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The L^2–(co)homology of groups with hierarchies

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We study group actions on manifolds that admit hierarchies, which generalizes the idea of Haken n–manifolds introduced by Foozwell and Rubinstein. We show that these manifolds satisfy the Singer conjecture in dimensions $n \leq 4$. Our main application is to Coxeter groups whose Davis complexes are manifolds; we show that the natural action of these groups on the Davis complex has a hierarchy. Our second result is that the Singer conjecture is equivalent to the cocompact action dimension conjecture, which is a statement about all groups, not just fundamental groups of closed aspherical manifolds.

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Introduction

In his PhD thesis [10], Foozwell introduced Haken n–manifolds as a higher dimensional analogue of Haken 3–manifolds. Loosely speaking, these are closed n–manifolds that can be cut inductively along codimension-1 submanifolds to a disjoint union of n–balls. The exact definition is somewhat technical. The resulting sequence of manifolds is called a hierarchy. Foozwell and Rubinstein have explored many properties of these manifolds, in particular, they have shown [11; 12] that their universal covers are homeomorphic to $\mathbb{R}^n$ and their fundamental groups have solvable word problem. Both of these properties show that Haken n–manifolds are a special class of aspherical manifolds; see Davis [4] and Mess [18].

The classical Euler characteristic conjecture, attributed to Hopf, predicts the sign of the Euler characteristic of a closed aspherical $2n$–dimensional manifold M^{2n}: $(-1)^n \chi(M^{2n}) \geq 0$. In a special case of right-angled Coxeter group manifolds, this conjecture becomes a purely combinatorial statement about flag simplicial triangulations of $(2n-1)$–spheres, known as the Charney–Davis conjecture [3].

Another classical conjecture about aspherical manifolds, the Singer conjecture, predicts that the reduced L^2–homology of the universal cover vanishes except possibly in the middle dimension. Since one can use L^2–Betti numbers to compute χ, the Singer conjecture immediately implies the Euler characteristic conjecture.

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Edmonds [9] proved the Euler characteristic conjecture for closed Haken 4–manifolds by showing that it was equivalent to the Charney–Davis conjecture for 3–spheres, which holds true by a result of Davis and the first author [8], where the Singer conjecture for 4–dimensional right-angled Coxeter group manifolds is proved. The equivalence of the two conjectures was extended by Davis and Edmonds [6] to all even dimensions. In fact, they showed this equivalence for *generalized Haken* $2n$*–manifolds*, where they allow the hierarchy to end in any compact contractible manifold.

The starting point of this paper was a question of Edmonds whether the Singer conjecture holds for Haken 4–manifolds.

One advantage of studying homological properties of Haken n–manifolds is that we can ignore most of the technicalities and study a more general class of manifolds that is closer to the loose definition above. Since we are interested in group actions that are not free, and because we think it is simpler, we build the hierarchies out of contractible manifolds with a proper and cocompact group action.

We say a group G *admits a hierarchy* if it acts on a contractible manifold M that can be cut inductively along codimension-1 contractible G–invariant submanifolds to a disjoint union of compact contractible manifolds. An example to keep in mind is $\mathbb{Z}^n$ acting on $\mathbb{R}^n$ with quotient the n–torus T^n. Cutting T^n along T^{n-1} corresponds to cutting along $\mathbb{Z}^n$–translates of $\mathbb{R}^{n-1}$ inside $\mathbb{R}^n$. In a similar way, hierarchies for Haken n–manifolds lift to our hierarchies on the universal covers.

The paper is organized as follows. We develop a general theory of group actions with hierarchies in Section 1. In Section 2 we prove that Coxeter group manifolds admit hierarchies. Section 3 recalls the necessary background material on L^2–(co)homology. Finally, in Section 4 we study various vanishing conjectures about L^2–homology.

Our first result is that the Singer conjecture holds for all groups that admit a hierarchy in dimension 4. Our main application of this result is to Coxeter groups: Theorem 4.16 generalizes the result in [8] for right-angled Coxeter groups and a later result of Schroeder [24] for even Coxeter groups.

We also introduce the notion of the *cocompact action dimension* of a group: the minimal dimension of a contractible manifold, possibly with boundary, which admits a proper cocompact action by the group. Our second result is that the Singer conjecture is actually a statement about all groups, not just about fundamental groups of closed aspherical manifolds. Namely, we show in the smooth or PL categories that the Singer conjecture is equivalent to the cocompact action dimension conjecture: the L^2–cohomology of a group vanishes above half of its cocompact action dimension. We also show that for type VF groups, the cocompact action dimension conjecture is equivalent to the action dimension conjecture [8, Conjecture 8.9.1].

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1 Hierarchies for group actions

Definition Let G be a discrete group. A *G–space* M is a topological space with a G–action. We say a G–space is *proper* or *cocompact* if the action of G is proper or cocompact respectively. If N is a G–invariant subspace of M then (M, N) is a *pair of G–spaces.*

Definition A *convex polyhedral cone* C in $\mathbb{R}^n$ is the intersection of a finite collection $\{B_i^+\}$ of linear half-spaces in $\mathbb{R}^n$ (a half-space is linear if its bounding hyperplane B_i is a linear subspace). C is *nondegenerate* if it has nonempty interior. A *hyperplane arrangement in a nondegenerate cone* C is a finite collection $\{A_i\}$ of linear hyperplanes such that each A_i intersects the interior of C.

We assume that our manifolds are topological and mention explicitly when we require a smooth or PL structure.

Definition Let M be a proper, cocompact G–manifold, and $\mathcal{E} = \{E_i\}_{i=0}^r$ a collection of codimension-1 G–submanifolds. $(M, \mathcal{E})$ is *tidy* if:

- The components of M are contractible.
- The components of any intersection of the E_i are either contractible, or contained in ∂M.
- $(M, \partial M, \mathcal{E})$ locally looks like a hyperplane arrangement in a nondegenerate cone in $\mathbb{R}^n$: every point in M has a chart which maps M into a nondegenerate cone in $\mathbb{R}^n$, the point to the origin, ∂M into the boundary of the cone, and the E_i into a hyperplane arrangement in the cone.

This local structure implies that each component L of any intersection of the E_i is a manifold and either $L \subset \partial M$ or $L \cap \partial M = \partial L$. In the first case we call L a boundary component, and in the second an interior component. Moreover, the condition that hyperplanes intersect the interior of the cone implies that each E_i has only interior components. Finally, note that if $x \in M - \partial M$, the local picture is that of a linear hyperplane arrangement in $\mathbb{R}^n$.

In the case where $\mathcal{E}$ consists of just one submanifold F, this definition is equivalent to requiring that F is locally flat as a submanifold with boundary (sometimes called a neat submanifold), and the components of both M and F are contractible. We will call such a pair (M, F) a *tidy pair*.

In this case, since the components of F are contractible, it admits a collar neighborhood, and since the components of M are contractible, F is separating. By *cutting* M along F we mean taking the disjoint union N of the closures in M of components of $M - F$. We say that N is M *cut-open* along F. The action of G on $M - F$ extends by continuity to a proper cocompact action on N. So $N = M \ominus F$ is a G–manifold with boundary $\partial M - \partial F$ union two copies of F.

Note that $N - \partial N$ is naturally identified with $M - \partial M - F$. Using this identification, we can cut any closed G–subspace L of M along F by taking the closure of $L - F$ in N: $L \ominus F := Cl_N(L - F)$. Note that we still have a natural identification $(L \ominus F) - \partial N = L - \partial M - F$.

Associated to the cut there is an exact sequence of a triple $(M, F \cup \partial M, \partial M)$:

$$\cdots \to H_c^{k-1}(F \cup \partial M, \partial M) \to H_c^k(M, F \cup \partial M) \\ \to H_c^k(M, \partial M) \to H_c^k(F \cup \partial M, \partial M) \to \cdots .$$

By excision, we have $H_c^k(F \cup \partial M, \partial M) \cong H_c^k(F, \partial F)$ and $H_c^k(M, F \cup \partial M) \cong H_c^k(N, \partial N)$, so the above sequence becomes

$$\cdots \to H_c^{k-1}(F, \partial F) \to H_c^k(N, \partial N) \to H_c^k(M, \partial M) \to H_c^k(F, \partial F) \to \cdots . \tag{1}$$

Finally, applying Poincaré duality and reindexing, we obtain a sequence

$$\cdots \to H_k(F) \to H_k(N) \to H_k(M) \to H_{k-1}(F) \to \cdots . \tag{2}$$

Lemma 1.1 *If (M, F) is a tidy pair and N is M cut-open along F, then the components of N are contractible manifolds.*

Notice that N may or may not have more G–orbits of components than does M.

Proof The van Kampen theorem implies that components of N are simply connected, and sequence (2) shows that N is acyclic. □

Lemma 1.2 *Suppose that $(M, \mathcal{E})$ is tidy, and let N be M cut-open along E_0. Then $(N, \{E_i \ominus E_0\}_{i=1}^r)$ is also tidy.*

Proof We check the conditions of tidiness. Contractibility of the components of N follows immediately from Lemma 1.1 since (M, E_0) is a tidy pair. After cutting, the local picture is mostly preserved, we just have to check near E_0. If $x \in E_0$, the new charts come from restricting the old chart to one of the two halfcones bounded by the hyperplane corresponding to E_0, and taking hyperplanes which pass through the interior of that halfcone. Note that this description of cutting an arrangement in a nondegenerate cone along one of the hyperplanes into two nondegenerate cones with arrangements agrees with the procedure of cutting M and the E_i by E_0 described above in terms of closures.

Next, we show that the interior components of an intersection $\bigcap(E_{i_\alpha} \ominus E_0)$ are contractible. Let L be the union of these interior components. It follows from the local structure that L is a manifold with $\partial L = L \cap \partial N$. Therefore, it's enough to show that $L - \partial L = L - \partial N$ has contractible components. These components come from cutting the components of $\bigcap(E_{i_\alpha} - \partial M)$ which are not contained in E_0. The local picture of $\bigcap(E_{i_\alpha} - \partial M)$ intersecting $E_0 - \partial M$ is of a hyperplane intersecting a subspace, and we ignore the case when the subspace is contained in the hyperplane, as this would produce a boundary component. Thus the intersections we are interested are transverse.

So, let D be the union of the interior components of $\bigcap E_{i_\alpha} - \partial M$ which are not contained in E_0. Since $E_0 \cap D$ has contractible components by hypothesis, it follows from the above that the pair $(D, E_0 \cap D)$ satisfies all conditions of tidiness except cocompactness and $L - \partial L = (D - \partial D) \ominus (E_0 \cap D)$. Therefore $L - \partial L$ has contractible components by Lemma 1.1. (The cutting procedure and the proof of the lemma did not use cocompactness.) □

Definition An *n–hierarchy* for an action of a discrete group G on a manifold M is a sequence

$$(M_0, F_0), (M_1, F_1), \dots, (M_m, F_m), (M_{m+1}, \varnothing),$$

such that

- $M_0 = M$,
- M_{m+1} is a disjoint union of compact contractible n–manifolds,
- (M_i, F_i) is a tidy pair for each i,
- M_{i+1} is M_i cut-open along F_i.

More generally, if (M, N) is a proper, cocompact G–pair of manifolds, we can define a *hierarchy ending in* N in the same way, with the one difference being that $M_{m+1} = N$.

Definition *G admits an n–hierarchy* if there exists a contractible, n–dimensional G–manifold M and a hierarchy for the action.

Lemma 1.3 *Let G act on M with a hierarchy, and let M_1^0 be a component of M_1. Then there is an induced hierarchy for the action of $\mathrm{St}_G(M_1^0)$ on M_1^0, where $\mathrm{St}_G(M_1^0)$ is the stabilizer of M_1^0.*

Proof We claim the following sequence is a hierarchy for M_1^0:

$$(M_1^0, F_1 \cap M_1^0), (M_2 \cap M_1^0, F_2 \cap M_1^0), \dots, (M_{m+1} \cap M_1^0, \varnothing).$$

We have that M_1^0 is a contractible $\mathrm{St}_G(M_1^0)$–manifold by Lemma 1.1. Since each F_i is G–invariant, $F_i \cap M_1^0$ is $\mathrm{St}_G(M_1^0)$–invariant, and the other conditions of our hierarchy follow immediately. □

Theorem 1.4 *Let M be a proper, cocompact G–manifold, and $\mathcal{E} = \{E_i\}_{i=0}^r$ a collection of submanifolds such that $(M, \mathcal{E})$ is tidy. If the components of the complement $M - \cup_i E_i$ have compact closure in M, then the action of G on M admits a hierarchy.*

Proof The proof is to apply Lemma 1.2 repeatedly, as this implies that if we cut along each E_i, we get a hierarchy ending in $M - \cup_i E_i$. To be precise, let $F_j = E_j$ cut-along by $E_0, E_1, \dots E_{j-1}$, and let $M_0 = M$ and $M_{j+1} = M_j$ cut along by F_j. Since each E_i is G–invariant, (M_j, F_j) is a tidy pair for all j. □

2 Coxeter groups

Recall that a Coxeter group W has generators s_i with relations $s_i^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$ for some $m_{ij} \in \mathbb{N} \cup \infty$. In other words, W is generated by reflections and each pair of reflections generates a dihedral subgroup (possibly D_∞). The *nerve* of a Coxeter group is a simplicial complex with vertices corresponding to generators s_i, and $s_{i_1}, \dots, s_{i_n}$ a simplex if and only if the subgroup generated by $s_{i_1}, \dots, s_{i_n}$ is finite. A Coxeter group is *right-angled* if $m_{ij} = 2$ or ∞ for all i, j.

Definition A *mirror structure* on a space X is an index set S and a collection of subspaces $\{X_s\}_{s \in S}$. For each $x \in X$, let

$$S(x) := \{s \in S \mid x \in X_s\}.$$

An example to keep in mind is a convex polytope in $\mathbb{E}^n$ or $\mathbb{H}^n$ with mirrors the codimension-1 faces. We will assume that our index set S is finite.

Definition Let X have a mirror structure, and let W be a Coxeter group with generators $s \in S$. Let W_T denote the subgroup generated by $s \in T \subset S$. Let $\sim$ denote the following equivalence relation on $W \times X$: $(w_1, x) \sim (w_2, y)$ if and only if $x = y$ and $w_1 w_2^{-1} \in W_{S(x)}$. The *basic construction* is the space

$$\mathcal{U}(W, X) := W \times X / \sim .$$

Therefore, $\mathcal{U}(W, X)$ is constructed by gluing together copies of X along its mirrors, with the exact gluing dictated by the Coxeter group. A standard example is where X is a right-angled pentagon in $\mathbb{H}^2$ with mirrors the edges of X, and W is the right-angled Coxeter group generated by reflections in these edges. Then $\mathcal{U}(W, X) \cong \mathbb{H}^2$.

Let W be a Coxeter group with nerve L. Again, L is the simplicial complex with vertex set corresponding to S and simplices corresponding to subsets of S that generate finite subgroups of W. Let K be the cone on the barycentric subdivision of L. K admits a natural mirror structure with K_s the closed star of the vertex corresponding to s in the barycentric subdivision of L. The *Davis complex* $\Sigma(W, S)$ is defined to be the simplicial complex $\mathcal{U}(W, K)$.

Lemma 2.1 [5] *$\Sigma(W, S)$ has the following properties:*

- *W acts properly and cocompactly on $\Sigma(W, S)$ with fundamental domain K.*
- *Σ admits a cellulation such that the link of every vertex can be identified with L. Therefore, if L is a triangulation of S^{n-1}, then $\Sigma(W, S)$ is an n–manifold.*
- *$\Sigma(W, S)$ admits a piecewise Euclidean metric that is* CAT(0).

We assume from now on that W is a Coxeter group with nerve a PL triangulation of S^{n-1}. If $w \in W$ acts as a reflection on $\Sigma(W, S)$, we call the fixed point set a wall, and denote it Σ^w.

Lemma 2.2 *Walls in $\Sigma(W, S)$ have the following properties:*

- *The stabilizer of each wall acts properly and cocompactly on the wall.*
- *Each wall and each half-space is a geodesically convex subset of $\Sigma(W, S)$.*
- *The collection of walls separates $\Sigma(W, S)$ into disjoint copies of the fundamental domain K.*
- *The stabilizer of each point in $\Sigma(W, S)$ is a finite Coxeter group, and the walls containing that point can be locally identified with the fixed hyperplanes of the standard action of this Coxeter group on $\mathbb{R}^n$.*

Though each wall of Σ is a contractible submanifold, a W–orbit of a wall has in general quite complicated topology. Even in the simple case where W is generated by reflections in a equilateral triangle in $\mathbb{R}^2$ the W–orbit of a wall is not contractible, as W–translates of a wall can intersect nontrivially. However, passing to a suitable subgroup fixes this problem.

Theorem 2.3 *W has a finite index torsion-free normal subgroup Γ, and the action of Γ on $\Sigma(W,S)$ admits a hierarchy.*

Proof The existence of such a subgroup Γ is well-known. The cutting submanifolds that we choose will be Γ–orbits of walls in $\Sigma(W,S)$.

A lemma of Millson and Jaffee [19] shows that any torsion-free normal subgroup of W has the trivial intersection property: for all $\gamma \in \Gamma$, either $\gamma\Sigma^s = \Sigma^s$ or $\gamma\Sigma^s \cap \Sigma^s = \varnothing$. Therefore, each Γ–orbit is a disjoint union of walls and has contractible components.

Once we have removed all the walls, we are left with disjoint copies of the fundamental domain K, and since Γ is of finite index in W, there are only finitely many orbits of walls to remove, so by Lemma 2.2 this is a tidy collection. Therefore, we are done by Theorem 1.4. □

Remark If W is a Coxeter group with nerve a PL triangulation of D^{n-1}, then $\Sigma(W,S)$ is an n–manifold with boundary, and these groups also virtually admit hierarchies.

3 L^2–homology

Let X be a proper, cocompact G–CW–complex, and let $C_*(X)$ denote the usual cellular chains of X, which we regard as left $\mathbb{Z}G$–modules. The *square-summable chains of X* are the tensor product

$$C_*^{(2)}(X) = L^2(G) \otimes_{\mathbb{Z}G} C_*(X),$$

where $L^2(G)$ is the Hilbert space of real-valued square-summable functions on G.

The usual boundary homomorphism $\partial\colon C_*(X) \to C_{*-1}(X)$ extends to a boundary operator $\partial\colon C_*^{(2)}(X) \to C_{*-1}^{(2)}(X)$ whose adjoint is the coboundary operator $\delta\colon C_*^{(2)}(X) \to C_{*+1}^{(2)}(X)$.

The (reduced) L^2–(co)homology groups can be defined as the kernel of the Laplacian operator:

$$L^2H_*(X;G) \cong L^2H^*(X;G) \cong \ker(\partial\delta + \delta\partial)\colon C_*^{(2)}(X) \to C_*^{(2)}(X).$$

These are Hilbert G–modules, and one defines L^2–Betti numbers as their von Neumann dimension. These definitions can be extended to arbitrary topological spaces with G–action using, for example, singular (co)chains, as follows; see [17, Chapter 6].

Let $\mathcal{N}(G)$ denote the von Neumann algebra of bounded G–equivariant operators on $L^2(G)$. As explained in [17], given an algebraic $\mathcal{N}(G)$–module A there is a well-behaved notion of dimension $\dim_{\mathcal{N}(G)}(A)$. The key feature of $\dim_{\mathcal{N}(G)}$ is *additivity* under short exact sequences.

Consider equivariant singular (co)homology with $\mathcal{N}(G)$ coefficients: $H_*^G(X,\mathcal{N}(G)) := H_*(\mathcal{N}(G)\otimes_{\mathbb{Z}G} C_*^{\text{sing}}(X))$ and $H_G^*(X,\mathcal{N}(G)) := H^*(\operatorname{Hom}_{\mathbb{Z}G}(C_*^{\text{sing}}(X),\mathcal{N}(G)))$. The i^{th} *L^2–Betti number* $b_i^{(2)}(X;G)$ is defined to be $\dim_{\mathcal{N}(G)}(H_i^G(X;\mathcal{N}(G)))$. We will also consider the cohomological version $b^i_{(2)}(X;G) := \dim_{\mathcal{N}(G)}(H^i_G(X;\mathcal{N}(G)))$.

Since the category of finitely generated projective $\mathcal{N}(G)$–modules is equivalent to the category of Hilbert G–modules (via completion), the resulting theory is equivalent to the combinatorial version for G–CW–complexes.

We record as a lemma some of the basic algebraic properties of L^2–homology that we will need. In the next section we will often use the fact that in the exact sequences a term between two zero-dimensional terms has to be zero-dimensional itself.

Lemma 3.1

- **Functoriality** *A G–equivariant map $f\colon (X_1,Y_1)\to(X_2,Y_2)$ between pairs of G–spaces induces a map $f_*\colon H_k^G(X_1,Y_1;\mathcal{N}(G))\to H_k^G(X_2,Y_2;\mathcal{N}(G))$. If f is a weak G–equivariant homotopy equivalence, then f_* is an isomorphism.*
- **Exact sequence of a pair** *Let (X,Y) be a pair of G–spaces, then the sequence*
$$\cdots\to H_i^G(Y;\mathcal{N}(G))\to H_i^G(X;\mathcal{N}(G))\to H_i^G(X,Y;\mathcal{N}(G))\to\cdots$$
is exact.
- **Multiplicativity** *Let $H<G$ be a subgroup of finite index. If X is a G–space then $b_i^{(2)}(X;H)=[G:H]b_i^{(2)}(X;G)$ and $b^i_{(2)}(X;H)=[G:H]b^i_{(2)}(X;G)$*
- **Excision** *Suppose (X,A,B) is a triple of G–spaces such that $Cl_X(B)\subset Int(A)$, then the map $H_*^G(X-B,A-B;\mathcal{N}(G))\to H_*^G(X,A;\mathcal{N}(G))$ is a isomorphism.*
- **Poincaré duality** *If G acts properly, cocompactly, and preserving orientation on an orientable n–manifold $(M,\partial M)$, then $H^i_G(M;\mathcal{N}(G))\cong H^G_{n-i}(M,\partial M;\mathcal{N}(G))$ and $H_i^G(M;\mathcal{N}(G))\cong H_G^{n-i}(M,\partial M;\mathcal{N}(G))$.*

- **Induction principle** *The L^2–homology of a G–space X is induced from the L^2–homology of its components:*

$$H_i^G(X;\mathcal{N}(G)) = \bigoplus_{[X^0]\in\pi_0(X)/G} \mathcal{N}(G)\otimes_{\mathcal{N}(\mathrm{St}_G X^0)} H_i^{\mathrm{St}_G X^0}(X^0;\mathcal{N}(\mathrm{St}_G X^0)),$$
$$b_i^{(2)}(X;G) = \sum_{[X^0]\in\pi_0(X)/G} b_i^{(2)}(X^0,\mathrm{St}_G X^0),$$

where the sums are over representatives of the orbits of the components of X.

- **Künneth formula** *If M is a G–space and Y is an H–space, then*

$$b_n^{(2)}(X\times Y) = \sum_{i+j=n} b_i^{(2)}(X) b_j^{(2)}(Y).$$

Remark The first four statements are quite standard and have analogous versions for L^2–cohomology. The proofs of the last two statements in [17] strongly depend on nice properties of $\dim_{\mathcal{N}(G)}$ with respect to tensor products and colimits. It's unclear whether their cohomological versions hold in this generality.

We will need the following version of Poincaré duality.

Lemma 3.2 *If $(M,\partial M)$ is an n–dimensional proper cocompact G–manifold with orientable components, then*

$$b_i^{(2)}(M,\partial M;G) = \sum_{[M^0]\in\pi_0(M)/G} b_{(2)}^{n-i}(M^0;\mathrm{St}_G M^0),$$

where the sums are over representatives of the orbits of the components of M.

Proof By the induction principle,

$$b_i^{(2)}(M,\partial M;G) = \sum_{[M^0]\in\pi_0(M)/G} b_i^{(2)}(M^0,\partial M;\mathrm{St}_G M^0).$$

Since each M^0 is contractible, it is orientable, and by taking, if necessary, an index 2 subgroup of $\mathrm{St}_G M^0$ we get a cocompact orientation preserving action on M^0. Thus we can apply Poincaré duality and multiplicativity to each M^0 to finish the proof. □

Definition For a discrete group G, define

$$b_k^{(2)}(G) := b_k^{(2)}(EG;G),$$
$$b_{(2)}^k(G) := b_{(2)}^k(EG;G).$$

By the functoriality property, these are well-defined. In fact, one can use L^2–(co)homology of any proper contractible G–space, since the chain complex of such a space still gives a projective resolution of $\mathbb{Q}$ over the group ring $\mathbb{Q}G$, and we are using $\mathcal{N}(G)$ coefficients anyway.

Note that in general the relation between the homological and cohomological versions of L^2–Betti numbers is unclear, however if X is a proper and cocompact G–CW–complex, a cellular version of the Hodge decomposition shows $b^k_{(2)}(X;G) = b_k^{(2)}(X;G)$. In the next two lemmas, we establish some partial results in this direction.

Lemma 3.3 *If X is a countable proper G–CW–complex, then $b^k_{(2)}(X;G) \geq b_k^{(2)}(X;G)$. In particular for any group G, $b^k_{(2)}(G) \geq b_k^{(2)}(G)$.*

Proof X is the colimit of a directed sequence of proper, cocompact G–complexes $\{X_i \mid i \in \mathbb{N}\}$. By [17, Theorems 6.13 and 6.18], we have

$$b_k^{(2)}(X;G) = \sup_i \inf_{j \geq i} \dim_{\mathcal{N}(G)}(\operatorname{im} i_{i,j} \colon H_k^G(X_i) \to H_k^G(X_j)),$$
$$\dim_{\mathcal{N}(G)} \varprojlim H^k_G(X_i;G) = \sup_i \inf_{j \geq i} \dim_{\mathcal{N}(G)}(\operatorname{im} i^{i,j} \colon H^k_G(X_j) \to H^k_G(X_i)).$$

Since X_i and X_j are cocompact proper G–complexes, the terms on the right-hand side are equal, and because $H^k_G(X)$ surjects onto $\varprojlim H^k_G(X_i)$, we have that $b^k_{(2)}(X;G) \geq b_k^{(2)}(X;G)$.

The last sentence follows, since the standard bar construction gives a countable model for EG. □

Lemma 3.4 *If G acts properly and cocompactly on an n–dimensional contractible manifold without boundary, then $b_k^{(2)}(G) = b^{n-k}_{(2)}(G) = b_{n-k}^{(2)}(G) = b^k_{(2)}(G)$.*

Proof Applying Lemma 3.3 and 3.2 twice we get

$$b_k^{(2)}(G) = b^{n-k}_{(2)}(G) \geq b_{n-k}^{(2)}(G) = b^k_{(2)}(G) \geq b_k^{(2)}(G).$$

Thus the inequalities above are equalities, and the result is proved. □

Remark In general, it is not true that a proper G–action on a manifold M is weak G–homotopy equivalent to a G–action on a countable CW–complex, even for finite groups. For example, Ancel and Guilbault [1] proved that doubling an open manifold along a $\mathcal{Z}$–boundary results in a closed manifold, and therefore any such $\mathcal{Z}$–boundary is the fixed point set of an involution acting on a closed manifold. The $\mathcal{Z}$–boundaries may have uncountable fundamental group.

We will also need the following version of excision.

Lemma 3.5 *Let (X, A, B) be a G–triple of spaces. Suppose that for every open U in X there is an excision isomorphism $H_*(U, U \cap A, \mathbb{R}) \cong H_*(U - B, (U \cap A) - B, \mathbb{R})$. Then we have an isomorphism $H_*^G(X, A, \mathcal{N}(G)) \cong H_*^G(X - B, A - B, \mathcal{N}(G))$.*

Proof For this proof, we will say a G–subspace Y of X "satisfies (A, B)–excision" if $H_*^G(Y, Y \cap A, \mathcal{N}(G)) \cong H_*^G(Y - B, (Y \cap A) - B, \mathcal{N}(G))$. If G is finite, we have $H_*^G(X, A, \mathcal{N}(G)) \cong H_*(X, A, \mathbb{R})$, so by assumption we get the desired isomorphism. Now, suppose G is infinite.

Consider the collection of open G–invariant subsets V satisfying (A, B)–excision, partially ordered by inclusion. Since singular homology commutes with direct limits, it follows by the Zorn lemma that there is a maximal such V. We claim that $V = X$.

Indeed, otherwise by properness there is a open set $U \not\subset V$ with finite stabilizer G_U, and for which $gU \cap U = \varnothing$ for $g \notin G_U$. We have the relative Mayer–Vietoris sequence with $\mathcal{N}(G)$ coefficients:

$$H_*^G(GU \cap V, A) \to H_*^G(GU, A) \oplus H_*^G(V, A) \to H_*^G(GU \cup V, A).$$

We have a corresponding Mayer–Vietoris sequences where we have excised B. Note that $GU \cong G \times_{G_U} U$ and $GU \cap V \cong G \times_{G_U} (U \cap V)$, since each element of G not in G_U moves U off of itself. Therefore, we have induction isomorphisms:

$$H_*^G(GU, \mathcal{N}(G)) = \mathcal{N}(G) \otimes_{\mathcal{N}(G_U)} H_*^{G_U}(U, \mathcal{N}(G_U))$$

and

$$H_*^G(GU \cap V, \mathcal{N}(G)) = \mathcal{N}(G) \otimes_{\mathcal{N}(G_U)} H_*^{G_U}(U \cap V, \mathcal{N}(G_U)).$$

By the finite group case, we have (A, B)–excision for U and $U \cap V$, and therefore, we have (A, B)–excision for GU and $GU \cap V$. Since we have (A, B)–excision for V, by the 5 lemma, this implies excision for $GU \cup V$, contradiction. □

4 Vanishing conjectures and results

Conjecture (Singer conjecture) *If G acts properly and cocompactly on a contractible n–manifold without boundary, then $b_i^{(2)}(G) = 0$ for $i \neq n/2$.*

In general, it seems this conjecture is stronger than the original Singer conjecture, which assumed G to be torsion-free. By the multiplicativity of L^2–Betti numbers, the two versions are equivalent for type VF groups (groups which are virtually finite type).

The conjecture holds for trivial reasons in dimensions ≤ 2. In dimension 3, Lott and Lück [16] proved the conjecture for all fundamental groups of manifolds that satisfy the geometrization conjecture, therefore by Perelman's work [21; 22; 23] it holds for all type VF groups acting properly and cocompactly on contractible 3–manifolds. We record this as a theorem.

Theorem 4.1 *The Singer conjecture is true for type VF groups in dimensions $n \leq 3$.*

Definition The *L^2–dimension* of a group G, $L^2\mathrm{dim}(G)$ is the largest degree n, such that $b^n_{(2)}(G) \neq 0$.

Definition The *action dimension* $\mathrm{actdim}(G)$ of a group G is the least dimension of a contractible manifold which admits a proper G–action.

Action dimension was introduced and studied by Bestvina, Kapovich, and Kleiner in [2]. One consequence of their work is that an n–fold product of nonabelian free groups does not act properly discontinuously on a contractible $(2n-1)$–manifold. Since nonabelian free groups have $b^1_{(2)}(F_n) \neq 0$, it follows from the Künneth formula that the n–fold products have nontrivial $b^n_{(2)}$. Therefore, as noted in [8], their result is implied by the following conjecture.

Conjecture (actdim conjecture) $\mathrm{actdim}(G) \geq 2L^2\mathrm{dim}(G)$.

Remark In [8], the conjecture is stated in terms of homology. Lemma 3.3 implies that the above version is potentially stronger than the original.

We note the following bounds, which are well-known for cellular actions.

Lemma 4.2 *We have* $\mathrm{actdim}(G) \geq \mathrm{cd}_{\mathbb{Q}}(G)$. *If G is virtually torsion free, then* $\mathrm{actdim}(G) \geq \mathrm{vcd}(G)$.

Proof We only prove the first inequality, the proof of the second one is entirely similar. Suppose G acts properly on a contractible n–manifold M^n, and let A be a $\mathbb{Q}[G]$–module. We need to show that $H^i(G; A) = 0$ for $i > n$. We will use equivariant sheaf cohomology; see [13, Chapter V].

Denote by $\mathcal{A}$ the constant sheaf on M with stalk A, and by $\mathcal{A}^G$ the sheaf on M/G whose sections over an open set in M/G are G–invariant sections of $\mathcal{A}$ over its preimage in M. Since the action is proper and the coefficients are rational, [13, Corollaire to Théorème 5.3.1, page 204] applies and we obtain a spectral sequence with E_2–term $H^i(G; H^j(M, \mathcal{A}))$ which converges to $H^{i+j}(M/G, \mathcal{A}^G)$. Since M is contractible, the spectral sequence collapses, and we get $H^i(G, A) = H^i(M/G, \mathcal{A}^G)$.

Since M is a manifold, it is a separable metrizable space, and [20, Theorem 4.3.4] implies that M/G is also metrizable, in particular paracompact, and $\dim M/G \leq n$. Finally, paracompactness of M/G allows us to use Čech cohomology to conclude that $H^i(M/G, \mathcal{A}^G) = 0$ for $i > n$. □

Definition The *cocompact action dimension* $\mathrm{cadim}(G)$ of a group G is the least dimension of a proper cocompact contractible G–manifold.

Obviously, $\mathrm{actdim}(G) \leq \mathrm{cadim}(G)$. We do not know of a type VF group G with $\mathrm{actdim}(G) < \mathrm{cadim}(G)$.

Conjecture (cadim conjecture) $\mathrm{cadim}(G) \geq 2L^2\mathrm{dim}(G)$.

Note that these conjectures all have smooth and PL versions.

Lemma 4.3 *The actdim conjecture implies the cadim conjecture, which in turn implies the Singer conjecture.*

Proof The first implication is trivial. By considering cohomology with compact support, we see that a group acting properly and cocompactly on a contractible n–manifold without boundary has $\mathrm{actdim} = \mathrm{cadim} = \mathrm{cd}_{\mathbb{Q}} = n$, so the second implication follows from Lemma 3.4. □

Shmuel Weinberger pointed out that a recent theorem of Craig Guilbault [14] implies that for type F groups the difference between cadim and actdim is at most 1, at least in high dimensions.

Theorem 4.4 [14] *For an open manifold M^n ($n \geq 5$), $M^n \times \mathbb{R}$ is homeomorphic to the interior of a compact $(n+1)$–manifold with boundary if and only if M^n has the homotopy type of a finite complex.*

If G acts freely and properly on a contractible manifold M, we can apply Guilbault's theorem to the interior of M/G to get the following.

Corollary 4.5 *If G is type F and* $\mathrm{actdim}(G) \geq 5$, *then* $\mathrm{cadim}(G) \leq \mathrm{actdim}(G) + 1$.

Although the precise relationship between actdim and cadim is unclear, we can still show the two conjectures are equivalent, at least for type VF groups.

Theorem 4.6 *The cadim and actdim conjectures are equivalent for type VF groups.*

Proof We need to show that the cadim conjecture implies the actdim conjecture. So, let G acting properly on a contractible n–manifold M be a counterexample to the actdim conjecture, ie $2L^2\mathrm{dim}(G) - n > 0$. By removing the boundary, we can assume that M is open. If H is finite index in G and type F, then M/H is an open aspherical manifold of finite homotopy type, and $2L^2\dim(H) - n > 0$. Note that by the Künneth formula, by taking direct products of H with itself, we can assume that H acts freely on a contractible n–manifold M with $2L^2\dim(H) - n$ arbitrarily large. (The Künneth formula applies here since H is type F and therefore homological and cohomological L^2–Betti numbers are the same.) By Theorem 4.4, $M/H \times \mathbb{R}$ is the interior of a compact manifold. Therefore, the action of H on the universal cover of this compact manifold is a counterexample to the cadim conjecture. □

Since Guilbault's result holds in the PL and smooth categories, the smooth and PL versions of these conjectures are also equivalent to each other.

Remark These conjectures put restrictions on the embedding dimension of a $K(G,1)$–space. For example, if $b^i_{(2)}(G) \neq 0$, the cadim conjecture implies that no $K(G,1)$ space can embed in $\mathbb{R}^{2i-1}$.

Since any contractible proper G–manifold can be used to compute $b^i_{(2)}(G)$, using Lemma 3.2 we obtain an equivalent series of conjectures in terms of manifolds.

Conjecture (cadim conjecture in dimension n) *Suppose $(M, \partial M)$ is an n–manifold with contractible components which admits a proper cocompact G–action. Then $b_i^{(2)}(M, \partial M; G) = 0$ for $i < n/2$.*

We now consider these conjectures in the context of manifolds with hierarchies. Excision (Lemma 3.5) allows us to apply the argument used to derive sequence (1) to L^2–homology to obtain the following.

Lemma 4.7 *If (M, F) is a tidy pair and N is M cut-open along F, there is an exact sequence with $\mathcal{N}(G)$ coefficients*

$$\cdots \to H^G_k(F, \partial F) \xrightarrow{i_*} H^G_k(M, \partial M) \to H^G_k(N, \partial N) \to H^G_{k-1}(F, \partial F) \to \cdots \tag{3}$$

Thus we have the following apparently weaker version of the cadim conjecture.

Conjecture (weak cadim conjecture) *If (M^{2k+1}, F^{2k}) is a tidy pair, then the map induced by inclusion $i_*\colon H^G_k(F, \partial; \mathcal{N}(G)) \to H^G_k(M, \partial; \mathcal{N}(G))$ has zero-dimensional image.*

Lemma 4.8 *Suppose that (M^n, F) is a tidy G–pair, N is M cut-open by F, and the cadim conjecture in dimension $(n-1)$ holds for F. Then the cadim conjecture in dimension n holds for M if and only if it holds for N and, if $n = 2k+1$ is odd, the weak cadim conjecture holds for (M, F).*

Proof First, suppose that the cadim conjecture holds for M. We have $b_i^{(2)}(M,\partial M)=0$ for $i<n/2$, and $b_i^{(2)}(F,\partial F)=0$ for $i<(n-1)/2$. Then the cadim conjecture holds for N by sequence (3).

Next, suppose the cadim conjecture holds for N, so that $b_i^{(2)}(N,\partial N)=0$ for $i<n/2$, and $b_i^{(2)}(F,\partial F)=0$ for $i<(n-1)/2$. By the same argument as above, we can say that $b_i^{(2)}(M,\partial M)=0$ for $i<(n+1)/2$.

Now, we only have to consider the case where $n=2k+1$ and $i=k+1$. The weak cadim conjecture says that the map $H_k^G(F,\partial F;\mathcal{N}(G))\to H_k^G(M,\partial M;\mathcal{N}(G))$ in sequence (3) has zero-dimensional image. The result follows. □

Theorem 4.9 *The cadim conjecture in dimension $2k-1$ implies the cadim conjecture in dimension $2k$ for manifolds with hierarchies. The cadim conjecture in dimension $2k$ and the weak cadim conjecture in dimension $2k+1$ imply the cadim conjecture in dimension $2k+1$ for manifolds with hierarchies.*

Proof This is immediate by induction on the length of the hierarchy, using Lemmas 1.3 and 4.8, and noting that the cadim conjecture holds for manifolds with compact components. □

A somewhat surprising result is a converse to the second implication in Lemma 4.3, at least if we somewhat restrict the category. By *the cadim conjecture with PL boundary*, we mean the version of the cadim conjecture where we only allow manifolds whose boundaries admit PL structures and actions restricted to the boundaries are PL.

Theorem 4.10 *The Singer conjecture and the cadim conjecture with PL boundary are equivalent.*

The result follows immediately from the following key lemma and induction.

Lemma 4.11 *The Singer conjecture in dimension n and the cadim conjecture with PL boundary in dimension $(n-1)$ imply the cadim conjecture with PL boundary in dimension n.*

In the proof of this lemma we will need the following result, which is probably well-known to the experts, but we could not find any reference for it.

Proposition 4.12 *Suppose a group G acts PL properly and cocompactly on a polyhedron M. Then M has a G–equivariant PL triangulation.*

Proof Using cocompactness choose a finite subpolyhedron $P \subset M$, so that $GP = M$, and set $F := \{f \in G \mid fP \cap P \neq \varnothing\}$. By properness, F is finite. Set $Q := \bigcup\{fP \mid f \in F\}$. Let $\mathcal{T}$ be a triangulation of Q such that each fP is a subcomplex of Q. The pullback $f^*\mathcal{T}$ is a triangulation of P such that $f\colon f^*\mathcal{T} \to \mathcal{T}$ is simplicial. Let $\mathcal{S}$ be a common subdivision of $\{f^*\mathcal{T} \mid f \in F\}$. Then each $f \in F$ is a linear map $\mathcal{S} \to \mathcal{T}$. For each $x \in M$ set

$$C_x = \bigcap\{g\sigma \mid g \in G,\ \sigma \in \mathcal{S},\ g\sigma \ni x\}.$$

Then the collection $\mathcal{C} = \{C_x \mid x \in M\}$ is a G–equivariant cover of M, which restricts to a cellulation of P, since for $x \in P$ the last two conditions imply that $g \in F$, and thus C_x is a finite intersection of linear simplices: a closed convex cell. Therefore, $\mathcal{C}$ is a cellulation of M, and taking the barycentric subdivision gives an equivariant triangulation of M. □

Proof of Lemma 4.11 We use the equivariant Davis reflection group trick as in [7; 5]. The idea is that the trick turns the input of the cadim conjecture (a contractible manifold with boundary and proper cocompact group action) into the input of the Singer conjecture (a contractible manifold without boundary and proper cocompact group action). In addition, the newly constructed manifold action admits a hierarchy ending at a disjoint union of copies of the original. Once this has been established, the proof is more or less the same as that of Theorem 4.9.

Suppose that G acts properly and cocompactly on a contractible n–manifold with boundary $(M, \partial M)$ (if $\partial M = \varnothing$, we are done since we are assuming the Singer conjecture holds). Let L be a flag PL triangulation of M that is equivariant with respect to the G–action. Suppose that the stabilizer of any simplex fixes the simplex pointwise and that $g.v \cap \mathrm{Lk}(v) = \varnothing$ for all $g \in G$ and and $v \in L^0$ (by subdividing, these triangulations can always be constructed). We can now apply the equivariant reflection group trick. Indeed, L determines a right-angled Coxeter group W, and we can form the basic construction $\mathcal{U} = \mathcal{U}(W, M)$. By the conditions imposed on L, there is an action of G on W which determines a semidirect product $W \rtimes G$. Since $\mathcal{U}/W \rtimes G \cong M/G$, $W \rtimes G$ acts cocompactly on $\mathcal{U}$. Here are some key properties of the reflection group trick:

- Each wall is a codimension-1 contractible submanifold of N.
- There are a finite number of $W \rtimes G$–orbits of walls, and each orbit is a disjoint union of walls.

- Any component of a nonempty intersection of orbits of walls is itself a Davis complex and is therefore contractible.
- The stabilizer of each wall acts properly and cocompactly on the wall.
- The collection of walls looks locally like a right-angled hyperplane arrangement in $\mathbb{R}^n$ (this is where we need the triangulation to be PL).

It follows, similarly to Theorem 1.4, that the $W \rtimes G$ action on $\mathcal{U}$ admits a hierarchy that ends in disjoint copies of M, where the cutting submanifolds are $W \rtimes G$–orbits of walls. Since $\mathcal{U}$ has no boundary, and we are assuming that the Singer conjecture holds for $\mathcal{U}$, the cadim conjecture in dimension n holds for $\mathcal{U}$. Since we are also assuming the cadim conjecture in dimension $n-1$, it follows by applying Lemma 4.8 inductively that the cadim conjecture holds for the original M. □

If M is a PL manifold, and the action of G is PL, then the basic construction $\mathcal{U}$ and the action of $W \rtimes G$ in the above argument are also PL.

If M is a smooth manifold, and the action of G is smooth, the existence of a smooth equivariant triangulation is part of the main result of [15]. Moreover, in this case the reflection trick produces a smooth manifold with the smooth action.

Thus we have the following.

Corollary 4.13 *The Singer conjecture and the cadim conjecture are equivalent in the smooth and the PL categories.*

Since for a type VF group the reflection trick produces another type VF group, Theorem 4.6 gives us a corollary.

Corollary 4.14 *For type VF groups the Singer conjecture and the action dimension conjecture are equivalent in the smooth and the PL categories.*

Since TOP = PL in dimension 2, Theorem 4.1 and Theorem 4.10 imply another.

Corollary 4.15 *The cadim conjecture holds for all type VF groups in dimensions less than or equal to* 3.

Now, Corollary 4.15, Theorem 4.9 and Lemma 4.3 imply our main theorem.

Theorem 4.16 *The Singer conjecture holds for all type VF groups that admit a hierarchy in dimensions less than or equal to* 4.

Theorem 4.16 and Theorem 2.3 now imply our main applications.

Theorem 4.17 *If W is a Coxeter group with nerve a triangulation of S^3, then the Singer conjecture holds for W acting on $\Sigma(W,S)$.*

Theorem 4.18 *If W is a Coxeter group with nerve a triangulation of D^3, then $b_i^{(2)}(W)=0$ for $i>2$.*

Remark The hierarchies for Coxeter groups have more structure in the following sense: the hierarchy for $\Sigma(W,S)$ induces a hierarchy on each wall. This means that if we restrict our attention to Coxeter groups, we can relax many of the assumptions. For instance, Theorem 4.9 restricted to Coxeter groups need only assume the cadim conjecture in dimension $2k-1$ for manifolds with hierarchies.

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