Obstructions to Lagrangian concordance

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We investigate the question of the existence of a Lagrangian concordance between two Legendrian knots in $\mathbb{R}^3$. In particular, we give obstructions to a concordance from an arbitrary knot to the standard Legendrian unknot, in terms of normal rulings. We also place strong restrictions on knots that have concordances both to and from the unknot and construct an infinite family of knots with nonreversible concordances from the unknot. Finally, we use our obstructions to present a complete list of knots with up to 14 crossings that have Legendrian representatives that are Lagrangian slice.

1 Introduction

In symplectic and contact topology, there has been a great deal of recent interest in the subject of Lagrangian cobordisms between Legendrian submanifolds; see, for example, Baldwin and the third author [1], Boranda, Traynor and Yan [3], Bourgeois, Sabloff and Traynor [5], Cao, Gallup, Hayden and Sabloff [6], Chantraine [8; 9], Ekholm [13], Ekholm, Honda and Kálmán [14], Golovko [26], Hayden and Sabloff [27], and Sabloff and Traynor [44]. A key motivation is that one can construct a category whose objects are Legendrian submanifolds and whose morphisms are exact Lagrangian cobordisms, and this category fits nicely into symplectic field theory; see Eliashberg, Givental and Hofer [17]. In particular, Legendrian contact homology gives a functor from the category of Legendrians to the category of differential graded algebras.

In this paper, we restrict ourselves to the setting of Legendrian knots in standard contact $\mathbb{R}^3$, and address the question of when there exists an exact Lagrangian concordance between two such knots. Let $\mathbb{R}^3$ be equipped with the standard contact structure $\ker\alpha$ with $\alpha = dz - y\,dx$, and let $\mathbb{R}^4 = \mathbb{R}_t \times \mathbb{R}^3$ be the symplectization of $\mathbb{R}^3$ with symplectic form $\omega = d(e^t\alpha)$. Recall that a knot $\Lambda \subset \mathbb{R}^3$ is *Legendrian* if $\alpha|_\Lambda = 0$, and a surface $L \subset \mathbb{R}^4$ is *Lagrangian* if $\omega|_L = 0$.

Definition 1.1 Let $\Lambda_-, \Lambda_+ \subset \mathbb{R}^3$ be Legendrian knots. A *Lagrangian cobordism from* Λ_- *to* Λ_+ is an embedded Lagrangian $L \subset \mathbb{R}^4$ such that

$$L \cap ((-\infty, -T] \times \mathbb{R}^3) = (-\infty, -T] \times \Lambda_-$$

$$L \cap ([T, \infty) \times \mathbb{R}^3) = [T, \infty) \times \Lambda_+$$

for some $T > 0$. This cobordism is *exact* if there exists $f : L \to \mathbb{R}$ such that $df = \alpha|_L$. A Lagrangian cobordism of genus 0 (ie a cylinder) is a *Lagrangian concordance*. Define a relation $\prec$ on the set of Legendrian knots by $\Lambda_- \prec \Lambda_+$ if there is a Lagrangian concordance from Λ_- to Λ_+.

We note that any Lagrangian concordance is automatically exact: since α vanishes on Λ_- and Λ_- generates $H_1(L)$ if L is a concordance, α must equal 0 in the de Rham cohomology of L.

It is clear that $\prec$ is transitive. If Λ_0 and Λ_1 are isotopic as Legendrian knots, then $\Lambda_0 \prec \Lambda_1$ and $\Lambda_1 \prec \Lambda_0$ as shown by Chantraine [8]. It follows that $\prec$ descends to a well-defined, reflexive relation on the set of isotopy classes of Legendrian knots. Chantraine [9] also showed that $\prec$ is not symmetric (see below for further discussion).

At present, it is unknown whether $\prec$ is antisymmetric: that is, if $\Lambda_0 \prec \Lambda_1$ and $\Lambda_1 \prec \Lambda_0$, must Λ_0 be Legendrian isotopic to Λ_1? We remark that although our definition of $\prec$ involves only concordances rather than general cobordisms, this is no restriction in this setting. By a result of Chantraine [8], if there is a Lagrangian cobordism L from Λ_0 to Λ_1, then $\mathrm{tb}(\Lambda_1) - \mathrm{tb}(\Lambda_0) = 2g(L) \geq 0$, where tb is the Thurston–Bennequin number, and so the existence of cobordisms in both directions between Λ_0 and Λ_1 implies that $\mathrm{tb}(\Lambda_0) = \mathrm{tb}(\Lambda_1)$ and that the cobordisms are concordances.

The special case when one of the Legendrian knots is the standard $\mathrm{tb} = -1$ unknot U is of particular interest. A cobordism from U to a Legendrian knot Λ can be filled at the negative end by a Lagrangian disk, resulting in a *Lagrangian filling* of Λ. Such fillings are relatively common; see eg Hayden and Sabloff [27]. In the case when the cobordism is a cylinder ($U \prec \Lambda$), the smooth knot type of Λ must be smoothly slice, and we say that Λ is *Lagrangian slice*. It is currently unknown whether any Lagrangian slice knot besides U is concordant *to* U. Chantraine proved in [9], using the augmentation category of Bourgeois and Chantraine [4], that there is a Lagrangian slice knot Λ of type $\overline{9_{46}}$ (the mirror of 9_{46}) such that $\Lambda \not\prec U$.

One of the goals of this paper is to give strong and easily computable obstructions to the existence of a concordance $\Lambda \prec U$. In particular, we show the following result.

Theorem 1.2 (See Theorem 2.7) *If* Λ *has at least two normal rulings, then* $\Lambda \not\prec U$.

In particular, the $\overline{9_{46}}$ knot considered by Chantraine satisfies this condition, and so we have a new, simpler proof of Chantraine's result. We also use this result in Theorem 2.10 to construct an infinite family of Legendrian knots Λ with $U \prec \Lambda$ and $\Lambda \not\prec U$. It should be noted that Baldwin and the third author [1] previously presented a different infinite family of nonreversible concordances involving stabilized unknots rather than U.

The proof of Theorem 1.2 involves two ingredients. One is the fact proved by Ekholm, Honda and Kálmán [14] that exact Lagrangian cobordisms induce maps on Legendrian contact homology and, in particular, that an exact Lagrangian filling of Λ_- induces an augmentation for the differential graded algebra of Λ_+. The second is a study of a particular 2–cable of Legendrian knots to prove Theorem 1.2. This study relies in turn on the following observation; see Section 2.2 for the definition of Legendrian satellite.

Theorem 1.3 (See Theorem 2.4) *If* Λ_-, Λ_+ *are Legendrian knots in* $\mathbb{R}^3$ *such that* $\Lambda_- \prec \Lambda_+$, *then their Legendrian satellites satisfy* $S(\Lambda_-, \Lambda') \prec S(\Lambda_+, \Lambda')$ *for any Legendrian solid torus knot* $\Lambda' \subset J^1(S^1)$.

Applying this result not just to Λ but to a particular family of satellites of Λ, we arrive at an infinite family of obstructions to $\Lambda \prec U$ that depend only on the underlying smooth knot type.

Theorem 1.4 (See Theorem 2.15) *Let* Λ *be a Legendrian knot of smooth knot type* K, *and let* K_n *denote the* 0–*framed* n–*cable of* K. *If* $U \prec \Lambda \prec U$, *and* $p_n(a, z)$ *is either the HOMFLY-PT or Kauffman (Dubrovnik) polynomial of* K_n, *then* $\max\text{-deg}_a\, p_n(a, z) = n - 1$, *and* $p_n(a, z)$ *has* a^{n-1}–*coefficient equal to* z^{1-n} *for all* $n \geq 1$.

In particular, one can use this result to obstruct $\Lambda \prec U$ when $U \prec \Lambda$ by finding two distinct rulings of a Legendrian representative $S(\Lambda, \text{tw}_n)$ of K_n; one ruling is guaranteed by the existence of a ruling on Λ, but the second violates the conclusion of Theorem 2.15. See Theorem 2.13 for more details and Section 4.4 for applications of this technique.

In a different direction, a result of Ekholm, Honda, and Kálmán [14] gives explicit exact Lagrangian cobordisms between Legendrian knots whose fronts are related by two elementary moves, unknot filling and pinch moves, which correspond topologically to 0–handle and 1–handle attachment. We call a Lagrangian cobordism *decomposable* if it can be broken into these elementary pieces; decomposable cobordisms currently form a central tool for constructing exact Lagrangian cobordisms.

Although we do not answer the general question of whether a nontrivial Legendrian knot Λ can satisfy $U \prec \Lambda \prec U$, we prove a special case of this in Section 3: no nontrivial

Λ can have a decomposable Lagrangian concordance to U; see Theorem 3.2. The proof of this is purely topological, relying on work of Kronheimer and Mrowka [31]. We do, however, expect that indecomposable Lagrangian concordances exist, and we exhibit a possible example in Conjecture 3.4, but we do not know of any potential examples from U to another knot.

Finally, in Section 4, we enumerate all knots up through 14 crossings with Legendrian representatives that are Lagrangian slice. Necessary conditions for a knot K to have such a Legendrian representative are that K must be smoothly slice and satisfy $\overline{\text{tb}}(K) = -1$, where $\overline{\text{tb}}$ is the maximal Thurston–Bennequin number. Through 14 crossings, we show that these conditions are sufficient as well, using explicit decomposable cobordisms for each knot type. To help with the census of Lagrangian slice knots, we prove that no nontrivial alternating knot can be Lagrangian slice (see Theorem 4.3), which has the side benefit of giving a new, contact-geometric proof of a result of Nakamura [34] that any reduced alternating diagram of a positive knot can only have positive crossings.

2 Legendrian satellites and concordance

In this section, we present new obstructions to the existence of a Lagrangian concordance between two Legendrian knots. We assume basic familiarity with the theory of Legendrian knots along the lines of Etnyre's survey [18]. Throughout this section (and, indeed, for the rest of the paper), we use U to denote the standard Legendrian unknot with tb $= -1$.

Normal rulings, introduced by Chekanov and Pushkar [38] and Fuchs [22], will play a significant role; to review their definition see eg Sabloff [43, Section 2.3]. While a ruling is defined in reference to a particular front projection of Λ, the number of rulings, and the more refined count given by the ruling polynomial, are invariants of Λ; see the footnote in Section 2.3.

A ruling of a front projection of Λ is determined by the set of crossings that are *switches* in the ruling. In figures, we specify a ruling by decorating those crossings. Sometimes we will depict a front projection by a grid diagram; the corresponding front is obtained by rotating 45° counterclockwise, making upper left and lower right corners into cusps, and smoothing other corners of the grid diagram.

2.1 Review of functoriality of Legendrian contact homology under cobordisms

Associated to any Legendrian knot in $\mathbb{R}^3$ is the Chekanov–Eliashberg differential graded algebra (DGA) [10; 15], whose homology is the Legendrian contact homology

of the knot. As part of the symplectic field theory package [17], this DGA behaves functorially under exact Lagrangian cobordism. Since this behavior underlies our study of obstructions, we briefly review the statement here, as proved by Ekholm, Honda, and Kálmán [14].

Proposition 2.1 [14] *If Λ_-, Λ_+ are Legendrian knots such that there is an exact Lagrangian cobordism from Λ_- to Λ_+, then there is a morphism of Legendrian contact homology DGAs*

$$\phi \colon \mathcal{A}_{\Lambda_+} \to \mathcal{A}_{\Lambda_-}.$$

For our purposes, it will be convenient to clarify the statement of Proposition 2.1 in two ways, which we present as the following two remarks.

Remark 2.2 (Coefficient ring) The morphism in Proposition 2.1 restricts to the identity map on the coefficient ring of the DGA, which is $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ in [14]. In fact, when the cobordism is a concordance, Proposition 2.1 can be extended to give a morphism of DGAs over $\mathbb{Z}[t, t^{-1}]$ where the coefficients are lifted to $\mathbb{Z}[t, t^{-1}]$ as in Etnyre, Ng and Sabloff [19]. As noted in [14], a proof over $\mathbb{Z}$ (or $\mathbb{Z}[t, t^{-1}]$) would entail a consideration of orientations of moduli spaces. However, working mod 2 and lifting Proposition 2.1 to DGAs over $\mathbb{F}[t, t^{-1}]$ simply entails choosing base points on both ends of the concordance, joining the base points by a path on the concordance cylinder, and keeping track of intersections of boundaries of holomorphic disks with this path. We omit the details of the proof here.

Remark 2.3 (Grading) The extent to which the morphism ϕ in Proposition 2.1 preserves the grading in the DGAs depends on the Maslov index of the Lagrangian cobordism L, defined to be the gcd of the Maslov indices of all closed curves in L, including Λ_- and Λ_+. If L is oriented, then ϕ preserves grading mod 2; if L is unoriented, then ϕ may not preserve the grading at all. There is no reason in general that ϕ needs to preserve the full $\mathbb{Z}$ grading, even if Λ_- and Λ_+ have rotation number 0. However, we will be particularly interested below in the special case where Λ_- is the standard Legendrian unknot U. In this case, since U can be filled in with a Lagrangian disk D and curves in L are null-homotopic in $L \cup D$, ϕ does preserve the full $\mathbb{Z}$ grading.

2.2 Solid torus knots and concordance

Our obstructions rely on considering satellites of Legendrian knots. We begin by reviewing Legendrian solid torus knots and the Legendrian satellite construction from

the second author and Traynor [37]. We identify the open solid torus $S^1 \times \mathbb{R}^2$ as the 1–jet space of the circle, ie as $J^1(S^1) \cong T^*S^1 \times \mathbb{R}$, which equips it with a natural contact form $\alpha = dz - y\,dx$; here, x and y are the base and fiber coordinates on T^*S^1, and z is the $\mathbb{R}$–coordinate. Just as in the case of $\mathbb{R}^3 \cong J^1(\mathbb{R})$, we can recover a Legendrian knot from its front projection onto $S^1_x \times \mathbb{R}_z$, which, in practice, is drawn by representing S^1 as an interval and identifying its endpoints.

Given a Legendrian companion knot $\Lambda \subset \mathbb{R}^3$ and a Legendrian pattern knot $\Lambda' \subset J^1(S^1)$, the contact neighborhood theorem (see eg Geiges [24, Corollary 2.5.9]) says that Λ has a standard neighborhood $N(\Lambda)$, and the zero section in $J^1(S^1)$ has a neighborhood $N(0)$, for which there is a contactomorphism $\varphi\colon N(0) \xrightarrow{\sim} N(\Lambda)$, and we define the *Legendrian satellite* $S(\Lambda, \Lambda') \subset \mathbb{R}^3$ to be the image $\varphi(\Lambda')$ (after suitably rescaling Λ' to lie inside $N(0)$). We remark that this requires a choice of framing for $N(\Lambda)$, which we fix to be the contact (Thurston–Bennequin) framing.

We can produce a front projection of $S(\Lambda, \Lambda')$ as follows. If the front projection of L intersects the ends of the S^1 interval in n points, then we produce a front for the n–*copy* of Λ by taking n copies of the front for Λ and shifting each one a very small distance in the z–direction. (Topologically, the n–copy is the $(n, n \cdot \mathrm{tb}(\Lambda))$–cable of Λ, in which the first coordinate denotes the longitudinal winding.) We then take a point where Λ is oriented from left to right, cut the front open along the n–copy of that point, and insert the front diagram for Λ'. See Figure 1 for an example satellite in which the companion Λ is a right handed trefoil with $(\mathrm{tb}, r) = (1, 0)$ and the pattern Δ_2 is a positive half twist on two strands, whose name is borrowed from work of the second author and Rutherford [36]. Topologically, $S(\Lambda, \Delta_2)$ is the $(2, 3)$–cable of the right handed trefoil.
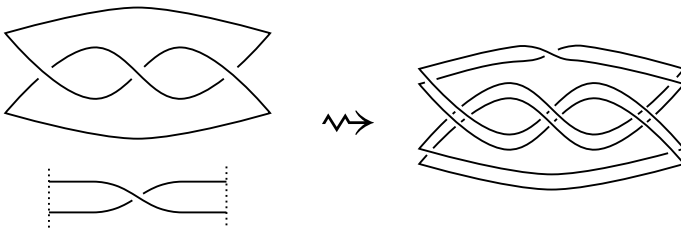

Figure 1: Using the companion Λ and pattern Δ_2 to produce the satellite $S(\Lambda, \Delta_2)$.

Our main theorem says that Lagrangian concordance is preserved by Legendrian satellite operations. This is an analogue of a well-known fact in classical knot concordance, and indeed, the proofs are nearly identical once we observe that Lagrangian cylinders have standard neighborhoods.

Theorem 2.4 *Let Λ' be a Legendrian knot in $J^1(S^1)$. If Λ_- and Λ_+ are Legendrian knots in $\mathbb{R}^3$ such that $\Lambda_- \prec \Lambda_+$, then $S(\Lambda_-, \Lambda') \prec S(\Lambda_+, \Lambda')$.*

Proof Let L be a Lagrangian concordance from Λ_- to Λ_+ in the symplectization $\mathbb{R} \times \mathbb{R}^3$, and restrict to some $[-T, T] \times \mathbb{R}^3$ such that L is a product cylinder outside this region. If $\Lambda'_- \subset J^1(S^1)$ is the core of the solid torus given by $y = z = 0$, then Λ'_- is Legendrian, and so the cylinder $[-T, T] \times \Lambda'_- \subset [-T, T] \times J^1(S^1)$ is also Lagrangian. The Weinstein neighborhood theorem thus provides a symplectomorphism $\varphi \colon N([-T, T] \times \Lambda'_-) \xrightarrow{\sim} N(L)$ between neighborhoods of the two Lagrangian cylinders, and since $[-T, T]$ is compact we can isotope Λ' close enough to Λ'_- to ensure that the Lagrangian $[-T, T] \times \Lambda'$ lies inside $N([-T, T] \times \Lambda'_-)$.

Choosing the Thurston–Bennequin framing on Λ_- for the neighborhood of $\{-T\} \times \Lambda_-$ in $\{-T\} \times \mathbb{R}^3$, it follows that $\varphi(\{-T\} \times \Lambda')$ is the Legendrian satellite $S(\Lambda_0, \Lambda')$ and that $\varphi([-T, T] \times \Lambda')$ is a Lagrangian cylinder. Its restriction to $\{T\} \times \mathbb{R}^3$ is a Legendrian satellite of Λ_1, and since $\Lambda_- \prec \Lambda_+$ implies that $\mathrm{tb}(\Lambda_-) = \mathrm{tb}(\Lambda_+)$ by [8], this satellite is also tb–framed; hence it is $S(\Lambda_1, \Lambda')$. We conclude, after gluing on cylindrical ends, that $\varphi([-T, T] \times \Lambda')$ is the desired concordance. $\square$

2.3 $A_{(2)}$–compatibility and the Legendrian unknot

In this section, we examine a particular obstruction to the existence of Lagrangian concordances. Following the second author and Rutherford [36], we say that a Legendrian knot Λ is $A_{(2)}$–*compatible* if the satellite $S(\Lambda, \Delta_2)$ admits a normal ruling or, equivalently, if it admits an augmentation, ie a DGA morphism $\mathcal{A}_{S(\Lambda, \Delta_2)} \to \mathbb{F} = \mathbb{Z}/2\mathbb{Z}$; see Fuchs [22], Fuchs and Ishkhanov [23], and Sabloff [43] for the proof of equivalence. The standard Legendrian unknot U is not $A_{(2)}$–compatible. This follows since the satellite $S(U, \Delta_2)$ is a topological unknot with $\mathrm{tb} = -3$, and is hence a stabilization.

Theorem 2.5 *Let Λ be a Legendrian knot. If $\Lambda \prec U$ then Λ is not $A_{(2)}$–compatible.*

Proof If $\Lambda \prec U$, then Theorem 2.4 says that $S(\Lambda, \Delta_2) \prec S(U, \Delta_2)$, and then Proposition 2.1 provides a morphism $\mathcal{A}_{S(U, \Delta_2)} \to \mathcal{A}_{S(\Lambda, \Delta_2)}$ between the Legendrian contact homology DGAs of the two satellites over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. If Λ were $A_{(2)}$–compatible, then composing this morphism with an augmentation $\mathcal{A}_{S(\Lambda, \Delta_2)} \to \mathbb{F}$ would give an augmentation of $S(U, \Delta_2)$, which does not exist. $\square$

Remark 2.6 We know from [36] that Λ is $A_{(2)}$–compatible if and only if the DGA $\mathcal{A}_\Lambda$ defined over $\mathbb{F}[t, t^{-1}]$ has a 2–dimensional representation sending t to $\begin{pmatrix} p & 1 \\ 1 & 0 \end{pmatrix}$ for some p. Since $\mathcal{A}_U$ has no such representations, Theorem 2.5 also follows from the

extension of Proposition 2.1 to $\mathbb{F}[t, t^{-1}]$ coefficients. This can be reworked to give an obstruction to concordance to U that does not explicitly mention $A_{(2)}$–compatibility. For example, for the $\overline{9_{46}}$ knot Λ considered by Chantraine [9] (see Example 2.8 below), $\Lambda \not\prec U$ since it can be checked that the DGA for Λ over $\mathbb{F}[t, t^{-1}]$ has a 2–dimensional representation sending t to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (and in fact another sending t to $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$), while the DGA for U does not.

In order to apply Theorem 2.5, it is convenient to have sufficient conditions for $A_{(2)}$–compatibility. The second author and Rutherford [36, Theorem 5.4] showed that if the ruling polynomial $R_\Lambda(z)$ has positive degree, then Λ is $A_{(2)}$–compatible.[1] The following result is similar but allows for $R_\Lambda(z)$ to be a constant as well.

Theorem 2.7 *If Λ has at least two normal rulings, then Λ is $A_{(2)}$–compatible, and thus $\Lambda \not\prec U$.*

Proof We will use two distinct rulings ρ_1 and ρ_2 of Λ to produce an explicit ruling of $S(\Lambda, \Delta_2)$. First, note that ρ_1 and ρ_2 induce a ruling of the 2–copy of Λ whose switches are the switches of ρ_1 in the first copy and the switches of ρ_2 in the second copy. We will construct a ruling of $S(\Lambda, \Delta_2)$ that agrees with this 2–copy ruling away from a neighborhood of the crossing in Δ_2.
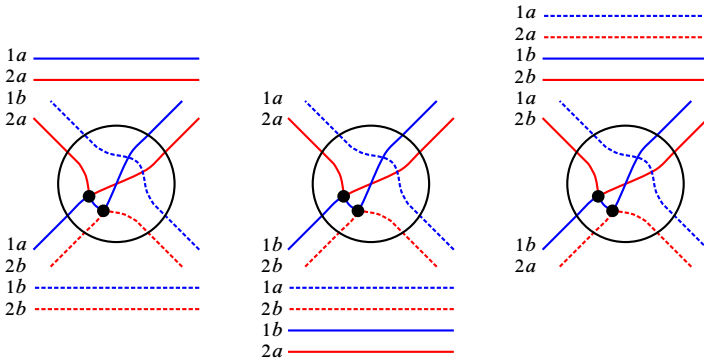

Figure 2: The switches placed at the 2–copy of c in the proof of Theorem 2.7. Strands with the same number and letter are paired, and the number indicates whether a strand corresponds to ρ_1 or to ρ_2.

Let c be the rightmost crossing of a front diagram for Λ where ρ_1 and ρ_2 differ, and label the rulings so that ρ_1 does not have a switch at c but ρ_2 does. The 2–copy of

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[1]The convention in [36] is that a ruling ρ with s switches and c right cusps contributes z^{s-c} to the ruling polynomial, so their condition is $\deg R_\Lambda(z) \geq 0$. We instead use the convention z^{s-c+1} of Rutherford [42], so that the 2–graded and ungraded ruling polynomials match the appropriate coefficients of the HOMFLY-PT and Kauffman polynomials, respectively.

the given front has four crossings for every crossing in the front of Λ; we place the half-twist Δ_2 inside the four crossings corresponding to c along the 2–copy of the undercrossing strand. At all other crossings of Λ, we place switches as in the 2–copy ruling: a switch at the corresponding north crossing of the 2–copy if ρ_1 has a switch, and likewise for the south crossing if ρ_2 has a switch.

At the 2–copy of the distinguished crossing c, we place switches at the south and west of these four crossings. This is illustrated in Figure 2 in three different cases. It then remains to check that the set of switches we have described provides a normal ruling of $S(\Lambda, \Delta_2)$. Indeed, outside of a neighborhood of c, this agrees with the 2–copy ruling, while inside the neighborhood, an inspection of Figure 2 shows that the depicted switches are normal. $\qquad\square$

Example 2.8 In Figure 3, we display two different rulings of a Legendrian knot Λ of topological type $\overline{9_{46}}$ and the corresponding ruling of $S(\Lambda, \Delta_2)$. (Here, $\overline{9_{46}}$ denotes the mirror of 9_{46}; see Remark 4.5.) By Theorem 2.7, it follows that there is no Lagrangian concordance from Λ to U, reproving the main result of Chantraine [9].
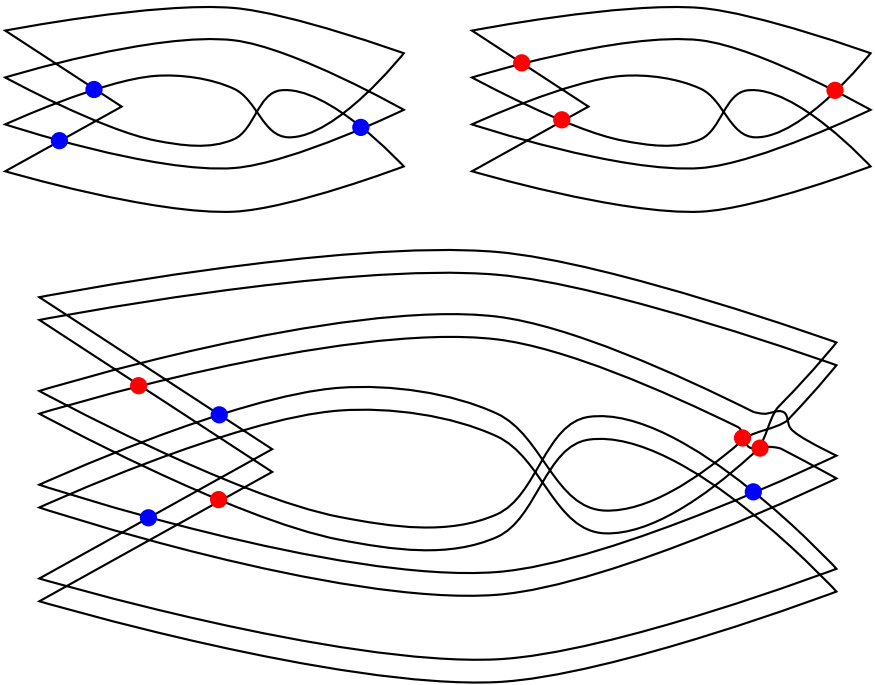
Figure 3: Constructing a normal ruling of $S(\Lambda, \Delta_2)$ from two different rulings of Λ.

Corollary 2.9 *If $\Lambda \prec U$, then Λ has ungraded ruling polynomial $R_\Lambda(z)$ equal to either 0 or 1. In particular, if $U \prec \Lambda$ as well, then the d–graded ruling polynomial of Λ is equal to 1 for all d.*

Proof Theorem 2.5 says that Λ is not $A_{(2)}$–compatible, so by [36, Theorem 5.4], the ungraded ruling polynomial $R_\Lambda(z)$ must have degree 0. If its constant term is at least 2, then it has two or more rulings, which is ruled out by Theorem 2.7, so the only remaining possibilities are 0 and 1.

If we also have $U \prec \Lambda$, then the morphism $\mathcal{A}_\Lambda \to \mathcal{A}_U \cong \mathbb{F}$ induced by such a concordance is by definition an augmentation of Λ. Thus Λ has at least one graded normal ruling, and any other d–graded ruling would also be a second ungraded ruling of Λ, which cannot exist. □

We can now generalize the $\overline{9_{46}}$ example of [9] and Example 2.8 by providing an infinite family of irreversible Lagrangian concordances. The simplest of these is a Legendrian $\overline{14n_{15581}}$ knot, which will appear later in Table 1.

Theorem 2.10 *There are infinitely many Legendrian knots Λ such that $U \prec \Lambda$ but $\Lambda \not\prec U$.*
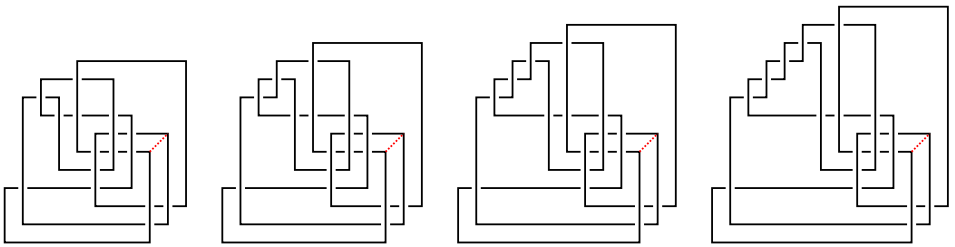
Figure 4: Grid diagrams for the first four members of an infinite family of Lagrangian slice knots, beginning with $\Lambda_1 = \overline{14n_{15581}}$.

Proof We can form an infinite family $\Lambda_1, \Lambda_2, \ldots$ of Legendrian knots by adding half-twists to the $\overline{14n_{15581}}$ knot of Table 1, as shown in Figure 4. Each Λ_n is Lagrangian slice: surgering along the dotted line indicated in the figure produces a two-component Legendrian unlink, and we cap off one component to get a concordance from the Legendrian unknot to Λ_n, so $U \prec \Lambda_n$. However, each Λ_n admits at least two normal rulings, as illustrated in Figure 5 for Λ_3; we generalize to all Λ_n by placing a switch at every one of the added half-twists and by using the same set of switches as in either ruling of Figure 5 away from the half-twists. We conclude that $\Lambda_n \not\prec U$ for all n. □
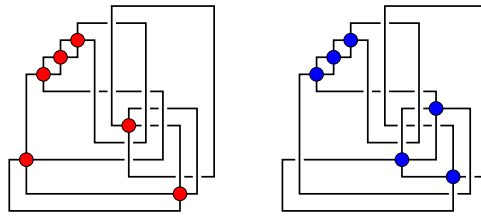
Figure 5: Two rulings of the third member Λ_3 of the family of Figure 4.

2.4 Satellites that fix the Legendrian unknot U

If Λ' is a Legendrian pattern for which $S(U, \Lambda')$ is Legendrian isotopic to U, then applying Theorem 2.4 to any concordance $U \prec \Lambda$ or $\Lambda \prec U$ tells us that $U \prec S(\Lambda, \Lambda')$ or $S(\Lambda, \Lambda') \prec U$, respectively. In this subsection we will investigate a family of such patterns.
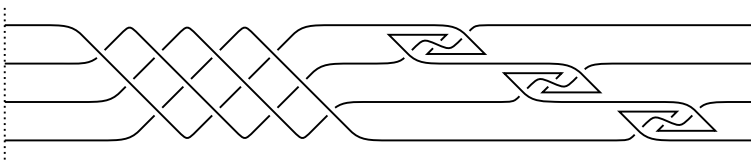

Figure 6: The Legendrian solid torus knot P_n, shown here for $n = 4$.

Let P_n denote the knot in $J^1(S^1)$ depicted in Figure 6, where n is the winding number of P_n around the solid torus. We can describe P_n as the concatenation of a full positive twist tw_n on n strands and a cascade of $n-1$ clasps, where the ith clasp connects the ith and $(i+1)^{\text{st}}$ strands as numbered from top to bottom.

Lemma 2.11 *The satellite $S(U, P_n)$ is Legendrian isotopic to U for all $n \geq 1$.*

Proof The lemma is true for $n = 1$ by inspection, and a straightforward computation shows that $\mathrm{tb}(S(U, P_n)) = -1$, so it suffices to check that $S(U, P_n)$ is topologically isotopic to $S(U, P_{n-1})$ for all $n \geq 2$. This is illustrated in Figure 7: the highlighted portion of $S(U, P_n)$ can be pushed back through the middle of the satellite, lifted behind it, and then twisted to remove the top clasp, leaving $S(U, P_{n-1})$. □

Theorem 2.12 *The d–graded ruling polynomials of $S(\Lambda, P_n)$ and $S(\Lambda, \mathrm{tw}_n)$ are related by*

$$R^d_{S(\Lambda, P_n)}(z) = z^{n-1} \cdot R^d_{S(\Lambda, \mathrm{tw}_n)}(z)$$

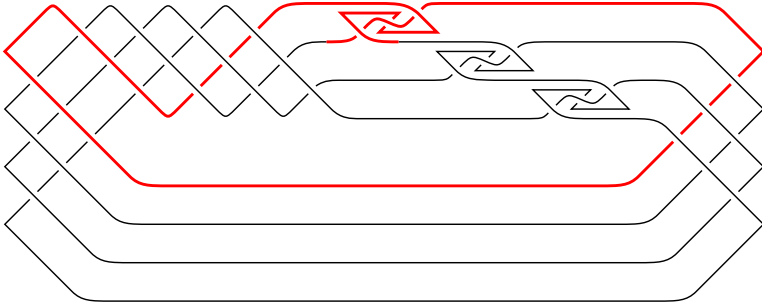*for any Legendrian knot Λ and any $d \mid 2r(\Lambda)$.*

Figure 7: The satellite $S(U, P_4)$ is isotopic to $S(U, P_3)$, as can be seen by moving the highlighted portion appropriately to simplify the knot diagram.

Proof Each of the $n-1$ clasps in P_n contains three crossings and two right cusps, and the gradings of these crossings are $0 \pmod{2r(\Lambda)}$ since the Maslov potentials on parallel strands coming from adjacent copies of Λ differ by $2r(\Lambda)$. In a clasp, each of the two right cusps is connected to a left cusp by a curve that participates in no crossings. Hence these left and right cusps must be paired by a normal ruling of $S(\Lambda, P_n)$, which, therefore, must have switches at all three crossings. Once we remove the pairs of short strands incident to these cusps from the ruling, then what remains is a normal ruling of $S(\Lambda, \mathrm{tw}_n)$. Conversely, any ruling of $S(\Lambda, \mathrm{tw}_n)$ gives rise to a ruling of $S(\Lambda, P_n)$ in this fashion, and so there is a bijection between the sets of rulings of the two satellites.

Let ρ_{tw} be a ruling of $S(\Lambda, \mathrm{tw}_n)$ with s switches, and suppose that $S(\Lambda, \mathrm{tw}_n)$ has c right cusps. Then ρ_{tw} contributes z^{s-c+1} to its ruling polynomial. The above bijection pairs ρ_{tw} with some ruling ρ_P of $S(\Lambda, P_n)$ which has $s + 3(n-1)$ switches, and $S(\Lambda, P_n)$ has $c + 2(n-1)$ right cusps, so ρ_P contributes

$$z^{(s+3(n-1))-(c+2(n-1))+1} = z^{n-1} \cdot z^{s-c+1}$$

to the ruling polynomial of $S(\Lambda, P_n)$. Since this is true for all rulings ρ_{tw} of $S(\Lambda, P_n)$, the ruling polynomials of the two satellites differ by a factor of z^{n-1} as desired. $\qquad\square$

Theorem 2.13 *Let Λ be a Legendrian knot such that $U \prec \Lambda \prec U$. Then $S(\Lambda, \mathrm{tw}_n)$ has d–graded ruling polynomial $R^d_{S(\Lambda, \mathrm{tw}_n)}(z) = z^{1-n}$ for all $n \geq 1$ and all d.*

Proof Taking satellites $S(\cdot, P_n)$ and applying Theorem 2.4 gives $U = S(U, P_n) \prec S(\Lambda, P_n)$ and $S(\Lambda, P_n) \prec S(U, P_n) = U$. This implies that $r(S(\Lambda, P_n)) = r(U) = 0$; hence from Corollary 2.9, we must have $R^d_{S(\Lambda, P_n)}(z) = 1$ for all d, and the conclusion about $R^d_{S(\Lambda, \mathrm{tw}_n)}(z)$ is a consequence of Theorem 2.12. $\qquad\square$

Remark 2.14 Under the hypotheses of Theorem 2.13, the satellite $S(\Lambda, \mathrm{tw}_n)$ has exactly one ruling for each n. We can construct such a ruling as the n–copy of the unique ruling of Λ by taking each switch of that ruling and placing switches among the corresponding n^2 crossings of $S(\Lambda, \mathrm{tw}_n)$ at exactly the n crossings where both strands belong to the same component. Thus if $U \prec \Lambda$, then we can prove $\Lambda \not\prec U$ by exhibiting a single ruling of some $S(\Lambda, \mathrm{tw}_n)$ (or even of the n–copy of Λ) that is not the n–copy of a ruling of Λ.

Since a knot Λ for which $U \prec \Lambda \prec U$ must have Thurston–Bennequin invariant -1, the n–copy of Λ is topologically the $(n, -n)$–cable of Λ. We obtain $S(\Lambda, \mathrm{tw}_n)$ from the n–copy by inserting a full positive twist, so the Legendrian satellite $S(\Lambda, \mathrm{tw}_n)$ is topologically the $(n, 0)$–cable of Λ, ie the Seifert-framed n–stranded cable of Λ. We will denote this n–cable by Λ_n.

In the cases $d = 1$ and $d = 2$, Rutherford [42] showed that the d–graded ruling polynomials of a Legendrian link Λ of topological type K are determined by its Kauffman polynomial $F_K(a, z)$ (the Dubrovnik version) and its HOMFLY-PT polynomial $P_K(a, z)$. More precisely, these polynomials both have maximum a–degree at most $-\overline{\mathrm{tb}}(K) - 1$ (see Rudolph [40], Franks and Williams [21], and Morton [32]), and in fact, he proved that $R_\Lambda^1(z)$ and $R_\Lambda^2(z)$ are the coefficients of $a^{-\mathrm{tb}(\Lambda)-1}$ in $F_K(a, z)$ and $P_K(a, z)$, respectively. We can compute that $\mathrm{tb}(S(\Lambda, \mathrm{tw}_n)) = n^2(\mathrm{tb}(\Lambda) + 1) - n$, so if $U \prec \Lambda \prec U$, then $\mathrm{tb}(S(\Lambda, \mathrm{tw}_n)) = -n$. Therefore, we have proved the following.

Theorem 2.15 *If a smooth knot K has a Legendrian representative Λ such that $U \prec \Lambda \prec U$, and K_n denotes the topological n–cable of K, then the HOMFLY-PT and Kauffman polynomials $P_{K_n}(a, z)$ and $F_{K_n}(a, z)$ both have maximal a–degree equal to $n - 1$ and a^{n-1}–coefficient equal to z^{1-n} for all $n \geq 1$.*

Remark 2.16 The conclusion of Theorem 2.15 depends only on the smooth knot type, so if the theorem can be used to prove that $\Lambda \not\prec U$ for one Legendrian representative Λ of K with $U \prec \Lambda$, then it proves $\Lambda \not\prec U$ for all such representatives.

3 Concordances to the unknot

We say that a Lagrangian concordance $\Lambda_0 \prec \Lambda_1$ is *decomposable* if it can be built as a composition of elementary moves, namely isotopies, minimum cobordisms, and saddle cobordisms in the language of Ekholm, Honda and Kálmán [14, Section 6]. Our goal in this section is to prove that Λ_1 can never be a topological unknot in such a concordance; the proof will make no use of any contact topology, proceeding instead

by considering branched double covers of knots. We thus begin with the following theorem, which may be of independent interest.

Theorem 3.1 Let $\Sigma(K)$ denote the double cover of S^3 branched over the knot K. Then the fundamental group of $\Sigma(K)$ admits a nontrivial representation

$$\rho: \pi_1(\Sigma(K)) \to \mathrm{SO}(3)$$

if and only if K is not an unknot.

Proof In the case where K is the unknot, this is immediate since $\Sigma(K) \cong S^3$ is simply connected.

For nontrivial K, a theorem of Kronheimer and Mrowka [31, Corollary 7.17] says that, given a meridian μ of K, there is an irreducible homomorphism $\varphi: \pi_1(S^3 \setminus K) \to \mathrm{SU}(2)$ such that $\varphi(\mu) = i$. (Here, we view $\mathrm{SU}(2)$ as the unit quaternions.) Since $\pi_1(S^3 \setminus K)$ is normally generated by μ, it has a unique normal subgroup N of index 2, namely the kernel of the composition of the abelianization and mod 2 reduction maps $\pi_1(S^3 \setminus K) \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, corresponding to the double cover of $S^3 \setminus K$. This subgroup contains μ^2, and in fact, $\pi_1(\Sigma(K)) = N/\langle \mu^2 \rangle$, so we wish to use φ to produce a map $N/\langle \mu^2 \rangle \to \mathrm{SO}(3)$.

Consider the composition $\widetilde{\varphi}: \pi_1(S^3 \setminus K) \to \mathrm{SO}(3)$ of φ with the quotient $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Since $\varphi(\mu^2) = i^2 = -1$ in $\mathrm{SU}(2)$, it follows that $\widetilde{\varphi}(\mu^2) = 1$ in $\mathrm{SO}(3)$, and so $\widetilde{\varphi}|_N$ descends to a map

$$\rho: \pi_1(\Sigma(K)) = N/\langle \mu^2 \rangle \to \mathrm{SO}(3).$$

We need to check that ρ is nontrivial. But since $\pi_1(S^3 \setminus K)$ is generated by meridians of K, and since φ is irreducible, there must be some meridian ν such that $\varphi(\nu) \neq \pm i$. The product $\mu\nu$ lies in N, and since $\varphi(\nu) \neq \pm i$, we cannot have $\varphi(\mu\nu) = \pm 1$, so $\rho(\mu\nu) \neq 1$ as desired. $\square$

Theorem 3.2 There are no decomposable Lagrangian concordances of the form $\Lambda \prec \widetilde{U}$, where Λ is a topologically nontrivial Legendrian knot and $\widetilde{U}$ is any topologically unknotted Legendrian knot.

Proof Given such a concordance L, we can take a front diagram for $\widetilde{U}$ and perform a sequence of pinch moves, isotopies, and capping off tb $= -1$ unknots to produce a front diagram for K, where K is the smooth knot type of Λ. Since L has Euler characteristic zero, there must be exactly as many pinch moves as there are capping moves; call this number n. Moreover, we can postpone any capping moves until the end

by taking tb $= -1$ unknots U which are about to be capped off and instead isotoping them far away from the rest of the diagram so that they no longer interact with any other components. Thus we can suppose that there is a sequence of Legendrian isotopies and n pinch moves which transforms a front diagram for $\widetilde{U}$ into a front diagram for $\Lambda \sqcup nU$.

A pinch move can be performed by taking a small ball B^3 with two unknotted arcs passing through it and replacing those arcs with a different pair of unknotted arcs. The branched double cover of B^3 over either pair of arcs is a solid torus, so pinch moves correspond to Dehn surgeries on the branched double cover. It follows that there must be an n–component link $L = L_1 \cup \cdots \cup L_n$ in $\Sigma(\widetilde{U}) = S^3$ upon which some Dehn surgery produces $\Sigma(K \sqcup nU) = \Sigma(K) \# n(S^1 \times S^2)$. Since $H_1(\Sigma(K) \# n(S^1 \times S^2)) = H_1(\Sigma(K)) \oplus \mathbb{Z}^n$ is presented by the $n \times n$ framing matrix of L, we must have $H_1(\Sigma(K)) = 0$, and the framing matrix must be identically zero. In particular, we must perform 0–surgery on each L_i.

We can now build a 4–dimensional handlebody X with boundary $\Sigma(K)$ by taking B^4, attaching n 0–framed 2–handles to its boundary along L to produce $\Sigma(K) \# n(S^1 \times S^2)$ on the boundary, and then attaching n 3–handles to eliminate each of the $S^1 \times S^2$ summands. Then X does not have any 1–handles, and an easy exercise shows that X is contractible since $H_1(\Sigma(K)) = 0$. Following an argument from Kirby [30, Section I.3], we now turn X upside down to construct it by attaching n 1–handles, n 2–handles, and a 4–handle to $\Sigma(K)$, and thus we see that the trivial group can be constructed from $\pi_1(\Sigma(K))$ by adding n generators and n relations. But this is ruled out by a theorem of Gerstenhaber and Rothaus [25, Theorem 3] since $\pi_1(\Sigma(K))$ admits a nontrivial map into the compact connected Lie group SO(3) by Theorem 3.1, so the claimed decomposable concordance cannot exist. $\qquad\square$

Since Legendrian representatives of the unknot are uniquely determined by their classical invariants tb and r, as proved by Eliashberg–Fraser [16], it follows that there can only be a decomposable Lagrangian concordance $\Lambda \prec \widetilde{U}$ if Λ is Legendrian isotopic to $\widetilde{U}$. In fact, the proof of Theorem 3.2 establishes more.

Corollary 3.3 *If L is a smooth concordance from a nontrivial knot to the unknot in $\mathbb{R} \times S^3$, then L must have a local maximum with respect to the $\mathbb{R}$–coordinate.*

Proof Suppose not; then the $\mathbb{R}$–coordinate can be C^2–approximated by a Morse function on L with no local maxima, providing a handle decomposition of L consisting only of 0–handles and 1–handles. The proof of Theorem 3.2 only used the fact that capping and pinch moves correspond to smooth 0–handles and 1–handles, so by repeating the same argument, we have a contradiction. $\qquad\square$

One might be tempted to conjecture that all Lagrangian concordances are decomposable, but we believe that this is not the case. If $W \subset J^1(S^1)$ denotes the Legendrian Whitehead double pattern W_0 of [37, Figure 23] and U is the standard Legendrian unknot as usual, then $S(U, W)$ is the Legendrian right-handed trefoil T with tb $= 1$; see Figure 8.
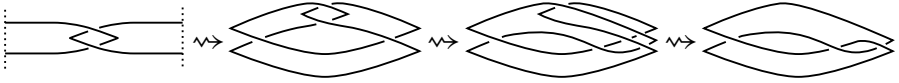

Figure 8: The satellite $S(U, W)$ is a right-handed trefoil with tb $= 1$.

Letting Λ be the Legendrian $\overline{9_{46}}$ knot of Example 2.8 for which $U \prec \Lambda$, we apply Theorem 2.4 to construct a Lagrangian concordance C from T to $S(\Lambda, W)$; these knots are displayed in Figure 9. Topologically, $S(\Lambda, W)$ is the positively clasped, -1–twisted Whitehead double of $\overline{9_{46}}$.
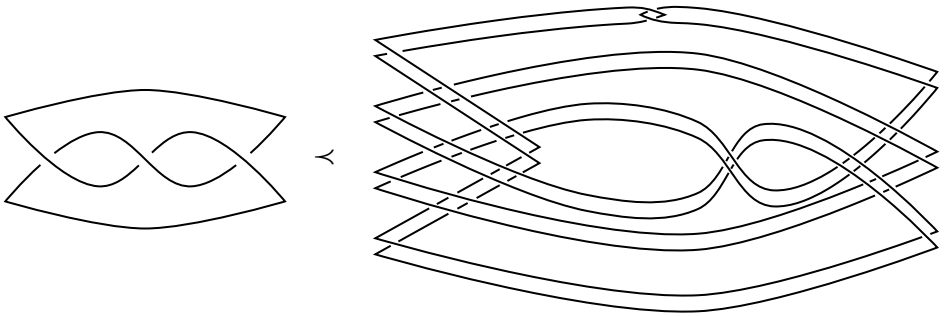
Figure 9: There is a Lagrangian concordance from T to $S(\Lambda, W)$.

Conjecture 3.4 *Let C be the Lagrangian concordance constructed above from T to $S(\Lambda, W)$. Then C is not decomposable.*

We believe that this conjecture may hold more generally if Λ is replaced with any other topologically nontrivial Legendrian knot with $U \prec \Lambda$.

4 A census of Lagrangian slice knots

Recall that a Legendrian knot Λ is said to be *Lagrangian slice* if it bounds a Legendrian disk in B^4 or, equivalently, if $U \prec \Lambda$ where U is the standard unknot. In this section, we provide a complete list of nontrivial topological knot types through 14 crossings that have Legendrian representatives that are Lagrangian slice. Our data support the following conjecture:

Conjecture 4.1 *If K is smoothly slice and $\overline{\mathrm{tb}}(K) = -1$, then K bounds a Lagrangian disk in B^4.*

Note that the converse of this conjecture is true.

Our results are summarized in Table 1, which provides a list of smoothly slice knots K with up to 14 crossings such that $\overline{\mathrm{tb}}(K) = -1$, along with a Legendrian representative that bounds a Lagrangian disk; this table verifies Conjecture 4.1 when K has up to 14 crossings.

In Section 4.1 and Section 4.2, we show that nontrivial alternating knots cannot be Lagrangian slice and then rule out all but 23 nonalternating knot types of crossing numbers ≤ 14. In Section 4.3, for each of the remaining 23 knot types, Table 1 gives a Legendrian representative Λ with $U \prec \Lambda$, completing the census through 14 crossings. We conclude by discussing in Section 4.4 the extent to which we can establish $\Lambda \nprec U$ for these knots.

4.1 Alternating knots

We will first show that no Lagrangian slice knot can be alternating unless it is an unknot, using the following description, by the second author, of $\overline{\mathrm{tb}}(K)$ for an alternating K.

Proposition 4.2 [35, page 1646] *Let L be an oriented, alternating, nonsplit link, and D a reduced alternating diagram for L with $n_-(D)$ negative crossings. Then*

$$\overline{\mathrm{tb}}(L) = \sigma(L) - n_-(D) - 1,$$

where $\sigma(L)$ denotes signature, normalized so that the right-handed trefoil has signature $+2$.

Theorem 4.3 *An alternating knot K bounds an oriented Lagrangian surface Σ in the standard symplectic 4–ball if and only if K is also a positive knot, in which case its Seifert, smooth, and topological 4–ball genera and the genus of Σ are all equal to $\sigma(K)/2$.*

Proof The "if" direction is due to Hayden and Sabloff [27], who showed that all positive links bound exact Lagrangian surfaces. Conversely, suppose that K bounds a Lagrangian surface Σ. Chantraine [8] showed that $\overline{\mathrm{tb}}(K) = 2g(\Sigma) - 1$ and $g_s(K) = g(\Sigma)$, where g_s denotes the smooth slice genus. Now Proposition 4.2 tells us that

$$2g(\Sigma) - 1 = \overline{\mathrm{tb}}(K) = \sigma(K) - n_-(D) - 1$$

for a reduced alternating diagram D of K; hence $\sigma(K) = 2g_s(K) + n_-(D)$. We apply Murasugi's bound $|\sigma(K)| \le 2g_4(K)$ [33], with $g_4(K)$ the topological 4–ball genus, to get

$$2g_s(K) + n_-(D) = \sigma(K) \le 2g_4(K) \le 2g_s(K),$$

and hence each inequality is an equality, implying that $n_-(D) = 0$ and $g_s(K) = g_4(K) = g(\Sigma)$ as desired. The claim that $g(K) = g_s(K)$ follows from Rasmussen [39, Theorem 4] because K is positive. $\qquad\square$

This gives a new, contact geometric proof of the result of Nakamura [34] that a reduced alternating diagram of a positive knot has no negative crossings.

Corollary 4.4 *Nontrivial alternating knots are not Lagrangian slice.*

Proof By Theorem 4.3, if K is alternating and Lagrangian slice, then $g(K) = 0$. $\square$

4.2 Knots with at most 14 crossings

By Corollary 4.4, to enumerate Lagrangian slice knots, it suffices to restrict to nonalternating knots. Here we narrow the list of candidates with up to 14 crossings to a list of 23, which we then show are all Lagrangian slice in Table 1 in Section 4.3.

4.2.1 Up to 12 crossings If a Legendrian knot Λ of smooth type K bounds a Lagrangian disk, then it must have $\mathrm{tb} = -1$ (see [8]) and be smoothly slice, and the latter implies $\overline{\mathrm{tb}}(K) = -1$ by the slice-Bennequin inequality $\mathrm{tb}(\Lambda) + |r(\Lambda)| \le 2g_s(K) - 1$, proved by Rudolph [41]. According to KnotInfo [7], there are exactly six nontrivial knot types of at most 12 crossings that are smoothly slice and have $\overline{\mathrm{tb}} = -1$, namely

$$\overline{9_{46}}, \overline{10_{140}}, 11n_{139}, 12n_{582}, \overline{12n_{768}}, 12n_{838}.$$

all of which are Lagrangian slice.

Remark 4.5 A note on chirality: We use $\overline{K}$ to denote the mirror of K. There is some discrepancy in the literature over which of each mirror pair is specified by a particular numbered knot. Our conventions align with the `KnotTheory` package, available from the Knot Atlas [2], which provides the Rolfsen knot tables for knots through 10 crossings and the Hoste–Thistlethwaite enumeration for 11 crossings and above. We note that this sometimes differs from KnotInfo; for instance, by the Thurston–Bennequin data in KnotInfo, 9_{46} rather than $\overline{9_{46}}$ has $\overline{\mathrm{tb}} = -1$. It may be helpful to note that the Lagrangian slice knots which we have labeled $\overline{9_{46}}, \overline{10_{140}}, 11n_{139}, 12n_{582}$ are unambiguously described as the $P(-m, -3, 3)$ pretzel knots for $m = 3, 4, 5, 6$.

4.2.2 13– and 14–crossing knots We searched in `KnotTheory` ` for nonalternating 13– and 14–crossing knots satisfying the conditions of the following lemma.

Lemma 4.6 *Suppose that K admits a Legendrian representative Λ which is Lagrangian slice. If $\Delta_K(t)$, $P_K(a,z)$, $F_K(a,z)$ and $Kh_K(q,t)$ denote the Alexander, HOMFLY-PT, and Kauffman (Dubrovnik) polynomials of K and the Poincaré polynomial of its Khovanov homology over $\mathbb{Q}$, respectively, then:*

(1) *$\det(K)$ is a perfect square, the signature $\sigma(K)$ is zero, and $\Delta_K(t) = f(t)f(t^{-1})$ for some polynomial f.*

(2) *We have max-deg$_a$ $P_K(a,z) = $ max-deg$_a$ $F_K(a,z) = 0$.*

(3) *If $p(z)$ and $f(z)$ denote the a^0–coefficients of $P_K(a,z)$ and $F_K(a,z)$, then $f(z) \geq p(z) \geq 0$, ie the coefficients of $f - p$ and of p are all nonnegative.*

(4) *We have min-deg$_q$ $Kh_K(q,t/q) \geq -1$.*

(5) *If the Khovanov homology of K has width at most 3, then $Kh_K(q,-q^{-4}) = q + q^{-1}$.*

Proof We know that K must be smoothly slice, that Λ admits a graded ruling, and that $\overline{\mathrm{tb}}(K) = \mathrm{tb}(\Lambda) = -1$. Thus (1) follows from K being topologically slice, and in particular, $\Delta_K(t) = f(t)f(t^{-1})$ is the Fox–Milnor condition [20], and $\det(K) = f(-1)^2$. Item (2) follows from achieving equality in the HOMFLY-PT [21; 32] and Kauffman [40] bounds on $\overline{\mathrm{tb}}(K)$, namely $\overline{\mathrm{tb}}(K) \leq -\,$max-deg$_a$ $P_K(a,z) - 1$, and likewise for $F_K(a,z)$ because Rutherford [42] showed that this equality follows from Λ admitting a ruling. Indeed, the leading coefficients $p(z)$ and $f(z)$ are then the 2–graded and ungraded ruling polynomials of Λ, and every 2–graded ruling is an ungraded ruling, so (3) is an immediate consequence.

Item (4) is the weak Khovanov bound $\overline{\mathrm{tb}}(K) \leq $ min-deg$_q$ $Kh_K(q,t/q)$ of the second author [35]. Finally, we observe that Rasmussen [39, Proposition 5.3] says that knots of width at most 3 satisfy

$$Kh_K(q,t) = q^{s(K)}(q + q^{-1}) + (1 + tq^4)Q_K$$

for some polynomial Q_K. Furthermore, $s(K) = 0$ since K is smoothly slice, and we conclude (5). $\qquad\square$

There are six 13–crossing knot types satisfying all of the conditions of Lemma 4.6, namely

$$\overline{13n_{579}}, \quad \overline{13n_{3158}}, \quad 13n_{3523}, \quad \overline{13n_{4236}}, \quad 13n_{4514}, \quad \overline{13n_{4659}};$$

and 17 such knot types with 14 crossings, namely $14n_m$ for m in

$$\overline{2459}, \quad 2601, \quad 8091, \quad \overline{8579}, \quad \overline{9271}, \quad \overline{12406}, \quad \overline{14251}, \quad 14799, \quad 15489,$$
$$\overline{15581}, \quad 17376, \quad 18212, \quad 21563, \quad \overline{22150}, \quad \overline{22789}, \quad \overline{24246}, \quad 25967.$$

Six of these knot types actually have $\overline{tb}(K) < -1$; we prove this by noting that they do not have arc index at most 10 (see Jin et al. [28]) or 11 (see Jin and Park [29]) and using Gridlink [11] to find grid diagrams of complexity 12 for each of them, which must then be minimal. We then apply Dynnikov and Prasolov's theorem that grid diagrams which minimize complexity maximize the Thurston–Bennequin invariant within a given knot type; see [12]. Specifically, the knot types

$$\overline{13n_{4236}}, \quad 14n_{8091}, \quad \overline{14n_{9271}}, \quad 14n_{14799}, \quad 14n_{15489}, \quad 14n_{17376}$$

all have arc index 12, and in each case we find that $\overline{tb}(K) = -2$.

The remaining knot types are

$$\overline{13n_{579}}, \quad \overline{13n_{3158}}, \quad 13n_{3523}, \quad 13n_{4514}, \quad \overline{13n_{4659}},$$
$$\overline{14n_{2459}}, \quad 14n_{2601}, \quad \overline{14n_{8579}}, \quad \overline{14n_{12406}}, \quad \overline{14n_{14251}}, \quad \overline{14n_{15581}},$$
$$14n_{18212}, \quad 14n_{21563}, \quad \overline{14n_{22150}}, \quad \overline{14n_{22789}}, \quad \overline{14n_{24246}}, \quad 14n_{25967},$$

and in Section 4.3 we verify that these are all Lagrangian slice.

Remark 4.7 Applying Lemma 4.6, we determine that there are at most 48 Lagrangian slice knot types with 15 crossings, namely $15n_m$ for m in
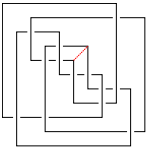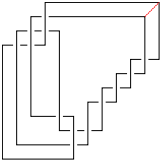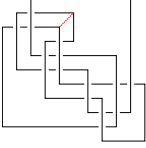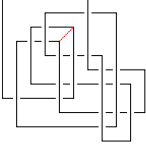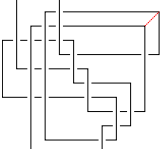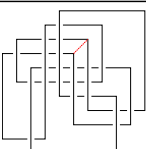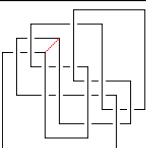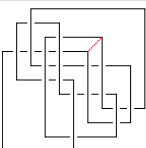
$$\overline{1481}, \quad \overline{11562}, \quad 11847, \quad 11848, \quad 38594, \quad 41697, \quad \overline{43982}, \quad 46734, \quad \overline{46855},$$
$$\overline{57450}, \quad 73973, \quad 77224, \quad \overline{77245}, \quad 77461, \quad 81490, \quad 83506, \quad \overline{83742}, \quad \overline{88825},$$
$$96161, \quad \overline{96452}, \quad \overline{96790}, \quad 103488, \quad \overline{104659}, \quad \overline{110305}, \quad 110461, \quad 112479,$$
$$127845, \quad 127852, \quad \overline{130682}, \quad 131344, \quad 132539, \quad 133913, \quad 134517, \quad 135516,$$
$$136561, \quad 138242, \quad 138810, \quad 139311, \quad \overline{144052}, \quad 145082, \quad 153611, \quad 153975,$$
$$154694, \quad 155137, \quad 155659, \quad 155828, \quad \overline{162371}, \quad \overline{164338}.$$

(The Khovanov homology of $15n_{115646}$ has width 4 but satisfies the weaker condition of Shumakovitch [45, Proposition 2.1], so we can still use the proof of item (5) to show that $s(15n_{115646}) = 2$ and thus rule it out.) However, we do not claim that all of these actually are Lagrangian slice.

4.3 The census

For each of the 23 knot types K described in the previous subsection, Table 1 provides one Legendrian knot Λ (depicted as a grid diagram), marked to indicate where one can perform surgery (a pinch move) to construct a Lagrangian concordance $U \prec \Lambda$. We also note the cases in which we can prove that $\Lambda \not\prec U$; see Section 4.4.

K	Λ	Coordinates	$\Lambda \not\prec U$?
$\overline{9_{46}}$		$X: 1, 6, 7, 5, 3, 4, 2, 8$ $O: 5, 2, 4, 8, 6, 1, 7, 3$	✓
$\overline{10_{140}}$		$X: 1, 2, 8, 6, 7, 4, 5, 3, 9$ $O: 6, 7, 4, 9, 3, 1, 2, 8, 5$	
$11n_{139}$		$X: 1, 8, 9, 7, 3, 4, 2, 6, 5, 10$ $O: 7, 2, 4, 10, 8, 1, 5, 3, 9, 6$	
$12n_{582}$		$X: 1, 2, 10, 8, 9, 4, 5, 3, 7, 6, 11$ $O: 8, 9, 4, 11, 3, 1, 2, 6, 5, 10, 7$	
$\overline{12n_{768}}$		$X: 2, 3, 10, 1, 9, 6, 7, 5, 8, 4$ $O: 6, 9, 7, 8, 5, 10, 4, 2, 3, 1$	✓
$12n_{838}$		$X: 8, 6, 9, 7, 4, 5, 2, 10, 1, 3$ $O: 5, 2, 3, 10, 8, 1, 9, 4, 6, 7$	✓
$\overline{13n_{579}}$		$X: 9, 7, 8, 6, 2, 11, 1, 10, 4, 5, 3$ $O: 4, 10, 1, 9, 7, 5, 6, 3, 11, 2, 8$	✓

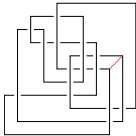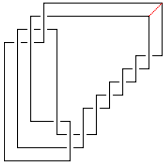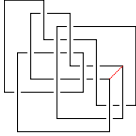| K	Λ	Coordinates	$\Lambda \not\prec U$?
$\overline{13n_{3158}}$		X : 11, 9, 10, 8, 6, 7, 5, 3, 4, 1, 2 O : 3, 1, 7, 2, 9, 4, 8, 6, 11, 5, 10	✓
$13n_{3523}$		X : 1, 10, 11, 9, 3, 4, 2, 6, 5, 8, 7, 12 O : 9, 2, 4, 12, 10, 1, 5, 3, 7, 6, 11, 8	
$13n_{4514}$		X : 2, 10, 11, 4, 9, 8, 6, 1, 7, 3, 5 O : 9, 6, 7, 8, 5, 10, 3, 4, 2, 11, 1	
$\overline{13n_{4659}}$		X : 4, 8, 5, 10, 3, 9, 11, 7, 2, 1, 6 O : 11, 2, 9, 7, 8, 4, 6, 1, 10, 5, 3	
$\overline{14n_{2459}}$		X : 5, 7, 1, 11, 12, 9, 4, 3, 2, 6, 10, 8 O : 9, 12, 10, 2, 8, 6, 7, 1, 5, 3, 4, 11	
$14n_{2601}$		X : 2, 9, 7, 10, 5, 8, 4, 6, 1, 3, 11 O : 8, 6, 1, 2, 11, 3, 9, 10, 5, 7, 4	✓
$\overline{14n_{8579}}$		X : 8, 5, 10, 2, 9, 6, 7, 4, 1, 3, 11 O : 1, 9, 7, 8, 3, 11, 2, 10, 5, 6, 4	✓
$\overline{14n_{12406}}$		X : 8, 6, 11, 2, 10, 1, 3, 9, 5, 7, 4 O : 1, 10, 7, 9, 5, 6, 8, 4, 2, 3, 11	✓
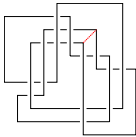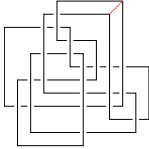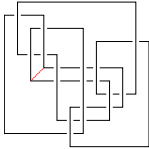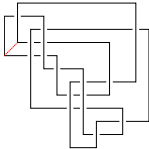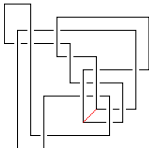
K	Λ	Coordinates	$\Lambda \not\prec U$?
$\overline{14n_{14251}}$		$X : 10, 4, 3, 9, 5, 7, 8, 6, 2, 11, 1$ $O : 5, 2, 8, 4, 11, 10, 1, 9, 7, 3, 6$	
$\overline{14n_{15581}}$		$X : 1, 9, 8, 5, 11, 7, 10, 4, 6, 2, 3$ $O : 4, 2, 10, 9, 6, 3, 5, 8, 1, 7, 11$	✓
$14n_{18212}$		$X : 1, 2, 12, 10, 11, 4, 5, 3, 7, 6, 9, 8, 13$ $O : 10, 11, 4, 13, 3, 1, 2, 6, 5, 8, 7, 12, 9$	
$14n_{21563}$		$X : 4, 1, 10, 11, 9, 6, 7, 8, 5, 2, 3$ $O : 11, 7, 5, 8, 2, 10, 4, 3, 1, 6, 9$	✓
$\overline{14n_{22150}}$		$X : 10, 6, 8, 5, 12, 7, 1, 9, 11, 4, 2, 3$ $O : 4, 1, 2, 11, 9, 10, 8, 6, 3, 12, 5, 7$	✓
$\overline{14n_{22789}}$		$X : 2, 12, 6, 11, 8, 4, 10, 9, 3, 7, 5, 1$ $O : 11, 8, 10, 7, 3, 1, 2, 5, 6, 4, 12, 9$	
$\overline{14n_{24246}}$		$X : 11, 9, 10, 7, 8, 6, 2, 1, 5, 4, 12, 3$ $O : 8, 12, 4, 11, 5, 1, 7, 3, 9, 2, 6, 10$	
$14n_{25967}$		$X : 9, 1, 12, 5, 8, 6, 7, 4, 2, 3, 10, 11$ $O : 12, 10, 2, 1, 11, 9, 3, 8, 5, 6, 4, 7$	

Table 1: Lagrangian slice knots through 14 crossings.

4.4 Nonreversible concordances

Here we study the question of which concordances $U \prec \Lambda$ from the census in Section 4.3 are nonreversible in the sense that $\Lambda \not\prec U$, ie there is a concordance from U to Λ but not vice versa. Our results are summarized in Table 2 below.

K	n	$z^{n-1} R^1_{\Lambda_n}(z)$
$\overline{9_{46}}$	1	2
$\overline{10_{140}}$	> 2	
$11n_{139}$	> 2	
$12n_{582}$	> 2	
$\overline{12n_{768}}$	2	$z^{12} + 12z^{10} + 49z^8 + 78z^6 + 41z^4 + 4z^2 + 1$
$12n_{838}$	1	2
$\overline{13n_{579}}$	2	$z^{12} + 9z^{10} + 25z^8 + 21z^6 + 4z^4 + 1$
$\overline{13n_{3158}}$	1	$z^4 + 3z^2 + 3$
$13n_{3523}$	> 2	
$13n_{4514}$	> 2	
$\overline{13n_{4659}}$	> 2	
$\overline{14n_{2459}}$	> 2	
$14n_{2601}$	1	$z^4 + 3z^2 + 3$
$\overline{14n_{8579}}$	2	$3z^{12} + 27z^{10} + 81z^8 + 93z^6 + 38z^4 + 4z^2 + 1$
$\overline{14n_{12406}}$	2	$3z^{12} + 26z^{10} + 72z^8 + 68z^6 + 17z^4 + 1$
$\overline{14n_{14251}}$	> 2	
$\overline{14n_{15581}}$	1	$z^4 + 2z^2 + 2$
$14n_{18212}$	> 2	
$14n_{21563}$	1	$z^4 + 3z^2 + 2$
$\overline{14n_{22150}}$	2	$z^{12} + 8z^{10} + 18z^8 + 8z^6 + 1$
$\overline{14n_{22789}}$	> 2	
$\overline{14n_{24246}}$	> 2	
$14n_{25967}$	> 2	

Table 2: For each Lagrangian slice knot K through 14 crossings with Legendrian representative Λ from Table 1, we give the least n for which the ruling polynomial of the n–cable of Λ is known to differ from 1. An entry of the form "$n > 2$" indicates that $z^{n-1} R^1_{\Lambda_n}(z) = 1$ for all $n \le 2$.

An obstruction to the existence of a concordance giving $\Lambda \prec U$ is provided by Corollary 2.9. For 6 of the 23 Legendrian knots Λ from Table 1 (of types $\overline{9_{46}}$, $12n_{838}$, $\overline{13n_{3158}}$, $14n_{2601}$, $\overline{14n_{15581}}$, $14n_{21563}$), the ungraded ruling polynomial

$R^1_{\Lambda(z)}$ is not equal to 1, and so $\Lambda \not\prec U$ by Corollary 2.9. In 5 additional cases ($\overline{12n_{768}}$, $\overline{13n_{579}}$, $\overline{14n_{8579}}$, $\overline{14n_{12406}}$, $\overline{14n_{22150}}$), we may similarly apply Theorem 2.13 to the 2–cable Λ_2 of Λ: the ungraded ruling polynomial for Λ_2 is not equal to z^{-1}, and hence Theorem 2.13 shows that $\Lambda \not\prec U$. Note that this argument shows $\Lambda \not\prec U$ for all Legendrian representatives Λ of these knot types, not just the particular ones from Table 1; see Remark 2.16.

We have not been able to rule out the possibility that $\Lambda \prec U$ for Λ representing the remaining 12 knot types. One indication that it may be difficult to do so for at least some of these knots is the curious family of Legendrian $P(-m, -3, 3)$ pretzel knots for $m \geq 4$, of which $\overline{10_{140}}$, $11n_{139}$, $12n_{582}$, $13n_{3523}$, $14n_{18212}$ are shown in Table 1; these have a natural generalization to all $m \geq 4$. For this family of Legendrian knots, it can be shown that the Legendrian contact homology DGA is stable tame isomorphic to the DGA for the unknot U (even over $\mathbb{Z}[t, t^{-1}]$), and so these knots are indistinguishable from U from the viewpoint of contact homology. It follows from this fact that any n–cable of these knots has the same ungraded ruling polynomial as the n–cable of the unknot, though we omit the proof.

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