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A self-pairing theorem for tangle Floer homology

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We show that for a tangle T with $-\partial^0 T \cong \partial^1 T$ the Hochschild homology of the tangle Floer homology $\widetilde{\mathrm{CT}}(T)$ is equivalent to the link Floer homology of the closure $T' = T/(-\partial^0 T \sim \partial^1 T)$ of the tangle, linked with the tangle axis. In addition, we show that the action of the braid group on tangle Floer homology is faithful.

1 Introduction

Tangle Floer homology is an invariant of tangles in 3–manifolds with boundary S^2 or $S^2 \sqcup S^2$, or in closed 3–manifolds, which takes the form of a differential graded module, bimodule, or a chain complex, respectively; see Petkova and Vértesi [9]. It behaves well under gluing and recovers knot Floer homology. Before we state the main results, we recall some definitions from [9] and make some new ones.

Definition 1.1 An *n–marked sphere* $\mathcal{S}$ is a sphere S^2 with n oriented points $t_1, \ldots, t_n$ on its equator $S^1 \subset S^2$ numbered respecting the orientation of S^1.

Definition 1.2 A *marked (m,n)–tangle* $\mathcal{T}$ in an oriented 3–manifold Y with two boundary components $\partial^0 Y \cong S^2$ and $\partial^1 Y \cong S^2$ is a properly embedded 1–manifold T with $(-\partial^0 Y, -(\partial^0 Y \cap \partial T))$ identified with an m–marked sphere and $(\partial^1 Y, \partial^1 Y \cap \partial T)$ identified with an n–marked sphere (via orientation-preserving diffeomorphisms). We denote $-(\partial^0 Y \cap \partial T)$ and $\partial^1 Y \cap \partial T$ along with the ordering information by $-\partial^0 \mathcal{T}$ and $\partial^1 \mathcal{T}$.

Definition 1.3 A *strongly marked (m,n)–tangle* $(Y, \mathcal{T}, \gamma)$ is a marked (m,n)–tangle $(Y, \mathcal{T})$, along with a framed arc γ connecting $\partial^0 Y$ to $\partial^1 Y$ in the complement of $\mathcal{T}$ such that γ and its framing λ_γ (viewed as a push-off of γ) have ends on the equators of the two marked spheres, and we see $-\partial^0 \mathcal{T}, -\partial^0 \gamma, -\partial^0 \lambda_\gamma$ and $\partial^1 \mathcal{T}, \partial^1 \gamma, \partial^1 \lambda_\gamma$ in this order along each equator. See Figure 1.
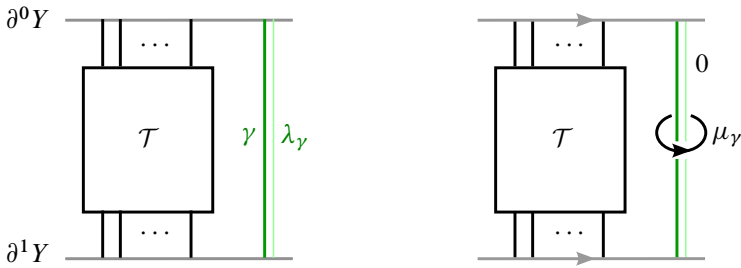
Figure 1: Left: a strongly marked tangle $(Y, \mathcal{T}, \gamma)$. Right: the surgered closure (Y_0, T_0) of the tangle $(Y, \mathcal{T}, \gamma)$.

As a special case, an (m, n)–tangle in $\mathbb{R}^2 \times I$ is a cobordism (contained in $[1, \infty) \times \mathbb{R} \times I$) from $\{1, \ldots, m\} \times \{0\} \times \{0\}$ to $\{1, \ldots, n\} \times \{0\} \times \{1\}$. A tangle in $\mathbb{R}^2 \times I$ can be thought of as a strongly marked tangle, by compactifying $\mathbb{R}^2 \times I$ to $S^2 \times I$, taking the images of $\mathbb{R} \times \{0\} \times \{0\}$ and $\mathbb{R} \times \{0\} \times \{1\}$ to be the equators of the marked spheres, and setting $(\gamma, \lambda_\gamma) := (\{(-1, 0)\} \times I, \{(0, 0)\} \times I)$.

We turn our attention to strongly marked tangles $(Y, \mathcal{T}, \gamma)$ with $-\partial^0 \mathcal{T} \cong \partial^1 \mathcal{T}$.

Definition 1.4 A strongly marked tangle $(Y, \mathcal{T}, \gamma)$ is called *closable* if $-\partial^0 \mathcal{T} \cong \partial^1 \mathcal{T}$. Given a closable tangle $(Y, \mathcal{T}, \gamma)$, we can glue it to itself to form its *closure* (Y', T', γ') by identifying the two boundary components of Y, $-\partial^0 Y$ and $\partial^1 Y$, with the same marked sphere. The *surgered closure* of $(Y, \mathcal{T}, \gamma)$ is the pair (Y_0, T_0), where the link T_0 is the union of T' and the negatively oriented meridian μ_γ of γ in the 0–surgery $Y_0 = Y'_0(\gamma')$ of Y' along the framed knot γ'. We call μ_γ the *tangle axis* of the tangle $\mathcal{T}$. See Figure 1.

When Y is $S^2 \times I$ and (γ, λ_γ) is a product as above, then $Y_0 \cong S^3$ and T_0 is the link formed by the closure of $T \subset \mathbb{R}^2 \times I \subset S^3$ and an unknot that is the boundary of a disk containing $-\partial^0 T \sim \partial^1 T$, see Figure 2. For example, for a braid $B \in \mathbb{R}^2 \times I$, the tangle axis is precisely the braid axis.

Main Theorem *Let $(Y, \mathcal{T}, \gamma)$ be a closable strongly marked tangle. Then there is an equivalence*

$$\mathrm{HH}(\widetilde{\mathrm{CT}}(Y, \mathcal{T}, \gamma)) \cong \widetilde{\mathrm{HFK}}(Y_0, T_0).$$

Here and throughout the paper, $\widetilde{\mathrm{CT}}$ stands for the *DA* bimodule $\widetilde{CFDTA}$ from [9, Section 10.3], and $\widetilde{\mathrm{HFK}}$ is the multipointed version of knot Floer homology; see for example [5; 4]. Note that both $\widetilde{\mathrm{CT}}$ and $\widetilde{\mathrm{HFK}}$ depend on the number of basepoints in a Heegaard diagram, and the precise formulation of the Main Theorem follows in Section 3, in the form of Theorems 3.1 and 3.2.
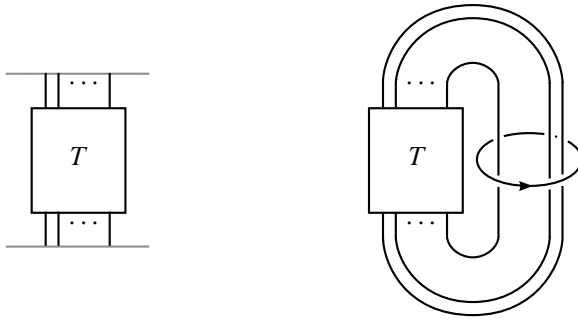
Figure 2: Left: a tangle $T \subset \mathbb{R}^2 \times I$. Right: the corresponding link $T_0 \subset S^3$.

Combined with a result of Baldwin and Grigsby [1], we get the following corollary.

Corollary *Tangle Floer homology restricts to a faithful linear-categorical action of the braid group.*

2 Algebra review

Let A be a unital differential graded algebra over a ground ring $\boldsymbol{k}$, where $\boldsymbol{k}$ is a direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. The unit gives a preferred map $\iota\colon \boldsymbol{k} \to A$. We assume that A is *augmented*, ie there is a map $\epsilon\colon A \to \boldsymbol{k}$ such that $\epsilon(1) = 1$, $\epsilon(ab) = \epsilon(a)\epsilon(b)$, and $\epsilon(\partial a) = 0$. The *augmentation ideal* $\ker \epsilon$ is denoted by A_+.

A *type DA bimodule over* (A, A) is a graded $\boldsymbol{k}$–bimodule N, together with degree 0 $\boldsymbol{k}$–linear maps

$$\delta^1_{1+j}\colon N \otimes A[1]^{\otimes j} \to A \otimes N[1],$$

satisfying a certain compatibility condition; see [3, Definition 2.2.42].

A *DA* bimodule is *bounded* if the structure maps behave in a certain nice way; see [3, Definition 2.2.45]. We will not recall the complete definition of boundedness here, but we point out that the structures arising from nice Heegaard diagrams are bounded, and moreover the only nonzero structure maps in that case are δ^1_1 and δ^1_2. We will call a *DA* bimodule *nice* if it is bounded and $\delta^1_i = 0$ for all $i > 2$.

Given a bounded type DA bimodule N over (A, A), one can define a chain complex $(N^\circ, \widetilde{\partial})$ whose homology agrees with the Hochschild homology of the $\mathcal{A}_\infty$–bimodule $A \boxtimes N$ corresponding to N; see [3, Section 2.3.5]. The vector space N°, called the *cyclicization* of N, is the quotient $N/[N, \boldsymbol{k}]$, where $[N, \boldsymbol{k}]$ is the submodule of N generated by elements $xk - kx$, for $x \in N$ and $k \in \boldsymbol{k}$. The differential $\widetilde{\partial}$ is easy to describe when N is nice. We recall the construction in this special case below.

Define a cyclic rotation map $R \colon (A \otimes N)^\circ \to (N \otimes A_+)^\circ$ by

$$R(a \otimes x) = x \otimes [(\mathrm{id} - \iota \circ \epsilon)(a)].$$

The map $\epsilon \otimes \mathrm{id} \colon A \otimes N \to \boldsymbol{k} \otimes N = N$ descends to a map $(A \otimes N)^\circ \to N^\circ$, which we will also denote ϵ. We denote the cyclizations of

$$\delta_1^1 \colon N \to A \otimes N \quad \text{and} \quad \delta_2^1 \colon N \otimes A_+ \to A \otimes N$$

by δ_1^1 and δ_2^1 as well. Finally, $\widetilde{\partial}$ is defined as

$$\widetilde{\partial} = \epsilon \circ \delta_1^1 + \epsilon \circ \delta_2^1 \circ R \circ \delta_1^1.$$

Given a tangle $(Y, \mathcal{T}, \gamma)$, one can represent it by a multipointed bordered Heegaard diagram $\mathcal{H}$ with two boundary components $\partial^0 \mathcal{H}$ and $\partial^1 \mathcal{H}$; see [9, Section 8.2]. To $-\partial^0 \mathcal{H}$ and $\partial^1 \mathcal{H}$ one associates differential algebras $\mathcal{A}(-\partial^0 \mathcal{T})$ and $\mathcal{A}(\partial^1 \mathcal{T})$, and to $\mathcal{H}$ a DA bimodule $\widetilde{\mathrm{CT}}(\mathcal{H})$ over $\mathcal{A}(-\partial^0 \mathcal{T})$ and $\mathcal{A}(\partial^1 \mathcal{T})$. The structure maps on the bimodule are obtained by counting certain holomorphic curves in $\mathcal{H} \times I \times \mathbb{R}$; see [9, Sections 7.2 and 10.3]. For a tangle in $\mathbb{R}^2 \times I$, the bimodule can also be defined in terms of sequences of strand diagrams corresponding to a decomposition of the tangle into elementary pieces; see [9, Sections 3 and 5.2]. We do not recall the two constructions here, but refer the reader to [9].

3 Proofs of the main results

We prove two versions of the Main Theorem, both via nice diagrams: the first, Theorem 3.1, deals with the general case (ungraded); the second, Theorem 3.2, deals with the case when the tangle is in $\mathbb{R}^2 \times I$.

Tangle Floer homology is an invariant of the tangle: if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Heegaard diagrams for $(Y, \mathcal{T}, \gamma)$ with $2k_1$ and $2k_2$ basepoints, respectively, and $k_1 \geq k_2$, then

$$\widetilde{\mathrm{CT}}(\mathcal{H}_1) \simeq \widetilde{\mathrm{CT}}(\mathcal{H}_2) \otimes (\mathbb{F}_2 \oplus \mathbb{F}_2)^{\otimes (k_1 - k_2)}.$$

For a closable tangle $-\partial^0 \mathcal{T}$ is isomorphic to $\partial^1 \mathcal{T}$, so the algebras $\mathcal{A}(-\partial^0 \mathcal{T})$ and $\mathcal{A}(\partial^1 \mathcal{T})$ are the same, and one can take the Hochschild homology of the bimodule.
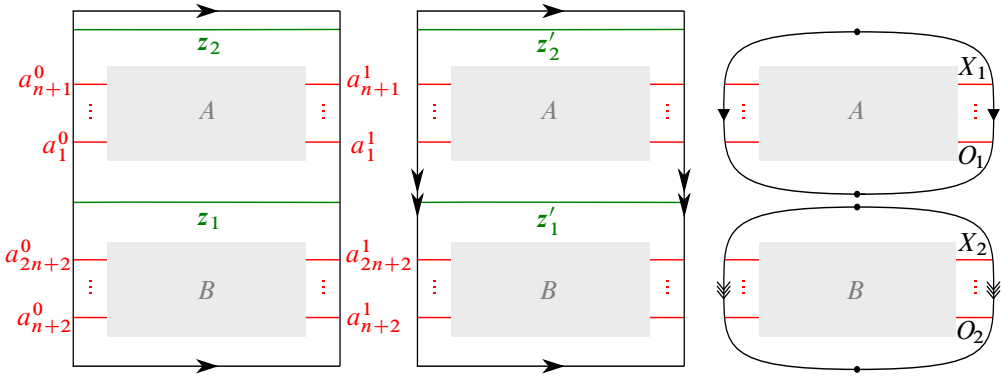
Figure 3: Left: a Heegaard diagram $\mathcal{H}$ for a closable tangle $(Y, \mathcal{T}, \gamma)$. The left edge is $\partial^0 \mathcal{H}$ and the right edge is $\partial^1 \mathcal{H}$. Middle: the diagram $\mathcal{H}'$ obtained by self-gluing $\mathcal{H}$. Right: the Heegaard diagram $\mathcal{H}^\circ$ obtained by surgery on the two green curves z_1' and z_2' on $\mathcal{H}'$.

Theorem 3.1 *If $\mathcal{H}$ is Heegaard diagram with $2k$ basepoints for a closable strongly marked tangle $(Y, \mathcal{T}, \gamma)$, and $\mathcal{H}^\circ$ is a diagram with $2k + 4$ basepoints for (Y_0, T_0), for some k, then*

$$\mathrm{HH}(\widetilde{\mathrm{CT}}(\mathcal{H})) \cong \widetilde{\mathrm{HFK}}(\mathcal{H}^\circ).$$

Note that in [9], gradings for tangle Floer homology are only defined when the underlying manifold is $S^2 \times I$ or B^3, so Theorem 3.1 only claims an ungraded isomorphism.

Proof The proof is very similar to that of [3, Theorem 14]. By invariance under Heegaard moves, it suffices to prove the theorem for one choice of $\mathcal{H}$, and one choice of $\mathcal{H}^\circ$ with 4 more basepoints than $\mathcal{H}$.

Let $(Y, \mathcal{T}, \gamma)$ be a closable strongly marked (n, n)–tangle, and $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbb{X}, \mathbb{O}, z)$ be a nice bordered Heegaard diagram for $(Y, \mathcal{T}, \gamma)$, as in [9, Proposition 12.1]. Glue $\mathcal{H}$ to itself by identifying $-\partial^0 \mathcal{H}$ and $\partial^1 \mathcal{H}$, and call the result $\mathcal{H}'$ (note that this is not a valid Heegaard diagram). Recall that $z = \{z_1, z_2\}$ is a set of two arcs in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ with boundary on $\partial \Sigma \setminus \boldsymbol{\alpha}$, oriented from the left to the right boundary, and let z_1' and z_2' be the resulting closed curves in $\mathcal{H}'$. Surger $\mathcal{H}'$ along z_1' and z_2', and place 4 basepoints in the 4 resulting regions: X_1, O_1, X_2, and O_2 in the region whose boundary contains $a_{n+1}^0, a_1^0, a_{2n+2}^0$, and a_{n+2}^0, respectively. The result is a diagram $\mathcal{H}^\circ = (\Sigma^\circ, \boldsymbol{\alpha}^\circ, \boldsymbol{\beta}^\circ, \mathbb{X}^\circ, \mathbb{O}^\circ)$; see Figure 3.

Recall that originally $(Y, \mathcal{T})$ was obtained by attaching a two handle to the pair $(Y_{\mathrm{dr}}, \mathcal{T}_{\mathrm{dr}})$ that we got from a drilled diagram on $\Sigma - N(z_2)$ by the usual handle attachments. Thus, the Heegaard diagram $\mathcal{H}''$ that we get by only doing surgery on $\mathcal{H}'$ along z_2'

describes $(Y' - N(\gamma'), T')$, with $\mathbf{z}_1'$ lying on the boundary of $N(\gamma')$. Now the surgery along $\mathbf{z}_1'$ on $\mathcal{H}''$ simply results in the Dehn filling of $(Y' - N(\gamma'), T')$ with framing $\mathbf{z}_1'$. The knot described by X_1, O_1, X_2, and O_2 goes through the former $\mathbf{z}_2$ and $\mathbf{z}_1$ on the Heegaard diagram, thus indeed describes a meridian for T'. This means that $\mathcal{H}^\circ$ is a Heegaard diagram for (Y_0, T_0).

Observe that the generators of $\mathcal{H}'$, or equivalently the generators of $\mathcal{H}^\circ$, correspond to generators $\mathbf{x}$ of $\mathcal{H}$ with $\bar{o}^0(\mathbf{x}) = o^1(\mathbf{x})$, whereas in [9] $o^1(\mathbf{x})$ (respectively $\bar{o}^0(\mathbf{x})$) denotes the set of $\boldsymbol{\alpha}$ arcs that are occupied (not occupied) by $\mathbf{x}$ on $\boldsymbol{\alpha}^1$ (and $\boldsymbol{\alpha}^0$).

Denote the algebra $\mathcal{A}(\partial^1 \mathcal{T}) \cong \mathcal{A}(-\partial^0 \mathcal{T})$ by $\mathcal{A}$, and its ring of idempotents by $\boldsymbol{k}$. Recall that $\mathcal{A}$ has a basis over $\mathbb{F}_2$ consisting of strand diagrams [9, Section 7]. We define the augmentation map $\epsilon\colon \mathcal{A} \to \boldsymbol{k}$ on this basis explicitly: it is the identity on generators in $\boldsymbol{k} \subset \mathcal{A}$ and zero on generators $a \notin \boldsymbol{k} \subset \mathcal{A}$. The structure maps on the DA bimodule $\widetilde{\mathrm{CT}}(\mathcal{H})$ count the following types of domains; see [9, Sections 10 and 12]:

(1) Empty provincial rectangles and bigons. These contribute to δ_1^1, with image in $\boldsymbol{k} \otimes \widetilde{\mathrm{CT}}(\mathcal{H}) \subset \mathcal{A} \otimes \widetilde{\mathrm{CT}}(\mathcal{H})$.

(2) Empty rectangles that intersect $\partial^0 \mathcal{H}$ (the left boundary of $\mathcal{H}$). These contribute to δ_1^1, with image in $\mathcal{A}_+ \otimes \widetilde{\mathrm{CT}}(\mathcal{H}) \subset \mathcal{A} \otimes \widetilde{\mathrm{CT}}(\mathcal{H})$.

(3) Sets of empty rectangles, each of which intersects $\partial^1 \mathcal{H}$ (the right boundary of $\mathcal{H}$). These comprise δ_2^1, whose image is entirely contained in $\boldsymbol{k} \otimes \widetilde{\mathrm{CT}}(\mathcal{H})$.

The differential on the Hochschild complex $(\widetilde{\mathrm{CT}}(\mathcal{H})^\circ, \tilde{\partial})$ then counts the following domains on $\mathcal{H}$. The map $\epsilon \circ \delta_1^1$ counts provincial rectangles and bigons, and then forgets the idempotent component of the output. These are exactly the empty rectangles and bigons in $\mathcal{H}^\circ$ that do not cross $-\partial^0 \mathcal{H} = \partial^1 \mathcal{H}$. The map $\epsilon \circ \delta_2^1 \circ R \circ \delta_1^1$ counts rectangles as follows. Since the rotation map R is zero on elements $e \otimes x$ with $e \in \boldsymbol{k}$, only the part of δ_1^1 that counts domains of Type (2) contributes. Thus, the image of $R \circ \delta_1^1$ is generated by elements of form $y \otimes a$, where $a \in \mathcal{A}_+$ is a generator with only one moving strand. Thus, the part of δ_2^1 that contributes to $\epsilon \circ \delta_2^1 \circ R \circ \delta_1^1$ counts individual empty rectangles that intersect $\partial^1 \mathcal{H}$. To sum up, $\epsilon \circ \delta_2^1 \circ R \circ \delta_1^1$ counts pairs of empty rectangles, one with an edge on $\partial^0 \mathcal{H}$ and one with an edge on $\partial^1 \mathcal{H}$, which glue up to a rectangle after the identification $-\partial^0 \mathcal{H} \sim \partial^1 \mathcal{H}$. These are exactly the empty rectangles in $\mathcal{H}^\circ$ that cross $-\partial^0 \mathcal{H} \sim \partial^1 \mathcal{H}$.

Thus, $(\widetilde{\mathrm{CT}}(\mathcal{H})^\circ, \tilde{\partial}) \cong \widetilde{\mathrm{CFK}}(\mathcal{H}^\circ)$. $\qquad\square$

For the special case of a tangle T in $\mathbb{R}^2 \times I$, we state a graded version. In this case, $\mathrm{HH}(\widetilde{\mathrm{CT}}(\mathcal{H}))$ inherits the Maslov and Alexander gradings M and A from $\widetilde{\mathrm{CT}}(\mathcal{H})$, and also carries a *strands* grading S (counting the number of occupied α–arcs that

touch $\partial^1 \mathcal{H}$). If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Heegaard diagrams for T with $2k_1$ and $2k_2$ basepoints, respectively, and $k_1 \geq k_2$, then $\widetilde{\mathrm{CT}}(\mathcal{H}_1) \simeq \widetilde{\mathrm{CT}}(\mathcal{H}_2) \otimes (\mathbb{F}_2 \oplus \mathbb{F}_2)^{\otimes(k_1 - k_2)}$, where each tensor factor $\mathbb{F}_2 \oplus \mathbb{F}_2$ has one summand in (M, A, S) trigrading $(0, 0, 0)$ and one in trigrading $(-1, -1, 0)$.

Let l be the number of components of T_0, and label the them $L_0 = \mu_\gamma, L_1, \ldots, L_{l-1}$. The link Floer homology of the Heegaard diagram $\mathcal{H}^\circ$ for T_0 described earlier, $\widetilde{\mathrm{HFL}}(\mathcal{H}^\circ)$, is multigraded, with Maslov grading M in $\mathbb{Z} + \frac{1}{2}(l-1)$ and Alexander multigrading $(A_0, \ldots, A_{l-1})$ in $(\frac{1}{2}\mathbb{Z})^l$, with each $\frac{1}{2}\mathbb{Z}$ factor corresponding to a component of the link [8]. Let $\widetilde{\mathrm{HFL}}(\mathcal{H}^\circ, 0)$ be $\widetilde{\mathrm{HFL}}(\mathcal{H}^\circ)$ with multigrading collapsed to a trigrading by M, A_0, and $A' := A_1 + \cdots + A_{l-1} = A - A_0$ (here $A = A_0 + \cdots + A_{l-1}$ is the Alexander grading on $\widetilde{\mathrm{HFK}}(\mathcal{H}^\circ)$).

Theorem 3.2 *Let $\mathcal{H}$ be a Heegaard diagram with $2k$ basepoints for a tangle T in $\mathbb{R}^2 \times I$ with $-\partial^0 T \cong \partial^1 T$, and let $\mathcal{H}^\circ$ be a diagram with $2k + 4$ basepoints for the surgered closure T_0, with 4 of the basepoints corresponding to the component μ_γ. Then there is an isomorphism*

$$\mathrm{HH}(\widetilde{\mathrm{CT}}(\mathcal{H})) \cong \widetilde{\mathrm{HFL}}(\mathcal{H}^\circ, 0)$$

which respects the trigrading in the following sense. If the isomorphism maps a homogeneous element $x \in \mathrm{HH}(\widetilde{\mathrm{CT}}(\mathcal{H}))$ to an element $y \in \widetilde{\mathrm{HFL}}(\mathcal{H}^\circ, 0)$, then y is homogeneous and

$$M(y) = M(x) + S(x) - a - 1,$$
$$A'(y) = A(x) - S(x) + \tfrac{1}{2}l + n - a - 1,$$
$$A_0(y) = S(x) - \tfrac{1}{2}(n+1),$$

where $n = |\partial^1 T|$, a is the number of positively oriented points in $\partial^1 T$, and l is the number of components of T_0.

Proof Again, by invariance under Heegaard moves, it suffices to prove the theorem for any one specific choice of $\mathcal{H}$ and $\mathcal{H}^\circ$ with the prescribed relative number of basepoints. We already discussed the isomorphism in the proof of Theorem 3.1. It remains to identify the gradings.

Let $\mathcal{H}$ be a Heegaard diagram for T obtained by plumbing annular bordered grid diagrams, as in [9, Section 4]. By gluing on a diagram for the straight strands $\partial^1 T \times I$ if necessary, we may assume that $\mathcal{H}$ has even genus, which we denote by $2g$ (this makes for an easier gradings argument). See the top diagram in Figure 4. We modify $\mathcal{H}$ to a diagram $\mathcal{H}^\circ$ for T_0, as in the proof of Theorem 3.1. See the bottom diagram in Figure 4.
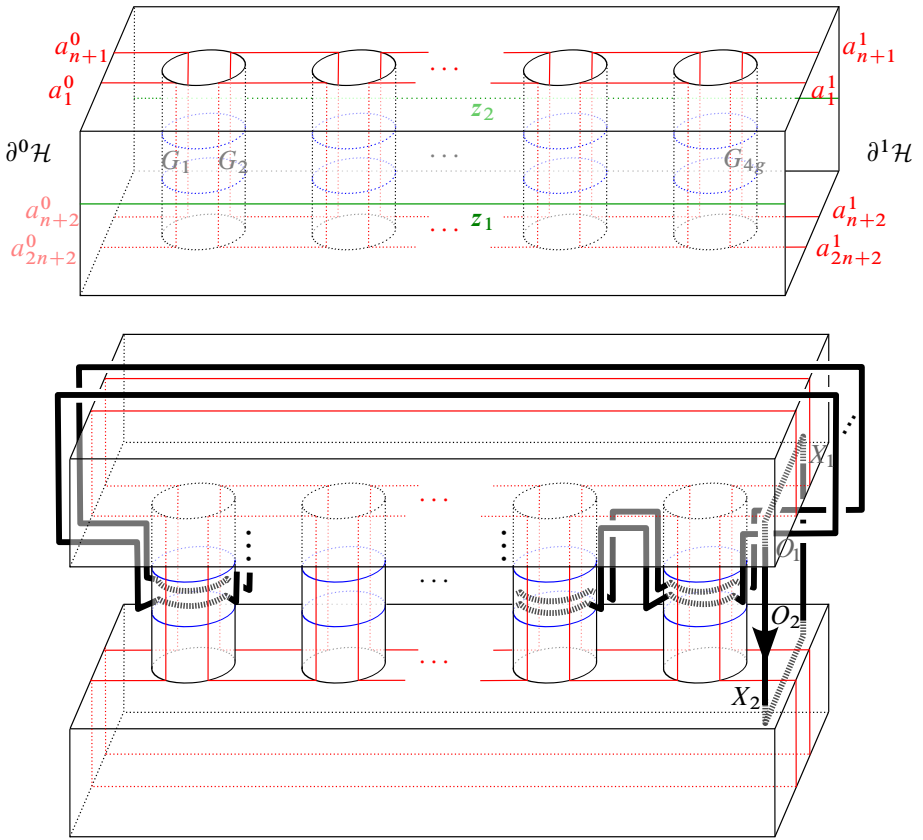
Figure 4: Top: a diagram for T in $S^2 \times I$. There are two grids on each vertical annulus, Xs and Os omitted for simplicity. Bottom: the corresponding diagram for T_0 in S^3, along with T_0 in bold black.

Call the part of $\mathcal{H}^\circ$ away from the four regions resulting from the surgery on z'_i the *nice* part of $\mathcal{H}^\circ$ (this is the part that is the plumbing of grid diagrams). As in Figure 3, we can draw $\mathcal{H}^\circ$ on the plane, as the union of two $2g$–punctured disks with certain identifications of the boundary, see Figure 7. As seen on Figure 7, we refer to the top/bottom disk as the *top/bottom half* of $\mathcal{H}^\circ$, respectively.

Denote the set of generators of $\mathcal{H}$ by $\mathfrak{S}(\mathcal{H})$, and the subset of generators with i occupied α–arcs on the right by $\mathfrak{S}_i(\mathcal{H})$. Denote the subsets of $\mathfrak{S}(\mathcal{H})$ and $\mathfrak{S}_i(\mathcal{H})$ that correspond to generators of $\mathcal{H}^\circ$ by $\mathfrak{S}(\mathcal{H})^\circ$ and $\mathfrak{S}_i(\mathcal{H})^\circ$, and the corresponding sets of generators of $\mathcal{H}^\circ$ by $\mathfrak{S}(\mathcal{H}^\circ)$ and $\mathfrak{S}_i(\mathcal{H}^\circ)$, respectively. For a generator $\boldsymbol{x} \in \mathfrak{S}(\mathcal{H})$, denote the corresponding generator in $\mathfrak{S}(\mathcal{H}^\circ)$ by $\boldsymbol{x}^\circ$. Define a *strands* grading on generators by $S(\boldsymbol{x}_i^\circ) = i$ for $\boldsymbol{x}_i^\circ \in \mathfrak{S}_i(\mathcal{H}^\circ)$.
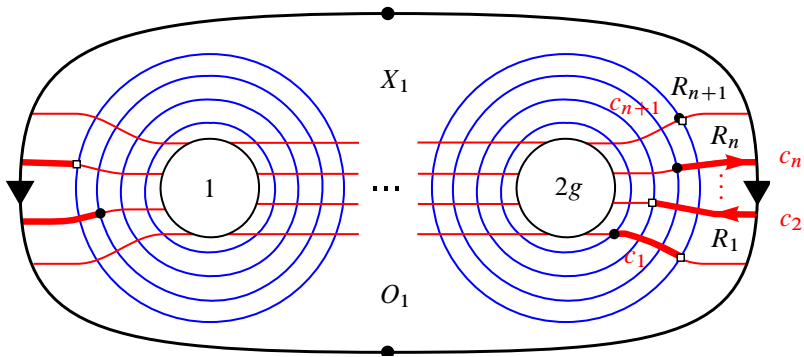
Figure 5: The top half of $\mathcal{H}^\circ$, along with the arcs c_i in thick red (c_{n+1} is just a point).

Since $\mathcal{H}^\circ$ is a diagram for S^3, any two generators are connected by a domain. Let $\boldsymbol{x}^\circ, \boldsymbol{y}^\circ$ be elements of $\mathfrak{S}(\mathcal{H}^\circ)$, and B an element of $\pi_2(\boldsymbol{x}^\circ, \boldsymbol{y}^\circ)$. By adding regions of $\Sigma^\circ \setminus \boldsymbol{\alpha}^\circ$, we can assume that the domain of B is contained entirely in the top half of $\mathcal{H}^\circ$ and has zero multiplicity in the lowest region (the one containing O_1) of the top half of $\mathcal{H}^\circ$. The oriented boundary of B splits into two pieces: $\partial^\alpha B \subset \boldsymbol{\alpha}^\circ$ and $\partial^\beta B \subset \boldsymbol{\beta}^\circ$. The piece $\partial^\alpha B$ is the union of arcs in $\boldsymbol{\alpha}^\circ$ such that $\partial(\partial^\alpha B) = \boldsymbol{y}^\circ - \boldsymbol{x}^\circ$.

Let $\alpha_1, \ldots, \alpha_{n+1}$ be the α–circles in $\mathcal{H}^\circ$ resulting from the gluing of the α–arcs in $\mathcal{H}$, labeled so that $a_i^0 \in \alpha_i$, and let $x_i = \boldsymbol{x}^\circ \cap \alpha_i$, $y_i = \boldsymbol{y}^\circ \cap \alpha_i$. Below, we turn our attention to the oriented arcs $c_i := \partial^\alpha B \cap \alpha_i$, and to each c_i we associate a number $t_i \in \{-1, 0, 1\}$. Since B is contained in the top half of $\mathcal{H}^\circ$, there are three possibilities for each c_i:

- c_i is contained in the rightmost/leftmost grid if and only if x_i and y_i are; in this case, define $t_i = 0$.

- c_i covers both the rightmost and the leftmost grid, and is oriented to the right, as seen on Figure 5, if and only if x_i is in the rightmost grid and y_i is in the leftmost grid; in this case, define $t_i = 1$.

- c_i covers both the rightmost and the leftmost grid, and is oriented to the left, as seen on Figure 5, if and only if x_i is in the leftmost grid and y_i is in the rightmost grid; in this case, define $t_i = -1$.

For $1 \leq i \leq n$, let R_i be the region of $\Sigma^\circ \setminus (\boldsymbol{\alpha}^\circ \cup \boldsymbol{\beta}^\circ)$ containing the image of the interval (a_i^0, a_{i+1}^0) in Σ°, and let R_{n+1} be the topmost region of the top half of $\mathcal{H}^\circ$. See Figure 5.

It is not hard to see that the multiplicity of B at each R_i is $t_1 + \cdots + t_i$, and that the multiplicity of B at R_{n+1} is zero if and only if the generators x and y of $\mathcal{H}$ corresponding to x° and y° occupy the same number of arcs on the right.

Suppose $x^\circ, y^\circ \in \mathfrak{S}_i(\mathcal{H}^\circ)$. By the above, x° and y° are connected by a domain B contained entirely in the nice part of $\mathcal{H}^\circ$, ie $n_{X_1}(B) = n_{X_2}(B) = n_{O_1}(B) = n_{O_2}(B) = 0$, so $A_0(x) = A_0(y)$. Then B is the result of self-gluing a domain B' in $\mathcal{H}$. Note that the left and right multiplicities of B' match up, ie if $p_0 \in \partial^0 \mathcal{H}$ and $p_1 \in \partial^1 \mathcal{H}$ are points that are identified in $\mathcal{H}^\circ$, then $m(\partial^0 B', p_0) = -m(\partial^1 B', p_1)$. Recall the definitions of the sets $S_{\mathbb{O}}^i, S_{\mathbb{X}}^i, S_{\boldsymbol{x}}^i, S_{\boldsymbol{y}}^i, \bar{S}_{\mathbb{O}}^i, \bar{S}_{\mathbb{X}}^i, \bar{S}_{\boldsymbol{x}}^i, \bar{S}_{\boldsymbol{y}}^i$, and of the gradings of domains from [9, Section 11.2]. For each subscript $\bullet \in \{\mathbb{O}, \mathbb{X}, \boldsymbol{x}, \boldsymbol{y}\}$, $p_0 \in \bar{S}_\bullet^0$ if and only if $p_1 \in S_\bullet^1$. Further, $e(B) = e(B')$, and $n_p(B) = n_p(B')$ for any point p. Then

$$
\begin{aligned}
M(B') &= -e(B') - n_{\boldsymbol{x}}(B') - n_{\boldsymbol{y}}(B') + \tfrac{1}{2} m([\partial^\partial B'], \bar{S}_{\boldsymbol{x}}^0 + \bar{S}_{\boldsymbol{y}}^0 + S_{\boldsymbol{x}}^1 + S_{\boldsymbol{y}}^1) \\
&\qquad\qquad\qquad\qquad - m([\partial^\partial B'], \bar{S}_O^0 + S_O^1) + 2n_{\mathbb{O}}(B') \\
&= -e(B) - n_{\boldsymbol{x}}(B) - n_{\boldsymbol{y}}(B) + 2n_{\mathbb{O}}(B) \\
&= M(B),
\end{aligned}
$$

$$
\begin{aligned}
A(B') &= \tfrac{1}{2} m([\partial^\partial B'], \bar{S}_{\mathbb{X}}^0 - \bar{S}_O^0 + S_{\mathbb{X}}^1 - S_O^1) + n_{\mathbb{O}}(B') - n_{\mathbb{X}}(B') \\
&= n_{\mathbb{O}}(B) - n_{\mathbb{X}}(B) \\
&= A(B).
\end{aligned}
$$

Thus, the relative (M, A) gradings are the same in $\widetilde{\mathrm{CT}}(\mathcal{H})$ as in $\widehat{\mathrm{CFK}}(\mathcal{H}^\circ)$, and the relative A_0 grading is zero, ie

$$
\begin{aligned}
M(x^\circ) - M(y^\circ) &= M(x) - M(y), \\
A(x^\circ) - A(y^\circ) &= A(x) - A(y), \\
A_0(x^\circ) - A_0(y^\circ) &= 0.
\end{aligned}
$$

Next, we compare the gradings of generators with distinct numbers of occupied arcs on the right. The plumbing of $4g$ grid diagrams for $\mathcal{H}$ corresponds to a sequence of $4g$ shadows $\mathcal{P}_1, \ldots, \mathcal{P}_{4g}$; see [9, Sections 3 and 4] and Figure 6. Suppose $\mathfrak{S}_i(\mathcal{H}^\circ) \neq \varnothing$ and $\mathfrak{S}_{i+j}(\mathcal{H}^\circ) \neq \varnothing$ for some i, j, and let $x_i \in \mathfrak{S}_i(\mathcal{H})^\circ, x_{i+j} \in \mathfrak{S}_{i+j}(\mathcal{H})^\circ$. The generator x_{i+j} has j more strands than x_i in each even-indexed shadow $\mathcal{P}_{2t}$. Choose one strand of x_{i+j} in each $\mathcal{P}_{2t}$, and let p_{2t-1} and p_{2t} be the endpoints of this strand in $\partial^0 \mathcal{P}_{2t}$ and $\partial^1 \mathcal{P}_{2t}$, respectively. Replacing these $2g$ strands with the strands from p_{2t} to p_{2t+1} produces a generator $x_{i+j-1} \in \mathfrak{S}_{i+j-1}(\mathcal{H})^\circ$. Repeating this procedure shows that $\mathfrak{S}_k(\mathcal{H})^\circ \neq \varnothing$ for every $i + 1 \leq k \leq i + j - 1$ as well.

Then it suffices to choose two generators $x_i \in \mathfrak{S}_i(\mathcal{H})^\circ$ and $x_{i+1} \in \mathfrak{S}_{i+1}(\mathcal{H})^\circ$ for each i such that $\mathfrak{S}_i(\mathcal{H})^\circ \neq \varnothing$ and $\mathfrak{S}_{i+1}(\mathcal{H})^\circ \neq \varnothing$, and understand the relative (M, A, A_0) grading for the corresponding generators x_i°, x_{i+1}°. Let $x_{i+1} \in \mathfrak{S}_{i+1}(\mathcal{H})^\circ$ and let $p_0, \ldots, p_{4g} \simeq p_0$ be as above. Modify x_{i+1} as follows. Let q_t be the topmost point in $\partial^1 \mathcal{P}_t$. The strand at q_t may be contained in $\mathcal{P}_t$ or in $\mathcal{P}_{t+1}$. Starting at $t = 1$, and moving up to $t = 4g$ (identify $\partial^1 \mathcal{P}_{4g}$ with $\partial^0 \mathcal{P}_1$), do the following exchanges of strands. If $q_t \neq p_t$, take the two distinct strands with ends at q_t and at p_t, and exchange their endpoints on $\partial^0 \mathcal{P}_t$. In other words, if one strand connects p_t to another point p_t', and the other strand connected q_t to another point q_t', then replace the two strands with a strand connecting p_t to q_t' and a strand connecting q_t to p_t'. In this modified x_{i+1}, there is a strand connecting q_{2t-1} to q_{2t}, for $1 \leq t \leq 2g$. Let x_i be the generator obtained from x_{i+1} by replacing these strands with strands from q_{2t} to q_{2t+1}, as above. Now $x_i \in \mathfrak{S}_i(\mathcal{H})^\circ$ and $x_{i+1} \in \mathfrak{S}_{i+1}(\mathcal{H})^\circ$ agree almost everywhere, except that x_{i+1} contains the strand at the very top of each even-indexed shadow, and x_i contains the strand at the very top of each odd-indexed shadow. The Maslov and Alexander gradings on strand generators are defined by counting various intersections of strands, see [9, Section 3.4], and one sees that $M(x_{i+1}) = M(x_i)$ and $A(x_{i+1}) = A(x_i)$.

Switching back to Heegaard diagrams, x_i and x_{i+1} differing in the above way is equivalent to saying that the $4g$–gon R_{n+1} connects x_{i+1}° to x_i°. Since $e(R_{n+1}) = 1 - g$, $n_{x_{i+1}^\circ}(R_{n+1}) = g/2 = n_{x_i^\circ}(R_{n+1})$, $n_{X_1}(R_{n+1}) = 1$ and $n_p(R_{n+1}) = 0$ for any other $p \in \mathbb{X} \cup \mathbb{O}$, we see that

$$M(x_{i+1}^\circ) - M(x_i^\circ) = 1,$$
$$A(x_{i+1}^\circ) - A(x_i^\circ) = 0,$$
$$A_0(x_{i+1}^\circ) - A_0(x_i^\circ) = 1.$$

So for $i < j$ and arbitrary $x_i^\circ \in \mathfrak{S}_i(\mathcal{H}^\circ)$, $x_j^\circ \in \mathfrak{S}_j(\mathcal{H}^\circ)$, we have

(1) $$M(x_j^\circ) - M(x_i^\circ) = M(x_j) - M(x_i) + j - i,$$

(2) $$A(x_j^\circ) - A(x_i^\circ) = A(x_j) - A(x_i),$$

(3) $$A_0(x_j^\circ) - A_0(x_i^\circ) = j - i.$$

The argument that the isomorphism respects the absolute gradings is analogous to the one from [9, Section 6]. With the $4g$ grids arranged as in Figure 6, indexed $G_1, \ldots, G_{4g}$ from left to right, let $x_{\mathbb{O}}^\circ$ be the generator formed by the bottom left corner x_j of each O_j in G_{4i} and G_{4i+1}, the top right corner x_j of each O_j in G_{4i+2} and G_{4i+3}, the very top right corner x'_{4i+1} of each grid G_{4i+1}, and the bottom left
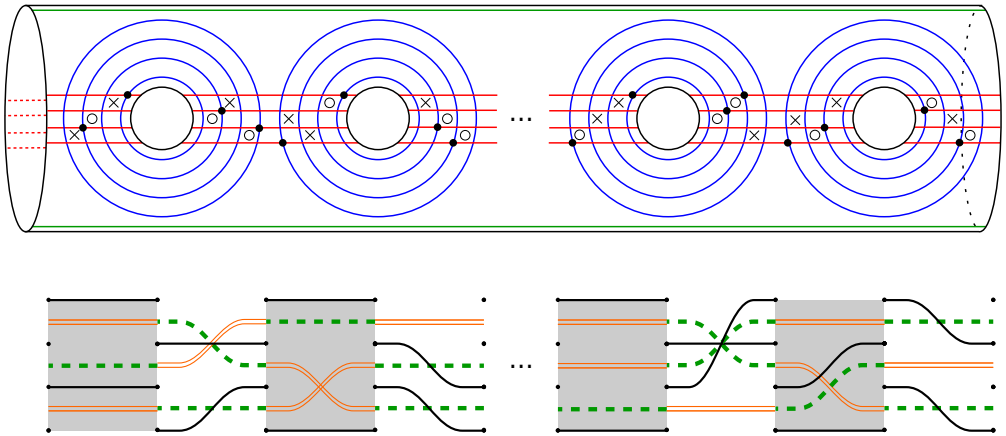
Figure 6: Top: a Heegaard diagram for T coming from a plumbing of grids and the "bottommost" generator $x_{\mathbb{O}}$. Bottom: the corresponding sequence of shadows for T, and the strands diagram for $x_{\mathbb{O}}$.

corner x'_{4i+3} of each grid G_{4i+3}. Define $x^{\circ}_{\mathbb{X}}$ analogously, by replacing O_j with X_j in the above definition.

Denote the β circle containing each x_i or x'_i by β_i or β'_i, respectively. Form a set of circles γ by performing handleslides (which are allowed to cross $\mathbb{X}$ but not $\mathbb{O}$) of all β_i and perturbations of all β'_i, as in Figure 7. We look at the holomorphic triangle map associated to $(\Sigma, \alpha, \beta, \gamma, \mathbb{O})$; see [2; 6; 7]. Let k be the number of Os in $\mathcal{H}$ (so the number of Os in $\mathcal{H}^{\circ}$ is $k+2$). Observe that $(\Sigma, \beta, \gamma, \mathbb{O})$ is a diagram for $(\#^{k+2}S^1 \times S^2)$, and let Θ be the top-dimensional generator. Let y be the generator of $(\Sigma, \alpha, \gamma, \mathbb{O})$ nearest to $x^{\circ}_{\mathbb{O}}$. There is a holomorphic triangle that maps $x^{\circ}_{\mathbb{O}} \otimes \Theta$ to y, so $M(x^{\circ}_{\mathbb{O}}) = M(y)$.

Observe that $(\Sigma, \alpha, \gamma, \mathbb{O})$ is a diagram for S^3 with $k+2$ basepoints, so, as a group graded by the Maslov grading, we have $\widehat{\mathrm{HF}}(\Sigma, \alpha, \gamma, \mathbb{O}) \cong H_{*+k+1}(T^{k+1})$. The diagram has 2^{k+1} generators, so they are a basis for the homology. Let y' be the generator obtained from y by replacing the intersection of γ'_{4i+1} and the topmost α of G_{4i+1} with the intersection of γ'_{4i+1} and the bottommost α of G_{4i+2}, and the intersection of γ'_{4i+3} and the bottommost α of G_{4i+3} with the intersection of γ'_{4i+3} and the topmost α of G_{4i+4}. There are k disjoint bigons going into y', so $M(y') \leq -k$. The shaded $4g$–gon on Figure 7 from y' to y shows that $M(y') - M(y) = 1$, so $M(y) \leq -k - 1$. But the 2^{k+1} generators are a basis for the homology, so $M(y) \in [-k-1, 0]$. Thus, $M(y) = -k - 1$, so $M(x^{\circ}_{\mathbb{O}}) = -k - 1$. We can also compute $M(x_{\mathbb{O}})$ using the definition from [9, Section 3.4]. The computation is analogous to the one from [9, Section 6], and we see that $M(x_{\mathbb{O}}) = -k$. Note that $S(x^{\circ}_{\mathbb{O}}) = a$, so
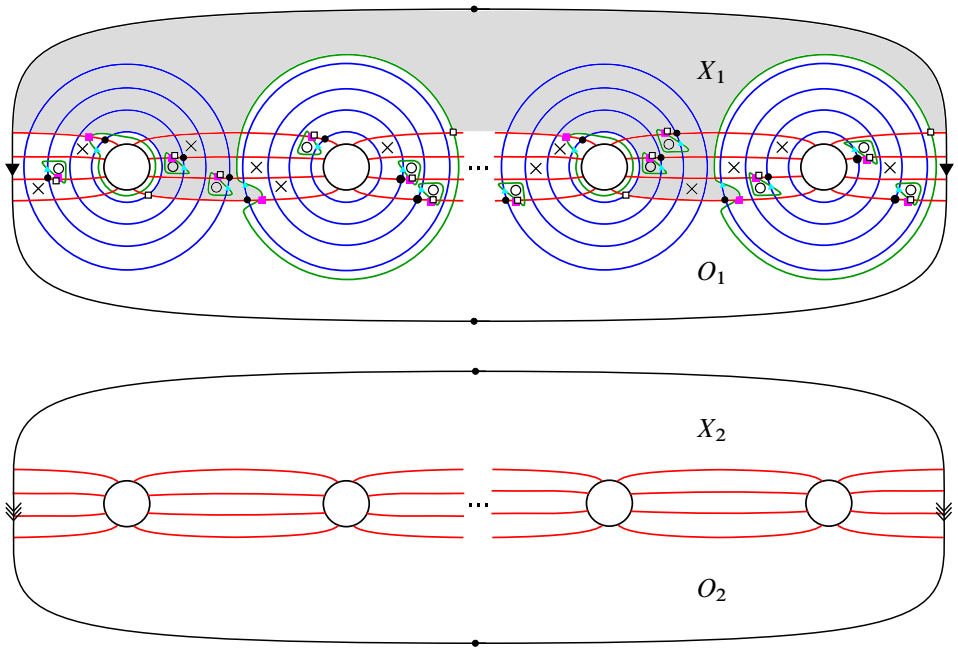
Figure 7: The Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbb{O})$, with the original $\mathbb{X}$ markings left in. The black dots form the generator $\boldsymbol{x}_{\mathbb{O}}$, the purple squares form $\boldsymbol{y}$, the white squares from $\boldsymbol{y}'$, and the cyan triangles form Θ.

by Equation (1), for an arbitrary generator $\boldsymbol{x}_i^{\circ} \in \mathfrak{S}_i(\mathcal{H}^{\circ})$ we have

$$M(\boldsymbol{x}_i^{\circ}) - M(\boldsymbol{x}_{\mathbb{O}}^{\circ}) = M(\boldsymbol{x}_i) - M(\boldsymbol{x}_{\mathbb{O}}) + i - a,$$

so

$$M(\boldsymbol{x}_i^{\circ}) = M(\boldsymbol{x}_i) + i - a - 1.$$

Similarly, for the z–normalized (or $\mathbb{X}$–normalized, to match the notation in this paper) grading N, we have $N(\boldsymbol{x}_{\mathbb{X}}^{\circ}) = -k - 1$. Since $N = M - 2A - (k + 2 - l)$, we get

$$A(\boldsymbol{x}_{\mathbb{X}}^{\circ}) = \tfrac{1}{2}(M(\boldsymbol{x}_{\mathbb{X}}^{\circ}) - N(\boldsymbol{x}_{\mathbb{X}}^{\circ}) - (k + 2 - l)) = \tfrac{1}{2}(M(\boldsymbol{x}_{\mathbb{X}}^{\circ}) + l - 1).$$

Again using the definition from [9, Section 3.4] as we do in [9, Section 6], we see that $M(\boldsymbol{x}_{\mathbb{O}}) = -k$, $N(\boldsymbol{x}_{\mathbb{X}}) = -k$, $A(\boldsymbol{x}_{\mathbb{X}}) = \tfrac{1}{2}M(\boldsymbol{x}_{\mathbb{X}})$. Since $S(\boldsymbol{x}_{\mathbb{X}}) = n - a$, we have $M(\boldsymbol{x}_{\mathbb{X}}^{\circ}) = M(\boldsymbol{x}_{\mathbb{X}}) + n - 2a - 1$, so we get

$$A(\boldsymbol{x}_{\mathbb{X}}^{\circ}) = A(\boldsymbol{x}_{\mathbb{X}}) + \tfrac{1}{2}(n - 2a + l - 2).$$

The Alexander multigrading on a generator $\boldsymbol{x}^{\circ}$ can be described by the relative Spin^c structure $\mathfrak{s}(\boldsymbol{x}^{\circ}) \in \overline{\mathrm{Spin}^c}(S^3, L)$; see [8]. In the case when the link is in S^3, one can think of A_i by looking at the projection of a Seifert surface for L_i onto $\mathcal{H}^{\circ}$.

Specifically, for a generator $\boldsymbol{x}_i^\circ \in \mathfrak{S}_i(\mathcal{H}^\circ)$, we can compute its A_0 grading in the following way. Connect X_1 to O_1 and X_2 to O_2 away from $\boldsymbol{\beta}^\circ$, and O_1 to X_2 and O_2 to X_1 away from $\boldsymbol{\alpha}^\circ$ to obtain a curve C on $\mathcal{H}^\circ$ representing L_0, so that C is negative the boundary of a disk D that is a neighborhood of the rightmost grid (in general C may be immersed but not necessarily embedded). Then

$$A_0(\boldsymbol{x}_i^\circ) = \tfrac{1}{2}(e(D) + 2n_{\boldsymbol{x}_i^\circ}(D) - n_{\mathbb{X}}(D) - n_{\mathbb{O}}(D)) = i - \tfrac{1}{2}(n+1).$$

It follows that

$$A'(\boldsymbol{x}_i^\circ) = A(\boldsymbol{x}_i) - S(\boldsymbol{x}_i) + \tfrac{1}{2}l + n - a - 1.$$

This completes the identification of gradings. $\square$

Remark The authors are in the process of upgrading the invariants in [9] to have Alexander multigradings, corresponding to different components of the tangle. The arguments in this paper automatically imply that the isomorphism from Theorem 3.2 respects the multigrading, with appropriate additive constants.

Last, we prove the Corollary to Main Theorem.

Proof of Corollary Fix n, and let $\mathbb{1}_n$ denote the trivial braid on n strands (oriented from top to bottom). Let B be an n–braid. Let $\mathcal{A} = \mathcal{A}(\partial^1 \mathbb{1}_n) = \mathcal{A}(\partial^1 B)$, and let $\mathrm{H}(\mathrm{Mod}_{\mathcal{A}})$ be the homotopy category of right type A modules over $\mathcal{A}$. Suppose that $-\boxtimes \widetilde{\mathrm{CT}}(B)$ acts as the identity on $\mathrm{H}(\mathrm{Mod}_{\mathcal{A}})$. Recall that one can recover the homotopy type of any AA bimodule, ie A_∞ bimodule, ${}_{\mathcal{A}}N_{\mathcal{A}}$ from the functor $-\widetilde{\otimes}N$ on $\mathrm{H}(\mathrm{Mod}_{\mathcal{A}})$, for example as $Q_{-\widetilde{\otimes}N}(\mathcal{A}_{\mathcal{A}}, \mathcal{A}_{\mathcal{A}})$ in [10, (7.23) and (7.24)]. Thus, as $-\boxtimes \widetilde{\mathrm{CT}}(B) = -\widetilde{\otimes}(\mathcal{A}\boxtimes \widetilde{\mathrm{CT}}(B))$ is the identity, $\mathcal{A} \simeq \mathcal{A}\boxtimes \widetilde{\mathrm{CT}}(B)$, so $\widetilde{\mathrm{CT}}(\mathbb{1}_n) \simeq \widetilde{\mathrm{CT}}(B)$. We show that the latter implies $B = \mathbb{1}_n$.

Let $k \geq 1$ be an integer such that B^k is pure. Then

$$\begin{aligned}
\widetilde{\mathrm{CT}}(B^k) &\simeq \widetilde{\mathrm{CT}}(B) \boxtimes \cdots \boxtimes \widetilde{\mathrm{CT}}(B) \\
&\simeq \widetilde{\mathrm{CT}}(\mathbb{1}_n) \boxtimes \cdots \boxtimes \widetilde{\mathrm{CT}}(\mathbb{1}_n) \\
&\simeq \widetilde{\mathrm{CT}}(\mathbb{1}_n^k) \simeq \widetilde{\mathrm{CT}}(\mathbb{1}_n).
\end{aligned}$$

Taking Hochschild homology, it follows that $\widehat{\mathrm{HFL}}((B^k)_0, 0) \cong \widehat{\mathrm{HFL}}((\mathbb{1}_n)_0, 0)$. Recall that, in our notation, [1, Theorem 1(b)] says that if there is a triply graded isomorphism $\widehat{\mathrm{HFL}}(T_0, 0) \cong \widehat{\mathrm{HFL}}((\mathbb{1}_n)_0, 0)$ and T is a pure braid, then $T = \mathbb{1}_n$. Thus, $B^k = \mathbb{1}_n$. Since the braid group is torsion-free, it follows that $B = \mathbb{1}_n$. $\square$

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