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Near-symplectic $2n$–manifolds

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We give a generalization of the concept of near-symplectic structures to $2n$ dimensions. According to our definition, a closed 2–form on a $2n$–manifold M is near-symplectic if it is symplectic outside a submanifold Z of codimension 3 where ω^{n-1} vanishes. We depict how this notion relates to near-symplectic 4–manifolds and broken Lefschetz fibrations via some examples. We define a generalized broken Lefschetz fibration as a singular map with indefinite folds and Lefschetz-type singularities. We show that, given such a map on a $2n$–manifold over a symplectic base of codimension 2, the total space carries such a near-symplectic structure whose singular locus corresponds precisely to the singularity set of the fibration. A second part studies the geometry around the codimension-3 singular locus Z. We describe a splitting property of the normal bundle N_Z that is also present in dimension four. A tubular neighbourhood theorem for Z is provided, which has a Darboux-type theorem for near-symplectic forms as a corollary.

53D35, 57R17; 57R45

1 Introduction

The motivation for near-symplectic manifolds arose from a program, initiated by Taubes [16], to study 4–manifolds equipped with symplectic forms that vanish on circles, with the goal of obtaining smooth invariants of nonsymplectic 4–manifolds. A 4–manifold is called near-symplectic if it is equipped with a closed 2–form that is nondegenerate outside a disjoint union of circles, where it vanishes. These structures were studied in detail in the work of Auroux, Donaldson and Katzarkov [2] using broken Lefschetz fibrations (bLfs). It was shown that there is a direct correspondence between bLfs and near-symplectic 4–manifolds. These results extended the theorems of Donaldson [6] and Gompf [10] on Lefschetz fibrations and symplectic manifolds, which in turn expanded Thurston's theorem on symplectic fibrations. Broken Lefschetz fibrations have found fruitful application in low-dimensional topology: for example, in holomorphic quilts (see eg Wehrheim and Woodward [17]) and Lagrangian matching invariants (see eg Perutz [15]). A relevant existence result states that every smooth closed oriented 4–manifold admits a bLf; see Gay and Kirby [8], Lekili [13], Baykur [3; 4], and Akbulut and Karakurt [1]. The geometric structure induced by a near-symplectic 4–manifold

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on the boundary of the tubular neighbourhood of its singular locus is an overtwisted structure as studied by Honda [12] and Gay and Kirby [7]. This shows that these manifolds are not fillable, as that would require removing all singular circles, which Perutz proved to be impossible [14].

This work aims to find a good notion to generalize near-symplectic structures on higher dimensions. We propose a definition on manifolds of dimensions $2n$ and use singular maps that resemble broken Lefschetz fibrations. We also study the underlying geometric structure, induced by the near-symplectic form, on the boundaries of tubular neighbourhoods, which are codimension-1 submanifolds in this setting.

In Section 2A, we suggest a definition of a near-symplectic structure in dimension $2n$. The goal is to relax the nondegeneracy condition of the symplectic form in a controlled way so that it degenerates exclusively on a certain submanifold. The idea starts by considering a closed 2–form ω on a smooth, orientable, $2n$–manifold M such that $\omega^n \geq 0$. At the points where the degeneracy occurs, that is, where $\omega^n = 0$, we impose a transversality condition on the gradient or differential map of ω. This transversality condition tells us that the singular locus Z is a submanifold of codimension 3 in M, where $\omega^n = 0$ and $\omega_p^{n-1} = 0$ for all $p \in Z$, but $\omega_p^{n-2} \neq 0$. We call these 2–forms near-symplectic. Examples of near-symplectic $2n$–manifolds are given in Section 2B

Next, we study the existence of these structures using singular fibrations, which are analogous to bLfs. We define a *generalized bLf* as a submersion $f\colon M^{2n} \to X^{2n-2}$ with two types of sets of singularities, both of which lie in M. First, we have codimension-4 submanifolds of extended Lefschetz type singularities, locally modelled by complex coordinate charts

$$\mathbb{C}^n \to \mathbb{C}^{n-1}, \quad (z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-2}, z_{n-1}^2 + z_n^2).$$

The second singularities are codimension-3 submanifolds Σ of indefinite folds, modelled by real coordinate charts

$$\mathbb{R}^{2n} \to \mathbb{R}^{2n-2}, \quad (t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \dots, t_{2n-3}, -x_1^2 + x_2^2 + x_3^2).$$

Using these maps, we obtain the following result.

Theorem 1.1 *Let $f\colon M \to X$ be a generalized bLf from a smooth closed oriented $2n$–manifold M to a compact symplectic $(2n-2)$–manifold (X, ω_X). Denote by Σ the set of fold singularities of f. Assume that there is a class $\alpha \in H^2(M)$ that pairs positively with every component of every fibre such that $\alpha|_\Sigma = [\omega_X|_\Sigma]$. Then there is a near-symplectic form ω on M, with singular locus Z equal to Σ, and with symplectic fibres outside Σ.*

We start the proof appearing in Section 3B by constructing an explicit closed 2–form on the fibres that vanishes at the set of singularities of the mapping. We then pull back the symplectic form of the base. Both 2–forms are combined and glued together into a global 2–form representing the class α. This statement follows a similar line of reasoning as the construction of Auroux, Donaldson, and Katzarkov [2], using bLfs in dimension 4.

Section 4 concerns the geometric structure on the boundary of the neighbourhood of the singular locus. We study two geometric structures that appear on a codimension-1 submanifold of M. Firstly, we look at Hamiltonian structures. A Hamiltonian structure on a $(2n-1)$–dimensional manifold N is a closed 2–form Ω such that $\Omega^{n-1} \neq 0$ everywhere. In the presence of a Hamiltonian structure, there is a 1–dimensional distribution associated to Ω through its kernel $\ker(\Omega)$. A 1–form λ is called a stabilizing 1–form if $\lambda \wedge \Omega^{n-1} > 0$ and $\ker(\Omega) \subset \ker(d\lambda)$. The pair (λ, Ω) is known as a stable Hamiltonian structure. A near-symplectic form naturally equips the singular locus Z with a Hamiltonian structure. Moreover, if Z carries a stable Hamiltonian structure, so does the boundary of a small tubular neighbourhood in the case that the normal bundle is trivial.

We conclude by examining the properties of the normal bundle of Z that are defined by the near-symplectic form. As in dimension 4, there is a decomposition of the normal bundle N_Z in two subbundles: a rank-1 bundle L^- and a rank-2 bundle L^+. In Section 4C, we prove the following neigbourhood theorem for near-symplectic forms around their singular loci.

Theorem 1.2 *Let (M_0, ω_0), (M_1, ω_1) be two near-symplectic manifolds with diffeomorphic singular loci $Z_0 \cong Z_1$ and equal symplectic forms $\omega_0|_{Z_0} = \omega_1|_{Z_1}$ on them. Assume that there is an isomorphism on the normal bundles $N_{Z_0} \simeq N_{Z_1}$ such that it restricts to an isomorphism on the positive subbundles $L_0^+ \simeq L_1^+$. Denote by $\mathcal{U}_0 \subset M_0$ and $\mathcal{U}_1 \subset M_1$ the corresponding tubular neighbourhoods of Z_0 and Z_1. Then there is a homeomorphism $\varphi\colon \mathcal{U}_0 \to \mathcal{U}_1$ that is a diffeomorphism away from Z, such that $\varphi^*\omega_1 = \omega_0$.*

As a corollary, we obtain a local Darboux-type theorem which describes a near-symplectic form around a point of Z.

Corollary 1.3 *Let (M, ω) be a near-symplectic manifold and p a point of the singular locus $Z \subset M$. There is a coordinate neighbourhood $U \subset M$ around p such that, on U,*

$$\begin{aligned}(1)\quad \omega = \omega_Z - 2x_1(dz_0 \wedge dx_1 + dx_2 \wedge dx_3)& \\ +x_2(dz_0 \wedge dx_2 - dx_1 \wedge dx_3) + x_3(dz_0 \wedge dx_3 + dx_1 \wedge dx_2)&,\end{aligned}$$

where $\omega_Z := i^*\omega$ is a closed 2–form of maximal rank on Z.

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2 Near-symplectic forms

We first recall the definition of near-symplectic forms on 4–manifolds [2].

Definition 2.1 Let X be a smooth oriented 4–manifold. Consider a closed 2–form $\omega \in \Omega^2(X)$ such that $\omega^2 \geq 0$ and such that ω_p only has rank 4 or rank 0 at any point $p \in X$, but never rank 2. The form ω is called *near-symplectic* if it is nondegenerate or if it vanishes transversally along circles. That is, for every $p \in X$, either

(1) $\omega_p^2 > 0$, or

(2) $\omega_p = 0$ and $\mathrm{Rank}(\nabla\omega_p) = 3$, where $\nabla\omega_p\colon T_pX \to \Lambda^2 T_p^*X$ denotes the intrinsic gradient of ω.

It follows from the condition on $\nabla\omega_p$ that the singular locus Z_ω is a smooth 1–submanifold of X [2; 14].

Example 2.2 A prototypical example of a near-symplectic 4–manifold is given by $X = S^1 \times Y^3$, where Y is a closed 3–manifold. Consider a closed 1–form $\alpha \in \Omega^1(Y)$ with indefinite Morse critical points, and let t be the parameter of S^1. The 2–form $\omega = dt \wedge \alpha + *(dt \wedge \alpha)$ is near-symplectic, where the Hodge $*$–operator is defined with respect to the product metric on S^1 and Y. The singular locus $Z_\omega = \{p \in X \mid \omega_p = 0\}$ is, in this case, $S^1 \times \mathrm{Crit}(\alpha)$.

2A Near-symplectic 2n–manifolds

The following definition of near-symplectic forms in higher dimensions is due to Tim Perutz. The author would also like to acknowledge that the coming exposition very closely follows a message from Perutz.

Let M be an oriented smooth $2n$–manifold, and $\omega \in \Omega^2(M)$ a closed 2–form such that

$$\omega^n \geq 0 \tag{2}$$

everywhere. Suppose that at some point p, the kernel K of ω,

$$K = \{v \in T_p M \mid \omega_p(v, \cdot) = 0\}, \tag{3}$$

seen as a subspace of the tangent space, has dimension 4. We have an intrinsic gradient $\nabla\omega\colon K \to \Lambda^2 T_p^* M$. We can restrict this map to bivectors in K to get a map $K \to \Lambda^2 T_p^* M \to \Lambda^2 K^*$, where the map $\Lambda^2 T_p^* M \to \Lambda^2 K^*$ corresponds to the dual of the inclusion $K \hookrightarrow T_p M$ in the corresponding exterior algebra. We denote this composition as

$$D_K\colon K \to \Lambda^2 K^*. \tag{4}$$

The wedge square gives us a nondegenerate quadratic form

$$q\colon \Lambda^2 K^* \otimes \Lambda^2 K^* \to \Lambda^4 K^*.$$

Proposition 2.3 *The image* $\mathrm{Im}(D_K)$ *has dimension at most* 3*. In local coordinates, this is a positive semidefinite subspace of* $\Lambda^2 K^*$ *with respect to the wedge square form, that is, the* 4*–form* $D_K(v) \wedge D_K(v) \geq 0$ *for* $v \in K$*.*

Proof Take an arbitrary tangent vector $v \in T_p M$, and choose coordinates such that $p = 0$ is the point at the origin. By (2) above, we have $\omega^n(t \cdot v) \geq 0$ for all scalars t, where $t \cdot v$ points into the manifold. Yet if we use a Taylor expansion to write $\omega(t \cdot v) = \omega(0) + t \cdot \nabla_v \omega(0) + O(t^2)$ and take $v \in K$, we have

$$\omega^n(t \cdot v) = \underbrace{\omega^n(0)}_{=0} + t\binom{n}{1}\underbrace{\omega^{n-1}(0)}_{=0} \wedge \nabla_v \omega(0) + t^2\binom{n}{2}\omega(0)^{n-2} \wedge (\nabla\omega_v(0))^2 + O(t^3).$$

The forms $\omega^n(0)$ and $\omega^{n-1}(0)$ vanish since they take vectors $\partial_{k_1}, \dots, \partial_{k_4}$ from K, whereas in the linear combination of ω^{n-2}, there will be vectors outside of K where the form remains nonzero. This gives us

$$\omega^n(t \cdot v) = \binom{n}{2} \cdot t^2 \cdot \omega(0)^{n-2} \wedge (\nabla\omega_v(0))^2 + O(t^3).$$

We work over a local coordinate system using the tangent space at $p = 0$. The space $T_p M / K$ has a symplectic structure, and we can combine an orientation on it with an orientation of K to get an orientation of $T_p M$, which has a natural orientation. With respect to this chosen orientation, we want to show that $D_K(v) \wedge D_K(v) \geq 0$ for $v \in K$. Let $e_i = (\partial/\partial x_i)_{1 \leq i \leq 2n}$ be an oriented basis. Since $\omega^n(tv) \geq 0$ from our original consideration (2), we have that $\omega^n(t \cdot v)(e_1, \dots, e_{2n}) \geq 0$; thus

$$\omega^n(t \cdot v) \approx C \cdot \omega(0)^{n-2} \wedge (\nabla\omega_v(0))^2 (e_1, \dots, e_{2n}) \geq 0$$

with the constant $C = \binom{n}{2} \cdot t^2$. The form $\omega(0)^{n-2}$ has a sign on the subspace complementary to K in T_pM since we have chosen an orientation. However from (4), by restricting to vectors in K, we have

$$\omega^n(t \cdot v) \approx C \cdot t^2 \cdot \omega(0)^{n-2}(e_1, \dots, e_{2n-4}) \wedge \underbrace{(\nabla\omega_v(0))^2}_{D_K(v) \wedge D_K(v)} (\partial_{k_1}, \dots, \partial_{k_4}) \geq 0.$$

We can see now that the image of D_K is a positive semidefinite subspace of $\Lambda^2 K^*$. Hence $\operatorname{Im}(D_K)$ has dimension at most 3. In particular, $D_K(v) \wedge D_K(v)$ is a nonnegative 4–form with respect to K. □

Definition 2.4 A 2–form $\omega \in \Omega^2(M^{2n})$ is *near-symplectic* if it is closed, $\omega^n \geq 0$ and, at a point p where $\omega^n = 0$, one has that the kernel $K = \{v \in T_pM \mid \omega_p(v, \cdot) = 0\}$ is 4–dimensional and that $\operatorname{Im}(D_K)$ has dimension 3.

Remark 2.5 Informally, the definition implies that a closed 2–form $\omega \in \Omega^2(M)$ is *near-symplectic* if, for every $p \in M$, either

(i) $\omega_p^n > 0$, or

(ii) $\omega_p^{n-1} = 0$, but $\omega_p^{n-2} \neq 0$ at a codimension-3 submanifold of M.

In the remaining part of this section, we will explain why the degeneracy locus is a codimension-3 submanifold.

The image of the map $D_K \colon K \to \Lambda^2 K^*$ is of dimension 3, thus it has a 1–dimensional kernel. If we look at ω^{n-1}, then it vanishes at p since it takes at least two vectors from K. Moreover, $G = \nabla\omega^{n-1}(p)$ is intrinsically defined. Choose coordinates (x_k) so that K is defined by the vanishing of all but the last four dx_k. Take the derivative of ω^{n-1} and apply the chain rule to obtain

$$G = (n-1)\omega(p)^{n-2}\nabla\omega_p,$$

where the gradient on the right is defined using the coordinates. The 2–form ω is symplectic on the submanifold Z where the last 4 coordinates are zero. We can adjust the coordinates to Darboux form so that ω is constant on Z; that is, for $p \in Z$ we have $\omega|_p = dx_1 \wedge dx_2 + \dots + dx_{2n-5} \wedge dx_{2n-4}$. Hence $\nabla\omega_p(\partial x_i) = 0$ for $i = 1, \dots, 2n-4$. However, we have

$$\ker G = \ker(\nabla\omega_p),$$

and now one sees that this is a codimension-3 subspace containing the line $\ker(D_K)$. Hence the degeneracy locus Z of the near-symplectic form is a codimension-3 submanifold of M.

Lemma 2.6 *The singular locus* $Z_\omega = \{p \in M \mid \omega_p^{n-1} = 0\}$ *is a codimension-3 submanifold of* M *.*

Remark 2.7 Let V be a $2n$–dimensional manifold and Z a $(2n-3)$–submanifold. The property of $\omega|^n_{V\setminus Z} > 0$ guarantees that the whole V^{2n} is orientable. This is due to the fact that Z is a submanifold of codimension 3. In fact, it follows from a standard algebraic topological argument that this orientability property is true for any dimension if the codimension of the submanifold is greater than or equal to two. That is to say, if ω is a 2–form on V, the submanifold $K \subset V$ is k–dimensional, and $\omega^n > 0$ on $V \setminus K$, then V is oriented if $\mathrm{codim}(K) \geq 2$.

Remark 2.8 In dimension 4, near-symplectic structures are related to self-dual harmonic forms. An obvious obstacle in dimensions 6 and above is that there is no analogue of self-duality for 2–forms. Some local models of near-symplectic forms on 6–manifolds M^6 seem to indicate that near-symplectic forms could be equivalent to $\omega = *\omega^2$ for some generic metric, outside the singular locus Z.

2B Examples

Example 2.9 On $\mathbb{R}^{2n}$ with coordinates $(q_1, p_1, \dots, q_{n-2}, p_{n-2}, x_0, x_1, x_2, x_3)$, the following 2–form is near-symplectic:

$$\begin{aligned}\omega = -2x_1(dx_0 \wedge dx_1 + dx_2 \wedge dx_3) + x_2(dx_0 \wedge dx_2 - dx_1 \wedge dx_3)&\\ +x_3(dx_0 \wedge dx_3 + dx_1 \wedge dx_2) + \sum_{i=1}^{n-2} dq_i \wedge dp_i&.\end{aligned}$$

The singular locus where $\omega^{n-1} = 0$ is given by $Z_\omega = \{p \in \mathbb{R}^{2n} \mid x_1 = x_2 = x_3 = 0\}$ and $\omega^n > 0$ away from Z_ω.

For the next example, let $(Q, \overline{\omega})$ be a symplectic manifold and $\phi\colon Q \to Q$ a symplectomorphism. Consider the mapping torus

$$N = Q(\phi) = (Q \times [0,1])/\sim,$$

where $(x,0) \sim (\phi(x), 1)$. The mapping torus is, in particular, a fibre bundle over S^1, and it carries a nonvanishing closed 1–form β. We can extend $\overline{\omega}$ from Q to N. There is a 2–form defined on $Q \times \mathbb{R}$. The $\mathbb{Z}$–action on this manifold given by $(x,t) \mapsto (\phi(x), t+1)$ leaves the 2–form invariant; hence it descends to the quotient. Thus, $\overline{\omega}$ is a well-defined 2–form on N that is symplectic on Q.

Example 2.10 Consider the $2n$–manifold $M = N \times Y$, where N is the mapping torus described in the previous paragraph, and Y is a closed, connected, orientable, smooth 3–manifold. Let $\alpha \in \Omega^1(Y)$ be a closed 1–form with indefinite Morse singular points (ie no maximum or minimum). By Calabi's and Honda's theorems [11; 12], this form can be replaced by an intrinsically harmonic 1–form lying in the same cohomology class and having the same Morse numbers. Thus we may assume that $\Delta\alpha = 0$ for some Riemannian metric on Y. Equip the $2n$–manifold with the 2–form

$$\omega = \beta \wedge \alpha + \overline{\omega} + (*_Y \alpha), \tag{5}$$

where $*_Y$ denotes the Hodge-$*$ operator. This 2–form is near-symplectic on M, and its singular locus is $Z_\omega = N \times \mathrm{Crit}(\alpha)$.

3 Fibrations

3A Near-symplectic fibrations

We recall the definition of broken Lefschetz fibrations on dimension four. On a smooth, closed 4–manifold X^4, a *broken Lefschetz fibration* or *bLf* is a smooth map to the 2–sphere, $f\colon X^4 \to S^2$, with two types of singularities:

(1) isolated *Lefschetz-type* singularities, which are contained in the finite subset of points $B \subset X^4$ and are locally modelled by complex charts
$$\mathbb{C}^2 \longrightarrow \mathbb{C}, \quad (z_1, z_2) \longmapsto z_1^2 + z_2^2;$$

(2) *indefinite fold* singularities, also called *broken*, which are contained in the smooth embedded 1–dimensional submanifold $\Gamma \subset X^4 \setminus B$ and are locally modelled by the real charts
$$\mathbb{R}^4 \longrightarrow \mathbb{R}^2, \quad (t, x_1, x_2, x_3) \longmapsto (t, -x_1^2 + x_2^2 + x_3^2).$$

In [2] these mappings were studied under the name of singular Lefschetz fibrations. It was shown that there is a natural connection between bLfs and near-symplectic manifolds. Up to blow-ups, a near-symplectic 4–manifold X can be decomposed into a bLf. The other direction is given by the following theorem.

Theorem 3.1 [2] *If we have a bLf with singularity set $\Gamma \sqcup B$ on a closed oriented 4–manifold X, with the property that there is a class $\alpha \in H^2(X)$ such that it pairs positively with every component of every fibre, then X carries a near-symplectic structure with zero-locus being equal to the set of broken singularities of f.*

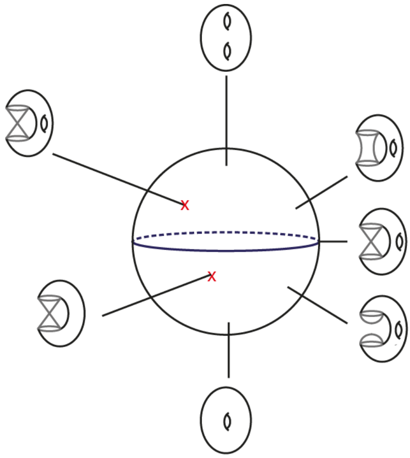

Figure 1: Example of a bLf with one circle of folds and two Lefschetz points.

Our Theorem 1.1 shows a similar statement in $2n$ dimensions. Now we define a map that will play an analogous role of a bLf two dimensions higher. This map is a submersion with folds and Lefschetz-type singularities. Notice that a submersion with folds is stable if the map f restricted to its fold set is an immersion with normal crossings [9]. By stable, we mean that any nearby map $\tilde{f}$ is identical to f after a change of coordinates.

Definition 3.2 Let M be a smooth, closed $2n$–manifold M and X a smooth, closed $(2n-2)$–manifold. By a *(generalized) broken Lefschetz fibration*, we mean a submersion $f\colon M \to X$ with two types of singularities:

(1) "Extended" *Lefschetz-type* singularities are locally modelled by

$$\mathbb{C}^n \to \mathbb{C}^{n-1}, \quad (z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-2}, z_{n-1}^2 + z_n^2).$$

These singularities are contained in codimension-4 submanifolds crossed with a Lefschetz singular point. Singular fibres present an isolated nodal singularity, but nearby fibres are smooth and convex.

(2) *Indefinite fold* singularities are locally modelled by

$$\mathbb{R}^{2n} \to \mathbb{R}^{2n-2}, \quad (t_1, \dots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \dots, t_{2n-3}, -x_1^2 + x_2^2 + x_3^2).$$

The fold locus is an embedded codimension-3 submanifold, and we denote it by Σ. Singular fibres present a nodal singularity, but this time crossing Σ changes the genus of the regular fibre by one.

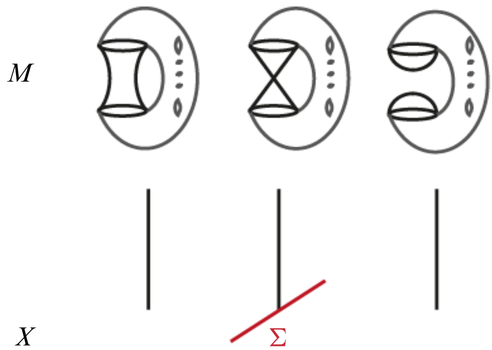

Figure 2: Fibres: regular (left, right) and singular (middle).

If we consider the total space to be near-symplectic with regular fibres being symplectic and the fold locus being Z_ω, then we will refer to the previous map $f\colon M \to X$ as a *near-symplectic fibration*.

3B Proof of Theorem 1.1

Step 1: Constructing the local 2–form We first want to define the local near-symplectic form near the singular sets $\Sigma \sqcup C$, where Σ denotes the singularity set of folds and C the set of extended Lefschetz-type singularities. We begin by defining a singular symplectic form vanishing at Σ, and then we pull back the symplectic form of the base. Let $(t_1, \dots, t_{2n-3}, x_1, x_2, x_3)$ be coordinates around a fold point $p \in \Sigma$ of index 1, locally modelled by $\tilde{f}\colon (t, x) \mapsto (t_1, \dots, t_{2n-3}, -x_1^2 + \frac{1}{2}(x_2^2 + x_3^2))$. Since the fibres are 2–dimensional, we can take a similar local model as the near-symplectic forms on 4–manifolds. Define the following 2–form on a piece of the tubular neighbourhood of Σ containing p:

$$\tau_p = d(\chi(t)x_1(x_2 dx_3 - x_3 dx_2)). \tag{6}$$

This 2–form is closed, vanishes at the singularity set, is nondegenerate outside Σ, and evaluates positive on the fibres. Here $\chi(t)$ is a smooth cut-off function depending on coordinates on Z. This cut-off function will help us in the gluing process when summing up the 2–forms τ_{p_i} to build a local 2–form on the whole tubular neighbourhood of Σ. We sum up the forms τ_{p_k} over a finite cover of Σ, and pullback the symplectic form from the base. We obtain

$$\omega_A = \sum_{p_k} \tau_{p_k} + f^*\omega_X. \tag{7}$$

This closed 2–form is defined on the tubular neighbourhood of Σ. It is nondegenerate outside Σ and positive on the fibres. At the degeneracy points, $K_p = N_p\Sigma \oplus \varepsilon_p$ is of dimension four, where $\varepsilon = \ker(f^*\omega_X) \subset T\Sigma$.

Around elements of C, where f is given by $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-2}, z_{n-1}^2 + z_n^2)$, we can choose disjoint neighbourhoods B_k such that $\omega_0|_{B_k} = \omega_{\mathbb{C}^{n-1}}$. Since, near C, we are in a situation similar to a Lefschetz fibration, we can proceed as in [2; 10]. For any $v_1, v_2 \in T_pF$, we get $\omega_0|_{B_k}(v_1, v_2) > 0$ away from the singularity. The symplectic form $\omega_A|_{B_k}$ can be extended to the fibre F_q as a symplectic form for all $q \in f(B_k) \subset X$.

Step 2: Extension over the neighbourhoods of the fibres In this step we want to construct local 2–forms on the neighbourhood of the fibres. We extend the 2–form to a local model over the neighbourhood of the fibres such that it agrees with ω_A near $\Sigma \sqcup C$. Let U be the tubular neighbourhood of $\Sigma \sqcup C$. Choose a closed 2–form $\zeta \in \Omega^2(M)$ with a class being represented by α. Since $\alpha|_\Sigma = \omega_X|_\Sigma \in H^2(\Sigma)$, over U there exists a 1–form $\bar{\mu} \in \Omega^1(U)$ such that $\omega_A - \zeta = d\bar{\mu}$. We now extend $\bar{\mu}$ to an arbitrary 1–form on the manifold, $\mu \in \Omega^1(M)$, supported in a neighbourhood W of U. By substituting $\eta = \zeta + d\mu$ on U, we can regard η to be ω_A when restricted to U.

By assumption, we have a positive pairing $\langle \alpha, F \rangle > 0$ over each component of the fibre, $[\eta] = \alpha$, and the fibres have a symplectic form σ_F. We equip the fibres with a closed singular 2–form σ_q with the following properties:

(a) $\sigma_q|_{F_q \cap U_1} = \eta$. By restricting σ_q to U, this 2–form is near-symplectic since $\eta|_U = \omega_A$. The form σ_q is defined on the fibre, so $\sigma_q|_{F_q \cap U_1}$ is near-symplectic.

(b) $\sigma_q|_{F_q}$ is positive on F_q, where the fibre is smooth. This can be seen by considering two subsets of the fibre. Take a small open neighbourhood around the singularity and a second larger one covering the rest of the fibre. On the first neighbourhood around the singularity, the 2–form ω_A evaluates positively except at the singular point. On the second subset where the fibre is smooth, the area form of F_q evaluates positively.

(c) $\int_F \sigma_q = \langle \alpha, F \rangle > 0$, since

$$[\sigma_q - \eta|_{F_q}] = 0 \quad \text{in} \quad H^2(F_q, F_q \cap U_1) \overset{PD}{\simeq} H_0(F_q, F_q \cap U_1) \simeq 0,$$

assuming F_q is connected. Then $(\sigma_q - \eta|_{F_q})$ is exact in $F_q \cap U_1$; that is, $[\sigma_q] = [\eta] = \alpha$.

We now describe some properties of the neighbourhood of the fibres in order to extend the 2–form. For any $q \in X$, we can find a tubular neighbourhood V_q of the fibre F_q

and neighbourhoods $U_2 \subset U_1 \subset U$ of the fold singularity set Σ. A $q \in X$ can be engulfed by an m–disk D^m. Around a fibre F_q, take $f^{-1}(D^m) = V_q$. After removing a small neighbourhood of the critical set, we have that $V_q \setminus (V_q \cap U_2)$ is diffeomorphic to $D^{m-1} \times (F_q \setminus (F_q \cap U_2))$. This follows from the Ehresmann theorem, since we locally have a nice smooth map without critical points.

To extend the 2–form on the neighbourhood of the fibre, we build a smooth map

$$\pi\colon V_q \to V_q$$

by interpolating between two maps as follows:

(i) Close to the singular point of the fibre inside the neighbourhood $V_q \cap U_1$, we use the identity map so that π is $\mathrm{id}_{V_q \cap U_1}$. Since V_q is a neighbourhood of a fibre F_q and $V_q \cap U_1$ retracts to $F_q \cap \Sigma$, we want that π maps down to the piece of the fibre close to the singularity together with the intersection of the neighbourhoods V_q and U_1; that is,

$$\mathrm{Im}(\pi) \subset F_q \cup (V_q \cap U_1).$$

(ii) Farther away from the singular region, that is, on the smooth part $F_q \setminus (F_q \cap U_2)$, we use the projection map pr: $V_q \setminus (V_q \cap U_2) \to F_q \setminus F_q \cap U_2$ that comes from the product structure.

We use the map π to construct a near-symplectic form β on V_q. With π, we pull back the 2–form η on $V_q \cap U_1$ and the 2–form σ_q on F_q to get

$$\beta = \pi^*\sigma_q + \pi^*\eta.$$

This 2–form has the following features:

(1) $d\beta = 0$ and $[\beta] = \alpha|_{V_q}$.

(2) $\beta|_{V_q \cap U_2} = \eta$.

(3) There exists a 1–form μ_q on V_q such that $\beta - \eta = d\mu_q$, since $[\beta - \eta] = 0$ in $H^2(V_q, V_q \cap U_2) \simeq H^2(F_q, F_q \cap U_2)$. Thus, on V_q,

$$\beta_q = \eta + d\mu_q.$$

(4) $\beta_q|_{F_q} > 0$ restricts positively to the fibre for every regular point $q \in V_q$.

Step 3: Patching into a global form We expand the near-symplectic form over the whole manifold M. Since our base is compact, we can find a finite subset $Q \subset X$ and choose a finite cover $\mathcal{D}$ with open subsets $(D_q)_{q \in Q}$ such that $f^{-1}(D_q) \subset V_q$ for each $q \in X$. Consider a smooth partition of unity ρ: $X \to [0,1]$, with $\sum_{q \in Q} \rho_q = 1$, subordinate to the cover $\mathcal{D}$ with $\mathrm{supp}(\rho_q) \subset D_q$. We build a global 2–form Ω on M by patching the local 1–forms μ_q previously defined on V_q. Thus, we define the following closed 2–form:

$$\Omega = \eta + d\Big(\sum_{q \in Q} (\rho_q \circ f)\mu_q\Big). \tag{8}$$

Since f is constant on the fibres, the 1–form $d((\rho_q \circ f)\mu_q) = 0$ when evaluated on the vectors tangent to the fibre. From the second step, η agrees with ω_A when restricted to U. Let $\overline{U}_2$ be the intersection of all neighbourhoods U_2 for all $q \in Q$; that is, $\overline{U}_2 = U_2 \cap \bigcap_{q \in Q} f^{-1}(D_q)$. The 2–form Ω agrees with η when restricted to $\overline{U}_2$, so it agrees with the local model of ω_A at U_2. Thus, Ω is globally well-defined over M.

The 2–form Ω restricts to a fibre F_q in the following way:

$$\begin{aligned}
\Omega|_{F_q} &= \eta|_{F_q} + \sum_{q \in Q} \rho \circ f(p) d\mu_q|_{F_q} \\
&= \sum_{q \in Q} \rho \circ f(p)(\eta + d\mu_q)|_{F_q} \\
&= \sum_{q \in Q} (\rho \circ f(p))\, \beta_q|_{F_q}.
\end{aligned}$$

This is a convex combination of near-symplectic 2–forms. On each fibre, Ω is closed, positive outside the singular locus, and degenerates at Σ, inducing a symplectic structure on each fibre outside the singularities.

Step 4: Positivity on vertical and horizontal tangent subspaces To conclude the global construction, we apply a similar argument as in the symplectic case. The 2–form Ω is positive on the vertical subspaces tangent to the fibre $\ker df(p) = T_pF \subset T_pM$, outside the singularity set. To guarantee positivity on the horizontal spaces, we multiply the pullback from the symplectic form of the base by a sufficiently large real number $K > 0$ to obtain the 2–form

$$\omega_K = \Omega + K \cdot f^*\omega_X. \tag{9}$$

If we restrict ω_K to the vertical subspaces tangent to the fibre, it agrees with Ω. The 2–form ω_K defines a near-symplectic structure on M. $\square$

3C Examples

Example 3.3 (pullback bundle) We can obtain examples of near-symplectic manifolds and near-symplectic fibrations via a pullback bundle construction. Suppose we have $n > 2$, M and X oriented, closed manifolds of dimension $(2n-2)$, B an oriented, closed, connected manifold of dimension $(2n-4)$, f and g smooth mappings, and $W = \{(x,m) \in X \times M \mid f(x) = g(m)\}$, fitting into a pullback diagram:

$$\begin{tikzcd} W \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] & M \arrow[d, "g"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

Before going to the near-symplectic case, we briefly comment on the symplectic one. A theorem from Thurston tells us that if g is a compact symplectic fibration over a closed connected symplectic manifold B, and there is a class $\alpha \in H^2(M)$ such that $\iota^*\alpha = [\sigma_b]$ for all $b \in B$, where $\sigma_b \in \Omega^2(F_b)$ is the canonical form of the fibre, then M is symplectic. We can pull back this information to W via $\tilde{f}$ and obtain a class $\tilde{\alpha} = \tilde{f}^*\alpha \in H^2(W)$ with the same property. Thus we only need X to be symplectic so that W is a symplectic manifold via the induced map $\tilde{g}$. We now discuss the near-symplectic scenario.

Throughout these examples, we assume that the critical set of g forms regular points for f, so that f behaves like a bundle near the critical sets by the Ehresmann theorem (whenever there is a critical set for g). The first example follows from Theorem 1.1. If g is a bLf (thus $\tilde{g}$ a generalized bLf), and X is symplectic, then W is near-symplectic via $\tilde{g}$ assuming that the cohomological condition of Theorem 1.1 is satisfied. A second case appears when the base X is near-symplectic. Keeping a vertical view of the diagram, we do not now consider g and $\tilde{g}$ to be bLfs. The following proposition explains this situation.

Proposition 3.4 *Let $g\colon M \to B$ be a compact symplectic fibration with symplectic total space M, and let (X, ω_X) be a closed, near-symplectic manifold over a closed connected symplectic base B of codimension 2. Let W be the pullback bundle as defined in the previous paragraph. Then W carries a near-symplectic structure induced by $\tilde{g}\colon W \to X$.*

Proof Let Γ be the singular locus of ω_X, which is a codimension-3 submanifold in X. Its preimage under $\tilde{g}$ is a surface bundle, and we will denote by Z its total space. This bundle will become the singular locus of the near-symplectic form of W. Let $\mathcal{U}$

be the tubular neighbourhood of Γ and let $E = \widetilde{g}^{-1}(\mathcal{U})$. E is a surface bundle. We will also consider a small tubular neighbourhood $\overline{E}$ inside E.

We now construct a closed 2–form $\widetilde{\eta} \in \Omega^2(W)$ that is positive on the fibres of $\widetilde{g}$ in W, whose wedge power $\widetilde{\eta}^{n-1}$ is zero on E. Since g is a symplectic fibration, we have a cohomology class $\alpha \in H^2(M)$ that pairs positively with the fibre class. We choose $\widetilde{\eta}$ such that $[\widetilde{\eta}] = \widetilde{f}^*\alpha \in H^2(W)$ with $\iota^*\widetilde{\alpha} = \widetilde{f}^*[\sigma]$. Secondly, as $\overline{E}$ and E are cohomologically $(2n-3)$–dimensional, we can select $\widetilde{\eta}$ with the property that $\widetilde{\eta}^{n-1}|_{\overline{E}} = 0$.

Let U_k be contractible open subsets of a cover of B with trivializations ϕ_k, such that $\phi_k \circ \phi_j^{-1}$ are symplectomorphisms over $U_k \cap U_j$. We bring these neighbourhoods to W as $(\widetilde{g} \circ f)^{-1}(U_k) = \widetilde{U}_k$. Define $\psi_k := (\mathrm{proj} \circ \widetilde{\phi}_k \circ \widetilde{f})\colon \widetilde{U}_k \to F$. Over $\widetilde{U}_k$, there is a 1–form μ_k such that $d\mu_k = \psi^*\widetilde{\sigma}_k - \widetilde{\eta}_k$, since $[\widetilde{\eta}] = \widetilde{f}^*|_F(\alpha) = [\psi^*\widetilde{\sigma}]$.

The rest of the proof follows similarly as in steps 3 and 4 of the proof of Theorem 1.1. Choose a partition of unity $\rho\colon W \to [0,1]$ in such a way that its open subsets do not touch $\overline{E}$, and with it define a closed 2–form $\beta = \widetilde{\eta} + \sum_k \rho_k d\mu_k$ on W. This form has the properties that $\beta|_{\overline{E}} = \widetilde{\eta}|_{\overline{E}}$ and $\beta|_F = \sigma_b$, where σ_b is the form of the fibre F_b. Finally, we build up our global form by adding $\widetilde{g}^*\omega_X$. If K is a sufficiently large positive real number, then we have a closed 2–form which is nondegenerate away from Z:

$$\omega_K = \beta + K \cdot \widetilde{g}^*\omega_X. \qquad \square$$

Example 3.5 (near-symplectic manifolds coming from bLfs) Broken Lefschetz fibrations also provide ways to obtain near-symplectic fibrations on $2n$–manifolds over near-symplectic $(2n-2)$–manifolds. Let $g\colon M \to B$ be a bLf as defined previously with singular fold set $\Sigma_{\widetilde{g}}$, where M is near-symplectic with $\dim(M) \geq 4$, and B is a closed, connected, symplectic manifold with $\dim(B) \geq 2$. Furthermore, consider (X, ω_X) to be a symplectic manifold with $\dim(X) \geq 4$. Assume that there is a class $\alpha \in H^2(M)$ such that $\langle \alpha, F \rangle > 0$ and $\widetilde{\alpha}|_{\Sigma_{\widetilde{g}}} = \omega_X|_{\Sigma_{\widetilde{g}}}$. Then W is near-symplectic via a generalized bLf $\widetilde{g}$.

If both $f\colon X \to B$ and $g\colon M \to B$ are two bLfs, then we require the intersection of their critical images to be transversal in B, but not necessarily disjoint. In that case, it follows from a standard differential topological argument that W is a $2n$–dimensional manifold. The maps $\widetilde{f}$ and $\widetilde{g}$ become near-symplectic fibrations, carrying the same type and number of fold and Lefschetz-type singularities as f and g, respectively. Around a critical point in f^*M, the maps $\widetilde{f}$ and $\widetilde{g}$ are locally modelled by coordinate charts φ and π respectively defined as

$$\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}, \qquad (r_1, \dots, r_{2n}) \mapsto (r_1^2 + r_2^2 - r_3^2, r_4, \dots, r_{2n}),$$
$$\pi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}, \qquad (r_1, \dots, r_{2n}) \mapsto (r_1, \dots, r_{2n-3}, -r_{2n-2}^2 + r_{2n-1}^2 + r_{2n}^2).$$

Assume the cohomological condition on the class $\widetilde{\alpha} \in H^2(W)$ as above, and denote by Γ the singular locus of ω_X, and Σ the singularity set of $\widetilde{g}$. The mapping $\widetilde{g}$ becomes a near-symplectic fibration over a near-symplectic base (X, ω_X), if $\widetilde{g}^{-1}(\Gamma) \not\subset \Sigma$ in W. This construction gives 2 generalized bLfs, one for each pullback mapping.

Remark 3.6 If we would like to consider deformations of near-symplectic fibrations, in a similar fashion as Lekili [13], then it would be necessary to consider all stable singularities of maps from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n-2}$. For maps going from a 6–dimensional source to a 4–dimensional target, there are 4 stable singularities: folds, cusps, swallowtails, and butterflies [9]. For higher dimensions, the list becomes longer and more complicated.

4 Geometry of the singular locus

In this section, we study the geometry around the singular locus induced by the near-symplectic form. First, we show that the singular locus carries a natural Hamiltonian structure. Then we show that if Z admits a stable Hamiltonian structure, so does its normal sphere bundle $Z \times S^2$ in the case where the normal bundle is trivial. In the second part, we describe the splitting property of the normal bundle following from a near-symplectic structure, similar to the 4–dimensional case. Then, we give a neighbourhood-type theorem. As a corollary, we find a local Darboux-type statement for near-symplectic forms.

4A Stable Hamiltonian structures

We present the next definitions as exposed by Cielebak and Volkov [5].

Definition 4.1 A *Hamiltonian structure* (HS) on an oriented $(2n-1)$–dimensional manifold M is a closed 2–form Ω such that $\Omega^{n-1} \neq 0$ everywhere. Associated to Ω is its 1–dimensional *kernel distribution* $\ker(\Omega) := \{v \in TM \mid \iota_v\Omega = 0\}$. We orient $\ker(\Omega)$ using the orientation on M together with the orientation on the local transversal to $\ker(\Omega)$ given by Ω^{n-1}.

A *stabilizing* 1*–form* for Ω is a 1–form λ such that

(1) $\lambda \wedge \Omega^{n-1} > 0$, and

(2) $\ker(\Omega) \subset \ker(d\lambda)$.

A Hamiltonian structure Ω is called *stabilizable* if it admits a stabilizing 1–form λ. A *stable Hamiltonian structure (SHS)* is the pair (Ω, λ).

An SHS (Ω, λ) induces a canonical *Reeb vector field* R generating $\ker(\Omega)$ and normalized by $\lambda(R) = 1$. Note that if (Ω, λ) is an SHS, then $(\Omega, -\lambda)$ is an SHS inducing the opposite orientation.

Example 4.2 (1) **Contact manifolds** (M, λ) is a contact manifold, R is the Reeb vector field, and $\Omega = \pm d\lambda$.

(2) **Mapping tori** $M := W_\phi = \mathbb{R} \times W/(t,x) \sim (t+1, \phi(x))$ is the mapping torus of a symplectomorphism ϕ of a symplectic manifold $(W, \overline{\omega})$, $R = \partial/\partial t$, $\lambda = dt$, and Ω is the form on M induced by $\overline{\omega}$. Note that $d\lambda = 0$, so $\ker(\lambda)$ defines a foliation. Notice that $W_\phi = [0,1] \times W/(0,x) \simeq (1, \phi(x))$.

(3) **Circle bundles** $\pi\colon M \to W$ is a principal circle bundle over a symplectic manifold $(W, \overline{\omega})$, R is the vector field generating the circle action, λ is the connection form, and $\Omega = \pi^*\overline{\omega}$.

The next class of examples follows directly from the definition of a near-symplectic form.

Proposition 4.3 *A near-symplectic structure induces a Hamiltonian structure on its singular locus Z_ω.*

Proposition 4.4 *Let $(Z \times \mathbb{R}^3, \omega)$ be a near-symplectic manifold with singular locus $Z \times \{0\}$, where Z is an oriented $(2n-1)$–manifold. If ε is a stabilizing 1–form for ω_Z on Z, then the normal sphere bundle $Z \times S^2$ has a stable Hamiltonian structure.*

Proof By assumption, we have that $\varepsilon \wedge \omega_Z^{n-2} > 0$ on Z and $\ker(\omega_Z) \subset \ker(d\varepsilon)$. Let σ_{S^2} be the symplectic form of S^2. The boundary of a piece of the tubular neighbourhood $\partial(Z \times B^3) = Z \times S^2$ can be equipped with a Hamiltonian structure by

$$\overline{\omega} = \omega_Z + \sigma_{S^2}. \tag{10}$$

This is a closed 2–form of maximal rank on $Z \times S^2$ since $\overline{\omega}^{n-1} = \omega_Z^{n-2} \wedge \sigma_{S^2} > 0$. The stabilizing 1–form on $Z \times S^2$ is defined by $\lambda = \varepsilon$. We have

$$\lambda \wedge \overline{\omega}^{n-1} = \varepsilon \wedge (\overline{\omega}^{n-2} \wedge \sigma_{S^2}) > 0.$$

This shows the first condition of an SHS. Now for the second property, observe that

$$\ker(\overline{\omega}) = \{v \in TM \mid \iota_v\overline{\omega} = \iota_v(\omega_Z + \sigma_{S^2}) = 0\} = \ker(\omega_Z).$$

In this case, $\ker(\overline{\omega}) \subset \ker(d\lambda)$. The pair $(\overline{\omega}, \lambda)$ is a stable Hamiltonian structure for $Z \times S^2 \subset (M, \omega_{\mathrm{ns}})$. □

Stable Hamiltonian in bLf case

Proposition 4.5 *Let $(Z, \xi_Z = \ker(\alpha_Z))$ be a contact manifold of $\dim(Z) = 2n-1$ and $(Z \times \mathbb{R}, \omega_B = d(e^t\alpha_Z))$ its symplectization. Let $f\colon Z \times \mathbb{R}^3 \to Z \times \mathbb{R}$ be a broken Lefschetz fibration. The total space $Z \times \mathbb{R}^3$ is near-symplectic inducing a stable Hamiltonian structure on $Z \times S^2$.*

Proof We now equip $M = Z \times \mathbb{R}^3$ with a near-symplectic form along the lines of [2] and Theorem 1.1. Over the regular neighbourhood of Z, using the coordinates (x_i) of the fibre, define the 2–form

$$\tau = d(x_1(x_2dx_3 - x_3dx_2)). \tag{11}$$

We obtain a closed 2–form that is positive on the fibres and nondegenerate outside Z. Define the 2–form $\omega \in \Omega^2(Z \times \mathbb{R}^3)$ as

$$\omega = \tau + f^*\omega_B.$$

At the points where $\omega^n = 0$, we have a 4–dimensional kernel

$$K = \{v \in T_pM \mid \omega_p(v, \cdot) = 0\} \simeq \varepsilon_p \oplus T_pY^3,$$

where $\varepsilon = \ker(f^*\omega_B)$. The 2–form ω defines a near-symplectic structure on $Z \times \mathbb{R}^3$.

Let $\mathcal{U}$ be the tubular neighbourhood of Z in M and σ_{S^2} the area form of S^2. Define on the boundary of $\mathcal{U}$ the 2–form

$$\overline{\omega} = d\alpha_Z + \sigma_{S^2}. \tag{12}$$

The contact form α_Z will work as the stabilizing 1–form $\lambda = \alpha_Z$. A simple computation shows that

$$\lambda \wedge \overline{\omega}^{n-1} = \alpha_Z \wedge d\alpha_Z^{n-2} \wedge \sigma_{S^2} > 0.$$

Moreover, since $\ker(\overline{\omega}) \simeq \varepsilon \simeq \ker(d\alpha_Z)$, the second property is also satisfied. Hence the pair $(\overline{\omega}, \alpha_Z)$ defines a stable Hamiltonian structure on the boundary of the singular locus $Z \times S^2$. □

4B Normal bundle of Z

In this section, we will first show that the definition of near-symplectic form reflects properties on the normal bundle of the singular locus similar to dimension 4. In particular, we obtain a splitting of the normal bundle N_Z into two subbundles.

Let $K := \varepsilon \oplus N_Z$ be defined by $\varepsilon = \ker(\omega_Z)$ and the normal bundle of Z, the singular locus of ω. Fix a metric g on K such that $\omega|_K$ is self-dual. Identify the intrinsic normal bundle N_Z with the complement $(TZ)^{\perp}$ using the metric g. From the transversality of ω, the image of the intrinsic gradient $D_K := \nabla\omega|_K$ is 3–dimensional. In fact, we have that $\mathrm{Im}(D_k) = \Lambda^2_+ K^*$. Thus we have a natural identification with the bundle of self-dual 2–forms. This implies that D_K defines an isomorphism

$$N_Z \to \Lambda^2_+ K^*.$$

Let $X = \partial/\partial z_0$ be the unit vector field defined on the line $\ker(\omega|_Z) \subset TZ$. The interior derivative defines a bundle isomorphism

$$\Lambda^2_+ K^* \to N_Z^*, \quad \beta \mapsto \iota_X \beta.$$

Its inverse $N_Z^* \to \Lambda^2_+$ is given by $\nu \mapsto \zeta \wedge \nu + *(\zeta \wedge \nu)$, where ζ is a 1–form that is nonvanishing on ε. Using the metric g we can define an isomorphism $N_Z^* \to N_Z$. The endomorphism

$$F\colon N_Z \to N_Z$$

defined by the composition

$$N_Z \xrightarrow{D_K} \Lambda_+ \xrightarrow{\iota_X} N_Z^* \xrightarrow{g} N_Z$$

is a self-adjoint, trace-free automorphism as in dimension 4 [11; 14]. The matrix A representing this map is symmetric and trace-free. Consequently, at each point $p \in Z$, A has three real eigenvalues, two of them positive and one negative, following the sign convention used in low dimensions [11; 14; 16]. We obtain a splitting of the normal bundle in two eigensubbundles defined by the negative and positive eigenspaces:

$$N_Z \simeq L_- \oplus L_+.$$

Here L_- is a rank-1 bundle, locally trivial, and L_+ is a rank-2 bundle, the orthogonal complement to L_-. After a choice of basis, the linear map F can be represented by a trace-free symmetric matrix $A = A_+ \oplus A_-$, where A_+ is a 2×2 positive-definite matrix, and $A_- < 0$.

4C Proof of Theorem 1.2

Step 1: Family of near-symplectic forms Define $\omega_t = (1-t)\omega_0 + t\cdot\omega_1$. We want to show that this is a family of near-symplectic forms. The closedness property follows from the fact that this family is a linear combination of closed 2–forms. The symplectic subspaces defined by ω_{Z_0} and ω_{Z_1} are the same on $TZ_0 \simeq \mathrm{Symp}_0 \oplus \varepsilon_0$ and $TZ_1 \simeq \mathrm{Symp}_1 \oplus \varepsilon_1$. This defines the same complementary line bundle $\varepsilon = \ker(\omega_{Z_0}) = \ker(\omega_{Z_1})$.

The kernels $K_0 \simeq \varepsilon \oplus N_{Z_0}$ and $K_1 \simeq \varepsilon \oplus N_{Z_0}$ are 4–dimensional. Interpolating between ω_0 and ω_1 leaves $\dim(K_t) = 4$ for all t. Thus up to scaling, the intrinsic gradients $D_{K_0} := \nabla\omega|_{K_0}$ and $D_{K_1} := \nabla\omega|_{K_1}$ agree, and so do their images. Hence, at a point $p = 0$ in Z, we have that $\omega_t^n = 0$. Notice that this property can also be computed directly by looking at the expansion

$$\omega_t^n(0) = c_n(t)\omega_0^n(0) + c_{n-1}(t)\binom{n}{1}\cdot\omega_0^{n-1}\wedge\omega_1(0) + c_{n-2}(t)\binom{n}{2}\omega_0^{n-2}\wedge\omega_1^2(0) + \cdots + c_0(t)\omega_1^n(0),$$

where $c_k(t) = (1-t)^k\cdot t^{n-k}$ for $k \in \{0,\dots,n\}$. In the previous expression, all terms vanish since each of them necessarily takes four vectors from K_t.

Now we show that ω_t^n is nonnegative. Let v be a vector in N_Z and $s \in \mathbb{R}$. Consider the Taylor expansion around $p \in Z$:

$$\begin{aligned}
\omega_t^n(s\cdot v) = {} & \underbrace{\omega_0^n(0)}_{=0} + s\cdot\underbrace{\omega_0^{n-1}}_{=0}\wedge\nabla_v\omega_0 + s^2\cdot\omega_0^{n-2}\wedge(\nabla_v\omega_0)^2 + \cdots + \underbrace{\omega_0^k\wedge\omega_1^{n-k}(0)}_{=0} \\
& + s\cdot\underbrace{\omega_0^{k-1}\wedge\omega_1^{n-k}}_{=0}\wedge\nabla\omega_0 + s\cdot\underbrace{\omega_0^k\wedge\omega_1^{n-k-1}}_{=0}\wedge\nabla\omega_1 \\
& + s^2\cdot\big(\omega_0^{k-2}\wedge\omega_1^{n-k}\wedge(\nabla_v\omega_0)^2 + \omega_0^{k-1}\wedge\omega_1^{n-k-1}\wedge(\nabla_v\omega_0\wedge\nabla_v\omega_1) \\
& + \omega_0^k\wedge\omega_1^{n-k-2}\wedge(\nabla_v\omega_1)^2\big) + \cdots \\
= {} & \underbrace{\omega_1^n(0)}_{=0} + s\cdot\underbrace{\omega_1^{n-1}}_{=0}\wedge\nabla_v\omega_1 + s^2\cdot\omega_1^{n-2}\wedge(\nabla_v\omega_1)^2 + \cdots
\end{aligned}$$

The terms of the form $\omega_0^k\wedge\omega_1^{n-k}$ for $k \in \{0,\dots,n\}$ vanish identically as explained in the previous paragraph. The linear terms of the form $\omega_0^{k-1}\wedge\omega_1^{n-k+1}\wedge\nabla_v\omega_i$ for $i \in \{0,1\}$ are also zero since, from the $2n-2$ vectors v_i which are allocated in $\omega_0^{k-1}\wedge\omega_1^{n-k+1}(v_1,\dots,v_{2n-2})$, at least two of those vectors should come from K_t.

This leaves us with the following expression with leading terms of order s^2:

$$\begin{aligned}\omega_t^n(s\cdot v) = s^2\cdot\big(&\omega_0^{n-2}(0)\wedge(\nabla_v\omega_0)^2+\cdots+\omega_0^{n-2}\wedge(\nabla_v\omega_0\wedge\nabla_v\omega_1)+\cdots\\ &+\omega_0^{n-3}\wedge\omega_1(\nabla_v\omega_0)^2+\cdots+\omega_0^{n-2}\wedge(\nabla_v\omega_1)^2+\cdots\\ &+\omega_0^{n-3}\wedge\omega_1(\nabla_v\omega_0\wedge\nabla_v\omega_1)+\cdots+\omega_0^{n-4}\wedge\omega_1^2\wedge(\nabla_v\omega_0)^2+\cdots\\ &+\omega_0^{n-3}\wedge\omega_1\wedge(\nabla_v\omega_0)^2+\cdots+\omega_0^{n-4}\wedge\omega_1^2(\nabla_v\omega_0\wedge\nabla_v\omega_1)+\cdots\\ &+\omega_0^{n-5}\wedge\omega_1^3\wedge(\nabla_v\omega_0)^2+\cdots+\omega_1^{n-2}\wedge(\nabla_v\omega_0\wedge\nabla_v\omega_1)+\cdots\\ &+\omega_0\wedge\omega_1^{n-3}(\nabla_v\omega_0)^2+\cdots+\omega_1^{n-2}(0)\wedge(\nabla_v\omega_1)^2+\cdots\big).\end{aligned}$$

Factorizing the $(n-2)$–forms which are symplectic on Z, we can rewrite the previous expression as

$$\begin{aligned}\text{(13)}\quad \omega_t^n(s\cdot v) = s^2\cdot\big(&\omega_0^{n-2}(0)\wedge\big((\nabla_v\omega_0)^2+\nabla_v\omega_0\wedge\nabla_v\omega_1+(\nabla_v\omega_1)^2\big)+\cdots\\ &+\omega_0^{n-k}\wedge\omega_1^k\wedge\big((\nabla_v\omega_0)^2+\nabla_v\omega_0\wedge\nabla_v\omega_1+(\nabla_v\omega_1)^2\big)+\cdots\\ &\qquad+\omega_1^{n-2}(0)\wedge\big((\nabla_v\omega_0)^2+\nabla_v\omega_0\wedge\nabla_v\omega_1+(\nabla_v\omega_1)^2\big)\big).\end{aligned}$$

As in Section 2, by restricting the terms $(\nabla_v\omega_0)^2$ and $(\nabla_v\omega_1)^2$ to vectors ∂_{k_i} in K_t, we have $(\nabla_v\omega_0)^2(\partial_{k_1},\dots,\partial_{k_4}) = D^2_{K_0}\geq 0$ and $(\nabla_v\omega_1)^2(\partial_{k_1},\dots,\partial_{k_4}) = D^2_{K_1}\geq 0$. Thus, in (13), the square binomial terms are nonnegative:

$$((\nabla_v\omega_0)|_K^2+\nabla_v\omega_0\wedge\nabla_v\omega_1|_K+(\nabla_v\omega_1)|_K^2) = (\nabla_v\omega_0|_K+\nabla_v\omega_1|_K)^2 := \nabla_v\omega_t^2|_K\geq 0.$$

Also, the forms $\omega_0^{n-k}\wedge\omega_1^k$ for $k\in\{0,1,\dots,n\}$ are positive on the symplectic subspace in Z, from which we conclude that $\omega_t^n\geq 0$ on the tubular neighbourhood of the singular locus.

Step 2: Poincaré lemma These next two steps follow the lines of Perutz [14], where we first use an application of the Poincaré lemma. The De Rham homotopy operator

$$Q\colon \Omega^k\to\Omega^{k-1},\quad Q\Omega=\int_0^1 h_t^*(\iota_R\Omega)\,dt$$

satisfies

$$\mathrm{Id}(\Omega)-(\iota\circ\pi)^*(\Omega) = dQ(\Omega)+Qd(\Omega).$$

Here we have $\pi\colon N_Z\to Z$ the bundle projection, $i\colon Z\to N_Z$ the zero section, $h_t\colon N_Z\to N_Z$, $x\to t\cdot x$ the fibrewise dilation, and R the Euler vector field. Applying this lemma to a neighbourhood of the zero section $\mathcal{U}_0\subset N_Z$, we find a 1–form $\lambda_t := Q(\omega_t)$ satisfying $d\lambda_t=\omega_t$ on $\mathcal{U}_0\setminus Z$. Moreover, notice that ω_t vanishes up to degree 1 on K_t. Inserting the Euler vector field R into ω_t adds one degree more and produces a 1–form $\iota_R\omega_t$ that vanishes on Z up to degree 2.

Step 3 We proceed in two parts. First we consider the case on $\mathcal{U}_0 \setminus Z$, where the argument is very similar as in dimension 4. Then we focus on the symplectic subspace inside Z.

On $\mathcal{U}_0 \setminus Z$, where ω_t is near-symplectic, introduce vector fields X_t defined by

$$\iota_{X_t}\omega_t + \lambda_t = 0. \tag{14}$$

We want to show that, on the tubular neighbourhood, X_t shrinks as it approaches Z. On the other four complementary directions defined by K_t, we have that $\nabla\lambda_t(u) = 0$ for all nonzero vectors $u \in N_{Z_0}$ since λ_t vanishes to the second order along Z. Furthermore, ω_t degenerates on K_t, and a Taylor expansion shows that $\nabla\omega_t \neq 0$ on K, so that $|X^t_K(x)| \leq C|x|$ for a constant C, as shown in [14].

On the symplectic subspace in Z, we have $\lambda_t|_{\mathrm{Symp}_Z} = 0$, but the restriction $\omega_t|_{\mathrm{Symp}_Z}$ is nondegenerate on this subspace. Thus, in order to satisfy (14), the vector field X_t needs to vanish on Symp_Z. In particular, the components of the vector field along the symplectic subspace satisfy $|X^t_{\mathrm{Symp}}(x)| \leq c|x|$ for a constant c.

The family $\{X_t\}_{t\in[0,1]}$ generates a flow $\{\psi_t\}_{t\in[0,1]}$ on $\mathcal{U}_0$ outside Z. A trajectory x_s defined on some interval $[0,\tilde{s}]$ satisfies $d(\log|x_s|)/ds \geq -C$. Integrating over $[0,\tilde{s}]$, we obtain $|x_{\tilde{s}}| \geq e^{C\tilde{s}}|x_0|$. This shows that the trajectory stays inside $\mathcal{U}_0 \setminus Z_0$; hence the flow ψ_s is well defined.

Step 4 Define on $\mathcal{U}_0 \setminus Z_0$

$$\tilde{\omega}_t := \psi_t^*\omega_t,$$

and on Z_0

$$\tilde{\omega}_t := \omega_{Z_0}.$$

Moser's argument shows that $\tilde{\omega}_t = \omega_t$ in some neighbourhood of Z. The diffeomorphism ψ_1 is not defined on Z. Extend it to Z by the identity. At the level of the singular locus, we can take the diffeomorphism to be the one from the theorem's assumption that $Z_0 \approx Z_1$. This leads to a homeomorphism, which is a diffeomorphism away from Z.

Finally, set $\varphi = \psi_1$ and $\psi_1(\mathcal{U}_0) = \mathcal{U}_1$. Then we have that $\varphi^*\omega_1 = \omega_0$ away from Z, but ω_1 and ω_0 agree on Z by assumption. □

4D Proof of Corollary 1.3

This proof uses the previous theorem and an adaptation of an argument from [14].

Proof First let γ be a closed interval inside the line $\varepsilon = \ker(\omega_Z)$. Now consider $\kappa := \gamma \times B^3 \subset K = \varepsilon \oplus N_Z$. Identify an open subset of Z with $V \times \{0\} \subset U \simeq V \times B^3_0(R)$ inside M, such that $\kappa \subset U$. Denote by z_0 the coordinate on γ and by ∂_{z_0} a positively oriented vector field on γ for the orientation determined by ω.

Take a metric g for which $\omega|_\kappa$ is self-dual. We can find an orthonormal frame (e_1, e_2, e_3) for N_Z such that $L_- = \mathrm{span}(e_1)$ and $L_+ = \mathrm{span}(e_2, e_3)$. The metric and the choice of e_i provide normal coordinates $(\overline{z}, z_0, x_1, x_2, x_3)$ on a small neighbourhood of p inside U, where $\overline{z}$ correspond to the $(2n-4)$ coordinates complementary to z_0 on Z. Using these coordinates, we can write three basis elements β_i of $\Lambda^2_+ K^*$:

$$\begin{aligned}\beta_1 &= dz_0 \wedge dx_1 + dx_2 \wedge dx_3,\\ \beta_2 &= dz_0 \wedge dx_2 - dx_1 \wedge dx_3,\\ \beta_3 &= dz_0 \wedge dx_3 + dx_1 \wedge dx_2.\end{aligned}$$

Let $\widetilde{F} = \widetilde{F}_- \oplus \widetilde{F}_+$ be a matrix representing the linear map $F \in \mathrm{End}(N_Z)$ with respect to (e_1, e_2, e_3), and let $x = (x_1, x_2, x_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)^T$. Expand ω near Z to obtain

$$\begin{aligned}\omega(z,x) &= \omega|_Z + (x \cdot \widetilde{F} \cdot \beta^T + O(x^2))\\ &= \omega|_Z + \left(x_1 \widetilde{F}_- \beta_1 + (x_2, x_3)\widetilde{F}_+ \begin{pmatrix}\beta_2\\ \beta_3\end{pmatrix} + O(x^2)\right).\end{aligned}$$

Define, on a small neighbourhood of Z, a family of near-symplectic forms with common singular locus Z by

$$\omega_t = (1-t)\omega + t \cdot (\omega|_Z - 2x_1\beta_1 + x_2\beta_2 + x_3\beta_3).$$

Following the same reasoning as in the proof of the previous theorem in a local setting, we can show that this is a family of near-symplectic forms with common degeneracy locus Z. The next steps follow as in the previous proof. □

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