

Homological stability for automorphism groups of RAAGs

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We show that the homology of the automorphism group of a right-angled Artin group stabilizes under taking products with any right-angled Artin group.

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Introduction

It has been conjectured that, for any (finitely generated) discrete group G , the homology groups $H_i(\text{Aut}(G^{*n}); \mathbb{Z})$ and $H_i(\text{Aut}(G^n); \mathbb{Z})$ should be independent of n , for $n \gg i$, generalizing the classical stability results for $\text{GL}_n(\mathbb{Z})$ and $\text{Aut}(F_n)$ when $G = \mathbb{Z}$. (See the conjectures of Hatcher and Wahl [9, Conjecture 1.4], Wahl and Randal-Williams [17, Conjecture 5.16], and the classical results in Charney [1], Hatcher [7], Hatcher and Vogtmann [8], Maazen [15], and van der Kallen [13].)

The stabilization of $H_i(\text{Aut}(G^{*n}); \mathbb{Z})$ for i large has been shown to hold for most groups by the main theorem of Collinet, Djament and Griffin [3], and [9, Corollary 1.3].¹ The stabilization of $H_i(\text{Aut}(G^n); \mathbb{Z})$ in contrast has so far only been known in two extreme cases: when G is abelian and when G has trivial center and does not factorize as a direct product. Indeed, in the first case $\text{Aut}(G^n)$ is isomorphic to $\text{GL}_n(\text{End}(G))$, which is known to stabilize (see Proposition 5.2), while in the second case, the group $\text{Aut}(G^n)$ is isomorphic to $\text{Aut}(G) \wr \Sigma_n$ (see Johnson [12]), a group that is also known to stabilize [9, Proposition 1.6].² In the present paper, we verify that the second conjecture holds for G any right-angled Artin group, possibly factorizable, possibly with a nontrivial center. This proves a first “mixed case” of the conjecture, which interpolates between the two previously known cases.

A *right-angled Artin group* (or *RAAG*) is a group with a finite set of generators s_1, \dots, s_n and relations that are commutation relations between the generators, ie relations of the

¹[3] gives stability for $\text{Aut}(G^{*n})$ with G any group with a finite free product decomposition (eg. a finitely generated group) without \mathbb{Z} factor, while [9] treats the cases with G arising as fundamental groups of certain 3-manifolds, allowing \mathbb{Z} factors in the free product decomposition.

²Slightly more generally, for the second case, one can get stability for $\text{Aut}(G^n)$ for G a product of certain such center-free groups using [12].

form $s_i s_j = s_j s_i$ for certain pairs $\{i, j\}$. The extreme examples of RAAGs are the free groups F_n if no commutation relation holds, and the free abelian groups \mathbb{Z}^n if all commutation relations hold. Given any two RAAGs A and B , their product is again a RAAG. We consider in the present paper the sequence of groups $G_n = \text{Aut}(A \times B^n)$ associated to A and B , and the sequence of maps

$$\sigma_n: G_n = \text{Aut}(A \times B^n) \rightarrow G_{n+1} = \text{Aut}(A \times B^{n+1})$$

taking an automorphism f of $A \times B^n$ to the automorphism $f \times B$ of $A \times B^{n+1}$ leaving the last B factor fixed. Note that when A is the trivial group, the group $G_n = \text{Aut}(B^n)$ is a group as in the second conjecture above.

Our main result is the following:

Theorem A (stability with constant coefficients) *Let A, B be any RAAGs. The map*

$$H_i(\text{Aut}(A \times B^n); \mathbb{Z}) \rightarrow H_i(\text{Aut}(A \times B^{n+1}); \mathbb{Z})$$

induced by σ_n is surjective for all $i \leq \frac{1}{2}(n - 1)$ and an isomorphism for $i \leq \frac{1}{2}(n - 2)$. If B has no \mathbb{Z} -factors, then surjectivity holds for $i \leq \frac{1}{2}(n)$ and injectivity holds for $i \leq \frac{1}{2}(n - 1)$.

We prove this stability theorem using the general method developed by Randal-Williams and the second author in [17]. This method provides a more general stability result, namely stability in homology not only with constant coefficients \mathbb{Z} as above, but also with both *polynomial* and *abelian* coefficients, and we establish our main result also in this level of generality as Theorem 5.1. Theorems B and C are further special cases of Theorem 5.1.

Stability for $\text{Aut}(A \times B^n)$ with the (abelian) coefficients $H_1(\text{Aut}(A \times B^n))$ implies the following:

Theorem B (stability for commutator subgroups) *Let A, B be any RAAGs and let $\text{Aut}'(A \times B^n)$ denote the commutator subgroup of $\text{Aut}(A \times B^n)$. The map*

$$H_i(\text{Aut}'(A \times B^n); \mathbb{Z}) \rightarrow H_i(\text{Aut}'(A \times B^{n+1}); \mathbb{Z})$$

induced by σ_n is surjective for all $i \leq \frac{1}{3}(n - 2)$ and an isomorphism for $i \leq \frac{1}{3}(n - 4)$. If B has no \mathbb{Z} -factors, then surjectivity holds for $i \leq \frac{1}{3}(n - 1)$ and injectivity holds for $i \leq \frac{1}{3}(n - 3)$.

An example of a polynomial coefficient system for the groups $\text{Aut}(A \times B^n)$ is the sequence of “standard” representations $H_1(A \times B^n)$, and stability with polynomial coefficients yields the following in that case:

Theorem C (stability with coefficients in the standard representation) *Let A, B be any RAAGs. Then the map*

$$H_i(\text{Aut}(A \times B^n); H_1(A \times B^n)) \rightarrow H_i(\text{Aut}(A \times B^{n+1}); H_1(A \times B^{n+1}))$$

is surjective for all $i \leq \frac{1}{2}(n-2)$ and an isomorphism for $i \leq \frac{1}{2}(n-3)$. If B has no \mathbb{Z} -factors, then surjectivity holds for $i \leq \frac{1}{2}(n-1)$ and injectivity for $i \leq \frac{1}{2}(n-2)$.

To prove the above theorems, we show that right-angled Artin groups under direct product fit in the setup of homogeneous categories developed in [17], and recalled here in Section 1. The main ingredient of stability is the high connectivity of certain semi-simplicial sets $W_n(A, B)$ associated to the sequence of groups $\text{Aut}(A \times B^n)$. We define and study those semi-simplicial sets in Section 2, together with three closely related simplicial complexes $I_n(A, B)$, $SI_n(A, B)$ and $S_n(A, B)$. Sections 1 and 2 are written in the general context of families of groups closed under direct product. In Section 3, we show that right-angled Artin groups admit a “prime decomposition” with respect to direct product, and we give a description of the automorphism group of such a group in terms of this decomposition. Section 4 then uses these results that are specific to RAAGs together with the complexes defined in Section 2 to prove that the semi-simplicial sets $W_n(A, B)$ are highly connected. For the connectivity results, we use *join complex* methods from [9], as well as an argument of Maazen [15] for the case $B = \mathbb{Z}$. Finally Section 5 states the general stability result, which, given the connectivity result, is a direct application of the main result in [17].

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1 Families of groups

We consider here families of groups \mathcal{F} which are closed under direct product. We say that \mathcal{F} satisfies *cancellation* if for all A, B, C in \mathcal{F} , we have that

$$A \times C \cong B \times C \implies A \cong B.$$

Cancellation is not satisfied for the family of all finitely generated groups; see eg [11, Section 3] or [10] for an example where cancellation with \mathbb{Z} fails. Cancellation holds for the family of all finitely generated abelian groups by their classification, the family

of all finite groups [10], or for the family of all right angled Artin groups as we will show in Section 3.

Given a family of groups \mathcal{F} , we let $\mathcal{G}_{\mathcal{F}}$ denote its associated groupoid, namely the groupoid with objects the elements of \mathcal{F} and morphisms all group isomorphisms. Let e denote the trivial group. When \mathcal{F} is closed under direct product, we have that $(\mathcal{G}_{\mathcal{F}}, \times, e)$ is a symmetric monoidal groupoid.

Recall from [17, Section 1.1] and [6, page 219] the category $U\mathcal{G}_{\mathcal{F}} = \langle \mathcal{G}_{\mathcal{F}}, \mathcal{G}_{\mathcal{F}} \rangle$ associated to $(\mathcal{G}_{\mathcal{F}}, \times, e)$: it has the same objects as $\mathcal{G}_{\mathcal{F}}$, namely the elements of \mathcal{F} , and morphisms from A to B given as pairs (X, f) , where $X \in \mathcal{F}$ and $f: X \times A \xrightarrow{\cong} B$ is an isomorphism, up to the equivalence relation that $(X, f) \sim (X', f')$ if there exists an isomorphism $\phi: X \rightarrow X'$ such that $f = f' \circ (\phi \times A)$.

Recall from [17, Definition 1.2] that a monoidal category $(\mathcal{C}, \oplus, 0)$ is called *homogeneous* if 0 is initial in \mathcal{C} and for every A, B in \mathcal{C} , the following two properties hold:

- H1** $\text{Hom}(A, B)$ is a transitive $\text{Aut}(B)$ -set under postcomposition.
- H2** The map $\text{Aut}(A) \rightarrow \text{Aut}(A \oplus B)$ taking f to $f \oplus B$ is injective with image $\text{Fix}(B, A \oplus B)$.

Here $\text{Fix}(B, A \oplus B)$ is the set of $\phi \in \text{Aut}(A \oplus B)$ satisfying that $\phi \circ (\iota_A \oplus B) = \iota_A \oplus B$ in $\text{Hom}(B, A \oplus B)$, for $\iota_A: 0 \rightarrow A$ the unique morphism.

Proposition 1.1 *If \mathcal{F} satisfies cancellation, then the category $U\mathcal{G}_{\mathcal{F}}$ is a symmetric monoidal homogeneous category whose underlying groupoid is $\mathcal{G}_{\mathcal{F}}$.*

Proof As $(\mathcal{G}_{\mathcal{F}}, \times, e)$ is symmetric monoidal, $U\mathcal{G}_{\mathcal{F}}$ is symmetric monoidal by [17, Proposition 1.6], and e is initial in $U\mathcal{G}_{\mathcal{F}}$. We have that $\mathcal{G}_{\mathcal{F}}$ satisfies cancellation by assumption, and for any $A, B \in \mathcal{F}$, the map $\text{Aut}_{\mathcal{G}_{\mathcal{F}}}(A) \rightarrow \text{Aut}_{\mathcal{G}_{\mathcal{F}}}(A \times B)$ taking f to $f \times B$ is injective. Then [17, Theorem 1.8] implies that $U\mathcal{G}_{\mathcal{F}}$ is a homogeneous category. Finally, if $A \times B \cong e$, we must have $A = B = e$ and the unit e has no nontrivial automorphisms. Hence $\mathcal{G}_{\mathcal{F}}$ satisfies the hypothesis of Proposition 1.10 in [17], which gives that $\mathcal{G}_{\mathcal{F}}$ is the underlying groupoid of $U\mathcal{G}_{\mathcal{F}}$. \square

Remark 1.2 If one wants to consider a family \mathcal{F} that does not satisfy cancellation, one can replace $\mathcal{G}_{\mathcal{F}}$ by a groupoid that does satisfy cancellation (by forgetting that certain objects are isomorphic) and obtain an associated homogeneous category. For simplicity, we will only consider families satisfying cancellation.

We end the section by showing that the homogeneous categories $UG_{\mathcal{F}}$ considered here are not pathological in the sense that they satisfy the following *standardness* property: Let $(\mathcal{C}, \oplus, 0)$ be a homogeneous category and (A, X) a pair of objects in \mathcal{C} . We say that \mathcal{C} is *locally standard at (A, X)* [17, Definition 2.5] if:

- LS1** The morphisms $\iota_A \oplus X \oplus \iota_X$ and $\iota_{A \oplus X} \oplus X$ are distinct in $\text{Hom}(X, A \oplus X^{\oplus 2})$.
- LS2** For all $n \geq 1$, the map $\text{Hom}(X, A \oplus X^{\oplus n-1}) \rightarrow \text{Hom}(X, A \oplus X^{\oplus n})$ taking f to $f \oplus \iota_X$ is injective.

Proposition 1.3 *For any family \mathcal{F} , the category $UG_{\mathcal{F}}$ is locally standard at any pair (A, X) .*

To prove this proposition, it is easiest to use an alternative description of the morphisms in the category $UG_{\mathcal{F}}$, given by the following:

Lemma 1.4 *The association $[X, f] \mapsto (f(X), f|_A)$ defines a one-to-one correspondence between $\text{Hom}_{UG_{\mathcal{F}}}(A, B)$ and the set of pairs (H, g) with $H \leq B$ and $g: A \rightarrow B$ an injective homomorphism such that $B = H \times g(A)$.*

Proof of Lemma 1.4 First note that both $f(X)$ and $f|_A$ are independent of the representative of $[X, f]$, so the association is well-defined.

Suppose $[X, f]$ and $[Y, g]$ are morphisms from A to B in $UG_{\mathcal{F}}$ satisfying that $(f(X), f|_A) = (g(Y), g|_A)$. Then $g^{-1}|_{f(X)} \circ f|_X: X \rightarrow Y$ is an isomorphism and $f = g \circ ((g^{-1}|_{f(X)} \circ f|_X) \times A)$ as both maps agree on their restrictions to X and A . Hence $[X, f] = [Y, g]$.

We are left to check that any (H, g) is in the image. This follows from the fact that, given such an (H, g) , the map $H \times g: H \times A \rightarrow B$ is an isomorphism. □

Proof of Proposition 1.3 We need to check the two axioms LS1 and LS2. For LS1, we need that the maps $\iota_A \times X \times \iota_X$ and $\iota_{A \times X} \times X$ from X to $A \times X^2$ in $UG_{\mathcal{F}}$ are distinct. From the definition of the monoidal structure in $UG_{\mathcal{F}}$ given in the proof of Proposition 1.6 of [17], we have that $\iota_A \times X \times \iota_X = [A \times X, A \times b_{X,X}^{-1}]$ and $\iota_{A \times X} \times X = [A \times X, \text{id}_{A \times X^2}]$, where $b_{X,X} = b_{X,X}^{-1}: X^2 \rightarrow X^2$ denotes the symmetry. The fact that they are distinct then follows from the lemma as, for example, $(A \times b_{X,X}^{-1})|_{e \times e \times X} \neq \text{id}_{A \times X^2}|_{e \times e \times X}$.

For LS2, we need to show that the map $- \times \iota_X: \text{Hom}(X, A \times X^{n-1}) \rightarrow \text{Hom}(X, A \times X^n)$ is injective. This follows again from Lemma 1.4 as $(H, f) \times \iota_X = (H \times i_n(X), f)$ in the description of the morphisms given by the lemma, where $i_n(X) \leq A \times X^n$ denotes the last X factor. This association is injective. □

2 Simplicial complexes and semi-simplicial sets associated to a family of groups

To a family of groups \mathcal{F} closed under direct product, we associated in the previous section a category $U\mathcal{G}_{\mathcal{F}}$ with objects the elements of \mathcal{F} . Using the morphism sets in this category, the paper [17] associates to any pair of objects $A, X \in \mathcal{F}$ and any $n \geq 0$, a semi-simplicial set $W_n(A, X)$ and a simplicial complex $S_n(A, X)$. In the present section, we recall the definitions of $S_n(A, X)$ and $W_n(A, X)$ and introduce new simplicial complexes $I_n(A, X)$ and $SI_n(A, X)$ likewise associated to $A, X \in \mathcal{F}$. We then study the relationship between these four different simplicial objects. To prove homological stability, we will need to show that the semi-simplicial sets $W_n(A, X)$ are highly connected. This will be done in Section 4 in the case of the family of all right-angled Artin groups using the three simplicial complexes introduced here. We give in the present section results that allow transfer of connectivity from one of the above spaces to another that work in a general context and that will be combined in Section 4 with results specific to right-angled Artin groups. For simplicity, we will again assume that \mathcal{F} satisfies cancellation.

Standing assumption for the section \mathcal{F} is a family of finitely generated groups, closed under direct product, and satisfying cancellation.

Given groups A, X in \mathcal{F} , we will consider injective maps $f: X^k \rightarrow A \times X^n$ so that there is a splitting $A \times X^n = f(X^k) \times H$ with H in \mathcal{F} . As \mathcal{F} satisfies cancellation, we always have that $H \cong A \times X^{n-k}$. We call such a map f an \mathcal{F} -split map, and we call the pair (f, H) an \mathcal{F} -splitting.

Recall that a *simplicial complex* Y is defined from a set of vertices Y_0 by giving a collection of finite subsets of Y_0 which are closed under taking subsets. The subsets of cardinality $p+1$ are called the p -simplices of Y . On the other hand, a *semi-simplicial set* W is a collection of sets W_p of p -simplices for each $p \geq 0$ related by boundary maps $d_i: W_p \rightarrow W_{p-1}$ for each $0 \leq i \leq p$ satisfying the simplicial identities. Both simplicial complexes and semi-simplicial sets admit a realization that has a copy of Δ^p for each p -simplex of the simplicial object. When we talk about connectivity of such objects, we always refer to the connectivity of their realization.

We define now three simplicial complexes and one semi-simplicial set whose objects are either \mathcal{F} -split maps or \mathcal{F} -splittings.

Definition 2.1 To a pair of groups $X, A \in \mathcal{F}$ and a natural number $n \geq 0$, we associate the following simplicial complexes:

- $I_n(A, X)$ A vertex in $I_n(A, X)$ is an \mathcal{F} -split map $f: X \rightarrow A \times X^n$. Distinct vertices f_0, \dots, f_p form a p -simplex in $I_n(A, X)$ whenever the map $(f_0, \dots, f_p): X^{p+1} \rightarrow A \times X^n$ is \mathcal{F} -split.
- $SI_n(A, X)$ A vertex in the simplicial complex $SI_n(A, X)$ is an \mathcal{F} -splitting (f, H) with $f \in I_n(A, X)$. Distinct vertices $(f_0, H_0), \dots, (f_p, H_p)$ form a p -simplex of $SI_n(A, X)$ if $\langle f_0, \dots, f_p \rangle$ is a p -simplex of $I_n(A, X)$ and $f_i(X) \leq H_j$ for each $i \neq j$.
- $S_n(A, X)$ The vertices of $S_n(A, X)$ are the same as those of $SI_n(A, X)$. Distinct vertices $(f_0, H_0), \dots, (f_p, H_p)$ form a p -simplex of $S_n(A, X)$ if there exists an \mathcal{F} -splitting (f, H) , with $f = (f_0, \dots, f_p): X^{p+1} \rightarrow A \times X^n$, such that $H_j = H \times \prod_{i \neq j} f_i(X)$ for each j .

We moreover associate the following semi-simplicial set:

- $W_n(A, X)$ A p -simplex in $W_n(A, X)$ is an \mathcal{F} -splitting (f, H) , with f a map $X^{p+1} \rightarrow A \times X^n$. The j^{th} face $d_j(f, H)$ is given by $(f \circ d^j, H \times f(i_j))$ for $d^j: X^p \rightarrow X^{p+1}$ which is the map skipping the $(j + 1)^{\text{st}}$ factor and $i_j = \iota_{X^j} \times X \times \iota_{X^{p-j}}: X \rightarrow X^{p+1}$.

Using Lemma 1.4, one checks immediately that $W_n(A, X)$ identifies with the semi-simplicial set of [17, Definition 2.1] associated to the category $UG_{\mathcal{F}}$, and $S_n(A, X)$ identifies with the simplicial complex of [17, Definition 2.8] likewise associated to $UG_{\mathcal{F}}$.

The following proposition shows that, in the context we work with, we can always approach the connectivity of $W_n(A, X)$ via that of $S_n(A, X)$.

Proposition 2.2 *Let \mathcal{F} be a family of groups satisfying cancellation and let $a, k \geq 1$. The simplicial complex $S_n(A, X)$ is $\frac{n-a}{k}$ -connected for all $n \geq 0$ if and only if the semi-simplicial set $W_n(A, X)$ is $\frac{n-a}{k}$ -connected for all $n \geq 0$.*

Proof As $UG_{\mathcal{F}}$ is symmetric monoidal, homogeneous (Proposition 1.1) and locally standard (Proposition 1.3), Proposition 2.9 of [17] yields that the semi-simplicial sets $W_n(A, X)$ satisfy condition (A) in that paper (see [17, Section 2.1]). The result then follows from [17, Theorem 2.10]. □

Note that there is an inclusion of simplicial complexes

$$S_n(A, X) \hookrightarrow SI_n(A, X).$$

Indeed, the two complexes have the same set of vertices, and simplices of $S_n(A, X)$ satisfy the condition for being a simplex of $SI_n(A, X)$. There is also a forgetful map

$$SI_n(A, X) \rightarrow I_n(A, X).$$

Recall from [9, Definition 3.2] that a *join complex* over a simplicial complex X is a simplicial complex Y together with a simplicial map $\pi: Y \rightarrow X$ satisfying the following properties:

- (1) π is surjective.
- (2) π is injective on individual simplices.
- (3) For each p -simplex $\sigma = \langle x_0, \dots, x_p \rangle$ of X the subcomplex $Y(\sigma)$ of Y consisting of all the p -simplices that project to σ is the join $Y_{x_0}(\sigma) * \dots * Y_{x_p}(\sigma)$ of the vertex sets $Y_{x_i}(\sigma) = Y(\sigma) \cap \pi^{-1}(x_i)$.

We say that Y is a *complete join* over X if $Y_{x_i}(\sigma) = \pi^{-1}(x_i)$ for each σ and each x_i .

Join complexes usually arise via *labeling systems* (see [9, Example 3.3]): a labeling system for a simplicial complex X is a collection of nonempty sets $L_x(\sigma)$ for each simplex σ of X and each vertex x of σ , satisfying $L_x(\tau) \supset L_x(\sigma)$ whenever $x \in \tau \subset \sigma$. One can think of $L_x(\sigma)$ as the set of labels of x that are compatible with σ . We can use the labeling system L to define a new simplicial complex X^L having vertices the pairs (x, l) with $x \in X$ and $l \in L_x(\langle x \rangle)$. A collection of pairs $((x_0, l_0), \dots, (x_p, l_p))$ then forms a p -simplex of X^L if and only if $\sigma = \langle x_0, \dots, x_p \rangle$ is a p -simplex of X and $l_i \in L_{x_i}(\sigma)$ for each i . Then the natural map $\pi: X^L \rightarrow X$ forgetting the labels represents X^L as a join complex over X .

Proposition 2.3 *The complex $SI_n(A, X)$ is a join complex over $I_n(A, X)$.*

Proof We check that $SI_n(A, X)$ can be constructed from $I_n(A, X)$ via a labeling system in the sense described above. For each simplex $\sigma = \langle f_0, \dots, f_p \rangle$ of $I_n(A, X)$ and each vertex f_i in σ , we define the set of labels of f_i compatible with σ as

$$L_{f_i}(\sigma) := \{H \leq A \times X^n : (f_i, H) \in SI_n(A, X), f_j(X) \leq H \text{ for each } f_j \neq f_i \in \sigma\}.$$

These sets are nonempty because the fact that $\langle f_0, \dots, f_p \rangle$ is a simplex of $I_n(A, X)$ implies that there exists an \mathcal{F} -splitting (f, H) with

$$f = (f_0, \dots, f_p): X^{p+1} \rightarrow A \times X^n = H \times f(X^{p+1}).$$

Let $H_i = H \times \prod_{j \neq i} f_j(X)$. Then $H_i \in L_{f_i}(\sigma)$. We clearly have that for any $f_i \in \tau \subset \sigma$, $L_{f_i}(\tau) \supset L_{f_i}(\sigma)$, and $SI_n(A, X) = (I_n(A, X))^L$. □

This will allow us to use results from [9] to obtain in good cases a connectivity bound for $SI_n(A, X)$ from one for $I_n(A, X)$.

We now show that, under one additional assumption, $S_n(A, X)$ and $SI_n(A, X)$ are isomorphic, in which case we will also get a connectivity result for S_n from that of SI_n .

Proposition 2.4 *Suppose that for any simplex $\langle (f_0, H_0), \dots, (f_p, H_p) \rangle$ of $SI_n(A, X)$ we have that $\bigcap_{i=0}^p H_i \in \mathcal{F}$. Then the inclusion $S_n(A, X) \hookrightarrow SI_n(A, X)$ is an isomorphism.*

Lemma 2.5 *Suppose A, B, A', B' are groups such that $A \times B = A' \times B'$ and $A' \leq A$. Then $A = A' \times (B' \cap A)$.*

Proof Consider the inclusion $A' \times (B' \cap A) \rightarrow A$. This is an injective group homomorphism. Now every $a \in A \leq A \times B = A' \times B'$ can be written as $a = a'b'$ with $a' \in A'$ and $b' \in B'$. But then $b' = (a')^{-1}a \in A$ and hence $a \in A' \times (B' \cap A)$ and the map is also surjective. \square

Proof of Proposition 2.4 Recall that $S_n(A, X)$ and $SI_n(A, X)$ have the same set of vertices, and that there is an inclusion $S_n(A, X) \hookrightarrow SI_n(A, X)$, that is simplices of $S_n(A, X)$ are also simplices in $SI_n(A, X)$. So we are left to check that simplices of $SI_n(A, X)$ are also always simplices in $S_n(A, X)$. So consider a p -simplex $\langle (g_0, K_0), \dots, (g_p, K_p) \rangle$ of $SI_n(A, X)$. We have that

$$g = (g_0, \dots, g_p): X^{p+1} \rightarrow A \times X^n$$

is split injective. To show that these vertices form a p -simplex in $S_n(A, X)$, we need to find a complement $K \leq A \times X^n$ for g with $K \in \mathcal{F}$ satisfying that

$$(1) \quad K_j = K \times \prod_{i \neq j} g_i(X).$$

Note that if K satisfies (1), it necessarily is a complement for g as $K_j \times g_j(X) = A \times X^n$ for each j . Let $K = \bigcap_j K_j \leq A \times X^n$. By the assumption, we have that $K \in \mathcal{F}$. We will now check that it satisfies (1), which will finish the proof. By renaming the factors, it is enough to prove that equation (1) holds for $j = 0$. We do it by induction: we start with $K_0 = K_0$. Suppose $r \geq 2$ and assume that we have proved that $K_0 = \bigcap_{j=0}^{r-1} K_j \times g_1(X) \times \dots \times g_{r-1}(X)$. We have that

$$\bigcap_{j=0}^{r-1} K_j \times g_1(X) \times \dots \times g_{r-1}(X) \times g_0(X) = K_0 \times g_0(X) = A \times X^n = K_r \times g_r(X).$$

Now $g_r(X) \leq K_j$ for all $j = 0, \dots, r - 1$. Applying the lemma, we thus get that $\bigcap_{j=0}^{r-1} K_j = g_r(X) \times ((\bigcap_{j=0}^{r-1} K_j) \cap K_r)$, which gives the induction step. \square

The following proposition will also be useful in the sequel:

Proposition 2.6 *The action of $\text{Aut}(A \times X^n)$ on $A \times X^n$ induces an action on the complexes $I_n(A, X)$ and $S_n(A, X)$ which is transitive on the set of p -simplices for every p in both cases, and the composed map $S_n(A, X) \rightarrow SI_n(A, X) \rightarrow I_n(A, X)$ is equivariant with respect to these actions.*

Proof The action is induced by postcomposition by automorphisms on the split maps $f: X^{p+1} \rightarrow A \times X^n$, and by evaluation on splittings $H \leq A \times X^n$. The map $S_n(A, X) \rightarrow I_n(A, X)$ forgets the choice of splitting and is hence equivariant. For $S_n(A, X)$, transitivity of the action is axiom H1 in the homogeneous category $U\mathcal{G}_{\mathcal{F}}$ [17, Definition 1.2], which is satisfied by Proposition 1.1. For $I_n(A, X)$, it follows from the corresponding fact for $S_n(A, X)$ and the fact that every simplex of $I_n(A, X)$ admits a lift in $S_n(A, X)$. \square

3 RAAGs and their groups of automorphisms

Now we consider the family \mathcal{F} of all right-angled Artin groups, and give in this section a few properties that are particular to these groups and that will allow us to prove the connectivity result necessary for stability. In particular, we show that the family of RAAGs satisfies cancellation and give a description of the automorphism group of a direct product of RAAGs in terms of the automorphism groups of its factors. We start by recalling what a RAAG is.

Given a finite simplicial graph Γ one can associate a group A_{Γ} with one generator v for each vertex of Γ and a commuting relation $vw = wv$ for each edge (v, w) in Γ . Such a group A_{Γ} is called a graph group or more commonly *right-angled Artin group*. The main theorem of [4] says that the graph describing such a group is unique in the sense that two such groups A_{Γ} and $A_{\Gamma'}$ are isomorphic if and only if the graphs Γ and Γ' are isomorphic.

The next proposition says that RAAGs admit a “prime decomposition” with respect to direct product.

Proposition 3.1 *Any RAAG A_{Γ} admits a maximal decomposition as*

$$A_{\Gamma} = A_{\Gamma_1} \times \dots \times A_{\Gamma_k}$$

with each A_{Γ_i} a RAAG, and this decomposition is unique up to isomorphism and permutation of the factors.

Proof From Droms’ theorem [4], we have that the group A_Γ splits as a direct product $A_{\Gamma_1} \times A_{\Gamma_2} = A_{\Gamma_1 * \Gamma_2}$ if and only if the graph Γ is isomorphic to the join $\Gamma_1 * \Gamma_2$. This reduces the proposition to the existence and uniqueness of the maximal decomposition of a finite graph as a join. Let X be a finite simplicial graph. Since the graph is finite there exists a maximal join decomposition $X = X_1 * X_2 * \dots * X_n$, with $n < \infty$. Now suppose $X = X_1 * X_2 * \dots * X_n = Y_1 * Y_2 * \dots * Y_m$ are two distinct such decompositions. Let $k \geq 0$ be maximal such that there is a permutation $\sigma \in \Sigma_m$ with $X_i = Y_{\sigma(i)}$ for each $1 \leq i \leq k$. By maximality, X_{k+1} is distinct from all the remaining Y_i . We have that X_{k+1} cannot be contained in some Y_i otherwise Y_i would decompose as a proper join, a contradiction to the maximality of the join decomposition $Y_1 * Y_2 * \dots * Y_m$. Then X_{k+1} must intersect nontrivially r of the remaining Y_i and so it itself must split as an r -join, contradicting the maximality of the decomposition $X_1 * X_2 * \dots * X_n$. \square

Corollary 3.2 *The family of RAAGs satisfies cancellation with respect to direct product.*

We are here interested in the automorphism groups of RAAGs. The papers [16; 14] establish that the automorphism group $\text{Aut}(A_\Gamma)$ is generated by the following automorphisms:

- (1) **Graph automorphisms** Automorphisms of the graph Γ via a permutation of its set of vertices V .
- (2) **Inversions** For $v \in V$, a map sending $v \rightarrow v^{-1}$ and fixing all other generators.
- (3) **Transvections** For $v \neq w \in V$ such that $\text{Link}(v) \subseteq \text{Star}(w)$, a map sending $v \rightarrow vw$ and fixing all other generators.
- (4) **Partial conjugations** For $v \in V$ and C a component of $\Gamma \setminus \text{Star}(v)$, the map sending $x \rightarrow vxv^{-1}$ for every vertex x of C and fixing all other generators.

The next proposition builds on the work of Fullarton [5] to show how automorphisms of RAAGs interact with the direct product decomposition of a RAAG.

Proposition 3.3 *Given a RAAG A_Γ with maximal decomposition*

$$A_\Gamma = \mathbb{Z}^d \times (A_{\Gamma_1})^{i_1} \times \dots \times (A_{\Gamma_k})^{i_k},$$

with the A_{Γ_i} distinct and not equal to \mathbb{Z} , we have

$$\begin{aligned} \text{Aut}(A_\Gamma) &\cong \mathbb{Z}^{d|\Gamma'|} \rtimes (\text{GL}_d(\mathbb{Z}) \times \text{Aut}(A_{\Gamma'})) \\ &\cong \mathbb{Z}^{d|\Gamma'|} \rtimes (\text{GL}_d(\mathbb{Z}) \times (\text{Aut}(A_{\Gamma_1}) \wr \Sigma_{i_1}) \times \dots \times (\text{Aut}(A_{\Gamma_k}) \wr \Sigma_{i_k})) \end{aligned}$$

where $\Gamma' = (*_{i_1} \Gamma_1) * \dots * (*_{i_k} \Gamma_k)$ with $|\Gamma'|$ its number of vertices, and where $\mathbb{Z}^{d|\Gamma'|}$ is generated by transvections $v \rightarrow vz$ for $v \in \Gamma'$ and $z \in \mathbb{Z}^d$.

Proof The first isomorphism is given by Propositions 3.1 and 3.2 in [5]. To get the second isomorphism, we are left to study $\text{Aut}(A_{\Gamma'})$, the automorphism group of a RAAG $A_{\Gamma'}$ with no \mathbb{Z} -factor. If $A_{\Gamma'}$ is unfactorizable, there is nothing to show, so we assume that it is factorizable. We have an inclusion

$$(\text{Aut}(A_{\Gamma_1}) \wr \Sigma_{i_1}) \times \cdots \times (\text{Aut}(A_{\Gamma_k}) \wr \Sigma_{i_k}) \hookrightarrow \text{Aut}(A_{\Gamma'})$$

so all we need to check is that every automorphism of $A_{\Gamma'}$ comes from the left-hand side. We do this by inspecting the generators in the classification recalled above. We see that type (2) automorphisms are internal to each factor, ie are elements of some $\text{Aut}(A_{\Gamma_i})$. Type (3) can only be internal to a factor for $A_{\Gamma'}$ because $A_{\Gamma'}$ has no \mathbb{Z} -factors, and likewise for type (4) because $A_{\Gamma'}$ is a direct product. Finally, type (1) automorphisms, the graph automorphisms, satisfy

$$\text{Aut}(\Gamma') = \text{Aut}((\ast_{i_1} \Gamma_1) \ast \cdots \ast (\ast_{i_k} \Gamma_k)) = (\text{Aut}(\Gamma_1) \wr \Sigma_{i_1}) \times \cdots \times (\text{Aut}(\Gamma_k) \wr \Sigma_{i_k}).$$

Indeed suppose ϕ is such a graph automorphism and let v be a vertex of some copy of Γ_i and suppose that $\phi(v)$ is a vertex of a copy of some Γ_j . As Γ_i is not a join, we must have that ϕ restricted to that Γ_i gives an injective map $\Gamma_i \hookrightarrow \Gamma_j$. If $i = j$, this map must be an isomorphism. If not, $\Gamma_j^{\ast i_j}$, which lies in the link of v , cannot be mapped to itself by ϕ . So there must be a vertex of some Γ_j mapped to some other Γ_k with a corresponding injection induced by ϕ . By the pigeonhole principle, the sequence of such graph injection will end in some copy of Γ_i after finitely many steps, which then implies that in fact $\Gamma_i \cong \Gamma_j \cong \Gamma_k \cong \cdots$. Hence each Γ_i has to be mapped by such a ϕ to some standard copy of Γ_i in the join and the automorphism group of the join is as described. \square

4 Connectivity of the simplicial complexes

In this section we show that the semi-simplicial sets $W_n(A, X)$ of Section 2 are highly connected for any unfactorizable X when \mathcal{F} is the family of all RAAGs. We will treat separately the cases $X \neq \mathbb{Z}$ and $X = \mathbb{Z}$. In both cases, we will deduce this result from a computation of the connectivity of the simplicial complexes $I_n(A, X)$. In the first case we will show that $I_n(A, X) \cong S_n(A, X)$ while for $X = \mathbb{Z}$, following [17, Section 5.3] in the case of $\text{GL}_n(R)$, we will show that $S_n(A, \mathbb{Z}) \cong SI_n(A, \mathbb{Z})$ and use that $SI_n(A, \mathbb{Z})$ is a join complex over $I_n(A, \mathbb{Z})$. The connectivity of $W_n(A, X)$ will then follow using Proposition 2.2.

The proof of connectivity of $I_n(A, X)$ when $X \neq \mathbb{Z}$ is a ‘‘coloring argument’’, while for $X = \mathbb{Z}$, we follow closely the work of Maazen [15]. The semi-simplicial set $W_n(e, \mathbb{Z})$ is essentially already in the work of Charney [2] under the name $\text{SU}(\mathbb{Z}^n)$. Charney’s proof

of connectivity can be adapted to the present setting and yields the same connectivity as we get.

Case $X \neq \mathbb{Z}$

The main result of the section is the following:

Theorem 4.1 *Let A, X be RAAGs such that $X \neq \mathbb{Z}$. Then the semi-simplicial set $W_n(A, X)$ is $(n-2)$ -connected.*

The proof of the theorem will use the following:

Proposition 4.2 *Let A, X be RAAGs such that $X \neq \mathbb{Z}$ is unfactorizable and A has no direct factor X . Then RAAG-split maps $f: X^p \rightarrow A \times X^n$ have unique complements. Moreover, the complexes $I_n(A, X)$ and $S_n(A, X)$ are isomorphic.*

Proof The map $S_n(A, X) \rightarrow I_n(A, X)$ forgetting the chosen complements is surjective. To show that it is also injective, it is enough to check that it is injective on vertices. Hence the first part of the statement in the proposition in the case $p = 1$ implies the second.

By Proposition 2.6, it is enough to check the uniqueness of complements for the standard p -simplex $\sigma_p = \langle f_{n-p}, \dots, f_n \rangle$ for each p , with $f_j: X \rightarrow A \times X^n$ including X as the j^{th} X -factor. The standard simplex σ_p admits as a complement the subgroup $H_p = A \times X^{n-p-1} \times e \leq A \times X^{n-p-1} \times X^{p+1}$. Again by Proposition 2.6, any other complement for σ_p can be obtained from H_p by acting by an automorphism of $A \times X^n$ fixing the last $p + 1$ factors X . But from the description of the automorphisms (Proposition 3.3), we see that H_p is fixed by all such automorphisms and hence H_p is the only possible complement. \square

Let $X = \langle v_1, v_2, \dots, v_r \rangle$ where v_1, \dots, v_r is the standard vertex generating set of X . A vertex $f \in I_n(X, A)$ is determined by the tuple $f = (f(v_1), \dots, f(v_r)) \in (A \times X^n)^r$. Write $X^n = X_1 \times \dots \times X_n$. By Proposition 2.6, we can write $f = \phi \circ f_1$ for $f_1: X \rightarrow A \times X^n$ the inclusion as X_1 , the first X -factor, and $\phi \in \text{Aut}(A \times X^n)$. By Proposition 3.3, it follows that $f(v_i) = wz_1^{i_1} \dots z_d^{i_d}$, for z_1, \dots, z_d the canonical generators of $\mathbb{Z}^d \leq A$ and $w \in X_j$ for some j independent of i . We say that j is the *color* of f . Note that the unique complement of f is $H = A \times \prod_{i \neq j} (X_i)$ if the color of f is j .

Lemma 4.3 *Let A, X be RAAGs such that $X \neq \mathbb{Z}$ is unfactorizable and is not a factor in A . Then vertices $f_0, \dots, f_p \in I_n(A, X)$ form a simplex if and only if the f_i have distinct colors in the above sense.*

Proof Simplices of $I_n(A, X)$ have this property by Proposition 2.6 and Proposition 3.3. Conversely, suppose f_0, \dots, f_p are vertices of I_n of distinct colors. Then the map $f = (f_0, \dots, f_p): X^{p+1} \rightarrow A \times X^n$ is an injective homomorphism and

$$H = A \times \prod_{i \neq \text{col}(f_0), \dots, \text{col}(f_p)} X_i$$

is a complement for it. Hence (f_0, \dots, f_p) is a p -simplex of I_n . □

Recall that a simplicial complex S is *Cohen–Macaulay* of dimension n if it has dimension n , is $(n-1)$ -connected, and the link of any p -simplex in S is $(n-p-2)$ -connected.

Proposition 4.4 *Let A, X be RAAGs such that $X \neq \mathbb{Z}$ is unfactorizable and it is not a factor in A . Then the simplicial complex $I_n(X, A)$ is Cohen–Macaulay of dimension $n - 1$. In particular, it is $(n-2)$ -connected.*

Proof Consider the map $\pi: I_n(A, X) \rightarrow \Delta^{n-1}$ taking a vertex f to its color. This is a simplicial map which exhibits $I_n(A, X)$ as a complete join over Δ^{n-1} in the sense of [9, Definition 3.2] (see also Section 2). Indeed, this map is surjective as well as injective in individual simplices. Also, for every simplex $\sigma = \langle i_0, \dots, i_p \rangle$ in Δ^{n-1} , we have that $\pi^{-1}(\sigma) = \pi^{-1}(i_0) * \dots * \pi^{-1}(i_p)$ as vertices of I_n form a simplex if and only if they have different colors by the lemma. The result is then a direct application of [9, Proposition 3.5] and the fact that Δ^{n-1} is Cohen–Macaulay of dimension $n - 1$. □

We are finally ready to prove the main result of the section:

Proof of Theorem 4.1 If $A \cong A' \times X^k$ for some $k > 0$, we replace $W_n(A, X)$ by the isomorphic complex $W_{n+k}(A', X)$. Hence we may assume that A has no X -factor. By Proposition 4.4, we have that $I_n(A, X)$ is $(n-2)$ -connected for all $n \geq 0$. Hence by Proposition 4.2, the same holds for $S_n(A, X)$. Finally by Proposition 2.2 with $k = 1$ and $a = 2$, we have that the same also holds for $W_n(A, X)$. □

Case $X = \mathbb{Z}$

The main result of the section is the following:

Theorem 4.5 *Let A be a RAAG. Then we have that the semi-simplicial set $W_n(A, \mathbb{Z})$ is $\frac{n-3}{2}$ -connected.*

Lemma 4.6 $I_n(A, \mathbb{Z}) \cong I_n(e, \mathbb{Z})$ for any RAAG A with no \mathbb{Z} -summand.

Proof There is an inclusion $\alpha: I_n(e, \mathbb{Z}) \rightarrow I_n(A, \mathbb{Z})$ induced by composing maps to \mathbb{Z}^n with the canonical inclusion $\mathbb{Z}^n \rightarrow A \times \mathbb{Z}^n$, given that a complement in $A \times \mathbb{Z}^n$ can be obtained from a complement in \mathbb{Z}^n by crossing with A . The map α is simplicial and injective, and we claim that it is also surjective. Indeed, by Proposition 2.6, vertices of $I_n(\mathbb{Z}, A)$ are maps $f: \mathbb{Z} \rightarrow A \times \mathbb{Z}^n$ that can be written as compositions $f = \phi \circ f_1$ for f_1 the canonical inclusion as first \mathbb{Z} -factor and ϕ an automorphism of $A \times \mathbb{Z}^n$. Now f_1 is in the image of α , and by the classification of the automorphisms (or Proposition 3.3), we can see that $f = \phi \circ f_1$ still is in the image of α : as automorphisms of $A \times \mathbb{Z}^n$ take \mathbb{Z}^n to itself, the map f has image in \mathbb{Z}^n . Moreover, a complement for f is of the form $A' \times H$ with $A' \cong A$ and $H \subset \mathbb{Z}^n$ a complement for $\alpha^{-1}(f)$. Likewise, if vertices form a simplex in $I_n(A, \mathbb{Z})$, they will also form a simplex in $I_n(e, \mathbb{Z})$. \square

Proposition 4.7 $I_n(A, \mathbb{Z})$ is Cohen–Macaulay of dimension $n - 1$.

Using the lemma, one can almost deduce the result from Corollary III.4.5 in [15], though Maazen works with posets instead of simplicial complexes, and checks the vanishing of the homology groups instead of the homotopy groups. The proof adapts to our situation without any difficulty. We give it here for completeness.

Proof From Lemma 4.6, we may assume that A is the trivial group. For the rest of the proof, we write I_n for $I_n(e, \mathbb{Z})$. We have that I_n has dimension $n - 1$. We need to show that it is $(n - 2)$ -connected, and that for every $p \geq 0$ the link of any p -simplex σ is $(n - p - 3)$ -connected. Allowing σ to be an empty “ (-1) -simplex”, we can also, and will, consider the connectivity of I_n itself as being that of such a link. The link of a p -simplex is nonempty whenever $n - p - 2 \geq 0$ and it has dimension $n - p - 2$. We prove that the connectivity holds for each link by induction on the pair of dimensions $(\dim(\text{Link}), \dim(\sigma))$ in lexicographic order. The cases of $\dim(\text{Link}) = n - p - 2 \leq 0$ are trivial as a nonempty space is (-1) -connected, and the empty space is (-2) -connected (which is defined as a non-condition). So we fix $n > p \geq -1$ with $n - p - 2 \geq 1$ and we assume that we have proved that $\text{Link}_{I_m}(\sigma)$ is $(m - k - 3)$ -connected for every k -simplex σ of I_m with $m > k \geq -1$ and $m - k - 2 \leq n - p - 2$, with $k < p$ if $m - k - 2 = n - p - 2$. Let σ be a p -simplex of I_n . By Proposition 2.6, we may assume that $\sigma = \sigma_p = \langle e_{n-p}, \dots, e_n \rangle$ is the $(p + 1)$ -tuple consisting of the last $p + 1$ standard generators of \mathbb{Z}^n , where we identify a map $f: \mathbb{Z} \rightarrow \mathbb{Z}^n$ with the element $f(1) \in \mathbb{Z}^n$. We will show that $\text{Link}_{I_n}(\sigma)$ is $(n - p - 3)$ -connected, which will give the induction step and prove the result.

A vertex v in $\text{Link}_{I_n}(\sigma_p)$ is given as an n -tuple of integers $v = ((v)_1, \dots, (v)_n)$. We filter the link using the absolute value of the last coordinate. Let

$$\mathcal{O}_q := \langle v \in \text{Link}_{I_n}(\sigma_p) : |(v)_n| \leq q \rangle \subset \text{Link}_{I_n}(\sigma_p),$$

that is, \mathcal{O}_q is the full subcomplex of $\text{Link}_{I_n}(\sigma_p)$ on the vertices whose last coordinate in \mathbb{Z}^n has absolute value at most q . If $p \geq 0$, we have that $\mathcal{O}_0 \cong \text{Link}_{I_{n-1}}(\sigma_{p-1})$, the link of the last p generators of \mathbb{Z}^{n-1} in I_{n-1} . Indeed, if $v_1, \dots, v_k \in \mathbb{Z}^n$ have their last coordinate equal to 0, then $\langle v_1, \dots, v_k, e_{n-p}, \dots, e_n \rangle$ is a simplex of I_n if and only if $\langle \bar{v}_1, \dots, \bar{v}_k, e_{n-p}, \dots, e_{n-1} \rangle$ is a simplex of I_{n-1} for $\bar{v}_i \in \mathbb{Z}^{n-1}$ the first $n-1$ coordinates of v_i . Hence by induction, \mathcal{O}_0 is $(n-p-3)$ -connected in that case. If $p = -1$, $\mathcal{O}_0 \cong I_{n-1}$ is $(n-3)$ -connected by induction. We will show that \mathcal{O}_1 is $(n-2)$ -connected when $p = -1$, so also $(n-p-3)$ -connected. Then we will show that in both cases, for every $q \geq 0$, if \mathcal{O}_q is $(n-p-3)$ -connected, then so is \mathcal{O}_{q+1} . This will prove the result given that \mathcal{O}_0 (or \mathcal{O}_1 if $p = -1$) is $(n-p-3)$ -connected, as by compactness, any map from a sphere into the link will have image in \mathcal{O}_q for some $q \geq 1$.

We start by showing that \mathcal{O}_1 is $(n-2)$ -connected when $p = -1$. Recall that in this case $\text{Link}_{I_n}(\sigma_p) = I_n$. We can construct \mathcal{O}_1 from \mathcal{O}_0 by attaching successively the vertices $v \in I_n$ with $|(v)_n| = 1$, along their link in \mathcal{O}_0 , then edges formed by such vertices along their links in the newly formed complex, and so on. Explicitly this gives

$$\mathcal{O}_1 = \mathcal{O}_0 \bigcup_{v_1 \in \mathcal{O}_1 \setminus \mathcal{O}_0} C(\langle v_1 \rangle) \bigcup_{\substack{\langle v_1, v_2 \rangle \\ \subset \mathcal{O}_1 \setminus \mathcal{O}_0}} C(\langle v_1, v_2 \rangle) \cdots \bigcup_{\substack{\langle v_1, \dots, v_n \rangle \\ \subset \mathcal{O}_1 \setminus \mathcal{O}_0}} C(\langle v_1, \dots, v_n \rangle)$$

where $C(\langle v_1, \dots, v_k \rangle) = \langle v_1, \dots, v_k \rangle * (\text{Link}_{I_n}(\langle v_1, \dots, v_k \rangle) \cap \mathcal{O}_0)$, attached successively along $L_k := \partial \langle v_1, \dots, v_k \rangle * (\text{Link}_{I_n}(\langle v_1, \dots, v_k \rangle) \cap \mathcal{O}_0)$. For $k = 1$, we have that $\text{Link}_{I_n}(\langle v_1 \rangle) \cap \mathcal{O}_0 = \mathcal{O}_0$ as the last coordinate of v_1 is ± 1 . Hence this link is $(n-3)$ -connected. Now pick a vertex $v \in \mathcal{O}_1 \setminus \mathcal{O}_0$. We can write

$$\mathcal{O}_0 \bigcup_{v_1 \in \mathcal{O}_1 \setminus \mathcal{O}_0} C(\langle v_1 \rangle) = \text{Star}(v) \bigcup_{\substack{v_1 \in \mathcal{O}_1 \setminus \mathcal{O}_0 \\ v_1 \neq v}} C(\langle v_1 \rangle)$$

where $\text{Star}(v) = \mathcal{O}_0 * v$ is the star of v within this complex. It follows that this second stage of the filtration is $(n-2)$ -connected, being homotopic to a wedge of suspensions of $(n-3)$ -connected spaces.

For $k > 1$, we have again that all the v_i have last coordinate ± 1 . Let $\varepsilon_i = +1$ if $(v_i)_n$ and $(v_1)_n$ have the same sign, and -1 otherwise. Then

$$\text{Link}_{I_n}(\langle v_1, \dots, v_k \rangle) \cap \mathcal{O}_0 \cong \text{Link}_{I_{n-1}}(\langle \overline{v_2 - \varepsilon_2 v_1}, \dots, \overline{v_k - \varepsilon_k v_1} \rangle),$$

with $\overline{v_i - \varepsilon_i v_1}$ denoting as above the first $n-1$ coordinates of this vector, noting that its last coordinate is zero. Indeed, a simplex $\langle w_1, \dots, w_q \rangle$ is in the first link if and only if the w_j have 0 as last coordinate and $\langle w_1, \dots, w_q, v_1, \dots, v_k \rangle$ is a partial basis of \mathbb{Z}^n , which is the case if and only if $\langle w_1, \dots, w_q, v_1, v_2 - \varepsilon_2 v_1, \dots, v_k - \varepsilon_k v_1 \rangle$ is a partial

basis of \mathbb{Z}^n , which is the case if and only if $\langle \overline{w_1}, \dots, \overline{w_q}, \overline{v_2 - \varepsilon_2 v_1}, \dots, \overline{v_k - \varepsilon_k v_1} \rangle$ is a partial basis of \mathbb{Z}^{n-1} . Hence this link is $(n-k-2)$ -connected by induction. So the space L_k , along which the cone $C(\langle v_1, \dots, v_k \rangle)$ is attached, is $(n-2)$ -connected. Hence attaching each $C(\langle v_1, \dots, v_k \rangle)$ keeps the space $(n-2)$ -connected.

We are left to show that if \mathcal{O}_q is $(n-p-3)$ -connected then \mathcal{O}_{q+1} is also $(n-p-3)$ -connected, where \mathcal{O}_q is now the q^{th} filtration of $\text{Link}_{I_n}(\sigma_p)$ without any special assumption on p . We construct \mathcal{O}_{q+1} from \mathcal{O}_q by successively attaching the missing vertices, edges, and so on, just like we constructed \mathcal{O}_1 from \mathcal{O}_0 above:

$$\mathcal{O}_{q+1} = \mathcal{O}_q \bigcup_{v_1 \in \mathcal{O}_{q+1} \setminus \mathcal{O}_q} C(\langle v_1 \rangle) \bigcup_{\substack{\langle v_1, v_2 \rangle \subset \\ \mathcal{O}_{q+1} \setminus \mathcal{O}_q}} C(\langle v_1, v_2 \rangle) \cdots \bigcup_{\substack{\langle v_1, \dots, v_{n-p-1} \rangle \\ \subset \mathcal{O}_{q+1} \setminus \mathcal{O}_q}} C(\langle v_1, \dots, v_{n-p-1} \rangle).$$

Again we need to compute the connectivity of the link of $\langle v_1, \dots, v_k \rangle$ in $\text{Link}_{I_n}(\sigma_p)$ intersected with \mathcal{O}_q . This link is a subcomplex of $\text{Link}_{\text{Link}_{I_n}(\sigma_p)}(\langle v_1, \dots, v_k \rangle)$. By Proposition 2.6, this last link is isomorphic to $\text{Link}_{I_n}(\sigma_{p+k})$, which by assumption is $(n-p-k-3)$ -connected.

Let $\kappa: \mathbb{Z} \rightarrow \mathbb{Z}$ be a map satisfying $\kappa(z) = 0$ if $|z| < q + 1$, and $|z - \kappa(z)(q + 1)| < q + 1$ for $|z| \geq q + 1$. We have $(v_1)_n = \varepsilon_1(q + 1)$ for $\varepsilon_1 = \pm 1$. Now define

$$\pi: \text{Link}_{\text{Link}_{I_n}(\sigma_p)}(\langle v_1, \dots, v_k \rangle) \rightarrow \text{Link}_{\text{Link}_{I_n}(\sigma_p)}(\langle v_1, \dots, v_k \rangle) \cap \mathcal{O}_q$$

to be the map taking a vertex w to $w - \varepsilon_1 \kappa((w)_n) v_1$. Then $w - \varepsilon_1 \kappa((w)_n) v_1 \in \mathcal{O}_q$ and lies in $\text{Link}_{\text{Link}_{I_n}(\sigma_p)}(\langle v_1, \dots, v_k \rangle)$ if w was in that link. Moreover π is simplicial and defines a retraction. It follows that $\text{Link}_{\text{Link}_{I_n}(\sigma_p)}(\langle v_1, \dots, v_k \rangle) \cap \mathcal{O}_q$ is also at least $(n-p-k-3)$ -connected. Hence attaching $C(\langle v_1, \dots, v_k \rangle)$ along $\partial \langle v_1, \dots, v_k \rangle * (\text{Link}_{\text{Link}(\sigma_p)}(\langle v_1, \dots, v_k \rangle) \cap \mathcal{O}_q)$ does not change the connectivity as the latter space is at least $(n-p-3)$ -connected. The result follows. \square

Remark 4.8 The existence of the function κ used in the proof makes \mathbb{Z} , together with the absolute value, a *Euclidean ring*. Maazen’s proof of the above statement was set in the more general context of Euclidean rings.

Proposition 4.9 *Let A be a right-angled Artin group with no \mathbb{Z} -summand and let $((K_0, f_0), \dots, (K_p, f_p))$ be a simplex of $SI_n(A, \mathbb{Z})$. Then $\bigcap_j K_j \cong A \times \mathbb{Z}^m$ for some m .*

Proof Suppose that A has generators a_1, \dots, a_r and \mathbb{Z}^n has generators z_1, \dots, z_n . We know that each K_i can be obtained from the standard $A \times \mathbb{Z}^{n-1} \times e \leq A \times \mathbb{Z}^n$ by applying an automorphism. From the description of the automorphisms of $A \times \mathbb{Z}^n$, it follows that K_i is generated by $a_1 w_1, \dots, a_r w_r, t_1, \dots, t_{n-1}$ for some $w_j, t_j \in \mathbb{Z}^n$.

Moreover, we know that \mathbb{Z}^n is generated by t_1, \dots, t_{n-1}, f_i , where $f_i := f_i(1)$. Hence we can rewrite the generators of K_i as $a_1 f_i^{m_{i,1}}, \dots, a_r f_i^{m_{i,r}}, t_1, \dots, t_{n-1}$ for some $m_{i,1}, \dots, m_{i,r} \in \mathbb{Z}$.

As $f_i \in K_j$ whenever $i \neq j$ and $a_k f_i^{m_{i,k}} \in K_i$, each $a_k f_0^{m_{0,k}} f_1^{m_{1,k}} \dots f_p^{m_{p,k}}$ lies in $\bigcap_i K_i$. Let $A' \cong A$ denote the subgroup of $A \times \mathbb{Z}^n$ generated by the elements $a_k f_0^{m_{0,k}} f_1^{m_{1,k}} \dots f_p^{m_{p,k}}$. We have that $A \times \mathbb{Z}^n = A' \times \mathbb{Z}^n$. We want to show that $\bigcap_i K_i = A' \times (\bigcap_i (K_i \cap \mathbb{Z}^n))$. The right side is included in the left side, so all we need to show is that the left side is included in the right side. Let $x \in \bigcap_i K_i$ be some element. As $x \in K_i$, we can write it as $x = x'_i x''_i$ with $x'_i \in A'$ and $x''_i \in K_i \cap \mathbb{Z}^n$. Now these different expressions of x are all equal, and all live in $A \times \mathbb{Z}^n = A' \times \mathbb{Z}^n$. It follows that $x'_i = x'_j$ and hence $x''_i = x''_j$ for each i, j . It follows that $x''_0 \in \bigcap_i (K_i \cap \mathbb{Z}^n)$ and $x = x'_0 x''_0 \in A' \times (\bigcap_i (K_i \cap \mathbb{Z}^n))$. As $\bigcap_i (K_i \cap \mathbb{Z}^n) \leq \mathbb{Z}^n$, the result follows. \square

Proof of Theorem 4.5 Just as in the proof of Theorem 4.1, we may assume that A has no \mathbb{Z} -factor. From Proposition 2.3, we have that $SI_n(A, \mathbb{Z})$ is a join complex over $I_n(A, \mathbb{Z})$ (in the sense of [9, Definition 3.2], see also Section 2). As $I_n(A, \mathbb{Z})$ is Cohen–Macaulay of dimension $n - 1$ by Proposition 4.7, Theorem 3.6 of [9] gives that $SI_n(A, \mathbb{Z})$ is $\frac{n-3}{2}$ -connected for all $n \geq 0$. By Proposition 4.9, the hypothesis of Proposition 2.4 is satisfied, and hence $S_n(A, \mathbb{Z})$ is isomorphic to $SI_n(A, \mathbb{Z})$. So the connectivity also holds for $S_n(A, \mathbb{Z})$. Hence by Proposition 2.2 with $k = 2$ and $a = 3$, we have that the same also holds for $W_n(A, X)$. \square

5 Stability theorem

We consider in this section the family of all right-angled Artin groups with $\mathcal{C} = \mathcal{UG}_{RAAGs}$ the associated homogeneous category, as defined in Section 1. Let A, X be RAAGs, and denote by $\mathcal{C}_{A,X}$ the full subcategory of \mathcal{C} on the objects $A \times X^n$ for all $n \geq 0$. Recall from [17, Section 4.2] the *lower suspension* functor $\Sigma_X: \mathcal{C}_{A,X} \rightarrow \mathcal{C}_{A,X}$ taking $A \times X^n$ to $A \times X^{n+1}$ and a morphism $f: A \times X^n \rightarrow A \times X^k$ to the composition $(b_{X,A} \times X^k) \circ (X \times f) \circ (b_{X,A}^{-1} \times X^n)$, where $b_{X,A}: X \times A \rightarrow A \times X$ denotes the symmetry. Recall from [17, Definition 4.10] that a functor

$$F: \mathcal{C}_{A,X} \rightarrow \mathbb{Z}\text{-Mod}$$

is a *coefficient system of degree r at N* if the kernel of the suspension map $F \rightarrow F \circ \Sigma_X$ is trivial when evaluated at $A \times X^n$ with $n \geq N$, and the cokernel is of degree $r - 1$ at $N - 1$, with degree -1 at N meaning taking the value 0 at $A \times X^n$ whenever $n \geq N$. In particular, constant coefficient systems are of degree 0 at 0. A coefficient system F is *split* if it splits as a functor.

Applying the main results of [17] to our situation, we get the following stability theorem:

Theorem 5.1 *Let A and X be RAAGs, where X is unfactorizable. Suppose that $F: C_{A,X} \rightarrow \mathbb{Z}$ -Mod is a coefficient system of degree r at N . Let $n > N$. Then the map $\text{Aut}(A \times X^n) \rightarrow \text{Aut}(A \times X^{n+1})$ taking an automorphism f to $f \times X$ induces a map*

$$H_i(\text{Aut}(A \times X^n); F(A \times X^n)) \rightarrow H_i(\text{Aut}(A \times X^{n+1}); F(A \times X^{n+1}))$$

which is surjective for all $i \leq \frac{1}{2}(n - 1) - r$ and an isomorphism for all $i \leq \frac{1}{2}(n - 3) - r$. If the coefficient system is split, this range improves to $i \leq \frac{1}{2}(n - r - 1)$ for surjectivity and $i \leq \frac{1}{2}(n - r - 3)$ for injectivity, and if the coefficient system is constant, the isomorphism holds for $i \leq \frac{1}{2}(n - 2)$.

Moreover, let $\text{Aut}'(A \times X^n)$ denote the commutator subgroup of $\text{Aut}(A \times X^n)$, and let $n > 2N$. Then the map

$$H_i(\text{Aut}'(A \times X^n); F(A \times X^n)) \rightarrow H_i(\text{Aut}'(A \times X^{n+1}); F(A \times X^{n+1}))$$

is surjective for all $i \leq \frac{1}{3}(n - 2) - r$ and an isomorphism for all $i \leq \frac{1}{3}(n - 5) - r$. If the coefficient system is split, this range improves to $i \leq \frac{1}{3}(n - 2r - 2)$ for surjectivity and $i \leq \frac{1}{3}(n - 2r - 5)$ for injectivity, and if the coefficient system is constant, the isomorphism holds for $i \leq \frac{1}{3}(n - 4)$.

If $X \neq \mathbb{Z}$ -summand, one can replace n by $n + 1$ in all the bounds of the theorem.

Proof By Proposition 1.1, the category $\mathcal{C} = U\mathcal{G}_{RAAGs}$ is symmetric monoidal homogeneous, and hence pre-braided and locally homogeneous at any (A, X) in the sense of [17, Definitions 1.1 and 1.4]. Theorem 4.1 gives that it satisfies LH3 of that paper with slope 2 at (A, X) for all A and all irreducible $X \neq \mathbb{Z}$ [17, Definition 2.2], and Theorem 4.5 that it satisfies LH3 with slope 2 at $(A \times \mathbb{Z}, \mathbb{Z})$ for all A . The result then follows from Theorems 3.1, 3.4 and 4.20 of [17] for (A, X) with X unfactorizable not equal to \mathbb{Z} , and for $(A \times \mathbb{Z}, \mathbb{Z})$, using the argument of [17, Corollary 3.9] for the second part of the statement. □

Theorems A and B are obtained by applying the above theorem to constant coefficient systems for each irreducible factor of $B = X_1 \times \dots \times X_k$. Theorem C is obtained likewise applying the theorem to the coefficient system defined by the abelianization, noting that this is a split coefficient system of degree 1 at 0.

Note that van der Kallen obtains better bounds for $\text{GL}_n(\mathbb{Z})$, which is the case when A is the trivial group and $B = X = \mathbb{Z}$. Most particularly, for Theorem B his bound has slope 2 instead of slope 3 as we have (see [13, Theorem 4.6] or Proposition 5.2) However his argument does not obviously extend to all RAAGs. (The argument of van der Kallen is explained at the end of Section 5.3 in [17].)

Finitely generated abelian groups

Homological stability for the automorphism groups of finitely generated abelian groups under taking direct product can be deduced directly from existing results in the literature, without needing to prove new connectivity results. We give the exact statement and its proof here for completeness.

Proposition 5.2 *Let G be a finitely generated abelian group. Then the homomorphism $\text{Aut}(G^n) \rightarrow \text{Aut}(G^{n+1})$ taking an automorphism f of G^n to the automorphism $f \times G$ of G^{n+1} fixing the last factor, induces maps*

$$\begin{aligned} H_i(\text{Aut}(G^n); \mathbb{Z}) &\rightarrow H_i(\text{Aut}(G^{n+1}); \mathbb{Z}), \\ H_i(\text{Aut}'(G^n); \mathbb{Z}) &\rightarrow H_i(\text{Aut}'(G^{n+1}); \mathbb{Z}) \end{aligned}$$

which are surjective for all $i \leq \frac{1}{2}n$ and isomorphisms for $i \leq \frac{1}{2}(n-1)$.

(As in the introduction, $\text{Aut}'(G^n)$ denotes the commutator subgroup of $\text{Aut}(G^n)$.)

Proof A finitely generated abelian group G is a \mathbb{Z} -module, and the automorphism group $\text{Aut}(G^n)$ is isomorphic to $\text{GL}_n(\text{End}(G))$, for $\text{End}(G)$ its ring of endomorphisms. By [18, Theorem 3.4], $\text{End}(G)$ has 2 in its stable range (in the terminology of [18], see Definition 1.5 in that paper), and so satisfies Bass' condition SR_3 or has $\text{sdim} = 1$ in the terminology of [13, Section 2.2]. The result then follows from [13, Theorem 4.11] using the fact that $\text{GL}'_n(R)$ is isomorphic to its subgroup of elementary matrices (Whitehead's lemma). \square

Theorem 5.6 of [13] and Theorem 5.10 of [17] can likewise be applied to show that homological stability for the groups $\text{Aut}(G^n)$ with G finitely generated abelian also holds with polynomial twisted coefficients.

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