The fattened Davis complex
and weighted L^2–(co)homology of Coxeter groups

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This article consists of two parts. First, we propose a program to compute the weighted L^2–(co)homology of the Davis complex by considering a thickened version of this complex. The program proves especially successful provided that the weighted L^2–(co)homology of certain infinite special subgroups of the corresponding Coxeter group vanishes in low dimensions. We then use our complex to perform computations for many examples of Coxeter groups. Second, we prove the weighted Singer conjecture for Coxeter groups in dimension three under the assumption that the nerve of the Coxeter group is not dual to a hyperbolic simplex, and in dimension four under the assumption that the nerve is a flag complex. We then prove a general version of the conjecture in dimension four where the nerve of the Coxeter group is assumed to be a flag triangulation of a 3–manifold.

1 Introduction

Given a Coxeter system (W, S) with nerve L, Davis defines a contractible simplicial complex Σ_L on which W acts properly and cocompactly. We provide the definition in Section 2, but more details can be found in [6; 5]. Given an S–tuple $\boldsymbol{q} = (q_s)_{s \in S}$ of positive real numbers, where $q_s = q_{s'}$ if s and s' are conjugate in W, one defines the weighted L^2–(co)chain complex $L^2_{\boldsymbol{q}} C_*(\Sigma_L)$ and the weighted L^2–(co)homology spaces $L^2_{\boldsymbol{q}} H_k(\Sigma_L)$; see Davis, Dymara, Januszkiewicz, and Okun [8]. They are special in the sense that they admit a notion of dimension: one can attach a nonnegative real number to each of the Hilbert spaces $L^2_{\boldsymbol{q}} H_k(\Sigma_L)$ called the von Neumann dimension. Hence one defines weighted L^2–Betti numbers, denoted by $L^2_{\boldsymbol{q}} b_k(\Sigma_L)$. We present a brief introduction to this theory, but more details can be found in [6; 8] and in Dymara [13].

In [8], weighted L^2–(co)homology was explicitly computed for CW complexes on which a Coxeter group acts properly and cocompactly by reflections whenever $\boldsymbol{q} \in \overline{\mathcal{R}} \cup \overline{\mathcal{R}}^{-1}$, where $\mathcal{R}$ denotes the region of convergence of the growth series of the Coxeter group. These formulas generalize those of Dymara [13] for Σ_L and

also compute the ordinary L^2–(co)homology of buildings of type (W, S) with large integer thickness vectors. Furthermore, the Singer conjecture for Coxeter groups was formulated for weighted L^2–(co)homology:

Conjecture 1.1 (weighted Singer conjecture) *Suppose that L is a triangulation of S^{n-1}. Then*

$$L_{\boldsymbol{q}}^2 H_k(\Sigma_L) = 0 \quad for \quad k > \frac{n}{2} \quad and \quad \boldsymbol{q} \leq \mathbf{1}.$$

By weighted Poincaré duality, this is equivalent to the conjecture that if $\boldsymbol{q} \geq \mathbf{1}$ and $k < n/2$, then $L_{\boldsymbol{q}}^2 H_k(\Sigma_L)$ vanishes. Conjecture 1.1 is true by [12, Theorem 2.1] for $n \leq 2$, and in [8], it was proved for the case where W is right-angled and $n \leq 4$. Furthermore, it was shown that Conjecture 1.1 for n odd implies Conjecture 1.1 for n even (also under the assumption that W is right-angled). This was done following the techniques of Davis and Okun [9] in their proof of the right-angled case of the conjecture when $\boldsymbol{q} = \mathbf{1}$. The progress for the conjecture when $\boldsymbol{q} = \mathbf{1}$ is as follows. A result of Lott and Lück [17], in conjunction with the validity of the geometrization conjecture for 3–manifolds due to Perelman [20], proves the case when $n = 3$. It was later proved for the case where W is an even Coxeter group and $n = 4$ by Schroeder [23], under the restriction that L is a flag complex. Recently, Okun and Schreve [19, Theorem 4.9] gave a proof for the case $n = 4$, so now Conjecture 1.1 is known to hold for dimensions $n \leq 4$ when $\boldsymbol{q} = \mathbf{1}$.

As evidenced above, there is little progress on the conjecture whenever $\boldsymbol{q} \neq \mathbf{1}$, and weighted L^2–Betti numbers have proven to be difficult to compute in general, with very little known when $\boldsymbol{q} \notin \overline{\mathcal{R}} \cup \overline{\mathcal{R}}^{-1}$. The goal of this article is to prove variations of Conjecture 1.1 in dimensions three and four, and to propose a method to compute weighted L^2–Betti numbers. Of note is that most of our computations are done for $\boldsymbol{q} \geq \mathbf{1}$, and hence they compute the ordinary L^2-(co)homology of the buildings associated to the corresponding Coxeter groups with thickness vector $\boldsymbol{q}$ [8, Theorem 13.8].

The article is structured as follows. We first introduce some basic notions and definitions, and then we proceed to construct what we call the fattened Davis complex. The idea is to "fatten" Σ_L to a (homology) manifold with boundary so that we have standard algebraic topology tools (such as Poincaré duality) at our disposal. We carefully perform this fattening so that we can understand the weighted L^2–(co)homology of the boundary; in fact, understanding this will simply amount to understanding the weighted L^2–(co)homology of certain infinite special subgroups of W. We then study the structure and algebraic topology of the fattened Davis complex and proceed to perform computations for many examples of Coxeter groups.

For the purpose of stating the theorems, we label the edges of the nerve as follows, resulting in what we call the *labeled nerve*. The vertices of the nerve are the generators for the Coxeter system, so if the vertices of an edge are s and t, then we put the corresponding Coxeter relation m_{st} on that edge, where $(st)^{m_{st}} = 1$. With this terminology we state the first main theorem.

Theorem 4.3 *Suppose that the labeled nerve L of a Coxeter group W is the one-skeleton of a cell complex that is a generalized homology n–sphere, where $n \geq 2$. Furthermore, suppose that the vertex set of every 2–cell generates a Euclidean reflection subgroup of W. If $\boldsymbol{q} \geq \mathbf{1}$ then $L_{\boldsymbol{q}}^2 b_*(\Sigma_L)$ is concentrated in degree 2.*

Once concentration has been established, $L_{\boldsymbol{q}}^2 b_2(\Sigma_L)$ is equal to the weighted Euler characteristic, so an explicit formula can be obtained using [13, Corollary 3.4] and [6, Theorem 17.1.9]. Also, if we place some restrictions on either the labels or the cell complex, then the formula for $L_{\boldsymbol{q}}^2 b_2(\Sigma_L)$ becomes relatively simple; see Corollary 4.5. In Theorem 5.3, we discuss how Theorem 4.3 can be used to produce Coxeter groups which satisfy the weighted Singer conjecture in dimensions three and four. We then turn our attention to a class of Coxeter groups which, in the literature, are sometimes called quasi-Lánner groups.

Theorem 4.10 *Suppose that W acts properly but not cocompactly on hyperbolic space $\mathbb{H}^n$ by reflections with fundamental chamber an n–simplex of finite volume. Then $L_{\boldsymbol{q}}^2 b_k(\Sigma_L) = 0$ whenever $k \geq n - 1$ and $\boldsymbol{q} \leq \mathbf{1}$, or when $k \leq 1$ and $\boldsymbol{q} \geq \mathbf{1}$.*

Recall that a Coxeter system is 2–*spherical* if the one-skeleton of the corresponding nerve is a complete graph. In other words, this is equivalent to the condition that, for any distinct $s, t \in S$, we have the Coxeter relation $(st)^{m_{st}} = 1$, where $m_{st} \geq 2$ is a finite natural number. A Coxeter group is *Euclidean* if it acts properly and cocompactly by reflections on a Euclidean space of some dimension.

Theorem 4.12 *Suppose that (W, S) is infinite 2–spherical with $|S| \geq 5$. Suppose furthermore that:*

(1) *For every $T \subseteq S$ with $|T| \geq 5$, vcd $W_T \leq |T| - 2$.*

(2) *Every infinite subgroup W_T, with $|T| = 3, 4$, is Euclidean or quasi-Lánner.*

If $\boldsymbol{q} \geq \mathbf{1}$, then $L_{\boldsymbol{q}}^2 b_k(\Sigma_L) = 0$ for $k < 2$.

The above theorem implies a specialized version of the weighted Singer conjecture for 2–spherical Coxeter groups when $n = 4$; see Corollary 4.13. The computations of the

above theorems rely not only on the fattened Davis complex, but also on Lemma 2.7. In many cases, this lemma allows us to "push" the vanishing of high-dimensional weighted L^2–Betti numbers for $q = 1$ to $q \leq 1$. In fact, with the help of the work of Okun and Schreve [19], we obtain the following theorem.

Theorem 5.2 *Suppose that the nerve L is an $(n-1)$–disk. Then*

$$L_q^2 H_k(\Sigma_L) = 0 \quad \text{for} \quad k \geq n-1 \quad \text{and} \quad q \leq 1.$$

Note that, when $n = 3$ or 4, this theorem proves a version of the weighted Singer conjecture for the case where Σ_L is a manifold with boundary. We then prove Conjecture 1.1 for $n = 3$ or 4, under some additional restrictions on the nerve L. First, recall that the nerve L of a Coxeter system (W, S) has a natural piecewise spherical structure, and under this structure, if $s, t \in S$ are connected by an edge in L, then the edge has length $\pi - \pi/m_{st}$, where $(st)^{m_{st}} = 1$. Hence L inherits the structure of a *metric flag complex* [6, Lemma 12.3.1], meaning that any collection of pairwise connected edges of L spans a simplex if and only if there exists a spherical simplex with the corresponding edge lengths.

Theorem 1.2 *Suppose that the nerve L of a Coxeter group is a triangulation of S^2 not dual to a hyperbolic 3–simplex. Then*

$$L_q^2 H_k(\Sigma_L) = 0 \quad \text{for} \quad k > 1 \quad \text{and} \quad q \leq 1.$$

Note that, in conjunction with weighted Poincaré duality and [13, Theorem 7.1], Theorem 1.2 yields ranges of concentration of the L_q^2–(co)homology groups as follows.

Corollary 1.3 *Suppose that the nerve L of a Coxeter group is a triangulation of S^2 not dual to a hyperbolic 3–simplex.*

- *If $q \in \overline{\mathcal{R}}$, then $L_q^2 H_*(\Sigma_L)$ is concentrated in dimension 0.*
- *If $q \notin \mathcal{R}$ and $q \leq 1$, then $L_q^2 H_*(\Sigma_L)$ is concentrated in dimension 1.*
- *If $q \notin \mathcal{R}^{-1}$ and $q \geq 1$, then $L_q^2 H_*(\Sigma_L)$ is concentrated in dimension 2.*
- *If $q \in \overline{\mathcal{R}}^{-1}$, then $L_q^2 H_*(\Sigma_L)$ is concentrated in dimension 3.*

In either case, one can use [13, Corollary 3.4], along with a standard computation for growth series [6, Theorem 17.1.9], to explicitly compute each L_q^2–Betti number. The Coxeter groups whose nerves are dual to hyperbolic 3–simplices are sometimes called Lánner groups in the literature, and there are only nine Lánner groups in dimension three; see Humphreys [16, Table 6.9]. So there are only nine groups standing in the

way of proving Conjecture 1.1 in full generality for dimension three. In dimension four, we prove the following theorem. First, recall that a subcomplex A of a simplicial complex L is *full* if, whenever the vertices of a simplex of L lie in A, the simplex must lie in A.

Theorem 1.4 *Suppose that the nerve L of a Coxeter group is a triangulation of S^3. Furthermore, suppose that there exists a vertex of L such that its link is a full subcomplex of L and not dual to a hyperbolic 3–simplex. Then*

$$L_q^2 H_k(\Sigma_L) = 0 \quad \text{for} \quad k > 2 \quad \text{and} \quad \boldsymbol{q} \leq \mathbf{1}.$$

We obtain the following corollary.

Corollary 1.5 *Suppose that the nerve L of a Coxeter group is a flag triangulation of S^3. Then*

$$L_q^2 H_k(\Sigma_L) = 0 \quad \text{for} \quad k > 2 \quad \text{and} \quad \boldsymbol{q} \leq \mathbf{1}.$$

Proof   Since L is flag, it follows that the link of every vertex is a full subcomplex of L. Furthermore, the link of every vertex is not the boundary of a 3–simplex (and in particular, not dual to a 3–simplex). Theorem 1.4 now completes the proof.   □

We conclude the article by proving the following generalization of the above corollary.

Theorem 1.6 *Suppose that the nerve L of a Coxeter group is a flag triangulation of a 3–manifold. Then*

$$L_q^2 H_k(\Sigma_L) = 0 \quad \text{for} \quad k > 2 \quad \text{and} \quad \boldsymbol{q} \leq \mathbf{1}.$$

2 Preliminaries

Coxeter systems and Coxeter groups

A *Coxeter matrix* $M = (m_{st})$ on a set S is an $S \times S$ symmetric matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that

$$m_{st} = 1 \quad \text{if} \quad s = t, \quad \text{and} \quad m_{st} \geq 2 \quad \text{otherwise.}$$

One can associate to M a presentation for a group W as follows. Let S be the set of generators and let $\mathcal{I} = \{(s,t) \in S \times S \mid m_{st} \neq \infty\}$. The set of relations for W is

$$R = \{(st)^{m_{st}}\}_{(s,t) \in \mathcal{I}}.$$

The group defined by the presentation $\langle S, R \rangle$ is a *Coxeter group* and the pair (W, S) is a *Coxeter system*. If all off-diagonal entries of M are either 2 or ∞, then W is *right-angled*.

Given a subset $T \subset S$, W_T is the subgroup of W generated by the elements of T. Then (W_T, T) is a Coxeter system. Subgroups of this form are *special subgroups*. W_T is a *spherical subgroup* if W_T is finite and, in this case, T is a *spherical subset*. If W_T is infinite, then T is *nonspherical*.

Let $\mathcal{S}$ be the poset of all spherical subsets of S, partially ordered by inclusion. Then $\mathcal{S}$ is an abstract simplicial complex with vertex set S. Let L be the geometric realization of the abstract simplicial complex $\mathcal{S}$. L is the *nerve* of (W, S).

Let c denote a point and let L be the labeled nerve. Consider the join $L' = c * L$, where all of the new edges are labeled by 2. L' is called the *right-angled cone on* L. Note that the corresponding Coxeter system to L' is $(W \times \mathbb{Z}_2, S \cup \{c\})$.

Mirrored spaces and mirrored homology manifolds with corners

A *mirror structure* over a set S on a space X is a family of subspaces $(X_s)_{s \in S}$ indexed by S. Then X is a *mirrored space over* S. Put $X_\varnothing = X$, and for each nonempty subset $T \subseteq S$, define the following subspaces of X:

$$X_T := \bigcap_{s \in T} X_s \quad \text{and} \quad X^T := \bigcup_{s \in T} X_s.$$

If (W, S) is a Coxeter system and X is a mirrored space over S, then the mirror structure $(X_s)_{s \in S}$ is W*–finite* if $X_T = \varnothing$ for all nonspherical $T \subseteq S$.

Suppose that X is a mirrored space over S with W–finite mirror structure. X is an S*–mirrored homology n–manifold with corners* if every nonempty X_T is a homology $(n-|T|)$–manifold with boundary $\partial X_T = \bigcup_{U \supsetneq T} X_U$. By taking $T = \varnothing$, this definition implies that the pair $(X, \partial X)$ is a homology n–manifold with boundary.

Set $S' = S \cup \{e\}$, where e is the identity element of W. We now say that $T \subseteq S'$ is spherical if and only if $T - \{e\}$ is spherical. A mirrored space X over the set S' with W–finite mirror structure $(X_s)_{s \in S'}$ is a *partially S–mirrored homology n–manifold with corners* if every nonempty X_T is a homology $(n-|T|)$–manifold with boundary $\partial X_T = \bigcup_{U \supsetneq T} X_U$. To summarize, we simply have defined the non-S–mirrored part of X to be an auxiliary mirror corresponding to the identity element of W.

Basic construction

Suppose that (W, S) is a Coxeter system and that X is a mirrored space over S. Put $S(x) := \{s \in S \mid x \in X_s\}$ and define an equivalence relation $\sim$ on $W \times X$ by $(w, x) \sim (w', y)$ if and only if $x = y$ and $w^{-1}w' \in W_{S(x)}$. Give $W \times X$ the product topology and let $\mathcal{U}(W, X)$ denote the quotient space

$$\mathcal{U}(W, X) = (W \times X)/\sim .$$

$\mathcal{U}(W, X)$ is the *basic construction* and X is the *fundamental chamber*. A version of the following proposition appears in [6, Proposition 10.7.5] without proof.

Proposition 2.1 *Suppose that (W, S) is a Coxeter system and suppose that X is a partially S–mirrored homology n–manifold with corners. Set $Y = X_e$ and give Y the induced mirror structure $(Y_s)_{s \in S}$, where $Y_s := Y \cap X_s$. Then $\mathcal{U}(W, X)$ is a homology n–manifold with boundary $\partial\mathcal{U}(W, X) = \mathcal{U}(W, Y)$.*

Proof Before proving the proposition, we first prove the following claim.

Claim *If X is an S–mirrored homology n–manifold with corners, then $\mathcal{U}(W, X)$ is a homology n–manifold.*

Proof of claim Without loss of generality suppose that $x \in X$. By excision, we need to show that the local homology groups $H_*(U, U - x)$ are correct for some neighborhood U of x in $\mathcal{U}(W, X)$. If $x \in X - \partial X$, then we are done since $X - \partial X$ is a homology n–manifold, and x does not lie in any mirror. So suppose that $|S(x)| \geq 1$.

Let V be a neighborhood of x in X. For each $s \in S(x)$, set $V_s = V \cap X_s$, and give V the mirror structure $\{V_s\}_{s \in S(x)}$. Note that $V_T = V \cap X_T$ for each $T \subseteq S(x)$. Now $x \in X_{S(x)}$, so $x \in \partial X_T$ for each $T \subset S(x)$ (X_T is, by assumption, a homology $(n-|T|)$–manifold with boundary, and $X_{S(x)} \subseteq \partial X_T$). Furthermore, x does not lie in $\partial X_{S(x)}$. Therefore, by excision, it follows that for each $T \subset S(x)$, the local homology groups $H_*(V_T, V_T - x)$ vanish, and so $H_*(V_{S(x)}, V_{S(x)} - x)$ is concentrated in dimension $n - |S(x)|$, and it equals $\mathbb{Z}$ in that dimension.

Now define

$$Z := V \cup \mathrm{Cone}(V - x) \quad \text{and} \quad Z_s := V_s \cup \mathrm{Cone}(V_s - x).$$

So Z has the mirror structure $\{Z_s\}_{s \in S(x)}$. Since V is a neighborhood of x in X, and $x \in \partial X$, it follows that the local homology groups $H_*(V, V - x)$ vanish. In particular, $H_*(V) \cong H_*(V - x)$, and the Mayer–Vietoris sequence, along with the five lemma,

implies that Z is acyclic. Similarly, for each $T \subset S(x)$, since the local homology groups $H_*(V_T, V_T - x)$ vanish, it follows that Z_T is acyclic. Since $H_*(V_{S(x)}, V_{S(x)} - x)$ is concentrated in dimension $n - |S(x)|$ and is $\mathbb{Z}$ in that dimension, that Mayer–Vietoris sequence again implies that the same is true for $H_*(Z_{S(x)})$. In particular, $Z_{S(x)}$ has the same homology as $S^{n-|S(x)|}$. We now finish the proof of the claim by applying the following lemma:

Lemma 2.2 [6, Corollary 8.2.5] *$\mathcal{U}(W_{S(x)}, Z)$ has the same homology as S^n if and only if there is a unique spherical subset $R \subseteq S(x)$ satisfying these conditions:*

(a) *$W_{S(x)}$ decomposes as $W_{S(x)} = W_R \times W_{S(x)-R}$.*

(b) *For all spherical $T' \subseteq S(x)$ with $T' \neq R$, we have that $(Z, Z^{T'})$ is acyclic.*

(c) *(Z, Z^R) has the same homology as (D^n, S^{n-1}).*

We apply the lemma to $R = S(x)$. Condition (a) is then satisfied vacuously, so we wish to show (b) and (c). For $T \subseteq R$, consider the cover of Z^T by the mirrors $\{Z_s\}_{s \in T}$. Note that, for each $U \subset R$, the intersection of mirrors Z_U is acyclic. The nerve of this cover is a simplex on U and, in particular, is contractible. The acyclic covering lemma [3, Theorem 4.4, Chapter VII] then implies that Z^U is acyclic. Note that Z_R has the same homology as $S^{n-|R|}$, so a similar spectral sequence argument also implies that Z^R has the same homology as S^{n-1}. Now set $U = \mathcal{U}(W_R, V)$. Since $\mathcal{U}(W_R, Z) = U \cup \mathrm{Cone}(U - x)$ and $\mathcal{U}(W_R, Z)$ has the same homology as S^n, it follows that $H_*(U, U - x)$ is concentrated in dimension n and is $\mathbb{Z}$ in that dimension. Therefore, U is our desired neighborhood. This concludes the proof of the claim. $\square$

We now prove the proposition. Set $\mathcal{U} = \mathcal{U}(W, X)$ and $\partial\mathcal{U} = \mathcal{U}(W, Y)$. It follows from the above claim that $\partial\mathcal{U}$ is a homology $(n-1)$–manifold. This is because $Y = X_e$, and X_e (with its induced S–mirror structure) is an S–mirrored homology $(n-1)$–manifold with corners. Similarly, the claim implies that $\mathcal{U} - \partial\mathcal{U}$ is a homology n–manifold since $\mathcal{U} - \partial\mathcal{U} = U(W, Z)$, where $Z = X - Y$ (with its induced S–mirror structure) is an S–mirrored homology n–manifold with corners. It remains to show that, for each $x \in \partial\mathcal{U}$, the local homology groups $H_*(\mathcal{U}, \mathcal{U} - x)$ vanish.

Without loss of generality, we can assume that $x \in Y \subset \partial X$. If x does not lie in any mirror $(X_s)_{s \in S}$, then we are done by excision. So suppose $|S(x)| \geq 1$, and let V be a neighborhood of x in X. We now give V the S–mirror structure as in the proof of the claim, noting that the only difference between that proof and the current situation is the fact that the local homology groups $H_*(V_{S(x)}, V_{S(x)} - x)$ vanish. This is because, since $x \in Y$ and $|S(x)| \geq 1$, it follows that $x \in \partial X_{S(x)}$. Now, following the proof of the above claim line by line, the only difference now is that $Z_{S(x)}$ is
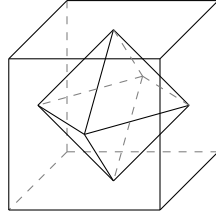
Figure 1: (Λ, S)–chamber when Λ is the boundary complex of an octahedron

acyclic (as opposed to having the homology of S^{n-1} as before). This then implies that $\mathcal{U}(W_{S(x)}, Z)$ is acyclic [6, Corollary 8.2.8], which in turn implies that the local homology groups $H_*(\mathcal{U}, \mathcal{U} - x)$ vanish. $\qquad\qquad\square$

The (Λ, S)–chamber

Suppose that Λ is a cell complex with vertex set S and let $\mathcal{F}(\Lambda)$ denote the poset of cells of Λ, including the empty set (here a *cell* is the convex hull of finitely many points in $\mathbb{R}^n$, and we always assume the cell complex to be locally finite). Let $P := |\mathcal{F}(\Lambda)|$ denote the geometric realization of the poset $\mathcal{F}(\Lambda)$. For each $T \in \mathcal{F}(\Lambda)$, define $P_T := |\mathcal{F}(\Lambda)_{\geq T}|$ and $\partial P_T := |\mathcal{F}(\Lambda)_{>T}|$, so each P_T is the cone on $b \operatorname{Lk}(T, \Lambda)$, the barycentric subdivision of $\operatorname{Lk}(T, \Lambda)$. In particular, taking $T = \varnothing$, we have that P is the cone on $b\Lambda$ with cone point corresponding to $\varnothing$. For each $s \in S$, put $P_s := P_{\{s\}}$. This endows P with the mirror structure $(P_s)_{s \in S}$. P is the (Λ, S)–*chamber*.

Recall that a space X is a *generalized homology n–sphere*, abbreviated GHS^n, if it is a homology n–manifold with the same homology as S^n. Similarly, the pair $(X, \partial X)$ is a *generalized homology n–disk*, abbreviated GHD^n, if it is a homology n–manifold with boundary with the same homology as the pair (D^n, S^{n-1}). Note that the cone on a generalized homology sphere is a generalized homology disk.

Now, if Λ is a GHS^{n-1}, then the link of every cell σ in Λ is a $\mathrm{GHS}^{n-\dim\sigma-2}$. It follows that P is a GHD^n, and that, for each $T \in \mathcal{F}(\Lambda)$, the pair $(P_T, \partial P_T)$ is a $\mathrm{GHD}^{n-\dim\sigma_T-1}$, where σ_T is the geometric cell in Λ spanned by T.

Let $b\sigma_T$ denote the barycentric subdivision of σ_T. By definition, $b\sigma_T$ is the $(\partial\sigma_T, T)$–chamber, and in particular, σ_T has a natural mirror structure over T. Now P is itself a flag simplicial complex, and P_T is a subcomplex of P for each $T \in \mathcal{F}(\Lambda)$. Hence $P_T - \bigcup_{U \supset T} P_U$ has a neighborhood of the form $\sigma_T * P_T$, the join of σ_T and P_T. Following the join lines for a little while, it follows that $P_T - \bigcup_{U \supset T} P_U$ has neighborhoods of the form $\mathrm{Cone}(\sigma_T) \times P_T$. We record this fact, as we will use it in an upcoming construction.

The Davis complex and ruins

If (W, S) is a Coxeter system, then by definition, the *Davis chamber* K is the (L, S)–chamber. The *Davis complex* Σ_L associated to the nerve L is

$$\Sigma_L := \mathcal{U}(W, K).$$

Note that Σ_L is naturally a simplicial complex, the simplicial structure induced by that of K, and moreover, it is proved in [5] that Σ_L is contractible. Furthermore, if L is a triangulation of an $(n-1)$–sphere (resp. $(n-1)$–disk), then Σ_L is an n–manifold (resp. n–manifold with boundary).

The Davis complex admits a decomposition into *Coxeter cells*. For each $T \in \mathcal{S}$, let v_T denote the corresponding barycenter in K. Let c_T denote the union of simplices $c \subset \Sigma_L$ such that $c \cap K_T = v_T$. The boundary of c_T is then cellulated by wc_U, where $w \in W_T$ and $U \subset T$. With its simplicial structure, the boundary ∂c_T is the Coxeter complex corresponding to the Coxeter system (W_T, T), which is a sphere since W_T is finite. It follows that c_T and its translates are disks, which are called *Coxeter cells of type T*. We denote Σ_L with this decomposition into Coxeter cells by Σ_{cc}. Note that Σ_{cc} is a regular CW complex with poset of cells that can be identified with $W\mathcal{S} := \{wW_U \mid w \in W, T \in \mathcal{S}\}$. The simplicial structure on Σ_L is the geometric realization of the poset $W\mathcal{S}$; hence Σ_L is the barycentric subdivision of Σ_{cc}.

Now, for $U \subset S$, set $\mathcal{S}(U) := \{T \in \mathcal{S} \mid T \subset U\}$. Define $\Sigma(U)$ to be the subcomplex of Σ_{cc} consisting of all (closed) Coxeter cells of type T with $T \in \mathcal{S}(U)$. Given $T \in \mathcal{S}(U)$, we define the following subcomplexes of $\Sigma(U)$:

$$\Omega_{UT} : \text{the union of closed cells of type } T', \text{ with } T' \in \mathcal{S}(U)_{\geq T},$$

$$\partial\Omega_{UT} : \text{the cells of } \Omega(U, T) \text{ of type } T'', \text{ with } T'' \notin \mathcal{S}(U)_{\geq T}.$$

The pair $(\Omega(U, T), \partial\Omega(U, T))$ is the (U, T)–*ruin*. For brevity, we sometimes write $(\Omega(U, T), \partial)$ to denote the (U, T)–ruin. Note that, if $T = \varnothing$, then $\Omega(U, T) = \Sigma(U)$ and $\partial\Omega(U, T) = \varnothing$.

For $s \in T$, set $U' = U - s$ and $T' = T - s$. As in [8, Proof of Theorem 8.3], we have the following weak exact sequence:

$$(1) \quad \cdots \to L_{\boldsymbol{q}}^2 H_*(\Omega(U', T'), \partial) \to L_{\boldsymbol{q}}^2 H_*(\Omega(U, T'), \partial) \to L_{\boldsymbol{q}}^2 H_*(\Omega(U, T), \partial) \to \cdots.$$

For the special case when $U = S$ and $T = \{s\}$, the above sequence becomes

$$\cdots \to L_{\boldsymbol{q}}^2 H_*(\Sigma(S - s)) \to L_{\boldsymbol{q}}^2 H_*(\Sigma(S)) \to L_{\boldsymbol{q}}^2 H_*(\Omega(S, s), \partial) \to \cdots.$$

Hecke–von Neumann algebras

Let (W, S) be a Coxeter system. For the remainder of this article, let $\boldsymbol{q} = (q_s)_{s \in S}$ denote an S–tuple of positive real numbers satisfying $q_s = q_{s'}$ whenever s and s' are conjugate in W. Set $\boldsymbol{q}^{-1} = (q_s^{-1})_{s \in S}$. If $w = s_1 \cdots s_n$ is a reduced expression for $w \in W$, we define $q_w := q_{s_1} \cdots q_{s_n}$.

Let $\mathbb{R}(W)$ denote the group algebra of W and let $\{e_w\}_{w \in W}$ denote the standard basis on $\mathbb{R}(W)$ (here e_w denotes the characteristic function of $\{w\}$). We deform the standard inner product on $\mathbb{R}(W)$ to an inner product

$$\langle e_w, e_{w'} \rangle_{\boldsymbol{q}} = \begin{cases} q_w & \text{if } w = w', \\ 0 & \text{otherwise.} \end{cases}$$

Using the multiparameter $\boldsymbol{q}$, one can give $\mathbb{R}(W)$ the structure of a *Hecke algebra*. We will denote $\mathbb{R}(W)$ with this inner product and Hecke algebra structure by $\mathbb{R}_{\boldsymbol{q}}(W)$, and $L_{\boldsymbol{q}}^2(W)$ will denote the Hilbert space completion of $\mathbb{R}_{\boldsymbol{q}}(W)$ with respect to $\langle \ , \ \rangle_{\boldsymbol{q}}$. There is a natural anti-involution on $\mathbb{R}_{\boldsymbol{q}}(W)$, which implies that there is an associated *Hecke–von Neumann algebra* $\mathcal{N}_{\boldsymbol{q}}(W)$ acting on the right on $L_{\boldsymbol{q}}^2(W)$. It is the algebra of all bounded linear endomorphisms of $L_{\boldsymbol{q}}^2(W)$ which commute with the left $\mathbb{R}_{\boldsymbol{q}}(W)$–action.

Define the *von Neumann trace* of $\phi \in \mathcal{N}_{\boldsymbol{q}}(W)$ by $\mathrm{tr}_{\mathcal{N}_{\boldsymbol{q}}}(\phi) := \langle e_1 \phi, e_1 \rangle_{\boldsymbol{q}}$ and similarly for an $(n \times n)$–matrix with coefficients in $\phi \in \mathcal{N}_{\boldsymbol{q}}(W)$ by taking the sum of the von Neumann traces of elements on the diagonal. This allows us to attribute an nonnegative real number called the *von Neumann dimension* for any closed subspace of an n–fold direct sum of copies of $L_{\boldsymbol{q}}^2(W)$ which is stable under the $\mathbb{R}_{\boldsymbol{q}}(W)$–action, called a *Hilbert $\mathcal{N}_{\boldsymbol{q}}$–module*. If $V \subseteq (L_{\boldsymbol{q}}^2(W))^n$ is a Hilbert $\mathcal{N}_{\boldsymbol{q}}$–module, and $p_V \colon (L_{\boldsymbol{q}}^2(W))^n \to (L_{\boldsymbol{q}}^2(W))^n$ is the orthogonal projection onto V (note that $p_V \in \mathcal{N}_{\boldsymbol{q}}(W)$), then define

$$\dim_{\mathcal{N}_{\boldsymbol{q}}} V := \mathrm{tr}_{\mathcal{N}_{\boldsymbol{q}}}(p_V).$$

Weighted L^2–(co)homology

Suppose (W, S) is a Coxeter system and that X is a mirrored finite CW complex over S. Set $\mathcal{U} = \mathcal{U}(W, X)$. We first orient the cells of X and extend this orientation to $\mathcal{U}$ in such a way so that if σ is a positively oriented cell of X, then $w\sigma$ is positively oriented for each $w \in W$. We define a measure on the i–cells σ of $\mathcal{U}$ by

$$\mu_{\boldsymbol{q}}(\sigma) = q_u,$$

where u is the element of shortest length in W such that $\sigma \subseteq uX$. Define the $\boldsymbol{q}-$ *weighted i–dimensional L^2–(co)chains* on $\mathcal{U}$ to be the Hilbert space

$$L^2_{\boldsymbol{q}} C_i(\mathcal{U}) = L^2_{\boldsymbol{q}} C^i(\mathcal{U}) = L^2(\mathcal{U}^{(i)}, \mu_{\boldsymbol{q}}).$$

These are infinite W–equivariant square summable (with respect to $\mu_{\boldsymbol{q}}$) real-valued i–chains. The inner product is given by

$$\langle f, g \rangle_{\boldsymbol{q}} = \sum_{\sigma} f(\sigma) g(\sigma) \mu_{\boldsymbol{q}}(\sigma),$$

and we denote the induced norm by $\| \ \|_{\boldsymbol{q}}$.

The boundary map $\partial_i \colon L^2_{\boldsymbol{q}} C_i(\mathcal{U}) \to L^2_{\boldsymbol{q}} C_{i-1}(\mathcal{U})$ and coboundary map $\delta^i \colon L^2_{\boldsymbol{q}} C_i(\mathcal{U}) \to L^2_{\boldsymbol{q}} C_{i+1}(\mathcal{U})$ are defined by the usual formulas, however there is one caveat: they are not adjoints with respect to this inner product whenever $\boldsymbol{q} \neq \mathbf{1}$. Thus one remedies this issue by perturbing the boundary map ∂_i to $\partial_i^{\boldsymbol{q}}$:

$$\partial_i^{\boldsymbol{q}}(f)(\sigma^{i-1}) = \sum_{\sigma^{i-1} \subset \alpha^i} [\sigma : \alpha] \mu_{\boldsymbol{q}}(\alpha) \mu_{\boldsymbol{q}}^{-1}(\sigma) f(\alpha).$$

A simple computation shows that $\partial_i^{\boldsymbol{q}}$ is the adjoint of δ^i with respect to the weighted inner product; thus $(L^2_{\boldsymbol{q}} C_*(\mathcal{U}), \partial_i^{\boldsymbol{q}})$ is a chain complex. We now define the *reduced $\boldsymbol{q}$–weighted L^2–(co)homology* by

$$L^2_{\boldsymbol{q}} H_i(\mathcal{U}) = \operatorname{Ker} \partial_i^{\boldsymbol{q}} / \overline{\operatorname{Im} \partial_{i+1}^{\boldsymbol{q}}} \quad \text{and} \quad L^2_{\boldsymbol{q}} H^i(\mathcal{U}) = \operatorname{Ker} \delta^i / \overline{\operatorname{Im} \delta^{i-1}}.$$

The Hodge decomposition implies that $L^2_{\boldsymbol{q}} H^i(\mathcal{U}) \cong L^2_{\boldsymbol{q}} H_i(\mathcal{U}) \cong \ker \partial_i^{\boldsymbol{q}} \cap \ker \delta^i$, and versions of Eilenberg–Steenrod axioms hold for this homology theory. There is also a weighted version of Poincaré duality: if $\mathcal{U}$ is a locally compact homology n–manifold with boundary $\partial\mathcal{U}$, then

$$L^2_{\boldsymbol{q}} H_i(\mathcal{U}) \cong L^2_{\boldsymbol{q}^{-1}} H_{n-i}(\mathcal{U}, \partial\mathcal{U}).$$

One can also assign the von Neumann dimension to each of the Hilbert spaces $L^2_{\boldsymbol{q}} H_i(\mathcal{U})$ (as they are Hilbert $\mathcal{N}_{\boldsymbol{q}}$–modules). We denote this by $L^2_{\boldsymbol{q}} b_i(\mathcal{U})$ and call it the ith *$L^2_{\boldsymbol{q}}$–Betti number of $\mathcal{U}$*.

Discussed in [10, Section 6], there is an alternate approach to defining $L^2_{\boldsymbol{q}}$–Betti numbers using the ideas of Lück [18]. The main point is that there is an equivalence of categories between the category of Hilbert $\mathcal{N}_{\boldsymbol{q}}$–modules and projective modules of $\mathcal{N}_{\boldsymbol{q}}$. Hence one can define $\dim_{\mathcal{N}_{\boldsymbol{q}}} M$ for a finitely generated projective $\mathcal{N}_{\boldsymbol{q}}$–module M which agrees with the dimension of the corresponding Hilbert $\mathcal{N}_{\boldsymbol{q}}$–module. So, $\dim_{\mathcal{N}_{\boldsymbol{q}}} M$ for an arbitrary $\mathcal{N}_{\boldsymbol{q}}$–module is then defined to be the dimension of its projective part.

As in [18], define $H_*^W(\mathcal{U}, \mathcal{N}_q(W))$ to be the homology of the $\mathcal{N}_q(W)$–chain complex $C_*^W(\mathcal{U}, \mathcal{N}_q(W)) := \mathcal{N}_q(W) \otimes_{\mathbb{R}_q(W)} C_*(\mathcal{U})$, where $C_*(\mathcal{U})$ is the cellular chain complex of $\mathcal{U}$ with the induced $\mathbb{R}_q(W)$–structure. We then define

$$L_q^2 b_i(\mathcal{U}) := \dim_{\mathcal{N}_q} H_i^W(\mathcal{U}, \mathcal{N}_q(W)).$$

This definition does in fact agree with the previous one, and the advantage of this definition is that we do not need to take closures of images as in the definition of reduced q–weighted L^2–(co)homology (this is particularly useful when dealing with spectral sequences).

Some new results and observations for L_q^2–(co)homology

We begin with the following lemma, which says that we can compute the weighted L^2–Betti numbers of any acyclic complex of the form $\mathcal{U}(W, X)$, with X finite, on which W acts properly, and get the same answer. Thus we will sometimes write $L_q^2 b_k(W)$ instead of $L_q^2 b_k(\Sigma_L)$ to denote the kth L_q^2–Betti number of W.

Lemma 2.3 *Let (W, S) be a Coxeter system and suppose that X and X' are finite mirrored CW complexes with $\mathcal{U}(W, X)$ and $\mathcal{U}(W, X')$ both acyclic and both admitting a proper W–action. Then for every $k \geq 0$,*

$$L_q^2 b_k(\mathcal{U}(W, X)) = L_q^2 b_k(\mathcal{U}(W, X')).$$

Proof Set $\mathcal{U} = \mathcal{U}(W, X)$ and $\mathcal{U}' = \mathcal{U}(W, X')$. Since $\mathcal{U}$ and $\mathcal{U}'$ are both acyclic, it follows that the respective cellular chain complexes $C_*(\mathcal{U})$ and $C_*(\mathcal{U}')$ are chain homotopic. This chain homotopy induces a chain homotopy of the chain complexes $C_*^W(\mathcal{U}, \mathcal{N}_q(W))$ and $C_*^W(\mathcal{U}', \mathcal{N}_q(W))$. $\qquad\square$

In fact, Bestvina constructed such a complex for any finitely generated Coxeter group.

Theorem 2.4 [2] *Let W be a finitely generated Coxeter group. Then W acts properly and cocompactly on an acyclic vcd W–dimensional complex of the form $\mathcal{U}(W, X)$.*

Corollary 2.5 *Let (W, S) be a Coxeter system. Then*

$$L_q^2 b_k(W) = 0 \quad for \quad k > \operatorname{vcd} W.$$

Proof We can use the acyclic vcd W–dimensional complex of Theorem 2.4 to compute the weighted L^2–Betti numbers of W. Lemma 2.3 now completes the proof. $\qquad\square$
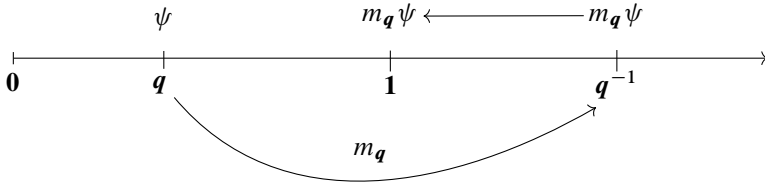
Figure 2: Schematic for the proof of Lemma 2.7

Remark 2.6 The similar technique of using the Bestvina construction for computations also appears in [14, Section 9].

We now prove a lemma which is crucial for later computations.

Lemma 2.7 Let $n = \mathrm{vcd}\, W$ and suppose and that $L^2_1 b_n(W) = 0$. Then

$$L^2_{\boldsymbol{q}} b_k(W) = 0 \quad \text{for} \quad k \geq n \quad \text{and} \quad \boldsymbol{q} \leq 1.$$

Proof By Corollary 2.5, we obtain vanishing for $k > n$. Now, suppose for a contradiction that $L^2_{\boldsymbol{q}} b_n(W) \neq 0$ for $\boldsymbol{q} < 1$. Let B_W denote the complex of Theorem 2.4. Lemma 2.3 says that we can compute weighted L^2–Betti numbers of W using the complex B_W. In particular, $L^2_{\boldsymbol{q}} b_n(W) = L^2_{\boldsymbol{q}} b_n(B_W)$ and we can choose a nontrivial element $\psi \in L^2_{\boldsymbol{q}} H_n(B_W)$. Thus ψ is a cycle under the weighted boundary map $\partial^{\boldsymbol{q}}$. Consider the isomorphism of Hilbert spaces

$$m_{\boldsymbol{q}} \colon L^2_{\boldsymbol{q}} C_n(B_W) \to L^2_{\boldsymbol{q}^{-1}} C_n(B_W)$$

defined by $m_{\boldsymbol{q}}(f(\sigma)) = \mu_{\boldsymbol{q}}(\sigma) f(\sigma)$. In particular, $m_{\boldsymbol{q}} \psi \in L^2_{\boldsymbol{q}^{-1}} C_n(B_W)$, and since $\boldsymbol{q}^{-1} > 1$,

$$\|m_{\boldsymbol{q}} \psi\|_1 \leq \|m_{\boldsymbol{q}} \psi\|_{\boldsymbol{q}^{-1}} < \infty.$$

Hence $m_{\boldsymbol{q}} \psi \in L^2_1 C_n(B_W)$.

Now, a simple computation shows that $\partial = m_{\boldsymbol{q}} \partial^{\boldsymbol{q}} m_{\boldsymbol{q}}^{-1}$ and since ψ is a cycle under $\partial^{\boldsymbol{q}}$, $m_{\boldsymbol{q}} \psi$ is a cycle under ∂, the standard L^2–boundary operator. Moreover, since B_W is n–dimensional, $m_{\boldsymbol{q}} \psi$ is trivially a cocycle. Thus we have produced a nontrivial element of $L^2_1 H_n(B_W)$, a contradiction. $\square$

3 The fattened Davis complex

We will now construct a complex which is a "fattened" version of the Davis complex. This thickened complex will be a homology manifold with boundary possessing the

Davis complex as a W–equivariant retract. For the remainder of this article we suppose that W is an infinite Coxeter group.

Construction

Given a Coxeter system (W, S), we find a compact P with mirror structure $(P_s)_{s \in S}$ as follows. Let P^* be a cell complex with vertex set S that is a GHS^{n-1}, with $n - 1 > \dim L$, such that the nerve L is a subcomplex of P^*. Take P to be the (P^*, S)–chamber.

Denote by $\mathcal{P}$ the collection of proper nonempty subsets T of S with $P_T \neq \varnothing$, and by $\mathcal{N}_P$ the subcollection of $\mathcal{P}$ corresponding to nonspherical subsets. For $T \in \mathcal{P}$, we denote a neighborhood of the face P_T by $N(P_T)$ and the corresponding closed neighborhood by $\bar{N}(P_T)$. We begin by building a regular neighborhood of ∂P in P. Start by choosing neighborhoods of codimension-n faces so that, for any two codimension-n faces P_U and P_V, we have $\bar{N}(P_U) \cap \bar{N}(P_V) = \varnothing$. Then we choose neighborhoods of codimension-$(n-1)$ faces so that, for any two codimension-$(n-1)$ faces P_U and P_V, we have

$$(2) \qquad \bar{N}(P_U) \cap \bar{N}(P_V) \subset N(P_U \cap P_V).$$

If $U \cup V \notin \mathcal{P}$, then we take $N(P_U \cap P_V) = \varnothing$. We proceed inductively, employing condition (2) at each step until we obtain the collection $\{N(P_T)\}_{T \in \mathcal{P}}$. This collection gives us a regular neighborhood of ∂P.

Finally, we realize the neighborhoods $\{N(P_T)\}_{T \in \mathcal{P}}$ in the above construction as $\{N_T \times P_T\}_{T \in \mathcal{P}}$, where N_T is a neighborhood of the cone point in $\mathrm{Cone}(\sigma_T)$, and σ_T is the geometric cell in P^* spanned by T (note that we can always do this; see the discussion in the previous section). We now define

$$K^f := P - \bigcup_{T \in \mathcal{N}_P} N(P_T).$$

We call K^f the *fattened Davis chamber*. Note that the mirror structure $(P_s)_{s \in S}$ on P induces a mirror structure $(K_s^f)_{s \in S}$ on K^f. The *fattened Davis complex* is now defined to be

$$\Phi_L := \mathcal{U}(W, K^f).$$

Given a $T \in \mathcal{N}_P$, we denote by $K^f(T)$ the fattened Davis chamber corresponding to σ_T and Coxeter system (W_T, T) (recall that the geometric cell σ_T has a natural W_T mirror structure).
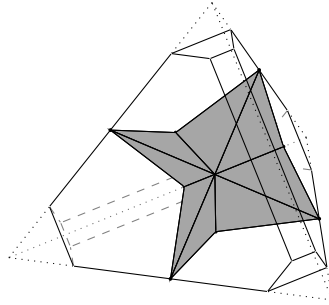
Figure 3: $K \subset K^f \subset P$ when $W = D_\infty \times D_\infty$ and $P = \Delta^3$

Remark 3.1 For any Coxeter system (W, S), one can always find a P^* for the above construction: simply let P^* be the boundary of the standard $(|S|-1)$–dimensional simplex $\Delta^{|S|-1}$. Then P is the barycentric subdivision of $\Delta^{|S|-1}$, and the Davis chamber K can then be viewed as a subcomplex of the barycentric subdivision of P spanned by the barycenters of spherical faces. One can see this using the language of posets. Note that K is the geometric realization of the poset $\mathcal{S}$ and P is the geometric realization of the poset of proper subsets of S. The natural inclusion of posets now induces the desired inclusion of K into P. The mirror structure $(K_s)_{s \in S}$ on K is now induced by the mirror structure $(P_s)_{s \in S}$ on P. In this case, $\mathcal{U}(W, P)$ is the traditional Coxeter complex, and we are essentially viewing Σ_L as a subcomplex of the barycentric subdivision of the Coxeter complex.

Remark 3.2 A variant of the above construction first appeared in [7] and was used for a different purpose. There, as in Remark 3.1, they only considered the case where $P = \Delta^{|S|-1}$, while our construction allows for different choices of P. The main difference between their construction and ours is that, instead of removing neighborhoods of faces with infinite stabilizers in $\mathcal{U}(W, P)$, they simply removed the faces (so in particular, their chamber was not compact). For reasons that will become clear in the coming sections, we not only need a compact P, but we must also be careful with how we chose the neighborhoods of faces.

Properties of Φ_L

W is assumed to be infinite, so via the choice of P for construction, the Davis chamber is the subcomplex of P spanned by vertices of P corresponding to spherical faces. Hence we have the following inclusions: $K \subset K^f \subset P$ (see Figure 3).

Note that there is a face preserving deformation retraction of K^f onto K, thus we have the following:

Proposition 3.3 Σ_L is a W–equivariant deformation retract of Φ_L.

Proposition 3.4 Φ_L is a locally compact contractible homology n–manifold with boundary $\partial\Phi_L$.

Proof Since Σ_L is contractible, it follows from Proposition 3.3 that Φ_L is contractible. Moreover, K^f is compact since it is closed in P (P is compact), so Φ_L is locally compact.

Now declare $K_e^f = \partial K^f - \bigcup_{T\in\mathcal{S}_{>\varnothing}}(K_T^f - \partial K_T^f)$, where e is the identity element of W. According to Proposition 2.1, it remains to show that K^f is a partially S–mirrored homology manifold with corners. Let $S' = S \cup \{e\}$, and note that, by construction, $K_T^f = \varnothing$ if and only if T is nonspherical. So we are done if we show that K_T^f has dimension $n - |T|$ for every spherical $T \subset S'$.

If $e \notin T$, then we are done since $(P_T, \partial P_T)$ is a GHD$^{n-|T|}$. This is because P is, by definition, the (P^*, S)–chamber, and the nerve L was assumed to be a subcomplex of P^*. Hence, since T is spherical, σ_T, the geometric cell in P^* corresponding to T, is a simplex of dimension $|T| - 1$. Therefore, the dimension of P_T is equal to $n - \dim\sigma_T - 1 = n - |T|$.

If $e \in T$, then $U = T - \{e\}$ is spherical, and by the above discussion, K_U^f has dimension $n - |U| = n - |T| + 1$. Then $K_T^f = K_U^f \cap K_e^f = \partial K_U^f$ has dimension $n - |T|$. $\qquad\square$

Remark 3.5 If $P = \Delta^{|S|-1}$, then the Coxeter complex $\mathcal{U}(W, P)$ is a PL–manifold away from faces with infinite stabilizers. This is because the links of faces corresponding to spherical subsets T are homeomorphic to the Coxeter complex of the corresponding group W_T. Since W_T is finite, this Coxeter complex is homeomorphic to a sphere of the appropriate dimension. Since we obtain Φ_L by removing neighborhoods of nonspherical faces (faces with infinite stabilizers), it follows that Φ_L is a PL–manifold with boundary.

The structure of $\partial\Phi_L$

The main goal of this section is to understand the structure of $\partial\Phi_L$. The first proposition will tell us that ∂K^f can be broken up into pieces, each of which has a nice product structure. This decomposition of ∂K^f then leads us to a cover of $\partial\Phi_L$ which will be used to study the algebraic topology of $\partial\Phi_L$.

For $T \in \mathcal{N}_P$, define

$$C_T = \partial N(P_T) - \bigcup_{U\in\mathcal{N}_P} N(P_U)$$

and

$$\Lambda_T = P_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ T \subset U}} N(P_U).$$

Proposition 3.6 (i) *Suppose that* $U, V \in \mathcal{N}_P$. *Then* $C_U \cap C_V \neq \varnothing$ *if and only if* $U \subset V$ *or* $V \subset U$.

(ii) *If* $T \in \mathcal{N}_P$, *then*

$$C_T \approx K^f(T) \times \Lambda_T.$$

(iii) *Suppose that* $T_1, T_2 \in \mathcal{N}_P$ *with* $T_1 \subset T_2$. *Then*

$$C_{T_1} \cap C_{T_2} \approx K^f(T_1) \times \Lambda_{T_2}.$$

Proof For (i), if $U \subset V$, then P_V is a face of P_U. Thus $C_U \cap C_V \neq \varnothing$. For the reverse implication, suppose that $U \not\subset V$ and $V \not\subset U$. By construction and condition (2), either $\bar{N}(P_U) \cap \bar{N}(P_V) = \varnothing$ or $\bar{N}(P_U) \cap \bar{N}(P_V) \subset N(P_U \cap P_V)$. The former case immediately implies that $C_U \cap C_V = \varnothing$, and the latter case implies that the intersection $\partial N(P_U) \cap \partial N(P_V)$ is removed at some point in the construction of the fattened Davis chamber; hence $C_U \cap C_V = \varnothing$.

For (ii), recall that we have realized the collection $\{N(P_T)\}_{T \in \mathcal{N}_P}$ as neighborhoods $\{N_T \times P_T\}_{T \in \mathcal{N}_P}$, where N_T is a neighborhood of the cone point in $\mathrm{Cone}(\sigma_T)$.

Now for each $U \subset T$, let α_U denote the face in σ_T corresponding to P_U. More precisely, σ_T has a W_T mirror structure, and α_U is the intersection of mirrors corresponding to $U \subset T$. We can express the neighborhoods in the construction of $K^f(T)$ as neighborhoods $\{\alpha_U \times N'_U\}_{U \in \mathcal{N}_P, U \subset T}$, where N'_U is a neighborhood of the cone point in $\mathrm{Cone}(\mathrm{Lk}(\alpha_U, \sigma_T))$. Here $\mathrm{Lk}(\alpha_U, \sigma_T)$ denotes the link of the face α_U in σ_T. In particular,

$$K^f(T) = \sigma_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \subset T}} \alpha_U \times N'_U.$$

Now, we have that $\mathrm{Lk}(\alpha_U, \sigma_T) \approx \sigma_U$, so $N'_U \approx N_U$. Hence

$$K^f(T) \approx \sigma_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \subset T}} P_U \times N_U.$$

Moreover, we can write Λ_T and C_T as

$$\Lambda_T = P_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ T \subset U}} P_U \times N_U \quad \text{and} \quad C_T = (\sigma_T \times P_T) - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \neq T}} P_U \times N_U.$$

We now show that $C_T \approx K^f(T) \times \Lambda_T$. Note that

$$K^f(T) \times \Lambda_T = (K^f(T) \times P_T) \cap (\sigma_T \times \Lambda_T),$$

so we begin unwinding definitions. We first observe that

$$K^f(T) \times P_T \approx \left(\sigma_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \subset T}} P_U \times N_U\right) \times P_T \approx (\sigma_T \times P_T) - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \subset T}} P_U \times N_U.$$

This is because P_T is a face of each P_U. Similarly, we have

$$\sigma_T \times \Lambda_T = \sigma_T \times \left(P_T - \bigcup_{\substack{U \in \mathcal{N}_P \\ T \subset U}} P_U \times N_U\right) \approx (\sigma_T \times P_T) - \bigcup_{\substack{U \in \mathcal{N}_P \\ T \subset U}} P_U \times N_U.$$

This follows from the fact that each P_U is a face of P_T. Thus we have shown that $K^f(T) \times \Lambda_T = (K^f(T) \times P_T) \cap (\sigma_T \times \Lambda_T) \approx C_T$, thereby proving (ii).

We now prove (iii). By (ii),

$$C_{T_1} \cap C_{T_2} \approx (K^f(T_1) \cap K^f(T_2)) \times (\Lambda_{T_1} \cap \Lambda_{T_2}).$$

It now simply remains to unwind the definitions. Since $T_1 \subset T_2$, it follows that P_{T_2} is a face of P_{T_1}. In particular, $\sigma_{T_1} \cap \sigma_{T_2} = \sigma_{T_1}$, and hence

$$K^f(T_1) \cap K^f(T_2) \approx \sigma_{T_1} \cap \sigma_{T_2} - \bigcup_{\substack{U,V \in \mathcal{N}_P \\ U \subset T_1, V \subset T_2}} N(P_U) \cup N(P_V)$$

$$\approx \sigma_{T_1} - \bigcup_{\substack{U \in \mathcal{N}_P \\ U \subset T_1}} N(P_U)$$

$$\approx K^f(T_1).$$

A similar computation shows that $\Lambda_{T_1} \cap \Lambda_{T_2} \approx \Lambda_{T_2}$, thus completing the proof of the proposition. $\qquad\square$

Proposition 3.7 Let $\mathcal{N}_P^{(j)} = \{T \in \mathcal{N}_P \mid \mathrm{Card}(T) = j\}$. *Then*

$$\partial \Phi_L = \bigcup_j \bigsqcup_{T \in \mathcal{N}_P^{(j)}} \mathcal{U}(W, C_T).$$

Proof The fact that one can decompose $\partial \Phi_L$ in this way is clear by construction, and the second union is, in fact, a disjoint union by Proposition 3.6(i). $\qquad\square$

Algebraic topology of Φ_L and $\partial\Phi_L$

We now turn our attention to studying the algebraic topology of Φ_L and $\partial\Phi_L$. We first begin with a corollary to Proposition 3.3.

Corollary 3.8 $L^2_{\boldsymbol{q}} H_*(\Phi_L) \cong L^2_{\boldsymbol{q}} H_*(\Sigma_L).$

Not only does Φ_L have the same weighted L^2–(co)homology as Σ_L, but Φ_L is a locally compact homology manifold with boundary by Proposition 3.4. Thus we have weighted Poincaré duality for Φ_L at our disposal. With this in mind, we prove the following lemma.

Lemma 3.9 *Suppose that (W, S) is a Coxeter system with vcd $W = m$ and that Φ_L is a homology n–manifold with boundary with $L^2_{\boldsymbol{q}} b_1(\partial\Phi_L) = 0$.*

(i)   *If $n - m = 1$ and $L^2_{\boldsymbol{q}^{-1}} b_m(\Phi_L) = 0$, then $L^2_{\boldsymbol{q}} b_1(\Sigma_L) = 0$.*

(ii) *If $n - m \geq 2$, then $L^2_{\boldsymbol{q}} b_1(\Sigma_L) = 0$.*

Proof Consider the long exact sequence for the pair $(\Phi_L, \partial\Phi_L)$:

$$\cdots \to L^2_{\boldsymbol{q}} H_1(\partial\Phi_L) \to L^2_{\boldsymbol{q}} H_1(\Phi_L) \to L^2_{\boldsymbol{q}} H_1(\Phi_L, \partial\Phi_L) \to \cdots.$$

By weighted Poincaré duality,

$$L^2_{\boldsymbol{q}} H_1(\Phi_L, \partial\Phi_L) \cong L^2_{\boldsymbol{q}^{-1}} H_{n-1}(\Phi_L).$$

Now by assumption, $L^2_{\boldsymbol{q}} H_1(\partial\Phi_L) = 0$, so by weak exactness, we must show that $L^2_{\boldsymbol{q}^{-1}} H_{n-1}(\Phi_L) = 0$. Since $L^2_{\boldsymbol{q}} H_1(\Sigma_L) = L^2_{\boldsymbol{q}} H_1(\Phi_L) = 0$ by Corollary 3.8, we will then be done.

For (i), we have that $L^2_{\boldsymbol{q}^{-1}} b_m(\Phi_L) = 0$. Since $n - m = 1$, we have that $m = n - 1$, so it follows that $L^2_{\boldsymbol{q}^{-1}} H_{n-1}(\Phi_L) = 0$. For (ii), we have that $n - m \geq 2$, so $n - 1 \geq m + 1$. Since vcd $W = m$, Corollary 2.5 implies that

$$L^2_{\boldsymbol{q}^{-1}} H_{n-1}(\Sigma_L) = L^2_{\boldsymbol{q}^{-1}} H_{n-1}(\Phi_L) = 0. \qquad \square$$

We devote the remainder of the section to studying the algebraic topology of $\partial\Phi_L$. The following is a corollary of Proposition 3.6.

Corollary 3.10   (i)   *If $T \in \mathcal{N}_P$, then for every $k \geq 0$,*

$$L^2_{\boldsymbol{q}} b_k(\mathcal{U}(W, C_T)) = L^2_{\boldsymbol{q}} b_k(\Phi_{L_T}) = L^2_{\boldsymbol{q}} b_k(\Sigma_{L_T}),$$

where L_T is the subcomplex of L corresponding to the subgroup W_T.

(ii) *Suppose that $T_1, T_2 \in \mathcal{N}_P$ with $T_1 \subset T_2$. Then for every $k \geq 0$,*

$$L^2_q b_k(\mathcal{U}(W, C_{T_1}) \cap \mathcal{U}(W, C_{T_2})) = L^2_q b_k(\Phi_{L_{T_1}}) = L^2_q b_k(\Sigma_{L_{T_1}}),$$

where L_{T_1} is the subcomplex of L corresponding to the subgroup W_{T_1}.

Remark 3.11 The L^2_q–Betti numbers in the center and on the right of the equations in (i) and (ii) are computed with respect to the special subgroups W_T (respectively W_{T_1}) of W, while the ones on the far left side of the equations are computed with respect to W.

Proof We prove only (i) as the proof of (ii) is similar. Proposition 3.6 implies that $C_T \approx K^f(T) \times \Lambda_T$ as mirrored spaces, where Λ_T is contractible and has no mirror structure. Therefore, $\mathcal{U}(W, C_T)$ is W–equivariantly homotopy equivalent to $\mathcal{U}(W, K^f(T))$. Now $L^2_q H_*(\mathcal{U}(W, K^f(T)))$ is just the completion of

$$L^2_q(W) \otimes_{\mathbb{R}_q(W_T)} L^2_q H_*(\mathcal{U}(W_T, K^f(T))),$$

so for every $k \geq 0$,

$$L^2_q b_k(\mathcal{U}(W, K^f(T))) = L^2_q b_k(\mathcal{U}(W_T, K^f(T))) = L^2_q b_k(\Phi_{L_T}). \qquad \square$$

Consider the cover $\mathcal{V} = \{\mathcal{U}(W, C_T)\}_{T \in \mathcal{N}_P}$ of $\partial \Phi_L$ in Proposition 3.7. The cover $\mathcal{V}$ will have intersections of variable depth, so we obtain a spectral sequence following [3, Chapter VII, Sections 3,4]:

Proposition 3.12 *There is a Mayer–Vietoris type spectral sequence converging to $H^W_*(\partial \Phi_L, \mathcal{N}_q(W))$ with E_1–term:*

$$E_1^{i,j} = \bigoplus_{\substack{\sigma \in \mathrm{Flag}(\mathcal{N}_P) \\ \dim \sigma = i}} H^W_j(\mathcal{U}(W, C_{\min \sigma}), \mathcal{N}_q(W)).$$

Proof Let $N(\mathcal{V})$ denote the nerve of the cover $\mathcal{V}$. It is the abstract simplicial complex whose vertex set is $\mathcal{N}_P$ and whose simplices are the nonempty subsets $\sigma \subset \mathcal{N}_P$ such that the intersection $V_\sigma = \bigcap_{T \in \sigma} \mathcal{U}(W, C_T)$ is nonempty. Following [3, Chapter VII, Sections 3,4], there is a Mayer–Vietoris type spectral sequence converging to $H^W_*(\partial \Phi_L, \mathcal{N}_q(W))$ with E_1–term:

$$E_1^{i,j} = \bigoplus_{\substack{\sigma \in N(\mathcal{V}) \\ \dim \sigma = i}} H^W_j(V_\sigma, \mathcal{N}_q(W)).$$

We have that $V_\sigma \neq \varnothing$ if and only if $\bigcap_{T \in \sigma} C_T \neq \varnothing$, and applying Proposition 3.6 inductively, this happens if and only if the vertices of σ form a chain $T_{i_1} \subset T_{i_2} \subset \cdots \subset T_{i_k}$.

This observation shows that $N(\mathcal{V}) = \mathrm{Flag}(\mathcal{N}_P)$. Now applying Proposition 3.6 inductively, it follows that $V_\sigma \approx \mathcal{U}(W, C_{T_{i_1}})$. Hence

$$H_*^W(V_\sigma, \mathcal{N}_{\boldsymbol{q}}(W)) \cong H_*^W(\mathcal{U}(W, C_{T_{i_1}}), \mathcal{N}_{\boldsymbol{q}}(W)),$$

so the terms in the spectral sequence are the ones claimed. $\qquad\square$

4 Computations

In this section we will use the fattened Davis complex to make concrete computations. We begin by considering the case where the nerve L of the Coxeter system (W, S) is a graph. Note that, for this special case, Σ_L is two-dimensional. We then direct our attention to quasi-Lánner groups, and finish with computations for 2–spherical Coxeter groups whose corresponding nerves are no longer restricted to be graphs. For the purpose of figures and examples, we will distinguish the special case where the labeled nerve $L = K_n(3)$, where $K_n(3)$ denotes the complete graph on n vertices with every edge labeled by 3.

Unless stated otherwise, the standing assumption in this section is that $\boldsymbol{q} \geq \mathbf{1}$.

The case where L is a graph

Before proving the main theorem of this section, we begin with a lemma. The special case of the lemma when $\boldsymbol{q} = \mathbf{1}$ is closely related to a result of Schroeder [23, Theorem 4.6]. We provide an argument which is analogous to that of Schroeder in his proof.

Lemma 4.1 *Suppose that the labeled nerve L is the one-skeleton of a cellulation of S^2. Then*

$$L_{\boldsymbol{q}}^2 b_2(\Sigma_L) = 0 \quad \text{for} \quad \boldsymbol{q} \leq \mathbf{1}.$$

Proof In light of Lemma 2.7, we must show that $L_{\mathbf{1}}^2 b_2(\Sigma_L) = 0$. We begin by building L to a triangulation of S^2 by coning on empty 2–cells and labeling each new edge by 2, at each step keeping track of the $L_{\mathbf{1}}^2$–(co)homology with a Mayer–Vietoris sequence. More precisely, start with $T_1 \subset S$ corresponding to an empty 2–cell L_{T_1} in L, and denote by CL_{T_1} the right-angled cone on L_{T_1}. The corresponding special subgroup W_{T_1} is infinite, and it acts properly and cocompactly by reflections on either $\mathbb{R}^2$ or $\mathbb{H}^2$. In both cases, $L_{\mathbf{1}}^2 H_2(\Sigma_{L_{T_1}}) = 0$, and hence the Künneth formula implies that $L_{\mathbf{1}}^2 H_2(\Sigma_{CL_{T_1}}) = 0$. We have the following Mayer–Vietoris sequence:

$$\cdots \longrightarrow L_{\mathbf{1}}^2 H_2(\Sigma_{L_{T_1}}) \longrightarrow L_{\mathbf{1}}^2 H_2(\Sigma_{CL_{T_1}}) \oplus L_{\mathbf{1}}^2 H_2(\Sigma_L) \xrightarrow{f_1} L_{\mathbf{1}}^2 H_2(\Sigma_{L \cup CL_{T_1}}) \longrightarrow \cdots.$$

In particular, the map f_1 is injective. We then choose another $T_2 \subset S$ corresponding to an empty 2–cell L_{T_2} in L and denote by CL_{T_2} the right-angled cone on L_{T_2}. By a similar argument, the map f_2 in the following Mayer–Vietoris sequence is injective:

$$\cdots \longrightarrow L_1^2 H_2(\Sigma_{CL_{T_2}}) \oplus L_1^2 H_2(\Sigma_{L \cup CL_{T_2}}) \xrightarrow{f_2} L_1^2 H_2(\Sigma_{L \cup CL_{T_1} \cup CL_{T_2}}) \longrightarrow \cdots .$$

Proceed inductively until all empty 2–cells have been coned off and denote the newly promoted nerve by L'. The f_i yield a sequence of injective maps:

$$L_1^2 H_2(\Sigma_L) \hookrightarrow L_1^2 H_2(\Sigma_{L \cup CL_{T_1}}) \hookrightarrow \cdots \hookrightarrow L_1^2 H_2(\Sigma_{L'}).$$

Since L' is a triangulation of S^2, it follows that $\Sigma_{L'}$ is a 3–manifold. Now a result of Lott and Lück [17], in conjunction with the validity of the geometrization conjecture for 3–manifolds [20], implies that $L_1^2 H_*(\Sigma_{L'})$ vanishes in all dimensions. In particular, $L_1^2 b_2(\Sigma_L) = 0$. $\qquad\square$

Remark 4.2 Schroeder proves a more general theorem for $q = 1$ [23, Theorem 4.6]. A metric flag complex L is *planar* if it can be embedded as a proper subcomplex of a triangulation of the 2–sphere. Schroeder proves that if the nerve L of a Coxeter system is planar, then $L_1^2 b_k(\Sigma_L) = 0$ for $k \geq 2$. If L is planar and W is the corresponding Coxeter group, then [6, Corollary 8.5.5] implies that vcd $W \leq 2$. Therefore, we can use Lemma 2.7 to deduce that $L_q^2 b_k(\Sigma_L) = 0$ for $k \geq 2$ and $q \leq 1$.

Theorem 4.3 *Suppose that the labeled nerve L of a Coxeter system (W, S) is the one-skeleton of a cell complex that is a generalized homology n–sphere, where $n \geq 2$. Furthermore, suppose that the vertex set of every 2–cell generates a Euclidean reflection subgroup of W. Then $L_q^2 b_*(\Sigma_L)$ is concentrated in degree 2.*

Note that a labeled nerve L satisfying the hypothesis of Theorem 4.3 can only have labels $m_{st} \in \{2, 3, 4, 6\}$.

Proof It follows from [13, Theorem 10.3] that $L_q^2 b_0(\Sigma_L) = 0$. We now turn our attention to showing that $L_q^2 b_1(\Sigma_L) = 0$. For the construction of the fattened Davis complex, we will use the given cell complex as P^*.

We prove the theorem by induction on n. For the base case $n = 2$, note that σ_T is Euclidean for every $T \in \mathcal{N}_P$. Hence Proposition 3.6 implies that each C_T appearing in ∂K^f corresponds to a set $T \in \mathcal{N}_P$ such that W_T is a Euclidean reflection group. Thus Corollary 3.10 and [8, Corollary 14.5] imply that $L_q^2 b_1(\mathcal{U}(W, C_T)) = 0$. This and [13, Theorem 10.3] imply that the $E_1^{0,1}$ and $E_1^{1,0}$ terms in the E_1 sheet of the spectral sequence in Proposition 3.12 are zero, which in turn implies that $L_q^2 b_1(\partial \Phi_L) = 0$.

Now note that Φ_L is three-dimensional, and vcd $W = 2$. Moreover, by Lemma 4.1, $L^2_{\boldsymbol{q}^{-1}} H_2(\Sigma_L) = 0$. Therefore, via Lemma 3.9(i), we reach the conclusion that $L^2_{\boldsymbol{q}} b_1(\Sigma_L) = 0$.

Now suppose the theorem is true for $m < n$. Since Σ_L is 2–dimensional, Lemma 3.9(ii) tells us that we are done if we show that $L^2_{\boldsymbol{q}} b_1(\partial \Phi_L) = 0$. Let $T \in \mathcal{N}_P$. Then σ_T is the (σ_T, T)–chamber, where σ_T is the geometric cell in P^* spanned by T. In particular, $\partial \sigma_T$ is a cell complex that is GHS^m for some $m < n$, and since all 2–cells of P^* are Euclidean, it follows that all 2–cells of $\partial \sigma_T$ are Euclidean. Hence, by induction and Corollary 3.10, it follows that $L^2_{\boldsymbol{q}} b_1(\mathcal{U}(W, C_T)) = L^2_{\boldsymbol{q}} b_1(\Sigma_{L_T}) = 0$ for every $T \in \mathcal{N}_P$. This and [13, Theorem 10.3] imply that the $E_1^{0,1}$ and $E_1^{1,0}$ terms in the E_1 sheet of the spectral sequence in Proposition 3.12 are zero, which in turn implies that $L^2_{\boldsymbol{q}} b_1(\partial \Phi_L) = 0$.   □

Consider the special case of Theorem 4.3 when $n = 2$. In this case, Theorem 4.3, along with Lemma 4.1, yields the following ranges of concentration for $L^2_{\boldsymbol{q}}$–Betti numbers.

Corollary 4.4 *Suppose that the labeled nerve L of a Coxeter system (W, S) is the one-skeleton of a cell complex that is a GHS^2 such that the vertex set of every 2–cell generates a Euclidean reflection subgroup of W.*

- *If $\boldsymbol{q} \in \overline{\mathcal{R}}$, then $L^2_{\boldsymbol{q}} H_*(\Sigma_L)$ is concentrated in dimension 0.*
- *If $\boldsymbol{q} \notin \mathcal{R}$ and $\boldsymbol{q} \leq \mathbf{1}$, then $L^2_{\boldsymbol{q}} H_*(\Sigma_L)$ is concentrated in dimension 1.*
- *If $\boldsymbol{q} \geq \mathbf{1}$, then $L^2_{\boldsymbol{q}} H_*(\Sigma_L)$ is concentrated in dimension 2.*

Once concentration has been established, an explicit formula for $L^2_{\boldsymbol{q}} b_*(\Sigma_L)$ can be obtained using [13, Corollary 3.4] and [6, Theorem 17.1.9]. Also, if we place some restrictions on either our labels or the cell complex, then the formula for $L^2_{\boldsymbol{q}} b_*(\Sigma_L)$ in Theorem 4.3 becomes relatively simple, as illustrated by the following corollary.

Corollary 4.5 *Let $L = K_n(3)$ with $n \geq 3$. Then $L^2_{\boldsymbol{q}} b_*(\Sigma_L)$ is concentrated in degree 2. Furthermore,*

$$L^2_q b_2(\Sigma_L) = 1 - \frac{nq}{1+q} + \frac{n(n-1)q^3}{2(1+2q+2q^2+q^3)}.$$

Remark 4.6 Note that, under the hypothesis of the above corollary, all generators in S are conjugate, so in this case, $\boldsymbol{q} = q$, where $q \geq 1$ is a positive real number.

We can also allow ourselves to remove some edges from $L = K_n(3)$. We denote by $K_n^l(3)$ the complete graph on n vertices with each edge labeled by 3 and with l edges removed. We have the following consequence of Corollary 4.5.

Corollary 4.7 *Suppose that $L = K_n^l(3)$, where $n \geq 5$ and $l \leq n-4$. Then $L_q^2 b_*(\Sigma_L)$ is concentrated in degree 2.*

Proof We first note that removing an edge from $K_n(3)$ splits the graph into two copies of $K_{n-1}(3)$ intersecting at $K_{n-2}(3)$. Thus we have a Mayer–Vietoris sequence:

$$(\star) \quad \cdots \to L_q^2 H_1(\Sigma_{K_{n-2}}) \to L_q^2 H_1(\Sigma_{K_{n-1}}) \oplus L_q^2 H_1(\Sigma_{K_{n-1}}) \to L_q^2 H_1(\Sigma_{K_n^1}) \to 0.$$

We first handle the case where $L = K_5^1(3)$. Removing an edge from $K_5(3)$ splits the graph into two copies of $K_4(3)$ intersecting at $K_3(3)$. Corollary 4.5 computes the L_q^2–(co)homology of each of the pieces in this decomposition, and applying the sequence $(\star)$ proves the assertion for the case $L = K_5^1(3)$.

The proof for $L = K_n^l(3)$ is now by induction, the above computation serving as the base case. Suppose that the theorem is true for $m < n$. Begin by removing an edge from $K_n(3)$, splitting it as two copies of $K_{n-1}(3)$ intersecting at $K_{n-2}(3)$. We now remove the remaining $l - 1 \leq n - 5$ edges from each of the graphs in the splitting, the worst case scenario being that we remove $l - 1$ edges from $K_{n-2}(3)$ (which in turn removes $l - 1$ edges from each copy of $K_{n-1}(3)$). Nevertheless, the inductive hypothesis is satisfied for each K_{n-1} in the splitting no matter how the remaining edges are removed. Applying a Mayer–Vietoris sequence analogous to $(\star)$ now shows that the theorem holds for $L = K_n^l(3)$. $\square$

With the help of ruins (defined in Section 2), we are also able to make computations when we change some labels on $L = K_n(3)$. For the proof that follows, note that $L_q^2 b_*(\Sigma(U)) = L_q^2 b_*(W_U)$ for every $U \subseteq S$; see the discussion before [8, Lemma 8.1].

Theorem 4.8 *Let $L = K_n$, the complete graph on n vertices, with $n \geq 5$. Let $k \leq n - 4$, and suppose that we label k edges of L with $m_{st} \in \mathbb{N} - \{1, 3\}$, and label the remaining edges by 3. Then $L_q^2 b_*(\Sigma_L)$ is concentrated in degree 2.*

Proof The proof is by induction on n. First consider the case where $L = K_5$ with one label $m_{st} \in \mathbb{N} - \{1, 3\}$. Then by Corollary 4.5, $L_q^2 b_1(\Sigma(S - s)) = L_q^2 b_1(\Sigma_{K_4(3)}) = 0$. According to sequence (1), it remains to show that $L_q^2 H_1(\Omega(S, s), \partial) = 0$. We turn our attention to sequence (1) with $U = S$, $T = \{s, t\}$, $U' = S - t$, and $T' = \{s\}$. By [8, Lemma 8.1], $L_q^2 H_1(\Omega(S, T), \partial) = 0$, so by weak exactness it remains to show that $L_q^2 H_1(\Omega(U', T'), \partial) = 0$. We consider the following version of sequence (1):

$$\cdots \to L_q^2 H_1(\Sigma(S - \{s, t\})) \to L_q^2 H_1(\Sigma(S - t)) \to L_q^2 H_1(\Omega(U', T'), \partial) \to \cdots.$$

Note that

$$L_q^2 b_0(\Sigma(S - \{s, t\})) = L_q^2 b_0(\Sigma_{K_3(3)}) = 0$$

and

$$L_{\boldsymbol{q}}^2 b_1(\Sigma(S-t)) = L_{\boldsymbol{q}}^2 b_1(\Sigma_{K_4(3)}) = 0,$$

by [8, Corollary 14.5] and Corollary 4.5, respectively. By weak exactness, we obtain that $L_{\boldsymbol{q}}^2 H_1(\Omega(U', T'), \partial) = 0$, and hence that $L_{\boldsymbol{q}}^2 H_1(\Omega(S, s), \partial) = 0$, thus proving the assertion for $L = K_5$.

Now suppose that the theorem is true for $L = K_m$ for all $m < n$. We wish to show the theorem is true for $L = K_n$. Begin by choosing an edge e with vertices s and t and label different from 3. We now observe that $L_{\boldsymbol{q}}^2 b_1(\Sigma(S-s)) = L_{\boldsymbol{q}}^2 b_1(\Sigma_{K_{n-1}}) = 0$ by the inductive hypothesis, since K_{n-1} now has at most $n-5$ edges with a label different from 3. Similarly, the inductive hypothesis implies $L_{\boldsymbol{q}}^2 b_1(\Sigma(S-t)) = 0$ and $L_{\boldsymbol{q}}^2 b_0(\Sigma(S-\{s,t\})) = 0$. Hence the weak exact sequences used in the proof for the case $L = K_5$ allow us to conclude that $L_{\boldsymbol{q}}^2 b_1(\Sigma_L) = L_{\boldsymbol{q}}^2 b_1(\Sigma(S)) = 0$. □

Remark 4.9 Note that, in conjunction with [8, Corollary 14.5] and Corollary 4.4, the above argument gives an alternate proof of Corollary 4.5.

Quasi-Lánner groups

A 2–spherical Coxeter group W is *quasi-Lánner* if it acts properly (but not cocompactly) on hyperbolic space $\mathbb{H}^n$ by reflections with fundamental chamber an n–simplex of finite volume. For brevity, we say that W is of type QL_n. Quasi-Lánner groups have been classified and only exist in dimensions 3 through 10. For a complete list, see [16, Section 6.9]. We note that the Coxeter group with corresponding nerve $L = K_4(3)$ is on the list.

All nonspherical proper special subgroups of a quasi-Lánner group are Euclidean and on the list appearing in [16, page 34]. Moreover, if W is of type QL_n, then the only proper infinite special subgroups are those W_T with $|T| = n-1$. Hence, by [6, Corollary 8.5.5], if W is of type QL_n, then vcd $W = n-1$. With this observation, we prove the following theorem.

Theorem 4.10 *Suppose that W is of type QL_n. Then $L_{\boldsymbol{q}}^2 b_k(\Sigma_L) = 0$ whenever $k \geq n-1$ and $\boldsymbol{q} \leq \mathbf{1}$, or $k \leq 1$ and $\boldsymbol{q} \geq \mathbf{1}$.*

Proof We first suppose that $\boldsymbol{q} = \mathbf{1}$. Since W is of type QL_n, we can realize a finite volume n–simplex in hyperbolic space $\mathbb{H}^n$, with W acting by reflections along codimension-one faces (note that this simplex has some ideal vertices). By a theorem of Cheeger–Gromov [4], $L_{\mathbf{1}}^2 H_k(\Sigma_L) \cong L_{\mathbf{1}}^2 \mathcal{H}^k(\mathbb{H}^n)$, where $L_{\mathbf{1}}^2 \mathcal{H}^k$ denotes the L^2 de Rham cohomology. By a theorem of Dodziuk [11], $L_{\mathbf{1}}^2 \mathcal{H}^k(\mathbb{H}^n) = 0$ for all $k \geq 0$ if
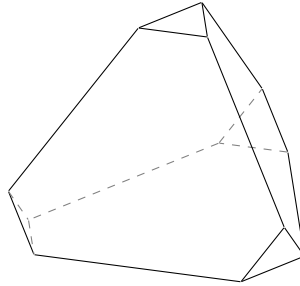
Figure 4: K^f when $L = K_4(3)$

n is odd, and is concentrated in dimension $n/2$ if n is even. Thus $L^2_1 b_{n-1}(\Sigma_L) = 0$. The result for $q \le 1$ now follows by Lemma 2.7 and the fact that vcd $W = n - 1$.

Now suppose that $q \ge 1$. Consider the fattened Davis complex Φ_L with respect to $P = \Delta^n$, the standard n–simplex; see Remark 3.1 and Figure 4.

Weighted Poincaré duality implies that

$$L^2_q H_1(\Phi_L, \partial\Phi_L) \cong L^2_{q^{-1}} H_{n-1}(\Phi_L) \cong L^2_{q^{-1}} H_{n-1}(\Sigma_L) = 0,$$

so by the long exact sequence for the pair $(\Phi_L, \partial\Phi_L)$, it remains to be shown that $L^2_q H_1(\partial\Phi_L) = 0$. Proposition 3.7 implies that each C_T appearing in ∂K^f corresponds to a set $T \in \mathcal{N}_P$ with W_T a Euclidean reflection group. In particular, Corollary 3.10 and [8, Corollary 14.5] imply that $L^2_q b_1(\mathcal{U}(W, C_T)) = 0$. Hence the $E_1^{0,1}$ term in the E_1 sheet of the spectral sequence of Proposition 3.12 is zero. By [13, Theorem 10.3], the first row of the E_1 sheet is also zero, and in particular, $E_1^{1,0}$ is zero. Therefore, $L^2_q b_1(\partial\Phi_L) = 0$. □

Other 2–spherical groups

We now perform computations for other 2–spherical groups, removing the restriction that the nerve L is a graph. Given a Coxeter system (W, S), we make a particular choice of P for the construction of Φ_L, namely $P = \Delta^{|S|-1}$, the standard $(|S|-1)$–simplex; see Remark 3.1.

Lemma 4.11 *Suppose that (W, S) is infinite 2–spherical with $|S| = 5$ and vcd $W \le 3$. Furthermore, suppose that every infinite special subgroup W_T, with $|T| = 3$ or 4, is Euclidean or QL_3, and that $L^2_1 b_3(\Sigma_L) = 0$. Then $L^2_q b_k(\Sigma_L) = 0$ for $k < 2$.*
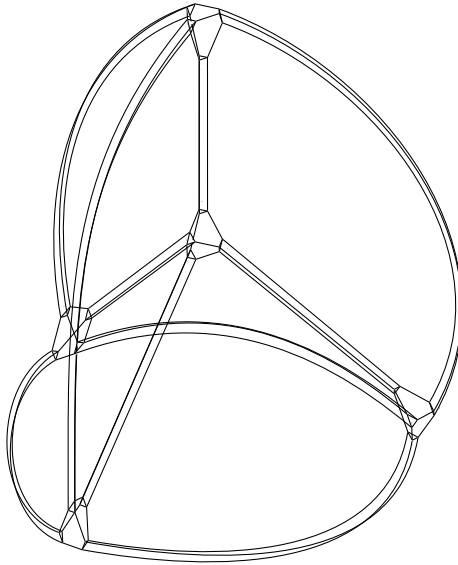
Figure 5: Fundamental chamber for $\partial\Phi_L$ when $L = K_5(3)$

Proof We wish to reduce the proof to showing that $L_q^2 b_1(\partial\Phi_L) = 0$. If we have vcd $W = 2$, then this is accomplished by Lemma 3.9(ii). If vcd $W = 3$, then according to Lemma 3.9(i), we accomplish this if we show that $L_{q^{-1}}^2 b_3(\Sigma_L) = 0$. Since, by assumption, $L_1^2 b_3(\Sigma_L) = 0$, we reach this conclusion by Lemma 2.7. So, to complete the proof, we must show that $L_q^2 b_1(\partial\Phi_L) = 0$.

We have that every C_T appearing in Proposition 3.6 corresponds to a set $T \in \mathcal{N}_P$ where W_T is Euclidean or QL$_3$. Thus [8, Corollary 14.5] and Theorem 4.10 imply that $L_q^2 b_1(\mathcal{U}(W, C_T)) = 0$. This, [13, Theorem 10.3], and the spectral sequence in Proposition 3.12 imply that $L_q^2 b_1(\partial\Phi_L) = 0$. □

Theorem 4.12 *Suppose that (W, S) is infinite 2–spherical with $|S| \geq 5$. Suppose furthermore that the following conditions hold.*

(1) *For every $T \subseteq S$ with $|T| \geq 5$, we have vcd $W_T \leq |T| - 2$.*

(2) *Every infinite subgroup W_T, with $|T| = 3$ or 4, is Euclidean or QL$_3$.*

Then $L_q^2 b_k(\Sigma_L) = 0$ for $k < 2$.

Proof First we prove the theorem under the assumption that $L_1^2 b_{|S|-2}(\Sigma_L) = 0$ (we will justify this at the end of the proof). By [13, Theorem 10.3], the statement

for $L^2_{\bm{q}}b_0(\Sigma_L)$ follows. So, we turn our attention to showing $L^2_{\bm{q}}b_1(\Sigma_L) = 0$. The proof of the theorem is now by induction on $|S|$, Lemma 4.11 serving as the base case. By Lemma 2.7, since vcd $W \leq |S| - 2$, it follows that $L^2_{\bm{q}^{-1}}b_{|S|-2}(\Sigma_L) = 0$. Furthermore, Φ_L has dimension $|S| - 1$, so by Lemma 3.9, it now suffices to show that $L^2_{\bm{q}}b_1(\partial\Phi_L) = 0$. By assumption, every nonspherical special subgroup W_U with $|U| = 3$ or 4 is Euclidean or QL_3. Thus every nonspherical special subgroup W_U, with $4 < |U| < |S|$, satisfies the inductive hypothesis. Therefore, by induction, [8, Corollary 14.5], and Theorem 4.10, for any $T \in \mathcal{N}_P$ we have that $L^2_{\bm{q}}b_1(\Sigma_{L_T}) = 0$ (here L_T is the subcomplex of L corresponding to the special subgroup W_T). Hence, by Corollary 3.10(i), for every $T \in \mathcal{N}_P$,

$$L^2_{\bm{q}}b_1(\mathcal{U}(W, C_T)) = L^2_{\bm{q}}b_1(\Sigma_{L_T}) = 0.$$

It follows that the $E_1^{0,1}$ term in the E_1 sheet of the spectral sequence of Proposition 3.12 is zero. By [13, Theorem 10.3], the first row of the E_1 sheet is also zero, and in particular, $E_1^{1,0}$ is zero. Therefore, $L^2_{\bm{q}}b_1(\partial\Phi_L) = 0$.

We now turn our attention to showing that $L^2_{\bm{1}}b_{|S|-2}(\Sigma_L) = 0$. We will use an argument analogous to the one in Lemma 4.1, invoking the help of Theorem 5.1. We first begin by coning empty 2–simplices of L, and then empty 3–simplices, and so on, until all empty simplices have been coned off. We then label all new edges by 2. In this way we obtain a newly promoted nerve L' which is a triangulation of $S^{|S|-2}$, and in particular, $\Sigma_{L'}$ is an $(|S|-1)$–manifold. By Theorem 5.1 (appearing in the next section), $L^2_{\bm{1}}b_{|S|-2}(\Sigma_{L'}) = 0$, and using the arguments of Lemma 4.1, we can conclude that $L^2_{\bm{1}}b_{|S|-2}(\Sigma_L) = 0$. $\qquad\square$

As a corollary to Theorem 4.12, we also obtain a specialized version of Conjecture 1.1 where $n = 4$ and W is 2–spherical.

Corollary 4.13 *Suppose that (W, S) is 2–spherical with $|S| \geq 6$ and that the nerve L is a triangulation of S^3. Furthermore, suppose that every infinite special subgroup W_T, with $|T| = 3$ or 4, is Euclidean or QL_3. Then*

$$L^2_{\bm{q}}b_k(\Sigma_L) = 0 \quad for \quad k < 2.$$

Proof Since L is a triangulation of S^3, it follows that vcd $W = 4$. In particular, W satisfies the hypothesis of Theorem 4.12. $\qquad\square$

Remark 4.14 Figure 6 gives examples of Coxeter diagrams whose corresponding Coxeter system (W, S) has $|S| = 6$ and satisfies the hypothesis of Corollary 4.13 (if two vertices are not connected, then the implied label between them is 2). The author does not know whether there exist examples whenever $|S| \geq 7$.
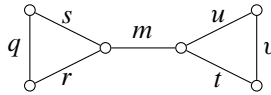
Figure 6: This is a 2–spherical Coxeter diagram satisfying the hypothesis of Corollary 4.13 provided that: (i) $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1$, $\frac{1}{t} + \frac{1}{u} + \frac{1}{v} = 1$, and $m = 2, 3, 4$; (ii) if $m = 3$, then $s, r, u, t \neq 6$ and either $s, r \neq 4$ or $u, t \neq 4$; (iii) if $m = 4$, then $s, r, u, t \neq 4, 6$.

5   The weighted Singer conjecture

For the rest of this article, we use the notation $\Sigma_L = X$ whenever Σ_L admits a W–invariant metric making it isometric to X.

The case where L is a disk

As discussed before, Conjecture 1.1 is now known to be true whenever $q = 1$ and $n \leq 4$ due to recent work of Okun–Schreve [19, Theorem 4.9]. In fact, induction and [19, Theorem 4.5, Lemma 4.6, Corollary 4.7] can be used to prove the following theorem.

Theorem 5.1   *Suppose that the nerve L is an $(n-1)$–sphere or an $(n-1)$–disk. Then*

$$L_1^2 H_k(\Sigma_L) = 0 \quad \text{for} \quad k \geq n - 1.$$

Note that if L is a triangulation of the $(n-1)$–disk, then Σ_L is an n–manifold with boundary. We now obtain the following theorem, which, whenever $n = 3$ or 4, can be thought of as a version of Conjecture 1.1 for the case where Σ_L is an n–manifold with boundary.

Theorem 5.2   *Suppose that the nerve L is an $(n-1)$–disk. Then*

$$L_q^2 H_k(\Sigma_L) = 0 \quad \text{for} \quad k \geq n - 1 \quad \text{and} \quad q \leq 1.$$

Proof   By Theorem 5.1, we have that $L_1^2 H_k(\Sigma_L) = 0$ for $k \geq n - 1$. Furthermore, [6, Corollary 8.5.5] implies that vcd $W \leq n - 1$, and hence we are done by Lemma 2.7. $\square$

We note that Theorem 4.3 provides convincing evidence for the validity of a weighted version of Theorem 5.1 when L is a triangulation of the $(n-1)$–sphere. Suppose that the labeled nerve L' is the one-skeleton of a cell complex that is a GHS^{n-1} for $n \geq 3$, where all 2–cells are Euclidean. Build L' to a triangulation L that is a GHS^{n-1} by coning on each empty cell and labeling new edges by 2. In other words, perform the

following sequence of right-angled cones: begin by coning on each empty 2–cell, then on each empty 3–cell, and so on, until each empty cell has been coned off (if $n = 3$, this process stops when each empty 2–cell has been coned off).

Theorem 5.3 *Suppose that the nerve L, a GHS^{n-1} for $n \geq 3$, is obtained via the above construction, and suppose that $q \geq 1$. Then*

$$L_q^2 b_k(\Sigma_L) = 0 \quad \text{for} \quad k \leq 1.$$

Proof The proof of the theorem follows the strategy of Lemma 4.1: one performs careful bookkeeping using Mayer–Vietoris sequences when constructing L from L'. The proof is by induction on n. To get started, let σ denote an empty 2–cell and $C\sigma$ denote the right-angled cone on σ. Then we have a Mayer–Vietoris sequence:

$$\cdots \longrightarrow L_q^2 H_1(\Sigma_\sigma) \longrightarrow L_q^2 H_1(\Sigma_{L'}) \oplus L_q^2 H_1(\Sigma_{C\sigma}) \longrightarrow L_q^2 H_1(\Sigma_{L' \cup C\sigma}) \longrightarrow 0.$$

As $q \geq 1$ and σ is Euclidean, it follows that $L_q^2 H_1(\Sigma_\sigma) = 0$, and hence, by the weighted Künneth formula, that $L_q^2 H_1(\Sigma_{C\sigma}) = 0$. By Theorem 4.3, $L_q^2 H_1(\Sigma_{L'}) = 0$, and hence, by weak exactness, $L_q^2 H_1(\Sigma_{L' \cup C\sigma}) = 0$. We then proceed to cone off the remaining empty 2–cells, at each step employing a similar Mayer–Vietoris sequence to obtain the theorem for the base case $n = 3$.

Now, suppose that the theorem is true for $k < n$. Let σ denote an empty m–cell with $m < n$, and suppose that all lower dimensional cells of σ have been coned off. Then σ satisfies the inductive hypothesis; hence $L_q^2 H_1(\Sigma_\sigma) = 0$, and therefore, $L_q^2 H_1(\Sigma_{C\sigma}) = 0$. Now suppose we are at the stage of the coning process where all cells of dimension less than m have been coned off, and let L_m denote the nerve at this stage. Then $L_q^2 H_1(\Sigma_{L_m}) = 0$, as we have been keeping track of the homology with Mayer–Vietoris sequences. As in the base case, start with an empty m–cell σ, apply the same Mayer–Vietoris sequence, and conclude that $L_q^2 H_1(\Sigma_{L_m \cup C\sigma}) = 0$. Proceed inductively, coning off each empty cell and applying a similar Mayer–Vietoris sequence to prove the theorem. □

A special case of Andreev's theorem

Suppose now that L is a triangulation of S^2. Let C be an empty circuit in L and suppose that C is not the boundary of two adjacent triangles. We say that C is a *Euclidean circuit* if the corresponding Coxeter group W_C is a Euclidean reflection group. It follows from L being a metric flag complex that a Euclidean circuit is always full subcomplex of L, and in particular, W_C is a special subgroup of W. The following theorem is now a special case of Andreev's theorem [1, Theorem 2].

Theorem 5.4 (compare [22, Section 3]) *Suppose that the nerve L is a triangulation of S^2, but not the boundary of a 3–simplex, and let (W, S) be the corresponding Coxeter system. Furthermore, suppose these conditions are met:*

- *For every $T \subset S$, W_T is not a Euclidean reflection group.*
- *$W \neq W_T \times D_\infty$, where $T \subset S$ spans an empty triangle in L, and D_∞ is the infinite dihedral group.*

Then $\Sigma_L = \mathbb{H}^3$.

Remark 5.5 Schroeder derived the above conditions from Andreev's theorem in [22, Section 3], where he used them as part of his proof of the Singer conjecture for Coxeter groups in dimension three whenever $q = 1$. Right-angled versions of Schroeder's argument and Theorem 5.4 also appeared earlier in [9].

Equidistant hypersurfaces

Suppose that the Coxeter group W has nerve L that is a triangulation of S^2 and that $\Sigma_L = \mathbb{H}^3$. Let D denote the Davis chamber (in $\mathbb{H}^3$) and let W_M be a special subgroup of W. We now consider the (possibly infinite) polytope $W_M D$ in $\mathbb{H}^3$. Note that $W_M D$ is convex: D is convex and has all dihedral angles $\leq \pi/2$, so the angles at each of the gluing edges of $W_M D$ are $\leq \pi$.

For $t > 0$, let S_t denote the t–distant surface from a component S of $\partial W_M D$. Then S_t is a piecewise smooth C^1 surface; see [15, Proposition II.2.2.1]. In fact, S_t is a union of pieces of which there are three types: hyperbolic, Euclidean, and spherical, each of which are the equidistant pieces from faces, edges, and vertices of S, respectively. The Euclidean pieces look like rectangles that are each adjacent to two hyperbolic pieces and two spherical pieces, and the spherical pieces are adjacent to Euclidean pieces.

As $W_M D$ is convex, the nearest point projection $p \colon \mathbb{H}^3 \cup \partial \mathbb{H}^3 \to W_M D$ is defined. If we fix $t > r > 0$, then p induces a map $p_{t,r} \colon S_t \to S_r$.

Lemma 5.6 *The map $p_{t,r} \colon S_t \to S_r$ that is induced by nearest point projection is $\tanh(t)/\tanh(r)$–quasiconformal.*

Proof It suffices to check what $p_{t,r}$ does on each of the three types of pieces. First note that a face of S is simply the intersection of $\partial W_M D$ with a hyperbolic plane in $\mathbb{H}^3$. Thus $p_{t,r}$ simply scales the corresponding hyperbolic pieces on S_t and S_r by a constant factor. Hence $p_{t,r}$ is conformal there. Similarly, the map $p_{t,r}$ is conformal on the spherical pieces. Second, we consider the Euclidean piece in S_t equidistant

from an edge of S. A Euclidean piece looks like a rectangle adjacent to two hyperbolic pieces at two parallel edges (parallel in the intrinsic Euclidean geometry), and the map induced by nearest point projection $S_t \to S$ scales by a factor of $1/\cosh(t)$ in the direction of those edges. The other two edges of the Euclidean piece are each adjacent to a spherical piece. An edge like this is the arc of a circle with radius t centered at a vertex in S. Thus the edge has length $\theta \sinh(t)$, where θ is the dihedral angle at the corresponding edge of S. Hence the map $p_{t,r}$ scales by a factor of $\cosh(r)/\cosh(t)$ in the direction of the edges adjacent to the hyperbolic pieces, and it scales the edges adjacent to the spherical pieces by a factor of $\sinh(r)/\sinh(t)$. Therefore, $p_{t,r}$ is $\tanh(t)/\tanh(r)$–quasiconformal on the Euclidean pieces. $\qquad\qquad\square$

Proof of Theorem 1.2

Suppose that M is a complete smooth Riemannian manifold. Given a nonnegative measurable function $f\colon M \to [0, \infty)$, we define a new norm on the C^∞ k–forms, called the L_f^2 norm, by

$$\|\omega\|_f^2 = \int_M \|\omega\|_p^2 f(p)\, dV,$$

where $\|\omega\|_p^2$ is the pointwise norm and dV is the volume form of M. Let $L_f^2 C^*(M)$ denote the weighted L^2 de Rham complex defined using the L_f^2 norm.

Lemma 5.7 *Let M and N be smooth surfaces and suppose that $\phi\colon M \to N$ is a K–quasiconformal diffeomorphism. Let $g\colon N \to [0, \infty)$ be the function defined by $g(p) = f(\phi^{-1}(p))$. Then, for every $\omega \in L_g^2 C^1(N)$, we have that*

$$\frac{1}{K}\|\omega\|_g^2 \le \|\phi^*(\omega)\|_f^2 \le K\|\omega\|_g^2.$$

Proof The pointwise norm of a 1–form is $\|\omega\|_p = \sup\{\omega(x) \mid x \in T_p M, \|x\| = 1\}$, where $T_p M$ is the tangent space of M at p. Since ϕ is K–quasiconformal, its differential $d\phi$ maps the circle $\{x \in T_p M \mid \|x\| = 1\}$ to an ellipse in $T_{\phi(p)}N$ with semiaxis $b(p) \le a(p)$ satisfying $a(p)/b(p) \le K$. Now, if $\omega \in L_q^2 C^1(N)$, then $\|\phi^*(\omega)\|_p = \sup\{\omega(d\phi(x)) \mid x \in T_p M, \|x\| = 1\}$; in particular, we are taking the supremum of ω over the ellipse in $T_{\phi(p)}N$. Thus, for any $\omega \in L_q^2 C^1(N)$,

$$b(p)\|\omega\|_{\phi(p)} \le \|\phi^*(\omega)\|_p \le a(p)\|\omega\|_{\phi(p)}.$$

Now, let dV_M and dV_N be the respective volume forms of M and N. We have that

$$(f dV_M)_p = \frac{(g(\phi)\phi^*(dV_N))_p}{a(p)b(p)},$$

so for L_f^2 norms, we have

$$\begin{aligned}
\|\phi^*(\omega)\|_f^2 &= \int_M \|\phi^*(\omega)\|_p^2 f(p) \, dV_M \\
&\leq \int_M \frac{a(p)}{b(p)} \|\omega\|_{\phi(p)}^2 g(\phi(p)) \phi^*(dV_N) \\
&\leq K \int_M \|\omega\|_{\phi(p)}^2 g(\phi(p)) \phi^*(dV_N) \\
&= K \int_N \|\omega\|_x^2 g(x) \, dV_N = K \|\omega\|_g^2.
\end{aligned}$$

The remaining inequality follows similarly. $\qquad\square$

Suppose that the nerve L of W is a triangulation of S^2 and that $\Sigma_L = \mathbb{H}^3$. Define f to be the function $f(p) = q_w$, where $w \in W_L$ is a word of shortest length such that $p \in wD$ (here D is the Davis chamber). Let $L_q^2 \mathcal{H}^*(\mathbb{H}^3)$ denote the weighted L^2 de Rham cohomology defined using this f.

Let W_M be an infinite special subgroup of W and let S be one of the components of $\partial W_M D$. Put coordinates (x, t) on $\mathbb{H}^3$ so that $t \in \mathbb{R}$ is the oriented distance from $p \in \mathbb{H}^3$ to the closest point $x \in S$. Fix $r > 0$, and for $t \geq r$, let S_t denote the hypersurface consisting of points of (oriented) distance t from S. Let $p_{t,r} : S_t \to S_r$ be the map induced by nearest point projection, and let $\phi_{t,r}$ denote the inverse of $p_{t,r}$. By [Lemma 5.6](#), $p_{t,r}$ is $K(t)$–quasiconformal, with $K(t) = \tanh(t)/\tanh(r)$, and hence so is its inverse $\phi_{t,r} : S_r \to S_t$. Let $i_r : S_r \to \mathbb{H}^3$ and $i_t : S_t \to \mathbb{H}^3$ be the inclusions. Then i_r and $i_t \circ \phi_{t,r}$ are properly homotopic.

We now adapt the argument after [8, Theorem 16.10] to prove the following lemma.

Lemma 5.8 If $q \geq 1$, then the map $i_r^* : L_q^2 \mathcal{H}^1(\mathbb{H}^3) \to L_q^2 \mathcal{H}^1(S_r)$ induced by the inclusion i_r is the zero map.

Proof Set $g(x, y) = f(x, 0)$, so $f(x, y) \geq g(x, y)$, and let ω be a closed L_f^2 1–form on $\mathbb{H}^3$. We now show that the restriction $i_r^*(\omega)$ to S_r represents the zero class in reduced L_f^2–cohomology. For the remainder of the proof, we will use the notation $\|[\alpha]\|_g$ and $\|[\alpha]\|_x$ to denote the respective L_g^2 norm and pointwise norm of the harmonic representative of the cohomology class $[\alpha]$.

Suppose for a contradiction that $[i_r^*(\omega)] \neq 0$. Then $\|i_r^*(\omega)\|_g \geq \|[i_r^*(\omega)]\|_g > 0$. By [Lemma 5.7](#), it follows that $\|\phi_{t,r}^*(i_t^*(\omega))\|_g^2 \leq K(t) \|i_t^*(\omega)\|_g^2$, and since i_r and $i_t \circ \phi_{t,r}$ are properly homotopic, $[i_r^*(\omega)] = [\phi_{t,r}^*(i_t^*(\omega))]$. Therefore,

$$K(t) \|i_t^*(\omega)\|_g^2 \geq \|[i_r^*(\omega)]\|_g^2 > 0.$$

Now we have the pointwise inequality $\|\omega\|_x \geq \|i_t^*(\omega)\|_x$, as $i_t^*(\omega)$ is just a restriction of ω. Using Fubini's theorem, we compute

$$\|\omega\|_g^2 = \int_{\mathbb{H}^3} \|\omega\|_x^2 g(x, y) \, dV$$

$$\geq \int_r^\infty \int_{S_t} \|\omega\|_x^2 g(x, y) \, dA \, dt \geq \int_r^\infty \int_{S_t} \|i_t^*(\omega)\|_x^2 g(x, y) \, dA \, dt$$

$$= \int_r^\infty \|i_t^*(\omega)\|_g^2 \, dt \geq \int_r^\infty \frac{\tanh(r)}{\tanh(t)} \|[i_r^*(\omega)]\|_g^2 \, dt = \infty.$$

Since $\|\omega\|_f \geq \|\omega\|_g$, this contradicts the assumption that the L_f^2 norm of ω is finite. $\square$

Suppose that L is the nerve of a Coxeter group W_L and that A is a full subcomplex of L. For the proofs that follow, note that $\dim_q L_q^2 H_k(W_L \Sigma_A) = L_q^2 b_k(\Sigma_A)$; see [6, page 352, property (vi)].

Lemma 5.9 *Suppose that the nerve L is a triangulation of S^2 and there exists a full subcomplex 1–sphere M of L that separates L into two full 2–disks L_1 and L_2 with boundary M. Furthermore, suppose that either* (i) $\Sigma_M = \mathbb{R}^2$ *or* (ii) $\Sigma_L = \mathbb{H}^3$. *Then*

$$L_q^2 H_k(\Sigma_L) = 0 \quad \text{for} \quad k \geq 2 \quad \text{and} \quad q \leq 1.$$

Proof Since Σ_L is a 3–manifold, it follows that $L_q^2 b_3(\Sigma_L) = 0$ [6, Proposition 20.4.1]. Hence we must show that $L_q^2 b_2(\Sigma_L) = 0$. Consider the following Mayer–Vietoris sequence applied to $L = L_1 \cup_M L_2$:

$$\cdots \to L_q^2 H_2(W_L \Sigma_{L_1}) \oplus L_q^2 H_2(W_L \Sigma_{L_2}) \to L_q^2 H_2(\Sigma_L) \to L_q^2 H_1(W_L \Sigma_M) \to \cdots$$

By Theorem 5.2, we have that $L_q^2 H_2(W_L \Sigma_{L_1}) = L_q^2 H_2(W_L \Sigma_{L_2}) = 0$. If (i) holds, then [8, Corollary 14.5] implies that $L_q^2 H_1(\Sigma_M) = 0$, and we are done. If (ii) holds, we argue that the connecting homomorphism $\partial_* \colon L_q^2 H_2(\Sigma_L) \to L_q^2 H_1(W_L \Sigma_M)$ is the zero map. By [8, Lemma 16.2], we reduce the proof to showing that the map induced by inclusion $i_* \colon L_{q^{-1}}^2 H_1(W_L \Sigma_M) \to L_{q^{-1}}^2 H_1(\Sigma_L)$ is the zero map, and since $W_L \Sigma_M$ is a disjoint union of copies of Σ_M, it is enough to show that the restriction of i_* to one summand $L_{q^{-1}}^2 H_1(\Sigma_M)$ is zero.

Consider the infinite convex polytope $W_M D$, where D is the Davis chamber for W. We have that W_M acts properly and cocompactly on $W_M D$ by isometries. In particular, if S is one of the components of $\partial W_M D$, then W_M acts properly and cocompactly on S, and therefore $L_{q^{-1}}^2 H^*(\Sigma_M) \cong L_{q^{-1}}^2 \mathcal{H}^*(S)$. Hence we are done if we show that map $i^* \colon L_{q^{-1}}^2 \mathcal{H}^1(\mathbb{H}^3) \to L_{q^{-1}}^2 \mathcal{H}^1(S)$ induced by the inclusion $i \colon S \to \mathbb{H}^3$ is the zero map.

Fix $r > 0$, and let S_r be the r–distant surface from S. S_r and S are properly homotopy equivalent, and this equivalence induces a weak isomorphism between $L^2_{\boldsymbol{q}^{-1}}\mathcal{H}^*(S)$ and $L^2_{\boldsymbol{q}^{-1}}\mathcal{H}^*(S_r)$. Thus we have reduced the proof to showing that the map $i_r^*\colon L^2_{\boldsymbol{q}^{-1}}\mathcal{H}^1(\mathbb{H}^3) \to L^2_{\boldsymbol{q}^{-1}}\mathcal{H}^1(S_r)$ induced by the inclusion $i_r\colon S_r \to \mathbb{H}^3$ is the zero map, and therefore we are done by Lemma 5.8. $\qquad\square$

Remark 5.10 In [8, Section 16], W is strictly assumed to be right-angled, but the proof of [8, Lemma 16.2] does not use this, as it only uses properties of weighted L^2–(co)homology.

Proof of Theorem 1.2 We first suppose that $\Sigma_L = \mathbb{H}^3$. We need to find a full subcomplex M of L satisfying the hypothesis of Lemma 5.9. First we suppose that L is a flag complex. Let v be a vertex of L and set $M = \mathrm{Lk}(v)$. Since L is flag, M is a full subcomplex of L, and since L is a triangulation of the 2–sphere, it follows that M is a 1–sphere, and we are done. Now suppose that L is not flag. Since L is not the boundary of a 3–simplex, there exists an empty 2–simplex in L. Let M denote this empty 2–simplex. Then M separates L into two full 2–disks, both with boundary M, and we are done. We now suppose that $\Sigma_L \neq \mathbb{H}^3$ and use Theorem 5.4 to perform a case-by-case analysis.

Case I First suppose that W contains a Euclidean special subgroup W_T. Let M be the full subcomplex of L corresponding to W_T. Then M separates L into two 2–disks both with boundary M, and hence Lemma 5.9(i) implies the assertion.

Case II Now suppose that $W = W_T \times D_\infty$, where $T \subset S$ spans empty triangle in L. Either $\Sigma_L = \mathbb{R}^3$ or $\Sigma_L = \mathbb{H}^2 \times \mathbb{R}$. In both cases we are done by the weighted Künneth formula.

Case III Lastly, suppose that L is the boundary of a 3–simplex. By assumption, L is not dual to a hyperbolic simplex, so $\Sigma_L = \mathbb{R}^3$. Therefore, we are done by [8, Corollary 14.5]. $\qquad\square$

Proof of Theorem 1.4

In this case, Σ_L is a 4–manifold, and hence $L^2_{\boldsymbol{q}}b_4(\Sigma_L) = 0$ [6, Proposition 20.4.1]. It remains to show that $L^2_{\boldsymbol{q}}b_3(\Sigma_L) = 0$. Suppose that the nerve L is a triangulation of S^3, and let $s \in L$ be a vertex. We make the following observations:

- The nerve L_{S-s} of the Coxeter system $(W_{S-s}, S - s)$ is a 3–disk.
- The nerve $\mathrm{St}(s)$ of the Coxeter group $W_{\mathrm{St}(s)}$ is a 3–disk.
- The nerve $\mathrm{Lk}(s)$ of the Coxeter group $W_{\mathrm{Lk}(s)}$ is a 2–sphere.

This is because the subcomplexes $\mathrm{St}(s)$, $\mathrm{Lk}(s)$, and L_{S-s} of L correspond to the closed star of the vertex s, the link of the vertex s, and the complement of the open star of s, respectively, which are all full subcomplexes of L by assumption. Consider the following Mayer–Vietoris sequence:

$$\cdots \to L^2_{\boldsymbol{q}} H_3(W_L \Sigma_{L_{S-s}}) \oplus L^2_{\boldsymbol{q}} H_3(W_L \Sigma_{\mathrm{St}(s)}) \to L^2_{\boldsymbol{q}} H_3(\Sigma_L) \to L^2_{\boldsymbol{q}} H_2(W_L \Sigma_{\mathrm{Lk}(s)}) \to \cdots.$$

By Theorem 5.2, $L^2_{\boldsymbol{q}} b_3(\Sigma_{\mathrm{St}(s)}) = 0$ and $L^2_{\boldsymbol{q}} b_3(\Sigma_{L_{S-s}}) = 0$, and by Theorem 1.2, $L^2_{\boldsymbol{q}} b_2(\Sigma_{\mathrm{Lk}(s)}) = 0$. Therefore, by the above sequence, $L^2_{\boldsymbol{q}} b_3(\Sigma_L) = 0$. $\square$

Proof of Theorem 1.6

Lemma 5.11 *Suppose that L is a flag triangulation of a 3–manifold. Then for every $t \in L$, we have $L^2_{\boldsymbol{q}} H_*(\Omega(S,t), \partial\Omega(S,t)) = 0$ for $* > 2$ and $\boldsymbol{q} \leq 1$.*

Proof First, for $t \in L$, note that the (S,t)–ruin has the property that $\Omega(S,t) = \Omega(\mathrm{St}(t), t)$, where $\mathrm{St}(t) = \{s \in S \mid m_{st} < \infty\}$. Set $\mathrm{Lk}(t) = \mathrm{St}(t) - t$, and so we have the following weak exact sequence:

$$\cdots \to L^2_{\boldsymbol{q}} H_*(\Sigma(\mathrm{Lk}(t))) \to L^2_{\boldsymbol{q}} H_*(\Sigma(\mathrm{St}(t))) \to L^2_{\boldsymbol{q}} H_*(\Omega(S,t), \partial\Omega(S,t)) \to \cdots.$$

Note that

$$L^2_{\boldsymbol{q}} b_*(\Sigma(\mathrm{St}(t))) = L^2_{\boldsymbol{q}} b_*(\Sigma_{\mathrm{St}(t)}) \quad \text{and} \quad L^2_{\boldsymbol{q}} b_*(\Sigma(\mathrm{Lk}(t))) = L^2_{\boldsymbol{q}} b_*(\Sigma_{\mathrm{Lk}(t)}),$$

where $\Sigma_{\mathrm{St}(t)}$ and $\Sigma_{\mathrm{Lk}(t)}$ are the Davis complexes corresponding to the subgroups $W_{\mathrm{St}(t)}$ and $W_{\mathrm{Lk}(t)}$, respectively; see the discussion before [8, Lemma 8.1]. Since L is flag, the respective nerves of the groups $W_{\mathrm{St}(t)}$ and $W_{\mathrm{Lk}(t)}$ are a 3–disk and a 2–sphere. Furthermore, the nerve of $W_{\mathrm{Lk}(t)}$ is not the boundary of a 3–simplex (again, L is flag). By Theorem 5.2, $L^2_{\boldsymbol{q}} b_k(\Sigma_{\mathrm{St}(t)}) = 0$ for $k > 2$, and by Theorem 1.2, $L^2_{\boldsymbol{q}} b_k(\Sigma_{\mathrm{Lk}(t)}) = 0$ for $k > 1$. Therefore, weak exactness of the sequence implies that, for $* > 2$, $L^2_{\boldsymbol{q}} H_*(\Omega(S,t), \partial) = 0$. $\square$

Lemma 5.12 (compare [23, Lemma 4.1]) *For every $T \in \mathcal{S}^{(2)}$ and $U \subset S$ with $T \subset U$, we have $L^2_{\boldsymbol{q}} H_4(\Omega(U,T), \partial\Omega(U,T)) = 0$ for $\boldsymbol{q} \leq 1$.*

Proof The proof of [21, Lemma 4.1] shows that $L^2_1 H_4(\Omega(U,T), \partial) = 0$, the main point being that L is a flag triangulation of a 3–manifold, and so it follows that Σ_{cc} is a 4–pseudomanifold; ie every 3–cell of Σ_L is contained in precisely two 4–cells. The argument of Lemma 2.7 now completes the proof. $\square$

Proof of Theorem 1.6 With the above lemmas, we now follow [23, Proof of the main theorem] line by line. For every $U \subset S$ and $t \in U$, we have this weak exact sequence:

$$\cdots \to L_q^2 H_*(\Sigma(U - t)) \to L_q^2 H_*(\Sigma(U)) \to L_q^2 H_*(\Omega(U, t), \partial) \to \cdots .$$

By Lemma 5.12 and [23, Proposition 4.2], $L_q^2 H_*(\Omega(U, t), \partial) = 0$ for $* > 2$, and hence by exactness,

$$L_q^2 H_*(\Sigma(U - t)) \cong L_q^2 H_*(\Sigma(U)) \quad \text{for} \quad * > 2.$$

It follows that $L_q^2 H_*(\Sigma(S)) \cong L_q^2 H_*(\Sigma(\varnothing))$ for $* > 2$, and hence the theorem. □

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