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# On $p$–almost direct products and residual properties of pure braid groups of nonorientable surfaces

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We prove that the $n^{\text{th}}$ pure braid group of a nonorientable surface (closed or with boundary, but different from $\mathbb{RP}^2$) is residually $2$–finite. Consequently, this group is residually nilpotent. The key ingredient in the closed case is the notion of $p$–almost direct product, which is a generalization of the notion of almost direct product. We also prove some results on lower central series and augmentation ideals of $p$–almost direct products.

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## 1 Introduction

Let $M$ be a compact, connected surface (orientable or not, possibly with boundary) and $F_n(M) = \{(x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$ its $n^{\text{th}}$ configuration space. The fundamental group $\pi_1(F_n(M))$ is called the $n^{\text{th}}$ *pure braid group* of $M$ and shall be denoted by $P_n(M)$.

The mapping class group $\Gamma(M)$ of $M$ is the group of isotopy classes of homeomorphisms $h\colon M \to M$ which act as the identity on the boundary. Let $\mathcal{X}_n = \{z_1, \dots, z_n\}$ be a set of $n$ distinguished points in the interior of $M$; the pure mapping class group $\mathrm{P}\Gamma(M, \mathcal{X}_n)$ relative to $\mathcal{X}_n$ is the group of isotopy classes of homeomorphisms $h\colon M \to M$ satisfying $h(z_i) = z_i$ for all $i$: since this group does not depend on the choice of the set $\mathcal{X}_n$ but only on its cardinality we can write $\mathrm{P}_n\Gamma(M)$ instead of $\mathrm{P}\Gamma(M, \mathcal{X}_n)$. Forgetting the marked points, we get a morphism $\mathrm{P}_n\Gamma(M) \to \Gamma(M)$ whose kernel is known to be isomorphic to $P_n(M)$ when $M$ is not a sphere, a torus, a projective plane or a Klein bottle (see Scott [28] and Guaschi and Juan-Pineda [20]).

Now, recall that if $\mathcal{P}$ is a group-theoretic property, then a group $G$ is said to be *residually* $\mathcal{P}$ if, for all $g \in G$, $g \neq 1$, there exists a group homomorphism $\varphi\colon G \to H$ such that $H$ satisfies $\mathcal{P}$ and $\varphi(g) \neq 1$. We are interested in the following properties: to be nilpotent, to be free and to be a finite $p$–group for a prime number $p$ (mostly $p = 2$). Recall that, if for subgroups $H$ and $K$ of $G$, $[H, K]$ is the subgroup generated

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by $\{[h,k] \mid (h,k) \in H \times K\}$ where $[h,k] = h^{-1}k^{-1}hk$, the lower central series of $G$, $(\Gamma_k G)_{k \geq 1}$, is defined inductively by $\Gamma_1 G = G$ and $\Gamma_{k+1} G = [G, \Gamma_k G]$. It is well known that $G$ is residually nilpotent if and only if $\bigcap_{k=1}^{\infty} \Gamma_k G = \{1\}$. From the lower central series of $G$ one can define another filtration $D_1(G) \supseteq D_2(G) \supseteq \cdots$ setting $D_1(G) = G$, and for $i \geq 2$ defining

$$D_i(G) = \{x \in G \mid \exists\, n \in \mathbb{N}^* \text{ with } x^n \in \Gamma_i(G)\}.$$

After Garoufalidis and Levine [13], this filtration is called the *rational lower central series* of $G$, and a group $G$ is residually torsion-free nilpotent if and only if $\bigcap_{i=1}^{\infty} D_i(G) = \{1\}$.

When $M$ is an orientable surface of positive genus (possibly with boundary) or a disc with holes, it is proved in Bellingeri, Gervais and Guaschi [6] and Bardakov and Bellingeri [1] that $P_n(M)$ is residually torsion-free nilpotent for all $n \geq 1$. The fact that a group is residually torsion-free nilpotent has several important consequences, notably that the group is bi-orderable (see Botto Mura and Rhemtulla [8]) and residually $p$–finite (see Gruenberg [19]). The goal of this article is to study the nonorientable case and, more precisely, to prove the following:

**Theorem 1.1** *The $n^{th}$ pure braid group of a nonorientable surface different from $\mathbb{RP}^2$ is residually 2–finite.*

In the case of $P_n(\mathbb{RP}^2)$ we give some partial results at the end of Section 4. Since a finite 2–group is nilpotent, a residually 2–finite group is residually nilpotent. Thus, we have:

**Corollary 1.2** *The $n^{th}$ pure braid group of a nonorientable surface different from $\mathbb{RP}^2$ is residually nilpotent.*

In González-Meneses [17] it was shown that the $n^{\text{th}}$ pure braid group of a nonorientable surface is not bi-orderable and therefore it is not residually torsion-free nilpotent. Our technique doesn't extend to $p \neq 2$; therefore the question if pure braid groups of nonorientable surfaces are residually $p$ for some $p \neq 2$ is still open (recall that there are groups residually $p$ for infinitely many primes $p$ which are not residually torsion-free nilpotent; see Hartley [21]).

One can prove that finite-type invariants separate classical braids using the fact that the pure braid group $P_n$ is residually nilpotent without torsion (see Papadima [25]). Moreover, using the residual properties discussed above it is possible to construct algebraically a universal finite-type invariant over $\mathbb{Z}$ for the classical braid group $B_n$ (see [25]). Similar constructions were afterwards proposed for braids on orientable surfaces (see Bellingeri and Funar [5] and González-Meneses and Paris [18]): in a further paper we will explore the relevance of Theorem 1.1 in the realm of finite-type invariants over $\mathbb{Z}/2\mathbb{Z}$ for braids on nonorientable surfaces.

From now on, $M = N_{g,b}$ is a nonorientable surface of genus $g$ with $b$ boundary components, simply denoted by $N_g$ when $b=0$. We will see $N_g$ as a sphere $S^2$ with $g$ open discs removed and $g$ Möbius strips glued on each circle (see Figure 4, where each crossed disc represents a Möbius strip). The surface $N_{g,b}$ is obtained from $N_g$ by removing $b$ open discs. The mapping class groups $\Gamma(N_{g,b})$ and pure mapping class group $P_n\Gamma(N_{g,b})$ will be denoted by $\Gamma_{g,b}$ and $\Gamma^n_{g,b}$, respectively.

The paper is organized as follows. In Section 2, we prove Theorem 1.1 for surfaces with boundary: following what the authors did in the orientable case (see [6]), we embed $P_n(N_{g,b})$ in a Torelli group. The difference here is that we must consider mod 2 Torelli groups. In Section 3 we introduce the notion of $p$–almost direct product, which generalizes the notion of almost direct product (see Definition 3.1) and we prove some results on lower central series and augmentation ideals of $p$–almost direct products (Theorems 3.3 and 3.6) that can be compared with similar results on almost direct products (Theorem 3.1 in Falk and Randell [12] and Theorem 3.1 in Papadima [25]).

In Section 4, the existence of a split exact sequence

$$1 \longrightarrow P_{n-1}(N_{g,1}) \longrightarrow P_n(N_g) \longrightarrow \pi_1(N_g) \longrightarrow 1$$

and results from Sections 2 and 3 are used to prove Theorem 1.1 in the closed case (Theorem 4.5). The method is similar to the one developed for orientable surfaces in [1]: the difference will be that the semi-direct product $P_{n-1}(N_{g,1}) \rtimes \pi_1(N_g)$ is a 2–almost-direct product (and not an almost-direct product as in the case of closed oriented surfaces).

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## 2 The case of non-empty boundary

In this section, $N = N_{g,b}$ is a nonorientable surface of genus $g \geq 1$ with boundary (ie $b \geq 1$). In this case, one has $P_n(N) = \mathrm{Ker}(\Gamma^n_{g,b} \to \Gamma_{g,b})$ for all $n \geq 1$.

## 2.1 Notation

We will follow notation from [27]. A simple closed curve in $N$ is an embedding $\alpha\colon S^1 \to N \setminus \partial N$; with a usual abuse of notation, we will call the image of a simple closed curve a simple closed curve also. Such a curve is said to be two-sided or one-sided if it admits a regular neighborhood homeomorphic to an annulus or a Möbius strip, respectively. We shall consider the following elements in $\Gamma_{g,b}$:

- If $\alpha$ is a two-sided simple closed curve in $N$ with a given orientation, $\tau_\alpha$ is a Dehn twist along $\alpha$.
- Let $\mu$ and $\alpha$ be two simple closed curves such that $\mu$ is one-sided, $\alpha$ is oriented and two-sided, and such that $|\alpha \cap \mu| = 1$. A regular neighborhood $K$ of $\alpha \cup \mu$ is diffeomorphic to a Klein bottle with one hole, and a regular neighborhood $M$ of $\mu$ is diffeomorphic to a Möbius strip. Pushing $M$ once along $\alpha$, we get a diffeomorphism of $K$ fixing the boundary (see Figure 1): it can be extended to $N$ by the identity. Such a diffeomorphism is called a crosscap slide, and denoted by $Y_{\mu,\alpha}$.

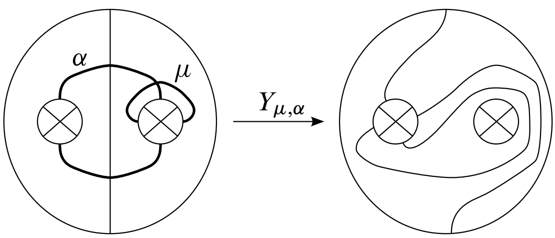


Figure 1: Crosscap slide.

## 2.2 Blowup homomorphism

Here we recall the construction of the blowup homomorphism $\eta^n_{g,b}\colon \Gamma^n_{g,b} \to \Gamma_{g+n,b}$ given in [30; 31] and [27].

Let $U = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and, for $i = 1, \dots, n$, fix an embedding $e_i\colon U \to N$ such that $e_i(0) = z_i$, $e_i(U) \cap e_j(U) = \varnothing$ if $i \neq j$ and $e_i(U) \cap \partial N = \varnothing$ for all $i$. If we remove the interior of each $e_i(U)$ (thus getting the surface $N_{g,b+n}$) and identify, for each $z \in \partial U$, $e_i(z)$ with $e_i(-z)$, we get a nonorientable surface of genus $g+n$ with $b$ boundary components, that is to say a surface homeomorphic to $N_{g+n,b}$. Let us denote by $\gamma_i = e_i(S^1)$ the boundary of $e_i(U)$, and by $\mu_i$ its image in $N_{g+n,b}$; it is a one-sided simple closed curve.

Now, let $h$ be an element of $\Gamma^n_{g,b}$. It can be represented by a homeomorphism $N_{g,b} \to N_{g,b}$, still denoted $h$, such that:

(1) $h(e_i(z)) = e_i(z)$ if $h$ preserves local orientation at $z_i$.

(2) $h(e_i(z)) = e_i(\bar{z})$ if $h$ reverses local orientation at $z_i$.

Such a homeomorphism $h$ commutes with the identification leading to $N_{g+n,b}$ and thus induces an element $\eta(h) \in \Gamma_{g+n,b}$. It is proved in [31] that the map

$$\eta^n_{g,b} = \eta\colon \Gamma^n_{g,b} \to \Gamma_{g+n,b}, \quad h \longmapsto \eta(h)$$

is well defined for $n = 1$, but the proof also works for $n > 1$. This homomorphism is called the *blowup* homomorphism.

**Proposition 2.1** *The blowup homomorphism $\eta^n_{g,b}\colon \Gamma^n_{g,b} \to \Gamma_{g+n,b}$ is injective if $(g+n,b) \neq (2,0)$.*

**Remark 2.2** This result is proved in [30] for $(g,b) = (0,1)$, but the proof can be adapted in our case as follows.

**Proof** Suppose that $h\colon N_{g,b} \to N_{g,b}$ is a homeomorphism satisfying $h(z_i) = z_i$ for all $i$ and that $\eta(h)\colon N_{g+n,b} \to N_{g+n,b}$ is isotopic to the identity. Then $h$ is isotopic to a map equal to the identity on $e_i(U)$ for all $i$. If not, $h$ reverses local orientation at $z_i$ and $h(\gamma_i)$ is isotopic to $\gamma_i^{-1}$. Then $\eta(h)(\gamma_i)$ is isotopic to $\mu_i$ and $\mu_i^{-1}$ and we get $2[\mu_i] = 0$ in $H_1(N_{g+n,b};\mathbb{Z})$, which is a contradiction. Consequently, $h$ lies in the kernel of the natural map $\Gamma_{g,b+n} \to \Gamma_{g+n,b}$ induced by gluing a Möbius strip onto $n$ boundary components. However, this kernel is generated by the Dehn twists along the curves $\gamma_i$ (see [29, Theorem 3.6[1]]). Now, any $\gamma_i$ bounds a disc with one marked point in $N_{g,b}$: the corresponding Dehn twist is trivial in $\Gamma_{g,b}$ and therefore $h$ is isotopic to the identity. □

## 2.3 Embedding $P_n(N_{g,b})$ in $\Gamma_{g+n+2(b-1),1}$

Since $b \geq 1$, we'll view $N_{g,b}$ as a disc $D^2$ with $g+b-1$ open discs removed and $g$ Möbius strips glued on $g$ boundary components so obtained (see Figure 2).

**Proposition 2.3** *For $g \geq 1$, $b \geq 1$ and $n \geq 1$, $P_n(N_{g,b})$ has the following complete set of generators (depicted in Figures 2 and 4):*

$$(B_{i,j})_{1 \leq i < j \leq n}, \quad (\rho_{k,l})_{\substack{1 \leq k \leq n \\ 1 \leq l \leq g}} \quad \text{and} \quad (x_{u,t})_{\substack{1 \leq u \leq n \\ 1 \leq t \leq b-1}}.$$

[1]This result is wrong when $(g+n,b) = (2,0)$.

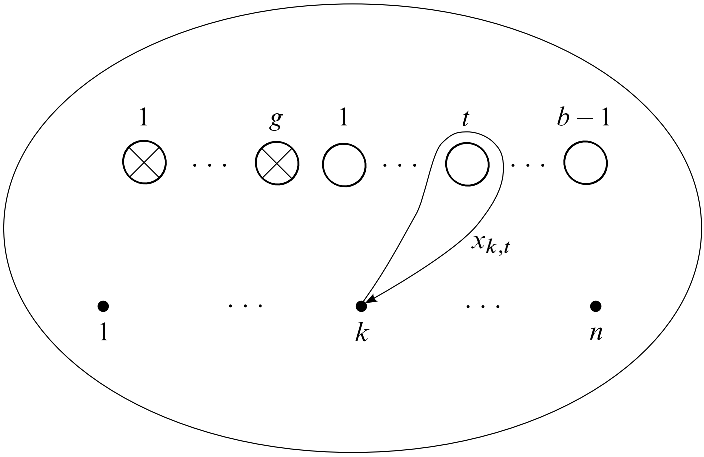


Figure 2: Generators $x_{k,t}$ for $P_n(N_{g,b})$, $b \geq 1$. See Figure 4 for a picture of generators $B_{i,j}$ and $\rho_{k,l}$.

**Proof** The proof works by induction and generalizes those of [16] (closed nonorientable case) and [4] (orientable case, possibly with boundary components). It uses the following short exact sequence obtained by forgetting the last strand (see [11]):

$$1 \longrightarrow \pi_1(N_{g,b} \setminus \{z_1, \dots, z_n\}, z_{n+1}) \xrightarrow{\alpha} P_{n+1}(N_{g,b}) \xrightarrow{\beta} P_n(N_{g,b}) \longrightarrow 1.$$

The set of generators is complete for $n = 1$: $P_1(N_{g,b}) = \pi_1(N_{g,b})$ is free on the $\rho_{1,l}$ and $x_{1,t}$ for $1 \leq l \leq g$ and $1 \leq t \leq b-1$. Suppose inductively that $P_n(N_{g,b})$ has the given complete set of generators. Then observe that

$$\{B_{i,n+1} \mid 1 \leq i \leq n\} \cup \{\rho_{n+1,l} \mid 1 \leq l \leq g\} \cup \{x_{n+1,t} \mid 1 \leq t \leq b-1\}$$

is a free generators set of $\mathrm{Im}(\alpha)$ and

$$(B_{i,j})_{1 \leq i < j \leq n}, \quad (\rho_{k,l})_{\substack{1 \leq k \leq n \\ 1 \leq l \leq g}} \quad \text{and} \quad (x_{u,t})_{\substack{1 \leq u \leq n \\ 1 \leq t \leq b-1}}$$

are coset representatives for the considered generators of $P_n(N_{g,b})$; this is a complete set of generators for $P_{n+1}(N_{g,b})$; see for instance [22, Theorem 1, Chapter 13]. Let us also remark that the above exact sequence could be used, as in [4] and [16], to find a complete set of relations for the group $P_n(N_{g,b})$. □

Gluing a one-holed torus onto $b-1$ boundary components of $N_{g,b}$ (recall that $b \geq 1$ in this second section), we get $N_{g,b}$ as a subsurface of $N_{g+2(b-1),1}$. This inclusion induces a homomorphism $\chi_{g,b}\colon \Gamma_{g,b} \to \Gamma_{g+2(b-1),1}$ which is injective (see [29]).

Thus, the composed map

$$\lambda^n_{g,b} = \chi_{g+n,b} \circ \eta^n_{g,b} \colon \Gamma^n_{g,b} \to \Gamma_{g+n+2(b-1),1}$$

is also injective.

Recall that the mod $p$ Torelli group $I_p(N_{g,1})$ is the subgroup of $\Gamma_{g,1}$ defined as the kernel of the action of $\Gamma_{g,1}$ on $H_1(N_{g,1};\mathbb{Z}/p\mathbb{Z})$. In the following we will consider in particular the case of the mod 2 Torelli group $I_2(N_{g,1})$.

**Proposition 2.4** *If $b \geq 1$, $\lambda^n_{g,b}(P_n(N_{g,b}))$ is a subgroup of the mod 2 Torelli subgroup $I_2(N_{g+n+2(b-1),1})$.*

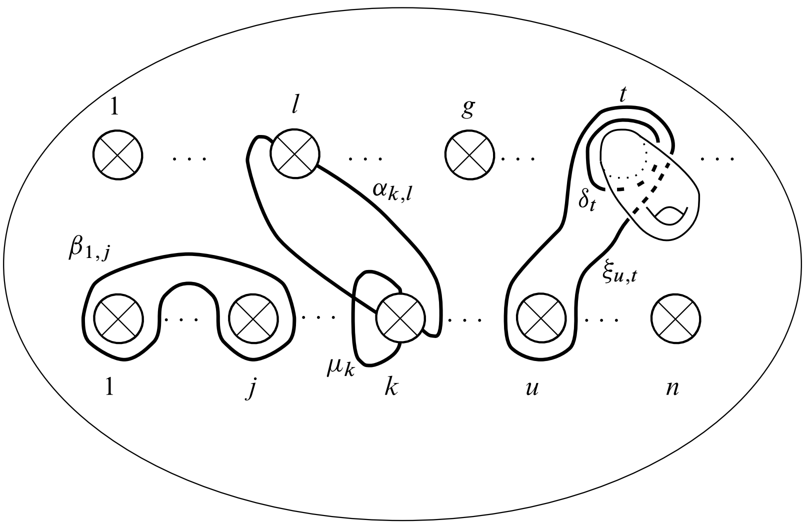


Figure 3: Image of the generators of $P_n(N_{g,b})$ in $\Gamma_{g+n+2(b-1),1}$.

**Proof** The image of the generators (see Figures 2, 4 and Proposition 2.3)

$$(B_{i,j})_{1\leq i<j\leq n}, \quad (\rho_{k,l})_{\substack{1\leq k\leq n\\ 1\leq l\leq g}} \quad \text{and} \quad (x_{u,t})_{\substack{1\leq u\leq n\\ 1\leq t\leq b-1}}$$

of $P_n(N_{g,b})$ under $\lambda^n_{g,b}$ are, respectively (see Figure 3):

- Dehn twists along curves $\beta_{i,j}$ which bound a subsurface homeomorphic to $N_{2,1}$.
- Crosscap slides $Y_{\mu_k,\alpha_{k,l}}$.
- The product $\tau_{\xi_{u,t}}\tau_{\delta_t}^{-1}$ of Dehn twists along the bounding curves $\xi_{u,t}$ and $\delta_t$.

According to [31], all of these elements are in $I_2(N_{g+n+2(b-1),1})$. □

**Remark 2.5** The embedding from Proposition 2.4 is invalid for $I_p(N_{g+n+2(b-1),1})$ when $p \neq 2$: for example, the crosscap slide $Y_{\mu_k,\alpha_{k,l}}$ is not in the mod $p$ Torelli subgroup since it sends $\mu_k$ to $\mu_k^{-1}$.

## 2.4 Conclusion of the proof

We shall use the following result, which is a straightforward consequence of a similar result for mod $p$ Torelli groups of orientable surfaces due to L Paris [26]:

**Theorem 2.6** *Let $g \geq 1$. The mod $p$ Torelli group $I_p(N_{g,1})$ is residually $p$–finite.*

**Proof** We use the Dehn–Nielsen–Baer theorem (see for instance [32, Theorem 5.15.3]), which states that $\Gamma_{g,1}$ embeds in $\mathrm{Aut}(\pi_1(N_{g,1}))$. Since $\pi_1(N_{g,1})$ is free we can apply [26, Theorem 1.4] which claims that, if $G$ is a free group, its mod $p$ Torelli group (ie the kernel of the canonical map from $\mathrm{Aut}(G)$ to $\mathrm{GL}(H_1(G,\mathbb{F}_p))$ is residually $p$–finite. Therefore $I_p(N_{g,1})$ is residually $p$–finite. □

**Theorem 2.7** *Let $g \geq 1$, $b > 0$, $n \geq 1$. Then $P_n(N_{g,b})$ is residually 2–finite.*

**Proof** The group $P_n(N_{g,b})$ is a subgroup of $I_2(N_{g+n+2(b-1),1})$ by Proposition 2.4 and by injectivity of the map $\lambda^n_{g,b}$. Then by Theorem 2.6 it follows that $P_n(N_{g,b})$ is residually 2–finite. □

# 3 *p*–almost direct products

## 3.1 On residually *p*–finite groups

Let $p$ be a prime number and $G$ a group. If $H$ is a subgroup of $G$, we denote by $H^p$ the subgroup generated by $\{h^p \mid h \in H\}$. Following [26], we define the lower $\mathbb{F}_p$–linear central filtration $(\gamma_n^p G)_{n\in\mathbb{N}^*}$ of $G$ as follows: $\gamma_1^p G = G$ and, for $n \geq 1$, $\gamma_{n+1}^p G$ is the subgroup of $G$ generated by $[G, \gamma_n^p G] \cup (\gamma_n^p G)^p$. Note that the subgroups $\gamma_n^p G$ are characteristic in $G$ and that the quotient group $G/\gamma_2^p G$ is nothing but the first homology group $H_1(G;\mathbb{F}_p)$. The following are proved in [26]:

- $[\gamma_m^p G, \gamma_n^p G] \subset \gamma_{m+n}^p G$ for $m, n \geq 1$.
- A finitely generated group $G$ is a finite $p$–group if and only if there exists some $N \geq 1$ such that $\gamma_N^p G = \{1\}$.
- A finitely generated group $G$ is residually $p$–finite if and only if $\bigcap_{n=1}^{\infty} \gamma_n^p G = \{1\}$.

Clearly, if $f\colon G \to G'$ is a group homomorphism, then $f(\gamma_n^p G) \subset \gamma_n^p G'$ for all $n \geq 1$.

**Definition 3.1** Let

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, hook] & B \arrow[r, "\lambda"] & C \arrow[r] \arrow[l, dotted, bend left, "\sigma"] & 1 \end{tikzcd}$$

be a split exact sequence.

- If the action of $C$ induced on $H_1(A;\mathbb{Z})$ is trivial (ie the action is trivial on $A^{\mathrm{Ab}} = A/[A,A]$), we say that B is an almost direct product of $A$ and $C$.
- If the action of $C$ induced on $H_1(A;\mathbb{F}_p)$ is trivial (ie the action is trivial on $A/\gamma_2^p A$), we say that $B$ is a $p$–almost direct product of $A$ and $C$.

Let us remark that, as in the case of almost direct products [7, Proposition 6.3], the property of being a $p$–almost direct product does not depend on the choice of section.

**Proposition 3.2** *Let* $1 \longrightarrow A \hookrightarrow B \xrightarrow{\lambda} C \longrightarrow 1$ *be a split exact sequence of groups. Let* $\sigma, \sigma'$ *be sections for* $\lambda$*, and suppose that the induced action of* $C$ *on* $A$ *via* $\sigma$ *on* $H_1(A;\mathbb{F}_p)$ *is trivial. Then the same is true for the section* $\sigma'$*.*

**Proof** Let $a \in A$ and $c \in C$. By hypothesis, $\sigma(c)a(\sigma(c))^{-1} \equiv a \mod \gamma_2^p A$. Since $\sigma'$ is also a section for $\lambda$, we have $\lambda \circ \sigma'(c) = \lambda \circ \sigma(c)$, and so $\sigma'(c)(\sigma(c))^{-1} \in \mathrm{Ker}(\lambda)$. Thus there exists $a' \in A$ such that $\sigma'(c) = a'\sigma(c)$, and hence

$$\sigma'(c)a(\sigma'(c))^{-1} \equiv a'\sigma(c)a(\sigma(c))^{-1}a'^{-1} \equiv a'aa'^{-1} \equiv a \mod \gamma_2^p A.$$

Thus the induced action of $C$ on $H_1(A;\mathbb{F}_p)$ via $\sigma'$ is also trivial. □

The first goal of this section is to prove the following theorem (see [12, Theorem 3.1] for an analogous result for almost direct products).

**Theorem 3.3** *Let*

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, hook] & B \arrow[r, "\lambda"] & C \arrow[r] \arrow[l, dotted, bend left, "\sigma"] & 1 \end{tikzcd}$$

*be a split exact sequence where* $B$ *is a* $p$*–almost direct product of* $A$ *and* $C$*. Then, for all* $n \geq 1$*, one has a split exact sequence*

$$\begin{tikzcd} 1 \arrow[r] & \gamma_n^p A \arrow[r, hook] & \gamma_n^p B \arrow[r, "\lambda_n"] & \gamma_n^p C \arrow[r] \arrow[l, dotted, bend left, "\sigma_n"] & 1, \end{tikzcd}$$

*where* $\lambda_n$ *and* $\sigma_n$ *are restrictions of* $\lambda$ *and* $\sigma$*.*

We shall need the following preliminary result.

**Lemma 3.4** *Under the hypotheses of Theorem 3.3, one has*

$$[\gamma^p_m C', \gamma^p_n A] \subset \gamma^p_{m+n} A \quad \text{for all } m,n \geq 1,$$

*where $C'$ denotes $\sigma(C)$.*

**Proof** First, we prove by induction on $n$ that $[C', \gamma^p_n A] \subset \gamma^p_{n+1} A$ for all $n \geq 1$. The case $n=1$ corresponds to the hypotheses: the action of $C$ on $H_1(A;\mathbb{F}_p) = A/\gamma^p_2 A$ is trivial if and only if $[C', A] \subset \gamma^p_2 A$. Thus, suppose that $[C', \gamma^p_n A] \subset \gamma^p_{n+1} A$ for some $n \geq 1$ and let us prove that $[C', \gamma^p_{n+1} A] \subset \gamma^p_{n+2} A$. In view of the definition of $\gamma^p_{n+1} A$, we have to prove that

$$\big[C', [A, \gamma^p_n A]\big] \subset \gamma^p_{n+2} A \quad \text{and} \quad [C', (\gamma^p_n A)^p] \subset \gamma^p_{n+2} A.$$

For the first case, we use a classical result (see [24, Theorem 5.2]) which says

$$\big[C', [A, \gamma^p_n A]\big] = \big[\gamma^p_n A, [C', A]\big]\big[A, [\gamma^p_n A, C']\big].$$

We have just seen that $[C', A] \subset \gamma^p_2 A$, thus

$$\big[\gamma^p_n A, [C', A]\big] \subset [\gamma^p_n A, \gamma^p_2 A] \subset \gamma^p_{n+2} A.$$

Then, the induction hypotheses says that $[\gamma^p_n A, C'] \subset \gamma^p_{n+1} A$, thus

$$\big[A, [\gamma^p_n A, C']\big] \subset [A, \gamma^p_{n+1} A] \subset \gamma^p_{n+2} A.$$

The second case works as follows: for $c \in C'$ and $x \in \gamma^p_n A$, one has, using the fact that $[u, vw] = [u,w][u,v]\big[[u,v],w\big]$ (see [24]),

$$\begin{aligned}[c, x^p] &= [c,x][c,x^{p-1}]\big[[c,x^{p-1}],x\big] \\ &= \cdots = [c,x]^p\big[[c,x],x\big]\big[[c,x^2],x\big]\cdots\big[[c,x^{p-1}],x\big].\end{aligned}$$

Since $c \in C'$ and $x \in \gamma^p_n A$, one has $[c, x^i] \in [C', \gamma^p_n A] \subset \gamma^p_{n+1} A$ for all $i$, $1 \leq i \leq p-1$, which leads to

$$[c,x]^p \in (\gamma^p_{n+1} A)^p \subset \gamma^p_{n+2} A \quad \text{and} \quad \big[[c,x^i],x\big] \in [\gamma^p_{n+1} A, A] \subset \gamma^p_{n+2} A.$$

Now, we suppose that $[\gamma^p_m C', \gamma^p_n A] \subset \gamma^p_{m+n} A$ for some $m \geq 1$ and all $n \geq 1$ and prove that $[\gamma^p_{m+1} C', \gamma^p_n A] \subset \gamma^p_{m+n+1} A$. As above, there are two cases which work in the same way:

(i) $$\begin{aligned}\big[[C',\gamma_m^p C'],\gamma_n^p A\big] &= \big[[\gamma_n^p A, C'],\gamma_m^p C'\big]\big[[\gamma_m^p C',\gamma_n^p A],C'\big]\\ &\subset \big[\gamma_{n+1}^p A,\gamma_m^p C'\big]\big[\gamma_{m+n}^p A, C'\big]\\ &\subset \gamma_{m+n+1}^p A.\end{aligned}$$

(ii) For $c\in\gamma_m^p C'$ and $x\in\gamma_n^p A$, one has

$$[c^p,x]=\big[c,[x,c^{p-1}]\big][c^{p-1},x][c,x]=\cdots=\big[c,[x,c^{p-1}]\big]\cdots\big[c,[x,c]\big][c,x]^p,$$

which is an element of $\gamma_{m+n+1}^p A$ by induction hypotheses. □

**Proof of Theorem 3.3** The restrictions of $\lambda$ and $\sigma$ give rise to morphisms

$$\lambda_n\colon \gamma_n^p B\to\gamma_n^p C \quad \text{and} \quad \sigma_n\colon \gamma_n^p C\to\gamma_n^p B$$

such that $\lambda_n\circ\sigma_n=\mathrm{Id}_{\gamma_n^p C}$, $\lambda_n$ is onto and $\mathrm{Ker}(\lambda_n)=A\cap\gamma_n^p B$. Thus, we need to prove that $A\cap\gamma_n^p B=\gamma_n^p A$ for all $n\geq 1$. Clearly one has $\gamma_n^p A\subset A\cap\gamma_n^p B$. In order to prove the reverse inclusion, we follow the method developed in [12] for almost semi-direct products and define $\tau\colon B\to B$ by $\tau(b)=(\sigma\lambda(b))^{-1}b$. This map has the following properties:

(i) Since $\lambda\sigma=\mathrm{Id}_C$, $\tau(B)\subset A$.

(ii) For $x\in B$, $\tau(x)=x$ if and only if $x\in A$.

(iii) For $(b_1,b_2)\in B^2$, $\tau(b_1b_2)=[\sigma\lambda(b_2),\tau(b_1)^{-1}]\tau(b_1)\tau(b_2)$.

(iv) For $b\in B$, setting $a=\tau(b)$ and $c=\sigma\lambda(b)$, we get a unique decomposition $b=ca$ with $c\in C'=\sigma(C)$ and $a\in A$.

We claim that $\tau(\gamma_n^p B)\subset\gamma_n^p A$ for all $n\geq 1$. From this, we easily conclude the proof: if $x\in A\cap\gamma_n^p B$, then $x=\tau(x)\in\gamma_n^p A$.

One has $\tau(\gamma_1^p B)\subset\gamma_1^p A$. Suppose inductively that $\tau(\gamma_n^p B)\subset\gamma_n^p A$ for some $n\geq 1$, and let us prove that $\tau(\gamma_{n+1}^p B)\subset\gamma_{n+1}^p A$. Suppose first that $x$ is an element of $\gamma_n^p B$. Then using (iii) we get

$$\begin{aligned}\tau(x^p)&=[\sigma\lambda(x),\tau(x^{p-1})^{-1}]\tau(x^{p-1})\tau(x)\\ &\ \ \vdots\\ &=[\sigma\lambda(x),\tau(x^{p-1})^{-1}][\sigma\lambda(x),\tau(x^{p-2})^{-1}]\cdots[\sigma\lambda(x),\tau(x)^{-1}]\tau(x)^p.\end{aligned}$$

Since $\sigma\lambda(x)\in\gamma_n^p C'$, and since $\tau(x^i)\in\gamma_n^p A$ for $1\leq i\leq p-1$ by the induction hypothesis, we get

$$\tau(x^p)\in[\gamma_n^p C',\gamma_n^p A]\cdot(\gamma_n^p A)^p\subset\gamma_{n+1}^p A$$

by Lemma 3.4: this proves that $\tau((\gamma_n^p B)^p) \subset \gamma_{n+1}^p A$. Next, let $b \in B$ and $x \in \gamma_n^p B$. Setting $a = \tau(b) \in A$, $y = \tau(x) \in \gamma_n^p A$ by the induction hypothesis, $c = \sigma\lambda(b) \in C'$ and $z = \sigma\lambda(x) \in \gamma_n^p C'$, we get

$$\begin{aligned}
\tau([b,x]) &= (\sigma\lambda([b,x]))^{-1}[b,x] \\
&= [\sigma\lambda(b), \sigma\lambda(x)]^{-1}[b,x] \\
&= [c,z]^{-1}[ca,zy] = [z,c]a^{-1}c^{-1}y^{-1}z^{-1}cazy \\
&= [z,c](a^{-1}c^{-1}y^{-1}cya)(a^{-1}y^{-1}c^{-1}z^{-1}czya)(a^{-1}y^{-1}z^{-1}azy) \\
&= [z,c](a^{-1}[c,y]a)(a^{-1}y^{-1}[c,z]ya)(a^{-1}y^{-1}ay)(y^{-1}a^{-1}z^{-1}azy) \\
&= [z,c](a^{-1}[c,y]a)(a^{-1}y^{-1}[c,z]ya)[a,y](y^{-1}[a,z]y) \\
&= \big[[c,z],(a^{-1}[y,c]a)\big](a^{-1}[c,y]a)[z,c](a^{-1}y^{-1}[c,z]ya)[a,y](y^{-1}[a,z]y) \\
&= \big[[c,z],(a^{-1}[y,c]a)\big](a^{-1}[c,y]a)\big[[c,z],ya\big][a,y](y^{-1}[a,z]y).
\end{aligned}$$

Now,

$$[c,z] \in [C', \gamma_n^p C'] \subset \gamma_{n+1}^p C' \quad \text{and} \quad [y,c] \in [\gamma_n^p A, C'] \subset \gamma_{n+1}^p A$$

(Lemma 3.4), thus $\big[[c,z],(a^{-1}[y,c]a)\big] \in \gamma_{n+1}^p A$. Then

$$\begin{aligned}
\big[[c,z],ya\big] &\in [\gamma_{n+1}^p C', A] \subset \gamma_{n+1}^p A, \\
[a,y] &\in [A, \gamma_n^p A] \subset \gamma_{n+1}^p A, \\
[a,z] &\in [A, \gamma_n^p C'] \subset \gamma_{n+1}^p A.
\end{aligned}$$

Thus, $\tau([b,x]) \in \gamma_{n+1}^p A$ and $\tau([B, \gamma_n^p B]) \subset \gamma_{n+1}^p A$. □

**Corollary 3.5** *Let*

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, hook] & B \arrow[r, "\lambda"] & C \arrow[r] \arrow[l, dotted, bend left, "\sigma"] & 1
\end{tikzcd}$$

*be a split exact sequence such that $B$ is a $p$–almost direct product of $A$ and $C$. If $A$ and $C$ are residually $p$–finite, then $B$ is residually $p$–finite.*

## 3.2 Augmentation ideals

Given a group $G$ and $\mathbb{K} = \mathbb{Z}$ or $\mathbb{F}_2$, we will denote by $\mathbb{K}[G]$ the group ring of $G$ over $\mathbb{K}$ and by $\overline{\mathbb{K}[G]}$ the augmentation ideal of $G$. The group ring $\mathbb{K}[G]$ is filtered by the powers $\overline{\mathbb{K}[G]}^j$ of $\overline{\mathbb{K}[G]}$, and we can define the associated graded algebra

$$\operatorname{gr}\mathbb{K}[G] = \bigoplus \overline{\mathbb{K}[G]}^j / \overline{\mathbb{K}[G]}^{j+1}.$$

The following theorem provides a decomposition formula for the augmentation ideal of a 2–almost direct product (see [25, Theorem 3.1] for an analogous result in the case of almost direct products).

Let $A \rtimes C$ be a semi-direct product between two groups $A$ and $C$. It is a classical result that the map $a \otimes c \mapsto ac$ induces a $\mathbb{K}$–isomorphism from $\mathbb{K}[A] \otimes \mathbb{K}[C]$ to $\mathbb{K}[A \rtimes C]$. Identifying these two $\mathbb{K}$–modules, we have the following:

**Theorem 3.6** *If $A \rtimes C$ is a 2–almost direct product, then*

$$\overline{\mathbb{F}_2[A \rtimes C]}^k = \sum_{i+h=k} \overline{\mathbb{F}_2[A]}^i \otimes \overline{\mathbb{F}_2[C]}^h \quad \text{for all } k.$$

**Proof** We sketch the proof, which is almost verbatim the same as the proof of [25, Theorem 3.1]. Let

$$R_k = \sum_{i+h=k} \overline{\mathbb{F}_2[A]}^i \otimes \overline{\mathbb{F}_2[C]}^h;$$

$R_k$ is a descending filtration on $\mathbb{F}_2[A] \otimes \mathbb{F}_2[C]$, and with the above identification, we get that $R_k \subset \overline{\mathbb{F}_2[A \rtimes C]}^k$. To verify the other inclusion we have to check that $\prod_{j=1}^k (a_j c_j - 1) \in R_k$ for every $a_1, \dots, a_k$ in $A$ and $c_1, \dots, c_k$ in $C$. Actually it is enough to verify that $e = \prod_{j=1}^k (e_j - 1) \in R_k$ where either $e_j \in A$ or $e_j \in C$ (see [25, Theorem 3.1] for a proof of this fact); we call $e$ a *special* element. We associate to a special element $e$ an element in $\{0,1\}^k$: let $\operatorname{type}(e) = (\delta(e_1), \dots, \delta(e_k))$, where $\delta(e_j) = 0$ if $e_j \in A$ and $\delta(e_j) = 1$ if $e_j \in C$. We will say that the special element $e$ is *standard* if

$$\operatorname{type}(e) = (\overbrace{0, \dots, 0}^{i}, \overbrace{1, \dots, 1}^{h}).$$

In this case $e \in \overline{\mathbb{F}_2[A]}^i \otimes \overline{\mathbb{F}_2[C]}^h \subset R_k$ and we are done. We claim that we can reduce all special elements to linear combinations of standard elements. If $e$ is not standard, then it must be of the form

$$e = \prod_{i=1}^{r} (a_i - 1) \prod_{j=1}^{s} (c_j - 1)(c-1)(a-1) \prod_{l=1}^{t} (e_l - 1),$$

where $a_1, \dots, a_r, a \in A$, $c_1, \dots, c_s, c \in C$, the element $\tilde{e} = \prod_{l=1}^t (e_l - 1)$ is special and $r + s + t + 2 = k$. Therefore

$$\operatorname{type}(e) = (\overbrace{0, \dots, 0}^{r}, \overbrace{1, \dots, 1}^{s}, 1, 0, \delta(e_1), \dots, \delta(e_t)).$$

Now we can use the assumption that $A \rtimes C$ is a 2–almost direct product to claim that one has commutation relations in $\mathbb{Z}[A \rtimes C]$ expressing the difference

$$(c-1)(a-1)-(a-1)(c-1)$$

as a linear combination of terms of the form

$$(a'-1)(a''-1)c \quad \text{with } a', a'' \in A,$$

for any $a \in A$ and $c \in C$. In fact,

$$(c-1)(a-1)-(a-1)(c-1) = ca - ac = (cac^{-1}a^{-1}-1)ac = (f-1)ac,$$

where $f = [c^{-1}, a^{-1}] \in [C, A] \subset \gamma_2^2(A)$ by Lemma 3.4. We can decompose $f$ as $f = h_1 k_1 \cdots h_m k_m$, where, for $j = 1, \dots, m$, $h_j$ belongs to $[A, A]$ and $k_j = (k'_j)^2$ for some $k'_j \in A$. One knows (see for instance [10, page 194]) that, for $j = 1, \dots, m$, $(h_j - 1)$ is a linear combination of terms of the form

$$(h'_j-1)(h''_j-1)\alpha_j \quad \text{with } h'_j, h''_j, \alpha_j \in A.$$

On the other hand, for $j = 1, \dots, m$, we have also that

$$(k_j - 1) = (k'_j-1)(k'_j-1) \quad \text{with } k'_j \in A, \qquad \text{since the coefficients are in } \mathbb{F}_2.$$

Then, recalling that $(hk-1) = (h-1)k + (k-1)$ for any $h, k \in A$, we can conclude that $f-1$ can be rewritten as a linear combination of terms of the form

$$(f'-1)(f''-1)\alpha \quad \text{with } f', f'', \alpha \in A$$

and that $(c-1)(a-1)-(a-1)(c-1)$ is a linear combination of terms of the form

$$(f'-1)(f''-1)\alpha c \quad \text{with } f', f'', \alpha \in A.$$

Rewriting $(f''-1)\alpha$ as $(f''\alpha-1)-(\alpha-1)$, we obtain that the difference

$$(c-1)(a-1)-(a-1)(c-1)$$

can be seen as a linear combination of terms of the form

$$(a'-1)(a''-1)c \quad \text{with } a', a'' \in A.$$

Therefore $e$ can be rewritten as a sum whose first term is the special element

$$e' = \prod_{i=1}^{r}(a_i-1)\prod_{j=1}^{s}(c_j-1)(a-1)(c-1)\prod_{l=1}^{t}(e_l-1)$$

and whose second term is a linear combination of elements of the form $e''c$, where

$$e'' = \prod_{i=1}^{r}(a_i - 1)\prod_{j=1}^{s}(c_j - 1)(a' - 1)(a'' - 1)\prod_{l=1}^{t}(ce_lc^{-1} - 1).$$

Using the lexicographic order from the left, one has $\mathrm{type}(e) > \mathrm{type}(e')$ and $\mathrm{type}(e) > \mathrm{type}(e'')$.

By induction on the lexicographic order, we infer that $e'$ and $e''$ belong to $R_k$; since $R_k \cdot c \subset R_k$ for any $c \in C$, it follows that $e$ belongs to $R_k$ and we are done. $\square$

# 4 The closed case

## 4.1 A presentation of $P_n(N_g)$ and induced identities

We recall a group presentation of $P_n(N_g)$ given in [16]; the geometric interpretation of generators is provided in Figure 4.

**Theorem 4.1** [16] *For $g \geq 2$ and $n \geq 1$, $P_n(N_g)$ has a presentation with generators*

$$(B_{i,j})_{1\leq i<j\leq n} \quad \text{and} \quad (\rho_{k,l})_{\substack{1\leq k\leq n\\ 1\leq l\leq g}}$$

*and relations of the following four types:*

(a) *For $1 \leq i < j \leq n$ and $1 \leq r < s \leq n$,*

$$B_{r,s}B_{i,j}B_{r,s}^{-1} = \begin{cases} B_{i,j} & \text{if } i<r<s<j \text{ or } r<s<i<j, & (\mathrm{a}_1)\\ B_{i,j}^{-1}B_{r,j}^{-1}B_{i,j}B_{r,j}B_{i,j} & \text{if } r<i=s<j, & (\mathrm{a}_2)\\ B_{s,j}^{-1}B_{i,j}B_{s,j} & \text{if } i=r<s<j, & (\mathrm{a}_3)\\ B_{s,j}^{-1}B_{r,j}^{-1}B_{s,j}B_{r,j}B_{i,j}B_{r,j}^{-1}B_{s,j}^{-1}B_{r,j}B_{s,j} & \text{if } r<i<s<j. & (\mathrm{a}_4)\end{cases}$$

(b) *For $1 \leq i < j \leq n$ and $1 \leq k, l \leq g$,*

$$\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1} = \begin{cases} \rho_{j,l} & \text{if } k<l. & (\mathrm{b}_1)\\ \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}^{2} & \text{if } k=l. & (\mathrm{b}_2)\\ \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}B_{i,j}^{-1}\rho_{j,l}B_{i,j}\rho_{j,k}^{-1}B_{i,j}\rho_{j,k} & \text{if } k>l. & (\mathrm{b}_3)\end{cases}$$

(c) *For $1 \leq i \leq n$,*

$$\rho_{i,1}^2 \cdots \rho_{i,g}^2 = T_i, \quad \text{where } T_i = B_{1,i}\cdots B_{i-1,i}B_{i,i+1}\cdots B_{i,n}. \qquad (\mathrm{c})$$

(d) *For* $1 \le i < j \le n$, $1 \le k \le n$, $k \ne j$ *and* $1 \le l \le g$,

$$\rho_{k,l} B_{i,j} \rho_{k,l}^{-1} = \begin{cases} B_{i,j} & \text{if } k < i \text{ or } j < k, & (\mathrm{d}_1) \\ \rho_{j,l}^{-1} B_{i,j}^{-1} \rho_{j,l} & \text{if } k = i, & (\mathrm{d}_2) \\ \rho_{j,l}^{-1} B_{k,j}^{-1} \rho_{j,l} B_{k,j}^{-1} B_{i,j} B_{k,j} \rho_{j,l}^{-1} B_{k,j} \rho_{j,l} & \text{if } i < k < j. & (\mathrm{d}_3) \end{cases}$$

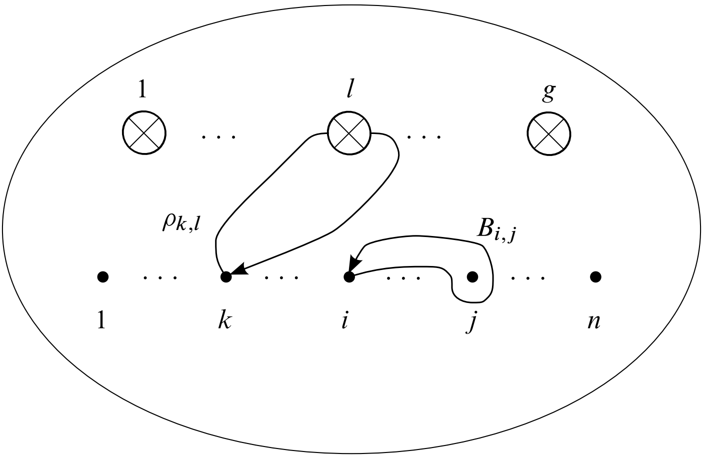


Figure 4: Generators of $P_n(N_g)$.

Let us denote by $U$ the element $\rho_{n,1}\rho_{n-1,1}\cdots\rho_{2,1}$ of $P_n(N_g)$.

**Lemma 4.2** *The following relations hold in* $P_n(N_g)$*:*

(1) $[\rho_{i,k}, \rho_{j,k}^{-1}] = B_{i,j}^{-1}$ *for* $1 \le i < j \le n$ *and* $1 \le k \le g$. (e)

(2) $[\rho_{k,1}, \rho_{1,1}\rho_{1,1}\rho_{1,2}T_1^{-1}] = 1$ *for* $2 \le k \le n$. (f)

(3) $[U, \rho_{1,1}\rho_{1,1}\rho_{1,2}T_1^{-1}] = 1$. (g)

(4) $U\rho_{1,1}U^{-1} = \rho_{1,1}T_1^{-1}$. (h)

**Proof** The first and second identities can be verified by drawing the corresponding braids (see Figure 5 and 6). The third one is a direct consequence of the second one and the definition of $U$. We prove the last one as follows:

$$\begin{aligned} \rho_{1,1}^{-1} U \rho_{1,1} &= (\rho_{1,1}^{-1}\rho_{n,1}\rho_{1,1})\cdots(\rho_{1,1}^{-1}\rho_{2,1}\rho_{1,1}) \\ &= (B_{1,n}^{-1}\rho_{n,1})\cdots(B_{1,2}^{-1}\rho_{2,1}) \quad \text{by (e)} \\ &= B_{1,n}^{-1}\cdots B_{1,2}^{-1}\rho_{n,1}\cdots\rho_{2,1} \quad \text{by } (\mathrm{d}_1) \\ &= T_1^{-1}U. \end{aligned}$$

□

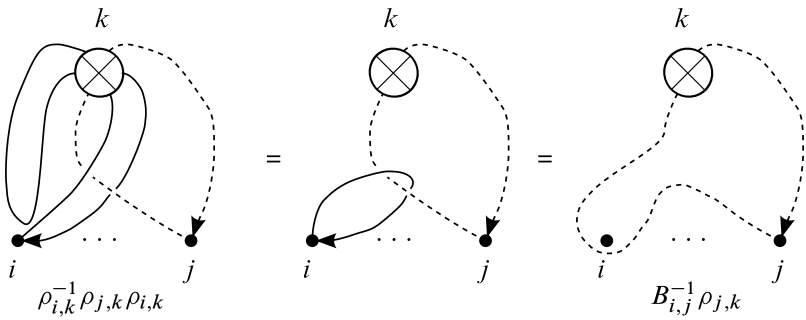


Figure 5: Identity (e).

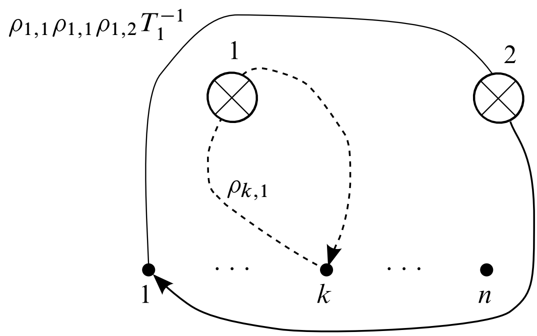


Figure 6: Identity (f).

## 4.2 The pure braid group $P_n(N_g)$ is residually 2–finite

Following [14], one has, for $g \geq 2$, a split exact sequence

$$(1) \qquad 1 \longrightarrow P_{n-1}(N_{g,1}) \xrightarrow{\mu} P_n(N_g) \xrightarrow{\lambda} P_1(N_g) = \pi_1(N_g) \longrightarrow 1,$$

where $\lambda$ is induced by the map which forgets all strands except the first one, and $\mu$ is defined by capping the boundary component by a disc with one marked point (the first strand in $P_n(N_g)$). According to the definition of $\mu$ and to Proposition 2.3, $\mathrm{Im}(\mu)$ is generated by $\{\rho_{i,k} \mid 2 \leq i \leq n,\ 1 \leq k \leq g\} \cup \{B_{i,j} \mid 2 \leq i < j \leq n\}$.

The section given in [14] is geometric, ie it is induced by a crossed section at the level of fibrations. In order to study the action of $\pi_1(N_g)$ on $P_{n-1}(N_{g,1})$, we need an algebraic section. Recall that $\pi_1(N_g)$ has a group presentation with generators $p_1, \dots, p_g$ and the single relation $p_1^2 \cdots p_g^2 = 1$. We define the set map $\sigma\colon \pi_1(N_g) \to P_n(N_g)$ by setting

$$\sigma(p_i) = \begin{cases} T_1^{-1} U \rho_{1,1} T_1 & \text{for } i = 1, \\ T_1^{-1} \rho_{1,1}^{-1} U^{-1} \rho_{1,1}\rho_{1,2} & \text{for } i = 2, \\ \rho_{1,i} & \text{for } 3 \leq i \leq g. \end{cases}$$

**Proposition 4.3** *The map $\sigma$ is a well-defined homomorphism satisfying $\lambda \circ \sigma = \mathrm{Id}_{\pi_1(N_g)}$.*

**Proof** Since $\lambda(\rho_{1,i}) = p_i$ for all $1 \le i \le g$ and $\lambda(U) = \lambda(T_1) = 1$, one clearly has $\lambda\sigma = \mathrm{Id}_{\pi_1(N_g)}$ if $\sigma$ is a group homomorphism. Thus, we just have to prove that $\sigma(p_1)^2 \cdots \sigma(p_g)^2 = 1$:

$$\begin{aligned}
\sigma(p_1)^2 \cdots \sigma(p_g)^2 &= T_1^{-1} U \rho_{1,1} T_1 T_1^{-1} U \rho_{1,1} T_1 T_1^{-1} \rho_{1,1}^{-1} U^{-1} \rho_{1,1} \rho_{1,2} T_1^{-1} \\
&\qquad \times \rho_{1,1}^{-1} U^{-1} \rho_{1,1} \rho_{1,2} \rho_{1,3}^2 \cdots \rho_{1,g}^2 \\
&= T_1^{-1} \underbrace{U \rho_{1,1} \rho_{1,1} \rho_{1,2} T_1^{-1}} \rho_{1,1}^{-1} U^{-1} \rho_{1,1} \rho_{1,2} \rho_{1,3}^2 \cdots \rho_{1,g}^2 \\
&= T_1^{-1} \rho_{1,1} \rho_{1,1} \rho_{1,2} T_1^{-1} \underbrace{U \rho_{1,1}^{-1} U^{-1}} \rho_{1,1} \rho_{1,2} \rho_{1,3}^2 \cdots \rho_{1,g}^2 \quad \text{by (g)} \\
&= T_1^{-1} \rho_{1,1} \rho_{1,1} \rho_{1,2} T_1^{-1} T_1 \rho_{1,1}^{-1} \rho_{1,1} \rho_{1,2} \rho_{1,3}^2 \cdots \rho_{1,g}^2 \quad \text{by (h)} \\
&= T_1^{-1} \rho_{1,1} \rho_{1,1} \rho_{1,2} \rho_{1,2} \rho_{1,3}^2 \cdots \rho_{1,g}^2 \\
&= 1 \quad \text{by (c).}
\end{aligned}$$

□

So, the exact sequence (1) splits. In order to apply Theorem 3.3, we have to prove that the action of $\pi_1(N_g)$ on $P_{n-1}(N_{g,1})$ is trivial on $H_1(P_{n-1}(N_{g,1}); \mathbb{F}_2)$. This is the claim of the following proposition.

**Proposition 4.4** *For all $x \in \mathrm{Im}(\sigma)$ and $a \in \mathrm{Im}(\mu)$, one has*

$$[x^{-1}, a^{-1}] = x a x^{-1} a^{-1} \in \gamma_2^2(\mathrm{Im}(\mu)).$$

**Proof** It is enough to prove the result for generators

$$a \in \{B_{j,k} \mid 2 \le j < k \le n\} \cup \{\rho_{j,l} \mid 2 \le j \le n \text{ and } 1 \le l \le g\}$$

and

$$x \in \{\sigma(p_1), \dots, \sigma(p_g)\}$$

of $\mathrm{Im}(\mu)$ and $\mathrm{Im}(\sigma)$, respectively. Suppose first that $2 \le j < k \le n$. One has:

- $[\sigma(p_i)^{-1}, B_{j,k}^{-1}] = [\rho_{1,i}^{-1}, B_{j,k}^{-1}] = 1$ for $3 \le i \le g$ by $(\mathrm{d}_1)$.
- Then, one has

$$\begin{aligned}
[\sigma(p_2)^{-1}, B_{j,k}^{-1}] &= [\rho_{1,2}^{-1} \rho_{1,1}^{-1} U \rho_{1,1} T_1, B_{j,k}^{-1}] \\
&= (\rho_{1,1}^{-1} U \rho_{1,1} T_1)^{-1} [\rho_{1,2}^{-1}, B_{j,k}^{-1}] (\rho_{1,1}^{-1} U \rho_{1,1} T_1) [\rho_{1,1}^{-1} U \rho_{1,1} T_1, B_{j,k}^{-1}] \\
&= [\rho_{1,1}^{-1} U \rho_{1,1} T_1, B_{j,k}^{-1}] \quad \text{by } (\mathrm{d}_1).
\end{aligned}$$

But $U$ and $T_1$ are elements of $\mathrm{Im}(\mu)$ and the latter being normal in $P_n(N_g)$, $\rho_{1,1}^{-1}U\rho_{1,1}T_1 \in \mathrm{Im}(\mu)$ thus $[\sigma(p_2)^{-1}, B_{j,k}^{-1}] \in \Gamma_2(\mathrm{Im}(\mu)) \subset \gamma_2^2(\mathrm{Im}(\mu))$.

- In the same way, one has

$$\begin{aligned}[\sigma(p_1)^{-1}, B_{j,k}^{-1}] = [T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1, B_{j,k}^{-1}] &= [\rho_{1,1}^{-1}\rho_{1,1}T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1, B_{j,k}^{-1}] \\ &= (\rho_{1,1}T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1)^{-1}[\rho_{1,1}^{-1}, B_{j,k}^{-1}](\rho_{1,1}T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1) \\ &\qquad \times [\rho_{1,1}T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1, B_{j,k}^{-1}] \\ &= [\rho_{1,1}T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1, B_{j,k}^{-1}] \quad \text{by } (\mathrm{d}_1),\end{aligned}$$

thus, as before, $[\sigma(p_1)^{-1}, B_{j,k}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$.

Now, let $j$ and $l$ be integers such that $2 \le j \le n$ and $1 \le l \le g$, and let us first prove that $[\rho_{1,i}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$ for all $i$, $1 \le i \le g$:

- This is clear for $i < l$ by $(\mathrm{b}_1)$.
- For $i = l$, the relation $(\mathrm{b}_2)$ gives $[\rho_{1,l}^{-1}, \rho_{j,l}^{-1}] = \rho_{j,l}^{-1}B_{1,j}^{-1}\rho_{j,l}$. But

$$B_{1,j}^{-1} = B_{2,j}\cdots B_{j-1,j}B_{j,j+1}\cdots B_{j,n}\rho_{j,g}^{-2}\cdots\rho_{j,1}^{-2} \quad \text{(relation (c))}$$

is an element of $\gamma_2^2(\mathrm{Im}(\mu))$ by (e), thus we get $[\rho_{1,l}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$.

- If $l < i$ then $[\rho_{1,i}^{-1}, \rho_{j,l}^{-1}] = [B_{1,j}\rho_{j,i}^{-1}B_{1,j}\rho_{j,i}, \rho_{j,l}^{-1}]$ by $(\mathrm{b}_3)$ so $[\rho_{1,i}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$ since $\rho_{j,l}$, $\rho_{j,i}$ and $B_{1,j}$ are elements of $\mathrm{Im}(\mu)$.

From this, we deduce the following facts.

(1) $[\sigma(p_i)^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$ for $i \ge 3$ since $\sigma(p_i) = \rho_{1,i}$.

(2) Next, one has

$$\begin{aligned}[\sigma(p_2)^{-1}, \rho_{j,l}^{-1}] &= [\rho_{1,2}^{-1}\rho_{1,1}^{-1}U\rho_{1,1}T_1, \rho_{j,l}^{-1}] \\ &= (\rho_{1,1}^{-1}U\rho_{1,1}T_1)^{-1}[\rho_{1,2}^{-1}, \rho_{j,l}^{-1}](\rho_{1,1}^{-1}U\rho_{1,1}T_1)[\rho_{1,1}^{-1}U\rho_{1,1}T_1, \rho_{j,l}^{-1}].\end{aligned}$$

Since $\rho_{1,1}^{-1}U\rho_{1,1}T_1$ and $\rho_{j,l}$ are elements of $\mathrm{Im}(\mu)$ and $[\rho_{1,2}^{-1}, \rho_{j,l}^{-1}]$ is an element of $\gamma_2^2(\mathrm{Im}(\mu))$, we get $[\sigma(p_2)^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$.

(3) In the same way, one has

$$\begin{aligned}[\sigma(p_1)^{-1}, \rho_{j,l}^{-1}] = [T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1, \rho_{j,l}^{-1}] &= [T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1\rho_{1,1}\rho_{1,1}^{-1}, \rho_{j,l}^{-1}] \\ &= \rho_{1,1}[T_1^{-1}\rho_{1,1}^{-1}U^{-1}T_1\rho_{1,1}, \rho_{j,l}^{-1}]\rho_{1,1}^{-1}[\rho_{1,1}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu)). \quad \square\end{aligned}$$

We are now ready to prove the main result of this section.

**Theorem 4.5** *For all $g \geq 2$ and $n \geq 1$, the pure braid group $P_n(N_g)$ is residually 2–finite.*

**Proof** Proposition 4.3 says that the sequence

$$1 \longrightarrow P_{n-1}(N_{g,1}) \longrightarrow P_n(N_g) \longrightarrow \pi_1(N_g) \longrightarrow 1$$

splits. Now $P_{n-1}(N_{g,1})$ is residually 2–finite (Theorem 2.7). It is proved in [2] and [3] that $\pi_1(N_g)$ is residually free for $g \geq 4$, so it is residually 2–finite. This result is proved in [23, Lemma 8.9] for $g = 3$. When $g = 2$, $\pi_1(N_2)$ has presentation $\langle a, b \mid aba^{-1} = b^{-1} \rangle$ so is a 2–almost direct product of $\mathbb{Z}$ by $\mathbb{Z}$. Since $\mathbb{Z}$ is residually 2–finite, $\pi_1(N_2)$ is residually 2–finite by Corollary 3.5. So, using Proposition 4.4 and Corollary 3.5, we can conclude that $P_n(N_g)$ is residually 2–finite. □

## 4.3 The case $P_n(\mathbb{RP}^2)$

The main reason to exclude $N_1 = \mathbb{RP}^2$ in Theorem 4.5 is that the exact sequence (1) doesn't exist in this case, but forgetting at most $n-2$ strands we get the following exact sequence ($1 \leq m \leq n-2$; see [9]):

$$1 \longrightarrow P_m(N_{1,n-m}) \longrightarrow P_n(\mathbb{RP}^2) \longrightarrow P_{n-m}(\mathbb{RP}^2) \longrightarrow 1.$$

This sequence splits if and only if $n = 3$ and $m = 1$ (see [15]). Thus, what we know is the following:

- $P_1(\mathbb{RP}^2) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$: $P_1(\mathbb{RP}^2)$ is a 2–group.
- $P_2(\mathbb{RP}^2) = Q_8$, the quaternion group (see [9]): $P_2(\mathbb{RP}^2)$ is a 2–group.
- One has the split exact sequence

$$1 \longrightarrow P_1(N_{1,2}) \longrightarrow P_3(\mathbb{RP}^2) \longrightarrow P_2(\mathbb{RP}^2) \longrightarrow 1,$$

  where $P_1(N_{1,2}) = \pi_1(N_{1,2})$ is a free group of rank 2, thus is residually 2–finite. Since $P_2(\mathbb{RP}^2)$ is 2–finite, we can conclude that $P_3(\mathbb{RP}^2)$ is residually 2–finite using [19, Lemma 1.5].

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