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L^2–invisibility of symmetric operad groups

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We show a homological result for the class of planar or symmetric operad groups: under certain conditions, group (co)homology of such groups with certain coefficients vanishes in all dimensions, provided it vanishes in dimension 0. This can be applied, for example, to l^2–homology or cohomology with coefficients in the group ring. As a corollary, we obtain explicit vanishing results for Thompson-like groups such as the Brin–Thompson groups nV.

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1 Introduction

In [9], Sauer and the author show that a certain class of groups acting on compact ultrametric spaces, the so-called dually contracting local similarity groups, are l^2–invisible. The latter means that group homology with group von Neumann algebra coefficients vanishes in every dimension; ie

$$H_k(G, \mathcal{N}(G)) = 0$$

for all $k \geq 0$, where $\mathcal{N}(G)$ denotes the group von Neumann algebra of G. If G is of type F_∞, ie there is a classifying space for G with finitely many cells in each dimension, then this is equivalent to

$$H_k(G, l^2(G)) = 0$$

for all $k \geq 0$, by Lück [8, Lemmas 6.98 and 12.3 on pages 286 and 438].

In [10], the author proposed to study fundamental groups of categories naturally associated to operads. This class of groups, called operad groups, contains a lot of Thompson-like groups already existent in the literature. Among these are the aforementioned local similarity groups; see [10, Section 3.5].

This article is mainly concerned with generalizing the results of [9] to the setting of symmetric operad groups, which form a much larger class of groups. The proof in [9] consists of constructing a suitable simplicial complex on which the group in question acts, and then applying a spectral sequence associated to this action which computes

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the homology of the group in terms of the homology of the stabilizer subgroups. The proof in the case of operad groups goes exactly the same way. However, it is a priori unclear how to construct the simplicial complex. The reason is the following: A local similarity group is defined as a representation, ie as a group of homeomorphisms of a compact ultrametric space. This space is used to construct the simplicial complex as a poset of partitions of this space. The case of operad groups is more abstract. A priori, there is no canonical space comparable to these ultrametric spaces on which an operad group acts. However, these spaces, called limit spaces, are conjectured to exist if the operad satisfies the calculus of fractions; see [10, Section 3.3] for the latter notion. We don't use these limit spaces here. Instead, we will take the conjectured correspondence between calculus of fractions operads and their limit spaces as a motivation to mimic the necessary notions for the construction of the desired simplicial complex in terms of the operad itself.

As in [9], we want to briefly discuss the relationship between these results and Gromov's *Zero-in-the-spectrum conjecture*; see [7]. The algebraic version of this conjecture states that if $\Gamma = \pi_1(M)$ is the fundamental group of a closed aspherical Riemannian manifold, then there always exists a dimension $p \geq 0$ such that $H_p(\Gamma, \mathcal{N}\Gamma) \neq 0$ or, equivalently, $H_p(\Gamma, l^2\Gamma) \neq 0$. Conjecturally, the fundamental groups of closed aspherical manifolds are precisely the Poincaré duality groups G of type F; ie there is a compact classifying space for G and a natural number $n \geq 0$ such that

$$H^i(G, \mathbb{Z}G) = \begin{cases} 0 & \text{if } i \neq n, \\ \mathbb{Z} & \text{if } i = n; \end{cases}$$

see Davis [5]. Dropping Poincaré duality and relaxing type F to type F_∞, we arrive at a more general question which has been posed by Lück [8, Remark 12.4 on page 440]: if G is a group of type F_∞, does there always exist a p with $H_p(G, \mathcal{N}G) \neq 0$? In [10], we discuss conditions for operads which imply that the associated operad groups are of type F_∞. Combining this with the results in the present article, we obtain a large class of groups of type F_∞ which are also l^2–invisible. This class contains the well-known symmetric Thompson group V, and consequently, Lück's question has to be answered in the negative. Unfortunately, none of these groups G are of type F, nor do they satisfy Poincaré duality since, with another corollary of our main theorem (Theorem 2.5), we can show $H^k(G, \mathbb{Z}G) = 0$ for all $k \geq 0$.

Prerequisites The present article is based on Sections 2 and 3 of [10].

Notation and conventions When $f \colon A \to B$ and $g \colon B \to C$ are two composable arrows, we write $f * g$ for the composite $A \to C$ instead of the usual notation $g \circ f$. Consequently, it is often better to plug in arguments from the left. When we do this, we use the notation $x \triangleright f$ for the evaluation of f at x. However, we won't entirely drop the usual notation $f(x)$ and use both notations side by side. Objects of type $\mathrm{Aut}(X)$ will be made into a group by the definition $fg = f \cdot g := f * g$. Conversely, a group G is considered as a groupoid with one object and arrows the elements in G together with the composition $f * g := f \cdot g$.

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2 Statement of the main theorem

Definition 2.1 Let $\mathcal{M}$ associate a $\mathbb{Z}G$–module $\mathcal{M}G$ to every group G.

- We say that $\mathcal{M}$ is *Künneth* if, for every two groups G_1, G_2, and $n_1, n_2 \in \mathbb{Z}$ with $n_i \geq -1$, the following is satisfied:
$$\left.\begin{matrix} \forall_{k \leq n_1}\ H_k(G_1, \mathcal{M}G_1) = 0 \\ \forall_{k \leq n_2}\ H_k(G_2, \mathcal{M}G_2) = 0 \end{matrix}\right\} \quad \Longrightarrow \quad \forall_{k \leq n}\ H_k(G, \mathcal{M}G) = 0,$$
where $G := G_1 \times G_2$ and $n := n_1 + n_2 + 1$.
- We say that $\mathcal{M}$ is *inductive* if, whenever H and G are groups with H a subgroup of G and $k \geq 0$, we have that
$$H_k(H, \mathcal{M}H) = 0 \quad \text{implies} \quad H_k(H, \mathcal{M}G) = 0.$$

Let $\mathfrak{P}$ be a property of groups. Then we say that $\mathcal{M}$ is $\mathfrak{P}$–Künneth if the Künneth property is satisfied for all $\mathfrak{P}$–groups G_1 and G_2. We say that $\mathcal{M}$ is $\mathfrak{P}$–inductive if the inductive property is satisfied for all $\mathfrak{P}$–subgroups H of the arbitrary group G (not necessarily for all subgroups of G). Furthermore, one can also formulate these two properties in the cohomological case.

Definition 2.2 Let $\mathcal{O}$ be a planar or symmetric or braided operad and X an object in $\mathcal{S}(\mathcal{O})$.

- We say that X is *split* if, in $\mathcal{S}(\mathcal{O})$, there are objects A_1, A_2, A_3 and an arrow
$$A_1 \otimes X \otimes A_2 \otimes X \otimes A_3 \to X.$$
- We say that X is *progressive* if, for every arrow $Y \to X$, there are objects A_1, A_2 and an arrow $A_1 \otimes X \otimes A_2 \to Y$ such that the coordinates of X are connected to only one operation in this arrow; see Figure 1.

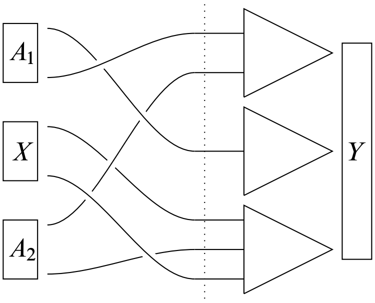


Figure 1: An arrow $A_1 \otimes X \otimes A_2 \to Y$ such that X is connected to only one operation

Remark 2.3 If X is just a single color, then X is split if and only if there is an operation with output color X and at least two inputs of color X. If $\mathcal{O}$ is monochromatic and $X \neq I$ is an object of $\mathcal{S}(\mathcal{O})$, then X is split if and only if there is at least one operation in $\mathcal{O}$ with at least two inputs. So in the monochromatic case, the split condition is in fact a property of $\mathcal{O}$.

Remark 2.4 If X is just a single color, then X is progressive if and only if, for every operation θ with output color X, there is another operation ϕ with at least one input of color X, and at least one input of θ has the same color as the output of ϕ. Now assume that $\mathcal{O}$ is monochromatic. Then an object $X \neq I$ in $\mathcal{S}(\mathcal{O})$ (which is just a natural number $X > 0$, eg $X = 3$) is progressive if and only if there is an operation in $\mathcal{O}$ with at least X inputs (eg three inputs). Note that $X = 1$ is always progressive in the monochromatic case.

Theorem 2.5 *Let $\mathcal{O}$ be a planar or symmetric operad which satisfies the calculus of fractions. Let $\mathcal{M}$ be a coefficient system which is Künneth and inductive. Let X be a split progressive object of $\mathcal{S}(\mathcal{O})$. Set $\Gamma := \pi_1(\mathcal{O}, X)$. Then*

$$H_0(\Gamma, \mathcal{M}\Gamma) = 0 \quad \Longrightarrow \quad H_k(\Gamma, \mathcal{M}\Gamma) = 0 \text{ for all } k \geq 0.$$

The same is true for cohomology. More generally, let $\mathfrak{P}$ be a property of groups which is closed under taking products. Then the statement is also true for coefficient systems $\mathcal{M}$ which are only $\mathfrak{P}$–Künneth and $\mathfrak{P}$–inductive, provided that Γ satisfies $\mathfrak{P}$.

Remark 2.6 Let X and Y be objects in $\mathcal{S}(\mathcal{O})$. Generalizing the notion of progressiveness, we say that X is Y–progressive if, for every arrow $Z \to X$, there is an arrow $A_1 \otimes Y \otimes A_2 \to Z$, and the coordinates of Y are connected to only one operation in this arrow (call this the *link condition*). In particular, there is an arrow $A_1 \otimes Y \otimes A_2 \to X$.

With this notion, we can formulate a slightly more general version of Theorem 2.5. Let $\mathcal{O}$, $\mathfrak{P}$ and $\mathcal{M}$ be as in the theorem. Let X be an object of $\mathcal{S}(\mathcal{O})$ and set $\Gamma = \pi_1(\mathcal{O}, X)$. Assume there is a split object Y such that X is Y–progressive, $\Upsilon := \pi_1(\mathcal{O}, Y)$ satisfies $\mathfrak{P}$, and $H_0(\Upsilon, \mathcal{M}\Upsilon) = 0$. Then $H_k(\Gamma, \mathcal{M}\Gamma) = 0$ for each $k \geq 0$. The same is true for cohomology.

3 Proof of the main theorem

We start with two general lemmas concerning the calculus of fractions.

Lemma 3.1 *Let $\mathcal{C}$ be a category satisfying the calculus of fractions. Then two square fillings of a given span can be combined to a common square filling. That is, let x and y be two arrows with the same codomain, and assume we have two square fillings as in the diagram:*

$$\begin{tikzcd}
\bullet \arrow[dd, "x"'] & & \bullet \arrow[ll, "i"'] \arrow[dd, "h"] \\
& \bullet \arrow[ul, "j"'] \arrow[dr, "g"] & \\
\bullet & & \bullet \arrow[ll, "y"]
\end{tikzcd}$$

Then we can complete this diagram to the commutative diagram:

$$\begin{tikzcd}
& & & \bullet \arrow[dlll, dashed, bend right, "\alpha"'] \arrow[dl, dashed, "\delta"] \arrow[ddl, dashed, "\epsilon"'] \arrow[ddd, dashed, bend left, "\beta"] \\
\bullet \arrow[dd] & & \bullet \arrow[ll] \arrow[dd] & \\
& \bullet \arrow[ul] \arrow[dr] & & \\
\bullet & & \bullet \arrow[ll] &
\end{tikzcd}$$

Proof Let (c, d) be a square filling of $a := ix = hy, b := jx = gy$; ie $ca = db$. Then ch and dg are two parallel arrows which are coequalized by y; ie $(ch)y = (dg)y$. By the equalization property, we find an equalizing arrow k with $k(ch) = k(dg)$. By the same reasoning, we find an arrow l with $l(ci) = l(dj)$. Let (m, n) be a square filling of l, k; ie $ml = nk =: p$. Then one can easily calculate that the arrows

$$\delta = pc \quad \alpha = pci \quad \beta = pch \quad \epsilon = pd$$

fill the diagram as required. □

Lemma 3.2 *Let $\mathcal{C}$ be a category satisfying the calculus of fractions. Let $\overline{x}$ and $\overline{y}$ be two arrows $A \to C$. Assume that there are arrows $x, y\colon A \to B$ and $a\colon B \to C$ such that $xa = \overline{x}$ and $ya = \overline{y}$:*

$$\begin{tikzcd}
C & B \arrow[l, "a"] & A \arrow[l, "x"] \arrow[r, "y"'] \arrow[ll, bend right, "\overline{x}"'] \arrow[rr, bend left, "\overline{y}"] & B \arrow[r, "a"'] & C.
\end{tikzcd}$$

Then the span $C \xleftarrow{\overline{x}} A \xrightarrow{\overline{y}} C$ is null-homotopic if and only if the span $B \xrightarrow{x} A \xrightarrow{y} B$ is null-homotopic.

Proof First note that a span like $B \xleftarrow{x} A \xrightarrow{y} B$ is null-homotopic if and only if the parallel arrows x and y are homotopic. Since $\mathcal{C}$ satisfies the calculus of fractions, this is the case if and only if there is an equalizing arrow, ie an arrow $d\colon D \to A$ with $dx = dy$. Now, if x and y are homotopic, then clearly $\overline{x}$ and $\overline{y}$ are also homotopic. On the other hand, assume that $\overline{x}$ and $\overline{y}$ are homotopic and $d\colon D \to A$ equalizes $\overline{x}$ and $\overline{y}$. Then we have

$$(dx)a = d(xa) = d\overline{x} = d\overline{y} = d(ya) = (dy)a.$$

Then by the equalization property, we find an arrow $e\colon E \to D$ with $e(dx) = e(dy)$. Consequently, the arrow ed equalizes x and y, and thus, x and y are homotopic. □

We now turn to the proof of Theorem 2.5. In the following, let $\mathcal{O}$ be a planar or symmetric operad satisfying the calculus of fractions with set of colors C, and let $\mathcal{S} := \mathcal{S}(\mathcal{O})$.

3A Marked objects

Let $c = (c_1, \dots, c_n)$ be an object of $\mathcal{S}$; ie $c_1, \dots, c_n$ are colors in C. First we define a marking on c in the symmetric case. It assigns to each coordinate of c a symbol. A symbol can be assigned several times and not every coordinate has to be marked by a symbol. More precisely, a marking of c is a set S of symbols together with a subset $I \subset \{1, \dots, n\}$ and a surjective function $f\colon I \to S$. In the planar case, we additionally require the marking to be ordered. This means that whenever $i \triangleright f = j \triangleright f$ for $i < j$ then also $i \triangleright f = k \triangleright f = j \triangleright f$ for $i < k < j$.

Let m_1 and m_2 be two markings of c with respective symbol sets S_1 and S_2. We say $m_1 \subset m_2$ if there is a function $i\colon S_1 \to S_2$ and every coordinate which is marked with $s_1 \in S_1$ is also marked with $s_1 \triangleright i \in S_2$. We say m_1 and m_2 are equivalent if $m_1 \subset m_2$ and $m_2 \subset m_1$. This means that there is a bijection $i\colon S_1 \to S_2$ and a coordinate is marked with $s_1 \in S_1$ if and only if it is marked with $s_1 \triangleright i \in S_2$. By

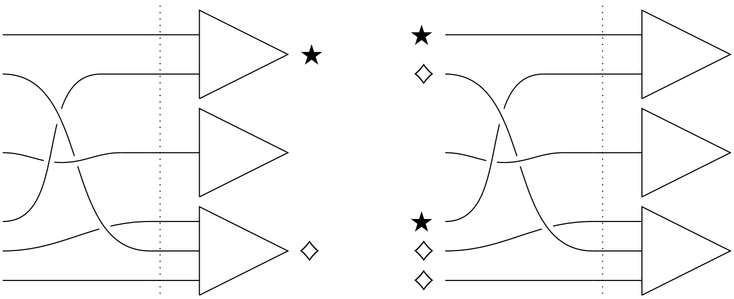


Figure 2: A comarking (left) and the pulled-back marking (right)

slight abuse of notation, we identify equivalent markings and write $m_1 = m_2$ if they are equivalent. Then $\subset$ becomes a partial order on the set of markings on c; see the first paragraph of Section 3C.

3B Marked arrows

Let $\alpha\colon c \to d$ be an arrow in $\mathcal{S}$ with objects $c = (c_1, \dots, c_n)$ and $d = (d_1, \dots, d_m)$. A marking on α is a marking on the domain c. A comarking on α is a marking on the codomain d. A comarking on α induces a marking on α: let (σ, X) be a representative of α where σ is either an identity or a colored permutation, depending on whether $\mathcal{O}$ is planar or symmetric. Write $X = (X_1, \dots, X_m)$. The comarking yields a marking on the operations X_i. Mark each input of X_i with the same symbol. Now push the markings through σ to obtain a marking on the domain c. Figure 2 illustrates this procedure. If m is the comarking, then we denote this pulled-back marking by $\alpha^*(m)$. Observe that this pull-back is functorial; ie we have

$$(\alpha\beta)^*(m) = \alpha^*(\beta^*(m)).$$

Furthermore, we have

$$m_1 \subset m_2 \quad \Longleftrightarrow \quad \alpha^*(m_1) \subset \alpha^*(m_2).$$

Now fix an object x in $\mathcal{S}$.

Let (α_1, m_1) and (α_2, m_2) be two marked arrows with codomain x; ie $\alpha_i\colon c_i \to x$ is an arrow and m_i is a marking on c_i. We write

$$(\alpha_1, m_1) \subset (\alpha_2, m_2)$$

if there is a square filling

$$\begin{tikzcd}
d \arrow[r, dotted, "\beta_2"] \arrow[d, dotted, "\beta_1"'] & c_2 \arrow[d, "\alpha_2"] \\
c_1 \arrow[r, "\alpha_1"'] & x
\end{tikzcd}$$

with $\beta_1^*(m_1) \subset \beta_2^*(m_2)$. Observe that, if this is the case, then it is true for *every* square filling: Let (γ_1, γ_2) be another square filling of (α_1, α_2). Then choose a common square filling (δ_1, δ_2) as in Lemma 3.1. It is not hard to see that the property $\delta_1^*(m_1) \subset \delta_2^*(m_2)$ is inherited from the square filling (β_1, β_2). On the other hand, this forces the property onto the square filling (γ_1, γ_2); ie we have $\gamma_1^*(m_1) \subset \gamma_2^*(m_2)$.

Remark 3.3 This observation also implies the following: Let $(\alpha_1, m_1) \subset (\alpha_2, m_2)$ and assume that $\alpha_1 = \alpha_2$. Then we necessarily have $m_1 \subset m_2$. Indeed, we can choose $\beta_1 = \mathrm{id} = \beta_2$ in the above square filling.

Proposition 3.4 *The relation $\subset$ on the set of marked arrows is reflexive and transitive.*

Proof Reflexivity is clear. For transitivity, assume $(\alpha_1, m_1) \subset (\delta, m)$ and $(\delta, m) \subset (\alpha_2, m_2)$. Choose two square fillings

$$\begin{tikzcd}
a_1 \arrow[r, dotted, "\delta_1"] \arrow[d, dotted, "\beta_1"'] & d \arrow[d, "\delta"] & a_2 \arrow[l, dotted, "\delta_2"'] \arrow[d, dotted, "\beta_2"] \\
c_1 \arrow[r, "\alpha_1"'] & x & c_2 \arrow[l, "\alpha_2"]
\end{tikzcd}$$

with $\beta_1^*(m_1) \subset \delta_1^*(m)$ and $\delta_2^*(m) \subset \beta_2^*(m_2)$. Choose a square filling of (δ_1, δ_2)

$$\begin{tikzcd}
 & e \arrow[dl, "\gamma_1"'] \arrow[d, "\eta"] \arrow[dr, "\gamma_2"] & \\
a_1 \arrow[r, dotted, "\delta_1"'] \arrow[d, dotted, "\beta_1"'] & d \arrow[d, "\delta"] & a_2 \arrow[l, dotted, "\delta_2"] \arrow[d, dotted, "\beta_2"] \\
c_1 \arrow[r, "\alpha_1"'] & x & c_2 \arrow[l, "\alpha_2"]
\end{tikzcd}$$

Now we have

$$\begin{aligned}
(\gamma_1\beta_1)^*(m_1) &= \gamma_1^*(\beta_1^*(m_1)) \\
&\subset \gamma_1^*(\delta_1^*(m)) = (\gamma_1\delta_1)^*(m) = \eta^*(m) = (\gamma_2\delta_2)^*(m) = \gamma_2^*(\delta_2^*(m)) \\
&\subset \gamma_2^*(\beta_2^*(m_2)) = (\gamma_2\beta_2)^*(m_2).
\end{aligned}$$

This proves $(\alpha_1, m_1) \subset (\alpha_2, m_2)$. □

3C Balls and partitions

A transitive and reflexive relation $\preccurlyeq$ on a set Z is not a poset in general since $a \preccurlyeq b$ together with $b \preccurlyeq a$ does not imply $a = b$ in general. We can repair this in the following way: Define $a, b \in Z$ to be equivalent if $a \preccurlyeq b$ and $b \preccurlyeq a$. This is indeed an equivalence relation because $\preccurlyeq$ is assumed to be reflexive and transitive. Now if $\mathfrak{a}$ and $\mathfrak{b}$ are two equivalence classes, we write $\mathfrak{a} \leq \mathfrak{b}$ if there are representatives a and b, respectively, with $a \preccurlyeq b$. One can easily show that then *any* two representatives satisfy this. Using this, it is not hard to see that $\leq$ is indeed a partial order on the set of equivalence classes. In particular, we have $\mathfrak{a} = \mathfrak{b}$ if and only if $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{b} \leq \mathfrak{a}$.

We want to apply this observation to the reflexive and transitive relation $\subset$ on the set of marked arrows. We say that two marked arrows (α_1, m_1) and (α_2, m_2) with common codomain x are equivalent if both $(\alpha_1, m_1) \subset (\alpha_2, m_2)$ and $(\alpha_2, m_2) \subset (\alpha_1, m_1)$ hold. We remark that this is equivalent to the existence of a square filling

$$\begin{tikzcd}
d \arrow[r, dotted, "\beta_2"] \arrow[d, dotted, "\beta_1"'] & c_2 \arrow[d, "\alpha_2"] \\
c_1 \arrow[r, "\alpha_1"'] & x
\end{tikzcd}$$

with $\beta_1^*(m_1) = \beta_2^*(m_2)$, and moreover, that *every* square filling satisfies this.

- A *semipartition* is an equivalence class of marked arrows.
- A *partition* is a semipartition with fully marked domain for some (and therefore, for every) representative of the semipartition. Here, an object in $\mathcal{S}$ is fully marked if every coordinate is marked.
- A *multiball* is a semipartition with a unimarked domain for some (and therefore for every) representative of the semipartition. Here, an object in $\mathcal{S}$ is unimarked if there is only one symbol in the marking.
- A *ball* is a semipartition such that there is a single-marked representative. Here, an object in $\mathcal{S}$ is single-marked if only one coordinate is marked.

Note that these definitions depend on the base point x. Following the remarks in the first paragraph, we write $\mathcal{P} \subset \mathcal{Q}$ for two semipartitions $\mathcal{P}$ and $\mathcal{Q}$ if there are representatives p of $\mathcal{P}$ and q of $\mathcal{Q}$ satisfying $p \subset q$. Then, for *all* such representatives p and q, we have $p \subset q$. It follows that $\subset$ is a partial order on the set of semipartitions. In particular, we have $\mathcal{P} = \mathcal{Q}$ if and only if $\mathcal{P} \subset \mathcal{Q}$ and $\mathcal{Q} \subset \mathcal{P}$.

We now investigate the relationship between semipartitions and multiballs. Let $\mathcal{P}$ be a semipartition with representative (α, m). Picking out a symbol s of m and removing all markings except those with the chosen symbol s gives a unimarked arrow (α, m^s). The corresponding equivalence class is a multiball and is independent of the chosen representative (α, m) in the following sense: If we choose another representative (β, n), then $(\alpha, m) \sim (\beta, n)$, and to the chosen symbol s of m corresponds a unique symbol r of n. Deleting all markings in n except those with the symbol r gives a unimarked arrow (β, n^r) which is equivalent to (α, m^s). Multiballs arising in this way are called submultiballs of $\mathcal{P}$, and we write $P \Subset \mathcal{P}$ for submultiballs. Note that Remark 3.3 implies that two submultiballs P_1 and P_2 coming from a representative of $\mathcal{P}$ by choosing two different symbols satisfy $P_1 \not\subset P_2$ and $P_2 \not\subset P_1$; in particular, $P_1 \neq P_2$. It follows that there is a canonical bijection between the set $\{P \Subset \mathcal{P}\}$ of submultiballs of $\mathcal{P}$ and the set of symbols of $\mathcal{P}$ (which is, by definition, the set of symbols of the marking of any representative for $\mathcal{P}$). Moreover, any two submultiballs $P_1, P_2 \Subset \mathcal{P}$ with $P_1 \neq P_2$ satisfy the stronger property $(P_1 \not\subset P_2) \wedge (P_2 \not\subset P_1)$. Equivalently, whenever $P_1 \subset P_2$ or $P_2 \subset P_1$, we already have $P_1 = P_2$.

Proposition 3.5 *Let $\mathcal{P}$ and $\mathcal{Q}$ be semipartitions. Then*

$$\mathcal{Q} \subset \mathcal{P} \iff \forall_{Q \Subset \mathcal{Q}} \exists_{P \Subset \mathcal{P}} \, Q \subset P.$$

In particular, $\mathcal{P} = \mathcal{Q}$ if and only if $\{Q \Subset \mathcal{Q}\} = \{P \Subset \mathcal{P}\}$.

Proof We first prove the last statement since it is a formal consequence of the previous statement and the remarks preceding the proposition. Recall that $\mathcal{P} = \mathcal{Q}$ is equivalent to $\mathcal{P} \subset \mathcal{Q}$ and $\mathcal{Q} \subset \mathcal{P}$. The first statement of the proposition says that there is a function $i\colon \{Q \Subset \mathcal{Q}\} \to \{P \Subset \mathcal{P}\}$ with the property that $Q \subset Q \triangleright i$ for each $Q \Subset \mathcal{Q}$. Since we also have $\mathcal{P} \subset \mathcal{Q}$, there is another function $j\colon \{P \Subset \mathcal{P}\} \to \{Q \Subset \mathcal{Q}\}$ with the property that $P \subset P \triangleright j$ for each $P \Subset \mathcal{P}$. We have

$$Q \subset Q \triangleright i \subset (Q \triangleright i) \triangleright j = Q \triangleright (ij)$$

for all $Q \Subset \mathcal{Q}$. Since both the left and right side are submultiballs of $\mathcal{Q}$, the remarks preceding the proposition imply $Q = Q \triangleright (ij)$ for all $Q \Subset \mathcal{Q}$. We then have

$$Q \subset Q \triangleright i \subset Q,$$

and therefore, $Q = Q \triangleright i$ for all $Q \Subset \mathcal{Q}$. This shows $\{Q \Subset \mathcal{Q}\} \subset \{P \Subset \mathcal{P}\}$. With a similar argument applied to ji, we also obtain $\{Q \Subset \mathcal{Q}\} \supset \{P \Subset \mathcal{P}\}$. So we have indeed $\{Q \Subset \mathcal{Q}\} = \{P \Subset \mathcal{P}\}$. The converse implication also follows easily from the first statement of this proposition.

Now let's turn to the first statement: assume $\mathcal{Q} \subset \mathcal{P}$. By the square filling technique, we know that we can choose a common arrow $\alpha\colon c \to x$ with markings $m_{\mathcal{Q}} \subset m_{\mathcal{P}}$ such that $[\alpha, m_{\mathcal{Q}}] = \mathcal{Q}$ and $[\alpha, m_{\mathcal{P}}] = \mathcal{P}$. If $Q \Subset \mathcal{Q}$, then we find a symbol s_Q of the marking $m_{\mathcal{Q}}$ which corresponds to Q. But since $m_{\mathcal{Q}} \subset m_{\mathcal{P}}$, there is a unique symbol $s_{\mathcal{P}}$ of the marking $m_{\mathcal{P}}$ such that the coordinates of c marked by $s_{\mathcal{Q}}$ are also marked by $s_{\mathcal{P}}$. The submultiball obtained from $(\alpha, m_{\mathcal{P}})$ corresponding to the symbol $s_{\mathcal{P}}$ is the one we are looking for.

Conversely, assume that there is a function $i\colon \{Q \in \mathcal{Q}\} \to \{P \in \mathcal{P}\}$ such that $Q \subset Q \triangleright i$ for every $Q \in \mathcal{Q}$. Using the square filling technique, we find a common arrow $\alpha\colon c \to x$ with markings $m_{\mathcal{Q}}$ and $m_{\mathcal{P}}$ such that $[\alpha, m_{\mathcal{Q}}] = \mathcal{Q}$ and $[\alpha, m_{\mathcal{P}}] = \mathcal{P}$. We want to show $m_{\mathcal{Q}} \subset m_{\mathcal{P}}$. Let s be any symbol of $m_{\mathcal{Q}}$. To this symbol corresponds exactly one submultiball $Q \in \mathcal{Q}$ such that $Q = [\alpha, m^s_{\mathcal{Q}}]$, where $m^s_{\mathcal{Q}}$ is the submarking of $m_{\mathcal{Q}}$ with all markings removed except those with the symbol s. To the submultiball $Q \triangleright i \in \mathcal{P}$ corresponds exactly one symbol r of $m_{\mathcal{P}}$ such that $Q \triangleright i = [\alpha, m^r_{\mathcal{P}}]$. Since $Q \subset Q \triangleright i$, we have $(\alpha, m^s_{\mathcal{Q}}) \subset (\alpha, m^r_{\mathcal{P}})$, and therefore, $m^s_{\mathcal{Q}} \subset m^r_{\mathcal{P}}$ by Remark 3.3. It follows that $m_{\mathcal{Q}} \subset m_{\mathcal{P}}$, and thus $\mathcal{Q} \subset \mathcal{P}$. □

3D The action on the set of semipartitions

Here we will define an action of $\Gamma = \pi_1(\mathcal{S}, x)$ on the set of semipartitions over x. Let $\gamma \in \Gamma$ and $\mathcal{P}$ be a semipartition over x. We will define another semipartition $\gamma \cdot \mathcal{P}$ over x. Recall that γ is represented by a span $x \xleftarrow{\gamma_d} a \xrightarrow{\gamma_n} x$ (the d refers to *denominator* and the n refers to *numerator*) and that $\mathcal{P}$ is represented by a marked arrow $(\alpha\colon c \to x, m)$. First, choose a square filling

$$\begin{tikzcd}
x & a \arrow[l, "\gamma_d"'] \arrow[r, "\gamma_n"] & x & c \arrow[l, "\alpha"'] \\
 & & b \arrow[ul, dotted, "\beta_1"] \arrow[ur, dotted, "\beta_2"'] \arrow[ull, dashed, "\delta"] &
\end{tikzcd}$$

of (γ_n, α), and then define $\delta := \beta_1 \gamma_d\colon b \to x$. Endow this arrow with the marking $\mu := \beta_2^*(m)$. Finally, define $\gamma \cdot \mathcal{P} := [\delta, \mu]$.

Claim *This is a well-defined action; ie the resulting class is independent of*

(1) *the square filling (β_1, β_2),*

(2) *the marked arrow (α, m) as a representative of $\mathcal{P}$,*

(3) *the span (γ_d, γ_n) as a representative of γ.*

Proof (1) Assume we have two square fillings of (γ_n, α) as in the following diagram:

$$\begin{tikzcd}
 & & b' \arrow[dl, dotted, "\beta'_1"'] \arrow[dr, dotted, "\beta'_2"] & \\
x & a \arrow[l, "\gamma_d"'] \arrow[r, "\gamma_n"] & x & c \arrow[l, "\alpha"'] \\
 & & b \arrow[ul, dotted, "\beta_1"] \arrow[ur, dotted, "\beta_2"'] &
\end{tikzcd}$$

Choose a common square filling as in Lemma 3.1:

$$\begin{tikzcd}
& & e \arrow[ddl, dashed, "\delta_1"'] \arrow[ddrr, dashed, bend left, "\delta_2"] \arrow[dr, dashed, "\eta'"] \arrow[dddr, dashed, "\eta"] & & \\
& & & b' \arrow[dll, dotted, "\beta'_1"'] \arrow[dr, dotted, "\beta'_2"] & \\
x & a \arrow[l, "\gamma_d"'] \arrow[rr, two heads, "\gamma_n"] & & x & c \arrow[l, "\alpha"'] \\
& & & b \arrow[ull, dotted, "\beta_1"] \arrow[ur, dotted, "\beta_2"'] &
\end{tikzcd}$$

Now the marked arrow $(\beta_1\gamma_d, \beta_2^*(m))$ is equivalent to the marked arrow $(\delta_1\gamma_d, \delta_2^*(m))$ via η. Analogously, the marked arrow $(\beta'_1\gamma_d, \beta_2'^*(m))$ is equivalent to $(\delta_1\gamma_d, \delta_2^*(m))$ via η' and, therefore, equivalent to $(\beta_1\gamma_d, \beta_2^*(m))$.

(2) Let (α', m') be another marked arrow equivalent to (α, m), and choose a square filling (β, β') such that $\beta^*(m) = \beta'^*(m') =: \mu$ as in the following diagram:

$$\begin{tikzcd}
& & & c \arrow[dl, "\alpha"'] & \\
x & a \arrow[l, "\gamma_d"'] \arrow[r, "\gamma_n"] & x & & e \arrow[ll, dotted, "\delta"'] \arrow[ul, dotted, "\beta"'] \arrow[dl, dotted, "\beta'"] \\
& & & c' \arrow[ul, "\alpha'"] &
\end{tikzcd}$$

First choose a square filling (η_1, η_2) of (γ_n, α), and then a square filling (ν_1, ν_2) of (η_2, β). Analogously, choose a square filling (η'_1, η'_2) of (γ_n, α'), and then a square filling (ν'_1, ν'_2) of (η'_2, β'):

$$\begin{tikzcd}
& & & z \arrow[dl, dashed, "\nu_1"'] \arrow[ddr, dashed, "\nu_2"] & \\
& & y \arrow[dl, dashed, "\eta_1"'] \arrow[r, dashed, "\eta_2"] & c \arrow[dl, "\alpha"'] & \\
x & a \arrow[l, "\gamma_d"'] \arrow[r, "\gamma_n"] & x & & e \arrow[ll, dotted, "\delta"'] \arrow[ul, dotted, "\beta"'] \arrow[dl, dotted, "\beta'"] \\
& & y' \arrow[ul, dashed, "\eta'_1"] \arrow[r, dashed, "\eta'_2"'] & c' \arrow[ul, "\alpha'"] & \\
& & & z' \arrow[ul, dashed, "\nu'_1"] \arrow[uur, dashed, "\nu'_2"'] &
\end{tikzcd}$$

The marked arrow $(\eta_1\gamma_d, \eta_2^*(m))$ is equivalent to $\Lambda := (\nu_1\eta_1\gamma_d, \nu_2^*(\mu))$ via ν_1. On the other side, the marked arrow $(\eta_1'\gamma_d, \eta_2'^*(m'))$ is equivalent to $\Lambda' := (\nu_1'\eta_1'\gamma_d, \nu_2'^*(\mu))$ via ν_1'. The marked arrows Λ and Λ' are both constructed from the same marked arrow (δ, μ) and so are equivalent by (1). Consequently, $(\eta_1\gamma_d, \eta_2^*(m))$ and $(\eta_1'\gamma_d, \eta_2'^*(m'))$ are equivalent.

(3) Let (γ_d', γ_n') be another representing span of γ homotopic to the span (γ_d, γ_n). Then recall that the two spans can be filled by a diagram as follows:

$$\begin{tikzcd}
 & a \arrow[dl, "\gamma_d"'] \arrow[dr, "\gamma_n"] & & \\
x & e \arrow[l, dotted, "\delta_d"'] \arrow[r, dotted, "\delta_n"] \arrow[u, dotted, "\eta"'] \arrow[d, dotted, "\eta'"] & x & c \arrow[l, "\alpha"'] \\
 & a' \arrow[ul, "\gamma_d'"] \arrow[ur, "\gamma_n'"'] & &
\end{tikzcd}$$

Now choose a square filling (ν_1, ν_2) of (δ_n, α), and note that (ϵ, ν_2), where $\epsilon := \nu_1\eta$, gives a square filling of (γ_n, α):

$$\begin{tikzcd}
 & & z \arrow[dl, "\epsilon"'] \arrow[ddl, dashed, "\nu_1"] \arrow[ddr, dashed, "\nu_2"] & \\
 & a \arrow[dl, "\gamma_d"'] \arrow[dr, "\gamma_n"] & & \\
x & e \arrow[l, dotted, "\delta_d"'] \arrow[r, dotted, "\delta_n"] \arrow[u, dotted, "\eta"'] \arrow[d, dotted, "\eta'"] & x & c \arrow[l, "\alpha"'] \\
 & a' \arrow[ul, "\gamma_d'"] \arrow[ur, "\gamma_n'"'] & &
\end{tikzcd}$$

The marked arrow $(\epsilon\gamma_d, \nu_2^*(m))$ is equivalent to $(\nu_1\delta_d, \nu_2^*(m))$. Similarly, define $\epsilon' = \nu_1\eta'$ and note that (ϵ', ν_2) gives a square filling of (γ_n', α). Again, the marked arrow $(\epsilon'\gamma_d', \nu_2^*(m))$ is equivalent to $(\nu_1\delta_d, \nu_2^*(m))$. Therefore, $(\epsilon\gamma_d, \nu_2^*(m))$ and $(\epsilon'\gamma_d', \nu_2^*(m))$ are equivalent. Thus $\gamma \cdot \mathcal{P}$ is well defined.

Now we want to show that this is indeed an action; ie $1 \cdot \mathcal{P} = \mathcal{P}$ and $\gamma^1 \cdot (\gamma^2 \cdot \mathcal{P}) = (\gamma^1\gamma^2) \cdot \mathcal{P}$. The first property is easy to see. The second property is not entirely trivial but straightforward. We will be explicit for completeness. Choose two representing spans (γ_d^1, γ_n^1) and (γ_d^2, γ_n^2) for γ^1 and γ^2 respectively. Let (α, m) represent $\mathcal{P}$. To get a representing span for the composition $\gamma_1\gamma_2$, choose a square filling (β_1, β_2) of (γ_n^1, γ_d^2) and take the span $(\beta_1\gamma_d^1, \beta_2\gamma_n^2)$. This span acts on (α, m) as before and is sketched diagrammatically as follows:

$$\begin{tikzcd}
 & & & & & z \arrow[dlll, "\eta"'] \arrow[dd, "\nu"] \\
 & & y \arrow[dll, "\delta_1"'] \arrow[dl, "\beta_1"] \arrow[dr, "\beta_2"'] \arrow[drr, "\delta_2"] & & & \\
x & a^1 \arrow[l, "\gamma_d^1"'] \arrow[r, "\gamma_n^1"'] & x & a^2 \arrow[l, "\gamma_d^2"'] \arrow[r, "\gamma_n^2"'] & x & c \arrow[l, "\alpha"']
\end{tikzcd}$$

So a representative of $(\gamma^1\gamma^2)\cdot\mathcal{P}$ is given by $(\eta\delta_1, \nu^*(m))$. Now a representative for $\gamma^2{\cdot}\mathcal{P}$ is given by $(\eta\beta_2\gamma_d^2, \nu^*(m))$ because $(\eta\beta_2, \nu)$ is a square filling for (γ_n^2, α). Since $(\eta\beta_1, \mathrm{id}_z)$ is a square filling for $(\gamma_n^1, \eta\beta_2\gamma_d^2)$, we obtain that $(\eta\beta_1\gamma_d^1, \mathrm{id}_z^*(\nu^*(m)))$ is a representative of $\gamma^1\cdot(\gamma^2\cdot\mathcal{P})$. But this last marked arrow is equal to $(\eta\delta_1, \nu^*(m))$. □

Remark 3.6 It is not hard to see that $\mathcal{P}\subset\mathcal{Q}$ implies $\gamma\cdot\mathcal{P}\subset\gamma\cdot\mathcal{Q}$.

Remark 3.7 The submultiballs of $\gamma\cdot\mathcal{P}$ are the multiballs $\gamma\cdot P$ with $P\in\mathcal{P}$.

3E Pointwise stabilizers of partitions

Let $\mathcal{P}$ be a partition over x. By the pointwise stabilizer of $\mathcal{P}$, we mean the subgroup

$$\Lambda := \{\gamma\in\pi_1(\mathcal{S}, x) \mid \gamma\cdot P = P \text{ for all } P\in\mathcal{P}\}.$$

Fix some representative (α, m) of $\mathcal{P}$. We can assume, without loss of generality, that the marking m on the domain c of α is ordered. That means that if $f\colon I\to S$ is the marking function of m, and whenever $i\triangleright f = j\triangleright f$ for $i<j$, then also $i\triangleright f = k\triangleright f = j\triangleright f$ for every k with $i<k<j$. This is true in the planar case by definition. In the symmetric case, we can choose a colored permutation $\sigma\in\mathfrak{Sym}(C)$ with $\sigma^*(m)$ ordered, and replace (α, m) by the equivalent marked arrow $(\sigma\alpha, \sigma^*(m))$.

Proposition 3.8 *Each symbol of the marking m determines a subword of the word $c=\mathrm{dom}(\alpha)$. Denote these subwords by $c_1,\dots,c_k$, and order them so that $c=c_1\otimes\cdots\otimes c_k$. Then we have a well-defined isomorphism of groups,*

$$\Xi\colon \pi_1(\mathcal{S}, c_1)\times\cdots\times\pi_1(\mathcal{S}, c_k)\to\Lambda,$$

which is given by applying the tensor product of paths and then conjugating with the arrow α. More explicitly, it is given by sending representing spans $p_1,\dots,p_k$ to the *homotopy class represented by the path*

$$\begin{tikzcd}
 & c_1 & & a_1 \arrow[ll, "p_1^{\prec}"'] \arrow[rr, "p_1^{\succ}"] & & c_1 & \\
 & \otimes & \otimes & \otimes & \otimes & \otimes & \\
x & \vdots \arrow[l, "\alpha"'] & \vdots & \vdots & \vdots & \vdots \arrow[r, "\alpha"] & x \\
 & \otimes & \otimes & \otimes & \otimes & \otimes & \\
 & c_k & & a_k \arrow[ll, "p_k^{\prec}"] \arrow[rr, "p_k^{\succ}"'] & & c_k &
\end{tikzcd}$$

where $p_i^{\prec}$ is the arrow pointing to the left and $p_i^{\succ}$ the arrow pointing to the right in the span p_i.

Proof It is not hard to see that the map is independent of the chosen representing spans p_i and that it is a group homomorphism. Injectivity follows from Lemmas 3.2 and 3.9 below. Before we prove surjectivity, we want to see that the image really lies in the subgroup Λ. We can use the representative (α, m) to extract representatives of submultiballs $P \in \mathcal{P}$. The subwords c_i are in one to one correspondence with the submultiballs $P \in \mathcal{P}$. A representative (α, m_i) of $P \in \mathcal{P}$ corresponding to c_i is obtained from (α, m) by removing all markings except the markings on the subword c_i. The representing span of $\Xi(p_1, \dots, p_k)$ pictured above can be written as $(p^{\prec}\alpha, p^{\succ}\alpha)$ where $p^{\prec} = p_1^{\prec} \otimes \cdots \otimes p_k^{\prec}$ and $p^{\succ} = p_1^{\succ} \otimes \cdots \otimes p_k^{\succ}$. Letting this span act on (α, m_i), we can choose $(\mathrm{id}, p^{\succ})$ as a square filling and the resulting representative is $(p^{\prec}\alpha, p^{\succ *}(m_i))$. But this is equivalent to (α, m_i) because $p^{\prec *}(m_i) = p^{\succ *}(m_i)$.

Now we prove surjectivity. Let $\gamma \in \Lambda$ which can be represented by a path of the form

$$x \xleftarrow{\alpha} c \xleftarrow{z^{\prec}} a \xrightarrow{z^{\succ}} c \xrightarrow{\alpha} x.$$

Observe the representatives (α, m_i) of the submultiballs $P \in \mathcal{P}$ from above. A representative of $\gamma \cdot [\alpha, m_i]$ is given by $(z^{\prec}\alpha, z^{\succ *}(m_i))$. So we have $(\alpha, m_i) \sim (z^{\prec}\alpha, z^{\succ *}(m_i))$. Of course, $(z^{\prec}, \mathrm{id})$ is a square filling of $(\alpha, z^{\prec}\alpha)$, and thus

$$z^{\prec *}(m_i) = z^{\succ *}(m_i).$$

Now assume for the moment that the operad $\mathcal{O}$ is planar. Then it follows easily from these equalities that the span $(z^{\prec}, z^{\succ})$ splits as a product according to the decomposition $c = c_1 \otimes \cdots \otimes c_k$; ie there are $z_i^{\prec}\colon a_i \to c_i$ and $z_i^{\succ}\colon a_i \to c_i$ with $z^{\prec} = z_1^{\prec} \otimes \cdots \otimes z_k^{\prec}$ and $z^{\succ} = z_1^{\succ} \otimes \cdots \otimes z_k^{\succ}$. By construction, the spans $(z_i^{\prec}, z_i^{\succ})$ give a preimage of γ under Ξ. If, on the other hand, $\mathcal{O}$ is symmetric, then there is colored permutation $\sigma \in \mathfrak{Sym}(C)$ such that the span $(\sigma z^{\prec}, \sigma z^{\succ})$, which is homotopic to $(z^{\prec}, z^{\succ})$, splits as above, and we can also finish the proof in this case. □

Lemma 3.9 *Let* $a \xleftarrow{q} b \xrightarrow{p} a$ *be a span in* $\mathcal{S}$ *which is a tensor product of* k *spans* $a_i \xleftarrow{q_i} b_i \xrightarrow{p_i} a_i$ *for* $i = 1, \dots, k$*, ie* $q = q_1 \otimes \cdots \otimes q_k$ *and* $p = p_1 \otimes \cdots \otimes p_k$*. Then the span* (q, p) *is null-homotopic if and only if each* (q_i, p_i) *is null-homotopic.*

Proof It is clear that if each (q_i, p_i) is null-homotopic, then (q, p) is null-homotopic. So we prove the converse. We can assume without loss of generality that $q_i \neq \mathrm{id}_I \neq p_i$ where I is the monoidal unit, ie the empty word, in $\mathcal{S}$. First observe that p and q are parallel arrows, and since $\mathcal{S}$ satisfies the calculus of fractions, they are homotopic if and only if there is an arrow $r\colon c \to b$ with $rq = rp$. Now, by precomposing with an arrow in $\mathfrak{Sym}(C)$ if necessary, we can assume that r is an arrow in $\mathcal{S}(\mathcal{O}_{\mathrm{pl}})$, ie a tensor product of operations in $\mathcal{O}$. Observe that, in $\mathcal{S}$, we have $\alpha_1 \otimes \cdots \otimes \alpha_l = \beta_1 \otimes \cdots \otimes \beta_m$ for arrows $\alpha_i \neq \mathrm{id}_I \neq \beta_i$ if and only if $l = m$ and $\alpha_i = \beta_i$ for each $i = 1, \dots, l$. Now it follows easily that r gives arrows $r_1, \dots, r_k$ such that $r_i q_i = r_i p_i$ for each $i = 1, \dots, k$. Thus, q_i is homotopic to p_i for each $i = 1, \dots, k$. □

3F The poset of partitions

From now on, fix some base object x which is split and progressive. More generally, in view of Remark 2.6:

Let y *be a split object such that* x *is* y*–progressive.*

Furthermore, let $n \in \mathbb{N}$.

Two objects a and b in $\mathcal{S}$ are called equivalent if they are isomorphic in $\pi_1(\mathcal{S})$, ie there is a path (equivalently, a span) between them in $\mathcal{S}$. Of course, $\pi_1(\mathcal{S}, a) \cong \pi_1(\mathcal{S}, b)$ in this case.

Let $c = (c_1, \dots, c_k)$ with $c_i \in C$ an object in $\mathcal{S}$ and m be a unimarking on c, ie there is only one symbol in m. Then m determines another object $\mathfrak{c}(m)$ by deleting all c_i which are not marked by m. If $\alpha\colon a \to c$ is an arrow, then $\mathfrak{c}(\alpha^*(m)) \sim \mathfrak{c}(m)$ in the above sense.

Let B be a multiball. If (α, m) and (α', m') are representatives, then $\mathfrak{c}(m) \sim \mathfrak{c}(m')$. Thus, each multiball B gives an equivalence class $\mathfrak{cc}(B)$ of objects.

We say that a partition $\mathcal{P}$ (over x) satisfies the n–condition with respect to y if at least n submultiballs $P \in \mathcal{P}$ satisfy $y \in \mathfrak{cc}(P)$. The n–condition with respect to y is preserved by the action of $\Gamma = \pi_1(\mathcal{S}, x)$ on the partitions: If $\mathcal{P}$ satisfies the n–condition with respect to y, then $\gamma \cdot \mathcal{P}$ also satisfies it.

We define a poset $(\mathbb{P}, \leq)$: The objects of $\mathbb{P}$ are partitions over x and $\mathcal{P} \leq \mathcal{Q}$ if and only if $\mathcal{P} \supset \mathcal{Q}$. The group $\Gamma = \pi_1(\mathcal{S}, x)$ acts on this poset via the action on partitions. Because of Remark 3.6, the action indeed respects the relation $\leq$.

Since the n–condition with respect to y is invariant under the action of Γ, we can define the invariant subposet $(\mathbb{P}_n, \leq)$ to be the full subposet consisting of partitions satisfying the n–condition with respect to y. Next, we want to show that

(1) $\mathbb{P}_n \neq \varnothing$, and

(2) $(\mathbb{P}_n, \leq)$ is filtered.

This will imply that the poset $\mathbb{P}_n$ is *contractible.*

(1) Since x is y–progressive, there is an arrow $a_1 \otimes y \otimes a_2 \to x$. Apply the split condition of y $n{-}1$ times to find an arrow $z \to x$ where z has a tensor product decomposition with at least n factors equal to y. Mark each of these factors with a different symbol and the rest with yet another symbol. This yields a partition $\mathcal{P} \in \mathbb{P}_n$.

(2) Let $\mathcal{P}, \mathcal{Q} \in \mathbb{P}_n$. We have to find $\mathcal{R} \in \mathbb{P}_n$ with $\mathcal{P}, \mathcal{Q} \leq \mathcal{R}$. First we find one in $\mathbb{P}$. Let $(\alpha_{\mathcal{P}}, m_{\mathcal{P}})$ and $(\alpha_{\mathcal{Q}}, m_{\mathcal{Q}})$ be representatives of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Choose a square filling $(\beta_{\mathcal{P}}, \beta_{\mathcal{Q}})$ of $(\alpha_{\mathcal{P}}, \alpha_{\mathcal{Q}})$, and set $\delta = \beta_{\mathcal{P}}\alpha_{\mathcal{P}} = \beta_{\mathcal{Q}}\alpha_{\mathcal{Q}}$. Now find a full marking μ with $\mu \subset \beta_{\mathcal{P}}^*(m_{\mathcal{P}})$ and $\mu \subset \beta_{\mathcal{Q}}^*(m_{\mathcal{Q}})$, for example, by marking each coordinate of $\mathrm{dom}(\delta)$ with a different symbol. Then $\mathcal{R} = [\delta, \mu]$ is a common refinement of $\mathcal{P}$ and $\mathcal{Q}$. Now use that x is y–progressive to find an arrow $\eta\colon z \to \mathrm{dom}(\delta)$, where z has a tensor product decomposition with at least one factor equal to y. Then apply the split condition of y $n{-}1$ times to obtain an arrow $\nu\colon w \to z$ where w has a tensor product decomposition with at least n factors equal to y. Observe the marked arrow $(\nu\eta\delta, (\nu\eta)^*(\mu))$. The so-called link condition in Remark 2.6 ensures that the n factors of w equal to y are marked with the same symbol in the marking $(\nu\eta)^*(\mu)$. Refine this marking such that these factors are marked with new different symbols. This gives a representative of a partition satisfying the n–condition with respect to y, refining $\mathcal{R}$, and thus refining both $\mathcal{P}$ and $\mathcal{Q}$.

A simplex σ in the poset $\mathbb{P}_n$ is a finite ascending sequence of objects, written $[\mathcal{P}_0 < \mathcal{P}_1 < \cdots < \mathcal{P}_p]$. We now observe the stabilizer subgroup Γ_σ of such a simplex. By definition, an element γ is in this stabilizer subgroup if and only if $\{\mathcal{P}_0, \ldots, \mathcal{P}_p\} = \{\gamma\cdot\mathcal{P}_0, \ldots, \gamma\cdot\mathcal{P}_p\}$. But since the action of γ respects $\leq$, this is equivalent to $\gamma\cdot\mathcal{P}_i = \mathcal{P}_i$ for each $i = 0, \ldots, p$. So each $\gamma \in \Gamma_\sigma$ fixes σ vertexwise. Observe the subgroup

$$\Lambda_\sigma := \{\gamma \in \Gamma \mid \gamma \cdot P = P \text{ for all } P \in \mathcal{P}_p\} < \Gamma.$$

By Proposition 3.8, we know that $\Lambda_\sigma \cong \pi_1(\mathcal{S}, c_1) \times \cdots \times \pi_1(\mathcal{S}, c_k)$ for appropriate objects c_i. Since $\mathcal{P}_p$ satisfies the n–condition with respect to y, at least n of these objects are equivalent to y, and thus at least n of the factors in the product decomposition of Λ_σ are isomorphic to $\Upsilon := \pi_1(\mathcal{S}, y)$. So we find a normal subgroup $\Lambda'_\sigma \lhd \Lambda_\sigma$

with $\Lambda'_\sigma \cong \Upsilon^n$. Below, we will show that Λ_σ is a normal subgroup of Γ_σ. So we arrive at the following situation:

$$\Upsilon^n \cong \Lambda'_\sigma \lhd \Lambda_\sigma \lhd \Gamma_\sigma.$$

Lemma 3.10 *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be semipartitions, and let $\mathcal{P}$ be a partition with $\mathcal{P} \subset \mathcal{R}_1$. Assume that*

$$\forall_{R_1 \in \mathcal{R}_1} \exists_{R_2 \in \mathcal{R}_2} \forall_{P \in \mathcal{P}}\ P \subset R_1 \Longrightarrow P \subset R_2.$$

Then we have $\mathcal{R}_1 \subset \mathcal{R}_2$.

Proof By applying the square filling technique twice, we find an arrow δ with three markings, $m_\mathcal{P}$, $m_{\mathcal{R}_1}$ and $m_{\mathcal{R}_2}$, on its domain such that $(\delta, m_\mathcal{P})$ represents $\mathcal{P}$ and $(\delta, m_{\mathcal{R}_i})$ represents $\mathcal{R}_i$. Since $\mathcal{P} \subset \mathcal{R}_1$, we have $(\delta, m_\mathcal{P}) \subset (\delta, m_{\mathcal{R}_1})$, and therefore $m_\mathcal{P} \subset m_{\mathcal{R}_1}$. Note that $\mathcal{P}$ is a partition, and therefore, $m_\mathcal{P}$ is a full marking. Now the assumption of the statement implies $m_{\mathcal{R}_1} \subset m_{\mathcal{R}_2}$, and thus $\mathcal{R}_1 \subset \mathcal{R}_2$. □

Claim *Λ_σ is contained in Γ_σ.*

Proof Let $\gamma \in \Lambda_\sigma$; ie $\gamma \cdot P = P$ for all $P \in \mathcal{P}_p$. In particular, we have $\gamma \cdot \mathcal{P}_p = \mathcal{P}_p$ (Proposition 3.5 and Remark 3.7). We also have to show $\gamma \cdot \mathcal{P}_i = \mathcal{P}_i$ for the other values of i. Write $\mathcal{P} := \mathcal{P}_p$ and $\mathcal{R} := \mathcal{P}_i$ for some $i \neq p$. Then we have $\mathcal{P} \subset \mathcal{R}$. We want to apply the above lemma to $\mathcal{R}_1 = \mathcal{R}$ and $\mathcal{R}_2 = \gamma \cdot \mathcal{R}$, and then deduce $\mathcal{R} \subset \gamma \cdot \mathcal{R}$. So let $R \in \mathcal{R}$, and observe $\gamma \cdot R \in \gamma \cdot \mathcal{R}$. Let $P \in \mathcal{P}$ with $P \subset R$. Then $P = \gamma \cdot P \subset \gamma \cdot R$, and the assumption of the lemma is satisfied. Similarly, we get $\mathcal{R} \subset \gamma^{-1} \cdot \mathcal{R}$ and thus $\gamma \cdot \mathcal{R} \subset \mathcal{R}$. This yields $\gamma \cdot \mathcal{R} = \mathcal{R}$. □

Claim *Λ_σ is normal in Γ_σ.*

Proof Let $\gamma \in \Gamma_\sigma$ and $\alpha \in \Lambda_\sigma$. We have to show $\gamma^{-1}\alpha\gamma \in \Lambda_\sigma$, ie $\gamma^{-1}\alpha\gamma \cdot P = P$ for all $P \in \mathcal{P}_p =: \mathcal{P}$, or equivalently, $\alpha \cdot (\gamma \cdot P) = \gamma \cdot P$ for all $P \in \mathcal{P}$. Since $\gamma \cdot \mathcal{P} = \mathcal{P}$, we have a bijection $f\colon \{P \in \mathcal{P}\} \to \{P \in \mathcal{P}\}$ such that $\gamma \cdot P = P \triangleright f$ for all $P \in \mathcal{P}$ by Proposition 3.5. Consequently, if $P \in \mathcal{P}$, then $\alpha \cdot (\gamma \cdot P) = \alpha \cdot (P \triangleright f) = P \triangleright f = \gamma \cdot P$ □

3G End of the proof

Let $\mathfrak{P}$ be a property of groups which is closed under taking products, and let $\mathcal{M}$ be a coefficient system which is $\mathfrak{P}$–Künneth and $\mathfrak{P}$–inductive. We will only give the proof for homology; using analogous devices for cohomology, we can obtain a proof of the cohomological version of the statement.

Our main tool will be a spectral sequence explained in Brown's book [2, Chapter VII.7]; see also [9, Section 4.1]. If we plug in our Γ–complex $(\mathbb{P}_n, \leq)$ and the $\mathbb{Z}\Gamma$–module $\mathcal{M}\Gamma$, we obtain a spectral sequence E^k_{pq} with

$$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma, \mathcal{M}\Gamma) \Rightarrow H_{p+q}(\Gamma, \mathcal{M}\Gamma),$$

where Σ_p is the set of p–cells representing the Γ–orbits of $(\mathbb{P}_n, \leq)$. This uses the fact that the poset $\mathbb{P}_n$ is contractible and that the cell stabilizers fix the cells pointwise.

We assumed that Υ satisfies $\mathfrak{P}$ and that $H_0(\Upsilon, \mathcal{M}\Upsilon) = 0$. Applying the $\mathfrak{P}$–Künneth property $n-1$ times, we obtain $H_k(\Upsilon^n, \mathcal{M}\Upsilon^n) = 0$ for $k \leq n-1$. So we have $H_k(\Lambda'_\sigma, \mathcal{M}\Lambda'_\sigma) = 0$ for $k \leq n-1$. The $\mathfrak{P}$–inductive property yields $H_k(\Lambda'_\sigma, \mathcal{M}\Gamma) = 0$ for $k \leq n-1$. Since $\Lambda'_\sigma \lhd \Lambda_\sigma \lhd \Gamma_\sigma$, we can apply the Hochschild–Serre spectral sequence twice to obtain $H_k(\Gamma_\sigma, \mathcal{M}\Gamma) = 0$ for $k \leq n-1$. The above spectral sequence now yields $H_k(\Gamma, \mathcal{M}\Gamma) = 0$ for $k \leq n-1$. Since n was arbitrary, the result follows. □

4 Nonamenability and infiniteness

In this section we use the techniques from Section 3 to prove nonamenability and infiniteness of some operad groups. Note that semipartitions and the action on the set of semipartitions can also be defined in the braided case.

Lemma 4.1 *If $\mathcal{O}$ satisfies the calculus of fractions, then the action of the colored permutations in $\mathrm{Aut}_{\mathfrak{Sym}(C)}(X)$ or the colored braids in $\mathrm{Aut}_{\mathfrak{Braid}(C)}(X)$ on the set of arrows $\mathrm{Hom}_{\mathcal{S}(\mathcal{O})}(X, Y)$ is free. In particular, in the operad $\mathcal{O}$, the action of the symmetric groups or the braid groups on the sets of operations is free.*

Proof Let $[\alpha, \Theta]$ be an element in $\mathrm{Hom}_{\mathcal{S}(\mathcal{O})}(X, Y)$ and $\sigma \in \mathrm{Aut}_{\mathfrak{Sym}(C)}(X)$ or $\sigma \in \mathrm{Aut}_{\mathfrak{Braid}(C)}(X)$. We have to show that $[\sigma, \mathrm{id}] * [\alpha, \Theta] = [\alpha, \Theta]$ implies that σ is trivial. From this equality and the equalization property of $\mathcal{S}(\mathcal{O})$, we obtain an arrow $z := [\delta, \Psi]$ with $z * [\sigma, \mathrm{id}] = z$. We can assume without loss of generality that $\delta = \mathrm{id}$. We then have $z * [\sigma, \mathrm{id}] = [\overline{\sigma}, \overline{\Psi}]$ with $\overline{\sigma} = \Psi \curvearrowright \sigma$ and $\overline{\Psi} = \Psi \curvearrowleft \sigma$. Consequently, the pairs $(\overline{\sigma}, \overline{\Psi})$ and (id, Ψ) are equivalent in $\mathfrak{Sym}(C) \times \mathcal{S}(\mathcal{O}_{\mathrm{pl}})$ or $\mathfrak{Braid}(C) \times \mathcal{S}(\mathcal{O}_{\mathrm{pl}})$. This is only possible if σ is trivial. □

Let $\mathcal{O}$ be a symmetric or braided operad. Let α be an arrow in $\mathcal{S}(\mathcal{O})$. For any colored permutation $\sigma \in \mathfrak{Sym}(C)$ or colored braid $\sigma \in \mathfrak{Braid}(C)$ with suitable domain and codomain, we can form the group element γ represented by the span $(\alpha, [\sigma, \mathrm{id}] * \alpha)$. Recall that the first arrow always denotes the denominator, ie points to the left.

Lemma 4.2 *Assume $\mathcal{O}$ satisfies the calculus of fractions. Then*

$$\sigma \neq 1 \quad \Longrightarrow \quad \gamma \neq 1.$$

Proof First consider the symmetric case. Observe the semipartition $\mathcal{R}$ represented by the marked arrow (α, m) where m is a marking on the domain of α with only one marked coordinate that is nontrivially permuted by σ^{-1}. It is easy to see that $\gamma \cdot \mathcal{R}$ is represented by $(\alpha, [\sigma, \mathrm{id}]^*(m))$. The marking $m' := [\sigma, \mathrm{id}]^*(m)$ is different from m because σ^{-1} maps the only marked coordinate of m to a different coordinate by assumption. From Remark 3.3, it follows that the marked arrow (α, m) cannot be equivalent to (α, m'), and thus $\gamma \cdot \mathcal{R} \neq \mathcal{R}$. Consequently, $\gamma \neq 1$.

Now if $\mathcal{O}$ is braided, we can apply the above argument verbatim if we require that the braid σ has a nontrivial permutation part. But there are, of course, nontrivial braids which are trivial as permutations (so-called pure braids). Assume that σ is such a pure braid and that $\gamma = 1$. Note that the latter means that the parallel arrows $\sigma\alpha := [\sigma, \mathrm{id}] * \alpha$ and α are homotopic. Since $\mathcal{S}(\mathcal{O})$ satisfies the calculus of fractions, this means that there is an arrow δ with $\delta * \sigma\alpha = \delta * \alpha$. We can assume without loss of generality that $\delta \in \mathcal{S}(\mathcal{O}_{\mathrm{pl}})$, ie that $\delta = [\mathrm{id}, \Theta]$. Composing in $\mathcal{S}(\mathcal{O})$, we get $\delta * [\sigma, \mathrm{id}] = [\overline{\sigma}, \overline{\Theta}]$, where $\overline{\sigma} = \Theta \curvearrowright \sigma$ and $\overline{\Theta} = \Theta \curvearrowleft \sigma$. Using that σ is pure we immediately see $\overline{\Theta} = \Theta$. So we have

$$[\overline{\sigma}, \mathrm{id}] * ([\mathrm{id}, \Theta] * \alpha) = (\delta * [\sigma, \mathrm{id}]) * \alpha = [\mathrm{id}, \Theta] * \alpha.$$

Lemma 4.1 now gives $\overline{\sigma} = 1$, and thus $\sigma = 1$. □

Denoting the element γ suggestively by $\xleftarrow{\alpha} \sigma \xrightarrow{\alpha}$, the lemma implies that two such group elements $\xleftarrow{\alpha} \sigma \xrightarrow{\alpha}$ and $\xleftarrow{\alpha} \sigma' \xrightarrow{\alpha}$ are equal if and only if $\sigma = \sigma'$. We will use this now to give a proof of the following proposition.

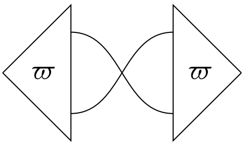
Proposition 4.3 *Let $\mathcal{O}$ be a symmetric operad satisfying the calculus of fractions, and let X be a split object of $\mathcal{S}(\mathcal{O})$. Then $\Gamma = \pi_1(\mathcal{O}, X)$ contains a nonabelian free subgroup and is therefore nonamenable.*

Proof Using the split condition on X, we will explicitly construct two nontrivial elements $\gamma_1, \gamma_2 \in \Gamma$ of order 2 and 3, respectively. Then we will define two disjoint subsets, A_1 and A_2, of the set of semipartitions over X such that $\gamma_1 \cdot A_2 \subset A_1$ and $\gamma_2^n \cdot A_1 \subset A_2$ for $n = 1, 2$. The ping-pong lemma then shows that the subgroup $\langle \gamma_1, \gamma_2 \rangle$ generated by the two elements γ_1 and γ_2 is isomorphic to the free product $\langle \gamma_1 \rangle * \langle \gamma_2 \rangle$. So we have found a subgroup which is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$. Since $\mathbb{Z}_2 * \mathbb{Z}_3$ contains a free nonabelian subgroup, the proof of the proposition is complete.

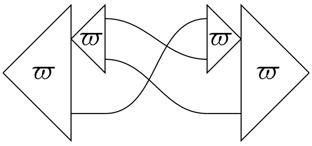
We now give the constructions. Because X is split, there is an arrow

$$\varpi\colon A_1 \otimes X \otimes A_2 \otimes X \otimes A_3 \to X.$$

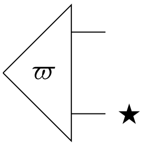
For better readability, we assume that X is a single color and $A_1 = A_2 = A_3 = I$. The construction goes the same way in the general case (with obvious modifications). So we assume that ϖ is just an operation with two inputs of color X and an output of color X. Respectively define γ_1 and γ_2 to be the following elements:



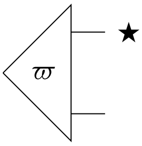
and



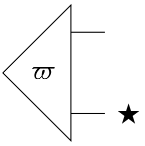
Lemma 4.2 implies that γ_1 is of order 2 and γ_2 is of order 3. Let B_1 and B_2 be the balls respectively represented by these marked arrows:

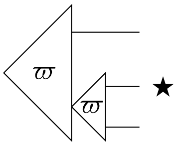


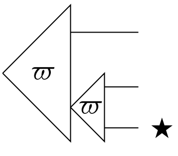
and



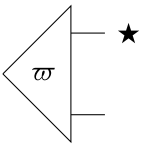
Composing the operation ϖ several times, one gets operations that look like binary trees. Call them ϖ–tree operations. Now define $\mathcal{A}_1$ to be the set of all balls $B \subset B_1$ which are represented by ϖ–tree operations. Similarly, define $\mathcal{A}_2$ to be the set of all balls $B \subset B_2$ which are represented by ϖ–tree operations. For example, the following marked arrows represent balls in $\mathcal{A}_1$:

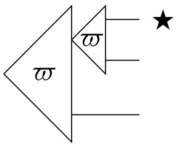


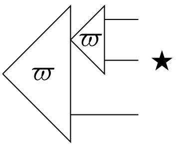




and the following marked arrows represent balls in $\mathcal{A}_2$:







It is straightforward to check that $\gamma_1 \cdot \mathcal{A}_2 \subset \mathcal{A}_1$, $\gamma_2 \cdot \mathcal{A}_1 \subset \mathcal{A}_2$ and $\gamma_2^2 \cdot \mathcal{A}_1 \subset \mathcal{A}_2$, so the proof is completed. □

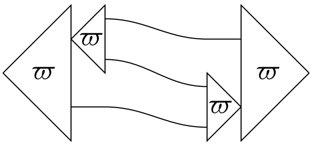
Next we give sufficient conditions for infiniteness of operad groups.

Proposition 4.4 *Let $\mathcal{O}$ be a planar, symmetric or braided operad satisfying the calculus of fractions, and let X be a split object of $\mathcal{S}(\mathcal{O})$. Then $\Gamma := \pi_1(\mathcal{O}, X)$ contains an infinite cyclic subgroup and is therefore infinite.*

Proof Because X is split, there is an arrow

$$\varpi\colon A_1 \otimes X \otimes A_2 \otimes X \otimes A_3 \to X.$$

For better readability, we assume that X is a single color and $A_1 = A_2 = A_3 = I$. The construction goes the same way in the general case (with obvious modifications). So we assume that ϖ is just an operation with two inputs of color X and an output of color X. Define γ to be the following element



Formally, γ is represented by the span $((\varpi, \mathrm{id}) * \varpi,\ (\mathrm{id}, \varpi) * \varpi)$. We claim that γ has infinite order. The element γ^n is represented by the span (for better readability, we use the same symbol id for different identities)

$$\big((\varpi, \mathrm{id}) * \cdots * (\varpi, \mathrm{id}) * \varpi,\ (\mathrm{id}, \varpi) * \cdots * (\mathrm{id}, \varpi) * \varpi\big).$$

By Lemma 3.2, this span is null-homotopic if and only if the span (remove the last ϖ in both arrows)

$$\big((\varpi, \mathrm{id}) * \cdots * (\varpi, \mathrm{id}),\ (\mathrm{id}, \varpi) * \cdots * (\mathrm{id}, \varpi)\big) =: (\varpi_1, \varpi_2)$$

is null-homotopic. This is true if and only if there is an arrow r with $r\varpi_1 = r\varpi_2$. The arrow r can be chosen to lie in $\mathcal{S}(\mathcal{O}_{\mathrm{pl}})$. But note that ϖ_1 splits as

$$\varpi_1 = ((\varpi, \mathrm{id}) * \cdots * (\varpi, \mathrm{id}) * \varpi) \otimes \mathrm{id}_X,$$

and ϖ_2 splits as

$$\varpi_2 = \mathrm{id}_X \otimes ((\mathrm{id}, \varpi) * \cdots * (\mathrm{id}, \varpi) * \varpi).$$

It can easily be seen that such an arrow r cannot exist, because otherwise operations with a different number of inputs must be equal. Consequently, each γ^n is nontrivial and therefore γ has infinite order. □

5 Applications

Observe the 1–dimensional planar cube cutting operads and the d–dimensional symmetric cube cutting operads from [10, Section 3.5]. They all satisfy the (cancellative) calculus of fractions. Furthermore, they are monochromatic and possess operations of arbitrarily high degree. From Remarks 2.3 and 2.4, it follows that all objects (except the uninteresting unit object) are split and progressive. So Theorem 2.5 is applicable to these operads. Furthermore, the corresponding operad groups are all infinite by Proposition 4.4 and nonamenable in the symmetric case by Proposition 4.3.

Observe now the local similarity operads. Let Sim_X be a finite similarity structure on the compact ultrametric space X. When choosing a ball in each Sim_X–equivalence of balls, we obtain a symmetric operad with transformations $\mathcal{O}$. The colors of $\mathcal{O}$ are the chosen balls. We choose X for the Sim_X–equivalence class $[X]$. We already know that $\mathcal{O}$ satisfies the (cancellative) calculus of fractions. In [9, Definition 3.1] we called Sim_X dually contracting if there are two disjoint proper subballs B_1 and B_2 of X together with similarities $X \to B_i$ in Sim_X. This easily implies that X is split.

Lemma 5.1 *The color X is progressive provided Sim_X is dually contracting.*

Proof Let $\theta = (f_1, \dots, f_l)$ be an operation with output X. This means that the $f_i\colon B_i \to X$ are Sim_X–embeddings (ie f_i yields a similarity in Sim_X when the codomain is restricted to the image) such that the images of the f_i are pairwise disjoint and their union is X. So the images $\mathrm{im}(f_i)$ form a partition $\mathcal{P}$ of X into balls. If we apply [9, Lemma 3.7] to this partition, we find a j and a small ball B which is Sim_X–equivalent to X such that $B \subset \mathrm{im}(f_j)$. Using this, we can construct an operation $\psi = (g_1, \dots, g_k)$ with codomain B_j such that $g_1\colon X \to B_j$. From Remark 2.4, it now follows that X is progressive. □

Consequently, Theorem 2.5 is applicable to dually contracting local similarity operads. Furthermore, the corresponding operad groups based at X are all infinite by Proposition 4.4 and nonamenable by Proposition 4.3.

5A L^2–homology

For a group G, let l^2G be the Hilbert space with Hilbert base G. Thus elements in l^2G are formal sums $\sum_{g\in G} \lambda_g g$ with $\lambda_g \in \mathbb{C}$ such that $\sum_{g\in G} |\lambda_g|^2 < \infty$. Left multiplication with elements in G induces an isometric G–action on l^2G. Denote the set of G–equivariant linear bounded operators $l^2G \to l^2G$ by $\mathcal{B}^G(l^2G)$, a subalgebra of the algebra of all bounded linear operators $\mathcal{B}(l^2G)$. *Right* multiplication with an element $\gamma \in G$ induces a G–equivariant linear bounded operator $\gamma \triangleright \rho\colon l^2G \to l^2G$. This induces a homomorphism $\rho\colon \mathbb{C}G \to \mathcal{B}(l^2G)$ from the group ring into the algebra of bounded linear operators, ie $1 \triangleright \rho = \mathrm{id}$, and $(\gamma_1\gamma_2) \triangleright \rho = (\gamma_1 \triangleright \rho) * (\gamma_2 \triangleright \rho)$. The closure of the image of this map with respect to the weak or strong operator topology is called the von Neumann algebra $\mathcal{N}G$ associated to G. It is equal to the subalgebra of all G–equivariant bounded linear operators $\mathcal{B}^G(l^2G) \subset \mathcal{B}(l^2G)$ [8, Example 9.7].

We will cite some known results about this von Neumann algebra in order to deduce a corollary for l^2–homology.

- **$\mathcal{N}$ is inductive** Let H be a subgroup of G, and let $A \in \mathcal{B}^H(l^2H)$. Then $\mathbb{C}G \otimes_{\mathbb{C}H} l^2H \subset l^2G$ is a dense G–invariant subspace, and
$$\mathrm{id}_{\mathbb{C}G} \otimes_{\mathbb{C}H} A\colon \mathbb{C}G \otimes_{\mathbb{C}H} l^2H \to \mathbb{C}G \otimes_{\mathbb{C}H} l^2H$$
is a G–equivariant linear map which is bounded with respect to the norm coming from l^2G. Consequently, it can be extended to an element in $\mathcal{B}^G(l^2G)$. We obtain a map $\mathcal{N}H \to \mathcal{N}G$ which is an injective ring homomorphism. So if $H < G$, then $\mathcal{N}H$ is a subring of $\mathcal{N}G$. Even more is true: it is a faithfully flat ring extension [8, Theorem 6.29]. From this, it follows easily that the coefficient system $\mathcal{N}$ is inductive.
- **$\mathcal{N}$ is Künneth** If H_1 and H_2 are two subgroups of G which commute in G, ie $h_1h_2 = h_2h_1$ for all $h_1 \in H_1$ and $h_2 \in H_2$, then $\mathcal{N}H_1$ and $\mathcal{N}H_2$ commute in $\mathcal{N}G$. In particular, $\mathcal{N}H_1 \otimes_{\mathbb{C}} \mathcal{N}H_2$ is a subring of $\mathcal{N}G$. This implies, using a standard homological algebraic argument [8, Lemma 12.11(3)], that $\mathcal{N}$ is Künneth.
- **H_0 and amenability** Going back to a result of Kesten, the 0th group homology of a group G with coefficients in the von Neumann algebra $\mathcal{N}G$ vanishes if and only if G is nonamenable [8, Lemma 6.36]. So we have
$$H_0(G, \mathcal{N}G) = 0 \quad \iff \quad G \text{ nonamenable}.$$
- **Relationship with l^2–homology** From either [8, Lemma 6.97] or [8, Theorem 6.24(3)] we get, for groups G of type F_∞ and for every $k \geq 0$,
$$H_k(G, \mathcal{N}G) = 0 \quad \iff \quad H_k(G, l^2G) = 0.$$

Applying Theorem 2.5 to these observations, we get the following corollary.

Corollary 5.2 *Let $\mathcal{O}$ be a planar or symmetric operad which satisfies the calculus of fractions. Let X be a split progressive object of $\mathcal{S}(\mathcal{O})$. Set $\Gamma := \pi_1(\mathcal{O}, X)$, and assume that Γ is nonamenable. Then*
$$H_k(\Gamma, \mathcal{N}\Gamma) = 0$$

for all $k \geq 0$. *If* Γ *is also of type* F_∞ *(eg if the conditions in [10, Theorem 4.3] are satisfied), we also have*

$$H_k(\Gamma, l^2\Gamma) = 0$$

for all $k \geq 0$.

From Proposition 4.3, we get the following corollary.

Corollary 5.3 *Let* $\mathcal{O}$ *be a symmetric operad which satisfies the calculus of fractions. Let* X *be a split progressive object of* $\mathcal{S}(\mathcal{O})$. *Set* $\Gamma := \pi_1(\mathcal{O}, X)$. *Then*

$$H_k(\Gamma, \mathcal{N}\Gamma) = 0$$

for all $k \geq 0$. *If* Γ *is also of type* F_∞, *we have*

$$H_k(\Gamma, l^2\Gamma) = 0$$

for all $k \geq 0$.

From the remarks at the beginning of this section and from [10, Section 4.6], we get the following corollary.

Corollary 5.4 *Let* $\mathcal{O}$ *be a symmetric cube cutting operad or a local similarity operad coming from a dually contracting finite similarity structure* Sim_X. *In the first case, let* A *be any object in* $\mathcal{S}(\mathcal{O})$ *different from the monoidal unit* I. *In the second case, let* A *be the object* X. *Set* $\Gamma = \pi_1(\mathcal{O}, A)$. *Then*

$$H_k(\Gamma, \mathcal{N}\Gamma) = 0$$

for all $k \geq 0$. *Assume furthermore that* Sim_X *is rich in ball contractions [6, Definition 5.12], in other words, the associated operad* $\mathcal{O}$ *is color-tame in the sense of [10, Definition 4.2]. Then we also have*

$$H_k(\Gamma, l^2\Gamma) = 0$$

for all $k \geq 0$.

In particular, we obtain that the Higman–Thompson groups $V_{n,r}$ and the higher-dimensional Thompson groups nV (see [1]) are l^2–invisible. This answers a question posed by Lück [8, Remark 12.4]: The *Zero-in-the-spectrum conjecture* by Gromov says that, whenever M is an aspherical closed Riemannian manifold, there is always a dimension p such that zero is contained in the spectrum of the minimal closure of the Laplacian acting on smooth p–forms on the universal covering of M:

$$\exists_{p \geq 0} \;\; 0 \in \mathrm{spec}(\mathrm{cl}(\Delta_p)\colon D \subset L^2\Omega^p(\widetilde{M}) \to L^2\Omega^p(\widetilde{M})).$$

By [8, Lemma 12.3], this is equivalent to

$$\exists_{p\geq 0}\ H_p(G,\mathcal{N}G)\neq 0$$

for $G=\pi_1(M)$. Dropping Poincaré duality from the assumptions, we arrive at the following question: if G is a group of type F (ie there exists a compact classifying space for G), then is there a p with $H_p(G,\mathcal{N}G)\neq 0$? Relaxing the assumption on the finiteness property, we arrive at the following question: if G is a group of type F_∞, then is there a p with $H_p(G,\mathcal{N}G)\neq 0$? Corollary 5.4 gives explicit counterexamples to this question.

5B Cohomology with coefficients in the group ring

We want to apply the cohomological version of Theorem 2.5 to $\mathcal{M}G:=\mathbb{Z}G$. To this end, we want to show that $\mathbb{Z}G$ is FP_∞–Künneth and FP_∞–inductive (in cohomology). The first property follows from [9, Proposition 4.3]. The second property follows from the observation that $\mathbb{Z}G$ is a free $\mathbb{Z}H$–module if $H<G$, and that group cohomology of groups of type FP_∞ commutes with direct limits in the coefficients [3, Theorem VIII.4.8]. From Theorem 2.5, Proposition 4.4 and the fact that $H^0(G,\mathbb{Z}G)=(\mathbb{Z}G)^G=0$ for infinite G, we obtain:

Corollary 5.5 *Let $\mathcal{O}$ be a planar or symmetric operad which satisfies the calculus of fractions. Let X be a split progressive object of $\mathcal{S}(\mathcal{O})$. Set $\Gamma:=\pi_1(\mathcal{O},X)$ and assume that Γ is of type FP_∞* (eg if the conditions in [10, Theorem 4.3] are satisfied). *Then*

$$H^k(\Gamma,\mathbb{Z}\Gamma)=0$$

for all $k\geq 0$.

Recall that type F_∞ implies type FP_∞, and note that, in this case, $H^k(\Gamma,\mathbb{Z}\Gamma)=0$ for all $k\geq 0$ implies that Γ has infinite cohomological dimension [3, Propositions VIII.6.1 and VIII.6.7]. Unfortunately, this tells us that none of these groups can be of type F, and consequently, we cannot find such a group which is also l^2–invisible.

From the remarks at the beginning of this section and from [10, Section 4.6], we get the following corollary.

Corollary 5.6 *Let $\mathcal{O}$ be a planar or symmetric cube cutting operad or a local similarity operad coming from a dually contracting finite similarity structure* Sim_X *which is also rich in ball contractions. In the first two cases, let A be any object in $\mathcal{S}(\mathcal{O})$ different from the monoidal unit I. In the last case, let A be the object X. Set $\Gamma=\pi_1(\mathcal{O},A)$. Then*

$$H^k(\Gamma,\mathbb{Z}\Gamma)=0$$

for all $k\geq 0$.

In particular, we obtain $H^k(F, \mathbb{Z}F) = 0$ and $H^k(V, \mathbb{Z}V) = 0$ for all $k \geq 0$. This has been shown before in [4, Theorem 7.2] and in [2, Theorem 4.21].

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