Algebraic & Geometric Topology 16 (2016) 397–426

Geometry of the SL(3, $\mathbb{C}$)–character variety of torus knots

VICENTE MUÑOZ
JOAN PORTI

Let G be the fundamental group of the complement of the torus knot of type (m,n). It has a presentation $G=\langle x,y \mid x^m=y^n\rangle$. We find a geometric description of the character variety $X(G)$ of characters of representations of G into $\mathrm{SL}(3,\mathbb{C})$, $\mathrm{GL}(3,\mathbb{C})$ and $\mathrm{PGL}(3,\mathbb{C})$.

14D20; 57M25, 57M27

1 Introduction

Since the foundational work of Culler and Shalen [1], the varieties of $\mathrm{SL}(2,\mathbb{C})$–characters have been extensively studied. Given a manifold M, the variety of representations of $\pi_1(M)$ into $\mathrm{SL}(2,\mathbb{C})$ and the variety of characters of such representations both contain information on the topology of M. This is especially interesting for 3–dimensional manifolds, where the fundamental group and the geometrical properties of the manifold are strongly related. This can be used to study knots $K\subset S^3$ by analyzing the $\mathrm{SL}(2,\mathbb{C})$–character variety of the fundamental group of the knot complement S^3-K (these are called *knot groups*).

For a very different reason, the case of fundamental groups of surfaces has also been extensively analyzed (see Hausel and Thaddeus [2], Hitchin [5], and Logares, Muñoz and Newstead [9]), in this situation focusing more on geometrical properties of the moduli space itself (cf nonabelian Hodge theory).

However, much less is known of the character varieties for other groups, notably for $\mathrm{SL}(r,\mathbb{C})$ with $r\geq 3$. The character varieties for $\mathrm{SL}(3,\mathbb{C})$ for free groups have been described by Lawton [7] and Lawton and Muñoz [8]. In the case of 3–manifolds, little has been done. In this paper, we study the case of the torus knots $K_{m,n}$ of any type (m,n), which are the first family of knots where the computations are rather feasible. The case of $\mathrm{SL}(2,\mathbb{C})$–character varieties of torus knots was carried out by Martín-Morales and Oller-Marcén [11] and Muñoz [12]. For $\mathrm{SL}(3,\mathbb{C})$, the torus knot $K_{2,3}$ has been treated by Heusener and Porti [3].

Published: 23 February 2016 DOI: 10.2140/agt.2016.16.397

In the case of $\mathrm{SL}(2,\mathbb{C})$–character varieties of torus knot groups, only one-dimensional irreducible components appear. However, when we move to $\mathrm{SL}(3,\mathbb{C})$, we see components of different dimensions. In the case of torus knots, we shall see components of dimension 4 and of dimension 2. Our main result is an explicit geometrical description of the $\mathrm{SL}(3,\mathbb{C})$–character variety of torus knots.

Theorem 1.1 *Let m and n be coprime positive integers. By swapping them if necessary, assume that m is odd. The $\mathrm{SL}(3,\mathbb{C})$–character variety $\mathcal{X}_3$ of the torus knot $K_{m,n}\subset S^3$ is stratified into the following components.*

- *There is one component consisting of totally reducible representations, isomorphic to $\mathbb{C}^2$.*
- *There are $[\frac{1}{2}(n-1)][\frac{1}{2}(m-1)]$ components consisting of partially reducible representations, each isomorphic to $(\mathbb{C}-\{0,1\})\times\mathbb{C}^*$.*
- *If n is even, there are $(m-1)/2$ extra components consisting of partially reducible representations, each isomorphic to $\{(u,v)\in\mathbb{C}^2 \mid v\neq 0, v\neq u^2\}$.*
- *There are $\frac{1}{12}(n-1)(n-2)(m-1)(m-2)$ components of dimension 4, consisting of irreducible representations, all isomorphic to each other, and which are described explicitly in Remark 8.5.*
- *There are $\frac{1}{2}(n-1)(m-1)(n+m-4)$ components consisting of irreducible representations, each isomorphic to $(\mathbb{C}^*)^2-\{x+y=1\}$.*

Moreover, m and n can be recovered from the above information.

We also geometrically describe how these components fit into the whole of the $\mathrm{SL}(3,\mathbb{C})$–character variety; that is, we describe the closure of each of the strata in $\mathcal{X}_3$. In particular, the total space is connected. In Section 10, we give also the corresponding results for the $\mathrm{GL}(3,\mathbb{C})$–character variety and the $\mathrm{PGL}(3,\mathbb{C})$–character variety of the torus knots. Finally, we compute the K–theory class (in the Grothendieck ring of varieties) of all these character varieties.

Potentially, the methods and results for a general r could be used to describe the geometry of the $\mathrm{SL}(r,\mathbb{C})$, $\mathrm{GL}(r,\mathbb{C})$ and $\mathrm{PGL}(r,\mathbb{C})$–character varieties for $r>3$, though the computations would become much more involved.

2 Moduli of representations and character varieties

Let Γ be a finitely presented group, and let $G=\mathrm{SL}(r,\mathbb{C})$, $\mathrm{GL}(r,\mathbb{C})$ or $\mathrm{PGL}(r,\mathbb{C})$. A *representation* of Γ in G is a homomorphism $\rho\colon \Gamma\to G$. Consider a presentation

$\Gamma = \langle x_1,\dots,x_k \mid r_1,\dots,r_s\rangle$. Then ρ is completely determined by the k–tuple $(A_1,\dots,A_k) = (\rho(x_1),\dots,\rho(x_k))$ subject to the relations $r_j(A_1,\dots,A_k) = \mathrm{Id}$, $1 \le j \le s$. The space of representations is

$$\begin{aligned} R(\Gamma,G) &= \mathrm{Hom}(\Gamma,G) \\ &= \{(A_1,\dots,A_k) \mid r_j(A_1,\dots,A_k) = \mathrm{Id}, 1 \le j \le s\} \subset G^k. \end{aligned}$$

Therefore $R(\Gamma,G)$ is an affine algebraic set.

We say that two representations ρ and ρ' are equivalent if there exists $P \in G$ such that $\rho'(g) = P^{-1}\rho(g)P$ for every $g \in \Gamma$. This corresponds to a change of basis in $\mathbb{C}^r$. Note that the action of G descends to an action of the projective group $PG = \mathrm{PGL}(r,\mathbb{C})$. This produces an action of $\mathrm{PGL}(r,\mathbb{C})$ on $R(\Gamma,G)$. The moduli space of representations is the GIT quotient

$$M(\Gamma,G) = R(\Gamma,G)/\!/G.$$

Recall that, by definition of the GIT quotient for an affine variety, if we write $R(\Gamma,G) = \mathrm{Spec}\, A$, then $M(\Gamma,G) = \mathrm{Spec}\, A^G$.

A representation ρ is *reducible* if there exists some proper subspace $V \subset \mathbb{C}^r$ such that for all $g \in G$ we have $\rho(g)(V) \subset V$; otherwise ρ is *irreducible*. Note that if ρ is reducible, then let $V \subset \mathbb{C}^r$ be an invariant subspace, and consider a complement $\mathbb{C}^r = V \oplus W$. Let $\rho_1 = \rho|_V$ and let ρ_2 be the induced representation on the quotient space $W = \mathbb{C}^r/V$. Then we can write $\rho = \left(\begin{smallmatrix}\rho_1 & 0\\ f & \rho_2\end{smallmatrix}\right)$, where $f\colon \Gamma \to \mathrm{Hom}(W,V)$. Let $k = \dim V$ and take

$$P_t = \begin{pmatrix} t^{r-k}\mathrm{Id} & 0 \\ 0 & t^{-k}\mathrm{Id}\end{pmatrix}.$$

Then

$$P_t^{-1}\rho P_t = \begin{pmatrix}\rho_1 & 0 \\ t^r f & \rho_2\end{pmatrix} \to \rho' = \begin{pmatrix}\rho_1 & 0 \\ 0 & \rho_2\end{pmatrix}$$

when $t \to 0$. Therefore ρ and ρ' define the same point in the quotient $M(\Gamma,G)$. Repeating this, we can substitute any representation ρ by some $\widetilde{\rho} = \bigoplus \rho_i$, where all ρ_i are irreducible representations. We call this process *semisimplification*, and $\widetilde{\rho}$ a semisimple representation; also ρ and $\widetilde{\rho}$ are called S-equivalent. The space $M(\Gamma,G)$ parametrizes semisimple representations; see Lubotzky and Magid [10, Theorem 1.28].

Suppose now that $G = \mathrm{SL}(r,\mathbb{C})$. Given a representation $\rho\colon \Gamma \to G$, we define its *character* as the map $\chi_\rho\colon \Gamma \to \mathbb{C}$, $\chi_\rho(g) = \mathrm{tr}\,\rho(g)$. Note that two equivalent representations ρ and ρ' have the same character. There is a character map $\chi\colon R(\Gamma,G) \to \mathbb{C}^\Gamma$, $\rho \mapsto \chi_\rho$, whose image

$$X(\Gamma,G) = \chi(R(\Gamma,G))$$

is called the *character variety of* Γ. Let us give $X(\Gamma, G)$ the structure of an algebraic variety. The traces χ_ρ span a subring $B \subset A$. Clearly $B \subset A^G$. As A is noetherian, we have that B is a finitely generated $\mathbb{C}$–algebra. Hence there exists a collection $g_1, \dots, g_a$ of elements of G such that χ_ρ is determined by $\chi_\rho(g_1), \dots, \chi_\rho(g_a)$ for any ρ. Such a collection gives a map

$$\Psi\colon R(\Gamma, G) \to \mathbb{C}^a, \quad \Psi(\rho) = (\chi_\rho(g_1), \dots, \chi_\rho(g_a)),$$

and $X(\Gamma, G) \cong \Psi(R(\Gamma, G))$. This endows $X(\Gamma, G)$ with the structure of an algebraic variety, which is independent of the chosen collection. The natural algebraic map

$$M(\Gamma, G) \to X(\Gamma, G)$$

is an isomorphism; see Sikora [14]. This is the same as to say that $B = A^G$; that is, the ring of invariant polynomials is generated by characters.

3 Grothendieck ring of varieties

Let $\mathcal{V}ar_{\mathbb{C}}$ be the category of quasiprojective complex varieties. We denote by $K(\mathcal{V}ar_{\mathbb{C}})$ the *Grothendieck ring* of $\mathcal{V}ar_{\mathbb{C}}$. This is the abelian group generated by elements $[Z]$ for $Z \in \mathcal{V}ar_{\mathbb{C}}$, subject to the relation $[Z] = [Z_1] + [Z_2]$ whenever Z can be decomposed as a disjoint union $Z = Z_1 \sqcup Z_2$ of a closed and a Zariski open subset.

There is a naturally defined product in $K(\mathcal{V}ar_{\mathbb{C}})$ given by $[Y] \cdot [Z] = [Y \times Z]$. Note that if $\pi\colon Z \to Y$ is an algebraic fiber bundle with fiber F, which is locally trivial in the Zariski topology, then $[Z] = [F] \cdot [Y]$.

We denote by $\mathbb{L} = [\mathbb{C}]$ the *Lefschetz object* in $K(\mathcal{V}ar_{\mathbb{C}})$. Clearly $\mathbb{L}^k = [\mathbb{C}^k]$. The following result will be useful later on. Let $\mu_r \subset \mathbb{C}^*$ denote the group of r^{th} roots of unity.

Proposition 3.1 *Let μ_r act on $X = (\mathbb{C}^*)^k$ by $(t_1, \dots, t_k) \mapsto (\xi^{a_1} t_1, \dots, \xi^{a_k} t_k)$, for $\xi = e^{2\pi\sqrt{-1}/r}$ and some weights $a_1, \dots, a_k \in \mathbb{Z}$. Then $X/\mu_r \cong (\mathbb{C}^*)^k$.*

As a consequence, for the same action on $Y = \mathbb{C}^k$, we have $[Y/\mu_r] = \mathbb{L}^k = [Y]$.

Proof If the action is not free, then it factors through some quotient of μ_r, and we substitute it by the action of the quotient group. So we can assume that the action is free, namely that the greatest common divisor of $a_1, \dots, a_k, r$ is 1. If $k = 1$ then the result is trivially true, since the quotient $\mathbb{C}^*/\mu_r$ is parametrized by $w = t^r \in \mathbb{C}^*$.

Suppose $k > 1$. There are integer numbers $b_1, \dots, b_k, b$ such that $a_1 b_1 + \dots + a_k b_k + rb = 1$. Consider the quotient

$$\mathbb{Z}_r^k \to \mathbb{Z}_r$$

given by $(x_1, \dots, x_k) \mapsto a_1x_1 + \cdots + a_kx_k$, where $\mathbb{Z}_r$ denotes the cyclic group of r elements $\mathbb{Z}/r\mathbb{Z}$. This is surjective, and the choice of $(b_1, \dots, b_k)$ gives a splitting. Let K be the kernel. Then $\mathbb{Z}_r^k \cong K \oplus \mathbb{Z}_r$. Clearly, it must be $K \cong \mathbb{Z}_r^{k-1}$. The inclusion $K \hookrightarrow \mathbb{Z}_r^k$ defines a collection of $k-1$ vectors $(c_{i1}, \dots, c_{ik})$, $i = 1, \dots, k-1$. Now, the matrix

$$M = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{k-1,1} & \cdots & c_{k-1,k} \\ b_1 & \cdots & b_k \end{pmatrix}$$

has determinant $d = \det(M)$, which is a unit in $\mathbb{Z}_r$. We want to modify the entries of the matrix M adding multiples of r so that we get $\det(M) = 1$, and so M is invertible over $\mathbb{Z}$. First, multiply the first row by an integer x so that $\det(M)x \equiv 1 \pmod r$. So M is still invertible modulo r and $\det(M) \equiv 1 \pmod r$. We can write $M = ABC$, where A, C are invertible over the integers, and $B = \operatorname{diag}(e_1, \dots, e_k)$, $e_1|e_2|\cdots|e_k$. Note that $\prod e_i \equiv 1 \pmod r$. If we add multiples of r to the entries of B, we do the same to the entries of M. So, without loss of generality, we can work with $M = B$.

If all $e_i = 1$, we have finished. Moreover, if all $e_i \equiv 1 \pmod r$, we can change each e_i by adding multiples of r to arrange $e_i = 1$. So now suppose that there is one entry $e_j \not\equiv 1 \pmod r$. As $\prod e_i \equiv 1 \pmod r$, there must be another entry $e_l \not\equiv 1 \pmod r$. Adding r to e_j, we have that $e_j + r$ and e_l are coprime (recall that $e_j|e_l$ or $e_l|e_j$, and all e_i are coprime to r). Then we can diagonalize this new M again, getting elementary divisors 1 and $(e_j + r)e_l$ instead of e_j and e_l. This increases the number of elements of the diagonal of M equal to 1. Repeating the process, we can finally get all diagonal entries equal to 1.

Now consider $u_j = t_1^{c_{j1}} \cdots t_k^{c_{jk}}$ for $j = 1, \dots, k-1$, and $u_k = t_1^{b_1} \cdots t_k^{b_k}$. As $\det M = 1$, this is a change of variables, so $(u_1, \dots, u_k)$ parametrizes $(\mathbb{C}^*)^k$, and the action of μ_r is given by $(u_1, \dots, u_{k-1}, u_k) \mapsto (u_1, \dots, u_{k-1}, \xi u_k)$. Therefore $X/\mu_r \cong (\mathbb{C}^*)^k/\mu_r = (\mathbb{C}^*)^{k-1} \times (\mathbb{C}^*/\mu_r) \cong (\mathbb{C}^*)^k$.

The last assertion follows by stratifying Y according to how many entries are zero. □

4 Character varieties of torus knots

Let $T^2 = S^1 \times S^1$ be the 2–torus and consider the standard embedding $T^2 \subset S^3$. Let m and n be a pair of coprime positive integers. Identifying T^2 with the quotient $\mathbb{R}^2/\mathbb{Z}^2$, the image of the straight line $y = (m/n)x$ in T^2 defines the *torus knot* of type (m, n), which we shall denote as $K_{m,n} \subset S^3$; see Rolfsen [13, Chapter 3].

For any knot $K \subset S^3$, we denote by Γ_K the fundamental group of the exterior $S^3 - K$ of the knot. It is known that

$$\Gamma_{m,n} = \Gamma_{K_{m,n}} \cong \langle x, y \mid x^n = y^m \rangle.$$

The purpose of this paper is to describe the character variety $X(\Gamma_{m,n}, G)$, for $G = \mathrm{SL}(r,\mathbb{C}), \mathrm{GL}(r,\mathbb{C})$ and $\mathrm{PGL}(r,\mathbb{C})$.

We introduce the notation

$$\begin{aligned}
\mathcal{X}_r &= X(\Gamma_{m,n}, \mathrm{SL}(r,\mathbb{C})),\\
\widetilde{\mathcal{X}}_r &= X(\Gamma_{m,n}, \mathrm{GL}(r,\mathbb{C})),\\
\overline{\mathcal{X}}_r &= X(\Gamma_{m,n}, \mathrm{PGL}(r,\mathbb{C})),
\end{aligned}$$

dropping the reference to m and n in the notation.

Lemma 4.1 $\widetilde{\mathcal{X}}_1 \cong \mathbb{C}^*$.

Proof Let $(\lambda, \mu) \in \widetilde{\mathcal{X}}_1$. Then $\lambda^n = \mu^m$, so there exists a unique $t \in \mathbb{C}^*$ such that $\lambda = t^m$ and $\mu = t^n$ (here we use that m and n are coprime). This means that $\widetilde{\mathcal{X}}_1 \cong \mathbb{C}^*$ via $t \mapsto (t^m, t^n)$. □

There is a map det: $\widetilde{\mathcal{X}}_r \to \widetilde{\mathcal{X}}_1$ given by $(\det \rho)(g) = \det(\rho(g))$ for any $g \in \Gamma$. Then $\mathcal{X}_r = \det^{-1}(\mathbf{1})$, where $\mathbf{1}$ is the trivial character. Otherwise said, if $\rho = (A, B) \in \widetilde{\mathcal{X}}_r$ then $(\det A, \det B) \in \widetilde{\mathcal{X}}_1$. Here $\det A = t^m$ and $\det B = t^n$ for some $t \in \mathbb{C}^*$. We shall write $\det \rho = t$. So $(A, B) \in \mathcal{X}_r$ when $t = 1$.

There is an action of $\mathbb{C}^*$ on $\widetilde{\mathcal{X}}_r$ given by

$$\gamma \cdot (A, B) = (\gamma^m A, \gamma^n B).$$

Note that $\det(\gamma \cdot \rho) = \gamma^r \det(\rho)$. The kernel of $\mathbb{C}^* \times \mathcal{X}_r \to \widetilde{\mathcal{X}}_r$ is given by the r^{th} roots of unity μ_r. So there is an isomorphism

$$\widetilde{\mathcal{X}}_r \cong (\mathcal{X}_r \times \mathbb{C}^*)/\mu_r. \tag{1}$$

Now let $([A], [B]) \in \overline{\mathcal{X}}_r$; that is, $[A], [B] \in \mathrm{PGL}(r,\mathbb{C})$ with $[A^m] = [B^n]$. There is a surjective map $\mathrm{SL}(r,\mathbb{C}) \to \mathrm{PGL}(r,\mathbb{C})$ with kernel μ_r. So we can assume $(A, B) \in \mathrm{SL}(r,\mathbb{C})$, and $A^n = \lambda B^m$ for some $\lambda \in \mathbb{C}^*$. Take determinants, we have $\lambda^r = 1$. The matrices A and B are well-defined up to multiplication by r^{th} roots of unity. Let $\epsilon, \varepsilon \in \mu_r$. Then $(\epsilon A)^n = \lambda'(\varepsilon B)^m$ with $\lambda' = \lambda \epsilon^n \varepsilon^{-m}$. As m and n are coprime, we can arrange $\lambda' = 1$ by choosing ϵ, ε suitably. Also, if $\lambda = 1$ then $\lambda' = 1$ means that $\epsilon = t^m$, $\varepsilon = t^n$ with $t \in \mu_r$. That is, we have the isomorphism

$$\overline{\mathcal{X}}_r \cong \mathcal{X}_r/\mu_r. \tag{2}$$

Equivalently,

$$\bar{\mathcal{X}}_r \cong \widetilde{\mathcal{X}}_r/\mathbb{C}^*.$$

Note that, in particular, this means that the representations of $\Gamma_{m,n}$ in $\mathrm{PGL}(r,\mathbb{C})$ all lift to $\mathrm{SL}(r,\mathbb{C})$ (this is not true for all fundamental groups of 3–manifolds).

Remark 4.2 The map $\widetilde{\mathcal{X}}_r \to \bar{\mathcal{X}}_r$ is a $\mathbb{C}^*$–fibration, locally trivial in the Zariski topology. Therefore the K–theory classes satisfy $[\widetilde{\mathcal{X}}_r] = (\mathbb{L}-1)[\bar{\mathcal{X}}_r]$. In particular, the Euler characteristic of $\widetilde{\mathcal{X}}_r$ is $\chi(\widetilde{\mathcal{X}}_r) = 0$ (this Euler characteristic is obtained by the substitution $\mathbb{L} \mapsto 1$ in the K–theory class).

Martín-Morales and Oller-Marcén [11] describe the character variety $\mathcal{X}_2$ by finding a set of equations satisfied by the traces of the matrices of the image by the representation. Muñoz [12] describes the same variety $\mathcal{X}_2$ by a geometric method based on the study of eigenvectors and eigenvalues of the matrices. Here we shall extend the latter to study the varieties $\mathcal{X}_3$, $\widetilde{\mathcal{X}}_3$, $\bar{\mathcal{X}}_3$.

5 Stratification of the character variety

We denote by $\pi = (r_1, \overset{(a_1)}{\ldots}, r_1, \ldots, r_s, \overset{(a_s)}{\ldots}, r_s)$ a partition of r; that is, $a_1 r_1 + \cdots + a_s r_s = r$ with $r_1 > \cdots > r_s > 0$ and $a_j \geq 1$. Let Π_r be the set of all partitions of r. We decompose the character variety

$$\widetilde{\mathcal{X}}_r = \bigsqcup_{\pi \in \Pi_r} \widetilde{X}_\pi$$

into locally closed subvarieties, where $\widetilde{X}_\pi$ corresponds to representations

$$\rho = \bigoplus_{t=1}^{s} \bigoplus_{l=1}^{a_t} \rho_{t,l}, \quad \rho_{t,l}\colon \Gamma \longrightarrow \mathrm{GL}(r_t, \mathbb{C}). \tag{3}$$

Also,

$$\mathcal{X}_r = \bigsqcup_{\pi \in \Pi_r} X_\pi,$$

where $X_\pi = \widetilde{X}_\pi \cap \mathcal{X}_r$; that is, it consists of those (3) with $\prod_{t,l} \det \rho_{t,l}(g) = 1$.

For $\mathrm{PGL}(r,\mathbb{C})$–representations, we have

$$\bar{\mathcal{X}}_r = \bigsqcup_{\pi \in \Pi_r} \bar{X}_\pi,$$

with $\bar{X}_\pi$ the image of X_π under the projection $\mathcal{X}_r \to \bar{\mathcal{X}}_r$.

The irreducible representations correspond to $\pi_0 = (r)$. We denote $\mathcal{X}_r^* = X_{\pi_0}$, $\widetilde{\mathcal{X}}_r^* = \widetilde{X}_{\pi_0}$ and $\bar{\mathcal{X}}_r^* = \bar{X}_{\pi_0}$.

Proposition 5.1 *We have* $\widetilde{X}_\pi = \prod_{t=1}^s \mathrm{Sym}^{a_t} \widetilde{\mathcal{X}}_{r_t}^*$.

The representations corresponding to $\pi_1 = (1, \overset{(r)}{\ldots}, 1)$ are called *totally reducible* since they have matrices that diagonalize simultaneously in the same basis. The corresponding sets will be denoted $X_{TR} = X_{\pi_1}$, $\widetilde{X}_{TR} = \widetilde{X}_{\pi_1}$ and $\bar{X}_{TR} = \bar{X}_{\pi_1}$.

Proposition 5.2 *We have* $\widetilde{X}_{TR} \cong \mathbb{C}^{r-1} \times \mathbb{C}^*$ *and* $X_{TR} \cong \mathbb{C}^{r-1}$.

In particular, $[\widetilde{X}_{TR}] = \mathbb{L}^{r-1}(\mathbb{L}-1)$, $[X_{TR}] = \mathbb{L}^{r-1}$ *and* $[\bar{X}_{TR}] = \mathbb{L}^{r-1}$.

Proof By Lemma 4.1, $\widetilde{\mathcal{X}}_1^* \cong \mathbb{C}^*$, and it is formed by representations (t^m, t^n). By Proposition 5.1, $\widetilde{X}_{TR} = \mathrm{Sym}^r \widetilde{\mathcal{X}}_1^*$, where (A, B) is given by $A = \mathrm{diag}(t_1^m, \ldots, t_r^m)$ and $B = \mathrm{diag}(t_1^n, \ldots, t_r^n)$ for $t_j \in \mathbb{C}^*$. Then

$$\widetilde{X}_{TR} \cong \mathrm{Sym}^r \mathbb{C}^* \cong \mathbb{C}^{r-1} \times \mathbb{C}^*.$$

Here the last isomorphism is given by $(t_1, \ldots, t_r) \mapsto (\sigma_1, \ldots, \sigma_r)$, where $\sigma_k = \sigma_k(t_1, \ldots, t_r)$ is the k^{th} elementary symmetric function on $t_1, \ldots, t_r$. Note that $t_i \neq 0, \forall i$ if and only if $\sigma_r \neq 0$.

The condition that $\det A = \det B = 1$ means that $\prod t_j^m = \prod t_j^n = 1$. So $\prod t_j = 1$, which is translated into $\sigma_r = 1$. Hence, $X_{TR} = \mathbb{C}^{r-1}$.

For analyzing the case of $\bar{X}_{TR}$, we look at the action of μ_r on X_{TR}. Note that $\epsilon \in \mu_r$ acts as $t_j \mapsto \epsilon t_j$; hence, it acts as $(\sigma_1, \sigma_2, \ldots, \sigma_{r-1}) \mapsto (\epsilon\sigma_1, \epsilon^2\sigma_2, \ldots, \epsilon^{r-1}\sigma_{r-1})$. Proposition 3.1 now gives the result. $\square$

Lemma 5.3 *Suppose that* $\rho = (A, B) \in \widetilde{\mathcal{X}}_r^*$. *Then* A *and* B *are both diagonalizable, and* $A^n = B^m = \varpi \mathrm{Id}$ *for some* $\varpi \in \mathbb{C}^*$. *Moreover, neither* A *nor* B *is a multiple of the identity. Furthermore, if* $(A, B) \in \mathcal{X}_r^*$, *then* $\varpi \in \mu_r$.

Proof Choose a suitable basis so that A is of Jordan form, with blocks $J_1, \ldots, J_k$. Let J_i be a Jordan block of size $m_i \geq 1$ and eigenvalue λ_i. Then A^n has blocks $J_1^n, \ldots, J_k^n$, and each J_i^n is conjugated to a Jordan matrix of size m_i with eigenvalue λ_i^n. In particular, if v_i is the eigenvector of J_i, then v_i is the eigenvector of J_i^n. The span of eigenvectors of A^n is $W = \langle v_1, \ldots, v_k \rangle$. This satisfies $A(W) \subset W$. In an analogous fashion, as $A^n = B^m$ and B and B^m have the same span of eigenvectors, we have that $B(W) \subset W$. By irreducibility, $W = \mathbb{C}^r$, so all $m_i = 1$. That is, both A and B are diagonalizable.

Now let $\varpi = \lambda_1^n$. Let W be the span of those v_i such that $\lambda_i^n = \varpi$. Then $A(W) \subset W$. In a similar fashion, $B(W) \subset W$, so $W = \mathbb{C}^r$. This means that $A^n = \varpi \mathrm{Id}$. Note that all eigenvalues λ_i of A and all eigenvalues μ_j of B satisfy $\lambda_i^n = \mu_j^m = \varpi$.

The last assertion is clear since $\det A = \det B = 1$ implies $\varpi^r = 1$. □

Remark 5.4 Lemma 5.3 is true for m and n not coprime.

Corollary 5.5 *The varieties $\mathcal{X}_r$, $\widetilde{\mathcal{X}}_r$ and $\bar{\mathcal{X}}_r$ are connected.*

Proof By (1) and (2), it is enough to see that $\mathcal{X}_r$ is connected. We just need to see that the closure of any component X_π intersects X_{TR}, which is connected by Proposition 5.2.

Let us focus first on $\mathcal{X}_r^*$. For $(A, B) \in \mathcal{X}_r^*$, we diagonalize A and B (by Lemma 5.3). This gives decompositions $\mathbb{C}^r = V_1 \oplus \cdots \oplus V_s$ and $\mathbb{C}^r = W_1 \oplus \cdots \oplus W_l$ into eigenspaces given for A and B, respectively. Let $v_i = \dim V_i$ and $w_j = \dim W_j$, where $r = \sum v_i = \sum w_j$. By Lemma 5.3, $A^n = B^m = \varpi \mathrm{Id}$ with $\varpi^r = 1$, and the eigenvalues $\epsilon_1, \ldots, \epsilon_s$ and $\varepsilon_1, \ldots, \varepsilon_l$ for A and B, respectively, satisfy $\epsilon_i^n = \varepsilon_j^m = \varpi$. So there are finitely many choices for $v_i, w_j, \epsilon_i, \varepsilon_j$. We denote

$$\kappa = ((\epsilon_1, \overset{(v_1)}{\ldots}, \epsilon_1, \ldots, \epsilon_s, \overset{(v_s)}{\ldots}, \epsilon_s), (\varepsilon_1, \overset{(w_1)}{\ldots}, \varepsilon_1, \ldots, \varepsilon_l, \overset{(w_l)}{\ldots}, \varepsilon_l)),$$

repeating eigenvalues according to multiplicity. This gives a collection of (disjoint) components

$$\mathcal{X}_r^* = \bigsqcup_\kappa \mathcal{X}_{r,\kappa}^*. \tag{4}$$

Fix a component $\mathcal{X}_{r,\kappa}^*$. To determine the pair (A, B), it is enough to give the eigenspaces $V_1, \ldots, V_s$ and $W_1, \ldots, W_l$. These are given by a point in the product $\prod \mathrm{Gr}(v_i, r) \times \prod \mathrm{Gr}(w_j, r)$. The set of possible points determining an irreducible representation is an open (if nonempty) subset $\mathcal{U}_\kappa \subset \prod \mathrm{Gr}(v_i, r) \times \prod \mathrm{Gr}(w_j, r)$, and

$$\mathcal{X}_{r,\kappa}^* = \mathcal{U}_\kappa / \mathrm{PGL}(r, \mathbb{C}). \tag{5}$$

This space is irreducible and hence connected. The choice of the subspaces $V_i = \langle e_{v_1+\cdots+v_{i-1}+1}, \ldots, e_{v_1+\cdots+v_i} \rangle$ and $W_j = \langle e_{w_1+\cdots+w_{j-1}+1}, \ldots, e_{w_1+\cdots+w_j} \rangle$ with respect to the standard basis $\{e_1, \ldots, e_r\}$ gives a representation in the closure of $\mathcal{X}_{r,\kappa}^*$, which is totally reducible. This completes the argument in this case.

Now consider another stratum X_π, for $\pi = (r_1, \overset{(a_1)}{\ldots}, r_1, \ldots, r_s, \overset{(a_s)}{\ldots}, r_s)$. We have, by Proposition 5.1, $\widetilde{X}_\pi = \prod_{t=1}^s \mathrm{Sym}^{a_t} \widetilde{\mathcal{X}}_{r_t}^*$. Take an irreducible component $Y_{t,l}$ of $\mathcal{X}_{r_t}^*$ for each $t = 1, \ldots, s$ and $l = 1, \ldots, a_t$. Let $\widetilde{Y}_{t,l}$ be the image of $\mathbb{C}^* \times Y_{t,l}$ in $\widetilde{\mathcal{X}}_{r_t}^*$.

Consider the map $\prod_{t,l}(\mathbb{C}^* \times Y_{t,l}) \to \widetilde{X}_\pi$. The condition of the determinant being 1 gives a hypersurface $W \subset (\mathbb{C}^*)^{a_1+\cdots+a_s}$, which is connected since $a_1 + \cdots + a_s > 1$. The image of $W \times \prod_{t,l} Y_{t,l}$ in X_π is connected, and all of them cover X_π. Now the closure of $Y_{t,l}$ contains elements in $X_{TR}^{r_t}$. Hence the closure of the image of $W \times \prod_{t,l} Y_{t,l}$ contains elements that are totally reducible, as required. □

6 Maximal dimensional components

We will now count the number of maximal dimensional irreducible components of $\mathcal{X}_r$ and give a geometric description of them.

Theorem 6.1 *The character variety $\mathcal{X}_r$ has dimension at most $(r-1)^2$. For $r \geq 3$, the number of irreducible components of this dimension is*

$$\frac{1}{r}\binom{n-1}{r-1}\binom{m-1}{r-1}.$$

In particular, there are no such components if either $n < r$ or $m < r$.

Proof We first bound the dimension of $\mathcal{X}_r^*$. As seen in the proof of Corollary 5.5, the space $\mathcal{X}_r^*$ consists of disjoint components $\mathcal{X}_{r,\kappa}^*$ as in (4), and each component $\mathcal{X}_{r,\kappa}^*$ is of the form (5), where $\mathcal{U}_\kappa \subset \prod \mathrm{Gr}(v_i, r) \times \prod \mathrm{Gr}(w_j, r)$. Now

$$\begin{aligned} \dim \mathcal{U}_\kappa &= \sum v_i(r - v_i) + \sum w_j(r - w_j) \\ &< \Big(\sum v_i\Big)(r-1) + \Big(\sum w_j\Big)(r-1) = 2r(r-1), \end{aligned}$$

unless all $v_i = w_j = 1$, in which case there is equality. We have to quotient by $\mathrm{PGL}(r, \mathbb{C})$, which has dimension $r^2 - 1$, and hence $\dim \mathcal{X}_{r,\kappa}^* \leq 2r^2 - 2r - (r^2 - 1) = r^2 - 2r - 1 = (r-1)^2$, with equality only if all $v_i = w_j = 1$.

In general, for a stratum X_π, using $\widetilde{X}_\pi = \prod_{t=1}^s \mathrm{Sym}^{a_t} \widetilde{\mathcal{X}}_{r_t}^*$ and $r = \sum_{t=1}^s a_t r_t$, we have

$$\dim X_\pi + 1 = \dim \widetilde{X}_\pi \leq \sum_{t=1}^s a_t(r_t^2 - 2r_t + 2) < r^2 - 2r + 2 = (r-1)^2 + 1,$$

unless $s = 1$, $a_1 = 1$ and $r_1 = r$, in which case there is equality. This corresponds to the component $\mathcal{X}_r^*$.

Now let us count the number of irreducible components of dimension $(r-1)^2$. It is the same as to count the number of $(\epsilon_1, \ldots, \epsilon_r)$ and $(\varepsilon_1, \ldots, \varepsilon_r)$ subject to

- $\epsilon_1 \cdots \epsilon_r = 1, \epsilon_i$ distinct,

- $\varepsilon_1 \cdots \varepsilon_r = 1, \varepsilon_j$ distinct,
- $\epsilon_i^n = \varepsilon_j^m = \varpi$,
- $\varpi^r = 1$.

Denote by

$$N(k_1,k_2) = \#\{(\epsilon_1,\dots,\epsilon_r) \text{ distinct}, (\varepsilon_1,\dots,\varepsilon_r) \text{ distinct} \\ \mid \epsilon_i^n = e^{2\pi\sqrt{-1}k_1/r}, \varepsilon_j^m = e^{2\pi\sqrt{-1}k_2/r}, \epsilon_1\cdots\epsilon_r = \varepsilon_1\cdots\varepsilon_r = 1\},$$

for any pair of integers k_1 and k_2. We have to compute the sum $T = \sum_{k=0}^{r-1} N(k,k)$.

As m and n are coprime, we have that the modulo-reduction map $\mathbb{Z}_{nm} \to \mathbb{Z}_n \times \mathbb{Z}_m$ is a bijection. Therefore,

$$\sum_{k=0}^{nm-1} N(k,k) = \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{m-1} N(k_1,k_2).$$

Now $N(k_1,k_2) = N_n(k_1)N_m(k_2)$, where

$$N_n(k_1) = \#\{(\epsilon_1,\dots,\epsilon_r) \text{ distinct} \mid \epsilon_i^n = e^{2\pi\sqrt{-1}k_1/r},\ \epsilon_1\cdots\epsilon_r = 1\},$$
$$N_m(k_2) = \#\{(\varepsilon_1,\dots,\varepsilon_r) \text{ distinct} \mid \varepsilon_j^m = e^{2\pi\sqrt{-1}k_2/r},\ \varepsilon_1\cdots\varepsilon_r = 1\}.$$

Writing $\epsilon_i = e^{2\pi\sqrt{-1}(k_1/r+a_i)/n}$ with $a_i \in \mathbb{Z}$, the equality $\epsilon_1\cdots\epsilon_r = 1$ is equivalent to $a_1+\cdots+a_r+k_1 \in n\mathbb{Z}$. Thus, the sum $\sum_{k_1=0}^{n-1} N_n(k_1)$ is the number of different $a_1,\dots,a_r \in \mathbb{Z}_n$; therefore, $\sum_{k_1=0}^{n-1} N_n(k_1) = n!/(n-r)!$ and $\sum_{k_2=0}^{m-1} N_m(k_2) = m!/(m-r)!$. So

$$\sum_{k=0}^{nm-1} N(k,k) = \frac{n!}{(n-r)!}\frac{m!}{(m-r)!}. \tag{6}$$

Note that the sum (6) is the same if we start at any other integer; ie $\sum_{k=a}^{nm-1+a} N(k,k)$ gives the same value. Therefore,

$$\sum_{k=0}^{rnm-1} N(k,k) = r\sum_{k=0}^{nm-1} N(k,k).$$

Also, $N(k_1,k_2) = N(k_1+ar, k_2+br)$ for all $a,b \in \mathbb{Z}$. Therefore, $\sum_{k=0}^{rnm-1} N(k,k) = nm\sum_{k=0}^{r-1} N(k,k) = nmT$. Thus,

$$T = \frac{r}{nm}\frac{n!}{(n-r)!}\frac{m!}{(m-r)!},$$

and the number of irreducible components is (taking into account the permutations of ϵ_i and of ε_j)

$$\frac{1}{(r!)^2}T=\frac{r}{nm}\binom{n}{r}\binom{m}{r}=\frac{1}{r}\binom{n-1}{r-1}\binom{m-1}{r-1}. \qquad \square$$

The proof of Theorem 6.1 gives us a way to label the maximal dimensional components of $\mathcal{X}_r^*$. Let

$$F'=\{((\epsilon_1,\dots,\epsilon_r),(\varepsilon_1,\dots,\varepsilon_r)) \text{ distinct} \mid \prod\epsilon_i=\prod\varepsilon_j=\omega,\omega^r=1\},$$

and define the set $F=F'/(\mathfrak{S}_r\times\mathfrak{S}_r)$, where $\mathfrak{S}_r$ denotes the symmetric group on r elements, the first $\mathfrak{S}_r$ acts by permutation on the components of $(\epsilon_1,\dots,\epsilon_r)$, and the second $\mathfrak{S}_r$ acts by permutation on the components of $(\varepsilon_1,\dots,\varepsilon_r)$. We shall denote a $\tau\in F$ as $\tau=[(\epsilon_1,\dots,\epsilon_r),(\varepsilon_1,\dots,\varepsilon_r)]$.

Now we geometrically describe the maximal dimensional components of $\mathcal{X}_r$. For this, we introduce some notation. Consider $\mathrm{GL}(r,\mathbb{C})$. Let $T\cong(\mathbb{C}^*)^r$ be the set of diagonal matrices, and consider the action of $T\times T$, where the first T acts on the left and the second T acts on the right on $\mathrm{GL}(r,\mathbb{C})$. The set $D=\{(\lambda\mathrm{Id},\lambda^{-1}\mathrm{Id})\}\subset T\times T$ acts trivially, so there is an effective action of $T\times_D T$.

Proposition 6.2 *For $\tau\in F$, the $(r-1)^2$–dimensional component $\mathcal{X}_{r,\tau}^*$ is isomorphic to*

$$\mathcal{X}_{r,\tau}^*\cong\mathcal{M}/(T\times_D T),$$

where $\mathcal{M}\subset\mathrm{GL}(r,\mathbb{C})$ is the open subset of stable points for the $(T\times_D T)$–action. In particular, all components for different $\tau\in F$ are isomorphic.

Proof A maximal dimensional component $\mathcal{X}_{r,\tau}^*$ is determined by $\tau=[(\epsilon_1,\dots,\epsilon_r),(\varepsilon_1,\dots,\varepsilon_r)]\in F$. We fix a lift to F'; that is, we fix an order of the eigenvalues throughout, say $((\epsilon_1,\dots,\epsilon_r),(\varepsilon_1,\dots,\varepsilon_r))$.

Given a pair $(A,B)\in\mathcal{X}_{r,\tau}^*$, recall that A and B are diagonalizable with the prescribed eigenvalues. Let $v_1,\dots,v_r$ be the eigenvectors of A and $w_1,\dots,w_r$ be the eigenvectors of B. These are well-defined up to scalar multiples. We use $v_1,\dots,v_r$ as a basis for $\mathbb{C}^r$ and write $w_j=(a_{1j},\dots,a_{rj})$ in these coordinates. This produces a matrix

$$M=\begin{pmatrix}a_{11}&\dots&a_{1r}\\ \vdots&\ddots&\vdots\\ a_{r1}&\dots&a_{rr}\end{pmatrix}. \tag{7}$$

Note that $\det(M)\neq 0$, since the vectors $w_1,\dots,w_r$ are linearly independent. Let $\mathcal{M}$ be the set of those M which yield irreducible representations (A,B). This is equivalent to the fact that there do not exist subcollections $v_{b_1},\dots v_{b_p}$ and $w_{a_1},\dots,w_{a_p}$ for $0<p<r$ such that $W=\langle w_{a_1},\dots,w_{a_p}\rangle=\langle v_{b_1},\dots v_{b_p}\rangle$ since, in this case, such W would be invariant. This condition translates to the fact that the subminor corresponding to $\{a_1,\dots,a_p\}\times(\{1,\dots,r\}-\{b_1,\dots,b_p\})$ is identically zero. This is equivalent to the condition that M is not a stable point for the action of $T\times_D T$; that is, the orbit of the point has another orbit in the closure, or the action is nonfree. Clearly, if (A,B) is as above, then acting by $(\mathrm{diag}(x_i),\mathrm{diag}(y_j))\in T\times_D T$, where

$$x_i=\begin{cases}\lambda & \text{if } i\in\{a_1,\dots,a_p\}\\ \lambda^{-1} & \text{if } i\notin\{a_1,\dots,a_p\}\end{cases}\quad\text{and}\quad y_j=\begin{cases}\lambda & \text{if } j\notin\{b_1,\dots,b_p\}\\ \lambda^{-1} & \text{if } j\in\{b_1,\dots,b_p\}\end{cases},$$

and then taking $\lambda\to 0$, we either get points in a different orbit or the action is not free (the complementary minor tends to zero).

Let $\mathcal{M}\subset\mathrm{GL}(r,\mathbb{C})$ be the open subset of stable points for the $(T\times_D T)$–action. Then a representation (A,B) is determined by a matrix $M\in\mathcal{M}$ modulo the possible rescaling of basis vectors $v_1,\dots,v_r$ (this corresponds to the action of T on the left) and of the eigenvectors $w_1,\dots,w_r$ (this corresponds to the action of T on the right). Note that the irreducible component $\mathcal{M}/(T\times_D T)$ has dimension $r^2-(2r-1)=(r-1)^2$, as expected. □

Remark 6.3 The closure of the stratum $\mathcal{X}^*_{r,\tau}$ is obtained by adding semisimple reducible representations (which are direct sums of irreducible representations of smaller rank). This corresponds to adding matrices M, as in (7), for which there is a subminor corresponding to $\{a_1,\dots,a_p\}\times(\{1,\dots,r\}-\{b_1,\dots,b_p\})$ identically zero, and at the same time the subminor $(\{1,\dots,r\}-\{a_1,\dots,a_p\})\times\{b_1,\dots,b_p\}$ also vanishes. We are therefore looking at a polystable point (direct sum of stable points) of the action of $T\times_D T$. This means that the closure of $\mathcal{X}^*_{r,\tau}$ is isomorphic to the GIT quotient

$$\mathrm{GL}(r,\mathbb{C})/\!/(T\times_D T).$$

Now we shall explain how to get the maximal dimensional components of $\widetilde{\mathcal{X}}_r$ and $\overline{\mathcal{X}}_r$, although we are not going to do the explicit count of them for general r (we shall do it later for $r=2,3$).

Let $\tau=[(\epsilon_1,\dots,\epsilon_r),(\varepsilon_1,\dots,\varepsilon_r)]\in F$. For $t\in\mu_r$, we have $t\cdot\tau=[(t\epsilon_1,\dots,t\epsilon_r),(t\varepsilon_1,\dots,t\varepsilon_r)]$, which gives another (or the same) component. The map $t\colon\mathcal{X}_r\to\mathcal{X}_r$ restricts to maps $t\colon\mathcal{X}^*_{r,\tau}\to\mathcal{X}^*_{r,t\cdot\tau}$.

The maximal dimensional components of $\overline{\mathcal{X}}_r$ are parametrized by the coset space F/μ_r. So the number of them is the cardinality of F/μ_r. For given $[\tau]\in F/\mu_r$, the corresponding component is

$$\overline{\mathcal{X}}^*_{r,[\tau]} = \Big(\bigsqcup_{t\in\mu_r} \mathcal{X}^*_{r,t\cdot\tau} \Big) / \mu_r. \tag{8}$$

Let $S = \mathrm{Stab}(\tau) \subset \mu_r$. If $t \in S$, then we have $t\colon \mathcal{X}^*_{r,\tau} \to \mathcal{X}^*_{r,\tau}$, and also there is an element $(f_t, g_t) \in \mathfrak{S}_r \times \mathfrak{S}_r$ so that we have $((t\epsilon_1,\dots,t\epsilon_r),(t\varepsilon_1,\dots,t\varepsilon_r)) = (f_t(\epsilon_1,\dots,\epsilon_r), g_t(\varepsilon_1,\dots,\varepsilon_r))$. Using the natural action of $\mathfrak{S}_r \times \mathfrak{S}_r$ on $\mathcal{M}$ (multiplication on the right and on the left by permutation matrices), we have that (8) is isomorphic to

$$\overline{\mathcal{X}}^*_{r,[\tau]} = \mathcal{X}^*_{r,\tau}/S = \mathcal{M}/(H\times S),$$

where $H = T\times_D T$.

The maximal dimensional components of $\widetilde{\mathcal{X}}_r$ are also parametrized by the coset space F/μ_r. They are of dimension $(r-1)^2+1$ and are isomorphic to

$$\widetilde{\mathcal{X}}^*_{r,[\tau]} = \Big(\mathbb{C}^* \times \Big(\bigsqcup_{t\in\mu_r} \mathcal{X}^*_{r,t\cdot\tau} \Big)\Big) / \mu_r \cong (\mathbb{C}^*\times \mathcal{X}^*_{r,\tau})/S = (\mathbb{C}^*\times\mathcal{M})/(H\times S).$$

7 Character varieties for SL(2, ℂ), GL(2, ℂ) and PGL(2, ℂ)

We can recover the results of Muñoz [12]. Actually, the arguments that we have developed here are an extension (and a refinement) of those in Muñoz [12] for $r \geq 2$.

Proposition 7.1 *The variety $\mathcal{X}_2$ consists of the following irreducible components:*

- *One component $X_{TR} \cong \mathbb{C}$.*
- *$(n-1)(m-1)/2$ components forming the irreducible locus, such that each is isomorphic to $\mathbb{C}-\{0,1\}$, and the closure of each component is $\mathbb{C}$ and intersects X_{TR} in two points.*

Proof Here the only possible representations are either totally reducible or irreducible. The totally reducible locus is given by Proposition 5.2. The irreducible representations must all be of type $\mathcal{X}^*_{2,\tau}$ for some $\tau = ((\epsilon_1,\epsilon_2),(\varepsilon_1,\varepsilon_2))$. The number of them is given in Theorem 6.1, and it is $(n-1)(m-1)/2$. Each of them is parametrized by $\mathcal{M}/H$, where $\mathcal{M}$ is formed by the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with all entries nonzero, and $H = T\times_D T$ (T are the diagonal matrices, and they act on the right and on the left on $\mathcal{M}$). Using the action of H, we can arrange $a=b=c=1$. Hence, the matrix is determined by $d \in \mathbb{C}-\{0,1\}$. However, we shall use the parameter $r = 1/(1-d) \in \mathbb{C}-\{0,1\}$.

The closure of $\mathcal{M}/H$ is given by $\mathrm{GL}(2,\mathbb{C})/\!/H$. By Remark 6.3, we have to add matrices with two entries that are zero. If $b=c=0$, then we have the reducible representation $(\mathrm{diag}(\epsilon_1,\epsilon_2),\mathrm{diag}(\varepsilon_1,\varepsilon_2))$. This corresponds to $r=0$. If $a=d=0$, then we have the reducible representation $(\mathrm{diag}(\epsilon_1,\epsilon_2),\mathrm{diag}(\varepsilon_2,\varepsilon_1))$. This corresponds to $r=1$. □

Note that $[\mathcal{X}_2]=\mathbb{L}+\frac{1}{2}(n-1)(m-1)(\mathbb{L}-2)$.

Proposition 7.2 *The variety $\bar{\mathcal{X}}_2$ consists of the following irreducible components:*

- *One component $\bar{X}_{TR}\cong\mathbb{C}$.*
- *$[\frac{1}{2}(n-1)][\frac{1}{2}(m-1)]$ components of the irreducible locus, such that each of them is isomorphic to $\mathbb{C}-\{0,1\}$, and the closure of each component is $\mathbb{C}$ and intersects $\bar{X}_{TR}$ in two points.*
- *If n is even and m is odd, $(m-1)/2$ components of the irreducible locus, such that each of them is isomorphic to $\mathbb{C}^*$, and the closure of each component is $\mathbb{C}$ and intersects $\bar{X}_{TR}$ in one point. (The case m even and n odd is analogous.)*

Proof We use the description $\bar{\mathcal{X}}_2\cong\mathcal{X}_2/\mu_2$ where $\mu_2=\{\pm 1\}$. For the reducible component, we have $\bar{X}_{TR}=\mathbb{C}/\mu_2\cong\mathbb{C}$.

The irreducible components of $\mathcal{X}_2$ are parametrized by the finite set F of eigenvalues of A and B. If $F'=\{((\epsilon_1,\epsilon_2),(\varepsilon_1,\varepsilon_2))$ distinct, $\epsilon_1\epsilon_2=\varepsilon_1\varepsilon_2=1$, $\epsilon_i^n=\varepsilon_j^m=\pm 1\}$, then μ_2 acts on F' by $((\epsilon_1,\epsilon_2),(\varepsilon_1,\varepsilon_2))\mapsto(((-1)^m\epsilon_1,(-1)^m\epsilon_2),((-1)^n\varepsilon_1,(-1)^n\varepsilon_2))$, and $F=F'/(\mathfrak{S}_2\times\mathfrak{S}_2)$. When both m and n are odd, there are no fixed points, because fixed points occur when $\epsilon_2=-\epsilon_1$ and $\varepsilon_2=-\varepsilon_1$ or, equivalently, when $\epsilon_1^2=\varepsilon_1^2=-1$, which contradicts $\epsilon_1^n=\varepsilon_1^m=\pm 1$. In this case, $\#(F/\mu_2)=\#F/2=\frac{1}{4}(n-1)(m-1)=(\frac{1}{2}(n-1))(\frac{1}{2}(m-1))$.

Assume now that n is even; then the same calculation shows that fixed points occur precisely when $\{\epsilon_1,\epsilon_2\}=\{\pm\sqrt{-1}\}$ for any admissible value of ε_j. This yields $(m-1)/2$ components. On those invariant components, the action of μ_2 permutes two rows or two columns of the matrix $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ in the proof of Proposition 7.1. This amounts to mapping the parameter d to $1/d$ or, equivalently, mapping the parameter r to $1-r$. So the component is isomorphic to $(\mathbb{C}-\{0,1\})/\mu_2\cong\mathbb{C}^*$. □

By Proposition 7.2, we have, for m and n odd, $[\bar{\mathcal{X}}_2]=\mathbb{L}+\frac{1}{4}(n-1)(m-1)(\mathbb{L}-2)$. For n even and m odd, we have that $[\bar{\mathcal{X}}_2]=\mathbb{L}+\frac{1}{4}(n-2)(m-1)(\mathbb{L}-2)+\frac{1}{2}(m-1)(\mathbb{L}-1)$.

Proposition 7.3 *The variety $\widetilde{\mathcal{X}}_2$ consists of the following irreducible components:*

- *One component $\widetilde{X}_{TR} \cong \mathbb{C} \times \mathbb{C}^*$.*
- *$[\frac{1}{2}(n-1)][\frac{1}{2}(m-1)]$ components of the irreducible locus, such that each of them is isomorphic to $(\mathbb{C} - \{0,1\}) \times \mathbb{C}^*$, and the closure of each component is $\mathbb{C} \times \mathbb{C}^*$ and intersects $\widetilde{X}_{TR}$ in two $\mathbb{C}^*$s.*
- *If n is even, $(m-1)/2$ extra components of the irreducible locus, such that each is isomorphic to $\{(u,v) \in \mathbb{C}^2 \mid v \neq 0, v \neq u^2\}$, and the closure of each component is $\mathbb{C} \times \mathbb{C}^*$ and intersects $\widetilde{X}_{TR}$ along a $\mathbb{C}^*$. (The case m even and n odd is analogous.)*

Proof The component $\widetilde{X}_{TR}$ is given in Proposition 5.2. Now we use the description $\widetilde{\mathcal{X}}_2 \cong (\mathcal{X}_2 \times \mathbb{C}^*)/\mu_2$. By the proof of Proposition 7.2, when m and n are odd, μ_2 switches components of $\mathcal{X}_2$ without preserving any of them, and the proposition follows in this case. When n is even, μ_2 preserves $(m-1)/2$ components of $\mathcal{X}_2$, and, for each such component, it maps the parameters $(r,\lambda) \in (\mathbb{C} - \{0,1\}) \times \mathbb{C}^*$ to $(1-r,-\lambda)$. The quotient is $((\mathbb{C} - \{0,1\}) \times \mathbb{C}^*)/\mu_2$. The compactification is given as $(\mathbb{C} \times \mathbb{C}^*)/\mu_2$ and can be coordinatized with $u = (2r-1)\lambda$ and $v = \lambda^2$. The quotient is given by $\{(u,v) \in \mathbb{C}^2 \mid v \neq 0\}$, and the image of the curves $(\{0,1\} \times \mathbb{C}^*)/\mu_2 \cong \mathbb{C}^*$ is given by $v = u^2$ with $u \neq 0$. Hence, the component is isomorphic to $\{(u,v) \in \mathbb{C}^2 \mid v \neq 0, v \neq u^2\}$. □

By Remark 4.2, we know that $[\widetilde{\mathcal{X}}_2] = (\mathbb{L}-1)[\overline{\mathcal{X}}_2]$. So, for m and n odd, we have $[\widetilde{\mathcal{X}}_2] = \big(\mathbb{L} + \frac{1}{4}(n-1)(m-1)(\mathbb{L}-2)\big)(\mathbb{L}-1)$. For n even and m odd, we have $[\widetilde{\mathcal{X}}_2] = \big(\mathbb{L} + \frac{1}{4}(n-2)(m-1)(\mathbb{L}-2) + \frac{1}{2}(m-1)(\mathbb{L}-1)\big)(\mathbb{L}-1)$.

8 Character varieties for $\mathrm{SL}(3,\mathbb{C})$

Now we move to the description of the $\mathrm{SL}(3,\mathbb{C})$–character variety $\mathcal{X}_3$.

Proposition 8.1 *The components of reducible representations of $\mathcal{X}_3$ are the following:*

- *The component of totally reducible representations $X_{TR} = \mathbb{C}^2$.*
- *$[\frac{1}{2}(n-1)][\frac{1}{2}(m-1)]$ components $X_{PR}^{1,i}$ of partially reducible representations, each isomorphic to $(\mathbb{C} - \{0,1\}) \times \mathbb{C}^*$.*
- *If n is even, $(m-1)/2$ extra components $X_{PR}^{2,i}$ of partially reducible representations, each isomorphic to $\{(u,v) \in \mathbb{C}^2 \mid v \neq 0, v \neq u^2\}$. (The case m even and n odd is analogous.)*

Proof The description of X_{TR} is in Proposition 5.2. Now we move to partially reducible representations. This corresponds to representations in $\mathrm{SL}(3,\mathbb{C})$ which split as $\mathbb{C}^3 = W \oplus W'$ where $\dim W = 2$ and $\dim W' = 1$. That is, the partition is $\pi = \{(2,1)\}$. Then $X_\pi \cong \widetilde{\mathcal{X}}_2^*$ since a representation $\rho \in X_\pi$ is determined by $\rho|_W$, because $\rho|_{W'}$ is fully determined as $(\det \rho|_W)^{-1}$. The description of the components now follows from Proposition 7.3. □

Proposition 8.2 *The set $\mathcal{X}_3^*$ of irreducible representations is composed by the following components $\mathcal{X}_{3,\pi}^*$:*

- $\frac{1}{12}(n-1)(n-2)(m-1)(m-2)$ *components of maximal dimension* 4*, which are isomorphic to $\mathcal{M}/(T \times_D T)$, where $\mathcal{M} \subset \mathrm{GL}(3,\mathbb{C})$ are the stable points for the $(T \times_D T)$–action.*
- $\frac{1}{2}(n-1)(m-1)(n+m-4)$ *components $\mathcal{X}_{3,\kappa}^*$, each isomorphic to $(\mathbb{C}^*)^2 - \{x+y=1\}$.*

Proof The number of irreducible components of maximal dimension is given by Theorem 6.1, and its geometric description by Proposition 6.2.

Now we look at the remaining components. According to (4), these are of the form $\mathcal{X}_{r,\kappa}^*$ where some eigenvalues in $\kappa = ((\epsilon_1,\epsilon_2,\epsilon_3),(\varepsilon_1,\varepsilon_2,\varepsilon_3))$ are repeated. From Lemma 5.3, neither A nor B are a multiple of the identity for any irreducible representation (A,B). Therefore, the three eigenvalues cannot be the same. Also, it cannot be that $\epsilon_1=\epsilon_2$ and $\varepsilon_1=\varepsilon_2$ since, in this case, the intersection of the two-dimensional eigenspace of A with the two-dimensional eigenspace of B would give an invariant nontrivial subspace of ρ. The only possibilities are as follows:

(1) $\epsilon_1=\epsilon_2\neq\epsilon_3$ and $\varepsilon_1,\varepsilon_2,\varepsilon_3$ are distinct,

(2) $\epsilon_1,\epsilon_2,\epsilon_3$ are distinct and $\varepsilon_1=\varepsilon_2\neq\varepsilon_3$.

Assume that case (1) holds. Then $\epsilon_3=\epsilon_1^{-2}$. As $\epsilon_1\neq\epsilon_3$, we have $\epsilon_1^3\neq 1$. On the other hand, since $\varpi=\epsilon_1^n$, we know $\epsilon_1^{3n}=1$, so there are $3n-3$ choices for ϵ_1.

Suppose $n\not\equiv 0 \pmod 3$. For each value of ϖ there are $n-1$ choices for ϵ_1. So there are a total of $(n-1)3(m^2-3m+2)/6=(n-1)(m-1)(m-2)/2$ possibilities. Suppose $n\equiv 0 \pmod 3$. Then for $\varpi=1$, there are $n-3$ choices for ϵ_1, and for $\varpi\neq 1$, there are n choices. Note that, in this case, $m\not\equiv 0 \pmod 3$. The total is, again, $(3n-3)(m^2-3m+2)/6=(n-1)(m-1)(m-2)/2$ possibilities.

Now fix one strata, ie the eigenvalues of A and B. Let L and v be the plane and vector which give eigenspaces of A, and let w_1,w_2 and w_3 be the eigenvectors of B.

Fix coordinates so that $w_1 = (1,0,0)$, $w_2 = (0,1,0)$ and $w_3 = (0,0,1)$. The plane L gives a line in $\mathbb{P}^2$. It does not contain any of the points $[w_1], [w_2], [w_3] \in \mathbb{P}^2$. Therefore, the line is given by $a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$; that is, its coordinates in the dual space $(\mathbb{P}^2)^*$, $[a_1, a_2, a_3]$, do not have any entry which is a zero. So we can rescale the coordinates to arrange $[a_1, a_2, a_3] = [1,1,1]$. Therefore, we only have the choice of $[v] \in \mathbb{P}^2$ with $[v] \notin L$. If $[v] \in R = \langle [w_1], [w_2] \rangle$, then take the intersection of L with R, say $L \cap R = \langle [u] \rangle$. Then $\langle u, v \rangle$ is an invariant subspace, and so ρ is reducible. So $[v]$ is not in any line $\langle [w_i], [w_j] \rangle$. The parameter space is thus $\mathbb{P}^2$ minus four lines: $(\mathbb{C}^*)^2 - \{x + y = 1\}$.

Case (2) is analogous to case (1), with $(m-1)(n-1)(n-2)/2$ strata. Note that $(m-1)(n-1)(n-2)/2 + (n-1)(m-1)(m-2)/2 = (n-1)(m-1)(n+m-4)/2$. □

We shall denote by G the index set of those κ parametrizing the components $\mathcal{X}^*_{3,\kappa}$ of dimension 2 in Proposition 8.2.

Theorem 8.3 *The K–theory class of the character variety $\mathcal{X}_3$ is as follows. For m and n both odd,*

$$\begin{aligned}[\mathcal{X}_3] = \tfrac{1}{12}(n-1)(n-2)(m-1)(m-2)(\mathbb{L}^4 + 4\mathbb{L}^3 - 3\mathbb{L}^2 - 15\mathbb{L} + 12)& \\ + \mathbb{L}^2 + \tfrac{1}{4}(n-1)(m-1)(\mathbb{L}^2 - 3\mathbb{L} + 2)& \\ + \tfrac{1}{2}(n-1)(m-1)(n+m-4)(\mathbb{L}^2 - 3\mathbb{L} + 3).&\end{aligned}$$

For n even and m odd, it is

$$\begin{aligned}[\mathcal{X}_3] = \tfrac{1}{12}&(n-1)(n-2)(m-1)(m-2)(\mathbb{L}^4 + 4\mathbb{L}^3 - 3\mathbb{L}^2 - 15\mathbb{L} + 12) \\ &+ \mathbb{L}^2 + \tfrac{1}{4}(n-2)(m-1)(\mathbb{L}^2 - 3\mathbb{L} + 2) \\ &+ \tfrac{1}{2}(m-1)(\mathbb{L}^2 - 2\mathbb{L} + 1) + \tfrac{1}{2}(n-1)(m-1)(n+m-4)(\mathbb{L}^2 - 3\mathbb{L} + 3).\end{aligned}$$

Proof We have to add the contributions from Proposition 8.1 and Proposition 8.2. Here, $[X_{TR}] = \mathbb{L}^2$, $[\mathcal{X}^*_{3,\kappa}] = (\mathbb{L}-1)^2 - (\mathbb{L}-2) = \mathbb{L}^2 - 3\mathbb{L} + 3$ for $\kappa \in G$, $[X^{1,i}_{PR}] = (\mathbb{L}-2)(\mathbb{L}-1) = \mathbb{L}^2 - 3\mathbb{L} + 2$ and $[X^{2,i}_{PR}] = (\mathbb{L}-1)^2 = \mathbb{L}^2 - 2\mathbb{L} + 1$. It only remains to compute the class $[\mathcal{M}/(T \times_D T)]$.

Consider the space $\mathcal{M}$. We stratify it as follows:

- If a_{11}, a_{21} and a_{31} are nonzero, then the first vector accounts for $(\mathbb{L}-1)^3$. The second and third vectors should be independent, and this accounts for $(\mathbb{L}^3 - \mathbb{L})(\mathbb{L}^3 - \mathbb{L}^2)$. However, they cannot lie in any coordinate plane, so accounting for $3(\mathbb{L}^2 - 1)(\mathbb{L}^2 - \mathbb{L})$. This gives a total of
$$(\mathbb{L}-1)^3\big((\mathbb{L}^3 - \mathbb{L})(\mathbb{L}^3 - \mathbb{L}^2) - 3(\mathbb{L}^2 - 1)(\mathbb{L}^2 - \mathbb{L})\big).$$

- Suppose one of a_{11}, a_{21} and a_{31} is zero (say $a_{31} = 0$, so we have to multiply by three at the end). Then the first vector accounts for $(\mathbb{L}-1)^2$. We have a matrix
$$M = \begin{pmatrix} a_{11} & b & c \\ a_{21} & d & e \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$
It must be that a_{32} and a_{33} are nonzero, so accounting for $(\mathbb{L}-1)^2$. The condition for $\det(M) = 0$ is linear on b, c, d and e, and all the coefficients of the linear equation are nonzero. So the choices for them yield $\mathbb{L}^4 - \mathbb{L}^3$. We have to subtract for accounting the cases where any column or row of $\left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right)$ is zero, that is, $4(\mathbb{L}^2-\mathbb{L}) - 4(\mathbb{L}-1)$. So the total is
$$3(\mathbb{L}-1)^4\big(\mathbb{L}^4 - \mathbb{L}^3 - 4(\mathbb{L}^2-\mathbb{L}) + 4(\mathbb{L}-1)\big).$$

Therefore,

$$\begin{aligned}
[\mathcal{M}/T \times_D T] &= \frac{[\mathcal{M}]}{(\mathbb{L}-1)^5} \\
&= \frac{(\mathbb{L}^3-\mathbb{L})(\mathbb{L}^3-\mathbb{L}^2) - 3(\mathbb{L}^2-1)(\mathbb{L}^2-\mathbb{L})}{(\mathbb{L}-1)^2} \\
&\qquad\qquad + 3\frac{\mathbb{L}^4 - \mathbb{L}^3 - 4(\mathbb{L}^2-\mathbb{L}) + 4(\mathbb{L}-1)}{\mathbb{L}-1} \\
&= \mathbb{L}^4 + 4\mathbb{L}^3 - 3\mathbb{L}^2 - 15\mathbb{L} + 12. \qquad \square
\end{aligned}$$

By substituting $\mathbb{L} \mapsto 1$ in Theorem 8.3, we obtain the Euler characteristic of $\mathcal{X}_3$,
$$\chi(\mathcal{X}_3) = -\tfrac{1}{12}(n-1)(n-2)(m-1)(m-2) + 1 + \tfrac{1}{2}(n-1)(m-1)(n+m-4).$$

Corollary 8.4 *The character variety $\mathcal{X}_3$ determines n and m up to order.*

Proof With $\mathcal{X}_3$ we have the class $[\mathcal{X}_3] \in K_0(\mathcal{V}ar_{\mathbb{C}})$ given in Theorem 8.3. The coefficient of $\mathbb{L}^4$ gives us $(n-1)(n-2)(m-1)(m-2)$. Now, we subtract the term $\frac{1}{12}(n-1)(n-2)(m-1)(m-2)(\mathbb{L}^4 + 4\mathbb{L}^3 - 3\mathbb{L}^2 - 15\mathbb{L} + 12)$. In the expression that we obtain, we substitute $\mathbb{L} \to 0$, producing $p = \frac{1}{2}(n-1)(m-1)(3n+3m-11)$. Substituting $\mathbb{L} \to 1$, it yields $q = 1 + \frac{1}{2}(n-1)(m-1)(n+m-4)$. The quantity $2p - 6q + 6 = (n-1)(m-1)$. Now we can recover $(n-2)(m-2)$ as well, and with this we get nm and $n+m$. This proves the result. $\square$

Remark 8.5 Note that we can describe the 4–dimensional components $\mathcal{M}/(T \times_D T)$ of Proposition 8.2 as follows. Each column of the matrix (7) gives a point $p_j =$

$[a_{1j}, a_{2j}, a_{3j}] \in \mathbb{P}^2$. These points must be independent, they cannot be equal to $q_1 = [1,0,0], q_2 = [0,1,0]$ or $q_3 = [0,0,1]$, and two of them cannot be simultaneously in a coordinate line $L_1 = \langle q_2, q_3 \rangle, L_2 = \langle q_1, q_3 \rangle$ or $L_3 = \langle q_1, q_2 \rangle$. Let

$$\begin{aligned}\mathcal{V} = \{(p_1, p_2, p_3) \in (\mathbb{P}^2 - \{q_1, q_2, q_3\})^3 &\text{ independent} \\ &\mid \text{no two of them lie simultaneously in either } L_1, L_2, L_3\}.\end{aligned}$$

Consider the $(\mathbb{C}^*)^2$ action given by $p_j = [a_{1j}, a_{2j}, a_{3j}] \mapsto [\alpha a_{1j}, \beta a_{2j}, a_{3j}]$ for $j = 1, 2, 3$. Then

$$\mathcal{M}/(T \times_D T) = \mathcal{V}/(\mathbb{C}^*)^2.$$

9 Intersection patterns

We have the stratification

$$\mathcal{X}_3 = X_{TR} \sqcup \Big(\bigsqcup X_{PR}^{1,i}\Big) \sqcup \Big(\bigsqcup X_{PR}^{2,j}\Big) \sqcup \Big(\bigsqcup_{\tau \in F} \mathcal{X}_{3,\tau}^*\Big) \sqcup \Big(\bigsqcup_{\kappa \in G} \mathcal{X}_{3,\kappa}^*\Big) \tag{9}$$

into totally reducible representations, partially reducible representations (of type 1 and type 2), and irreducible representations (of type I and of type II). The set X_{TR} is closed, and so we want to describe how the closures $\overline{X_{PR}^{1,i}}$, $\overline{X_{PR}^{2,j}}$, $\overline{\mathcal{X}_{3,\kappa}^*}$ and $\overline{\mathcal{X}_{3,\tau}^*}$ intersect the other strata.

The component $X_{TR} \cong \mathbb{C}^2$ is closed. By Proposition 5.2, it is parametrized by (x, y) where $x = t_1 + t_2 + t_3$, $y = t_1 t_2 + t_1 t_3 + t_2 t_3$ and $t_1 t_2 t_3 = 1$, with the matrices (A, B) being $A = \mathrm{diag}(t_1^m, t_2^m, t_3^m)$ and $B = \mathrm{diag}(t_1^n, t_2^n, t_3^n)$.

Proposition 9.1 *The partially reducible components of both types, $X_{PR}^{1,i} = \mathbb{C}^* \times (\mathbb{C} - \{0,1\})$ and $X_{PR}^{2,j} = \mathbb{C}^* \times \mathbb{C}^*$, have closure $\mathbb{C}^* \times \mathbb{C}$. Their intersections with X_{TR} are the curves with equation*

$$x^2 y^2 - (c_k + 2)(x^3 + y^3) + (c_k^2 + 5c_k + 4)xy - (c_k + 1)^3 = 0,$$

where $c_k = 2\cos(2\pi k/(mn))$ for $k \in \mathbb{Z}$, $k \notin m\mathbb{Z}$, and $k \notin n\mathbb{Z}$.

Two curves, indexed by k and k', belong to the closure of the same component if and only if $k' \equiv \pm k \pmod m$ and $k' \equiv \pm k \pmod n$. A curve belongs to the closure of a type 2 component precisely when, with n even, $k \equiv n/2 \pmod n$.

Proof The first assertion follows from the description of the closure of irreducible components in $\mathcal{X}_2$ and from the relationship between $\mathcal{X}_2$, $\bar{\mathcal{X}}_2$ and $\widetilde{\mathcal{X}}_2$, as $\widetilde{\mathcal{X}}_2^*$ is isomorphic to the locus of partially reducible representations of $\mathcal{X}_3$.

To describe the incidence of those components, we first discuss the incidence in $\mathcal{X}_2$ of reducible and irreducible components. According to Heusener, Porti and Suárez [4], the diagonal representation, with $A = \mathrm{diag}(s^m, s^{-m})$ and $B = \mathrm{diag}(s^n, s^{-n})$, can be deformed into an irreducible representation in $\mathrm{SL}(2,\mathbb{C})$ if and only if we have $\Delta_{m,n}(s^2) = 0$, where

$$\Delta_{m,n} = \frac{(t^{mn}-1)(t-1)}{(t^m-1)(t^n-1)}$$

is the Alexander polynomial of the torus knot. For matrices of $\mathrm{GL}(3,\mathbb{C})$, this leads to curves in $\overline{X^{\bullet,i}_{PR}} \cap X_{TR}$, determined by matrices $\mathrm{diag}(\delta t, \delta^{-1} t, t^{-2})$ with $\Delta_{m,n}(\delta^2) = 0$. In the coordinates of $\mathcal{X}_{3,TR}$, the equations with a parameter $t \in \mathbb{C}^*$ are

$$x = (\delta + \delta^{-1})t + t^{-2}, \qquad y = (\delta + \delta^{-1})t^{-1} + t^2,$$

with $\Delta_{m,n}(\delta^2) = 0$. Setting $c = \delta^2 + \delta^{-2}$, we obtain the curves of the statement. This gives $(m-1)(n-1)/2$ curves, hence we obtain all of them.

The assertion on the components is proved by using the argument of Proposition 7.1. Namely, the components of $\mathcal{X}_2$ are determined by the eigenvalues of A and B, which are $\{\delta^m, \delta^{-m}\}$ and $\{\delta^n, \delta^{-n}\}$, respectively, and then it follows from the discussion for the components of $\widetilde{\mathcal{X}}^*_2$ and the partially reducible components of $\mathcal{X}_3$. $\square$

Proposition 9.2 *The irreducible components of type I, $\mathcal{X}^*_{3,\tau} = \mathcal{M}/H$ for $H = T \times_D T$, have closure $\overline{\mathcal{X}^*_{3,\tau}} = \mathrm{GL}(3,\mathbb{C})/\!/H$. The boundary strata are given as follows.*

- *Orbits coming from 3×3 matrices M with two zeros in a row or a column. The representation is S-equivalent to $V \oplus W$, so it lies in some $X^{\bullet,i}_{PR}$.*
- *Orbits come from 3×3 matrices M with three zeros, two in a row, and two in a column (one of these zeros in common). This is S-equivalent to a diagonal matrix, ie a totally reducible representation $V \oplus V' \oplus V''$ lying in X_{TR}.*

The boundary strata consist of 9 lines of partially reducible representations that intersect each other following the pattern of the full graph $K_{3,3}$. In addition, the intersection points are precisely the totally reducible representations.

Proof The fact that the closure is $\mathrm{GL}(3,\mathbb{C})/\!/H$ is explained in Remark 6.3. The choice of two or three zeros as in the statement determines an invariant subspace or an invariant flag of the representation. Furthermore, in the closure of the orbit of this representation, we find another representation that is direct sum of irreducible ones (a semisimple one). This corresponds to the partially reducible and the totally reducible representations, and the eigenvalues of each factor can be computed. In particular, for the partially reducible representations, we obtain a whole component of $\mathcal{X}_2^*$ isomorphic to $\mathbb{C}-\{0,1\}$, whose closure is $\mathbb{C}$, by adding totally reducible representations.

The pattern of the compactification locus is the following:

- Draw a point for each totally reducible representation. This is the same as selecting three entries in the matrix not in the same row or column. There is a total of 6.
- Draw a line for each partially reducible representation. This is the same as fixing one entry in the matrix. The 2–dimensional representation is given by the 2×2 minor associated to it. There is a total of 9.
- Every line contains two points. Every point is in three lines.
- The pattern is the full graph $K_{3,3}$. It consists of all the edges connecting 3 points at the top with 3 points at the bottom in all possible ways. □

Proposition 9.3 *The irreducible components of type II, $\mathcal{X}_{3,\kappa}^*=\mathbb{C}^2-(\{x=0\}\cup\{y=0\}\cup\{x+y=1\})$, have closure $\overline{\mathcal{X}_{3,\kappa}^*}=\mathbb{C}^2$. The closure consists of adding three lines of partially reducible representations that intersect pairwise. The three intersection points are precisely the totally reducible representations.*

Proof Recall that, in this component, A has an eigenvalue with multiplicity two, and the eigenvalues of B are different (they could be the same changing the roles of A and B; the analysis would be similar in that case). This gives a point $p_0\in\mathbb{P}^2$ and a line $l_0\subset\mathbb{P}^2$ fixed by A, and three points $p_1,p_2,p_3\in\mathbb{P}^2$ fixed by B, with all points and lines in generic position. Putting l_0 as the line at infinity, p_1, p_2 and p_3 define an affine frame of $\mathbb{C}^2$, and the position of p_0 parametrizes the component. The closure is obtained by allowing p_0 to be at any position in $\mathbb{C}^2$. Notice that if p_0 belongs to the line through p_1 and p_2, then, since this line also meets l_0, it is preserved by both A and B; hence, we obtain a partially reducible representation. Total reducibility is obtained when p_0 equals one of $p_1,p_2,p_3\in\mathbb{P}^2$, because there are two invariant projective lines. □

Next, we want to understand the intersections of the components of $X_{PR}^{\bullet,i}$ with the closures of the components of irreducible representations. A component $X_{PR}^{\bullet,i}$ is determined by eigenvalues $\{\epsilon,1/\epsilon\}$ and $\{\varepsilon,1/\varepsilon\}$ corresponding to an irreducible representation in $\mathrm{SL}(2,\mathbb{C})$. The twisted Alexander polynomial for the representation in $\mathrm{SL}(2,\mathbb{C})$ is constant on each component, and it equals

$$\Delta_{\epsilon,\varepsilon}(t)=\frac{(t^{mn}-\epsilon^n)^2}{(t^m-\epsilon)(t^m-1/\epsilon)(t^n-\varepsilon)(t^n-1/\varepsilon)}.$$

See Kitano and Morifuji [6] for the computation of $\Delta_{\epsilon,\varepsilon}(t)$, or Heusener and Porti [3] for the trefoil.

Recall that $X_{PR}^{1,i} \cong (\mathbb{C}-\{0,1\})\times\mathbb{C}^*$ and that $X_{PR}^{2,i} \cong ((\mathbb{C}-\{0,1\})\times\mathbb{C}^*)/\mu_2$. Let $z\in\mathbb{C}^*$ denote the coordinate in the factor $\mathbb{C}^*$.

Proposition 9.4 *A component $X_{PR}^{\bullet,i}$ intersects the closure $\mathcal{X}_3^*$ precisely at the curves $\{z=z_0\}$ where $\Delta_{\epsilon,\varepsilon}(z_0^3)=0$. In addition, a curve defined by $z=z_0$ lies in the closure of a four dimensional component if z_0^3 is a root of multiplicity two of $\Delta_{\epsilon,\varepsilon}$, and a two dimensional component if it is a simple root.*

Proof By Heusener and Porti [3], $\Delta_{\epsilon,\varepsilon}(z^3)=0$ is a necessary condition for a representation with second coordinate $z\in\mathbb{C}^*$ to be deformed to irreducible representations. In addition, for simple roots, this condition is also sufficient, and the component of the character variety of irreducible ones has dimension two. Thus, we prove the proposition by counting the number of such curves obtained from the roots, and the number of curves in the closure of the variety of irreducible characters. Firstly, the degree of each polynomial $\Delta_{\epsilon,\varepsilon}(t^3)$ is $3(2mn-2m-2n)$. For m and n odd there are $\frac{1}{2}(m-1)\frac{1}{2}(n-1)$ possibilities for the eigenvalues $\{\epsilon,1/\epsilon\}$ and $\{\varepsilon,1/\varepsilon\}$; thus, we find at most $3\frac{1}{2}(m-1)\frac{1}{2}(n-1)2(mn-m-n)$ curves in the closure, counted twice for double roots. When n is even, we have $\frac{1}{2}(m-1)\frac{1}{2}(n-2)$ components as above but also $(m-1)/2$ components that are the quotient of a component of $\mathcal{X}_2\times\mathbb{C}^*$ by μ_2. The contribution of those components is half the contribution of the other components; thus, we get the same upper bound: $3(\frac{1}{2}(m-1)\frac{1}{2}(n-2)+\frac{1}{4}(m-1))2(mn-m-n) = 3\frac{1}{2}(m-1)\frac{1}{2}(n-1)2(mn-m-n)$.

On the other hand, there are $\frac{1}{12}(n-1)(n-2)(m-1)(m-2)$ components of irreducible representation of dimension 4 that have 9 lines in the adherence and that contribute to double roots. There are also $\frac{1}{2}(n-1)(m-2)(m+n-4)$ components of dimension two that contribute with three lines each. Thus, the total number of such lines (counted twice in the closure of a four dimensional component) is

$$2\cdot 9\tfrac{1}{12}(n-1)(n-2)(m-1)(m-2)+3\tfrac{1}{2}(n-1)(m-2)(m+n-4) = \tfrac{3}{2}(n-1)(m-1)(mn-m-n).$$

As we get precisely the previous upper bound, the proposition follows. □

10 Character varieties for GL(3, ℂ) and PGL(3, ℂ)

Now we describe the GL(3, ℂ) and PGL(3, ℂ)–character varieties $\widetilde{\mathcal{X}}_3$ and $\overline{\mathcal{X}}_3$.

Proposition 10.1 *The components of the* $\mathrm{PGL}(3,\mathbb{C})$*–character variety* $\bar{\mathcal{X}}_3$ *are:*

- *There is the component of totally reducible representations, which is isomorphic to* $\mathbb{C}^2/\mu_3 \cong \{(x,y,z)\in\mathbb{C}^3 \mid xy=z^3\}$.
- *There are* $[\frac{1}{2}(n-1)][\frac{1}{2}(m-1)]$ *components of partially reducible representations, each isomorphic to* $(\mathbb{C}-\{0,1\})\times\mathbb{C}^*$.
- *When* n *is even, there are* $(m-1)/2$ *additional components of partially reducible representations, each isomorphic to* $\{(u,v)\in\mathbb{C}^2 \mid v\neq 0, v\neq u^2\}$.
- *When* $m,n\notin 3\mathbb{Z}$, *there are these components of irreducible representations:*
  - $(n-1)(m-1)(n+m-4)/6$ *components isomorphic to* $(\mathbb{C}^*)^2-\{x+y=1\}$,
  - $(m-1)(m-2)(n-1)(n-2)/36$ *components of maximal dimension isomorphic to* $\mathcal{M}/(T\times_D T)$.
- *When* $n\in 3\mathbb{Z}$, *there are the following components of irreducible representations:*
  - $(m-1)(mn+n^2-5n-m+2)/6$ *components isomorphic to* $(\mathbb{C}^*)^2-\{x+y=1\}$,
  - $m-1$ *components isomorphic to* $\{(x,y,z)\in\mathbb{C}^3 \mid xy=z^3,\ x+y+3z\neq 1\}$,
  - $(m-1)(m-2)n(n-3)/36$ *components of maximal dimension isomorphic to* $\mathcal{M}/(T\times_D T)$,
  - $(m-1)(m-2)/6$ *components of maximal dimension that are isomorphic to* $\mathcal{M}/(T\times_D T\rtimes\mu_3)$, *where* μ_3 *acts by cyclic permutation of columns in* $\mathcal{M}$.

The case $m\in 3\mathbb{Z}$ *is symmetric.*

Proof Use the isomorphism $\bar{\mathcal{X}}_3=\mathcal{X}_3/\mu_3$ by (2) where $\mu_3=\{1,\varpi,\varpi^2\}$, and the stratification (9). The component $X_{TR}\cong\mathbb{C}^2$ is invariant by μ_3, and ϖ maps a point with coordinates (x,y) to $(\varpi x,\varpi^2 y)$. Hence, the quotient is $\mathbb{C}^2/\mu_3\cong\{(u,v,w)\in\mathbb{C}^3 \mid w^3=uv\}$, where $u=x^3, v=y^3$ and $w=xy$.

The components $X_{PR}^{1,i}$ correspond to components of $\widetilde{\mathcal{X}}_2^*\times\mathbb{C}^*$ that are not preserved by μ_2. Here ϖ acts trivially on $\widetilde{\mathcal{X}}_2^*$ and by multiplication by a third rood of unity on $\mathbb{C}^*$. This yields $[\frac{1}{2}(n-1)][\frac{1}{2}(m-1)]$ components isomorphic to $((\mathbb{C}-\{0,1\})\times\mathbb{C}^*)/\mu_3\cong(\mathbb{C}-\{0,1\})\times\mathbb{C}^*$.

For n even, the components $X_{PR}^{2,j}$ are isomorphic to $\{(u,v)\in\mathbb{C}^2 \mid v\neq 0, v\neq u^3\}$. To understand the action of μ_3, we build on the proof of Proposition 7.3. Recall from the proof of Proposition 7.3 that $u=(2r-1)\lambda$ and $v=\lambda^2$, where $r\in\mathbb{C}-\{0,1\}$ is the coordinate of a component $\mathcal{X}_2$ and $\lambda\in\mathbb{C}^*$. Since ϖ acts trivially on r and maps λ to $\varpi\lambda$, the action of ϖ on those coordinates is $(u,v)\mapsto(\varpi u,\varpi^2 v)$. The quotient is isomorphic to $\{(z,w)\in\mathbb{C}^2 \mid w\neq 0, w\neq z^2\}$, via $z=u/v^2$ and $w=1/v^3$.

We next discuss the irreducible components. The two-dimensional components, $\mathcal{X}^*_{3,\kappa}$ for $\kappa \in G$, are parametrized by the eigenvalues of A and B: $\{\epsilon, \epsilon, 1/\epsilon^2\}$ and $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ subject to $\epsilon^n = \varepsilon_i^m = \varpi^k$, $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$, $\epsilon \neq 1/\epsilon^2$, and $\varepsilon_1, \varepsilon_2$ and ε_3 being distinct. Then ϖ maps those eigenvalues to $\{\varpi^m\epsilon, \varpi^m\epsilon, 1/(\varpi^m\epsilon)^2\}$ and $\{\varepsilon_1\varpi^n, \varepsilon_2\varpi^n, \varepsilon_3\varpi^n\}$. They happen to be the same set of eigenvalues precisely when $m \in 3\mathbb{Z}$ and $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{1, \varpi, \varpi^2\}$. Therefore μ_3 permutes all components $\mathcal{X}^*_{3,\kappa}$ except when $m \in 3\mathbb{Z}$, those $n-1$ components are preserved. Thus, when $m, n \notin 3\mathbb{Z}$ we obtain $(n-1)(m-1)(n+m-4)/6$ components isomorphic to $(\mathbb{C}^*)^2 - \{x + y = 1\}$. When $m \in 3\mathbb{Z}$ we obtain $(n-1)(mn + m^2 - 5m - n + 2)/6$ such components, and $n-1$ additional components isomorphic to $((\mathbb{C}^*)^2 - \{x + y = 1\})/\mu_3$, where μ_3 acts linearly on $\mathbb{C}^2$ and cyclically permutes the three lines that we have removed; that is, $(x, y) \mapsto (1 - x - y, x)$. The quotient is isomorphic to $\{(u, v, w) \in \mathbb{C}^3 \mid uv = w^3,\ u + v + 3w \neq 1\}$, by taking coordinates $u = \frac{1}{9}((3x-1) - \varpi(3y-1))^3$, $v = \frac{1}{9}((3x-1) - \varpi^2(3y-1))^3$ and $w = 9^{-2/3}((3x-1) - \varpi(3y-1))((3x-1) - \varpi^2(3y-1))$.

The four dimensional components $\mathcal{X}^*_{3,\kappa}$ are parametrized by $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ and $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ distinct, subject to $\epsilon_i^n = \varepsilon_j^m = \varpi^k$ and $\epsilon_1\epsilon_2\epsilon_3 = \varepsilon_1\varepsilon_2\varepsilon_3 = 1$, being the eigenvalues of A and B, respectively. The generator of the cyclic group maps those eigenvalues to $\{\epsilon_1\varpi^m, \epsilon_2\varpi^m, \epsilon_3\varpi^m\}$ and $\{\varepsilon_1\varpi^n, \varepsilon_2\varpi^n, \varepsilon_3\varpi^n\}$. An elementary computation proves that a component is invariant precisely when $n \in 3\mathbb{Z}$ and $\{\epsilon_1, \epsilon_2, \epsilon_3\} = \{1, \varpi, \varpi^2\}$, or $m \in 3\mathbb{Z}$ and $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{1, \varpi, \varpi^2\}$. Hence, when $m, n \notin 3\mathbb{Z}$, we obtain $(m-1)(m-2)(n-1)(n-2)/36$ components isomorphic to the components of $\mathcal{X}^*_{3,\kappa}$. When $n \in 3\mathbb{Z}$, we obtain $(m-1)(m-2)n(n-3)/36$ such components and $(m-1)(m-2)/6$ components that are quotiented by the action of μ_3, which can be interpreted as cyclic permutation of columns in $\mathcal{M}$. $\square$

Remark 10.2 The closure of each component can be easily deduced from the closures of the components of $\mathcal{X}_3$. Just notice that, when $n \in 3\mathbb{Z}$, the closure of each component isomorphic to $\{(x, y, z) \in \mathbb{C}^3 \mid xy = z^3,\ x + y + 3z \neq 1\}$ is the hypersurface $xy = z^3$, and the curve of reducible representations $x + y + 3z = 1$ is singular exactly at $-x = -y = z = 1$, the totally reducible representation.

Corollary 10.3 *The K–theory class of the character variety $\overline{\mathcal{X}}_3$ is as follows.*

- *If $n, m \equiv 1, 5 \pmod 6$, then*

$$[\overline{\mathcal{X}}_3] = P_0 + \tfrac{1}{36}(m-1)(m-2)(n-1)(n-2)P_1 + \tfrac{1}{6}(n-1)(m-1)(n+m-4)P_3 + \tfrac{1}{4}(n-1)(m-1)P_5.$$

- *If* $n \equiv 2, 4 \pmod 6$, $m \equiv 1, 5 \pmod 6$, *then*

$$[\bar{\mathcal{X}}_3] = P_0 + \tfrac{1}{36}(m-1)(m-2)(n-1)(n-2)\,P_1 \\ + \tfrac{1}{6}(n-1)(m-1)(n+m-4)\,P_3 + \tfrac{1}{4}(n-2)(m-1)\,P_5 + \tfrac{1}{2}(m-1)\,P_6.$$

- *If* $n \equiv 3 \pmod 6$, $m \equiv 1, 5 \pmod 6$, *then*

$$[\bar{\mathcal{X}}_3] = P_0 + \tfrac{1}{36}(m-1)(m-2)n(n-3)\,P_1 + \tfrac{1}{6}(m-1)(m-2)\,P_2 \\ + \tfrac{1}{6}(m-1)(mn+n^2-5n-m-2)\,P_3 + (m-1)\,P_4 + \tfrac{1}{4}(n-1)(m-1)\,P_5.$$

- *If* $n \equiv 0 \pmod 6$, $m \equiv 1, 5 \pmod 6$, *then*

$$[\bar{\mathcal{X}}_3] = P_0 + \tfrac{1}{36}(m-1)(m-2)n(n-3)\,P_1 + \tfrac{1}{6}(m-1)(m-2)\,P_2 \\ + \tfrac{1}{6}(m-1)(mn+n^2-5n-m-2)\,P_3 \\ + (m-1)\,P_4 + \tfrac{1}{4}(n-2)(m-1)\,P_5 + \tfrac{1}{2}(m-1)\,P_6.$$

- *If* $n \equiv 2, 4 \pmod 6$, $m \equiv 3 \pmod 6$, *then*

$$[\bar{\mathcal{X}}_3] = P_0 + \tfrac{1}{36}m(m-3)(n-1)(n-2)\,P_1 + \tfrac{1}{6}(n-1)(n-2)\,P_2 \\ + \tfrac{1}{6}(n-1)(mn+m^2-n-5m-2)\,P_3 \\ + (n-1)\,P_4 + \tfrac{1}{4}(n-2)(m-1)\,P_5 + \tfrac{1}{2}(m-1)\,P_6.$$

Here, $P_0 = \mathbb{L}^2$, $P_1 = \mathbb{L}^4 + 4\mathbb{L}^3 - 3\mathbb{L}^2 - 15\mathbb{L} + 12$, $P_2 = \mathbb{L}^4 + 2\mathbb{L}^3 - 3\mathbb{L}^2 - \mathbb{L} + 4$, $P_3 = \mathbb{L}^2 - 3\mathbb{L} + 3$, $P_4 = \mathbb{L}^2 - \mathbb{L} + 1$, $P_5 = \mathbb{L}^2 - 3\mathbb{L} + 2$ *and* $P_6 = \mathbb{L}^2 - 2\mathbb{L} + 1$.

Proof The proof is analogous to that of Theorem 8.3. First, $[\mathbb{C}^2/\mu_3] = \mathbb{L}^2$ by Proposition 3.1. Second, $[((\mathbb{C}^*)^2 - \{x+y=1\})/\mu_3] = \mathbb{L}^2 - (\mathbb{L}-1)$, since the three lines in the quotient produce a single line with two points identified. Finally, it remains to compute the K–theory class $[\mathcal{M}/(T \times_D T \rtimes \mu_3)]$.

For this, we use the description of $\mathcal{M}/(T \times_D T)$ given in Remark 8.5. The action of μ_3 is given by cyclic permutation of p_1, p_2 and p_3. We divide the computation in two cases:

- All of p_1, p_2 and p_3 lie off L_1. Then $p_1, p_2, p_3 \in \mathbb{C}^2 = \mathbb{P}^2 - L_1$. The quotient $(\mathbb{C}^2)^3/\mu_3$ has class $\mathbb{L}^6$ by Proposition 3.1. Now we have to remove various contributions as given in the definition of $\mathcal{V}$:

  (1) If the p_i are equal, we get $\mathbb{L}^2$.

  (2) Suppose the p_i are not equal but lie on a line. The space parametrizing lines is a $\mathbb{P}^2$ minus a point corresponding to the line of infinity, hence giving $[\mathbb{P}^2] - 1 = \mathbb{L}^2 + \mathbb{L}$. The space parametrizing triples of points in a line is $[\mathbb{C}^3/\mu_3] = \mathbb{L}^3$

by Proposition 3.1. As we are considering unequal points, we have $\mathbb{L}^3-\mathbb{L}$. Multiplying, we get $(\mathbb{L}^2+\mathbb{L})(\mathbb{L}^3-\mathbb{L})$.

(3) Suppose the p_i are independent and that one is equal to q_1. Using μ_3, we can suppose $p_1=q_1$. Then $p_2\in\mathbb{C}^2-\{q_1\}$, and p_3 lies off the line $\langle q_1,p_2\rangle$. This yields $(\mathbb{L}^2-1)(\mathbb{L}^2-\mathbb{L})$.

(4) Suppose two of p_1, p_2 and p_3 lie on L_1 or L_2 (and that the three are independent and none is q_1). Using the cyclic permutation, we can assume this happens to p_1 and p_2. Then we have $2(\mathbb{L}-1)(\mathbb{L}-2)(\mathbb{L}^2-\mathbb{L})$, the last factor accounting for the fact that $p_3\notin\langle p_1,p_2\rangle$.

This gives a total of

$$\mathbb{L}^6-(\mathbb{L}^2+(\mathbb{L}^2+\mathbb{L})(\mathbb{L}^3-\mathbb{L})+(\mathbb{L}^2-1)(\mathbb{L}^2-\mathbb{L})+2(\mathbb{L}-1)(\mathbb{L}-2)(\mathbb{L}^2-\mathbb{L}))$$
$$=(\mathbb{L}-1)^2(\mathbb{L}^4+\mathbb{L}^3-3\mathbb{L}^2+3\mathbb{L}).$$

• If one of p_1, p_2 and p_3 lie in L_1, the cyclic permutation allows us to assume that $p_1\in L_1-\{q_2,q_3\}$. This produces a factor $(\mathbb{L}-1)$. Also, $p_2,p_3\in\mathbb{C}^2$, producing $\mathbb{L}^4$. We remove:

(1) If p_i are equal, we get $\mathbb{L}^2$.

(2) Suppose the p_i are not equal but lie on a line with direction given by p_1. This gives $\mathbb{L}(\mathbb{L}^2-\mathbb{L})$.

(3) Suppose one p_i is equal to q_1, and the remaining p_j is not collinear with the other two. This yields $2(\mathbb{L}^2-\mathbb{L})$.

(4) Suppose p_2 and p_3 lie on L_1 or L_2 (and neither is q_1). Then we have $2(\mathbb{L}-1)(\mathbb{L}-2)$.

This total is

$$(\mathbb{L}-1)(\mathbb{L}^4-(\mathbb{L}^2+\mathbb{L}(\mathbb{L}^2-\mathbb{L})+2(\mathbb{L}^2-\mathbb{L})+2(\mathbb{L}-1)(\mathbb{L}-2)))$$
$$=(\mathbb{L}-1)^2(\mathbb{L}^3-4\mathbb{L}+4).$$

Adding both contributions and dividing by $(\mathbb{L}-1)^2$, we get

$$[\mathcal{M}/(T\times_D T\rtimes\mu_3)]=\mathbb{L}^4+2\mathbb{L}^3-3\mathbb{L}^2-\mathbb{L}+4. \qquad \square$$

By substituting $\mathbb{L}\mapsto 1$ in Corollary 10.3, we obtain the Euler characteristic of $\bar{\mathcal{X}}_3$. For m and n coprime, we have:

• If $m,n\not\equiv 0,3 \pmod 6$, then

$$\chi(\bar{\mathcal{X}}_3)=1-\tfrac{1}{36}(m-1)(m-2)(n-1)(n-2)+\tfrac{1}{6}(n-1)(m-1)(n+m-4).$$

- If $n \equiv 0, 3 \pmod 6$, then

$$\chi(\overline{\mathcal{X}}_3) = 1 - \tfrac{1}{36}(m-1)(m-2)n(n-3) + \tfrac{1}{2}(m-1)(m-2) + \tfrac{1}{6}(m-1)(mn+n^2-5n-m-2) + (m-1).$$

An argument similar (but longer) to that in Corollary 8.4 proves that one can recover n and m up to order from the K–theory class $[\overline{\mathcal{X}}_3]$.

Proposition 10.4 *The components of the* $\mathrm{GL}(3,\mathbb{C})$*–character variety* $\widetilde{\mathcal{X}}_3$ *are:*

- *There is the component of totally reducible representations, which is isomorphic to* $\mathbb{C}^2 \times \mathbb{C}^*$.
- *There are* $[\frac{1}{2}(n-1)][\frac{1}{2}(m-1)]$ *components of partially reducible representations, each isomorphic to* $(\mathbb{C}-\{0,1\}) \times (\mathbb{C}^*)^2$.
- *When* n *is even, there are* $(m-1)/2$ *additional components of partially reducible representations, each isomorphic to* $\{(x,y,z) \in \mathbb{C}^3 \mid y,z \neq 0, y \neq z^2\}$.
- *When* $m,n \notin 3\mathbb{Z}$*, there are these components of irreducible representations:*
  - $(n-1)(m-1)(n+m-4)/6$ *components, each isomorphic to* $((\mathbb{C}^*)^2 - \{x+y=1\}) \times \mathbb{C}^*$.
  - $(m-1)(m-2)(n-1)(n-2)/36$ *components of maximal dimension, each isomorphic to* $\mathcal{M}/(T \times_D T) \times \mathbb{C}^*$.
- *When* $n \in 3\mathbb{Z}$*, there are the following components of irreducible representations:*
  - $(m-1)(mn+n^2-5n-m+2)/6$ *components, each isomorphic to* $((\mathbb{C}^*)^2 - \{x+y=1\}) \times \mathbb{C}^*$.
  - $m-1$ *components isomorphic to* $\{(u,v,w) \in \mathbb{C}^3 \mid u^3+v^3+3uv-w \neq 0, w \neq 0\}$.
  - $(m-1)(m-2)n(n-3)/36$ *components of maximal dimension isomorphic to* $\mathcal{M}/(T \times_D T) \times \mathbb{C}^*$.
  - $(m-1)(m-2)/6$ *components of maximal dimension, each isomorphic to* $(\mathcal{M}/(T \times_D T) \times \mathbb{C}^*)/\mu_3$*, where* μ_3 *acts by cyclic permutation of columns in* $\mathcal{M}$ *and multiplication on* $\mathbb{C}^*$.

  The case $m \in 3\mathbb{Z}$ *is symmetric.*

To prove this proposition, one may use the isomorphism between $\widetilde{\mathcal{X}}_3$ and $(\mathcal{X}_3 \times \mathbb{C}^*)/\mu_3$ and the proof is analogous to the discussion in the proof of Proposition 10.1. Namely, the components of $\mathcal{X}_3$ that are invariant (or not) by the action of μ_3 correspond to the components of $\mathcal{X}_3 \times \mathbb{C}^*$ that are invariant (or not), and the computation of the quotients of the invariant ones is changed by the factor $\mathbb{C}^*$.

Finally, the K–theory class is $[\widetilde{\mathcal{X}}_3] = (\mathbb{L}-1)[\overline{\mathcal{X}}_3]$, by Remark 4.2.

Acknowledgements We are grateful to Marina Logares, Jorge Martín-Morales, Maite Lozano, Javier Martínez and Sean Lawton for their interest in this work. Thanks to the referee for useful comments. The contribution of Henrik Rüping on MathOverflow was useful in our argument in Proposition 3.1. The second author is partially supported by Mineco through grant MTM2012-34834.

References

[1] **M Culler**, **P B Shalen**, *Varieties of group representations and splittings of 3–manifolds*, Ann. of Math. 117 (1983) 109–146 MR683804

[2] **T Hausel**, **M Thaddeus**, *Mirror symmetry, Langlands duality, and the Hitchin system*, Invent. Math. 153 (2003) 197–229 MR1990670

[3] **M Heusener**, **J Porti**, *Representations of knot groups into* $\mathrm{SL}_n(\mathbb{C})$ *and twisted Alexander polynomials*, Pacific J. Math. 277 (2015) 313–354 MR3402353

[4] **M Heusener**, **J Porti**, **E Suárez Peiró**, *Deformations of reducible representations of 3–manifold groups into* $\mathrm{SL}_2(\mathbf{C})$, J. Reine Angew. Math. 530 (2001) 191–227 MR1807271

[5] **N J Hitchin**, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. 55 (1987) 59–126 MR887284

[6] **T Kitano**, **T Morifuji**, *Twisted Alexander polynomials for irreducible* $\mathrm{SL}(2,\mathbb{C})$*–representations of torus knots*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 11 (2012) 395–406 MR3011996

[7] **S Lawton**, *Minimal affine coordinates for* $\mathrm{SL}(3,\mathbb{C})$ *character varieties of free groups*, J. Algebra 320 (2008) 3773–3810 MR2457722

[8] **S Lawton**, **V Muñoz**, E*–polynomial of the* $\mathrm{SL}(3,\mathbb{C})$*–character variety of free groups* (2014) `arXiv:1405.0816`

[9] **M Logares**, **V Muñoz**, **P E Newstead**, *Hodge polynomials of* $\mathrm{SL}(2,\mathbb{C})$*–character varieties for curves of small genus*, Rev. Mat. Complut. 26 (2013) 635–703 MR3068615

[10] **A Lubotzky**, **A R Magid**, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. 336, Amer. Math. Soc., Providence, RI (1985) MR818915

[11] **J Martín-Morales**, **A M Oller-Marcén**, *Combinatorial aspects of the character variety of a family of one-relator groups*, Topology Appl. 156 (2009) 2376–2389 MR2547783

[12] **V Muñoz**, *The* $\mathrm{SL}(2,\mathbb{C})$*–character varieties of torus knots*, Rev. Mat. Complut. 22 (2009) 489–497 MR2553945

[13] **D Rolfsen**, *Knots and links*, Mathematics Lecture Series 7, Publish or Perish, Houston, TX (1990) MR1277811

[14] **A S Sikora**, *Generating sets for coordinate rings of character varieties*, J. Pure Appl. Algebra 217 (2013) 2076–2087 MR3057078

VM: *Facultad de Matemáticas, Universidad Complutense de Madrid*
Plaza de Ciencias 3, 28040 Madrid, Spain

VM: *Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM)*
C/ Nicolás Cabrera 15, 28049 Madrid, Spain

JP: *Departament de Matemàtiques, Universitat Autònoma de Barcelona*
Cerdanyola del Valles, 08193 Barcelona, Spain

vicente.munoz@mat.ucm.es, porti@mat.uab.cat

http://www.mat.ucm.es/~vmunozve/, http://mat.uab.es/~porti/

Received: 29 September 2014 Revised: 7 June 2015