\mathrm{Emb}(M,N) \arrow[r] & \mathrm{Map}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM),\mathrm{Fr}(TN))
\end{tikzcd}$$

The right-hand side map is obtained as the composite

$$\mathrm{Map}(M,N) \to \mathrm{Map}_{\mathrm{GL}(d)}(M\times\mathrm{GL}(d), N\times\mathrm{GL}(d)) \cong \mathrm{Map}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM),\mathrm{Fr}(TN)),$$

where the first map is obtained by taking the product with $\mathrm{GL}(d)$ and the second map comes from the identifications $\mathrm{Fr}(TM)\cong M\times\mathrm{GL}(d)$ and $\mathrm{Fr}(TN)\cong N\times\mathrm{GL}(d)$ induced by our choice of framing on M and N.

Andrade explains in [1, Definition V.10.1] that there are well-defined composition maps

$$\mathrm{Emb}_f(M,N)\times\mathrm{Emb}_f(N,P)\to\mathrm{Emb}_f(M,P)$$

allowing the construction of a topological category $f\mathbf{Man}_d$.

We denote by D the open disk of dimension d.

Proposition 1.5 *The evaluation at the center of the disks induces a weak equivalence*

$$\mathrm{Emb}_f(D^{\sqcup p},M)\to\mathrm{Conf}(p,M).$$

Proof See [1, Proposition V.4.5] or Proposition 6.6 of Horel [10]. □

Definition 1.6 The *little d–disk operad* $\mathcal{E}_d$ is the one-color operad whose n^{th} space is

$$\mathcal{E}_d(n)=\mathrm{Emb}_f(D^{\sqcup n},D)$$

and whose composition is induced by composition of embeddings. We denote by $\boldsymbol{E}_d$ the PROP of the operad $\mathcal{E}_d$.

Remark 1.7 This model of the little d–disk operad was introduced by Andrade [1]. Using Proposition 1.5, it is not hard to show that this definition is weakly equivalent to any other definition of the little d–disk operad.

Factorization homology

From now on, until we say otherwise, we denote by $(\boldsymbol{C}^+, \otimes, \mathbb{I})$ the symmetric monoidal category $\mathbf{Mod}_E$ with its positive model structure and by $\boldsymbol{C}$ the same category equipped with the absolute model structure. We do this partly to simplify the notations but mostly to emphasize that our arguments hold in greater generality modulo a few easy modifications.

Definition 1.8 Let A be a cofibrant object of $\boldsymbol{C}^+[\mathcal{E}_d]$. We define the *factorization homology with coefficients in* A by the coend

$$\int_M A := \mathrm{Emb}_f(-, M) \otimes_{\boldsymbol{E}_d} A.$$

This functor sends weak equivalences between cofibrant algebras to weak equivalences.

Proposition 1.9 *The functor $M \mapsto \int_M A$ is a simplicial and symmetric monoidal functor from the category $f\mathbf{Man}_d$ to the category $\boldsymbol{C}$.*

Proof See [10, Definition 7.3] and the paragraph following it. $\square$

Let M be an object of $f\mathbf{Man}_d$. Let $\boldsymbol{D}(M)$ be the poset of subsets of M that are diffeomorphic to a disjoint union of disks. Let us choose for each object V of $\boldsymbol{D}(M)$ a framed diffeomorphism $V \cong D^{\sqcup n}$ for some uniquely determined n. Each inclusion $V \subset V'$ in $\boldsymbol{D}(M)$ induces a morphism $D^{\sqcup n} \to D^{\sqcup n'}$ in $\boldsymbol{E}_d$ by composing with the chosen parametrization. Therefore, each choice of parametrization induces a functor $\boldsymbol{D}(M) \to \boldsymbol{E}_d$. Up to homotopy this choice is unique, since the space of automorphisms of D in $\boldsymbol{E}_d$ is contractible.

In the following we assume that we have one of these functors $\delta\colon \boldsymbol{D}(M) \to \boldsymbol{E}_d$. We fix a cofibrant algebra $A\colon \boldsymbol{E}_d \to \boldsymbol{C}$.

Proposition 1.10 *There is a weak equivalence*

$$\mathrm{hocolim}_{V \in \boldsymbol{D}(M)} A(\delta V) \simeq \int_M A.$$

Proof See [10, Corollary 7.7]. $\square$

2 Modules over $\mathcal{E}_d$–algebras

We define the notion of an S_τ–shaped module. These are modules over $\mathcal{E}_d$–algebras that are studied in detail in Horel [11].

Definition 2.1 A *d–framing* of a closed $(d-1)$–manifold S is a trivialization of the d–dimensional bundle $TS \oplus \mathbb{R}$, where $\mathbb{R}$ is a trivial line bundle.

For M a d–manifold with boundary and m a point of ∂M, we say that a vector $u \in T_m M$ is pointing inward if it is not in $T_m \partial M$ and there is a curve $\gamma\colon [0,1) \to M$ whose derivative at 0 is u.

Definition 2.2 Let S be a closed $(d-1)$–manifold. An *S–manifold* is a d–manifold with boundary M together with the data of

- a diffeomorphism $f\colon S \to \partial M$,
- a non-vanishing section ϕ of the restriction of the vector bundle TM on ∂M which is such that $\phi(m)$ is pointing inward for any m in ∂M.

Definition 2.3 Let τ be a d–framing of S. Let $i\colon T\partial M \to TM_{|\partial M}$ be the obvious inclusion. A *framed S_τ–manifold* is an S–manifold (M, f, ϕ) with the data of a framing of TM such that the composite

$$TS \oplus \mathbb{R} \xrightarrow{Tf \oplus \mathbb{R}} T(\partial M) \oplus \mathbb{R} \xrightarrow{i \oplus \phi} TM_{|\partial M}$$

sends τ to the given framing on the right-hand side.

For $E \to M$ a d–dimensional vector bundle, we denote by $\mathrm{Fr}(E)$ the $\mathrm{GL}(d)$–bundle over M whose fiber over m is the space of bases of the vector space E_m. Note that a trivialization of E is exactly the data of a section of $\mathrm{Fr}(E)$.

For (M, f, ϕ) and (M, g, ψ) two framed S_τ–manifolds, we denote by

$$\mathrm{Map}^{S_\tau}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM), \mathrm{Fr}(TN))$$

the space of morphisms of $\mathrm{GL}(d)$–bundles whose underlying map $M \to N$ sends the boundary to the boundary and whose restriction to the boundary is fiberwise the identity (via the identification of both boundaries with S and of both tangent bundles with $TS \oplus \mathbb{R}$).

Definition 2.4 Let (M, f, ϕ) and (M, g, ψ) be two framed S_τ–manifolds. Let $\mathrm{Map}^S(M, N)$ be the topological space of maps between M and N that commute with the maps $f\colon S \to M$ and $g\colon S \to N$. Similarly, let $\mathrm{Emb}^S(M, N)$ be the topological space of embeddings that commute with the maps from S. The *topological space of framed embeddings from M to N*, denoted by $\mathrm{Emb}_f^{S_\tau}(M, N)$, is the following homotopy pullback taken in the category of topological spaces over $\mathrm{Map}^S(M, N)$:

$$\begin{tikzcd}
\mathrm{Emb}_f^{S_\tau}(M, N) \arrow[r] \arrow[d] & \mathrm{Map}^S(M, N) \arrow[d] \\
\mathrm{Emb}^S(M, N) \arrow[r] & \mathrm{Map}^{S_\tau}_{\mathrm{GL}(d)}(\mathrm{Fr}(TM), \mathrm{Fr}(TN))
\end{tikzcd}$$

Recall that a right module over an operad $\mathcal{M}$ is an $\boldsymbol{S}$–enriched functor $\boldsymbol{M}^{\mathrm{op}} \to \boldsymbol{S}$. We denote by $\mathbf{Mod}_{\mathcal{M}}$ the category of right modules over $\mathcal{M}$.

Definition 2.5 Let (S, τ) be a d–framed $(d-1)$–manifold. We define a right $\mathcal{E}_d$–module S_τ by the formula

$$S_\tau(n) = \mathrm{Emb}_f^{S_\tau}\big(D^{\sqcup n} \sqcup (S \times [0,1)), S \times [0,1)\big).$$

Recall, that there is a symmetric monoidal structure on $\mathbf{Mod}_{\mathcal{E}_d}$. If F and G are two objects of $\mathbf{Mod}_{\mathcal{E}_d}$, we can view them as contravariant functors on the groupoid Σ of finite sets and bijections. Then their tensor product is the left Kan extension of the functor

$$(n, m) \mapsto F(n) \times G(m)$$

along the functor $\Sigma^{\mathrm{op}} \times \Sigma^{\mathrm{op}} \to \Sigma^{\mathrm{op}}$ sending a pair of finite sets to their disjoint union.

Construction 2.6 We give S_τ the structure of an associative algebra in $\mathbf{Mod}_{\mathcal{E}_d}$. Let ϕ be an element of $S_\tau(m)$ and ψ be an element of $S_\tau(n)$. Let ψ^S be the restriction of ψ to $S \times [0,1)$. We define $\psi \Box \phi$ to be the element of $S_\tau(m+n)$ whose restriction to $S \times [0,1) \sqcup D^{\sqcup m}$ is $\psi^S \circ \phi$ and whose restriction to $D^{\sqcup n}$ is $\psi_{|D^{\sqcup n}}$.

The operation

$$-\Box-\colon S_\tau(n) \times S_\tau(m) \to S_\tau(n+m)$$

makes S_τ into an associative algebra in the symmetric monoidal category of right $\mathcal{E}_d$–modules.

Definition 2.7 The colored operad $S_\tau\mathcal{M}od$ has two colors a and m. Its only non-empty spaces of operations are

$$S_\tau\mathcal{M}od(\underbrace{a,\dots,a}_{n};a)=\mathcal{E}_d(n) \quad \text{and} \quad S_\tau\mathcal{M}od(\underbrace{a,\dots,a}_{n},m;m)=S_\tau(n).$$

The composition involves the operad structure on $\mathcal{E}_d$, the right $\mathcal{E}_d$–module structure on S_τ and the associative algebra structure on S_τ.

Again, $(\boldsymbol{C}^+,\otimes,\mathbb{I})$ denotes the symmetric monoidal model category $\mathbf{Mod}_E^+$ and $\boldsymbol{C}$ denotes the same category but with its absolute model structure. An algebra in $\boldsymbol{C}$ over $S_\tau\mathcal{M}od$ consists of a pair of objects (A,M) where A is an $\mathcal{E}_d$–algebra and M is equipped with an action of A of the form

$$\mathrm{Emb}_f^{S_\tau}(S\times[0,1)\sqcup D^{\sqcup n},S\times[0,1))\otimes M\otimes A^{\otimes n}\to M.$$

Definition 2.8 Let A be an $\mathcal{E}_d$–algebra in $\boldsymbol{C}$. We define the *category of S_τ–shaped modules* over A, denoted by $S_\tau\mathbf{Mod}_A$, to be the category whose objects are $S_\tau\mathcal{M}od$–algebras whose restriction to the color a is the $\mathcal{E}_d$–algebra A and whose morphisms are morphisms of $S_\tau\mathcal{M}od$–algebra inducing the identity map on A.

Remark 2.9 More generally, for any operad $\mathcal{O}$, and any right module P over $\mathcal{O}$, the above construction gives a notion of modules over $\mathcal{O}$–algebras. This construction is studied in detail in [11, Section 3].

Proposition 2.10 *Let A be an $\mathcal{E}_d$–algebra in $\boldsymbol{C}$. The coend*

$$U_A^{S_\tau}=S_\tau\otimes_{\boldsymbol{E}_d}A$$

inherits an associative algebra structure from the one on S_τ and there is an equivalence of categories between the category of left modules over $U_A^{S_\tau}$ and the category $S_\tau\mathbf{Mod}_A$.

Proof See [11, Proposition 3.9]. □

This proposition lets us put a model structure on $S_\tau\mathbf{Mod}_A$ in which the weak equivalences and fibrations are the maps that are sent to weak equivalences and fibrations by the forgetful functor $S_\tau\mathbf{Mod}_A\to\boldsymbol{C}$. Moreover, since $\boldsymbol{C}$ is a closed symmetric model category, the model category $S_\tau\mathbf{Mod}_A$ is a $\boldsymbol{C}$–enriched model category.

Example 2.11 The unit sphere inclusion $S^{d-1}\to\mathbb{R}^d$ has a trivial normal bundle. This induces a d–framing on S^{d-1}, which we denote by κ. On the other hand we have the notion of an operadic module over an $\mathcal{E}_d$–algebra A. This is an object M of $\boldsymbol{C}$ with multiplication maps

$$\mathcal{E}_d(n+1) \to \mathrm{Map}_{\boldsymbol{C}}(A^{\otimes n} \otimes M, M)$$

that are compatible with the $\mathcal{E}_d$–structure on A in a suitable way (see Definition 1.1 of Berger and Moerdijk [5]). We denote the category of such modules by $\mathbf{Mod}_A^{\mathcal{E}_d}$. The two notions are related by the following theorem:

Theorem 2.12 *For a cofibrant $\mathcal{E}_d$–algebra A, there is a Quillen equivalence*

$$S_\kappa \mathbf{Mod}_A \leftrightarrows \mathbf{Mod}_A^{\mathcal{E}_d}.$$

Moreover, the right adjoint of this equivalence commutes with the forgetful functor of both categories to $\boldsymbol{C}$.

Proof This is done in [11, Proposition 4.12]. The second claim follows from the fact that this equivalence is induced by a weak equivalence of associative algebras

$$U_A^{S_\kappa^{d-1}} \to U_A^{\mathcal{E}_d[1]},$$

where $U_A^{\mathcal{E}_d[1]}$ is the enveloping algebra of $\mathbf{Mod}_A^{\mathcal{E}_d}$ (ie it is an associative algebra such that there is an equivalence of categories $\mathbf{Mod}_{U_A^{\mathcal{E}_d[1]}} \simeq \mathbf{Mod}_A^{\mathcal{E}_d}$). □

Let S be a closed $(d-1)$–manifold and let τ be a d–framing of S. There is a map $S_\tau \to \mathrm{Emb}_f(-, S \times (0,1))$ sending an embedding $S \times [0,1) \sqcup D^{\sqcup n} \to S \times [0,1)$ to its restriction to $D^{\sqcup n}$.

Proposition 2.13 *The map $S_\tau \to \mathrm{Emb}_f(-, S \times (0,1))$ is a weak equivalence of right $\mathcal{E}_d$–modules.*

Proof This follows from [11, Proposition A.3] □

Corollary 2.14 *For a cofibrant $\mathcal{E}_d$–algebra A, there is a weak equivalence*

$$U_A^{S_\tau} \xrightarrow{\sim} \int_{S\times(0,1)} A.$$

Proof By the previous proposition, there is a weak equivalence of right $\mathcal{E}_d$–modules

$$S_\tau \xrightarrow{\sim} \mathrm{Emb}_f(-, S \times (0,1)).$$

We prove in [10, Proposition 2.8] that, for A cofibrant, the functor $- \otimes_{\boldsymbol{E}_d} A$ preserves all weak equivalences of right $\mathcal{E}_d$–modules. □

If A is an $\mathcal{E}_d$–algebra, then the object $\int_{S\times(0,1)} A$ is an $\mathcal{E}_1$–algebra. Indeed, any embedding $(0,1)^{\sqcup n} \to (0,1)$ induces an embedding $(0,1)\times S^{\sqcup n} \to (0,1)\times S$ by taking the product with S. Applying $\int_{\_} A$ to this last embedding, we get maps

$$\mathrm{Emb}^f((0,1)^{\sqcup n},(0,1)) \to \mathrm{Map}_{\boldsymbol{C}}\left(\left(\int_{S\times(0,1)} A\right)^{\otimes n}, \int_{S\times(0,1)} A\right).$$

We would like to say that the weak equivalence of the previous proposition is an equivalence of $\mathcal{E}_1$–algebras, but it is not one on the nose. However, we show in the next proposition that this is a map of S_τ–shaped modules.

Proposition 2.15 *There is an S_τ–shaped module structure on $\int_{S\times(0,1)} A$ such that the map*

$$U_A^{S_\tau} \to \int_{S\times(0,1)} A$$

is a weak equivalence of S_τ–shaped modules.

Proof Let us describe the S_τ–shaped module structure on $\int_{S\times(0,1)} A$. Let ϕ be a point in $\mathrm{Emb}_f^{S_\tau}(S\times[0,1)\sqcup D^{\sqcup n}, S\times[0,1))$. By forgetting about the boundary, ϕ defines a point in $\mathrm{Emb}_f(S\times(0,1)\sqcup D^{\sqcup n}, S\times(0,1))$ that induces a map

$$\left(\int_{S\times(0,1)} A\right)\otimes A^{\otimes n} \to \int_{S\times(0,1)} A.$$

Letting ϕ vary, this gives $\int_{S\times(0,1)} A$ the structure of an S_τ–shaped module. Moreover, the map $U_A^{S_\tau} \to \int_{S\times(0,1)} A$ is a map of S_τ–shaped modules. Since we already know that it is a weak equivalence, we are done. □

3 Higher Hochschild cohomology

In this section, we construct a geometric model for higher Hochschild cohomology. We still denote by $(\boldsymbol{C},\otimes,\mathbb{I})$ the symmetric monoidal model category $\mathbf{Mod}_E$. Our construction remains valid in other contexts (spaces, chain complexes, simplicial modules) modulo a few obvious modifications. We denote by $\underline{\mathrm{Hom}}$ the inner Hom in the category $\boldsymbol{C}$. This functor is uniquely determined by the fact that we have a natural isomorphism

$$\boldsymbol{C}(X\otimes Y, Z) \cong \boldsymbol{C}(X, \underline{\mathrm{Hom}}(Y,Z)).$$

For any associative R algebra in $\boldsymbol{C}$, the $\boldsymbol{C}$–enrichment of $\boldsymbol{C}$ induces to a $\boldsymbol{C}$–enrichment of $\mathbf{Mod}_R$. We denote by $\underline{\mathrm{Hom}}_R$ the homomorphisms object in $\mathbf{Mod}_R$.

Let A be an $\mathcal{E}_d$–algebra that we assume to be cofibrant. Our goal is to construct a functor

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{S\times[0,1]}\colon S_\tau\mathbf{Mod}_A^{\mathrm{op}}\times S_\tau\mathbf{Mod}_A\to \boldsymbol{C}$$

that is weakly equivalent to $\mathbb{R}\underline{\mathrm{Hom}}_{S_\tau\mathbf{Mod}_A}(-,-):=\mathbb{R}\underline{\mathrm{Hom}}_{U_A^{S_\tau}}(-,-)$ but which is closer to the factorization homology philosophy.

For (S,τ) a d–framed $(d-1)$–manifold, we denote by $-\tau$ the d–framing on S obtained by pulling back τ along the isomorphism of the vector bundle $TS\oplus\mathbb{R}$ that is the identity on the first summand and multiplication by -1 on the second summand.

In particular, $S\times[0,1)$ is naturally an S_τ–manifold and $S\times(0,1]$ is an $S_{-\tau}$–manifold.

Definition 3.1 We denote by $\mathbf{Disk}_d^{S_\tau\sqcup S_{-\tau}}$ the topological category whose objects are the $S_\tau\sqcup S_{-\tau}$–manifolds of the form $S\times[0,1)\sqcup D^{\sqcup n}\sqcup S\times(0,1]$ with n in $\mathbb{Z}_{\geq 0}$ and whose morphisms are given by the spaces $\mathrm{Emb}_f^{S_\tau\sqcup S_{-\tau}}$.

Construction 3.2 We define a functor

$$\mathcal{F}(M,A,N)\colon (\mathbf{Disk}_d^{S_\tau\sqcup S_{-\tau}})^{\mathrm{op}}\to\boldsymbol{C}.$$

Its value on $S\times[0,1)\sqcup D^{\sqcup n}\sqcup S\times(-1,0]$ is $\underline{\mathrm{Hom}}(M\otimes A^{\otimes n},N)$.

Notice that any map in $(S_\tau\sqcup S_{-\tau})\mathbf{Mod}$ can be decomposed as a disjoint union of embeddings of the following three types:

- $S\times[0,1)\sqcup D^{\sqcup k}\to S\times[0,1)$.
- $D^{\sqcup l}\to D$ (where l is possibly zero).
- $D^{\sqcup m}\sqcup S\times(0,1]\to S\times(0,1]$.

Let ϕ be an embedding $S\times[0,1)\sqcup D^{\sqcup n}\sqcup S\times(0,1]\to S\times[0,1)\sqcup D^{\sqcup m}\sqcup S\times(0,1]$ and let

$$\phi=\phi_+\sqcup\psi_1\sqcup\cdots\sqcup\psi_r\sqcup\phi_-$$

be its decomposition with ϕ_+ of the first type, ϕ_- of the third type and ψ_i of the second type for each i. We need to extract from this data a map

$$\underline{\mathrm{Hom}}(M\otimes A^{\otimes m},N)\to\underline{\mathrm{Hom}}(M\otimes A^{\otimes n},N).$$

The action of ϕ_+ and of the ψ_i are constructed in an obvious way from the $\mathcal{E}_d$–structure of A and the S_τ–shaped module structure on M. The only non-trivial part is the action of ϕ_-. We can hence assume that $\phi=\mathrm{id}_{S\times[0,1)\sqcup D^{\sqcup p}}\sqcup\phi_-$, where ϕ_- is an embedding $D^{\sqcup n}\sqcup S\times(0,1]\to S\times(0,1]$. We want to construct

$$\underline{\mathrm{Hom}}(M\otimes A^{\otimes p},N)\to\underline{\mathrm{Hom}}(M\otimes A^{\otimes p}\otimes A^{\otimes n},N).$$

First, observe that there is a diffeomorphism $S\times[0,1)\to S\times(0,1]$ sending (s,t) to $(s,1-t)$. This diffeomorphism sends the framing τ on $S\times[0,1)$ to the framing $-\tau$ on $S\times(0,1]$. Similarly, reflexion about the hyperplane $x_d=0$ induces a diffeomorphism $D\to D$. Conjugating by this diffeomorphism, the embedding ϕ_- induces an embedding

$$\widetilde{\phi}_-\colon S\times[0,1)\sqcup D^{\sqcup n}\to S\times[0,1).$$

In fact, this construction induces a homeomorphism

$$\mathrm{Emb}_f^{S_{-\tau}}(S\times(0,1]\sqcup D^{\sqcup n},S\times(0,1])\to\mathrm{Emb}_f^{S_\tau}(S\times[0,1)\sqcup D^{\sqcup n},S\times[0,1)).$$

Now, notice that $\underline{\mathrm{Hom}}(M\otimes A^{\otimes p},N)$ has the structure of an S_τ–shaped A module induced from the one on N. Thus, the map $\widetilde{\phi}_-$ induces a map

$$\underline{\mathrm{Hom}}(M\otimes A^{\otimes p},N)\otimes A^{\otimes n}\to\underline{\mathrm{Hom}}(M\otimes A^{\otimes p},N).$$

This map is adjoint to a map

$$\underline{\mathrm{Hom}}(M\otimes A^{\otimes p},N)\to\underline{\mathrm{Hom}}(M\otimes A^{\otimes p}\otimes A^{\otimes n},N),$$

which we define to be the action of ϕ.

Remark 3.3 In order to be homotopically meaningful, we need a derived version of $\mathcal{F}(M,A,N)$. We claim that the homotopy type of $\mathcal{F}(M,A,N)$ only depends on the homotopy type of M, A and N as long as A is a cofibrant $\mathcal{E}_d$–algebra, M is a cofibrant object of $S_\tau\mathbf{Mod}_A$ and N is a fibrant object of $S_\tau\mathbf{Mod}_A$. Indeed, these conditions imply that

- the object M is cofibrant in $\boldsymbol{C}$, because the forgetful functor $S_\tau\mathbf{Mod}_A\to\boldsymbol{C}$ preserves cofibrations,
- A is cofibrant in $\boldsymbol{C}$,
- M is cofibrant in $\boldsymbol{C}$,
- N is fibrant in $\boldsymbol{C}$.

This implies that for all k, $\underline{\mathrm{Hom}}(M\otimes A^{\otimes k},N)\simeq\mathbb{R}\underline{\mathrm{Hom}}(M\otimes A^{\otimes k},N)$.

We denote by $\underline{\mathrm{hom}}$ the functor $\boldsymbol{S}^{\mathrm{op}}\times\boldsymbol{C}\to\boldsymbol{C}$ sending (X,C) to $\underline{\mathrm{Hom}}(X\otimes\mathbb{I},C)$. Equivalently, this is the cotensor of $\boldsymbol{C}$ with $\boldsymbol{S}$ induced from the simplicial structure. For $\boldsymbol{A}$ a small simplicial category, F a functor from $\boldsymbol{A}$ to $\boldsymbol{S}$ and G a functor from $\boldsymbol{A}$ to $\boldsymbol{C}$, we denote by $\underline{\mathrm{hom}}_{\boldsymbol{A}}(F,G)$ the end

$$\int_{\boldsymbol{A}}\underline{\mathrm{hom}}(F(-),G(-)).$$

We denote by $\mathbb{R}\underline{\mathrm{hom}}_{\boldsymbol{A}}(F,G)$ the derived functor obtained by taking a cofibrant replacement of the source and a fibrant replacement of the target in the projective model structure of functors on $\boldsymbol{A}$.

Definition 3.4 We define $\mathbb{R}\underline{\mathrm{Hom}}_A^{S\times[0,1]}(M,N)$ to be the homotopy end

$$\mathbb{R}\underline{\mathrm{hom}}_{(\mathbf{Disk}_d^{S_\tau\sqcup S_{-\tau}})^{\mathrm{op}}}\big(\mathrm{Emb}_f^{S_\tau\sqcup S_{-\tau}}(-,S\times[0,1]),\mathcal{F}(QM,A,RN)\big),$$

where $QM\to M$ is a cofibrant replacement in $S_\tau\mathbf{Mod}_A$ and $N\to RN$ is a fibrant replacement.

We can now formulate the main theorem of this section.

Theorem 3.5 *There is a weak equivalence*

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{S\times[0,1]}(M,N)\simeq\mathbb{R}\underline{\mathrm{Hom}}_{S_\tau\mathbf{Mod}_A}(M,N).$$

The rest of this section is devoted to the proof of this theorem. The reader willing to accept this result can safely skip the proof and move directly to the last subsection of this section.

Case of $\mathcal{E}_1$–algebras

The one-point space is a 0–manifold. This manifold has two 1–framings, which we call the negative and positive framing. By definition, a 1–framing of the point is the data of a basis of $\mathbb{R}$ as a $\mathbb{R}$–vector space. The positive framing is the one given by 1 and the negative framing is the one given by -1. Thus, by Definition 2.5, we get two right modules over $\mathcal{E}_1$. We denote by $\mathcal{R}$ the one corresponding to the negative framing and $\mathcal{L}$ the one corresponding to the positive framing.

Definition 3.6 A *left module* over an $\mathcal{E}_1$–algebra A is an object of the category $\mathcal{L}\mathbf{Mod}_A$. Similarly, a *right module* over A is an object of $\mathcal{R}\mathbf{Mod}_A$.

More explicitly, an object of $\mathcal{L}\mathbf{Mod}_A$ is an object of $\boldsymbol{C}$, M together with multiplication maps

$$A^{\otimes n}\otimes M\to M$$

for each embedding $[0,1)\sqcup(0,1)^{\sqcup n}\to[0,1)$ These maps are moreover supposed to satisfy a unitality and associativity condition.

We denote by $\mathbf{Disk}_1^{-+}$ the one-dimensional version of the category $\mathbf{Disk}^{S_\tau\sqcup S_{-\tau}}$ defined in Definition 3.1. As a particular case of Definition 3.4, given a cofibrant $\mathcal{E}_1$–algebra A and two left modules M and N, we can define $\underline{\mathrm{Hom}}_A^{[0,1]}(M,N)$ and this is given by natural transformations between contravariants functors on $\mathbf{Disk}_1^{-+}$.

Definition 3.7 The *category of non-commutative intervals*, denoted by $\mathbf{Ass}^{-+}$, is a skeleton of the category whose objects are finite sets containing $\{-,+\}$ and whose morphisms are maps of finite sets f preserving $-$ and $+$ together with the extra data of a linear ordering of each fiber which is such that $-$ (resp. $+$) is the smallest (resp. largest) element in the fiber over $-$ (resp. $+$).

Note that the functor π_0, sending a disjoint union of intervals to the set of connected components, is an equivalence of topological categories from $\mathbf{Disk}_1^{-+}$ to $\mathbf{Ass}^{-+}$. In fact, we could have defined $\mathbf{Ass}^{-+}$ as the homotopy category of $\mathbf{Disk}_1^{-+}$.

Let A be an associative algebra and M and N be left modules over it. We define $F(M,A,N)$ to be the obvious functor $(\mathbf{Ass}^{-+})^{\mathrm{op}} \to \boldsymbol{C}$ sending $\{-,1,\dots,n,+\}$ to $\underline{\mathrm{Hom}}(A^{\otimes n} \otimes M, N)$. The functoriality is defined analogously to Construction 3.2.

Recall that Δ^{op} can be described as a skeleton of the category whose objects are linearly ordered sets with at least two elements and morphisms are order-preserving morphisms that preserve the minimal and maximal element.

With this description, there is an obvious functor $\Delta^{\mathrm{op}} \to \mathbf{Ass}^{-+}$ sending a totally ordered set with minimal element $-$ and maximal element $+$ to the underlying finite set and sending an order-preserving map to the underlying map with the data of the induced linear ordering of each fiber.

Recall that given a triple (M,A,N) consisting of an associative algebra A and two left modules M and N, we can form the cobar construction $C^\bullet(M,A,N)$. It is a cosimplicial object of $\boldsymbol{C}$ whose value at $[n]$ is $\underline{\mathrm{Hom}}(A^{\otimes n} \otimes M, N)$. It is classical that if A and M are cofibrant and N is fibrant, then $C^\bullet(M,A,N)$ is Reedy fibrant and its totalization is a model for the derived Hom $\mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{Mod}_A}(M,N)$.

Proposition 3.8 *Let A be an associative algebra and let M and N be left modules over it. The composition of $F(M,A,N)$ with the functor $\Delta \to (\mathbf{Ass}^{-+})^{\mathrm{op}}$ is the cobar construction $C^\bullet(M,A,N)$*

Proof This is a straightforward computation. □

We denote by $P\colon (\mathbf{Ass}^{-+})^{\mathrm{op}} \to \boldsymbol{S}$ the left Kan extension of the cosimplicial space that is levelwise a point along the map $\Delta \to (\mathbf{Ass}^{-+})^{\mathrm{op}}$. Concretely, P sends a finite set with two distinguished elements $-$ and $+$ to the set of linear orderings of that set whose smallest element is $-$ and largest element is $+$, seen as a discrete space.

Corollary 3.9 *Let A be a cofibrant associative algebra and let M and N be left modules over it. Then*

$$\mathbb{R}\underline{\mathrm{Hom}}_A(M,N) \simeq \mathbb{R}\underline{\mathrm{hom}}_{\mathbf{Ass}^{-+}}(P, F(M,A,N)).$$

Proof Assume that M is cofibrant and N is fibrant. If they are not, we take an appropriate replacement. The left-hand side is

$$\mathrm{Tot}\big([n] \to C^n(M, A, N) = \underline{\mathrm{Hom}}(M \otimes A^{\otimes n}, N)\big).$$

According to the cofibrancy/fibrancy assumption, this cosimplicial functor is Reedy fibrant, therefore the totalization coincides with the homotopy limit. Hence we have

$$\mathbb{R}\underline{\mathrm{Hom}}_A(M, N) \simeq \mathbb{R}\underline{\mathrm{hom}}_\Delta(*, C^\bullet(M, A, N)) \simeq \mathbb{R}\underline{\mathrm{hom}}_{\mathbf{Ass}^{-+}}(P, F(M, A, N)). \quad \square$$

Proposition 3.10 *Let A be a cofibrant associative algebra and let M and N be left modules over it. Then there is a weak equivalence*

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{[0,1]}(M, N) \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{Hom}}_A(M, N).$$

Proof Again, we can assume that M is cofibrant and N is fibrant. By the previous corollary, the right-hand side is the derived end

$$\mathbb{R}\underline{\mathrm{hom}}_{\mathbf{Ass}^{-+}}(P, F(M, A, N)),$$

which can be computed as the totalization of the Reedy fibrant cosimplicial object

$$C^\bullet(P, \mathbf{Ass}^{-+}, F(M, A, N)).$$

Similarly, the left-hand side is the totalization of the Reedy fibrant cosimplicial object

$$C^\bullet\big(\mathrm{Emb}^{-+}(-, [0,1]), \mathbf{Disk}^{-+}, \mathcal{F}(M, A, N)\big).$$

There is an obvious map of cosimplicial objects

$$C^\bullet\big(\mathrm{Emb}^{-+}(-, [0,1]), \mathbf{Disk}^{-+}, \mathcal{F}(M, A, N)\big) \to C^\bullet(P, \mathbf{Ass}^{-+}, F(M, A, N)),$$

which is degreewise a weak equivalence. Therefore, there is a weak equivalence between the totalizations

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{[0,1]}(M, N) \xrightarrow{\sim} \mathbb{R}\underline{\mathrm{Hom}}_A(M, N). \quad \square$$

If A is an $\mathcal{E}_1$–algebra, it can be seen as an object of $\mathcal{L}\mathbf{Mod}_A$ as follows. The map

$$A \otimes A^{\otimes n} \to A,$$

corresponding to an embedding

$$\phi\colon [0,1) \sqcup (0,1)^{\sqcup n} \to [0,1)$$

is defined to be the multiplication map $A^{\otimes n+1} \to A$ corresponding to the restriction of ϕ to its interior.

We denote by (A, A^m) the $\mathcal{LM}od$–algebra consisting of A acting on itself in the above way.

Corollary 3.11 *Let A be a cofibrant $\mathcal{E}_1$–algebra and N a left module. Then*

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{[0,1]}(A^m, N) \simeq N.$$

Proof The pair (A, N) forms an algebra over $\mathcal{LM}od$. The operad $\mathcal{LM}od$ is weakly equivalent to the operad $L\mathcal{M}od$ parameterizing strictly associative algebras and left modules. This implies that we can find a pair (A', N') consisting of an associative algebra and a left module together with a weak equivalence of $\mathcal{LM}od$–algebra

$$(A, N) \xrightarrow{\sim} (A', N').$$

Using the previous proposition, we have

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{[0,1]}(A^m, N) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{A'}(A', N') \simeq N' \simeq N. \qquad \square$$

Let $\boldsymbol{D}([0,1])$ be the poset of open sets of $[0,1]$ that are diffeomorphic to

$$[0,1) \sqcup (0,1)^{\sqcup n} \sqcup (0,1]$$

for some n. Let us choose a functor

$$\delta\colon \boldsymbol{D}([0,1]) \to \mathbf{Disk}^{-+}$$

by picking a diffeomorphism of each object of $\boldsymbol{D}([0,1])$ with an object of $\mathbf{Disk}^{-+}$.

Proposition 3.12 *There is a weak equivalence*

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{[0,1]}(M, N) \simeq \mathrm{holim}_{U \in \boldsymbol{D}([0,1])^{\mathrm{op}}} \mathcal{F}(M, A, N)(\delta U).$$

Proof We can assume that M is cofibrant and N is fibrant. First, by [10, Lemma 7.8], we have a weak equivalence

$$\mathrm{Emb}_f^{S^0}(-, [0,1]) \simeq \mathrm{hocolim}_{U \in \boldsymbol{D}([0,1])} \mathrm{Emb}_f^{S^0}(-, U).$$

It follows that there is an equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{[0,1]}(M, N) \simeq \mathrm{holim}_{U \in \boldsymbol{D}([0,1])^{\mathrm{op}}} \mathbb{R}\underline{\mathrm{Hom}}_A^{\delta U}(M, N).$$

Then we notice, using the Yoneda lemma, that $U \mapsto \mathbb{R}\underline{\mathrm{Hom}}_A^{\delta U}(M, N)$ is weakly equivalent as a functor to $U \mapsto \mathcal{F}(M, A, N)(\delta U)$. $\square$

Comparison with the actual homomorphisms

In this subsection, A is a cofibrant $\mathcal{E}_d$–algebra. We will compare $\mathbb{R}\underline{\mathrm{Hom}}_A^{S\times[0,1]}(M,N)$ with $\mathbb{R}\underline{\mathrm{Hom}}_{S_\tau\mathbf{Mod}_A}(M,N)$.

Construction 3.13 Let M be an S_τ–shaped module over an $\mathcal{E}_d$–algebra A. We give M the structure of a left module over the $\mathcal{E}_1$–algebra $\int_{S\times(0,1)} A$. Let

$$(0,1)^{\sqcup n}\sqcup[0,1)\to[0,1)$$

be a framed embedding. We can take the product with S and get an embedding in $f\mathbf{Man}_d^{S_\tau}$,

$$(S\times(0,1))^{\sqcup n}\sqcup S\times[0,1)\to S\times[0,1).$$

Evaluating $\int_{\_}(M,A)$ over this embedding, we find a map

$$\left(\int_{S\times(0,1)} A\right)^{\otimes n}\otimes M\to M.$$

All these maps give M the structure of a left $\left(\int_{S\times(0,1)} A\right)$–module.

Proposition 3.14 *Let M and N be two S_τ–shaped modules over A. There is a weak equivalence*

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{S\times[0,1]}(M,N)\simeq\mathrm{holim}_{U\in\boldsymbol{D}([0,1])^{\mathrm{op}}}\mathcal{F}\left(M,\int_{S\times(0,1)}A,N\right)(S\times U),$$

where M and N are given the structure of left $\left(\int_{S\times(0,1)} A\right)$–modules using the previous construction.

Proof This is a variant of Proposition 3.12. We first prove that

$$\mathbb{R}\underline{\mathrm{Hom}}_A^{S\times[0,1]}(M,N)\simeq\mathrm{holim}_{U\in\boldsymbol{D}([0,1])^{\mathrm{op}}}\mathbb{R}\underline{\mathrm{Hom}}_A^{S\times U}(M,N).$$

This follows from the equivalence

$$\mathrm{hocolim}_{U\in\boldsymbol{D}([0,1])}\mathrm{Emb}_f^{S_\tau\sqcup S_{-\tau}}(-,S\times U)\simeq\mathrm{Emb}_f^{S_\tau\sqcup S_{-\tau}}(-,S\times[0,1])$$

in the category $\mathrm{Fun}((\mathbf{Disk}^{S_\tau\sqcup S_{-\tau}})^{\mathrm{op}},\boldsymbol{S})$. Then, using the Yoneda lemma, we see that the functor

$$U\mapsto\mathbb{R}\underline{\mathrm{Hom}}_A^{S\times U}(M,N)$$

is weakly equivalent to

$$U\mapsto\mathcal{F}\left(M,\int_{S\times(0,1)}A,N\right)(U).\qquad\square$$

Corollary 3.15 *There is a weak equivalence*

$$\mathbb{R}\underline{\mathrm{Hom}}^{[0,1]}_{\int_{S\times(0,1)}A}(M,N)\simeq\mathbb{R}\underline{\mathrm{Hom}}^{S\times[0,1]}_{A}(M,N).$$

Proof Both sides are weakly equivalent to

$$\mathrm{holim}_{U\in\boldsymbol{D}([0,1])^{\mathrm{op}}}\mathcal{F}\left(M,\int_{S\times(0,1)}A,N\right)(S\times U),$$

one side by the previous proposition and the other by Proposition 3.12. □

Proof of Theorem 3.5 We fix A and a fibrant S_τ–shaped module N and we let M vary. We want to compare two contravariant functors from $S_\tau\mathbf{Mod}_A$ to $\boldsymbol{C}$. Both functors preserve weak equivalences between cofibrant objects and turn homotopy colimits into homotopy limits; therefore, it suffices to check that both functors are weakly equivalent on the generator of the category of S_τ–shaped modules. In other words, it is enough to prove that

$$\mathbb{R}\underline{\mathrm{Hom}}^{S\times[0,1]}_{A}(U_A^{S_\tau},N)\simeq\mathbb{R}\underline{\mathrm{Hom}}_{S_\tau\mathbf{Mod}_A}(U_A^{S_\tau},N).$$

The right-hand side of the above equation can be rewritten as $\mathbb{R}\underline{\mathrm{Hom}}_{U_A^{S_\tau}}(U_A^{S_\tau},N)$, which is trivially weakly equivalent to N.

We know from Proposition 2.15 that, as S_τ–shaped modules, there is a weak equivalence

$$U_A^{S_\tau}\to\int_{S\times(0,1)}A;$$

therefore, it is enough to prove that there is a weak equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}^{S\times[0,1]}_{A}\left(\int_{S\times(0,1)}A,N\right)\simeq N.$$

According to Corollary 3.15, it is equivalent to prove that there is a weak equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}^{[0,1]}_{\int_{S\times[0,1]}A}\left(\int_{S\times(0,1)}A,N\right)\simeq N.$$

This follows directly from Corollary 3.11. □

A generalization

We can generalize Definition 3.4. In [11, Construction 6.9], given the data of a framed bordism W between d–framed manifolds of dimension $d-1$, S_σ and T_τ, we construct a left Quillen functor

$$P_W\colon S_\sigma\mathbf{Mod}_A\to T_\tau\mathbf{Mod}_A.$$

The best way to think of this functor is as follows. Factorization homology of A over W is a $U_A^{S_\sigma}$-$U_A^{T_\tau}$–bimodule. Thus, tensoring with it induces a left Quillen functor

$$S_\sigma\mathbf{Mod}_A \to T_\tau\mathbf{Mod}_A.$$

Construction 3.16 Let W be bordism from S_σ to T_τ. Let M be an S_σ–shaped module over A and let N be a T_τ–shaped module. We can construct a functor $\mathcal{F}(M,A,N)$ as in Construction 3.2 from $(\mathbf{Disk}^{S_\sigma\sqcup T_{-\tau}})^{\mathrm{op}}$ to $\boldsymbol{C}$ that sends $S\times[0,1)\sqcup D^{\sqcup n}\sqcup T\times(0,1]$ to $\underline{\mathrm{Hom}}(A^{\otimes n}\otimes M, N)$. We define $\mathbb{R}\underline{\mathrm{Hom}}_A^W(M,N)$ to be the homotopy end

$$\mathbb{R}\underline{\mathrm{Hom}}_A^W(M,N)=\mathbb{R}\underline{\mathrm{hom}}_{(\mathbf{Disk}^{S_\sigma\sqcup T_{-\tau}})^{\mathrm{op}}}(\mathrm{Emb}_f^{S_\sigma\sqcup T_{-\tau}}(-,W),\mathcal{F}(M,A,N)).$$

This construction has the following nice interpretation:

Theorem 3.17 *Let W be a bordism from S_σ to T_τ. There is a weak equivalence*

$$\mathbb{R}\underline{\mathrm{Hom}}_A^W(M,N)\simeq\mathbb{R}\underline{\mathrm{Hom}}_A^{T\times[0,1]}(\mathbb{L}P_W(M),N).$$

Proof The proof is very analogous to the proof of Theorem 3.5. $\square$

We can now introduce our definition of higher Hochschild cohomology.

Definition 3.18 Let A be a cofibrant $\mathcal{E}_d$–algebra in $\boldsymbol{C}$ and let M be an S_κ^{d-1}–shaped module over A. The *$\mathcal{E}_d$–Hochschild cohomology* of A with coefficients in M is defined as

$$\mathrm{HH}_{\mathcal{E}_d}(A,M)=\mathbb{R}\underline{\mathrm{Hom}}_{S_\kappa^{d-1}\mathbf{Mod}_A}(A,M).$$

We now compare this definition to a more traditional definition. Let A be a cofibrant $\mathcal{E}_d$–algebra and let M be an object of $\mathbf{Mod}_A^{\mathcal{E}_d}$. By Theorem 2.12, we can see M as an S_κ^{d-1}–shaped module over A.

Proposition 3.19 *For A a cofibrant $\mathcal{E}_d$–algebra and M an object of $\mathbf{Mod}_A^{\mathcal{E}_d}$, we have a weak equivalence*

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{Mod}_A^{\mathcal{E}_d}}(A,M)\simeq\mathbb{R}\underline{\mathrm{Hom}}_{S_\kappa\mathbf{Mod}_A}(A,M).$$

Proof By Theorem 2.12, we have a Quillen equivalence

$$u_!\colon S_\kappa^{d-1}\mathbf{Mod}_A\leftrightarrows\mathbf{Mod}_A^{\mathcal{E}_d}:u^*.$$

Therefore, we have a weak equivalence $\mathbb{L}u_!u^*A\to A$ in $\mathbf{Mod}_A^{\mathcal{E}_d}$. This gives us a weak equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{Mod}_A^{\mathcal{E}_d}}(A,M)\to\mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{Mod}_A^{\mathcal{E}_d}}(\mathbb{L}u_!u^*A,M)\simeq\mathbb{R}\underline{\mathrm{Hom}}_{S_\kappa\mathbf{Mod}_A}(u^*A,u^*M).\ \square$$

Thus, our definition of $\mathrm{HH}_{\mathcal{E}_d}(A,M)$ coincides with the more traditional definition that we gave in the first paragraph of the introduction. According to Theorem 3.5, we have a weak equivalence $\mathrm{HH}_{\mathcal{E}_d}(A,M)\simeq\mathbb{R}\underline{\mathrm{Hom}}_A^{S^{d-1}\times[0,1]}(A,M)$. As usual, we write $\mathrm{HH}_{\mathcal{E}_d}(A)$ for $\mathrm{HH}_{\mathcal{E}_d}(A,A)$.

Proposition 3.20 *Let $\bar{D}$ be the closed unit ball in $\mathbb{R}^d$ seen as a bordism from the empty manifold to S^{d-1}_κ. There is a weak equivalence*

$$\mathrm{HH}_{\mathcal{E}_d}(A,M)\simeq\mathbb{R}\underline{\mathrm{Hom}}_A^{\bar{D}}(\mathbb{I},M).$$

Proof $\mathbb{I}$, the unit of $\boldsymbol{C}$, is an object of $\varnothing\mathbf{Mod}_A$ (note that $\varnothing\mathbf{Mod}_A$ is equivalent to the category $\boldsymbol{C}$) and $\mathbb{L}P_{\bar{D}}(\mathbb{I})$ is weakly equivalent to A. Then it suffices to apply Theorem 3.17. □

This has the following surprising consequence:

Corollary 3.21 *The group $\mathrm{Diff}_f^{S^{d-1}}(\bar{D})$ acts on $\mathrm{HH}_{\mathcal{E}_d}(A,M)$.*

Remark 3.22 The group $\mathrm{Diff}_f^{S^{d-1}}(\bar{D})$ is weakly equivalent to the homotopy fiber of the inclusion

$$\mathrm{Diff}^{S^{d-1}}(\bar{D})\to\mathrm{Imm}^{S^{d-1}}(\bar{D},\bar{D}),$$

where the S^{d-1} superscript means that we are restricting to the diffeomorphisms or immersions which are the identity outside on $S^{d-1}=\partial\bar{D}$. In fact, the action of $\mathrm{Diff}_f^{S^{d-1}}(\bar{D})$ factors through the inverse limit of the embedding calculus tower computing this group. Since we are in the codimension-0 case, the embedding calculus tower should not be expected to converge. Even if it does not converge, it is an interesting mathematical object. In particular, using the work of Arone and Turchin [3] and Willwacher [19, Theorem 1.2], we get an action of the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$ on the $\mathcal{E}_2$–Hochschild cohomology of an algebra over $H\mathbb{Q}$. We hope to study this action further in future work.

4 Higher Hochschild homology

Let R be a commutative graded ring. We denote by $\mathbf{Ch}_{\geq 0}(R)$ the category of nonnegatively graded chain complexes. This has a model category structure in which the weak equivalences are the quasi-isomorphisms, the cofibrations are the degreewise monomorphisms with degreewise projective cokernel and the fibrations are the epimorphisms. In particular, any object is fibrant and the cofibrant objects are the degreewise projective chain complexes.

The model category $\mathbf{Ch}_{\geq 0}(R)$ is cofibrantly generated. Thus, we have the projective model category structure on functors $\mathbf{Fin} \to \mathbf{Ch}_{\geq 0}(R)$, in which weak equivalences and fibrations are objectwise. The following definition is due to Pirashvili [15, Introduction, page 151] (see also Definition 2 of Ginot, Tradler and Zeinalian [8]).

Definition 4.1 Let A be a degreewise projective commutative algebra in $\mathbf{Ch}_{\geq 0}(R)$ and let X be a simplicial set. We denote by $\mathrm{HH}^X(A|R)$ the homotopy coend

$$\mathrm{Map}(-, X) \otimes^{\mathbb{L}}_{\mathbf{Fin}} A.$$

Remark 4.2 In practice, we can take $\mathrm{HH}^X(A|R)$ to be the realization of the simplicial object

$$\mathrm{B}_\bullet(\mathrm{Map}(-, X), \mathbf{Fin}, A).$$

This construction preserves quasi-isomorphism between degreewise projective commutative algebras. In the following, $\mathrm{HH}^X(A|R)$ will be taken to be this explicit model.

This construction also sends a weak equivalence $X \xrightarrow{\sim} Y$ to a weak equivalence

$$\mathrm{HH}^X(A|R) \xrightarrow{\sim} \mathrm{HH}^Y(A|R).$$

Proposition 4.3 *Let A be a degreewise projective commutative algebra in $\mathbf{Ch}_{\geq 0}(R)$; then the functor $X \mapsto \mathrm{HH}^X(A|R)$ lifts to a functor from $\boldsymbol{S}$ to the category of commutative algebras in $\mathbf{Ch}_{\geq 0}(R)$.*

Proof The category $\mathrm{Fun}(\mathbf{Fin}^{\mathrm{op}}, \boldsymbol{S})$ equipped with the convolution tensor product is a symmetric monoidal model category (see [13, Proposition 2.2.15]). It is easy to check that there is an isomorphism

$$\mathrm{Map}(-, X) \otimes \mathrm{Map}(-, Y) \cong \mathrm{Map}(-, X \sqcup Y).$$

Moreover, since $A\colon \mathbf{Fin} \to \mathbf{Ch}_{\geq 0}(R)$ is a commutative algebra for the convolution tensor product, the object $\mathrm{HH}^X(A|R)$ is a symmetric monoidal functor in the X variable. To conclude, it suffices to observe that any simplicial set is a commutative monoid with respect to the disjoint union in a unique way and that this structure is preserved by maps in $\boldsymbol{S}$. Therefore, $\mathrm{HH}^X(A|R)$ is a commutative algebra functorially in X. $\square$

Proposition 4.4 *Let A be a degreewise projective commutative algebra in $\mathbf{Ch}_{\geq 0}(R)$. Let*

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & P \end{array}$$

be a homotopy pushout in the category of simplicial sets. Then there is a weak equivalence

$$\mathrm{HH}^P(A|R) \simeq \big|\mathrm{B}_\bullet\big(\mathrm{HH}^Y(A|R), \mathrm{HH}^X(A|R), \mathrm{HH}^Z(A|R)\big)\big|.$$

Proof First, notice that the maps $X \to Z$ and $X \to Y$ induce commutative algebra maps $\mathrm{HH}^X(A|R) \to \mathrm{HH}^Y(A|R)$ and $\mathrm{HH}^X(A|R) \to \mathrm{HH}^Z(A|R)$. In particular, $\mathrm{HH}^Z(A|R)$ and $\mathrm{HH}^Y(A|R)$ are modules over $\mathrm{HH}^X(A|R)$. This explains the bar construction in the statement of the proposition.

We can explicitly construct P as the realization of the simplicial space

$$[p] \mapsto Y \sqcup X^{\sqcup p} \sqcup Z,$$

where the face maps are induced by the codiagonals and the maps $X \to Y$ and $X \to Z$ and the degeneracies are induced by the maps from the empty simplicial set to X, Y and Z.

For a finite set S, and any simplicial space $U_\bullet$, there is an isomorphism

$$|U_\bullet^S| \cong |U_\bullet|^S.$$

Therefore, there is a weak equivalence of functors on **Fin**,

$$\mathrm{Map}(-, P) \simeq \big|\mathrm{B}_\bullet\big(\mathrm{Map}(-, Y), \mathrm{Map}(-, X), \mathrm{Map}(-, Z)\big)\big|,$$

where the bar construction on the right-hand side is in the category Fun(**Fin**, $\boldsymbol{S}$) with the convolution tensor product.

We can form the following bisimplicial object in $\mathbf{Ch}_{\geq 0}(R)$:

$$\mathrm{B}_\bullet\big(\mathrm{B}_\bullet(\mathrm{Map}(-, Y), \mathrm{Map}(-, X), \mathrm{Map}(-, Z)), \mathbf{Fin}, A\big).$$

By the previous observation, if we realize first with respect to the inner simplicial variable and then the outer one, we find something equivalent to $\mathrm{HH}^P(A|R)$. If we first realize with respect to the outer variable, we find

$$\mathrm{B}_\bullet\big(\mathrm{HH}^Y(A|R), \mathrm{HH}^X(A|R), \mathrm{HH}^Z(A|R)\big).$$

The two realizations are equivalent. This concludes the proof. □

Corollary 4.5 *Let A be a degreewise projective commutative algebra in $\mathbf{Ch}_{\geq 0}(R)$, then $\mathrm{HH}^{S^1}(A)$ is quasi-isomorphic to the Hochschild chains on A.*

Proof We can write S^1 as the homotopy pushout of:

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & \mathrm{pt} \\ \mathrm{pt} & \end{tikzcd}$$

If S is a finite set $\mathrm{HH}^S(A) = A^{\otimes S}$ with the obvious commutative algebra structure. In particular, the previous theorem gives

$$\mathrm{HH}^{S^1}(A) \simeq |\mathrm{B}_\bullet(A, A \otimes A, A)|.$$

Since $A = A^{\mathrm{op}}$, the right-hand side is quasi-isomorphic to $A \otimes^{\mathbb{L}}_{A \otimes A^{\mathrm{op}}} A$. □

5 The spectral sequence

We construct a spectral sequence converging to factorization homology. Its E^2–page is identified with higher Hochschild homology. For R a $\mathbb{Z}$–graded ring, we denote by $\mathbf{GrMod}_R$ the category of $\mathbb{Z}$–graded left R–modules.

Definition 5.1 Let $\boldsymbol{I}$ be a small discrete category and let $F\colon \boldsymbol{I} \to \mathbf{GrMod}_R$ be a functor landing in the category of graded modules over R. We define *the homology of $\boldsymbol{I}$ with coefficients in F* to be the homology groups of the homotopy colimit of F seen as a functor concentrated in homological degree 0 from $\boldsymbol{I}$ to $\mathbf{Ch}_{\geq 0}(\mathbf{GrMod}_R)$. We write $\mathrm{H}^R_*(\boldsymbol{I}, F)$ for the homology of $\boldsymbol{I}$ with coefficients in F.

Note that since we consider graded modules, the chain complexes are graded chain complexes. This means that each homology group is graded. We denote by $\mathrm{H}^R_{s,t}(\boldsymbol{I}, F)$ the degree-t part of the s^{th} homology group. The index s lives in $\mathbb{Z}_{\geq 0}$ and the index t lives in $\mathbb{Z}$. There is an explicit model for this homology. We construct the simplicial object of $\mathbf{GrMod}_R$ whose p–simplices are

$$\mathrm{B}_p(R, \boldsymbol{I}, F) = \bigoplus_{i_0 \to \cdots \to i_p} F(i_0).$$

We can form the normalized chain complex associated to this simplicial object in $\mathbf{GrMod}_R$ and we get a non-negatively graded chain complex in $\mathbf{GrMod}_R$. Its homology groups are the homology groups of $\boldsymbol{I}$ with coefficients in F.

Recall that if E is an associative algebra in symmetric spectra, then $E_* = \pi_*(E)$ is an associative ring in graded abelian groups and, if M is a left E–module, then $\pi_*(M)$ is an object of $\mathbf{GrMod}_{E_*}$.

Proposition 5.2 *Let $F\colon \boldsymbol{I} \to \mathbf{Mod}_E$ be a functor from a discrete category to the category of left modules over an associative algebra in symmetric spectra E. There is a spectral sequence of E_*–modules*

$$\mathrm{E}^2_{s,t} \cong \mathrm{H}^{E_*}_{s,t}(\boldsymbol{I}, \pi_*(F)) \Longrightarrow \pi_{s+t}(\mathrm{hocolim}_{\boldsymbol{I}} F).$$

Proof The homotopy colimit can be computed by taking an objectwise cofibrant replacement of F and then the realization of the bar construction

$$\mathrm{hocolim}_{\boldsymbol{I}} F \simeq |\mathrm{B}_\bullet(*, \boldsymbol{I}, QF(-))|.$$

We can then use the standard spectral sequence associated to a simplicial object □

Now assume that E is commutative. Let A be an $\mathcal{E}_d$–algebra in $\mathbf{Mod}_E$. Let M be a framed d–manifold and let $\boldsymbol{D}(M)$ be the poset of open sets of M that are diffeomorphic to a disjoint union of copies of D. We know from Proposition 1.10 that the factorization homology of A over M can be computed as the homotopy colimit of the composition

$$\boldsymbol{D}(M) \xrightarrow{\delta} \boldsymbol{E}_d \xrightarrow{A} \mathbf{Mod}_E.$$

Hence, we are in a situation where we can apply the previous proposition. We get a spectral sequence of E_*–modules

$$\mathrm{H}^{E_*}_{s,t}(\boldsymbol{D}(M), \pi_*(A \circ \delta)) \Longrightarrow \pi_{s+t}\left(\int_M A\right).$$

We want to exploit the fact that A is a monoidal functor to obtain a more explicit model for the left-hand side in some cases.

From now on, K denotes an associative algebra in spectra whose associated homology theory has a $\mathbb{Z}/2$–equivariant Künneth isomorphism. That is, we assume that the obvious map

$$K_*(X) \otimes_{K_*} K_*(Y) \to K_*(X \wedge Y)$$

is an isomorphism of functors of the pair (X, Y) that is equivariant with respect to the obvious $\mathbb{Z}/2$–action on both sides. Examples of such ring spectra are the Eilenberg–MacLane spectra Hk for any field k and $K(n)$, the Morava K–theory of height n at odd primes.

We just smash the simplicial object computing $\mathrm{hocolim}_{\boldsymbol{D}(M)} A(\delta -)$ with K in each degree and take the associated spectral sequence. We then get a spectral sequence of $K_*(E)$–modules

$$\mathrm{H}^{K_*E}_*(\boldsymbol{D}(M), K_*(A \circ \delta)) \Longrightarrow K_*\left(\int_M A\right).$$

Now we want to identify $K_*(A \circ \delta)$ as a functor on $\boldsymbol{D}(M)$.

Proposition 5.3 *If $d = 1$, $K_*(A)$ is an associative algebra in K_*E–modules. If $d > 1$, $K_*(A)$ is a commutative algebra in the category of K_*E–modules.*

Proof An $\mathcal{E}_1$ algebra in $\mathbf{Mod}_E$ is in particular an associative algebra in $\mathrm{Ho}(\mathbf{Mod}_E)$ and an $\mathcal{E}_d$–algebra with $d > 1$ is a commutative algebra in $\mathrm{Ho}(\mathbf{Mod}_E)$. The result then follows from the fact that the functor

$$K_*\colon \mathrm{Ho}(\mathbf{Mod}_E) \to \mathbf{GrMod}_{K_*E}$$

is symmetric monoidal. □

Now, we focus on the case where $d > 1$. We have an obvious functor $\alpha\colon \boldsymbol{D}(M) \to \mathbf{Fin}$ that sends a configuration of disks on M to its set of connected components. In particular, we can consider the functor

$$\boldsymbol{D}(M) \xrightarrow{\alpha} \mathbf{Fin} \xrightarrow{K_*(A)} \mathbf{GrMod}_{K_*E},$$

where the second map is induced by the commutative algebra structure on $K_*(A)$ that we have constructed in the previous proposition. It is clear that this functor coincides with the functor obtained by applying K_* to the composite

$$\boldsymbol{D}(M) \xrightarrow{\delta} \boldsymbol{E}_d \xrightarrow{A} \mathbf{Mod}_E.$$

From this, we deduce the following proposition:

Proposition 5.4 *There is an isomorphism*

$$\mathrm{H}_*^{K_*E}(\boldsymbol{D}(M), K_*(A \circ \delta)) \cong \mathrm{HH}_*^{\mathrm{Sing}(M)}(K_*A|K_*E).$$

In particular, there is a spectral sequence

$$\mathrm{HH}_s^{\mathrm{Sing}(M)}(K_*A|K_*E)_t \Longrightarrow K_{s+t}\Big(\int_M A\Big).$$

Proof The first claim immediately implies the second.

In order to prove the first claim, we observe that we have weak equivalences

$$* \otimes^{\mathbb{L}}_{\boldsymbol{D}(M)} K_*(A \circ \delta) \simeq \mathbb{L}\alpha_! * \otimes^{\mathbb{L}}_{\mathbf{Fin}} K_*(A),$$

where $*$ denotes the constant functor with value $*$.

We have $\mathbb{L}\alpha_! * (S) = \mathrm{hocolim}_{U \in \boldsymbol{D}(M)} \mathbf{Fin}(S, \pi_0(U))$. By [11, Proposition 5.3], this contravariant functor on $\mathbf{Fin}$ coincides up to weak equivalences with $S \mapsto \mathrm{Sing}(M)^S$. □

Remark 5.5 The spectral sequence above still exists if K does not have a Künneth isomorphism as long as K_*A is flat as a K_*–module. We leave the details to the interested reader.

Multiplicative structure

Let us start with the general homotopy colimit spectral sequence.

Proposition 5.6 *Let $F\colon \boldsymbol{I} \to \mathbf{Mod}_E$ and $G\colon \boldsymbol{J} \to \mathbf{Mod}_E$ be functors. We have the equivalence*

$$\mathrm{hocolim}_{\boldsymbol{I}\times\boldsymbol{J}} F \otimes_E G \simeq (\mathrm{hocolim}_{\boldsymbol{I}} F) \otimes_E (\mathrm{hocolim}_{\boldsymbol{J}} G).$$

Proof Assume F and G are objectwise cofibrant. The right-hand side is the homotopy colimit over $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ of

$$\mathrm{B}_\bullet(*, \boldsymbol{I}, F) \otimes \mathrm{B}_\bullet(*, \boldsymbol{J}, G).$$

The diagonal of this bisimplicial object is exactly

$$\mathrm{B}_\bullet(*, \boldsymbol{I}\times\boldsymbol{J}, F \otimes_E G).$$

Since $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is homotopy cofinal, we are done. □

We denote by $\mathrm{E}^r_{**}(\boldsymbol{I}, F)$ and $\mathrm{E}^r_{**}(\boldsymbol{J}, G)$ the spectral sequence computing the homotopy colimit of $F\colon \boldsymbol{I} \to \mathbf{Mod}_E$ and $G\colon J \to \mathbf{Mod}_E$. Then there is a pairing of spectral sequences of E_*–modules

$$\mathrm{E}^r_{**}(\boldsymbol{I}, F) \otimes_{E_*} \mathrm{E}^r_{**}(\boldsymbol{J}, G) \to \mathrm{E}^r_{**}(\boldsymbol{I}\times\boldsymbol{J}, F \otimes_E G).$$

Let us specialize to the case of factorization homology. We consider an $\mathcal{E}_d$–algebra A in $\mathbf{Mod}_E$, a homology theory with $\mathbb{Z}/2$–equivariant Künneth isomorphism K and a framed manifold M of dimension d. We denote by $\mathrm{E}^r_{**}(M, A, K)$ the spectral sequence of the previous section.

Proposition 5.7 *Let M and N be two framed d–manifolds. There is a pairing of spectral sequences*

$$\mathrm{E}^r_{**}(M, A, K) \otimes_{K_*E} \mathrm{E}^r_{**}(N, A, K) \to \mathrm{E}^r_{**}(M \sqcup N, A, K).$$

Proof We observe that $\boldsymbol{D}(M \sqcup N) \cong \boldsymbol{D}(M) \times \boldsymbol{D}(N)$ and that $A \otimes_E A$ as a functor on $\boldsymbol{D}(M) \times \boldsymbol{D}(N)$ is equivalent to A as a functor on $\boldsymbol{D}(M \sqcup N)$. Then the pairing of spectral sequences of the previous paragraph reduces exactly to the desired result. □

The topological category $f\mathbf{Man}_d$ of framed d–manifolds and framed embeddings has a symmetric monoidal structure given by the disjoint union operation. This induces a symmetric monoidal structure on the ordinary category $\pi_0 f\mathbf{Man}_d$ which is the category obtained by applying π_0 to each mapping space of $f\mathbf{Man}_d$. We say that a framed d–manifold is an associative algebra up to isotopy if it has the structure of an associative algebra in $\pi_0 f\mathbf{Man}_d$. Examples of manifolds with such a structure are obtained by starting with a d–framed $(d-1)$–manifold N and then constructing the framed d–manifold $M = N \times (-1, 1)$. This manifold M has the structure of an $\mathcal{E}_1$–algebra in $f\mathbf{Man}_d$. In particular, it is an associative algebra up to isotopy.

There is a similar story in $\boldsymbol{S}$. This category has a symmetric monoidal structure with respect to the coproduct $\sqcup$. Any object has a unique commutative algebra structure given by the codiagonal $X \sqcup X \to X$. In particular, if M is an associative algebra up to isotopy, this structure reduces to the canonical multiplication on $\mathrm{Sing}(M)$.

Proposition 5.8 *Let M be a framed manifold of dimension $d \geq 2$ with the structure of an associative algebra up to isotopy. Let A be an $\mathcal{E}_d$–algebra. The spectral sequence $\mathrm{E}^r_{**}(M, A, K)$ has a commutative multiplicative structure converging to the associative algebra structure on $K_* \int_M A$. On the E^2–page, the multiplication is induced by the unique commutative algebra structure on $\mathrm{Sing}(M)$ in the category $(\boldsymbol{S}, \sqcup)$. Moreover, this structure is functorial with respect to embeddings of d–manifolds $M \to M'$ preserving the multiplication up to isotopy.*

Proof According to the previous proposition there is a multiplicative structure on the spectral sequence converging to the associative algebra structure on $K_* \int_M A$.

It is easy to see that the multiplication on the E^2–page is what is stated in the proposition. Since $\mathrm{Sing}(M)$ is commutative, the multiplication on the E^2–page is commutative. The homology of a commutative differential graded algebra is a commutative algebra, therefore the multiplication is commutative on each page.

The functoriality is clear. □

Now we want to construct an edge homomorphism. Let S be a $(d-1)$–manifold with a d–framing τ. Let ϕ be a framed embedding of $\mathbb{R}^{d-1} \times \mathbb{R}$ into $S \times \mathbb{R}$ commuting with the projection to $\mathbb{R}$. Applying factorization homology, we get a map of $\mathcal{E}_1$–algebras

$$u_\phi\colon A \cong \int_{\mathbb{R}^{d-1}\times\mathbb{R}} A \to \int_{S\times\mathbb{R}} A.$$

On the other hand, for any point x of $S \times \mathbb{R}$ we get a morphism of commutative algebras over K_*E,

$$u_x\colon K_*(A) \cong \mathrm{HH}^{\mathrm{pt}}(K_*A|K_*E) \to \mathrm{HH}^{\mathrm{Sing}(S)}(K_*A|K_*E).$$

Proposition 5.9 *For any framed embedding* $\phi\colon \mathbb{R}^{d-1}\times\mathbb{R}\to S\times\mathbb{R}$, *there is an edge homomorphism*

$$K_*A \to \mathrm{E}^r_{0,*}(S\times\mathbb{R}, A, K).$$

On the E^2*–page it is identified with the* K_*E*–algebra homomorphism*

$$u_{\phi(0,0)}\colon K_*(A)\to \mathrm{HH}^{\mathrm{pt}}(K_*A|K_*E)\to \mathrm{HH}^{\mathrm{Sing}(S)}(K_*A|K_*E)$$

and it converges to the K_*E*–algebra homomorphism*

$$K_*(u_\phi)\colon K_*A\to K_*\int_{N\times\mathbb{R}} A.$$

Proof The spectral sequence computing $K_*\int_{\mathbb{R}^{d-1}\times\mathbb{R}} A$ has its E^2–page K_*A concentrated on the 0^{th} column. For degree reasons, it degenerates. Then the result follows directly from the functoriality of the spectral sequence applied to the map ϕ. □

Note that the edge homomorphism only depends on the connected component of the image of ϕ. In the case of the sphere $S^{d-1}\times\mathbb{R}$ with the framing κ, we have a stronger result:

Lemma 5.10 *For any framed embedding* $\phi\colon \mathbb{R}^{d-1}\times\mathbb{R}\to (S^{d-1}\times\mathbb{R})_\kappa$ *commuting with the projection to* $\mathbb{R}$, *the map*

$$u_\phi\colon A\to \int_{S^{d-1}\times\mathbb{R}} A$$

has a section in the homotopy category of $\mathbf{Mod}_E$.

Proof There is an embedding

$$S^{d-1}\times\mathbb{R}\to\mathbb{R}^d$$

sending (θ,x) to $e^x\theta$. This embedding preserves the framing up to isotopy. Moreover, since $\mathrm{Emb}_f(\mathbb{R}^d,\mathbb{R}^d)$ is contractible, the composite

$$\mathbb{R}^d\xrightarrow{\phi} S^{d-1}\times\mathbb{R}\to\mathbb{R}^d$$

is isotopic to the identity. We can apply $\int_{\_} A$ to this sequence of morphisms of framed manifolds and we obtain the desired section. □

Although we will not need it, this has the following immediate corollary:

Corollary 5.11 *The image of the edge homomorphism in* $\mathrm{E}^r_{**}((S^{d-1}\times\mathbb{R})_\kappa, A, K)$ *consists of permanent cycles.*

Remark 5.12 Our geometric description of higher Hochschild cohomology (Definition 3.4) can be used to construct a similar spectral sequence calculating $K_*\mathrm{HH}_{\mathcal{E}_d}(A)$ whose E_2–page is a cohomological version of the higher Hochschild cohomology defined by Ginot [7]. However, this spectral sequence does not always converge.

6 Computations

Proposition 6.1 *Let A_* be a degreewise projective commutative graded algebra over a commutative graded ring R_*. Assume that A_* is a filtered colimit of étale algebras over R_*. Then, for all $d \geq 1$, the unit map*

$$A_* \to \mathrm{HH}^{S^d}(A_*|R_*)$$

is a quasi-isomorphism of commutative R_*–algebras.

Proof We proceed by induction on d. For $d = 1$, $\mathrm{HH}^{S^1}(A_*|R_*)$ is quasi-isomorphic to the ordinary Hochschild homology $\mathrm{HH}(A_*|R_*)$ by Corollary 4.5. If A_* is étale, the result is well known (see for instance [18, Étale descent theorem, page 368]). If A_* is a filtered colimit of étale algebras, the result follows from the fact that Hochschild homology commutes with filtered colimits.

Now assume that $A_* \to \mathrm{HH}^{S^{d-1}}(A_*|R_*)$ is a quasi-isomorphism of commutative algebras. The sphere S^d is part of the following homotopy pushout diagram:

$$\begin{tikzcd} S^{d-1} \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & S^d \end{tikzcd}$$

Applying Proposition 4.4, we find

$$\mathrm{HH}^{S^d}(A_*|R) \simeq |\mathrm{B}_\bullet(A_*, \mathrm{HH}^{S^{d-1}}(A_*|R_*), A_*)|.$$

The quasi-isomorphism $A_* \to \mathrm{HH}^{S^{d-1}}(A_*|R_*)$ induces a degreewise quasi-isomorphism between Reedy cofibrant simplicial objects:

$$\mathrm{B}_\bullet(A_*, A_*, A_*) \to \mathrm{B}_\bullet(A_*, \mathrm{HH}^{S^{d-1}}(A_*|R_*), A_*).$$

This induces a quasi-isomorphism between their realizations,

$$A_* \simeq \mathrm{HH}^{S^d}(A_*|R_*). \qquad \square$$

Corollary 6.2 *Let A be an $\mathcal{E}_d$–algebra in $\mathbf{C}$ such that $K_*(A)$ is a filtered colimits of étale algebras over K_*; then the unit map*

$$A \to \int_{S^{d-1}\times\mathbb{R}} A$$

is a K–local equivalence.

Proof The K–homology of this map can be computed as the edge homomorphism of the spectral sequence $\mathrm{E}^2(S^{d-1}\times\mathbb{R}, A, K)$. By the previous proposition, the edge homomorphism is an isomorphism on the E^2–page. Therefore, the spectral sequence collapses at the E^2–page for degree reasons. □

Let us fix a prime p. We denote by E_n the Lubin–Tate ring spectrum of height n at p and by K_n the 2–periodic Morava K–theory of height n. Recall that

$$(E_n)_* \cong \mathbb{W}(\mathbb{F}_{p^n})[\![u_1,\dots,u_{n-1}]\!][u^{\pm 1}], \qquad |u_i| = 0,\ |u| = 2,$$
$$(K_n)_* \cong \mathbb{F}_{p^n}[u^{\pm 1}] = (E_n)_*/(p, u_1, \dots, u_{n-1}).$$

The spectrum E_n is known to have a unique $\mathcal{E}_1$–structure inducing the correct multiplication on homotopy groups (this is a theorem of Hopkins and Miller; see [16]) and a unique commutative structure (see [9, Corollary 7.6]). As far as we know, there is no published proof that the space of $\mathcal{E}_d$–structure for $d \geq 2$ is contractible, although evidence suggests that this is the case. The ring spectrum K_n has a $\mathbb{Z}/2$–equivariant Künneth isomorphism if p is odd. If $p = 2$, the equivariance is not satisfied in general but it is true if we restrict $(K_n)_*$ to spectra whose K_n–homology is concentrated in even degree, like E_n. Our argument works at $p = 2$ modulo this minor modification.

Corollary 6.3 *For any positive integer n and any $\mathcal{E}_d$–algebra structure on E_n inducing the correct multiplication on homotopy groups, the unit map*

$$E_n \to \int_{S^{d-1}\times\mathbb{R}} E_n$$

induces an isomorphism in K_n–homology.

Proof By [12, Corollary 4.10], for any such $\mathcal{E}_d$–structure on E we have

$$(K_n)_*(E_n) \cong C(\Gamma, (K_n)_*).$$

Here the right-hand side denotes the set of continuous maps $\Gamma \to (K_n)_*$, where Γ is the Morava stabilizer group with its profinite topology and $(K_n)_*$ is given the discrete topology. By definition of a profinite group, the group Γ is an inverse limit

$\Gamma = \lim_U \Gamma/U$ taken over the filtered poset of open finite index subgroups U of Γ. Thus, we have

$$C(\Gamma, (K_n)_*) = \operatorname{colim}_U C(\Gamma/U, (K_n)_*).$$

This expresses $(K_n)_* E_n$ as a filtered colimit of étale algebras over $(K_n)_*$. Using Corollary 6.2, we get the desired result. $\square$

Proposition 6.4 *With the same notations, the map* $\mathrm{HH}_{\mathcal{E}_d}(E_n) \to E_n$ *is an equivalence.*

Proof We have

$$\mathrm{HH}_{\mathcal{E}_d}(E_n) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{\int_{S^{d-1}\times\mathbb{R}} E_n}(E_n, E_n).$$

This can be computed as the end

$$\underline{\hom}_{\mathbf{Disk}^{-+}}\left(\mathrm{Emb}^{S^0}(-,[0,1]), \mathcal{F}\left(E_n, \int_{S^{d-1}\times\mathbb{R}} E_n, E_n\right)\right).$$

The spectrum E_n is $K(n)$–local; therefore, $\underline{\mathrm{Hom}}(-, E_n)$ sends $K(n)$–equivalences to equivalences. This implies that

$$\mathcal{F}\left(E_n, \int_{S^{d-1}\times\mathbb{R}} E_n, E_n\right) \simeq \mathcal{F}(E_n, E_n, E_n).$$

Therefore, we have

$$\mathrm{HH}_{\mathcal{E}_d}(E_n) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{E_n}(E_n, E_n). \qquad \square$$

We can prove a variant of the previous result. Let $E(n) = BP/(v_{n+1}, v_{n+2}, \dots)[v_n^{-1}]$ be the Johnson–Wilson spectrum and let $K(n)$ be the v_n periodic Morava K–theory with $K(n)_* = E(n)/(p, v_1, \dots, v_{n-1}) = \mathbb{F}_p[v_n^{\pm 1}]$. Let $\widehat{E}(n)$ be $L_{K(n)}E(n)$.

Proposition 6.5 *For any* $\mathcal{E}_d$*–algebra structure on* $\widehat{E}(n)$ *inducing the correct multiplication on homotopy groups, the action map*

$$\mathrm{HH}_{\mathcal{E}_d}(\widehat{E}(n)) \to \widehat{E}(n)$$

is a weak equivalence.

Proof The proof is exactly the same once we know that $K(n)_*\widehat{E}(n)$ is the commutative ring

$$K(n)_*\widehat{E}(n) = C(\Gamma, K(n)_*),$$

where Γ is again the Morava stabilizer group. $\square$

7 Étale base change for Hochschild cohomology

In this section we put the previous result in the wider context of derived algebraic geometry over $\mathcal{E}_d$–algebras. This section is inspired by Francis [6].

We let $(\boldsymbol{C}, \otimes, \mathbb{I})$ denote the category $\mathbf{Mod}_E$ but the arguments hold more generally. Note however that we need $\boldsymbol{C}$ to be stable in this section.

There is a "polar coordinate" embedding $S^{d-1} \times (0,1) \to D$ sending (θ, r) to $e^{r-1}\theta$.

Definition 7.1 Let A be an $\mathcal{E}_d$–algebra in $\boldsymbol{C}$. The cotangent complex L_A of A is defined to be the n–fold desuspension of the cofiber of the map

$$\int_{S^{d-1}\times\mathbb{R}} A \to \int_{\mathbb{R}^{\mathbb{d}}} A \cong A$$

induced by the polar coordinate embedding.

Proposition 7.2 *This coincides with the cotangent complex of A defined by Francis.*

Proof Both sides of the map commute with homotopy colimits of $\mathcal{E}_d$–algebras; therefore, it suffices to check the claim for free $\mathcal{E}_d$–algebras. Let $A = F_{\mathcal{E}_d}(V)$. Using [4, Proposition 5.5], we see that

$$\int_{S^{d-1}\times(0,1)} F_{\mathcal{E}_d}(V) \simeq \bigvee_{i\geq 0} \mathrm{Conf}(i, S^{d-1}\times(0,1)) \otimes_{\Sigma_i} V^{\otimes i}$$

and, similarly,

$$\int_D F_{\mathcal{E}_d}(V) \simeq \bigvee_{i\geq 0} \mathrm{Conf}(i, D) \otimes_{\Sigma_i} V^{\otimes i}.$$

On the other hand, it is proved in [6, Theorem 2.26] that there is a cofiber sequence

$$\int_{S^{d-1}\times(0,1)} A \to A \to L_A[n].$$

Moreover, the proof of [6, Theorem 2.26] is based on an explicit computation in the free case and an inspection of this proof shows that the first map in the above cofiber sequence coincides with the polar embedding map. □

Remark 7.3 The above definition is a bit ad hoc. Francis actually defines in [6, Definition 2.10] the cotangent complex as the object representing the $\mathcal{E}_d$–derivations. That is, we have a weak equivalence

$$\mathbb{R}\underline{\mathrm{Hom}}_{S_\kappa^{d-1}\mathbf{Mod}_A}(L_A, M) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{\boldsymbol{C}[\mathcal{E}_d]/A}(A, A\oplus M) := \mathrm{Der}(A, M).$$

The fact that the two definitions coincide is [6, Theorem 2.26].

Definition 7.4 We say that an $\mathcal{E}_d$–algebra A is étale if L_A is contractible. More generally, given an object Z in $\boldsymbol{C}$, we say that A is Z–locally étale if $Z \otimes L_A$ is contractible.

We say that a map $X \to Y$ in $\boldsymbol{C}$ is a Z–local weak equivalence if the induced map $X \otimes^{\mathbb{L}} Z \to Y \otimes^{\mathbb{L}} Z$ is a weak equivalence.

An equivalent formulation of the previous definition is that A is (Z–locally) étale if the unit map $A \to \int_{S^{d-1}\times(0,1)} A$ is a (Z–local) equivalence. Indeed we have shown in Lemma 5.10 that the unit map is a section of $\int_{S^{d-1}\times(0,1)} A \to A$.

Proposition 7.5 *If A is a commutative algebra and is (Z–locally) étale as an $\mathcal{E}_d$–algebra, then it is (Z–locally) étale as an $\mathcal{E}_{d+1}$–algebra.*

Proof We have proved in [11, Theorem 5.8] that, for A a commutative algebra, $\int_M A$ is equivalent to $\mathrm{Sing}(M) \otimes A$ (ie the tensor in the category of commutative algebras in $\mathbf{Mod}_E$). Then the proof is the same as the proof of Proposition 6.1. □

Remark 7.6 More generally, using the excision property for factorization homology (see [4, Lemma 3.18]), we can prove that if A is $\mathcal{E}_{d+1}$ and is (Z–locally) étale as an $\mathcal{E}_d$–algebra, it is (Z–locally) étale as an $\mathcal{E}_{d+1}$–algebra.

Remark 7.7 If A is a commutative algebra, then A is étale as an $\mathcal{E}_2$–algebra if and only if it is formally THH–étale (ie if the map $A \to \mathrm{THH}(A)$ is an equivalence). Indeed, for commutative algebras (and in fact for $\mathcal{E}_3$–algebras), $\mathrm{THH}(A)$ coincides with $\int_{S^1\times\mathbb{R}} A$. Note that this is *not* true for $\mathcal{E}_2$–algebras, as the product framing on $S^1 \times \mathbb{R}$ is not connected to the κ–framing in the space of framings of $S^1 \times \mathbb{R}$.

Recall that an object U of $\boldsymbol{C}$ is said to be Z–local if, for all Z–local weak equivalences $X \to Y$, the induced map

$$\mathbb{R}\underline{\mathrm{Hom}}(Y, U) \to \mathbb{R}\underline{\mathrm{Hom}}(X, U)$$

is a weak equivalence in $\boldsymbol{C}$.

Lemma 7.8 *Let $u\colon R \to S$ be a map of cofibrant associative algebras in $\boldsymbol{C}$ that is a Z–local weak equivalence and let M and N be two left modules over S with N Z–local in $\boldsymbol{C}$. Then the map*

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{Mod}_S}(M, N) \to \mathbb{R}\underline{\mathrm{Hom}}_{\mathbf{Mod}_R}(u^*M, u^*N)$$

is a weak equivalence.

Proof The left-hand side can be computed as the homotopy limit of the cobar construction

$$[n] \mapsto \underline{\mathrm{Hom}}(S^{\otimes n} \otimes M, N).$$

Similarly, the left-hand side can be computed as the homotopy limit of

$$[n] \mapsto \underline{\mathrm{Hom}}(R^{\otimes n} \otimes M, N).$$

Since $R \to S$ is a Z–local weak equivalence, so is $R^{\otimes n} \otimes M \to S^{\otimes n} \otimes M$ for each n. Thus, since N is Z–local, the two cosimplicial objects are weakly equivalent. This implies that they have weakly equivalent homotopy limits. □

We can now state and prove the main theorem of this section.

Theorem 7.9 *Let T be a commutative algebra in $\boldsymbol{C}$ that is (Z–locally) étale as an $\mathcal{E}_d$–algebra over $\mathbb{I}$. Then, for any $\mathcal{E}_d$–algebra A over T (that is Z–local as an object of $\boldsymbol{C}$), the base-change map*

$$\mathrm{HH}_{\mathcal{E}_d}(A) \to \mathrm{HH}_{\mathcal{E}_d}(A|T)$$

is a weak equivalence.

Proof We write $A|T$ whenever we want to emphasize the fact that we are viewing A as an $\mathcal{E}_d$–algebra over T.

By Proposition 2.11 of Francis [6], there is cofiber sequence

$$u_! L_T \to L_A \to L_{A|T},$$

where $u\colon T \to A$ is the unit map and $u_!$ is the corresponding functor

$$u_!\colon S_\kappa^{d-1}\mathbf{Mod}_T \to S_\kappa^{d-1}\mathbf{Mod}_A.$$

By hypothesis, L_T is (Z–locally) contractible; therefore, $L_A \to L_{A|T}$ is a (Z–local) equivalence. We have a base-change map of cofiber sequences:

$$\begin{tikzcd}
\Sigma^{d-1} L_A \arrow[r] \arrow[d] & \int_{S^{d-1}\times(0,1)} A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{id}"] & \Sigma^d L_A \arrow[d] \\
\Sigma^{d-1} L_{A|T} \arrow[r] & \int_{S^{d-1}\times(0,1)} A|T \arrow[r] & A \arrow[r] & \Sigma^d L_{A|T}
\end{tikzcd}$$

This implies that $\int_{S^{d-1}\times(0,1)} A \to \int_{S^{d-1}\times(0,1)} A|T$ is a (Z–local) equivalence.

We can form the commutative diagram

$$\begin{tikzcd} U_A^{S_\kappa^{d-1}} \arrow[r] \arrow[d] & \int_{S^{d-1}\times(0,1)} A \arrow[d] \\ U_{A|T}^{S_\kappa^{d-1}} \arrow[r] & \int_{S^{d-1}\times(0,1)} A|T, \end{tikzcd}$$

where the horizontal maps are the maps of Corollary 2.14. These maps are weak equivalences by Corollary 2.14. Thus, the map $U_A^{S_\kappa^{d-1}} \to U_{A|T}^{S_\kappa^{d-1}}$ is a (Z–local) weak equivalence of associative algebras. The theorem follows from this fact and the previous lemma. □

Remark 7.10 The computation of Section 6 implies that $S \to E_n$ is $K(n)$–locally an étale morphism of $\mathcal{E}_d$–algebras for all d. Therefore, given a $K(n)$–local E_n–algebra A, we can compute its (higher) Hochschild cohomology over E_n or over S without affecting the result. This fact is used by Angeltveit [2, Theorem 6.9] in the case of ordinary Hochschild cohomology.

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