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Rectification of weak product algebras over an operad in $\mathcal{C}at$ and $\mathcal{T}op$ and applications

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We develop an alternative to the May–Thomason construction used to compare operad-based infinite loop machines to those of Segal, which rely on weak products. Our construction has the advantage that it can be carried out in $\mathcal{C}at$, whereas their construction gives rise to simplicial categories. As an application we show that a simplicial algebra over a Σ–free $\mathcal{C}at$ operad $\mathcal{O}$ is functorially weakly equivalent to a $\mathcal{C}at$ algebra over $\mathcal{O}$. When combined with the results of a previous paper, this allows us to conclude that, up to weak equivalences, the category of $\mathcal{O}$–categories is equivalent to the category of $B\mathcal{O}$–spaces, where $B\colon \mathcal{C}at \to \mathcal{T}op$ is the classifying space functor. In particular, n–fold loop spaces (and more generally E_n spaces) are functorially weakly equivalent to classifying spaces of n–fold monoidal categories. Another application is a change of operads construction within $\mathcal{C}at$.

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1 Introduction

In [14], May and Thomason compared infinite loop machines based on spaces with an operad acting on them to the Segal machine, which involves weakening the notion of Cartesian product to that of a product up to equivalence. In the process they introduced a hybrid notion of an algebra over a category of operators and created a rectification construction to pass from this to an equivalent space with an operad action. Their rectification is a 2–sided bar construction, which is simplicial in nature. Schwänzl and Vogt gave an alternative comparison of the two infinite loop space machines in [16], which is based on the fact that for a strong deformation retract $A \subset X$ the space of deformation retractions of X onto A is contractible. Neither approach translates directly to $\mathcal{C}at$, the category of small categories, with realization equivalences as weak equivalences: the May–Thomason construction would convert categories into simplicial categories, and there is no apparent candidate to replace the space of strong deformation retractions in the Schwänzl–Vogt construction.

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Similarly, the change of operads construction used in May [13], if applied to operads in $\mathcal{C}at$, ends up in simplicial categories.

In this paper we offer a comparatively simple third rectification which has the advantages that it can be carried out in $\mathcal{C}at$ and that a change of operads functor based on it stays in $\mathcal{C}at$.

Our main motivation for this paper is to realize a program started in Balteanu, Fiedorowicz, Schwänzl and Vogt [2], where a notion of n–fold monoidal category was introduced whose structure is codified by a Σ–free operad $\mathcal{M}_n$ in $\mathcal{C}at$. The classifying space functor $B\colon \mathcal{C}at \to \mathcal{T}op$ maps $\mathcal{M}_n$ to a topological operad $B\mathcal{M}_n$, and it was shown in [2] that there is a topological operad $\mathcal{D}$ and equivalences of operads

$$B\mathcal{M}_n \leftarrow \mathcal{D} \to \mathcal{C}_n,$$

where $\mathcal{C}_n$ is the little n–cubes operad. A change of operads construction for topological operads then implies that the classifying space $B\mathcal{A}$ of any n–fold monoidal category $\mathcal{A}$ is weakly equivalent to a $\mathcal{C}_n$–space and hence to an n–fold loop space up to group completion. It was conjectured that any n–fold loop space can be obtained up to equivalence in this way.

More generally, let $\mathcal{O}$ and $\mathcal{P}$ be Σ–free operads in $\mathcal{C}at$ and $\mathcal{T}op$, respectively, and let $\mathcal{O}$–$\mathcal{C}at$ and $\mathcal{P}$–$\mathcal{T}op$ be their associated categories of algebras. Taking $\mathcal{P} = B\mathcal{O}$, one might be tempted to conjecture that the classifying space functor induces an equivalence of categories

$$\mathcal{O}\text{–}\mathcal{C}at[\mathrm{we}^{-1}] \simeq B\mathcal{O}\text{–}\mathcal{T}op[\mathrm{we}^{-1}],$$

where $\mathrm{we} \subset B\mathcal{O}$–$\mathcal{T}op$ is the class of all homomorphisms whose underlying maps are weak homotopy equivalences and $\mathrm{we} \subset \mathcal{O}$–$\mathcal{C}at$ is the class of all homomorphisms which are mapped to weak equivalences in $B\mathcal{O}$–$\mathcal{T}op$. To ensure the existence of the localized categories $B\mathcal{O}$–$\mathcal{T}op[\mathrm{we}^{-1}]$ and $\mathcal{O}$–$\mathcal{C}at[\mathrm{we}^{-1}]$ we can use Grothendieck's language of universes [1, Appendix], where they exist in some higher universe.

A partial step towards a proof was accomplished in Fiedorowicz and Vogt [9], where it was shown that the classifying space functor followed by the topological realization functor induces an equivalence of categories

$$\mathcal{O}\text{–}\mathcal{SC}at[\mathrm{we}^{-1}] \simeq B\mathcal{O}\text{–}\mathcal{T}op[\mathrm{we}^{-1}],$$

where $\mathcal{O}$–$\mathcal{SC}at$ is the category of simplicial $\mathcal{O}$–algebras in $\mathcal{C}at$ and the weak equivalences in $\mathcal{O}$–$\mathcal{SC}at$ are those homomorphisms which are mapped to weak equivalences in $B\mathcal{O}$–$\mathcal{T}op$. In particular, each E_n–space is, up to equivalence, the classifying space of a simplicial n–fold monoidal category. As far as E_n–spaces are concerned the full program was finally realized in Fiedorowicz, Stelzer and Vogt [8] where a homotopy colimit construction for categories of algebras over a Σ–free operad in $\mathcal{C}at$ provided a passage from simplicial $\mathcal{O}$–algebras to $\mathcal{O}$–algebras. If the morphisms of the operad $\mathcal{O}$ satisfy a certain factorization condition this passage induces an equivalence of categories

$$\mathcal{O}\text{–}\mathcal{SC}at[\mathrm{we}^{-1}] \simeq \mathcal{O}\text{–}\mathcal{C}at[\mathrm{we}^{-1}],$$

and the operads codifying n–fold monoidal categories, strictly associative braided monoidal categories, and permutative categories satisfy this condition. For these operads it was also shown that there is an equivalence of categories

$$(*) \qquad \mathcal{O}\text{–}\mathcal{C}at[\widetilde{\mathrm{we}^{-1}}] \simeq B\mathcal{O}\text{–}\mathcal{T}op[\mathrm{we}^{-1}]$$

in the foundational setting of Gödel–Bernays, where $\mathcal{O}\text{–}\mathcal{C}at[\widetilde{\mathrm{we}^{-1}}]$ is a localization of $\mathcal{O}\text{–}\mathcal{C}at$ up to equivalence (for a definition see [8, Definition 7.3]).

The main application of the construction developed in this paper is the full proof of the above conjecture in the foundational setting of Gödel–Bernays with no restrictions on the operad $\mathcal{O}$ in $\mathcal{C}at$ apart from Σ–freeness. For the existence of the genuine localizations we use an observation of Schlichtkrull and Solberg [15, Proposition A.1], and we thank them for communicating this to us. As far as E_n–spaces are concerned, the present paper offers an alternative simpler proof, because it avoids the comparatively complicated homotopy colimit construction in $\mathcal{O}\text{–}\mathcal{C}at$, which is of independent interest. In particular, it considerably simplifies the part of the proof of the main result of Thomason in [20] (the special case of $(*)$ for the operad encoding permutative categories), which relies on the homotopy colimit construction in Thomason [19].

The genesis of this paper stems from a previous paper of Fiedorowicz, Gubkin and Vogt [7, Section 4], where a similar problem involved the rectification of a weak monoidal structure on a category, without passing to simplicial categories. It was observed there that the classical M–construction of Boardman and Vogt [5, Theorem 1.26], used for this kind of rectification in $\mathcal{T}op$, could be carried out in $\mathcal{C}at$. This led us to seek a modification of this construction for the purpose of rectifying weak product algebras in $\mathcal{C}at$.

This paper is organized as follows: In Section 2 we recall some basic notions of operads and their associated categories of operators. In Section 3 we recall free operad constructions and the language of trees, which underlie our rectification constructions. In Section 4 we construct a modification of the M–construction in $\mathcal{T}op$ which allows weak product algebras over an operad as inputs. In Sections 5 and 6 we recast our modified M–construction as a homotopy colimit of a diagram in $\mathcal{T}op$. Building upon work of Thomason [18] we then show that the Grothendieck construction on the same diagram in $\mathcal{C}at$ provides the requisite rectification of weak product algebras over an operad in $\mathcal{C}at$. The remaining sections are then devoted to various applications of our rectification construction.

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Note The first two authors were saddened by the death of the third author during the final stages of production of this manuscript. Rainer Vogt was a close friend and collaborator to both of us. We will miss him.

2 Operads and their categories of operators

For the reader's convenience we recall the notions of an operad and its associated category of operators.

Let $\mathcal{S}$ be either the category $\mathcal{C}at$ of small categories, or the category $\mathcal{S}ets$ of sets, or the category $\mathcal{SS}ets$ of simplicial sets, or the category $\mathcal{T}op$ of (not necessarily Hausdorff) k–spaces. Then $\mathcal{S}$ is a self-enriched symmetric monoidal category with the product as structure functor and the terminal object $*$ as unit. In what follows, for an object X in $\mathcal{S}$, it will be convenient to refer to *elements* in X. If X is a topological space, this will mean a point in X. If X is a simplicial set, this will mean a simplex in X. If X is a category, then this will mean either an object or morphism in X. We will also use the following notions of equivalence in $\mathcal{S}$. In $\mathcal{T}op$ an equivalence will mean a strict homotopy equivalence. An equivalence between simplicial sets will mean a simplicial map whose geometric realization is a homotopy equivalence. Lastly, in $\mathcal{C}at$ we will call a functor $F\colon \mathcal{C} \to \mathcal{D}$ an equivalence if it induces a homotopy equivalence on the geometric realizations of the nerves.

2.1 Definition An *operad* $\mathcal{O}$ in $\mathcal{S}$ is a collection $\{\mathcal{O}(k)\}_{k\geq 0}$ of objects in $\mathcal{S}$ equipped with symmetric group actions $\mathcal{O}(k)\times\Sigma_k \to \mathcal{O}(k)$, composition maps

$$\mathcal{O}(k)\times(\mathcal{O}(j_1)\times\cdots\times\mathcal{O}(j_k)) \to \mathcal{O}(j_1+\cdots+j_k),$$

and a unit $\mathrm{id}\in\mathcal{O}(1)$ satisfying the appropriate equivariance, associativity and unitality conditions; see [13] for details.

An operad in $\mathcal{T}op$ is called well-pointed if $\{\mathrm{id}\}\subset\mathcal{O}(1)$ is a closed cofibration.

2.2 Remark We often find it helpful to think of an operad in the following equivalent way. An operad $\mathcal{O}$ in $\mathcal{S}$ is an $\mathcal{S}$–enriched symmetric monoidal category $(\mathcal{O},\oplus,0)$ such that:

(i) $\operatorname{ob} \mathcal{O} = \mathbb{N}$ and $m \oplus n = m + n$.

(ii) $\oplus$ is a strictly associative $\mathcal{S}$–functor with strict unit 0.

(iii) The map

$$\coprod_{r_1+\cdots+r_n=k} \mathcal{O}(r_1,1)\times\cdots\times\mathcal{O}(r_n,1)\times_{\Sigma_{r_1}\times\cdots\times\Sigma_{r_n}}\Sigma_k \to \mathcal{O}(k,n),$$

$$((f_1,\dots,f_n),\tau) \longmapsto (f_1\oplus\cdots\oplus f_n)\circ\tau$$

is an isomorphism in $\mathcal{S}$. (Note in particular that $\mathcal{O}(n,0)=\varnothing$ for $n>0$. By contrast, there are no a priori restrictions on $\mathcal{O}(0,1)$.)

In the topological case, "well-pointed" translates to $\{\mathrm{id}\}\subset\mathcal{O}(1,1)$ is a closed cofibration.

Each such category determines an operad in the sense of Definition 2.1 by taking the collection $\{\mathcal{O}(k,1)\}_{k\geq 0}$. Conversely, each operad determines such a category by property (iii).

The symmetric monoidal category associated to the trivial operad $\mathcal{C}om$ parametrizing commutative monoid structures can be identified with a skeletal category of unbased finite sets $\mathcal{F}$. Here we identify the natural number n with the set $\{1,2,\dots,n\}$, which may be viewed as an object in any of our categories $\mathcal{S}$. In particular, we identify 0 with the empty set. For any operad $\mathcal{O}$, the natural map $\mathcal{O}\to\mathcal{C}om$ induces a symmetric monoidal functor $\epsilon\colon \mathcal{O}\to\mathcal{F}$. This functor induces an equivalence on $\mathcal{S}$–enriched morphism sets $\coprod_{m,n}\mathcal{O}(m,n)\to\coprod_{m,n}\mathcal{F}(m,n)$ for any E_∞ operad $\mathcal{O}$. More generally, for any morphism $\phi\colon m\to n$ in $\mathcal{F}$ and any operad $\mathcal{O}$, $\epsilon^{-1}(\phi)$ is isomorphic to the product $\prod_{i=1}^n\mathcal{O}(|\phi^{-1}(i)|)$, where $|S|$ denotes the cardinality of the set S.

2.3 Definition Let $\mathcal{O}$ and $\mathcal{P}$ be operads in $\mathcal{S}$.

(1) In the cases $\mathcal{S}=\mathcal{C}at$, $\mathcal{S}ets$ or $\mathcal{SS}ets$, $\mathcal{O}$ is called *Σ–free* if the Σ_n–action on $\mathcal{O}(n)$ is free for each n. If $\mathcal{S}=\mathcal{T}op$ we require that $\mathcal{O}(n)\to\mathcal{O}(n)/\Sigma_n$ is a numerable principal Σ_n–bundle for each n.

(2) An *operad map* $\mathcal{O}\to\mathcal{P}$ is a collection of equivariant maps $\mathcal{O}(n)\to\mathcal{P}(n)$ in $\mathcal{S}$, compatible with the operad structure.

(3) An *$\mathcal{O}$–structure* on an object X in $\mathcal{S}$ is an operad map $\mathcal{O}\to\mathcal{E}_X$ into the endomorphism operad $\mathcal{E}_X$ of X, which is defined by $\mathcal{E}_X(n)=\mathcal{S}(X^n,X)$ with the obvious Σ_n–action and the obvious composition maps and unit. We say that $\mathcal{O}$ *acts on* X, or that X is an $\mathcal{O}$–algebra; if $\mathcal{S}=\mathcal{T}op$ we also call X an $\mathcal{O}$–space.

If we interpret an operad as a symmetric monoidal category as in Remark 2.2, an $\mathcal{O}$–algebra is the same as a strict symmetric monoidal $\mathcal{S}$–functor $\mathcal{O} \to \mathcal{S}$ taking n to X^n. Here strict monoidal means that we use the canonical isomorphisms in $\mathcal{S}$ to identify X^{m+n} with $X^m \times X^n$.

(4) An operad map is called a *weak equivalence* if each map $\mathcal{O}(n) \to \mathcal{P}(n)$ is an equivariant homotopy equivalence (in $\mathcal{C}at$ or $\mathcal{SS}ets$ this means that each map is an equivariant homotopy equivalence after applying the classifying space functor or the topological realization functor, respectively).

(5) Two operads are called *equivalent* if there is a finite chain of weak equivalences connecting them.

We denote the category of $\mathcal{O}$–algebras in $\mathcal{S}$ by $\mathcal{O}$–$\mathcal{S}$.

2.4 Let $\mathcal{O}$ be an operad in $\mathcal{S}$, interpreted as in Remark 2.2. As is shown in [5, Chapter II], the symmetric monoidal category $\mathcal{O}$ can be enlarged into an $\mathcal{S}$–enriched category with products $\Theta_{\mathcal{O}}$, such that $n = 1 \times 1 \times \cdots \times 1$. This category $\Theta_{\mathcal{O}}$ is called the *theory* associated to $\mathcal{O}$ and is determined up to isomorphism by the requirement that an $\mathcal{O}$–structure on an object X extend uniquely to a product-preserving functor $\widetilde{X}\colon \Theta_{\mathcal{O}} \to \mathcal{S}$. The category $\Theta_{\mathcal{O}}$ contains $\mathcal{O}$ and Π, the category of projections, as subcategories, and $\mathcal{O} \cap \Pi = \Sigma_*$, the subcategory of bijections. We define the *category of operators* $\widehat{\mathcal{O}}$ as the subcategory of $\Theta_{\mathcal{O}}$ generated by $\mathcal{O}$ and Π, and note that the symmetric monoidal structure on $\Theta_{\mathcal{O}}$ restricts to $\widehat{\mathcal{O}}$. For X an $\mathcal{O}$–algebra, the functor $\widetilde{X}\colon \Theta_{\mathcal{O}} \to \mathcal{S}$ restricts to a strict symmetric monoidal functor $\widehat{X}\colon \widehat{\mathcal{O}} \to \mathcal{C}$.

A more explicit description of $\widehat{\mathcal{O}}$ can be obtained as follows. First observe that for any set S, a projection $S^l \to S^k$ corresponds to an injection $k \to l$ of finite sets. Thus the category of projections Π can be identified with $\mathrm{Inj}^{\mathrm{op}}$, the opposite of the category of injections in $\mathcal{F}$. Then

$$\widehat{\mathcal{O}}(l,n) = \coprod_{0 \le k \le l} \mathcal{O}(k,n) \times_{\Sigma_k} \mathrm{Inj}(k,l).$$

In particular, $\widehat{\mathcal{O}}(l,0)$ consists of a single morphism, the nullary projection. Composition of $(f,\sigma) \in \mathcal{O}(k,n) \times_{\Sigma_k} \mathrm{Inj}(k,l)$ with $(g_1 \oplus \cdots \oplus g_l, \tau) \in \mathcal{O}(p,l) \times_{\Sigma_p} \mathrm{Inj}(p,q)$, where $g_i \in \mathcal{O}(r_i,1)$ and $p = r_1 + \cdots + r_l$, is defined by

$$(f,\sigma) \circ (g_1 \oplus \cdots \oplus g_l, \tau) = (f \circ (g_{\sigma(1)} \oplus \cdots \oplus g_{\sigma(k)}), \tau \circ \sigma(r_1, \ldots, r_l)),$$

where

$$\sigma(r_1, \ldots, r_l)\colon \underline{r} = \underline{r_{\sigma(1)}} + \cdots + \underline{r_{\sigma(k)}} \to \underline{p}$$

is the following block injection: $\underline{r}$ and $\underline{p}$ are the ordered disjoint unions

$$\underline{r} = \underline{r_{\sigma(1)}} \sqcup \cdots \sqcup \underline{r_{\sigma(k)}} \quad \text{and} \quad \underline{r_1} \sqcup \cdots \sqcup \underline{r_l};$$

the block injection $\sigma(r_1, \dots, r_l)$ maps the block $\underline{r_{\sigma(i)}}$ identically onto the corresponding block in $\underline{p}$. For a comparison of this description of $\widehat{\mathcal{O}}$ with that given in May and Thomason [14], the reader is referred to the proof of Lemma 5.7 in that paper.

We will often denote the morphisms $(\mathrm{id}_k, \sigma) \in \widehat{\mathcal{O}}(l, k)$ by σ^*.

2.5 Remark (1) If $\mathcal{O} = \mathcal{C}om$, then $\widehat{\mathcal{O}}$ can be identified with $\mathcal{F}_*$, the skeletal category of *based* finite sets, with objects $n_+ = \{0, 1, 2, \dots, n\}$. The inclusion $\mathcal{O} \subset \widehat{\mathcal{O}}$ can be identified with the functor $\mathcal{F} \to \mathcal{F}_*$ which adjoins a disjoint basepoint 0 to the finite set $n = \{1, 2, \dots, n\}$. The theory $\Theta_{\mathcal{C}om}$ can be identified with the category whose objects are the natural numbers with morphisms $m \to n$ being $n \times m$ matrices with entries in the natural numbers, with composition given by multiplication of matrices. We can then identify $\widehat{\mathcal{O}} \cong \mathcal{F}_*$ with the subcategory of $\Theta_{\mathcal{C}om}$ whose morphisms are matrices with entries in $\{0, 1\}$, with at most one non-zero entry in each column.

(2) The unique map of operads $\mathcal{O} \to \mathcal{C}om$ induces functors $\hat{\epsilon}\colon \widehat{\mathcal{O}} \to \mathcal{F}_*$ and $\Theta_\epsilon\colon \Theta_{\mathcal{O}} \to \Theta_{\mathcal{C}om}$, and there is a pullback diagram of $\mathcal{S}$–enriched categories

$$\begin{tikzcd}
\widehat{\mathcal{O}} \arrow[r, hook] \arrow[d, "\hat{\epsilon}"] & \Theta_{\mathcal{O}} \arrow[d, "\Theta_\epsilon"] \\
\mathcal{F}_* \arrow[r, hook] & \Theta_{\mathcal{C}om}
\end{tikzcd}$$

For any E_∞ operad $\mathcal{O}$, $\hat{\epsilon}$ induces an equivalence on $\mathcal{S}$–enriched morphism sets

$$\coprod_{m,n} \widehat{\mathcal{O}}(m,n) \to \coprod_{m,n} \mathcal{F}_*(m,n).$$

More generally, for any morphism $\phi\colon m \to n$ in $\mathcal{F}_*$ and any operad $\mathcal{O}$, $\epsilon^{-1}(\phi)$ is isomorphic to the product $\prod_{i=1}^n \mathcal{O}(|\phi^{-1}(i)|)$. Moreover, $\mathcal{F}_*$ is the largest subcategory of $\Theta_{\mathcal{C}om}$ containing $\mathcal{F}$ with this property. For other morphisms in $\Theta_{\mathcal{C}om}$ the inverse image under Θ_ϵ is the quotient of such a product by a stabilizing group of permutations. For instance,

$$\Theta_\epsilon^{-1}\big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\colon 2 \to 2\big) \cong \big(\mathcal{O}(a+b)/\Sigma_a \times \Sigma_b\big) \times \big(\mathcal{O}(c+d)/\Sigma_c \times \Sigma_d\big).$$

2.6 Definition An $\widehat{\mathcal{O}}$–diagram in $\mathcal{S}$ is an $\mathcal{S}$–enriched functor $G\colon \widehat{\mathcal{O}} \to \mathcal{S}$. Such a diagram is called *special* if the injections $\iota_k\colon \underline{1} \to \underline{n}$ sending 1 to k define a homotopy equivalence $(\iota_1^*,\dots,\iota_n^*)\colon G(n)\to G(1)^n$ for each n, ie G is a weakly symmetric monoidal functor.

We denote the category of $\widehat{\mathcal{O}}$–diagrams in $\mathcal{S}$ by $\mathcal{S}^{\widehat{\mathcal{O}}}$.

As we noted above in 2.4, there is an obvious functor

$$\widehat{(-)}\colon \mathcal{O}\text{–}\mathcal{S}\to\mathcal{S}^{\widehat{\mathcal{O}}},$$

given by extending a symmetric monoidal functor $X\colon\mathcal{O}\to\mathcal{S}$ to a product-preserving functor $\widetilde{X}\colon\Theta_{\mathcal{O}}\to\mathcal{S}$, and then restricting this extension to $\widehat{X}\colon\widehat{\mathcal{O}}\to\mathcal{S}$. Explicitly, $\widehat{X}\colon\widehat{\mathcal{O}}\to\mathcal{S}$ is defined by $\widehat{X}(n)=X^n$, $\widehat{X}((f,\mathrm{id}))=X(f)$, and $\widehat{X}(\sigma^*)\colon X^l\to X^k$ being the projection

$$(x_1,\dots,x_l)\mapsto(x_{\sigma(1)},\dots,x_{\sigma(k)}).$$

By construction, $\widehat{X}$ is special.

We recall that the classifying space functor $B\colon\mathcal{C}at\to\mathcal{T}op$ is the composite

$$B\colon\mathcal{C}at\xrightarrow{N_*}\mathcal{S}Sets\xrightarrow{|-|}\mathcal{T}op$$

of the nerve functor N_* and the topological realization. The classifying space functor preserves products, which implies the following result:

2.7 Lemma *Let $\mathcal{O}$ be an operad in $\mathcal{C}at$, let X be an $\mathcal{O}$–algebra, and let $G\colon\widehat{\mathcal{O}}\to\mathcal{C}at$ be an $\widehat{\mathcal{O}}$–diagram. Then:*

(1) *$B\mathcal{O}$ is an operad in $\mathcal{T}op$ and BX is a $B\mathcal{O}$–space.*
(2) *$B(\widehat{\mathcal{O}})\cong\widehat{B\mathcal{O}}$ and $BG\colon\widehat{B\mathcal{O}}\to\mathcal{T}op$ is a $\widehat{B\mathcal{O}}$–diagram. If G is special, so is BG.*
(3) *$\widehat{BX}\cong B\widehat{X}$.*

If we want to determine the homotopy types of our categorical constructions, we usually have to assume that all operads we consider are Σ–free. The reason we need to make this assumption is that our constructions will require us to take quotients, by permutation groups, of categories which are products of various $\mathcal{O}(k)$ categories together with other categories. Under this hypothesis the classifying spaces of the resulting quotient categories will be homeomorphic to the quotients of the classifying spaces of the product categories, due to the fact that the classifying space functor preserves finite products and the following elementary result.

2.8 Lemma *Let a discrete group Γ act freely on a small category $\mathcal{C}$. Then $B\mathcal{C}$ is a free Γ–space and*

$$B(\mathcal{C}/\Gamma)\cong(B\mathcal{C})/\Gamma.$$

Proof Since $B\mathcal{C}$ is a Γ–CW complex with a free Γ–action, $B\mathcal{C} \to (B\mathcal{C})/\Gamma$ is a numerable principal bundle. We have $\mathrm{ob}(\mathcal{C}/\Gamma) = \mathrm{ob}(\mathcal{C})/\Gamma$ and $\mathrm{mor}(\mathcal{C}/\Gamma) = \mathrm{mor}(\mathcal{C})/\Gamma$. Composition in $\mathcal{C}/\Gamma$ is defined by lifting to $\mathcal{C}$: given composable morphisms $[f]\colon [A] \to [B]$ and $[g]\colon [B] \to [C]$ in $\mathcal{C}/\Gamma$, choose a representative object A in $\mathcal{C}$. Then there are unique morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\mathcal{C}$ representing $[f]$ and $[g]$, and $[g][f] = [gf]$. Hence any simplex in the nerve of $\mathcal{C}/\Gamma$ has a unique lift to the nerve of $\mathcal{C}$ once we choose a lift of the initial vertex. It follows that the nerve of $\mathcal{C}/\Gamma$ is the quotient of the nerve of $\mathcal{C}$ by the action of Γ, which implies the result. □

This result fails to hold if the action of Γ on $\mathcal{C}$ is not free. For instance, if H is a group regarded as a category with one object, $\mathcal{C} = H \times H$ and $\Gamma = \mathbb{Z}/2$ acts on $\mathcal{C}$ by permuting the factors, then $B(\mathcal{C}/\Gamma) \cong BA$, where A is the abelianization of H. This is clearly different from $B(H \times H)/\Gamma = (BH \times BH)/(\mathbb{Z}/2)$, particularly if H is perfect.

3 Free operads

Since our constructions start with free operads, we recall their construction for the convenience of the reader and to fix notation. We follow the expositions [3, Section 5.8] and [4, Section 3], because they are the most convenient ones for our purposes. We recommend [14, Part I, Section 2] for background on the language of trees.

Recall that a *collection* $\mathcal{K}$ in one of our categories $\mathcal{S}$ is an $\mathbb{N}$–indexed family of objects $\mathcal{K}(n)$ with a right Σ_n–action. Let $\mathcal{O}pr(\mathcal{S})$ and $\mathcal{C}oll(\mathcal{S})$ denote the categories of operads and collections in $\mathcal{S}$. Then there is the obvious forgetful functor

$$R\colon \mathcal{O}pr(\mathcal{S}) \to \mathcal{C}oll(\mathcal{S}),$$

and we are interested in its left adjoint

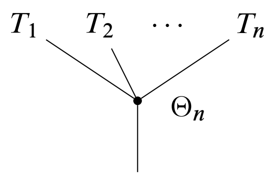
$$L\colon \mathcal{C}oll(\mathcal{S}) \to \mathcal{O}pr(\mathcal{S}),$$

the *free operad functor*.

Let $\mathbb{T}$ denote the groupoid of planar trees and non-planar isomorphisms. Its objects are finite directed rooted planar trees (see [12, pages 85–87] for a formal definition). A tree can have three types of edges: *internal edges* with a node on each end, *input edges* with a node only at the end, and one outgoing edge, called the *root*, with a node only at its beginning. Each node ν has a finite totally ordered poset $\underline{\mathrm{In}(\nu)}$ of incoming edges, also called *inputs* of ν, and exactly one outgoing edge, called its *output*. The cardinality $\mathrm{In}(\nu)$ of $\underline{\mathrm{In}(\nu)}$ is called the *valence* of ν. We allow *stumps*, ie nodes of valence 0, and the *trivial tree* consisting of a single edge. The poset of input edges of a tree T is denoted by $\underline{\mathrm{In}(T)}$, and its cardinality by $\mathrm{In}(T)$. We say that a subtree T' of a tree T is a *subtree above a node* v of T if T' consists of an incoming edge of v and all nodes and edges of T lying above that edge. Note that if T' is such a subtree, then $\underline{\mathrm{In}(T')}$ forms a (possibly empty) subinterval of $\underline{\mathrm{In}(T)}$.

3.1 Definition A morphism $\phi\colon T \to T'$ in $\mathbb{T}$ is an isomorphism of trees after forgetting their planar structures. So ϕ preserves inputs and hence induces a bijection $\mathrm{In}(\phi)\colon \underline{\mathrm{In}(T)} \to \underline{\mathrm{In}(T')}$. If $\mathrm{in}_1, \dots, \mathrm{in}_n$ are the inputs of T and $\mathrm{in}'_1, \dots, \mathrm{in}'_n$ are the inputs of T' counted from left to right, then ϕ has an associated permutation $\phi^\Sigma \in \Sigma_n$ defined by $\phi^\Sigma(k) = l$ if $\phi(\mathrm{in}_l) = \mathrm{in}'_k$. Note that $\phi \mapsto \phi^\Sigma$ is covariant: $(\psi\phi)^\Sigma = \psi^\Sigma\phi^\Sigma$.

Let Θ_n denote the tree with exactly one node and n inputs. Any tree T with a root node of valence n decomposes uniquely into n trees $T_1, \dots, T_n$ whose outputs are grafted onto the inputs of Θ_n as in the diagram

$T =$ 

We denote this grafting operation by

$$T = \Theta_n \circ (T_1 \oplus \cdots \oplus T_n).$$

Any isomorphism $\phi\colon T \to T'$ has a similar decomposition

$$\phi = \sigma \circ (\phi_1 \oplus \cdots \oplus \phi_n)$$

into isomorphisms $\sigma\colon \Theta_n \to \Theta_n$ and $\phi_i\colon T_{\sigma}(i) \to T'_i$. Since σ only permutes the inputs of Θ_n we usually denote σ^Σ simply by σ.

Since the number of nodes and edges in each T_i is strictly less than the number of nodes and edges in T, this decomposition is suited for inductive procedures.

For any collection $\mathcal{K}$ we define a functor $\underline{\mathcal{K}}\colon \mathbb{T}^{\mathrm{op}} \to \mathcal{S}$ inductively by mapping the trivial tree to the terminal object and putting

$$\underline{\mathcal{K}}(T) = \underline{\mathcal{K}}(\Theta_n \circ (T_1 \oplus \cdots \oplus T_n)) = \mathcal{K}(n) \times \underline{\mathcal{K}}(T_1) \times \cdots \times \underline{\mathcal{K}}(T_n).$$

On morphisms $\phi\colon T \to T'$ we define $\phi^*\colon \underline{\mathcal{K}}(T') \to \underline{\mathcal{K}}(T)$ inductively by

$$\phi^* = (\sigma \circ (\phi_1 \oplus \cdots \oplus \phi_n))^* = \sigma^* \times \phi^*_{\sigma(1)} \times \cdots \times \phi^*_{\sigma(n)},$$

which is determined by setting

$$\sigma^*\colon \mathcal{K}(n) \to \mathcal{K}(n), \quad a \mapsto a \cdot \sigma.$$

There is also a functor $\bar{\lambda}\colon \mathbb{T} \to \mathcal{S}ets$ associating with each tree T the set $\bar{\lambda}(T)$ of bijections $\tau\colon \{1, 2, \dots, \mathrm{In}(T)\} \to \underline{\mathrm{In}(T)}$. On morphisms $\phi\colon T \to T'$ we define

$$\bar{\lambda}(\phi)\colon \bar{\lambda}(T) \to \bar{\lambda}(T), \quad \tau \mapsto \mathrm{In}(\phi) \circ \tau.$$

Since $\mathcal{S}ets$ is canonically included in $\mathcal{C}at$, $\mathcal{SS}ets$ and $\mathcal{T}op$ as the full subcategory of discrete objects, we can consider $\bar{\lambda}$ as a functor $\bar{\lambda}\colon \mathbb{T} \to \mathcal{S}$. The groupoid $\mathbb{T}$ is the disjoint sum of the groupoids $\mathbb{T}(n) = \{T \in \mathbb{T} : \mathrm{In}(T) = n\}$, and the free operad functor

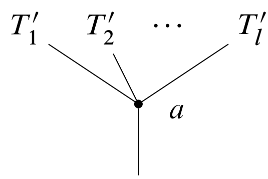
$$L\colon \mathcal{C}oll(\mathcal{S}) \to \mathcal{O}pr(\mathcal{S})$$

sends the collection $\mathcal{K}$ to the operad whose underlying collection is the family of coends

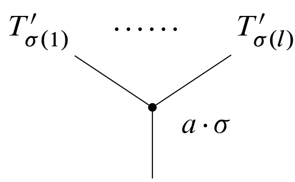
$$L\mathcal{K}(n) = \underline{\mathcal{K}} \otimes_{\mathbb{T}(n)} \bar{\lambda}, \quad n \in \mathbb{N}.$$

Before we define the operad structure, let us give an explicit description of $L\mathcal{K}(n)$. An element of $L\mathcal{K}(n)$ is represented by a triple (T, f, τ) consisting of a tree T with n inputs, a function f assigning to each node ν of T an element $a \in \mathcal{K}(\mathrm{In}(\nu))$, and a bijection $\tau\colon \underline{n} = \{1, 2, \dots, n\} \to \underline{\mathrm{In}(T)}$. We call a the *decoration* of ν and i the *label* of the input $\tau(i)$. We usually suppress f and τ and speak of a decorated tree T with input labels.

3.2 Equivariance relation We impose the following relation on the set of decorated trees T with input labels. Let

$T' =$ 

be a subtree of T above a node ν with decoration $a \in \mathcal{K}(l)$, and let $\sigma \in \Sigma_l$. Then T is equivalent to the decorated tree ${}^{\sigma}T$ obtained from T by replacing T' by

$T'' =$ 

The elements of $L\mathcal{K}(n)$ are the equivalence classes of decorated trees with input labels with respect to this relation.

If (T, f, τ) represents an element x in $L\mathcal{K}(n)$ and if $\sigma \in \Sigma_n$, we define $x \cdot \sigma$ to be represented by $(T, f, \tau \circ \sigma)$. This defines the right action of Σ_n on $L\mathcal{K}(n)$. Operad composition is defined by grafting decorated trees with input labels according to the labels: $T \circ (T_1 \oplus \cdots \oplus T_n)$ is obtained by grafting T_i on the input of T labeled by i.

Let $\tau\colon \underline{n} \to \underline{\mathrm{In}(T)}$ be an input labeling of T, and suppose $\tau(i)$ is the k^{th} input of T counted from left to right. Then we identify τ with the permutation $\tau \in \Sigma_n$ sending i to k. Using this identification we obtain the following:

3.3 Proposition *We have*

$$L\mathcal{K}(n) = \underline{\mathcal{K}} \otimes_{\mathbb{T}(n)} \bar{\lambda} = \coprod_{[T]} \underline{\mathcal{K}}(T) \otimes_{\mathrm{Aut}(T)} \Sigma_n, \quad n \geq 0,$$

where the sum is indexed by isomorphism classes of trees in $\mathbb{T}(n)$.

For later use we observe that Proposition 3.3 is a special case of a more general result.

3.4 Let $\mathbb{G}$ be a groupoid and let $\underline{F}\colon \mathbb{G}^{\mathrm{op}} \to \mathcal{S}$ and $\bar{\lambda}\colon \mathbb{G} \to \mathcal{S}$ be functors. Then

$$\underline{F} \otimes_{\mathbb{G}} \bar{\lambda} = \coprod_{[G]} \underline{F} \otimes_{[G]} \bar{\lambda} \cong \coprod_{[G]} \underline{F}(G) \otimes_{\mathrm{Aut}(G)} \bar{\lambda}(G),$$

where the sum is indexed by isomorphism classes in $\mathbb{G}$. The coend $\underline{F} \otimes_{[G]} \bar{\lambda}$ is taken over the elements in the class $[G]$. The isomorphism depends on the choice of representatives G in the class $[G]$.

4 Rectifying $\widehat{\mathcal{O}}$–spaces

We start with our rectification construction for $\widehat{\mathcal{O}}$–spaces, which is easier to describe than the version we use for the $\mathcal{C}at$ case. Although this space version is simpler, it uses some of the same ingredients as our subsequent rectification construction for $\widehat{\mathcal{O}}$–categories and will help to motivate that construction. In the process we give a simple variant of a rectification result of May and Thomason [14, Theorem 4.5] We should also note that the construction we define here is a variant of the M–construction of Boardman and Vogt [5, page 134ff].

4.1 Let $\mathcal{O}$ be an arbitrary operad in $\mathcal{T}op$. We are going to define a rectification functor

$$M\colon \mathcal{T}op^{\widehat{\mathcal{O}}} \to \mathcal{O}\text{–}\mathcal{T}op.$$

Our construction starts with a modification of the free operad construction. We inductively define a functor $\mathcal{L}\mathcal{O}\colon \mathbb{T}^{\mathrm{op}} \to \mathcal{T}op$ by mapping the trivial tree to a point and putting

$$\mathcal{L}\mathcal{O}(T) = \mathcal{L}\mathcal{O}(\Theta_n \circ (T_1 \oplus \cdots \oplus T_n)) = \mathcal{O}(n) \times I^n \times \mathcal{L}\mathcal{O}(T_1) \times \cdots \times \mathcal{L}\mathcal{O}(T_n),$$

where I is the unit interval. On morphisms $\phi\colon T \to T'$ the functor is given by

$$\phi^* = (\sigma \circ (\phi_1 \oplus \cdots \oplus \phi_n))^* = \sigma^* \times \phi^*_{\sigma(1)} \times \cdots \times \phi^*_{\sigma(n)},$$

with

$$\sigma^*\colon \mathcal{O}(n) \times I^n \to \mathcal{O}(n) \times I^n, \quad (a; t_1, \dots, t_n) \mapsto (a \cdot \sigma; t_{\sigma(1)}, \dots, t_{\sigma(n)}).$$

For $G\colon \widehat{\mathcal{O}} \to \mathcal{T}op$ there is a functor $\lambda = \lambda_G\colon \mathbb{T} \to \mathcal{T}op$, sending the trivial tree to $G(\mathcal{O}(1))$ and $\Theta_n \circ (T_1 \oplus \cdots \oplus T_n)$ to $G(\mathrm{In}(T_1)) \times \cdots \times G(\mathrm{In}(T_n))$. In particular, $\lambda(\Theta_n) = G(\mathcal{O}(1))^n$. On morphisms $\sigma\colon \Theta_n \to \Theta_n$ it is defined by

$$\lambda(\sigma)\colon G(1)^n \to G(1)^n, \quad (g_1, \dots, g_n) \mapsto (g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(n)}),$$

and for $\phi = \sigma \circ (\phi_1 \oplus \cdots \oplus \phi_n)\colon T \to T'$ by

$$\begin{aligned} \lambda(\phi)\colon G(\mathrm{In}(T_1)) \times \cdots \times G(\mathrm{In}(T_n)) &\to G(\mathrm{In}(T'_1)) \times \cdots \times G(\mathrm{In}(T'_n)), \\ (g_i)_{i=1}^n &\mapsto \big(G(\phi^{\Sigma}_{\sigma^{-1}(i)})(g_{\sigma^{-1}(i)})\big)_{i=1}^n. \end{aligned}$$

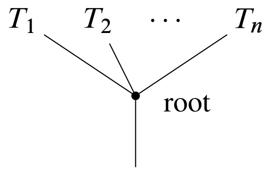
Here recall that $\phi_{\sigma^{-1}(i)}\colon T_{\sigma^{-1}(i)} \to T''_i$ is in $\mathbb{T}$ and $\phi^{\Sigma}_{\sigma^{-1}(i)}$ is the induced inputs permutation (see Definition 3.1). A natural transformation $G \to G'$ induces a natural transformation $\lambda_G \to \lambda_{G'}$.

Let $\widetilde{\mathbb{T}} \subset \mathbb{T}$ be the full subgroupoid of non-trivial trees. Restricting our functors to $\widetilde{\mathbb{T}}$, the coend construction defines a functor

$$\mathcal{L}\mathcal{O} \otimes_{\widetilde{\mathbb{T}}} \lambda_{(-)}\colon \mathcal{T}op^{\widehat{\mathcal{O}}} \to \mathcal{T}op, \quad G \mapsto \mathcal{L}\mathcal{O} \otimes_{\widetilde{\mathbb{T}}} \lambda_G.$$

The functor $M\colon \mathcal{T}op^{\widehat{\mathcal{O}}} \to \mathcal{O}\text{–}\mathcal{T}op$ will be a quotient of this functor.

4.2 We find it helpful to view an element of $\mathcal{L}\mathcal{O}(T)$ as a triple (T, f, h) consisting of a tree (T, f) with vertex decorations as in Section 3, and a length function h assigning to each internal edge of T a length in I. We usually suppress f and h and speak of a decorated tree T with lengths whose nodes are decorated by elements in $\mathcal{O}$ and whose internal edges have a length label. It will be clear from the context whether T denotes a decorated tree with lengths or just a tree. Let T have the form

(4.3) $T =$ 

Here T_i is allowed to be the trivial tree.

We define

$$V(G,T) = \mathcal{LO}(T) \times G(\mathrm{In}(T)),$$
$$U(G,T) = \mathcal{LO}(T) \times G(\mathrm{In}(T_1)) \times \cdots \times G(\mathrm{In}(T_n)),$$
$$\mathcal{LO} \otimes_{\widetilde{\mathbb{T}}} \lambda_G = \Big(\coprod_T U(G,T) \Big) / \sim,$$

where the unions is taken over all trees in $\widetilde{\mathbb{T}}$ and the relations are as follows:

4.4 Equivariance relations It is helpful to consider $U(G,T)$ as

$$U(G,T) = \mathcal{O}(n) \times I^n \times V(G,T_1) \times \cdots \times V(G,T_n),$$

where $(t_1, \dots, t_n) \in I^n$ are the lengths of the incoming edges of the root from left to right and $\mathcal{O}(n)$ is the space of root decorations.

(1) *Root equivariance* Let $\sigma \in \Sigma_n$. Then

$$(a; t_1, \dots, t_n; (T_1, g_1), \dots, (T_n, g_n))$$
$$\sim (a \cdot \sigma; t_{\sigma(1)}, \dots, t_{\sigma(n)}; (T_{\sigma(1)}, g_{\sigma(1)}), \dots, (T_{\sigma(n)}, g_{\sigma(n)})).$$

(2) *T_i–equivariance* T_i–equivariance is a relation on the factor $V(G,T_i)$. We use the notation of 3.2 with the difference that the internal edges of our trees have a length label. As in 3.2, let T' be the subtree above a node v of valence l of T_i decorated by a. Let $\sigma \in \Sigma_l$ and let ${}^\sigma T_i$ be obtained from T_i as in 3.2. Then σ determines an isomorphism $\phi\colon T_i \to {}^\sigma T_i$ of underlying trees in $\widetilde{\mathbb{T}}$, and T_i–equivariance is the relation

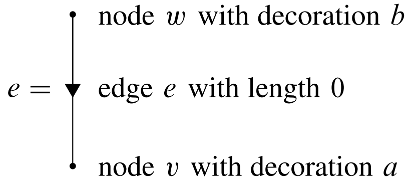
$$(T_i; g_i) \sim ({}^\sigma T_i; G(\phi^\Sigma)(g_i)).$$

4.5 Definition The functor

$$M\colon \mathcal{T}op^{\widehat{\mathcal{O}}} \to \mathcal{O}\text{–}\mathcal{T}op$$

is obtained from the functor $\mathcal{LO} \otimes_{\widetilde{\mathbb{T}}} \lambda_{(-)}$ by imposing the following relations. Let T be a decorated tree with lengths of the form (4.3).

(1) *Shrinking an internal edge* An internal edge e of length 0 may be shrunk:

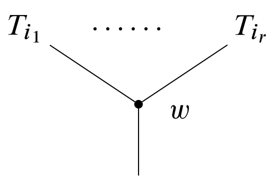


Let T' be obtained from T by shrinking e. If e is the i^{th} input of v counted from left to right, the new node in T' is decorated by $a \circ (\mathrm{id}_{i-1} \oplus b \oplus \mathrm{id}_{\mathrm{In}(v)-i})$.

(a) *v is not the root* Then

$$(T; g_1, \dots, g_n) \sim (T'; g_1, \dots, g_n), \quad g_i \in G(\mathrm{In}(T_i)).$$

(b) *v is the root* Then T_i has the form



and e is the outgoing edge of w. If w is not a stump, shrinking e makes the incoming edges of w into incoming edges of the root of T'. Let $\tau_j \colon \underline{\mathrm{In}(T_{i_j})} \subset \underline{\mathrm{In}(T_i)}$ be the inclusion. Then there is a map

$$\tau^* \colon G(\mathrm{In}(T_i)) \to \prod_{j=1}^{r} G(\mathrm{In}(T_{i_j}))$$

whose j^{th} component is $G(\tau_j^*)$. We have the relation

$$(T; g_1, \dots, g_n) \sim (T'; g_1, \dots, g_{i-1}, \tau^*(g_i), g_{i+1}, \dots, g_n).$$

If w is a stump, $\mathrm{In}(T_i) = \varnothing$ and we impose the relation

$$(T; g_1, \dots, g_n) \sim (T'; g_1, \dots, g_n).$$

(2) *Chopping an internal edge* An internal edge e of length 1 may be chopped off. Let e be as above, but of length 1. Let T'' be the subtree of T with root w. Then

T'' is a subtree of some T_i. Let T' be obtained from T by deleting the subtree T''. Composing all node decorations of T'' using the operad composition gives us an element $c \in \mathcal{O}(\mathrm{In}(T''))$. We label the inputs of T_i from left to right by 1 to $\mathrm{In}(T_i)$. Then the inputs of T'' form a subinterval $s+1, s+2, \dots, s+t$ with $t = \mathrm{In}(T'')$. Define

$$\hat{c} = \mathrm{id}_s \oplus c \oplus \mathrm{id}_{\mathrm{In}(T_i)-s-t} \in \mathcal{O} \subset \widehat{\mathcal{O}}.$$

We have the relation

$$(T; g_1, \dots, g_k) \sim (T'; g_1, \dots, g_{i-1}, G(\hat{c})(g_i), g_{i+1}, \dots, g_k).$$

In particular, if w is a stump then $c = b \in \mathcal{O}(0)$, $\mathrm{In}(T'') = \varnothing$ so that $t = 0$, and $\mathrm{In}(T') = \mathrm{In}(T) + 1$.

4.6 Proposition *$M(G)$ has an $\mathcal{O}$–algebra structure, and we obtain a functor*

$$M\colon \mathcal{T}op^{\widehat{\mathcal{O}}} \to \mathcal{O}\text{–}\mathcal{T}op.$$

Proof Let $x_i \in M(G)$, $i = 1, \dots, n$, be represented by $(T_i; g_{i1}, \dots, g_{ik_i})$ and let $a \in \mathcal{O}(n)$. Then $a(x_1, \dots, x_n)$ is represented by

$$(T; g_{11}, \dots, g_{1k_1}, \dots, g_{n1}, \dots, g_{nk_n}),$$

where T is obtained from $T_1, \dots, T_n$ by grafting the roots of the T_i together to a single root. If the root of T_i is decorated by b_i, the new root is decorated by

$$a \circ (b_1 \oplus \cdots \oplus b_n). \qquad \square$$

We want to compare the $\widehat{\mathcal{O}}$–space $\widehat{M(G)}$ associated with the $\mathcal{O}$–algebra $M(G)$ with the original $\widehat{\mathcal{O}}$–space G. For this purpose we define an $\widehat{\mathcal{O}}$–space

$$Q(G)\colon \widehat{\mathcal{O}} \to \mathcal{T}op, \quad n \mapsto Q_n(G)$$

by

$$Q_n(G) = \Big(\coprod \mathcal{L}\mathcal{O}(T_1) \times \cdots \times \mathcal{L}\mathcal{O}(T_n) \times G(\mathrm{In}(T_1) + \cdots + \mathrm{In}(T_n))\Big)\Big/\sim,$$

where the union is taken over all n–tuples $(T_1, \dots, T_n)$ of trees in $\widetilde{\mathbb{T}}$. The relations are:

(1) *Shrinking an internal edge* An internal edge e of length 0 in any of the trees may be shrunk as explained in Definition 4.5(1a), which makes sense even if e is a root edge.

(2) *Chopping an internal edge* Any internal edge e of length 1 in any of the trees may be chopped as explained in Definition 4.5(2) with the difference that $\hat{c}$ is formed using all inputs rather than only the ones of T_i.

(3) *Equivariance* T_i–equivariance, as explained in 4.4(2), holds for each tree $T_1, \dots, T_n$, and the relation reads

$$(T_1, \dots, T_n; g) \sim (T_1, \dots, {}^{\sigma}T_i, \dots, T_n; G(\mathrm{id} \times \dots \times \phi^{\Sigma} \times \dots \times \mathrm{id})(g)).$$

This defines $Q(G)$ on objects.

For $\sigma \in \mathrm{Inj}(k,l)$ the map $Q(G)(\sigma^*)\colon Q_l(G) \to Q_k(G)$ is given by the projections

$$\mathcal{LO}(T_1) \times \dots \times \mathcal{LO}(T_l) \to \mathcal{LO}(T_{\sigma(1)}) \times \dots \times \mathcal{LO}(T_{\sigma(k)})$$

and the map $G(\sigma(\mathrm{In}(T_1), \dots, \mathrm{In}(T_l))^*)$ defined in 2.4. If $k_1 + \dots + k_r = m$ and

$$\alpha = (\alpha_1, \dots, \alpha_r) \in \mathcal{O}(k_1, 1) \times \dots \times \mathcal{O}(k_r, 1) \subset \widehat{\mathcal{O}}(m, r),$$

then $Q(G)(\alpha)\colon Q_m(G) \to Q_r(G)$ maps a representing tuple $(T_1, \dots, T_m; g)$, where each T_i is a decorated tree with lengths and $g \in G(\mathrm{In}(T_1) + \dots + \mathrm{In}(T_m))$, to the element represented by $(T'_1, \dots, T'_r; g)$. If $p = k_1 + \dots + k_{i-1}$ then T'_i is obtained from $T_{p+1}, \dots, T_{p+k_i}$ by grafting their roots together and decorating the root of T'_i by $\alpha_i \circ (\beta_1 \oplus \dots \oplus \beta_{k_i})$, where β_j is the root decoration of T_{p+j}.

Like $M(G)$, the space $Q_n(G)$ is the quotient of a coend, namely the coend of the functor

$$\mathcal{LO}_{Q_n}\colon (\widetilde{\mathbb{T}}^{\mathrm{op}})^n \xrightarrow{\mathcal{LO}^n} \mathcal{T}op^n \xrightarrow{\text{product}} \mathcal{T}op$$

and the functor

$$\lambda_{Q_n}\colon \widetilde{\mathbb{T}}^n \to \mathcal{T}op, \quad (T_1, \dots, T_n) \mapsto G(\mathrm{In}(T_1) + \dots + \mathrm{In}(T_n)).$$

4.7 Theorem *There are maps of $\widehat{\mathcal{O}}$–spaces, natural in G,*

$$\widehat{M(G)} \xleftarrow{\tau} Q(G) \xrightarrow{\varepsilon} G,$$

such that

(1) *each $\varepsilon_n\colon Q_n(G) \to G(n)$ is a homotopy equivalence, and*

(2) *if $\mathcal{O}$ is Σ–free and G is special, each $\tau_n\colon Q_n(G) \to M(G)^n$ is a homotopy equivalence.*

Proof The map $\varepsilon_n\colon Q_n(G) \to G(n)$ is defined by chopping the roots of each tree. This makes sense in this case although roots are not internal edges. By construction, the ε_n define a map $\varepsilon\colon Q(G) \to G$ of $\widehat{\mathcal{O}}$–spaces. Each $\varepsilon_n\colon Q_n(G) \to G(n)$ has a section

$$s_n\colon G(n) \to Q_n(G), \quad g \mapsto (\Theta_1, \dots, \Theta_1; g),$$

with $\mathrm{id} \in \mathcal{O}(1)$ as node decoration of Θ_1. Let $T(t)$ be the tree obtained from T by putting T on top of Θ_1 and giving the newly created internal edge the length t. Then for

$$(T_1, \dots, T_n; g) \in \mathcal{LO}(T_1) \times \dots \times \mathcal{LO}(T_n) \times G(\mathrm{In}(T_1) + \dots + \mathrm{In}(T_n))$$

we have

$$(T_1(0), \dots, T_n(0); g) \sim (T_1, \dots, T_n; g)$$

by the shrinking relation, and

$$(T_1(1), \dots, T_n(1); g) \sim s_n \varepsilon_n (T_1, \dots, T_n; g)$$

by the chopping relation. Hence $t \mapsto (T_1(t), \dots, T_n(t); g)$ defines a homotopy from $\mathrm{id}_{Q_n(G)}$ to $s_n \circ \varepsilon_n$.

We define

$$\tau\colon Q_n(G) \to M(G)^n, \quad (T_1, \dots, T_n; g) \mapsto \big(T_i; G(\sigma_{i,1}^*)(g), \dots, G(\sigma_{i,k_i}^*)(g)\big)_{i=1}^n$$

if T_i is of the form $T_i = \Theta_{k_i} \circ (T_{i,1} \oplus \dots \oplus T_{i,k_i})$, and where

$$\sigma_{i,j}\colon \underline{\mathrm{In}(T_{i,j})} \subset \underline{\mathrm{In}(T_i)} \subset \underline{\mathrm{In}(T_1) + \dots + \mathrm{In}(T_n)}$$

is the canonical inclusion. By construction, the τ_n define a map of $\widehat{\mathcal{O}}$–spaces.

We now prove the second statement of the theorem. So assume that G is special and $\mathcal{O}$ is Σ–free. Then the map

$$\pi_n = (G(\sigma_{1,1}^*), \dots, G(\sigma_{n,k_n}^*))\colon G(\mathrm{In}(T_1) + \dots + \mathrm{In}(T_n)) \to \prod_{i=1}^{n} \prod_{j+1}^{k_i} G(\mathrm{In}(T_{i,j}))$$

is a homotopy equivalence. For notational convenience we denote

$$\begin{cases} G(\mathrm{In}(T_1) + \dots + \mathrm{In}(T_n)) \\ \prod_{i=1}^n \prod_{j+1}^{k_i} G(\mathrm{In}(T_{i,j})) \\ \prod_{i=1}^n \mathrm{Aut}(T_i) \end{cases} \quad \text{by} \quad \begin{cases} G_Q(T_1, \dots, T_n), \\ G_M(T_1, \dots, T_n), \\ \mathrm{Aut}(T_1, \dots, T_n). \end{cases}$$

Similarly, we denote

$$\mathcal{LO}_{Q_n}(T_1, \dots, T_n) = \mathcal{LO}(T_1) \times \dots \times \mathcal{LO}(T_n) \quad \text{by } \mathcal{LO}(T_1, \dots, T_n).$$

By 3.4, $M(G)^n$ is a quotient of

$$\coprod_{([T_1], \dots, [T_n])} (\mathcal{LO} \times_{\mathrm{Aut}} G_M)(T_1, \dots, T_n),$$

where

$$(\mathcal{LO} \times_{\mathrm{Aut}} G_M)(T_1, \dots, T_n) = \mathcal{LO}(T_1, \dots, T_n) \times_{\mathrm{Aut}(T_1,\dots,T_n)} G_M(T_1, \dots, T_n),$$

and $Q_n(G)$ is a quotient of

$$\coprod_{([T_1],\dots,[T_n])} (\mathcal{LO} \times_{\mathrm{Aut}} G_Q)(T_1, \dots, T_n),$$

where

$$(\mathcal{LO} \times_{\mathrm{Aut}} G_Q)(T_1, \dots, T_n) = \mathcal{LO}(T_1, \dots, T_n) \times_{\mathrm{Aut}(T_1,\dots,T_n)} G_Q(T_1, \dots, T_n).$$

In both cases the sum is indexed by the isomorphism classes in $\widetilde{\mathbb{T}}^n$.

Hence the proof reduces to showing that

$$\mathrm{id} \times_{\mathrm{Aut}} \pi_n \colon \coprod_{([T_1],\dots,[T_n])} (\mathcal{LO} \times_{\mathrm{Aut}} G_Q)(T_1, \dots, T_n) \to \coprod_{([T_1],\dots,[T_n])} (\mathcal{LO} \times_{\mathrm{Aut}} G_M)(T_1, \dots, T_n)$$

induces a homotopy equivalence $Q_n(G) \to M(G)^n$. (This part of the proof relies on certain technical facts about numerable principal bundles that we list in 4.8.)

We choose a representative T in each isomorphism class $[T]$ and filter both spaces. Let $F_r(Q)$ and $F_r(M)$ be the subspaces of $Q_n(G)$ and $M(G)^n$ of those points which can be represented by elements for which the $(T_1, \dots, T_n)$–part consists of trees whose total number of internal edges is less than or equal to r. We prove by induction that the above map induces a homotopy equivalence $F_r(Q) \to F_r(M)$ for all r, which in turn implies the result.

$F_0(Q)$ is the disjoint union of spaces

$$(\mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n)) \times_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_n}} G(k_1 + \cdots + k_n)$$

and $F_0(M)$ is the disjoint union of spaces

$$(\mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n)) \times_{\Sigma_{k_1} \times \cdots \times \Sigma_{k_n}} G(1)^{k_1 + \cdots + k_n}.$$

Here observe that $\mathrm{Aut}(\Theta_k) = \Sigma_k$. Since $\mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n)$ is a numerable principal $(\Sigma_{k_1} \times \cdots \times \Sigma_{k_n})$–space, $\mathrm{id} \times \pi_n$ defines a homotopy equivalence $F_0(Q) \to F_0(M)$ by 4.8(2).

Now assume that we have shown that $\mathrm{id} \times_{\mathrm{Aut}} \pi_n$ induces a homotopy equivalence $F_{r-1}(Q) \to F_{r-1}(M)$. We obtain $F_r(Q)$ from $F_{r-1}(Q)$ and $F_r(M)$ from $F_{r-1}(M)$ by attaching spaces $(\mathcal{LO} \times_{\mathrm{Aut}} G_Q)(T_1, \dots, T_n)$ and $(\mathcal{LO} \times_{\mathrm{Aut}} G_M)(T_1, \dots, T_n)$, respectively, where $(T_1, \dots, T_n)$ have exactly a total of r internal edges. In both cases, an element in the attached space represents an element of lower filtration if and only if an internal edge in $(T_1, \dots, T_n)$ is of length 0 or 1, because in these cases the shrinking and chopping relations apply, respectively. Let $D(T_1,\dots,T_n)) \subset \mathcal{L}\mathcal{O}(T_1,\dots,T_n))$ be the subspace of such decorated trees. The inclusion of this subspace is an $\mathrm{Aut}(T_1,\dots,T_n)$–equivariant cofibration (see the product pushout theorem [5, page 233]). Consider the diagram

$$\begin{tikzcd}
F_{r-1}(Q) \arrow[d] & (D \times_{\mathrm{Aut}} G_Q)(T_1,\dots,T_n) \arrow[l, "\mathrm{proj}"'] \arrow[r, "i_Q"] \arrow[d] & (\mathcal{L}\mathcal{O} \times_{\mathrm{Aut}} G_Q)(T_1,\dots,T_n) \arrow[d] \\
F_{r-1}(M) & (D \times_{\mathrm{Aut}} G_M)(T_1,\dots,T_n) \arrow[l, "\mathrm{proj}"'] \arrow[r, "i_M"] & (\mathcal{L}\mathcal{O} \times_{\mathrm{Aut}} G_M)(T_1,\dots,T_n)
\end{tikzcd}$$

where the vertical maps are induced by $\mathrm{id} \times_{\mathrm{Aut}} \pi_n$. The maps i_Q and i_M are closed cofibrations (see [5, page 232]). We will prove in Lemma 4.9 that $\underline{\mathcal{O}}(T)$ is a numerable principal $\mathrm{Aut}(T)$–space. (Recall that $\underline{\mathcal{O}}\colon \mathbb{T}^{\mathrm{op}} \to \mathcal{T}op$ was defined in Section 3 for the collection $\mathcal{K} = \mathcal{O}$.) Hence $\underline{\mathcal{O}}(T_1) \times \cdots \times \underline{\mathcal{O}}(T_n)$ is a numerable principal $\mathrm{Aut}(T_1,\dots,T_n)$–space (see 4.8(4)). Since there are equivariant maps

$$D(T_1,\dots,T_n) \to \mathcal{L}\mathcal{O}(T_1,\dots,T_n) \xrightarrow{\text{forget}} \underline{\mathcal{O}}(T_1) \times \cdots \times \underline{\mathcal{O}}(T_n),$$

both $D(T_1,\dots,T_n)$ and $\mathcal{L}\mathcal{O}(T_1,\dots,T_n)$ are numerable principal $\mathrm{Aut}(T_1,\dots,T_n)$–spaces (see 4.8(1)). Hence the vertical maps of the diagram are homotopy equivalences (see 4.8(2)), and the gluing lemma implies that $F_r(Q) \to F_r(M)$ is a homotopy equivalence. □

4.8 Facts about numerable principal Γ–spaces The following results are either fairly obvious or can be found in the appendix of [5]. Let Γ be a discrete group and X a numerable principal right Γ–space.

(1) If $f\colon Y \to X$ is a Γ–equivariant map, then Y is a numerable principal Γ–space. Moreover, f is an equivariant homotopy equivalence if and only if it is an ordinary homotopy equivalence of underlying spaces.

(2) If $f\colon Y \to Z$ is an equivariant map of left Γ–spaces, which is an ordinary homotopy equivalence of underlying spaces, then $\mathrm{id} \times_\Gamma f\colon X \times_\Gamma Y \to X \times_\Gamma Z$ is a homotopy equivalence.

(3) If H is a subgroup of Γ, then X is a numerable principal H–space.

(4) If Y is a numerable principal right Γ'–space, then $X \times Y$ is a numerable principal right $\Gamma \times \Gamma'$–space.

4.9 Lemma *If $\mathcal{O}$ is a Σ–free topological operad, then $\underline{\mathcal{O}}(T)$ is a numerable principal $\mathrm{Aut}(T)$–space.*

Proof We prove this by an inductive argument. If $T = \Theta_n$, then $\mathrm{Aut}(T) = \Sigma_n$ and $\underline{\mathcal{O}}(T) = \mathcal{O}(n)$. By assumption, $\mathcal{O}(n)$ is a numerable principal Σ_n–space.

Now let $T = \Theta_n \circ (T_1 \oplus \cdots \oplus T_n)$. The following calculation of $\mathrm{Aut}(T)$ is taken from [4, page 815]. By choosing T appropriately in $[T]$ we may assume that it has the form

$$T = \Theta_n \circ (T_1^1 \oplus \cdots \oplus T_{k_1}^1 \oplus \cdots \oplus T_1^l \oplus \cdots \oplus T_{k_l}^l),$$

where $T_1^i, \ldots, T_{k_i}^I$ are copies of a planar tree T^i, and T^i and T^j are not isomorphic in $\mathbb{T}$ for $i \neq j$. Then $\mathrm{Aut}(T)$ is the semi-direct product

$$\mathrm{Aut}(T) \cong (\mathrm{Aut}(T^1)^{k_1} \times \cdots \times \mathrm{Aut}(T^l)^{k_l}) \rtimes (\Sigma_{k_1} \times \cdots \times \Sigma_{k_l}) = \Gamma_T \rtimes \Sigma_T,$$

where Σ_{k_i} acts on $\mathrm{Aut}(T^i)^{k_i}$ by permuting the factors.

$\mathcal{O}(n)$ is a numerable principal Σ_T–space because Σ_T is a subgroup of Σ_n. By induction, $\underline{\mathcal{O}}(T^1)^{k_1} \times \cdots \times \underline{\mathcal{O}}(T^l)^{k_l}$ is a numerable principal Γ_T–space. Denote Θ_n by T^0. By [5, Appendix, Lemma 3.2] there are open covers $\mathcal{U}^i = \{U_\alpha^i;\ \alpha \in A^i\}$ of $\underline{\mathcal{O}}(T^i)$, $i = 0, \ldots, l$, with subordinate partitions of unity $\{f_\alpha^i\colon \underline{\mathcal{O}}(T^i) \to [0,1],\ \alpha \in A^i\}$ such that $U_\alpha^i \cdot h \cap U_\alpha^i = \varnothing$ for all $h \in H_i$ different from the unit, where $H_0 = \Sigma_T$ and $H_i = \mathrm{Aut}(T^i)$ for $i = 1, \ldots, l$. The open cover

$$\mathcal{V} = \{U_1^0 \times U_1^1 \times \cdots \times U_{k_1}^1 \times \cdots \times U_1^l \times \cdots \times U_{k_l}^l\}$$

of $\underline{\mathcal{O}}(T)$, where U_j^i runs through the elements of $\mathcal{U}^i$, satisfies the condition that

$$(U_1^0 \times U_1^1 \times \cdots \times U_{k_l}^l) \cdot h \cap (U_1^0 \times U_1^1 \times \cdots \times U_{k_l}^l) = \varnothing$$

for all $h \in \Gamma_T \rtimes \Sigma_T$ different from the unit. The product numeration obtained from the f_α^i provides a partition of unity subordinate to $\mathcal{V}$. Now the lemma follows from [5, Appendix 3.2]. □

If X is an $\mathcal{O}$–algebra and $G = \widehat{X}$ then, by inspection, $Q_n(G) \cong Q_1(G)^n$, and the $\widehat{\mathcal{O}}$–structure on $Q(G)$ defines an $\mathcal{O}$–algebra structure on $Q_1(G)$. The map $\tau_1\colon Q_1(G) \to M(G)$ is a homeomorphism of $\mathcal{O}$–algebras, and $\varepsilon_1\colon Q_1(G) \to G(1) = X$ is a weak equivalence of $\mathcal{O}$–algebras. Composing ε_1 with the inverse of τ_1, we obtain:

4.10 Proposition *If X is an $\mathcal{O}$–algebra in $\mathcal{T}op$ then there is a natural weak equivalence of $\mathcal{O}$–algebras*

$$\overline{\varepsilon}\colon M(\widehat{X}) \to X.$$

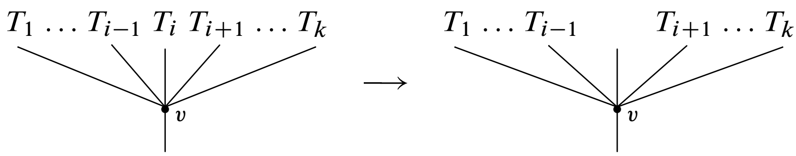
5 Tree-indexed diagrams

In order to adapt the rectification construction described in Section 4 to the case of categories, we will recast the topological version described there into a homotopy colimit construction of a certain diagram. The same diagram makes sense in $\mathcal{C}at$, where we will apply the Grothendieck construction, which is the analog of the homotopy colimit in $\mathcal{C}at$.

5.1 The indexing category $\mathcal{T}$ As in the previous section, our construction involves trees with a root vertex. So the objects of $\mathcal{T}$ are the isomorphism classes $[T]$ of planar trees in $\widetilde{\mathbb{T}}$. The shrinking and chopping relations of Definition 4.5 correspond to morphisms in the diagram to be constructed. So the generating morphisms of $\mathcal{T}$ are of two types:

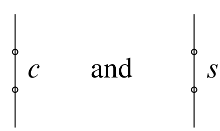
(1) Shrinking an internal edge.

(2) Chopping off a subtree above any node v of a tree:



That is, the subtree T_i in the original tree is replaced by a single input edge in the new tree.

To define a general morphism in $\mathcal{T}$ we introduce the notion of a *marked tree*. A marked tree is a planar tree S with a marking of some (possibly none) of its internal edges with either the symbol s or the symbol c, subject to the constraint that an edge which is anywhere above an edge marked c is left unmarked. A morphism in $\mathcal{T}$ is an isomorphism class of marked trees with respect to non-planar isomorphisms respecting the marking. The source of such a morphism f is the isomorphism class of the underlying unmarked planar tree. Let S be a marked tree representing f and let T be the unmarked tree obtained from S by first chopping off the branches above every edge marked c. (Note that this map would discard any markings of edges above such an edge, which accounts for the constraint.) Then one shrinks all edges marked s. The isomorphism class of T is the target of f. By construction, $[T]$ is independent of the choice of the representative S of f. In most cases, there is at most one morphism between objects of $\mathcal{T}$. However, there are exceptions. For instance,



represent distinct morphisms with the same source and target.

The following remark will make the definition of the composition easy.

5.2 Remark In the sequel we will need to prescribe a consistent way of representing simplices in the nerve of $\mathcal{T}$ by a chain of marked planar trees. Given an n–simplex

$$[T_0] \to [T_1] \to \cdots \to [T_n]$$

in the nerve of $\mathcal{T}$, pick a planar representative $T_0 \in [T_0]$. Any morphism $[T_0] \to [T_1]$ is represented by a marking of T_0. By applying the edge shrinking and chopping specified by the marking of T_0, we obtain a well-defined planar representative T_1 of $[T_1]$. Now apply the same procedure to the map $[T_1] \to [T_2]$ and carry on to obtain a sequence

$$T_0 \to T_1 \to \cdots \to T_n \quad \text{representing} \quad [T_0] \to [T_1] \to \cdots \to [T_n],$$

where the maps $T_i \to T_{i+1}$ are given by a marking of T_i. If we had picked a different representative $T_0' \in [T_0]$, then there is an isomorphism $\phi\colon T_0 \to T_0'$ in $\widetilde{\mathbb{T}}$, which transports the marking of T_0 to a marking of T_0', and the marked tree T_0' also represents the morphism $[T_0] \to [T_1]$. Clearly ϕ can be extended to the whole sequence of representatives in a unique way.

To define the composition of $f\colon [S] \to [T]$ and $g\colon [T] \to [U]$, we take a representing chain $S \to T \to U$ with a marking of S and a marking of T. Let $E(S)$ and $E(T)$ be the sets of internal edges of S and T, respectively. Since T is obtained from S by shrinking and chopping off internal edges, we may consider $E(T)$ as a subset of $E(S)$. Observe that the marked edges of S do not lie in $E(T)$. So the marking of T defines a marking on edges of S which have not been marked before. We now erase in this larger marking any mark above an edge marked with c to satisfy our constraint. The resulting marking of S represents the composition $g \circ f$.

5.3 The diagram Let $\mathcal{O}$ be an arbitrary operad in $\mathcal{S}$ and let $G\colon \widehat{\mathcal{O}} \to \mathcal{S}$ be an $\widehat{\mathcal{O}}$–diagram in $\mathcal{S}$, where $\mathcal{S}$ is $\mathcal{C}at$, $\mathcal{T}op$, $\mathcal{S}ets$ or $\mathcal{SS}ets$. We are going to define a diagram

$$F^G\colon \mathcal{T} \to \mathcal{S}.$$

We recall the functor $\underline{\mathcal{O}}\colon \mathbb{T}^{\mathrm{op}} \to \mathcal{S}$ from Section 3. The definition of the functor $\lambda_G\colon \widetilde{\mathbb{T}} \to \mathcal{T}op$ in 4.1 also makes sense if we replace $\mathcal{T}op$ by $\mathcal{S}$. We define

$$F^G([T]) = \underline{\mathcal{O}} \otimes_{[T]} \lambda_G,$$

the coend obtained by restricting of the functors to the isomorphism class $[T] \subset \widetilde{\mathbb{T}}$.

As in 4.2 we have the following explicit description of an element in $F^G([T])$. The object

$$W(G,T) = \underline{\mathcal{O}}(T) \times G(\mathrm{In}(T))$$

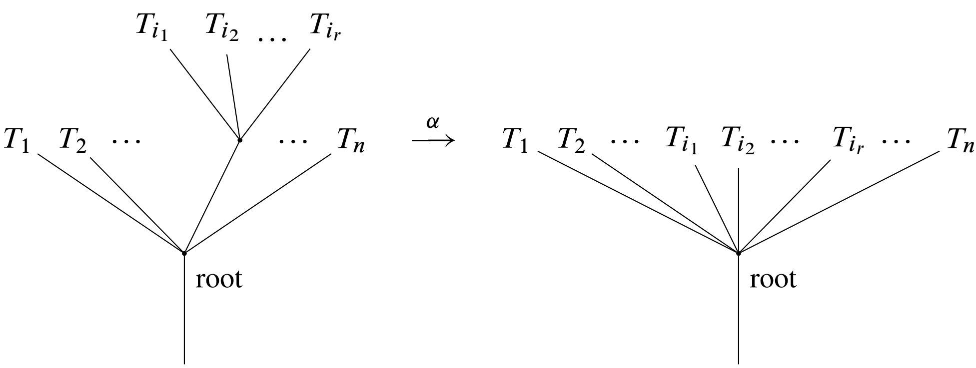
replaces $V(G,T)$. If T has the form (4.3) we define

$$F^G([T]) = \Big(\coprod_{T\in[T]} \mathcal{O}(n) \times \prod_{i=1}^{n} W(G,T_i)\Big)/\!\sim\; = \Big(\coprod_{T\in[T]} \underline{\mathcal{O}}(T) \times \prod_{i=1}^{n} G(\mathrm{In}(T_i))\Big)/\!\sim,$$

where the relation is the equivariance relation 4.4 with the factor I^n dropped.

Next we describe F^G on the generating morphisms of $\mathcal{T}$.

(1) Suppose $\alpha\colon [T] \to [T']$ is shrinking a bottom edge of $[T]$. So α is represented by T with a single marking s of the edge connecting the root node to a subtree T_i of T:



The corresponding morphism $F^G(\alpha)$ is induced by the map

$$\underline{\mathcal{O}}(T) \times \prod_{j=1}^{n} G(\mathrm{In}(T_j)) \to \underline{\mathcal{O}}(T') \times \prod_{j=1}^{i-1} G(\mathrm{In}(T_j)) \times \prod_{k=1}^{r} G(\mathrm{In}(T_{i_k})) \times \prod_{j=i+1}^{n} G(\mathrm{In}(T_j))$$

which sends a decorated tree $T \in \underline{\mathcal{O}}(T)$ to the decorated tree T' obtained from T as in the shrinking relation of Definition 4.5(1b), disregarding lengths. On the other factors the map is given by identities and the map τ^* of Definition 4.5(1b).

(2) Shrinking a nonbottom edge corresponds under F^G to the map $(T; g_1, \dots, g_n) \mapsto (T'; g_1, \dots, g_n)$, where T' is obtained from T as in (1).

(3) Let $\tau\colon [T] \to [T']$ be a chopping morphism, represented by a tree T of the form (4.3) with exactly one marked edge e with marking c. This edge belongs to some

subtree T_i of T; it could be its root. Then $F^G(\tau)$ is induced by the map

$$H\colon \underline{\mathcal{O}}(T)\times\prod_{j=1}^{n} G(\mathrm{In}(T_j)) \to \underline{\mathcal{O}}(T')\times\prod_{j=1}^{i-1} G(\mathrm{In}(T_j))\times G(\mathrm{In}(T_i'))\times\prod_{j=i+1}^{n} G(\mathrm{In}(T_j)),$$

where T' is obtained from T and T_i' from T_i by deleting the subtrees with root edge e (if e is the root of T_i then T_i' is the trivial tree). The map H is given on $\underline{\mathcal{O}}(T)\to\underline{\mathcal{O}}(T')$ by the projection (the set of decorated nodes in T' is a subset of the set of decorated nodes in T), and on the other factors by the identities and the map $G(\hat{c})$ of Definition 4.5(2).

In each case the equivariance relations on the operad $\mathcal{O}$ and the functoriality of G imply that the definition of F^G on the morphisms of $\mathcal{T}$ does not depend on the choice of representatives and that F^G is a well-defined functor.

5.4 A relative version There is a relative version of this construction with respect to a map of operads $\varphi\colon \mathcal{O}\to\mathcal{P}$ in $\mathcal{S}$. Again, let $G\colon \widehat{\mathcal{O}}\to\mathcal{S}$ be an $\widehat{\mathcal{O}}$–diagram in $\mathcal{S}$. We then define the functor $F^G_\varphi\colon \mathcal{T}\to\mathcal{S}$ in exactly the same way as we defined F^G, except that for $F^G_\varphi[T]$ the bottom node of a representing decorated T is decorated with an element of $\mathcal{P}(k)$ instead of $\mathcal{O}(k)$. Thus

$$F^G_\varphi([T]) = \Big(\coprod_{T\in[T]}\mathcal{P}(k)\times\prod_{i=1}^{k} W(G,T_i)\Big)/\sim,$$

with the equivariance relation as above. On morphisms F^G_φ is defined in the same way as F^G, except that when we shrink a bottom edge we apply φ to the element of $\mathcal{O}$ decorating the node at the top of the edge before we compose it with the element of $\mathcal{P}$ decorating the bottom node.

6 Homotopy colimits

For a diagram $D\colon \mathcal{C}\to\mathcal{C}at$ in $\mathcal{C}at$ the Grothendieck construction $\mathcal{C}\int D$ is the category whose objects are pairs (c,X) with $c\in\mathrm{ob}\,\mathcal{C}$ and $X\in\mathrm{ob}\,D(c)$. A morphism $(c,X)\to(c',X')$ is a pair (j,f) consisting of a morphism $j\colon c\to c'$ in $\mathcal{C}$ and a morphism $f\colon D(j)(X)\to X'$ in $D(c')$. Composition is the obvious one.

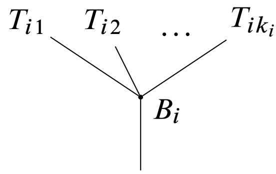
6.1 Proposition *If $\mathcal{O}$ is an operad in $\mathcal{C}at$ and $G\colon \widehat{\mathcal{O}}\to\mathcal{C}at$ is an $\widehat{\mathcal{O}}$–category then $\mathcal{T}\int F^G$ is an $\mathcal{O}$–algebra.*

Proof We define

$$(*) \qquad \mathcal{O}(m) \times_{\Sigma_m} \big(\mathcal{T} \textstyle\int F^G\big)^m \to \mathcal{T} \textstyle\int F^G$$

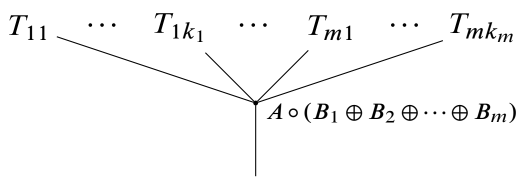
as follows. A planar representative of an object on the left side of (∗) looks like

$(A, \{(T_i)\}_{i=1}^m, \overline{C}_i)$, with $T_i =$



where A is an object in $\mathcal{O}(m)$, T_i is a planar tree whose nodes are decorated by objects in the appropriate $\mathcal{O}(k)$, and $\overline{C}_i$ is an object in $G(\mathrm{In}(T_{i1})) \times \cdots \times G(\mathrm{In}(T_{ik_i}))$. The underlying tree of T_i represents an object $[T_i]$ in $\mathcal{T}$ and the pair $X_i = (T_i, \overline{C}_i)$ an object in $F^G([T_i])$. We send this object to the object represented by the pair $(T, (\overline{C}_1, \overline{C}_2, \dots, \overline{C}_m))$, where

$T =$



For later use we denote this representative by $\overline{\mu}(A; X_1, \dots, X_m)$. This map extends to morphisms: the $\mathcal{O}(m)$ factor only affects A, while morphisms in $\mathcal{T} \int F^G$ may result in operad compositions from the right of the B_i with other node decorations of T_i, chopping or shrinking of internal edges and their effects on the $\overline{C}_i$. □

6.2 Definition For a diagram $F\colon \mathcal{I} \to \mathcal{T}op$ of topological spaces we define $\mathrm{hocolim}_{\mathcal{I}} F$ to be the 2–sided bar construction

$$\mathrm{hocolim}_{\mathcal{I}} F = B(*, \mathcal{I}, F),$$

where $*\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{T}op$ is the constant diagram on a point (see [11, Proposition 3.1] for a list of properties of the 2–sided bar construction). More explicitly, $B(*, \mathcal{I}, F)$ is the topological realization of the simplicial space

$$[n] \mapsto B_n(*, \mathcal{I}, F) = \coprod_{A,B} \mathcal{I}_n(A, B) \times F(A),$$

where $\mathcal{I}_n(A, B) \subset (\mathrm{mor}\, \mathcal{I})^n$ is the subset of composable morphisms

$$A \xrightarrow{f_1} \cdots \xrightarrow{f_n} B.$$

The degeneracy maps are defined as in the nerve of $\mathcal{I}$, the boundary maps

$$d^i\colon B_n(*,\mathcal{I},F) \to B_{n-1}(*,\mathcal{I},F)$$

are defined as in the nerve for $i > 0$, while $d^0(f_1,\ldots,f_n;x)) = (f_2,\ldots,f_n;F(f_1)(x))$.

6.3 Proposition *If $\mathcal{O}$ is an operad in $\mathcal{T}op$ and $G\colon \widehat{\mathcal{O}} \to \mathcal{T}op$ is an $\widehat{\mathcal{O}}$–space then* $\operatorname{hocolim}_{\mathcal{T}} F^G$ *is an $\mathcal{O}$–algebra.*

Proof Since the classifying space functor preserves products it suffices to show that $B_*(*,\mathcal{T},F^G)$ is a simplicial object in the category of $\mathcal{O}$–algebras. By Remark 5.2, an m–tuple of elements in $B_p(*,\mathcal{T},F^G)$ can be represented by sequences of marked trees

$$\big\{\big(T_{j0} \xrightarrow{t_{j1}} \cdots \xrightarrow{t_{jp}} T_{jp}; X_j\big)\big\}_{1\le j\le m},$$

where t_{jk} is T_{jk} with a marking. An operation $a \in \mathcal{O}(m)$ maps this m–tuple of elements to the element represented by

$$\big(\mu(T_{10},\ldots,T_{n0}) \xrightarrow{\mu(t_{11},\ldots,t_{n1})} \cdots \xrightarrow{\mu(t_{1p},\ldots,t_{np})} \mu(T_{1p},\ldots,T_{np}); \bar{\mu}(a;X_1,\ldots,X_m)\big),$$

where $\mu(T_1,\ldots,T_n)$ is the tree obtained from $T_1,\ldots,T_n$ by gluing their roots together and $\mu(t_1,\ldots,t_n)$ the corresponding marked tree, while $\bar{\mu}(a;X_1,\ldots,X_n)$ is defined as in the proof of Proposition 6.1. □

If $\mathcal{O}$ is an operad in $\mathcal{C}at$, then $B\mathcal{O}$ is an operad in $\mathcal{T}op$ by Lemma 2.7.

6.4 Proposition *If $\mathcal{O}$ is an operad in $\mathcal{C}at$ and $G\colon \widehat{\mathcal{O}} \to \mathcal{C}at$ is an $\widehat{\mathcal{O}}$–category then* $\operatorname{hocolim}_{\mathcal{T}} B(F^G)$ *is a $B\mathcal{O}$–space.*

Proof By definition, $\operatorname{hocolim}_{\mathcal{T}} B(F^G)$ is the topological realization of the bisimplicial set

$$([p],[q]) \mapsto N_p(\mathcal{T}) \times N_q(F^G([T_0])),$$

where N is the nerve functor. An element in $N_p(\mathcal{T}) \times N_q(F^G([T_0]))$ is a pair

$$\big([T_0] \xrightarrow{[t_1]} \cdots \xrightarrow{[t_p]} [T_p], X_0 \xrightarrow{x_1} \cdots \xrightarrow{x_q} X_q\big),$$

where $[T_0] \xrightarrow{t_1} \cdots \xrightarrow{t_p} [T_p]$ is a sequence of morphisms in $\mathcal{T}$ and $X_0 \xrightarrow{x_1} \cdots \xrightarrow{x_q} X_q$ is a sequence of morphisms in the category $F^G([T_0])$. We define an operation of $N_*\mathcal{O}$ on its diagonal: the element

$$\big(A_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_p} A_p\big) \in N_p\mathcal{O}(n)$$

maps the n–tuple represented by

$$\left\{\begin{pmatrix} T_{j0} \xrightarrow{t_{j1}} \cdots \xrightarrow{t_{jp}} T_{jp} \\ X_{j0} \xrightarrow{x_{j1}} \cdots \xrightarrow{x_{jp}} X_{jp} \end{pmatrix}\right\}_{1\le j\le n}$$

to the object represented by the pair of sequences:

$$\begin{pmatrix} \mu(T_{10},\dots,T_{n0}) \xrightarrow{\mu(t_{11},\dots,t_{n1})} \cdots \xrightarrow{\mu(t_{1p},\dots,t_{np})} \mu(T_{1p},\dots,T_{np}) \\ \bar\mu(A_0;X_{10},\dots,X_{n0}) \xrightarrow{\bar\mu(\alpha_1;x_{11},\dots,x_{n1})} \cdots \xrightarrow{\bar\mu(\alpha_p;x_{1p},\dots,x_{np})} (A_p;X_{1p},\dots,X_{np}) \end{pmatrix} \qquad \square$$

In degree p, the nerve $N_*(\mathcal{T}\int F^G)$ consists of diagrams

$$([T_0],X_0) \xrightarrow{([t_1],x_1)} \cdots \xrightarrow{([t_p],x_p)} ([T_p],X_p)$$

with

$$X_i \in F^G([T_i]), \quad [t_i]\colon [T_{i-1}] \to [T_i] \text{ in } \mathcal{T}, \quad \text{and} \quad x_i\colon [t_i](X_{i-1}) \to X_i \text{ in } F^G([T_i]).$$

We always tacitly assume that the representing trees T_i and the marked trees t_i are chosen as in Remark 5.2. The $\mathcal{O}$–structure on $\mathcal{T}\int F^G$ defined in the proof of Proposition 6.1 translates to an $N_*\mathcal{O}$–structure on $N_*(\mathcal{T}\int F^G)$ as follows: if

$$A_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_p} A_p$$

is an element in $N_p\mathcal{O}(n)$, it maps an n–tuple

$$\{([T_{j0}],X_{j0}) \xrightarrow{([t_{j1}],x_{j1})} \cdots \xrightarrow{([t_{jp}],x_{jp})} ([T_{jp}],X_{jp})\}_{1\le j\le n}$$

to

$$\begin{aligned}&\big([\mu(T_{10},\dots,T_{n0})],\bar\mu(A_0;X_{10},\dots,X_{n0})\big)\\ &\qquad\qquad \to \cdots \to \big([\mu(T_{1p},\dots,T_{np})],\bar\mu(A_p;X_{1p},\dots,X_{np})\big)\end{aligned}$$

in the notation above with the obvious maps.

Thomason [18] constructed a natural weak equivalence

$$\eta\colon \operatorname{hocolim}_{\mathcal{T}} B(F^G) \to B(\mathcal{T}\textstyle\int F^G)$$

defined on nerves by mapping

$$\begin{pmatrix} [T_0] \xrightarrow{[t_1]} \cdots \xrightarrow{[t_p]} [T_p] \\ X_0 \xrightarrow{x_1} \cdots \xrightarrow{x_p} X_p \end{pmatrix}$$

to

$$([T_0], X_0) \xrightarrow{([t_1],[t_1](x_1))} \cdots \xrightarrow{([t_p],[t_p]\circ\cdots\circ[t_1](x_p))} ([T_p], [t_p]\circ\cdots\circ[t_1]X_p).$$

6.5 Proposition *The map η: $\operatorname{hocolim}_{\mathcal{T}} B(F^G) \to B(\mathcal{T}\int F^G)$ is a weak equivalence of $B\mathcal{O}$–spaces natural in G.*

Proof We prove this on the level of nerves. So let

$$\overline{(T,X)} = \left\{ \begin{pmatrix} [T_{j0}] \xrightarrow{[t_{j1}]} \cdots \xrightarrow{[t_{jp}]} [T_{jp}] \\ X_{j0} \xrightarrow{x_{j1}} \cdots \xrightarrow{x_{jp}} X_{jp} \end{pmatrix} \right\}_{1\le j\le n}$$

be an element in $\prod_{j=1}^n N_p(\mathcal{T}) \times N_p(F^G([T_{j0}]))$ and

$$\bar{A} = \big(A_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_p} A_p\big) \in N_p\mathcal{O}(n).$$

We have to show that $\bar{A} * \eta^n(\overline{(T,X)}) = \eta(\bar{A} * \overline{(T,X)})$, where $\bar{A} * -$ stands for the operation of $\bar{A}$.

To avoid a multitude of indices we do this for $n=2$ and $p=1$; the general case is analogous. Then

$$\eta^2(\overline{(T,X)}) = \big\{([T_{i0}], X_{j0}) \xrightarrow{([t_{j1}],[t_{j1}](x_{j1}))} ([T_{j1}], [t_{j1}](X_{j1}))\big\}_{j=1,2}$$

and

$$\begin{aligned}\bar{A} * \eta^2(\overline{(T,X)}) = \big([\mu(T_{10}, T_{20})], \bar{\mu}(A_0; X_{10}, X_{20})\big)& \\ \to \big([\mu(T_{11}, T_{21})], \bar{\mu}(A_1; [t_{11}](X_{11}), [t_{21}](X_{21}))\big).&\end{aligned}$$

Now

$$\bar{A} * \overline{(T,X)} = \left\{ \begin{pmatrix} [\mu(T_{10}, T_{20})] \xrightarrow{[\mu(t_{11},t_{21})]} [\mu(T_{11}, T_{21})] \\ \bar{\mu}(A_0; X_{10}, X_{20}) \xrightarrow{\bar{\mu}(\alpha_1; x_{11}, x_{21})} \bar{\mu}(A_1; X_{11}, X_{21}) \end{pmatrix} \right\},$$

which is mapped by η to

$$\big([\mu(T_{10}, T_{20})], \bar{\mu}(A_0; X_{10}, X_{20})\big) \to \big([\mu(T_{11}, T_{21})], [\mu(t_{11}, t_{21})](\bar{\mu}(A_1; X_{11}, X_{21}))\big).$$

So we have to show that

$$\begin{aligned}[\mu(t_{11}, t_{21})](\bar{\mu}(A_1; X_{11}, X_{21})) &= \bar{\mu}\big(A_1; [t_{11}](X_{11}), [t_{21}](X_{21})\big), \\ [\mu(t_{11}, t_{21})](\bar{\mu}(\alpha_1; x_{11}, x_{21})) &= \bar{\mu}\big(\alpha_1; [t_{11}](x_{11}), [t_{21}](x_{21})\big).\end{aligned}$$

These equations hold because the operation of $\bar{A}$ is defined by composition from the left with the sum of the appropriate root labels, while the t_{ij} shrink edges of trees or are evaluations which could at most result in compositions with root labels from the right. □

6.6 Proposition *If $\mathcal{O}$ is Σ–free, there is a natural homeomorphism $B(F^G)([T]) \cong F^{BG}([T])$ and hence a homeomorphism*

$$\mathrm{hocolim}_{\mathcal{T}}\, B(F^G) \cong \mathrm{hocolim}_{\mathcal{T}}\, F^{BG}$$

of $B\mathcal{O}$–spaces natural in G. In particular, Thomason's map induces a weak equivalence of $B\mathcal{O}$–spaces $\mathrm{hocolim}_{\mathcal{T}} F^{BG} \to B(\mathcal{T}\int F^G)$.

Proof We have

$$F^G([T]) \cong \underline{\mathcal{O}}(T) \times_{\mathrm{Aut}(T)} \lambda_G(T) \quad \text{and} \quad F^{BG}([T]) \cong \underline{B\mathcal{O}}(T) \times_{\mathrm{Aut}(T)} \lambda_{BG(T)}$$

by 3.4. Since B is product-preserving, there is a natural homeomorphism $\lambda_{BG} \to B\lambda_G$. Since $\mathrm{Aut}(T)$ acts freely on $\underline{\mathcal{O}}(T)$ it acts freely on $\underline{\mathcal{O}}(T) \times \lambda_G(T)$. Hence there is a natural homeomorphism $B(\underline{\mathcal{O}}(T) \times_{\mathrm{Aut}(T)} \lambda_G(T)) \to B\underline{\mathcal{O}}(T) \times_{\mathrm{Aut}(T)} B\lambda_G(T)$ by Lemma 2.8. Here we also use that $B(\underline{\mathcal{O}}(T) \times \lambda_G(T)) \cong B(\underline{\mathcal{O}}(T)) \times B(\lambda_G(T))$. □

A map $\tau\colon G_1 \to G_2$ of $\widehat{\mathcal{O}}$–categories induces a map $F^\tau\colon F^{G_1} \to F^{G_2}$ of $\mathcal{T}$–diagrams in $\mathcal{C}at$ and hence a map

$$\mathcal{T}\textstyle\int F^\tau\colon \mathcal{T}\int F^{G_1} \to \mathcal{T}\int F^{G_2}$$

of $\mathcal{O}$–algebras.

6.7 Proposition *If $\mathcal{O}$ is a Σ–free operad in $\mathcal{S}$ and $\tau\colon G_1 \to G_2$ is a map of $\widehat{\mathcal{O}}$–diagrams in $\mathcal{S}$ which is objectwise a weak equivalence, then:*

(1) *If $\mathcal{S} = \mathcal{C}at$, the functor $\mathcal{T}\int F^\tau\colon \mathcal{T}\int F^{G_1} \to \mathcal{T}\int F^{G_2}$ is a weak equivalence of $\mathcal{O}$–algebras.*

(2) *If $\mathcal{S} = \mathcal{T}op$, the map* $\mathrm{hocolim}_{\mathcal{T}} F^{G_1} \to \mathrm{hocolim}_{\mathcal{T}} F^{G_2}$ *is a weak equivalence of $\mathcal{O}$–spaces.*

Proof By Proposition 6.6 it suffices to prove (2) because weak equivalences in $\mathcal{C}at$ are detected by the classifying space functor B.

We have a commutative diagram

$$\begin{tikzcd}
F^{G_1}([T]) \arrow[r, "F^\tau([T])"] \arrow[d, "\cong"] & F^{G_2}([T]) \arrow[d, "\cong"] \\
\underline{\mathcal{O}}(T) \times_{\mathrm{Aut}(T)} \lambda_{G_1}(T) \arrow[r, "\mathrm{id} \times_{\mathrm{Aut}} \lambda_\tau(T)"] & \underline{\mathcal{O}}(T) \times_{\mathrm{Aut}(T)} \lambda_{G_2}(T)
\end{tikzcd}$$

Since $\underline{\mathcal{O}}(T)$ is a numerable principal $\mathrm{Aut}(T)$ space by Lemma 4.9 and $\lambda_\tau(T)$ is a homotopy equivalence, the map $\mathrm{id}\times_{\mathrm{Aut}}\lambda_\tau(T)$ is a homotopy equivalence by 4.8(2). Hence $F^\tau\colon F^{G_1}\to F^{G_2}$ is objectwise a weak equivalence, inducing a homotopy equivalence $\operatorname{hocolim} F^{G_1}\to\operatorname{hocolim} F^{G_2}$. □

7 Change of operads

Let X be an $\mathcal{O}$–algebra in $\mathcal{C}at$ and $\widehat{X}\colon\widehat{\mathcal{O}}\to\mathcal{C}at$ its associated $\widehat{\mathcal{O}}$–diagram. Then we have a map

$$\varepsilon\colon \mathcal{T}\textstyle\int F^{\widehat{X}}\to X$$

induced by

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "p"] \arrow[dr, "F^{\widehat{X}}"'] & \arrow[d, phantom, "\overset{e}{\Rightarrow}"] & * \arrow[dl, "X"] \\ & \mathcal{C}at & \end{tikzcd}$$

where $e([T])$ is the composite

$$e([T])\colon F^{\widehat{X}}([T])\to\mathcal{O}(\mathrm{In}(T))\times_{\Sigma_{\mathrm{In}(T)}}X^{\mathrm{In}(T)}\to X.$$

If T is of the form (4.3) then the first map shrinks all edges of all decorated trees T_i. The second map is the $\mathcal{O}$–action on X.

By construction, ε is a homomorphism of $\mathcal{O}$–algebras.

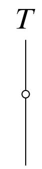
7.1 Proposition *The homomorphism $\varepsilon\colon\mathcal{T}\int F^{\widehat{X}}\to X$ is a weak equivalence of $\mathcal{O}$–algebras.*

Proof Note that $F^{\widehat{X}}([T])=W(\widehat{X},[T])$ modulo the equivariance relation, because $\widehat{X}(n)=X^n$. There is a section $s\colon X\to\mathcal{T}\int F^{\widehat{X}}$ of ε, which is not a map of algebras. It is induced by

$$\begin{tikzcd} * \arrow[rr, "i"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{\iota}{\Rightarrow}"] & \mathcal{T} \arrow[dl, "F^{\widehat{X}}"] \\ & \mathcal{C}at & \end{tikzcd}$$

where i takes $*$ to the tree Θ_1 which in turn is mapped to $\mathcal{O}(1)\times X$ by $F^{\widehat{X}}$, and $\iota(*)\colon X\to F^{\widehat{X}}\circ i$ is the inclusion $X=\{\mathrm{id}\}\times X\subset\mathcal{O}(1)\times X$.

Let $j\colon \mathcal{T} \to \mathcal{T}$ be the functor which maps $[T]$ to the isomorphism class represented by:



Now let

$$J\colon \mathcal{T}\textstyle\int F^{\widehat{X}} \to \mathcal{T}\textstyle\int F^{\widehat{X}}$$

be the functor sending an object $([T], X)$ with $[T] \in \mathcal{T}$ and $X \in F^{\widehat{X}}([T])$ to the pair $(j([T]), j(X))$, where $j(X)$ has the decoration of X on the T–part and id as decoration of the root of $j([T])$. This definition extends canonically to morphisms with the root of $j([T])$ always decorated by the identity.

Shrinking and chopping the incoming edge of the root of $j([T])$ define natural transformations $J \Rightarrow \mathrm{Id}$ and $J \Rightarrow s \circ \varepsilon$, respectively. The classifying space functor turns these transformations into homotopies $\mathrm{Id} \simeq BJ \simeq Bs \circ B\varepsilon$. □

7.2 Corollary *Let $\varphi\colon \mathcal{O} \to \mathcal{P}$ be a weak equivalence of Σ–free operads and let X be an $\mathcal{O}$–algebra. Then there are natural weak equivalences of $\mathcal{O}$–algebras*

$$X \leftarrow \mathcal{T}\textstyle\int F^{\widehat{X}} \to \mathcal{T}\textstyle\int F_{\varphi}^{\widehat{X}}.$$

In particular, X is weakly equivalent to a $\mathcal{P}$–algebra.

Proof The left map is a weak equivalence of $\mathcal{O}$–algebras by Proposition 7.1; the right map is a weak equivalence of $\mathcal{O}$–algebras since $F^{\widehat{X}} \to F_{\varphi}^{\widehat{X}}$ is objectwise a weak equivalence. □

The analogous results hold in $\mathcal{T}op$: by [11, Proposition 3.1] the functors and natural transformations constructed in the proof of Proposition 7.1 imply the following:

7.3 Proposition (1) *Let $\mathcal{O}$ be a Σ–free topological operad, X an $\mathcal{O}$–space and $\widehat{X}\colon \widehat{\mathcal{O}} \to \mathcal{T}op$ its associated $\widehat{\mathcal{O}}$–diagram. Then there is a natural homomorphism of $\mathcal{O}$–spaces $\varepsilon\colon \operatorname{hocolim} F^{\widehat{X}} \to X$, which is a weak equivalence.*

(2) *If $\varphi\colon \mathcal{O} \to \mathcal{P}$ is a weak equivalence of Σ–free topological operads, then there are natural weak equivalences of $\mathcal{O}$–spaces*

$$X \leftarrow \operatorname{hocolim} F^{\widehat{X}} \to \operatorname{hocolim} F_{\varphi}^{\widehat{X}}.$$

In particular, X is weakly equivalent to a $\mathcal{P}$–space.

8 Comparing $\widehat{\mathcal{O}}$–categories G with $\mathcal{T} \int F^G$

Let $G\colon \widehat{\mathcal{O}} \to \mathcal{C}at$ be an $\mathcal{O}$–category. We define a functor

$$\lambda_n = \lambda_{nG}\colon \widetilde{\mathbb{T}}^n \to \mathcal{C}at$$

by sending $(T_1, \dots, T_n)$ to $G(\mathrm{In}(T_1) + \cdots + \mathrm{In}(T_n))$ and a morphism $(\phi_1, \dots, \phi_n)$ to $G(\phi_1^{\Sigma} \oplus \cdots \oplus \phi_n^{\Sigma})$. Let

$$W_n^G\colon \mathcal{T}^n \to \mathcal{C}at$$

be the diagram given on objects by the coend

$$W_n^G([T_1], \dots, [T_n]) = \underline{\mathcal{O}} \times \cdots \times \underline{\mathcal{O}} \otimes_{[T_1] \times \cdots \times [T_n]} \lambda_n$$

obtained by restricting the functors λ_n and

$$\underline{\mathcal{O}} \times \cdots \times \underline{\mathcal{O}}\colon \widetilde{\mathbb{T}}^n \to \mathcal{C}at, \quad (T_1, \dots, T_n) \mapsto \underline{\mathcal{O}}(T_1) \times \cdots \times \underline{\mathcal{O}}(T_n)$$

to $[T_1] \times \cdots \times [T_n] \subset \widetilde{\mathbb{T}}^n$. On generating morphisms W_n^G is defined as follows:

- Shrinking an internal edge: $W_n^G(-)$ is defined in the same way as $F^G(-)$ for shrinking a nonbottom edge.
- Chopping off a branch: $W_n^G(-)$ is defined in the same way as $F^G(-)$ with the difference that $G(\hat{c})$ is defined with respect to the union of all inputs of $T_1, \dots, T_n$.

As before, an object or morphism of $W_n^G([T_1], \dots, [T_n])$ is represented by a tuple $(T_1, \dots, T_n, C)$ consisting of trees $T_i \in [T_i]$ decorated by objects or morphisms in $\mathcal{O}(\mathrm{In}(T_i))$, respectively, and an object or morphism $C \in G(\mathrm{In}(T_1) + \cdots + \mathrm{In}(T_n))$, respectively. The appropriate equivariance relations hold.

8.1 Remark Note that, unlike in the F^G construction, the bottom edges of trees play no special role in the W_n^G construction. Also note that $W_1^G([T])$, modulo the equivariance relation, coincides with the construction $W(G, T)$ used as a stepping stone for the F^G construction.

8.2 Lemma *The correspondence*

$$n \mapsto \mathcal{T}^n \int W_n^G$$

extends to an $\widehat{\mathcal{O}}$–category

$$M = \mathcal{T}^* \int W_*^G\colon \widehat{\mathcal{O}} \to \mathcal{C}at.$$

Proof If $\sigma \in \mathrm{Inj}(k,l)$ then $M(\sigma^*)\colon M(l) \to M(k)$ is induced by the projection

$$\mathcal{T}^l \to \mathcal{T}^k, \quad ([T_1],\ldots,[T_l]) \mapsto ([T_{\sigma(1)}],\ldots,[T_{\sigma(k)}])$$

and the map $G(\sigma(\mathrm{In}(T_1),\ldots,\sigma(\mathrm{In}(T_l))^*)$ (see 2.4). If

$$\alpha = (\alpha_1,\ldots,\alpha_r) \in \mathcal{O}(k_1,1)\times\cdots\times\mathcal{O}(k_r,1) \quad \text{and} \quad m = k_1+\cdots+k_r,$$

then $M(\alpha)\colon M(m)\to M(r)$ maps a representing tuple $(T_1,\ldots,T_m,C)$, where each T_i is a decorated tree and $C \in G(\mathrm{In}(T_1+\cdots+\mathrm{In}(T_m))$, to the element represented by $(T_i',\ldots,T_r',C)$. Here T_i' is obtained by grafting the roots of $T_{p+1},\ldots,T_{p+k_i}$ together, where $p=k_1+\cdots+k_{i-1}$, and decorating the root of T_i' by $\alpha_i\circ(\beta_1\oplus\cdots\oplus\beta_{k_i})$ if β_j is the root node decoration of T_{p+j}. □

8.3 Lemma *If $\mathcal{O}$ is a Σ–free operad, there is a map of $\widehat{\mathcal{O}}$–categories*

$$\tau\colon M = \mathcal{T}^*\textstyle\int W_*^G \to \widehat{\mathcal{T}\textstyle\int F^G},$$

natural in G, which is objectwise a weak equivalence if G is special.

Proof The map $\tau(n)\colon M(n) \to (\mathcal{T}\int F^G)^n$ sends a representative $(T_1,\ldots,T_n,C)$ to $((T_1,G(\sigma_1^*)(C)),\ldots,(T_n,G(\sigma_n^*)(C))$, where $\sigma_i\colon \underline{\mathrm{In}(T_i)} \to \underline{\mathrm{In}(T_1)+\cdots+\mathrm{In}(T_n)}$ is the canonical inclusion.

By construction, this defines a map of $\widehat{\mathcal{O}}$–categories.

If G is special, the map

$$(G(\sigma_1^*),\ldots,G(\sigma_n^*))\colon G(\mathrm{In}(T_1)+\cdots+\mathrm{In}(T_n)) \to \prod_{i=1}^n G(\mathrm{In}(T_i))$$

is a weak equivalence. Consequently, τ is a weak equivalence because $\mathcal{O}$ is Σ–free (see Lemma 2.8 and also 4.8(2)). □

8.4 Lemma *There is a map of $\widehat{\mathcal{O}}$–categories*

$$\varepsilon\colon M = \mathcal{T}^*\textstyle\int W_*^G \to G,$$

natural in G, which is objectwise a weak equivalence.

Proof The map $\varepsilon(n)\colon M(n) \to G(n)$ is defined on $(T_1,\ldots,T_n,C)$ by chopping off the roots of $T_1,\ldots,T_n$ as explained in the definition of F^G. To prove that $\varepsilon(n)$ is a weak equivalence we proceed as in the proof of Proposition 7.1. The functor $\varepsilon(n)$ has a section $s_n\colon G(n) \to M(n)$ sending $C \in G(n)$ to $(\Theta_1,\ldots,\Theta_1,C) \in M(n)$, where the node of Θ_1 is decorated by the identity.

We define a functor $J\colon \mathcal{T}^n \int W_n^G \to \mathcal{T}^n \int W_n^G$ in the same way as in Proposition 7.1: we map $([T_1], \dots, [T_n], X)$ to $(j[T_1], \dots, j[T_n]; j(X))$ with the difference that $j(X)$ is obtained from X by decorating each of the n new root nodes by the identity. By shrinking and chopping the incoming edges to the root nodes we again obtain natural transformations $J \Rightarrow \mathrm{Id}$ and $J \Rightarrow s_n \circ \varepsilon(n)$. $\square$

Combining the preceding three lemmas we obtain:

8.5 Theorem *Let $\mathcal{O}$ be a Σ–free operad in $\mathcal{C}at$. Then there are functors*

$$\mathcal{T} \int F^{(-)}\colon \mathcal{C}at^{\widehat{\mathcal{O}}} \to \mathcal{O}\text{–}\mathcal{C}at \quad \text{and} \quad \mathcal{T}^* \int W_*^{(-)}\colon \mathcal{C}at^{\widehat{\mathcal{O}}} \to \mathcal{C}at^{\widehat{\mathcal{O}}}$$

and natural transformations of functors $\mathcal{C}at^{\widehat{\mathcal{O}}} \to \mathcal{C}at^{\widehat{\mathcal{O}}}$

$$\widehat{\mathcal{T} \int F^{(-)}} \xleftarrow{\tau} \mathcal{T}^* \int W_*^{(-)} \xrightarrow{\varepsilon} \mathrm{Id}$$

such that

(1) *each $\varepsilon(G)(n)$ is a weak equivalence, and*

(2) *if G is special, then each $\tau(G)(n)$ is a weak equivalence.* $\square$

9 Comparing $\widehat{\mathcal{O}}$–spaces G with $\mathrm{hocolim}_{\mathcal{T}} F^G$

As one would expect, there are topological versions of the constructions and results of Section 8. We will give a short account of these. In the process we will use the following properties of the homotopy colimit construction (eg see [11, Proposition 3.1]):

9.1 Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, $F, G\colon \mathcal{C} \to \mathcal{D}$ functors, $X\colon \mathcal{C} \to \mathcal{T}op$ and $Y\colon \mathcal{D} \to \mathcal{T}op$ diagrams,

$$\tau\colon F \to G, \quad \alpha\colon X \to Y \circ F, \quad \beta\colon X \to Y \circ G$$

natural transformations such that $(Y \star \tau) \circ \alpha = \beta$:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F{,}G"] \arrow[dr, "X"'] & & \mathcal{D} \arrow[dl, "Y"] \\ & \mathcal{T}op & \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Y \circ F \arrow[rr, "Y \star \tau"] & & Y \circ G \end{tikzcd}$$

Then F and α induce a map

$$B(*, F, \alpha)\colon \mathrm{hocolim}_{\mathcal{C}} X \to \mathrm{hocolim}_{\mathcal{D}} Y$$

while τ defines a homotopy

$$B(*, F, \alpha) \simeq B(*, G, \beta)$$

(see Definition 6.2 and [11, Proposition 3.1]).

Let $\mathcal{O}$ be an operad in $\mathcal{T}op$ and $G\colon \widehat{\mathcal{O}} \to \mathcal{T}op$ an $\widehat{\mathcal{O}}$–space. Let

$$V_n^G\colon \mathcal{T}^n \to \mathcal{T}op$$

be the topological version of the diagram W_n^G, defined in exactly the same way. For a morphism $f \in \widehat{\mathcal{O}}(l,k)$, the corresponding map described in the proof of Lemma 8.2 defines a functor $\mathcal{T}(f)\colon \mathcal{T}^l \to \mathcal{T}^k$ together with a natural transformation

$$\bar{f}\colon V_l^G \to V_k^G \circ \mathcal{T}(f)$$

which induces a map

$$\operatorname{hocolim}_{\mathcal{T}^l} V_l^G \to \operatorname{hocolim}_{\mathcal{T}^k} V_k^G,$$

and we obtain an $\widehat{\mathcal{O}}$–space

$$\operatorname{hocolim}_{\mathcal{T}^*} V_*^G\colon \widehat{\mathcal{O}} \to \mathcal{T}op, \quad n \mapsto \operatorname{hocolim}_{\mathcal{T}^n} V_n^G.$$

9.2 Theorem *If $\mathcal{O}$ is a Σ–free operad in $\mathcal{T}op$, then there are functors*

$$\operatorname{hocolim}_{\mathcal{T}} F^{(-)}\colon \mathcal{T}op^{\widehat{\mathcal{O}}} \to \mathcal{O}\text{–}\mathcal{T}op \quad \text{and} \quad \operatorname{hocolim}_{\mathcal{T}^*} V_*^{(-)}\colon \mathcal{T}op^{\widehat{\mathcal{O}}} \to \mathcal{T}op^{\widehat{\mathcal{O}}}$$

and natural transformations of functors $\mathcal{T}op^{\widehat{\mathcal{O}}} \to \mathcal{T}op^{\widehat{\mathcal{O}}}$

$$\widehat{\operatorname{hocolim}_{\mathcal{T}} F^{(-)}} \xleftarrow{\tau} \operatorname{hocolim}_{\mathcal{T}^*} V_*^{(-)} \xrightarrow{\varepsilon} \mathrm{Id}$$

such that

(1) *each $\varepsilon(G)(n)$ is a weak equivalence, and*

(2) *if G is special, each $\tau(G)(n)$ is a weak equivalence.*

Proof The maps τ_n, defined on representatives like the maps $\tau(n)$ in the proof of Lemma 8.3, define a map from the diagram V_n^G to the diagram

$$(F^G, \dots, F^G)\colon \mathcal{T}^n \to \mathcal{T}op.$$

Since topological realization preserves products, it induces a map

$$\tau_n\colon \operatorname{hocolim}_{\mathcal{T}^n} V_n^G \to (\operatorname{hocolim}_{\mathcal{T}} F^G)^n.$$

By construction, the τ_n define a map of $\widehat{\mathcal{O}}$–spaces. If G is special, the map of diagrams $V_n^G \to (F^G, \dots, F^G)$ is objectwise a homotopy equivalence inducing a homotopy equivalence τ_n.

Let $*$ stand for the category with one object 0 and the identity morphism, let $G(n)\colon * \to \mathcal{T}op$ be the functor sending 0 to $G(n)$, and let $P_n\colon \mathcal{T}^n \to *$ be the projection. The maps ε_n of the proof of Lemma 8.4 define natural transformations $\beta_n\colon V_n^G \to G(n) \circ P_n$, thus inducing maps

$$\varepsilon_n = B(*, P_n, \beta_n)\colon \operatorname{hocolim}_{\mathcal{T}^n} V_n^G \to G(n)$$

which define a map of $\widehat{\mathcal{O}}$–spaces.

Let $j\colon \mathcal{T} \to \mathcal{T}$ be the functor defined in the proof of Proposition 7.1. Let $S\colon * \to \mathcal{T}^n$ send 0 to $([\Theta_1], \dots, [\Theta_1])$ and let $\gamma\colon G(n) \to V_n^G \circ S$ be the natural transformation sending $C \in G(n)$ to the element represented by $(\Theta_1, \dots, \Theta_1, C)$, where the root nodes are decorated by identities. The pair (S, γ) induces a section

$$s = B(*, S, \gamma)\colon G(n) \to \operatorname{hocolim}_{\mathcal{T}^n} V_n^G$$

of ε_n. The functor $j^n\colon \mathcal{T}^n \to \mathcal{T}^n$ together with the natural transformation $\alpha\colon V_n^G \to V_n^G \circ j^n$ sending a representative $(T_1, \dots, T_n, C)$ to $(j(T_1), \dots, j(T_n), C)$, where the added root nodes are decorated by identities, define a selfmap

$$J\colon \operatorname{hocolim}_{\mathcal{T}^n} V_n^G \to \operatorname{hocolim}_{\mathcal{T}^n} V_n^G.$$

There are natural transformations $\mathrm{sh}\colon j^n \to \mathrm{Id}$ and $\mathrm{ch}\colon j^n \to S \circ P_n$ defined by shrinking and chopping the incoming edges to the root nodes, respectively. Since

$$(V_n^G \star \mathrm{sh}) \circ \alpha = \mathrm{id} \quad \text{and} \quad (V_n^G \star \mathrm{ch}) \circ \alpha = \gamma \circ \beta_n,$$

there are homotopies $J \simeq \mathrm{id}$ and $J \simeq S \circ P_n$. □

9.3 Corollary *Let $\mathcal{O}$ be a Σ–free operad in $\mathcal{T}op$. Then there is a chain of natural transformations of functors $\mathcal{T}op^{\widehat{\mathcal{O}}} \to \mathcal{O}$–$\mathcal{T}op$ connecting the functor M of Section 4 and the functor* $\operatorname{hocolim}_{\mathcal{T}} F^{(-)}$, *which are weak equivalences when evaluated at special $\widehat{\mathcal{O}}$–spaces.*

Proof We apply the rectification of Section 4 to the diagram in Theorem 9.2 to obtain a diagram of weakly equivalent $\mathcal{O}$–algebras

$$M(G) \leftarrow M(\operatorname{hocolim}_{\mathcal{T}^*} V_*^G) \to M(\widehat{\operatorname{hocolim}_{\mathcal{T}} F^G}).$$

By Proposition 4.10, the $\mathcal{O}$–algebras $M(\widehat{\operatorname{hocolim}_{\mathcal{T}} F^G})$ and $\operatorname{hocolim}_{\mathcal{T}} F^G$ are weakly equivalent. □

10 From simplicial $\mathcal{O}$–algebras to $\mathcal{O}$–algebras

Given a simplicial category $\mathcal{C}_*\colon \Delta^{\mathrm{op}} \to \mathcal{C}at$ there are the Bousfield–Kan map and the Thomason map

$$|B(\mathcal{C}_*)| \xleftarrow{\rho} \operatorname{hocolim}_{\Delta^{\mathrm{op}}} B(\mathcal{C}_*) \xrightarrow{\eta} B\big(\Delta^{\mathrm{op}} \textstyle\int \mathcal{C}_*\big) \tag{10.1}$$

which are natural maps known to be homotopy equivalences by [6, Section XII.3.4] or [10, Theorem 18.7.4] and [18, Theorem 1.2], respectively. So Thomason's homotopy colimit construction in $\mathcal{C}at$ replaces a simplicial category by a category in a nice way: their realizations in $\mathcal{T}op$ via the classifying space functor are homotopy equivalent.

In this section we want to lift this result to simplicial $\mathcal{O}$–algebras over a Σ–free operad $\mathcal{O}$ in $\mathcal{C}at$.

We start with the right-hand map in (10.1), for which there is a more general version. Let $\mathcal{L}$ be a small indexing category. Let $\mathcal{O}$ be an operad in $\mathcal{S}$, where $\mathcal{S}$ is $\mathcal{C}at$ or $\mathcal{T}op$, and let $\mathcal{O}$–$\mathcal{S}^{\mathcal{L}}$ denote the category of $\mathcal{L}$–diagrams of $\mathcal{O}$–algebras in $\mathcal{S}$. We have functors

$$H_{\mathrm{top}}\colon \mathcal{O}\text{–}\mathcal{T}op^{\mathcal{L}} \to \mathcal{O}\text{–}\mathcal{T}op \quad \text{if } \mathcal{O} \text{ is an operad in } \mathcal{T}op,$$
$$H_{\mathrm{cat}}\colon \mathcal{O}\text{–}\mathcal{C}at^{\mathcal{L}} \to \mathcal{O}\text{–}\mathcal{C}at \quad \text{if } \mathcal{O} \text{ is an operad in } \mathcal{C}at,$$

defined by

$$H_{\mathrm{top}}(X) = \operatorname{hocolim}_{\mathcal{T}} F^{\operatorname{hocolim}_{\mathcal{L}} \hat{X}} \quad \text{for an } \mathcal{L}\text{–diagram } X\colon \mathcal{L} \to \mathcal{O}\text{–}\mathcal{T}op$$

and

$$H_{\mathrm{cat}}(D) = \mathcal{T} \textstyle\int F^{\mathcal{L} \int \hat{D}} \quad \text{for an } \mathcal{L}\text{–diagram } D\colon \mathcal{L} \to \mathcal{O}\text{–}\mathcal{C}at,$$

where $\hat{X}\colon \hat{\mathcal{O}} \to \mathcal{T}op^{\mathcal{L}}$ and $\hat{D}\colon \hat{\mathcal{O}} \to \mathcal{C}at^{\mathcal{L}}$ are induced by X and D, respectively.

10.2 Proposition *Let $\mathcal{O}$ be a Σ–free operad in $\mathcal{C}at$. Then there is a natural weak equivalence*

$$\eta\colon H_{\mathrm{top}} \circ B^{\mathcal{L}} \Rightarrow B \circ H_{\mathrm{cat}}$$

of functors $\mathcal{O}$–$\mathcal{C}at^{\mathcal{L}} \to B\mathcal{O}$–$\mathcal{T}op$.

Proof Let $D\colon \mathcal{L} \to \mathcal{O}\text{–}\mathcal{C}at$ be an $\mathcal{L}$–diagram of $\mathcal{O}$–algebras. We have natural maps of $B\mathcal{O}$–spaces

$$\begin{aligned} H_{\mathrm{top}}(BD) &= \operatorname{hocolim}_{\mathcal{T}} F^{\operatorname{hocolim}_{\mathcal{L}} \widehat{BD}} \\ &\to \operatorname{hocolim}_{\mathcal{T}} F^{B(\mathcal{L}\int \widehat{D})} \\ &\cong \operatorname{hocolim}_{\mathcal{T}} B(F^{\mathcal{L}\int \widehat{D}}) \\ &\to B\big(\mathcal{T}\textstyle\int F^{\mathcal{L}\int \widehat{D}}\big) = B(H_{\mathrm{cat}}(D)). \end{aligned}$$

By [18, Theorem 1.2] Thomason's map defines a pointwise weak equivalence

$$\operatorname{hocolim}_{\mathcal{L}} \widehat{BD} \to B\big(\mathcal{L}\textstyle\int \widehat{D}\big),$$

so the first map is a weak equivalence by Proposition 6.7. For the isomorphism see Proposition 6.6, and the second map is a weak equivalence by Proposition 6.5. □

In general we cannot say much about the homotopy type of $H_{\mathrm{top}}(X)$ and $H_{\mathrm{cat}}(D)$. This is different if $\mathcal{L} = \Delta^{\mathrm{op}}$ and X is proper. Here we call a simplicial space *proper* if the inclusions $sX_n \subset X_n$ of the subspaces sX_n of the degenerate elements of X_n are closed cofibrations for all n, and we call a simplicial $\mathcal{O}$–space proper if its underlying space is proper.

10.3 Proposition *Let $\mathcal{O}$ be a Σ–free operad in $\mathcal{T}op$ and let X_* be a proper simplicial $\mathcal{O}$–space. Then there is a weak equivalence of $\mathcal{O}$–spaces*

$$\rho\colon H_{\mathrm{top}}(X_*) \to |X_*|$$

natural with respect to homomorphisms of proper simplicial $\mathcal{O}$–spaces.

Proof We have natural maps of $\mathcal{O}$–spaces

$$H_{\mathrm{top}}(X_*) = \operatorname{hocolim}_{\mathcal{T}} F^{\operatorname{hocolim}_{\Delta^{\mathrm{op}}} \widehat{X_*}} \to \operatorname{hocolim}_{\mathcal{T}} F^{|\widehat{X_*}|} \to |X_*|.$$

The Bousfield–Kan map $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} \widehat{X_*} \to |\widehat{X_*}|$ is pointwise a weak equivalence provided X_* is proper, so that the first map is a weak equivalence by Proposition 6.7. Since topological realization preserves colimits and finite products, we have a natural isomorphism $|\widehat{X_*}| \cong |\widehat{X_*}|$, and the second map is a weak equivalence by Proposition 7.3. □

Combining these results we obtain the passage from simplicial $\mathcal{O}$–algebras in $\mathcal{C}at$ to $\mathcal{O}$–algebras in $\mathcal{C}at$.

10.4 Theorem *Let $\mathcal{O}$ be a Σ–free $\mathcal{C}at$–operad, and let $B\mathcal{O}\text{–}p\mathcal{T}op^{\Delta^{\mathrm{op}}}$ be the full subcategory of $B\mathcal{O}\text{–}\mathcal{T}op^{\Delta^{\mathrm{op}}}$ of proper simplicial $B\mathcal{O}$–spaces. Then there is a diagram*

$$\begin{tikzcd}
\mathcal{O}\text{–}\mathcal{C}at^{\Delta^{\mathrm{op}}} \arrow[r, "B^{\Delta^{\mathrm{op}}}"] \arrow[d, "H_{\mathrm{cat}}"'] \arrow[dr, phantom, "\Leftarrow"] & B\mathcal{O}\text{–}p\mathcal{T}op^{\Delta^{\mathrm{op}}} \arrow[r, phantom, "\subset"] \arrow[d, "H_{\mathrm{top}}"'] & B\mathcal{O}\text{–}\mathcal{T}op^{\Delta^{\mathrm{op}}} \arrow[dl, "|-|"] \arrow[dl, phantom, shift right=6, "\Rightarrow"] \\
\mathcal{O}\text{–}\mathcal{C}at \arrow[r, "B"] & B\mathcal{O}\text{–}\mathcal{T}op &
\end{tikzcd}$$

commuting up to natural weak equivalences.

Let const: $\mathcal{C}at \to \mathcal{C}at^{\Delta^{\mathrm{op}}}$ be the constant simplicial object functor and let $\mathcal{C}$ be an $\mathcal{O}$–algebra. Since $|B(\mathrm{const}\ \mathcal{C})| \cong B(\mathcal{C})$, the functor $H \circ \mathrm{const}$ preserves the homotopy type. But we can do better. The diagram

$$\begin{tikzcd}
\widehat{\mathcal{O}} \arrow[r, "\widehat{\mathcal{C}}"] \arrow[dr, "\widehat{\mathrm{const}\,\mathcal{C}}"'] & \mathcal{C}at \arrow[d, "\mathrm{const}"] \arrow[dr, "\Delta^{\mathrm{op}}\times -"] & \\
& \mathcal{C}at\Delta^{\mathrm{op}} \arrow[r, "\Delta^{\mathrm{op}}\int -"] & \mathcal{C}at
\end{tikzcd}$$

commutes.

10.5 Proposition *Let $\mathcal{O}$ be a Σ–free $\mathcal{C}at$–operad and let $\mathcal{C}$ be an $\mathcal{O}$–algebra. Then there are weak equivalences of $\mathcal{O}$–algebras*

$$H(\mathrm{const}\ \mathcal{C}) = \mathcal{T}\textstyle\int F^{\Delta^{\mathrm{op}}\int\widehat{\mathrm{const}\ \mathcal{C}}} = \mathcal{T}\int F^{\Delta^{\mathrm{op}}\times\widehat{\mathcal{C}}} \xrightarrow{\pi} \mathcal{T}\int F^{\widehat{\mathcal{D}}} \xrightarrow{\varepsilon} \mathcal{C},$$

where π is induced by the projection $\Delta^{\mathrm{op}} \times \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ and ε is the homomorphism of Proposition 7.1.

Proof This follows from Propositions 7.1 and 6.7, because the projection $\Delta^{\mathrm{op}} \times \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ is objectwise a weak equivalence. □

10.6 Remark Our passage from simplicial algebras to algebras translates verbatim to $\mathcal{T}op$, but, of course, topological realization is the preferred passage: it is well known that the topological realization of a simplicial $\mathcal{O}$–space is an $\mathcal{O}$–space in a canonical way.

11 An application

11.1 Definition Let $\mathcal{O}$ be an operad in $\mathcal{C}at$ and let $\mathcal{P}$ be an operad in $\mathcal{T}op$. In this section a homomorphism of $\mathcal{P}$–spaces is called a *weak equivalence* if its underlying map of spaces is a weak homotopy equivalence, and a homomorphism of $\mathcal{O}$–algebras $f\colon \mathcal{A} \to \mathcal{B}$ is called a *weak equivalence* if Bf is a weak equivalence of $B\mathcal{O}$–spaces. Two $\mathcal{P}$–spaces X and Y are called *weakly equivalent* if there is a chain of weak equivalences connecting X and Y, and similarly for two $\mathcal{O}$–algebras $\mathcal{A}$ and $\mathcal{B}$.

Let $\mathcal{O}$ be a Σ–free operad in $\mathcal{C}at$. We want to compare the categories $\mathcal{O}$–$\mathcal{C}at$ and $B\mathcal{O}$–$\mathcal{T}op$. The classifying space functor maps an $\mathcal{O}$–algebra $\mathcal{C}$ to the $B\mathcal{O}$–algebra $B\mathcal{C}$. In [9] we showed that for each $B\mathcal{O}$–space X there is a simplicial $\mathcal{O}$–algebra $\mathcal{A}_*$ and a sequence of natural weak equivalences of $B\mathcal{O}$–spaces connecting X and $|B\mathcal{A}_*|$. By Theorem 10.4 there is an $\mathcal{O}$–algebra $\mathcal{C}$ such that $B\mathcal{C}$ and $|B\mathcal{A}_*|$ are weakly equivalent $B\mathcal{O}$–spaces. So after localization with respect to the weak equivalences the categories $\mathcal{O}$–$\mathcal{C}at$ and $B\mathcal{O}$–$\mathcal{T}op$ are equivalent.

Since $B\mathcal{O}$–$\mathcal{T}op$ carries a Quillen model structure with the weak equivalences of Definition 11.1, its localization $B\mathcal{O}$–$\mathcal{T}op[\mathrm{we}^{-1}]$ with respect to these weak equivalences exists [17, Theorem B]. We do not know whether or not $\mathcal{O}$–$\mathcal{C}at$ carries a model structure, but combining our previous results with a result of Schlichtkrull and Solberg [15] we obtain:

11.2 Theorem *Let $\mathcal{O}$ be a Σ–free operad in $\mathcal{C}at$. Then the localization $\mathcal{O}$–$\mathcal{C}at[\mathrm{we}^{-1}]$ exists and the classifying space functor induces an equivalence of categories*

$$\mathcal{O}\text{–}\mathcal{C}at[\mathrm{we}^{-1}] \simeq B\mathcal{O}\text{–}\mathcal{T}op[\mathrm{we}^{-1}].$$

Proof Let $T = R \circ S\colon \mathcal{T}op \to \mathcal{T}op$ be the standard CW-approximation functor, ie the composite of the singular functor S and the topological realization functor, which we denote by R in this proof. Let X be a $B\mathcal{O}$–space. Then TX is a $B\mathcal{O}$–space and the natural map $TX \to X$ is a weak equivalence of $B\mathcal{O}$–spaces. To see this, recall that $B\mathcal{O} = RN_*\mathcal{O}$. Consider the diagram

$$\begin{tikzcd}
RN_*(\mathcal{O}(n)) \times RS(X)^n \arrow[r, "R\mu N_*\mathcal{O}(n)\times \mathrm{id}"] \arrow[d, "\mathrm{id}\times\varepsilon^n"'] & RSRN_*(\mathcal{O}(n)) \times RS(X)^n \arrow[r, "RS\alpha"] \arrow[dl, "\varepsilon RN_*\mathcal{O}(n)\times\varepsilon^n"] & RS(X) \arrow[d, "\varepsilon"] \\
RN_*(\mathcal{O}(n)) \times X^n \arrow[rr, "\alpha"] & & X
\end{tikzcd}$$

where $\alpha\colon RN_*(\mathcal{O}(n)) \times X^n \to X$ is a structure map, and μ and ε are the unit and counit of the adjunction

$$R\colon \mathcal{S}Sets \leftrightarrows \mathcal{T}op\ :S.$$

Since all functors are product-preserving, the right square commutes by naturality, and the triangle commutes because $\varepsilon R \circ R\mu = \mathrm{id}$.

In [9, Section 5] we constructed a functor $\widehat{D}_\bullet\colon B\mathcal{O}\text{–}\mathcal{T}op \to \mathcal{O}\text{–}\mathcal{C}at^{\Delta^{\mathrm{op}}}$ and showed that there is a sequence of natural weak equivalences in $\mathcal{O}\text{–}\mathcal{C}at^{\Delta^{\mathrm{op}}}$ joining $\mathcal{C}_\bullet$ and $\widehat{D}_\bullet(B\mathcal{C})$, where $\mathcal{C}$ is an $\mathcal{O}$–algebra and $\mathcal{C}_\bullet$ is the constant simplicial $\mathcal{O}$–algebra on $\mathcal{C}$ [9, Lemma 6.6]. Here we call a map $f_\bullet\colon X_\bullet \to Y_\bullet$ of simplicial $B\mathcal{O}$–spaces a weak equivalence if its realization $|f_\bullet|\colon |X_\bullet| \to |Y_\bullet|$ is a weak equivalence, and a map $g_\bullet\colon \mathcal{A}_\bullet \to \mathcal{B}_\bullet$ of simplicial $\mathcal{O}$–algebras a weak equivalence if $B(g_\bullet)$ is a weak equivalence in $B\mathcal{O}-\mathcal{T}op^{\Delta^{\mathrm{op}}}$. If X is a $B\mathcal{O}$–space whose underlying space is a CW-complex, we also showed that there is a sequence of natural weak equivalences in $B\mathcal{O}\text{–}p\mathcal{T}op^{\Delta^{\mathrm{op}}}$ joining $B\widehat{D}_\bullet(X)$ and the constant simplicial $B\mathcal{O}$–space $X_\bullet$ [9, Lemma 6.3].

We define

$$F = H_{\mathrm{cat}} \circ \widehat{D}_\bullet \circ T\colon\ B\mathcal{O}\text{–}\mathcal{T}op \to \mathcal{O}\text{–}\mathcal{C}at.$$

Let X be a $B\mathcal{O}$–space and $X_\bullet$ the constant simplicial $B\mathcal{O}$–space on X. Theorem 10.4 implies that we have a natural weak equivalence

$$H_{\mathrm{top}}(TX_\bullet) \to |TX_\bullet| \cong TX \to X.$$

By Theorem 10.4, H_{top} maps weak equivalences in $B\mathcal{O}\text{–}p\mathcal{T}op^{\Delta^{\mathrm{op}}}$ to weak equivalences in $B\mathcal{O}\text{–}\mathcal{T}op$. So if we apply H_{top} to the second sequence of weak equivalences we obtain a sequence of weak equivalences joining $H_{\mathrm{top}}(TX_\bullet)$ and $H_{\mathrm{top}}(B\widehat{D}_\bullet(TX))$, and, again by Theorem 10.4, there is a weak equivalence $H_{\mathrm{top}}(B\widehat{D}_\bullet(TX)) \to B(H_{\mathrm{cat}}(\widehat{D}_\bullet(TX))$. So there is a sequence of natural weak equivalences joining $B \circ F$ and Id.

Let $\mathcal{C}$ be an $\mathcal{O}$–algebra. By Proposition 10.5 there is a weak equivalence $H_{\mathrm{cat}}(\mathcal{C}_\bullet) \to \mathcal{C}$, and since $B\mathcal{C}$ is a CW-complex, the natural map $TB\mathcal{C} \to B\mathcal{C}$ induces a weak equivalence $\widehat{D}_\bullet(TB\mathcal{C}) \to \widehat{D}_\bullet(B\mathcal{C})$. By applying H_{cat} to the first sequence of weak equivalences above we obtain a sequence of natural weak equivalences joining $H_{\mathrm{cat}}(\mathcal{C}_\bullet)$ and $H_{\mathrm{cat}}(\widehat{D}_\bullet(B\mathcal{C}))$, because H_{cat} maps weak equivalences to weak equivalences. Altogether we obtain a sequence of natural weak equivalences in $\mathcal{O}\text{–}\mathcal{C}at$ joining $F \circ B$ and Id.

Then by [15, Proposition A.1], the existence of the localization $B\mathcal{O}\text{–}\mathcal{T}op[\mathrm{we}^{-1}]$ implies the existence of the localization $\mathcal{O}\text{–}\mathcal{C}at[\mathrm{we}^{-1}]$ and the equivalence

$$\mathcal{O}\text{–}\mathcal{C}at[\mathrm{we}^{-1}] \simeq B\mathcal{O}\text{–}\mathcal{T}op[\mathrm{we}^{-1}]. \qquad \square$$

From Theorem 11.2 we obtain the results about iterated loop spaces of [8, Section 8] without referring to the fairly complicated homotopy colimit construction in categories of algebras over Σ–free operads in $\mathcal{C}at$. We include a short summary of these applications, because we now have statements about genuine localizations rather than localizations up to equivalence. For further details, in particular the group completion functors, see [8].

11.3 Notation

- $\mathcal{B}r$ denotes the operad codifying strict braided monoidal categories, ie braided monoidal categories which are strictly associative and have a strict 2–sided unit (recall that any braided monoidal category is equivalent to a strict one).
- $\mathcal{M}_n$ denotes the operad codifying n–fold monoidal categories, $1 \leq n \leq \infty$, introduced in [2].
- $\mathcal{P}erm$ denotes the operad codifying permutative categories.
- $\mathcal{C}_n$, denotes the little n–cubes operad, $1 \leq n \leq \infty$.

11.4 Theorem *The composites of the classifying space functors and the change of operads functors induce equivalences of categories*

$$\begin{aligned} \mathcal{M}_n\text{–}\mathcal{C}at[\mathrm{we}^{-1}] &\simeq B\mathcal{M}_n\text{–}\mathcal{T}op[\mathrm{we}^{-1}] \simeq \mathcal{C}_n\text{–}\mathcal{T}op[\mathrm{we}^{-1}], \qquad 1 \leq n \leq \infty, \\ \mathcal{B}r\text{–}\mathcal{C}at[\mathrm{we}^{-1}] &\simeq B\mathcal{B}r\text{–}\mathcal{T}op[\mathrm{we}^{-1}] \simeq \mathcal{C}_2\text{–}\mathcal{T}op[\mathrm{we}^{-1}], \\ \mathcal{P}erm\text{–}\mathcal{C}at[\mathrm{we}^{-1}] &\simeq B\mathcal{P}erm\text{–}\mathcal{T}op[\mathrm{we}^{-1}] \simeq \mathcal{C}_\infty\text{–}\mathcal{T}op[\mathrm{we}^{-1}]. \end{aligned}$$

It is well known that the group completion of a $\mathcal{C}_n$–space is an n–fold loop space for $1 \leq n \leq \infty$. Let Ω^n-$\mathcal{T}op$ denote the category of n–fold loop spaces and n–fold loop maps. A *weak equivalence* in Ω^n-$\mathcal{T}op$ is an n–fold loop map whose underlying map is a weak homotopy equivalence, or equivalently whose May delooping [13] is an equivalence. Again by [15, Proposition A.1] the localization with respect to these weak equivalences exists. Let we_g denote the classes of morphisms in $\mathcal{B}r$-$\mathcal{C}at$, $\mathcal{M}_n$-$\mathcal{C}at$ and $\mathcal{P}erm$-$\mathcal{C}at$ which are mapped to weak equivalences by the composites of the classifying space functors, the change of operads functors, and the group completion functors. The localizations with respect to these weak equivalences exist by the same argument and we have:

11.5 Theorem *The composites of the classifying space functors, the change of operads functors, and the group completion functors induce equivalences of categories*

$$\begin{aligned} \mathcal{M}_n\text{-}\mathcal{C}at[{\mathrm{we}_g}^{-1}] &\simeq \Omega^n\mathcal{T}op[\mathrm{we}^{-1}], \quad 1 \leq n \leq \infty, \\ \mathcal{B}r\text{-}\mathcal{C}at[{\mathrm{we}_g}^{-1}] &\simeq \Omega^2\mathcal{T}op[\mathrm{we}^{-1}], \\ \mathcal{P}erm\text{-}\mathcal{C}at[{\mathrm{we}_g}^{-1}] &\simeq \Omega^\infty\mathcal{T}op[\mathrm{we}^{-1}]. \end{aligned}$$

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