

The η -inverted \mathbb{R} -motivic sphere

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We use an Adams spectral sequence to calculate the \mathbb{R} -motivic stable homotopy groups after inverting η . The first step is to apply a Bockstein spectral sequence in order to obtain h_1 -inverted \mathbb{R} -motivic Ext groups, which serve as the input to the η -inverted \mathbb{R} -motivic Adams spectral sequence. The second step is to analyze Adams differentials. The final answer is that the Milnor–Witt $(4k-1)$ -stem has order 2^{u+1} , where u is the 2-adic valuation of $4k$. This answer is reminiscent of the classical image of J . We also explore some of the Toda bracket structure of the η -inverted \mathbb{R} -motivic stable homotopy groups.

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1 Introduction

The first exotic property of motivic stable homotopy groups is that the Hopf map η is not nilpotent. This means that inverting η can be useful for understanding the global structure of motivic stable homotopy groups.

In Andrews and Miller [3] and Guillou and Isaksen [5], the η -inverted \mathbb{C} -motivic 2-completed stable homotopy groups $\hat{\pi}_{*,*}^{\mathbb{C}}[\eta^{-1}]$ were explicitly computed to be

$$\mathbb{F}_2[\eta^{\pm 1}][[\mu, \varepsilon]/\varepsilon^2].$$

This result naturally suggests that one should study the structure of η -inverted motivic stable homotopy groups over other fields.

In the present article, we consider the η -inverted \mathbb{R} -motivic 2-completed stable homotopy groups $\hat{\pi}_{*,*}^{\mathbb{R}}[\eta^{-1}]$. Our main tool is the motivic Adams spectral sequence, which takes the form

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}] \implies \hat{\pi}_{*,*}^{\mathbb{R}}[\eta^{-1}].$$

Here $\mathcal{A}^{\mathbb{R}}$ is the \mathbb{R} -motivic Steenrod algebra, and $\mathbb{M}_2^{\mathbb{R}}$ is the motivic \mathbb{F}_2 -cohomology of \mathbb{R} . We will exhaustively compute this spectral sequence.

We begin with computing the Adams E_2 -page $\text{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}]$ using the ρ -Bockstein spectral sequence; see Hill [6] and Dugger and Isaksen [4]. This spectral sequence takes the form

$$\text{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho][h_1^{-1}] \implies \text{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}],$$

where $\mathcal{A}^{\mathbb{C}}$ is the \mathbb{C} -motivic Steenrod algebra and $\mathbb{M}_2^{\mathbb{C}}$ is the motivic \mathbb{F}_2 -cohomology of \mathbb{C} .

The input to the ρ -Bockstein spectral sequence is completely known from Guillou and Isaksen [5]. In order to deduce differentials, one first observes, as in Dugger and Isaksen [4], that the groups

$$\text{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[\rho^{-1}, h_1^{-1}]$$

with ρ and h_1 both inverted are easy to describe. Then there is only one pattern of ρ -Bockstein differentials that is consistent with this ρ -inverted calculation.

Having obtained the Adams E_2 -page $\text{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}]$, the next step is to compute Adams differentials. The extension of scalars functor from \mathbb{R} -motivic homotopy theory to \mathbb{C} -motivic homotopy theory induces a map

$$\begin{array}{ccc} \text{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}] & \implies & \widehat{\pi}_{*,*}^{\mathbb{R}}[\eta^{-1}] \\ \downarrow & & \downarrow \\ \text{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[h_1^{-1}] & \implies & \widehat{\pi}_{*,*}^{\mathbb{C}}[\eta^{-1}] \end{array}$$

of Adams spectral sequences. The bottom Adams spectral sequence is completely understood; see Andrews and Miller [3] and Guillou and Isaksen [5]. The Adams d_2 differentials in the top spectral sequence can then be deduced by the comparison map.

This leads to a complete description of the h_1 -inverted \mathbb{R} -motivic Adams E_3 -page. Over \mathbb{C} , it turns out that the h_1 -inverted Adams spectral sequence collapses at this point. However, over \mathbb{R} , there are higher differentials that we deduce from manipulations with Massey products and Toda brackets.

In the end, we obtain an explicit description of the h_1 -inverted \mathbb{R} -motivic Adams E_{∞} -page, from which we can read off the η -inverted stable motivic homotopy groups over \mathbb{R} .

In order to state the result, we need a bit of terminology. Because η belongs to $\widehat{\pi}_{1,1}^{\mathbb{R}}$, it makes sense to use a grading that is invariant under multiplication by η . The Milnor–Witt n -stem is the direct sum $\Pi_n = \bigoplus_p \widehat{\pi}_{p+n,p}^{\mathbb{R}}$. Then multiplication by η is an endomorphism of the Milnor–Witt n -stem.

- Theorem 1.1** (1) The η -inverted Milnor–Witt 0–stem $\Pi_0[\eta^{-1}]$ is $\mathbb{Z}_2[\eta^{\pm 1}]$, where \mathbb{Z}_2 is the ring of 2–adic integers.
- (2) If $k > 1$, then the η -inverted Milnor–Witt $(4k-1)$ –stem $\Pi_{4k-1}[\eta^{-1}]$ is isomorphic to $\mathbb{Z}/2^{u+1}[\eta^{\pm 1}]$ as a module over $\mathbb{Z}_2[\eta^{\pm 1}]$, where u is the 2–adic valuation of $4k$.
- (3) The η -inverted Milnor–Witt n –stem $\Pi_n[\eta^{-1}]$ is zero otherwise.

For degree reasons, the product structure on $\widehat{\pi}_{*,*}^{\mathbb{R}}[\eta^{-1}]$ is very simple. However, there are many interesting Toda brackets. We explore much of the 3–fold Toda bracket structure in this article. In particular, we will show that all of $\widehat{\pi}_{*,*}^{\mathbb{R}}[\eta^{-1}]$ can be constructed inductively via Toda brackets, starting from just 2 and the generator of the Milnor–Witt 3–stem.

Theorem 1.1 gives a familiar answer. These groups have the same order as the classical image of J . For example, Π_3 consists of elements of order 8, which is the same as the order of the image of J in the classical 3–stem. Similarly, Π_7 consists of elements of order 16, which is the same as the order of the image of J in the classical 7–stem. One might expect a geometric proof that directly compares the classical image of J spectrum with the η -inverted \mathbb{R} -motivic sphere. However, higher structure in the form of Toda brackets suggests that such a direct proof is not possible.

We also observe that our calculations are reminiscent of the classical Adams spectral sequence for v_1 -periodic homotopy at odd primes, as carried out in Andrews [2]. We are not aware of a structural reason why the calculations are so similar.

The calculation of the η -inverted \mathbb{R} -motivic homotopy groups leads to questions about η -inverted motivic homotopy groups over other fields. We leave it to the reader to speculate on the behavior of these η -inverted groups over other fields.

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2 Preliminaries

2.1 Notation

We continue with notation from [4] as follows:

- (1) $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$ is the motivic cohomology of \mathbb{C} with \mathbb{F}_2 coefficients, where τ has bidegree $(0, 1)$.
- (2) $\mathbb{M}_2^{\mathbb{R}} = \mathbb{F}_2[\tau, \rho]$ is the motivic cohomology of \mathbb{R} with \mathbb{F}_2 coefficients, where τ and ρ have bidegrees $(0, 1)$ and $(1, 1)$, respectively.
- (3) \mathcal{A}^{cl} is the classical mod 2 Steenrod algebra.

- (4) $\mathcal{A}^{\mathbb{C}}$ is the mod 2 motivic Steenrod algebra over \mathbb{C} .
- (5) $\mathcal{A}^{\mathbb{R}}$ is the mod 2 motivic Steenrod algebra over \mathbb{R} .
- (6) Ext_{cl} is the trigraded ring $\text{Ext}_{\mathcal{A}^{\text{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$.
- (7) $\text{Ext}_{\mathbb{C}}$ is the trigraded ring $\text{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$.
- (8) $\text{Ext}_{\mathbb{R}}$ is the trigraded ring $\text{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$.
- (9) $\widehat{\pi}_{*,*}^{\mathbb{C}}$ is the motivic stable homotopy ring of the 2-completed motivic sphere spectrum over \mathbb{C} .
- (10) $\widehat{\pi}_{*,*}^{\mathbb{R}}$ is the motivic stable homotopy ring of the 2-completed motivic sphere spectrum over \mathbb{R} .
- (11) Π_n is the Milnor–Witt n -stem $\bigoplus_p \widehat{\pi}_{p+n,p}^{\mathbb{R}}$.
- (12) $\mathcal{R} = \mathbb{F}_2[\rho, h_1^{\pm 1}]$.
- (13) The symbols v_1^4 and P are used interchangeably for the Adams periodicity operator.

2.2 Grading conventions

We follow [7] in grading Ext according to (s, f, w) , where:

- (1) f is the Adams filtration, ie the homological degree.
- (2) $s + f$ is the internal degree, ie that corresponding to the first coordinate in the bidegree of the Steenrod algebra.
- (3) s is the stem, ie the internal degree minus the Adams filtration.
- (4) w is the weight.

Following this grading convention, the elements τ and ρ , as elements of $\text{Ext}_{\mathbb{R}}$, have degrees $(0, 0, -1)$ and $(-1, 0, -1)$ respectively.

We will consider the groups $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$ in which h_1 has been inverted. The degree of h_1 is $(1, 1, 1)$. As in [5], for this purpose it is convenient to introduce the following gradings whose values are zero for h_1 :

- (5) $mw = s - w$ is the Milnor–Witt degree.
- (6) $c = s + f - 2w$ is the Chow degree.

In order to avoid notational clutter, we will often drop h_1 from the notation. Since h_1 is a unit, no information is lost by doing this. The correct powers of h_1 can always be recovered by checking degrees.

For example, in Lemma 3.1 below, we claim that there is a differential $d_3^\rho(v_1^4) = \rho^3 v_2$ in the ρ -Bockstein spectral sequence. Strictly speaking, this formula is nonsensical because $d_3^\rho(v_1^4)$ has Adams filtration 5 while v_2 has Adams filtration 1. The correct full formula is $d_3^\rho(v_1^4) = \rho^3 h_1^4 v_2$.

If we are to ignore multiples of h_1 , we must rely on gradings that take value 0 on h_1 . This explains our preference for Milnor–Witt degree mw and Chow degree c .

3 The ρ -Bockstein spectral sequence

Recall [6; 4] that the ρ -Bockstein spectral sequence takes the form

$$\text{Ext}_{\mathbb{C}}[\rho] \implies \text{Ext}_{\mathbb{R}}.$$

After inverting h_1 , by [5, Theorem 1.1] this takes the form

$$\mathcal{R}[v_1^4, v_2, v_3, \dots] \implies \text{Ext}_{\mathbb{R}}[h_1^{-1}],$$

where $\mathcal{R} = \mathbb{F}_2[\rho, h_1^{\pm 1}]$. Table 1 lists the generators of the Bockstein E_1 -page.

(mw, c)	generator
$(0, 1)$	ρ
$(4, 4)$	v_1^4
$(3, 1)$	v_2
$(7, 1)$	v_3
$(15, 1)$	v_4
$(2^n - 1, 1)$	v_n

Table 1: Bockstein E_1 -page generators

Lemma 3.1 *In the ρ -Bockstein spectral sequence, there are differentials*

$$d_{2^n-1}^\rho(v_1^{2^n}) = \rho^{2^n-1} v_n \quad \text{for } n \geq 2.$$

All other nonzero differentials follow from the Leibniz rule.

The first few examples of these differentials are $d_3(v_1^4) = \rho^3 v_2$, $d_7(v_1^8) = \rho^7 v_3$ and $d_{15}(v_1^{16}) = \rho^{15} v_4$.

Proof Inverting ρ induces a map

$$\begin{array}{ccc} \text{Ext}_{\mathbb{C}}[h_1^{-1}][\rho] & \xrightarrow{\rho\text{-Bss}} & \text{Ext}_{\mathbb{R}}[h_1^{-1}] \\ \rho\text{-inv} \downarrow & & \downarrow \rho\text{-inv} \\ \text{Ext}_{\mathbb{C}}[h_1^{-1}][\rho^{\pm 1}] & \xrightarrow{\rho\text{-Bss}} & \text{Ext}_{\mathbb{R}}[h_1^{-1}, \rho^{-1}] \end{array}$$

of ρ -Bockstein spectral sequences. We will establish differentials in the ρ -inverted spectral sequence. The map of spectral sequences then implies that the same differentials occur when ρ is not inverted.

Recall [4, Theorem 4.1] there is an isomorphism $\text{Ext}_{\mathbb{C}l}[\rho^{\pm 1}] \cong \text{Ext}_{\mathbb{R}}[\rho^{-1}]$ sending the classical element h_0 to the motivic element h_1 . Using also that $\text{Ext}_{\mathbb{C}l}[h_0^{-1}] = \mathbb{F}_2[h_0^{\pm 1}]$, it follows $\text{Ext}_{\mathbb{R}}[h_1^{-1}, \rho^{-1}]$ is isomorphic to $\mathcal{R}[\rho^{-1}]$. Then the ρ -inverted ρ -Bockstein spectral sequence takes the form

$$\mathcal{R}[\rho^{-1}][v_1^4, v_2, v_3, \dots] \xrightarrow{\rho\text{-Bss}} \mathcal{R}[\rho^{-1}].$$

Because the target of the ρ -inverted spectral sequence is very small, essentially everything must either support a differential or be hit by a differential.

The ρ -Bockstein differentials have degree $(-1, 0)$ with respect to the grading (mw, c) used in Table 1. The elements $\rho^k v_2$ cannot support differentials because there are no elements in the Milnor–Witt 2-stem. The only possibility is that after inverting ρ , there is a ρ -Bockstein differential $d_3(v_1^4) = \rho^3 v_2$.

Then the ρ -inverted E_4 -page is $\mathcal{R}[v_1^8, v_3, v_4, \dots]$. The elements $\rho^k v_3$ cannot support differentials because the ρ -inverted E_4 -page has no elements in the Milnor–Witt 6-stem. The only possibility is that after inverting ρ , there is a ρ -Bockstein differential $d_7(v_1^8) = \rho^7 v_3$.

In general, the ρ -inverted E_{2n-1} -page is $\mathcal{R}[v_1^{2^n}, v_n, v_{n+1}, \dots]$. The elements $\rho^k v_n$ cannot support differentials because the ρ -inverted E_{2n-1} -page has no elements in the Milnor–Witt $(2^n - 2)$ -stem. The only possibility is that after inverting ρ , there is a ρ -Bockstein differential $d_{2n-1}(v_1^{2^n}) = \rho^{2^n-1} v_n$. □

The ρ -Bockstein E_{∞} -page can be directly computed from the Leibniz rule and the differentials in Lemma 3.1. For example, $d_3(v_1^4) = \rho^3 v_2$, so $d_3(v_1^{4+8k}) = \rho^3 v_1^{8k} v_2$. This establishes the relation $\rho^3 v_1^{8k} v_2 = 0$.

To ease the notation in Proposition 3.2, we write P rather than v_1^4 .

Proposition 3.2 The ρ -Bockstein E_∞ -page is the \mathcal{R} -algebra on the generators $P^{2^{n-1}k}v_n$ for $n \geq 2$ and $k \geq 0$ (see Table 2), subject to the relations

$$\rho^{2^n-1} P^{2^{n-1}k} v_n = 0$$

for $n \geq 2$ and $k \geq 0$, and

$$P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m + P^{2^{n-1}(k+2^{m-n}j)} v_n \cdot v_m = 0$$

for $m \geq n \geq 2$, $k \geq 0$ and $j \geq 0$.

(mw, c)	generator	ρ -torsion
$(0, 1)$	ρ	∞
$(0, 0)$	h_1	∞
$(3, 1) + k(8, 8)$	$P^{2k}v_2$	3
$(7, 1) + k(16, 16)$	$P^{4k}v_3$	7
$(15, 1) + k(32, 32)$	$P^{8k}v_4$	15
$(2^n - 1, 1) + k(2^{n+1}, 2^{n+1})$	$P^{2^{n-1}k}v_n$	$2^n - 1$

Table 2: Bockstein E_∞ -page generators

Remark 3.3 In practice, the relations mean that every P can be shifted onto the v_n with minimal n in any monomial. Thus an \mathcal{R} -module basis is given by monomials of the form $P^{2^{n-1}k}v_n \cdot v_{m_1} \cdots v_{m_a}$, where $n \leq m_1 \leq \cdots \leq m_a$. For example,

$$P^2v_2 \cdot P^4v_2 = P^6v_2 \cdot v_2, \quad P^4v_2 \cdot P^8v_3 = P^{12}v_2 \cdot v_3, \quad P^4v_3 \cdot P^{48}v_5 = P^{52}v_3 \cdot v_5.$$

4 The Adams E_2 -page

Having obtained the ρ -Bockstein E_∞ -page in Section 3, our next task is to consider hidden extensions in $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$. We will show that there are no hidden relations. This will require some careful analysis of degrees, as well as some manipulations with Massey products.

The ρ -Bockstein E_∞ -page is an associated graded object of $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$. Elements of the E_∞ -page only determine elements of $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$ up to higher filtration. Therefore, we must be careful about choosing specific generators of $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$.

We will show in Lemma 4.1 that $P^{2^{n-1}k}v_n$ detects a unique element of $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$. Therefore, we may unambiguously use the same notation $P^{2^{n-1}k}v_n$ for an element of $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$.

In general, the ρ -Bockstein spectral sequence does not allow for hidden extensions by ρ . More precisely, if x is an element of the ρ -Bockstein E_∞ -page such that $\rho^k x = 0$, then x detects an element of $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$ that is also annihilated by ρ^k . Beware that x might detect more than one element of $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$, and some such elements might not be annihilated by ρ^k . Nevertheless, there is always at least one element that is annihilated by ρ^k .

For example, the relation $\rho^{2^{n-1}} P^{2^{n-1}k} v_n = 0$ in the ρ -Bockstein E_∞ -page lifts to give the same relation in $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$.

Lemma 4.1 *For each $n \geq 2$ and $k \geq 0$, the element $P^{2^{n-1}k} v_n$ of the Bockstein E_∞ -page detects a unique element of $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$.*

Proof We need to show that in the ρ -Bockstein E_∞ -page, $P^{2^{n-1}k} v_n$ does not share bidegree with an element of higher filtration.

First suppose that $P^{2^{n-1}k} v_n$ has the same bidegree as $\rho^b P^{2^{m-1}j} v_m$. Then

$$(2^n - 1, 1) + k(2^{n+1}, 2^{n+1}) = (2^m - 1, 1) + j(2^{m+1}, 2^{m+1}) + b(0, 1).$$

Considering only the Milnor–Witt degree, we have

$$2^n(2k + 1) = 2^m(2j + 1).$$

Therefore, $n = m$ and $k = j$, so $b = 0$.

Suppose that $P^{2^{n-1}k} v_n$ shares bidegree with some element x . By Remark 3.3, we may assume that x is of the form $\rho^b P^{2^{m_1-1}j} v_{m_1} \cdot v_{m_2} \cdots v_{m_a}$, where $m_1 \leq m_2 \leq \cdots \leq m_a$. Since $\rho^{2^{m_1-1}} P^{2^{m_1-1}j} v_{m_1} = 0$, we may also assume that $b \leq 2^{m_1} - 2$. Because of the previous paragraph, we may assume that $a \geq 2$. We wish to show that $b = 0$.

We first show that $n \geq m_a$. Let $u(x)$ be the difference $mw - c$. We have that $u(P^{2^{m_1-1}j} v_{m_1}) = 2^{m_1} - 2$ and $u(\rho) = -1$. Since $b \leq 2^{m_1} - 2$, it follows that $u(\rho^b P^{2^{m_1-1}j} v_{m_1}) \geq 0$. Thus

$$2^n - 2 = u(P^{2^{n-1}k} v_n) = u(\rho^b P^{2^{m_1-1}j} v_{m_1}) + u(v_{m_2} \cdots v_{m_a}) \geq u(v_{m_a}) = 2^{m_a} - 2,$$

so that $n \geq m_a$.

Now consider the Milnor–Witt and Chow degrees modulo 4. We have

$$(-1, 1) \equiv (-a, a + b) \pmod{4},$$

so $a \equiv 1 \pmod{4}$ and $b \equiv 0 \pmod{4}$. Thus either $b = 0$, which was what we wanted to show, or $b \geq 4$.

We may now assume that $b \geq 4$. Since $\rho^4 P^{2j} v_2 = 0$, we must have $m_1 \geq 3$, so that all m_i , and also n , are at least 3.

Next, consider degrees modulo 8. Comparing degrees gives

$$(-1, 1) \equiv (-a, a + b) \pmod{8}.$$

Thus $b \equiv 0 \pmod{8}$, so that $b \geq 8$. Since $\rho^8 P^{4j} v_3 = 0$, we must have $j_1 \geq 4$, and therefore n and all other j_i are also at least 4. This argument can be continued to establish that b and n must be arbitrarily large under the assumption that $b > 0$. \square

Lemma 4.2 For each $n \geq 2$ and $k \geq 0$, the element $P^{2^{n-1}k} v_n \cdot v_n$ of the Bockstein E_∞ -page detects a unique element of $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$.

Proof The Milnor–Witt degree of $P^{2^{n-1}k} v_n \cdot v_n$ is even, while the Milnor–Witt degree of $\rho^b P^{2^{m-1}j} v_m$ is odd. Therefore, these elements cannot share bidegree.

Now suppose that the element $P^{2^{n-1}k} v_n \cdot v_n$ has the same bidegree as the element $\rho^b P^{2^{m_1-1}j} v_{m_1} \cdot v_{m_2} \cdots v_{m_a}$, with $m_1 \leq m_2 \leq \cdots \leq m_a$, $b \leq 2^{m_1} - 2$ and $a \geq 2$. The rest of the proof is essentially the same as the proof of Lemma 4.1. Consider $u = mw - c$ to get that $n \geq m_a$. Then consider congruences $(-2, 2) \equiv (-a, a + b)$ modulo higher and higher powers of 2 to obtain that $b = 0$. \square

Remark 4.3 The obvious generalization of Lemma 4.2 to elements of the form $P^{2^{n-1}k} v_n \cdot v_m$ is false. For example, $P^2 v_2 \cdot v_5$ has the same degree as $\rho^4 v_3^6$.

Remark 4.4 Lemmas 4.1 and 4.2 are equivalent to the claim that there are no ρ multiples in the ρ -Bockstein E_∞ -page in the same bidegrees as either $P^{2^{n-1}k} v_n$ or $P^{2^{n-1}k} v_n \cdot v_n$. This implies that there are also no ρ multiples in $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$ that share bidegree with these elements; we will need this fact later.

Lemma 4.5 $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$ is zero when the Milnor–Witt stem mw and the Chow degree c are both equal to $2i$ with $i \geq 1$.

Proof Under the condition $mw = c = 2i$, inspection of Table 1 shows the ρ -Bockstein E_1 -page consists of products of elements of the form v_1^4 or $\rho^{2^n+2^{m-4}} v_n v_m$. In the E_∞ -page, $\rho^{2^n+2^{m-4}} v_n v_m = 0$ since $\rho^{2^n-1} v_n = 0$. Also, v_1^{4k} supports a differential for all $k \geq 0$. \square

Lemma 4.6 For each $n \geq 2$, $k \geq 0$ and $m > n$, we have a Massey product

$$P^{2^{n-1}k+2^{m-2}} v_n = \langle \rho^{2^m-2^n} v_m, \rho^{2^n-1}, P^{2^{n-1}k} v_n \rangle$$

in $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$ with no indeterminacy.

Proof The Bockstein differential $d_{2^m-1}^\rho(P^{2^{m-2}}) = \rho^{2^m-1}v_m$ and May’s convergence theorem [8, Theorem 4.1] imply that the Massey product is detected by $P^{2^{n-1}k+2^{m-2}}v_n$ in the ρ -Bockstein E_∞ -page. There are no crossing Bockstein differentials as all classes are in nonnegative ρ -filtration. Lemma 4.1 says that this ρ -Bockstein E_∞ -page element detects a unique element of $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$.

The indeterminacy of the bracket is generated by products of the form $\rho^{2^m-2^n}v_m \cdot x$ and $y \cdot P^{2^{n-1}k}v_n$, where x and y have appropriate bidegrees. We showed in Lemma 4.5 that 0 is the only possibility for x or y . □

Remark 4.7 Lemma 4.6 gives many different Massey products for the same element. For example,

$$P^8v_2 = \langle \rho^4v_3, \rho^3, P^6v_2 \rangle = \langle \rho^{12}v_4, \rho^3, P^4v_2 \rangle = \langle \rho^{28}v_5, \rho^3, v_2 \rangle.$$

Lemma 4.8 For $m > n \geq 2$, there is a Massey product

$$P^{2^{n-1}k+2^{m-2}}v_n = \langle P^{2^{n-1}k}v_n, \rho^{2^m-2}v_m, \rho \rangle$$

in $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$ with no indeterminacy.

Proof The Massey product formula follows from the Bockstein differential

$$d_{2^m-1}^\rho(P^{2^{m-2}}) = \rho^{2^m-1}v_m$$

and May’s convergence theorem [8, Theorem 4.1]. There are no crossing Bockstein differentials as all classes are in nonnegative ρ -filtration. As in the proof of Lemma 4.6, we need Lemma 4.1 to tell us that the element $P^{2^{n-1}k+2^{m-2}}v_n$ of the ρ -Bockstein E_∞ -page detects a unique element of $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$.

The indeterminacy of the bracket is generated by products of the form $P^{2^{n-1}k}v_n \cdot x$ and $y \cdot \rho$. We showed in Lemma 4.5 that 0 is the only possibility for x . We observed in Remark 4.4 that $y \cdot \rho$ must be zero because there are no multiples of ρ in the appropriate bidegree. □

The relations in the Bockstein E_∞ -page given in Proposition 3.2 may lift to $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$ with additional terms that are multiples of ρ . In other words, there may be hidden relations in the Bockstein spectral sequence. For example, for degree reasons it is possible that $P^2v_2 \cdot P^{16}v_5 + P^{18}v_2 \cdot v_5$ equals $\rho^4P^{16}v_3 \cdot v_3^5$. Proposition 4.9 shows that there are no such hidden terms in the relations in $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$.

Proposition 4.9 There are no hidden relations in the Bockstein spectral sequence.

Proof The relation $\rho^{2^{n-1}} P^{2^{n-1}k} v_n = 0$ in the ρ -Bockstein E_∞ -page lifts to give the same relation in $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$, as we observed in the discussion preceding [Lemma 4.1](#). Therefore, we need only compute the products $P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m$ in $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$ for $m \geq n$.

[Lemma 4.6](#) implies that $P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m$ equals

$$P^{2^{n-1}k} v_n \langle \rho^{2^m} v_{m+1}, \rho^{2^{m-1}}, P^{2^{m-1}(j-1)} v_m \rangle.$$

Shuffle to obtain

$$\langle P^{2^{n-1}k} v_n, \rho^{2^m} v_{m+1}, \rho^{2^{m-1}} \rangle P^{2^{m-1}(j-1)} v_m.$$

This expression is contained in

$$\langle P^{2^{n-1}k} v_n, \rho^{2^{m+1}-2} v_{m+1}, \rho \rangle P^{2^{m-1}(j-1)} v_m,$$

which equals $P^{2^{n-1}k+2^{m-1}} v_n \cdot P^{2^{m-1}(j-1)} v_m$ by [Lemma 4.8](#).

By induction, $P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m$ equals $P^{2^{n-1}(k+2^{m-n}j)} v_n \cdot v_m$. □

Theorem 4.10 $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$ is the \mathcal{R} -algebra on the generators $P^{2^{n-1}k} v_n$ for $n \geq 2$ and $k \geq 0$ (see [Table 2](#)), subject to the relations

$$\rho^{2^{n-1}} P^{2^{n-1}k} v_n = 0$$

for $n \geq 2$ and $k \geq 0$, and

$$P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m + P^{2^{n-1}(k+2^{m-n}j)} v_n \cdot v_m = 0$$

for $m \geq n \geq 2$, $k \geq 0$ and $j \geq 0$.

Proof This follows immediately from [Propositions 3.2](#) and [4.9](#). □

Remark 4.11 Analogously to [Remark 3.3](#), an \mathcal{R} -module basis for $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$ is given by monomials of the form $P^{2^{n-1}k} v_n \cdot v_{m_1} \cdots v_{m_a}$, where $n \leq m_1 \leq \cdots \leq m_a$.

5 Adams differentials

Before computing with the h_1 -inverted \mathbb{R} -motivic Adams spectral sequence, we will consider convergence. A priori, there could be an infinite family of homotopy classes linked together by infinitely many hidden η multiplications. These classes would not be detected in $\text{Ext}_{\mathbb{R}}[h_1^{-1}]$. [Lemma 5.1](#) implies that this cannot occur for degree reasons.

Lemma 5.1 *Let $m > 0$ be a fixed Milnor–Witt stem. There exists a constant A such that $\text{Ext}_{\mathbb{R}}^{(s,f,w)}$ vanishes when $s - w = m$, s is nonzero, $f > A$ and $f > s + 1$.*

Lemma 5.1 can be restated in the following more casual form: within a fixed Milnor–Witt stem, there exists a horizontal line and a line of slope 1 such that $\text{Ext}_{\mathbb{R}}$ vanishes in the region above both lines, except in the 0–stem. Figure 1 depicts the shape of the vanishing region.

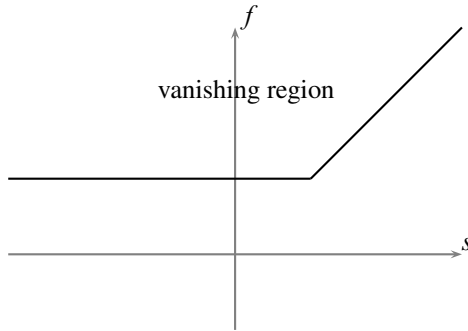


Figure 1: The vanishing region in a Milnor–Witt stem

Proof This argument occurs in $\text{Ext}_{\mathbb{R}}$, where h_1 has not been inverted.

As explained in [4, Theorem 4.1], the elements in the m –stem of the classical Ext groups Ext_{cl} correspond to elements of $\text{Ext}_{\mathbb{R}}$ in the Milnor–Witt m –stem that remain nonzero after ρ is inverted, ie that support infinitely many multiplications by ρ . Each stem of Ext_{cl} is finite except for the 0–stem. For $m > 0$, choose A to be larger than the Adams filtrations of all of the elements in the m –stem of Ext_{cl} . Then A is larger than the Adams filtrations of every element of $\text{Ext}_{\mathbb{R}}$ in the Milnor–Witt m –stem that remain nonzero after ρ is inverted.

Let x be a nonzero element of $\text{Ext}_{\mathbb{R}}^{(s,f,w)}$ such that $s - w = m$, $f > A$ and $f > s + 1$. We will show that s must equal zero.

The choice of A guarantees that x is annihilated by some positive power of ρ . Suppose that $\rho^k x = 0$ but $\rho^{k-1} x$ is nonzero, for some $k > 0$. Then there must be a differential in the ρ –Bockstein spectral sequence of the form $d_k(y) = \rho^k x$, where y is an element of $\text{Ext}_{\mathbb{C}}$ in degree $(s - k + 1, f - 1, w - k)$.

The argument from [1] establishes a vanishing line of slope 1 in the nonzero stems of $\text{Ext}_{\mathbb{C}}$. The conditions $f > s + 1$ and $k > 0$ imply that the element y lies strictly above this vanishing line, so it must be of the form $\tau^a h_0^b$ with $b \geq 1$. The only ρ –Bockstein differentials on such classes are $d_1(\tau^{2c+1} h_0^b) = \rho \tau^{2c} h_0^{b+1}$, which implies that x must be of the form $\tau^{2c} h_0^b$. This shows that $s = 0$. □

The h_1 -inverted motivic Adams spectral sequence over \mathbb{C} was studied in [5; 3]. It takes the form

$$\mathbb{F}_2[h_1^{\pm 1}, P, v_2, v_3, \dots] \implies \widehat{\pi}_{*,*}^{\mathbb{C}}[\eta^{-1}],$$

where $\widehat{\pi}_{*,*}^{\mathbb{C}}[\eta^{-1}]$ is the η -inverted motivic stable homotopy ring of the 2-completed motivic sphere spectrum over \mathbb{C} . This spectral sequence has differentials

$$d_2(P^k v_n) = P^k v_{n-1}^2$$

for all $k \geq 0$ and all $n \geq 3$. As usual, we omit any powers of h_1 .

Lemma 5.2 *In the h_1 -inverted \mathbb{R} -motivic Adams spectral sequence, there are differentials*

$$d_2(P^{2^{n-1}k} v_n) = P^{2^{n-1}k} v_{n-1}^2$$

for all $k \geq 0$ and all $n \geq 3$.

Proof There is an extension of scalars functor from \mathbb{R} -motivic homotopy theory to \mathbb{C} -motivic homotopy theory. This functor induces a map

$$\begin{array}{ccc} \text{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}] & \implies & \widehat{\pi}_{*,*}^{\mathbb{R}}[\eta^{-1}] \\ \downarrow & & \downarrow \\ \text{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[h_1^{-1}] & \implies & \widehat{\pi}_{*,*}^{\mathbb{C}}[\eta^{-1}] \end{array}$$

from the \mathbb{R} -motivic Adams spectral sequence to the \mathbb{C} -motivic Adams spectral sequence. This map takes ρ to zero.

The above map of spectral sequences implies that the \mathbb{R} -motivic Adams differential $d_2(P^{2^{n-1}k} v_n)$ equals $P^{2^{n-1}k} v_{n-1}^2$ plus terms that are divisible by ρ . Lemma 4.2 implies that there are no possible additional terms in the relevant bidegree. \square

Our next task is to completely describe the Adams E_3 -page. First, we explore some elements that survive to the E_3 -page. We will consider these elements more carefully in Proposition 5.4.

Despite the differential $d_2(P^{4k} v_3) = P^{4k} v_2^2$, the element $\rho^3 P^{4k} v_3$ survives to the E_3 -page because $\rho^3 P^{4k} v_2^2$ is zero. Similarly, $\rho^{2^{n-1}-1} P^{2^{n-1}k} v_n$ survives to the E_3 -page.

The element $P^2 v_2^2$ looks like it should be hit by an Adams d_2 differential on $P^2 v_3$. However, $P^2 v_3$ did not survive the ρ -Bockstein spectral sequence. Therefore, there is nothing to hit $P^2 v_2^2$ and it survives to the Adams E_3 -page. The same observation applies to the elements $P^{2^{n-1}(2j+1)} v_n^2$.

We record the following simple computation, as we will employ it several times.

Lemma 5.3 *Let S be an \mathbb{F}_2 -algebra. Let $B = S[w_1, w_2, \dots]$ be a polynomial ring in infinitely many variables, and define a differential on B by $\partial(w_n) = w_{n-1}^2$ for $n \geq 2$. Then $H^*(B, \partial) \cong S[w_1]/w_1^2$.*

In fact, we will use a slight generalization of Lemma 5.3 in which $\partial(w_n)$ is equal to $u_n w_{n-1}^2$, where u_n is a unit in S . This generalization implies, for example, that the h_1 -inverted \mathbb{C} -motivic Adams E_3 -page is $\mathbb{F}_2[h_1^{\pm 1}, P, v_2]/v_2^2$.

Proposition 5.4 *The h_1 -inverted \mathbb{R} -motivic Adams E_3 -page is free as an \mathcal{R} -module on the generators listed in Table 3 for $n \geq 2$, $k \geq 0$ and $j \geq 0$. Almost all products of these generators are zero, except that*

$$P^{4k} v_2 \cdot P^{4j+2} v_2 = P^{4k+4j+2} v_2^2$$

and for $n \geq 3$,

$$\rho^{2^{n-1}-1} P^{2^{n-1} \cdot 2k} v_n \cdot \rho^{2^{n-1}-1} P^{2^{n-1}(2j+1)} v_n = \rho^{2^n-2} P^{2^{n-1}(2k+2j+1)} v_n^2.$$

(mw, c)	generator	ρ -torsion
$(0, 0)$	1	∞
$(3, 1) + k(8, 8)$	$P^{2k} v_2$	3
$(7, 4) + k(16, 16)$	$\rho^3 P^{4k} v_3$	4
$(15, 8) + k(32, 32)$	$\rho^7 P^{8k} v_4$	8
$(2^n - 1, 2^{n-1}) + k(2^{n+1}, 2^{n+1})$	$\rho^{2^{n-1}-1} P^{2^{n-1}k} v_n$	2^{n-1}
$(6, 2) + (2j + 1)(8, 8)$	$P^{2(2j+1)} v_2^2$	3
$(14, 2) + (2j + 1)(16, 16)$	$P^{4(2j+1)} v_3^2$	7
$(30, 2) + (2j + 1)(32, 32)$	$P^{8(2j+1)} v_4^2$	15
$(2^{n+1} - 2, 2) + (2j + 1)(2^{n+1}, 2^{n+1})$	$P^{2^{n-1}(2j+1)} v_n^2$	$2^n - 1$

Table 3: \mathcal{R} -module generators for the Adams E_3 -page

Remark 5.5 The relations in Proposition 5.4 are just the ones that are obvious from the notation. For example,

$$v_2 \cdot P^2 v_2 = P^2 v_2^2, \quad \rho^3 P^4 v_3 \cdot \rho^3 P^8 v_3 = \rho^6 P^{12} v_3^2.$$

Proof of Proposition 5.4 Let $\text{Ext}\langle k, b \rangle$ be the $\mathbb{F}_2[h_1^{\pm 1}]$ -submodule of the h_1 -inverted \mathbb{R} -motivic Adams E_2 -page on generators of the form $\rho^b P^k v_{m_1} v_{m_2} \cdots v_{m_a}$ such that $m_1 \leq m_2 \leq \cdots \leq m_a$. Note that $b \leq 2^{m_1} - 2$ in this situation, since $\rho^{2^{m_1}-1} P^k v_{m_1} = 0$.

Also, k must be a multiple of 2^{m_1-1} . By Lemma 5.2 and the fact that ρ is a permanent cycle, each $\text{Ext}\langle k, b \rangle$ is a differential graded submodule. Thus it suffices to compute the cohomology of each $\text{Ext}\langle k, b \rangle$.

We start with $\text{Ext}\langle 0, b \rangle$, which is equal to $\rho^b \cdot \mathbb{F}_2[h_1^{\pm 1}, v_m, v_{m+1}, \dots]$ as a differential graded $\mathbb{F}_2[h_1^{\pm 1}]$ -module, where m is the smallest integer such that $b \leq 2^m - 2$. Now Lemma 5.3 implies that $H^*(\text{Ext}\langle 0, b \rangle, d_2)$ is a free $\mathbb{F}_2[h_1^{\pm 1}]$ -module on two generators ρ^b and $\rho^b v_m$.

So far, we have demonstrated that the powers of ρ and the elements

$$v_2, \quad \rho v_2, \quad \rho^2 v_2, \quad \rho^3 v_3, \dots, \rho^6 v_3, \quad \rho^7 v_4, \dots$$

are present in the h_1 -inverted \mathbb{R} -motivic Adams E_3 -page.

The module $\text{Ext}\langle k, b \rangle$ is zero when k is odd.

Now assume that k is equal to 2 modulo 4. If $b \leq 2$, then $\text{Ext}\langle k, b \rangle$ is equal to $\rho^b P^k v_2 \cdot \mathbb{F}_2[h_1^{\pm 1}, v_2, v_3, \dots]$ as a differential graded $\mathbb{F}_2[h_1^{\pm 1}]$ -module. Lemma 5.3 implies that $H^*(\text{Ext}\langle k, b \rangle, d_2)$ is a free $\mathbb{F}_2[h_1^{\pm 1}]$ -module on two generators $\rho^b P^k v_2$ and $\rho^b P^k v_2^2$. If $b \geq 3$, then $\text{Ext}\langle k, b \rangle$ is zero because $\rho^3 P^k v_2 = 0$.

We have now shown that the elements

$$P^k v_2, \quad \rho P^k v_2, \quad \rho^2 P^k v_2, \quad P^k v_2^2, \quad \rho P^k v_2^2, \quad \rho^2 P^k v_2^2$$

are present in the h_1 -inverted \mathbb{R} -motivic Adams E_3 -page for all k congruent to 2 modulo 4.

Next assume that k is equal to 4 modulo 8. If $b \leq 2$, then $\text{Ext}\langle k, b \rangle$ is the free $\mathbb{F}_2[h_1^{\pm 1}]$ -module on generators $\rho^b P^k v_{m_1} \cdots v_{m_a}$ such that m_1 equals 2 or 3, and $m_1 \leq \cdots \leq m_a$. There is a short exact sequence

$$0 \rightarrow \text{Ext}\langle k, b \rangle \rightarrow \rho^b P^k \cdot \mathbb{F}_2[h_1^{\pm 1}, v_2, v_3, \dots] \rightarrow \rho^b P^k \cdot \mathbb{F}_2[h_1^{\pm 1}, v_4, v_5, \dots] \rightarrow 0,$$

where the differential is defined on the second and third terms in the obvious way. By Lemma 5.3, the homology of the middle term has two generators $\rho^b P^k$ and $\rho^b P^k v_2$, while the homology of the right term has two generators $\rho^b P^k$ and $\rho^b P^k v_4$. Analysis of the long exact sequence in homology shows that $H^*(\text{Ext}\langle k, b \rangle, d_2)$ has two generators $\rho^b P^k v_2$ and $\rho^b P^k v_2^3$.

Now assume that $3 \leq b \leq 6$. Since $\rho^b P^k v_2 = 0$, we get that $\text{Ext}\langle k, b \rangle$ is equal to $\rho^b P^k v_3 \cdot \mathbb{F}_2[h_1^{\pm 1}, v_3, v_4, \dots]$. Lemma 5.3 implies that $H^*(\text{Ext}\langle k, b \rangle, d_2)$ is a free $\mathbb{F}_2[h_1^{\pm 1}]$ -module on two generators $\rho^b P^k v_3$ and $\rho^b P^k v_3^2$.

Finally, if $b \geq 7$, then $\text{Ext}\langle k, b \rangle$ is zero because $\rho^7 P^k v_2 = 0$ and $\rho^7 P^k v_3 = 0$. This finishes the argument when k is equal to 4 modulo 8, and we have shown that $\text{Ext}_{\mathbb{R}}[h_1^{\pm 1}]$ contains the elements

$$\begin{aligned} &P^k v_2, \rho P^k v_2, \rho^2 P^k v_2, \\ &\rho^3 P^k v_3, \dots, \rho^6 P^k v_3, \\ &P^k v_3^2, \rho P^k v_3^2, \dots, \rho^6 P^k v_3^2. \end{aligned}$$

Analysis of the other cases is the same as the argument for $k \equiv 4$ modulo 8. The details depend on the value of k modulo 2^i and inequalities of the form $2^j - 1 \leq b \leq 2^{j+1} - 2$. In each case there is a short exact sequence of differential graded modules whose first term is $\text{Ext}\langle k, b \rangle$ and whose other two terms have homology that is computed by Lemma 5.3. □

We have now calculated the h_1 -inverted \mathbb{R} -motivic E_3 -page. This E_3 -page is displayed in Figure 2. Beware that the grading on this chart is not the same as in a standard Adams chart. The Milnor–Witt stem $mw = s - w$ is plotted on the horizontal axis, while the Chow degree $c = s + f - 2w$ is plotted on the vertical axis. As a result, an Adams d_r differential has slope $-r + 1$, rather than slope $-r$. Vertical lines in Figure 2 represent multiplications by ρ .

Our next goal is to establish the Adams d_3 differentials. Inspection of Figure 2 reveals that the only possible nonzero d_3 differentials might be supported on elements of the form $\rho^b P^{2^{n-1}k} v_n$ for $n \geq 4$. In fact, these differentials all occur, as indicated in Figure 2 by lines that go left one unit and up two units. We will establish these d_3 differentials by first proving a homotopy relation in Lemma 5.6.

Lemma 5.6 *For each $n \geq 2$ and $j \geq 0$, the element $P^{2^{n-1}(2j+1)} v_n^2$ is a permanent cycle that detects a ρ -divisible element of the η -inverted \mathbb{R} -motivic homotopy groups.*

Proof Inspection of Figure 2 shows that $P^{2^{n-1}(2j+1)} v_n^2$ cannot support a differential.

Lemma 4.8 implies that

$$P^{2^{n-1}(2j+1)} v_n^2 \in \langle \rho, \rho^{2^{n+1}-2} v_{n+1}, P^{2^n j} v_n^2 \rangle \text{ in } \text{Ext}_{\mathbb{R}}[h_1^{-1}].$$

In fact, the Massey product has no indeterminacy because of Remark 4.4 and Lemma 4.5.

We will now apply Moss’s convergence theorem [10, Theorem 1.2] to this Massey product. There is an Adams differential $d_2(P^{2^n j} v_{n+1}) = P^{2^n j} v_n^2$, so $P^{2^n j} v_n^2$ detects the homotopy element 0. By inspection of Figure 2, $\rho^{2^{n+1}-2} v_{n+1}$ is a permanent

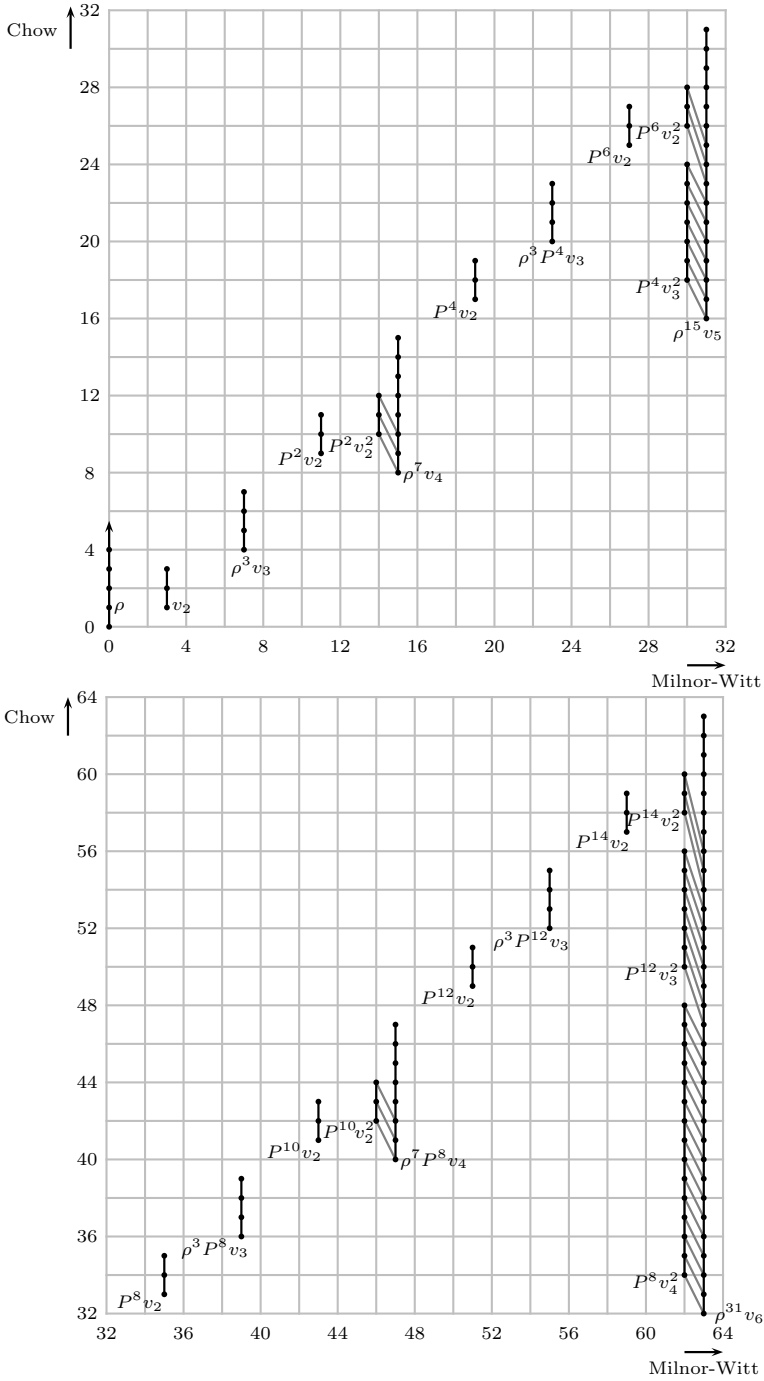


Figure 2: The η -inverted \mathbb{R} -motivic Adams E_3 -page

cycle; let α be a homotopy element detected by it. Moreover, $\rho\alpha$ is zero in homotopy because there are no classes in higher filtration that could detect it.

Moss’s convergence theorem says that the Toda bracket $\langle \rho, \alpha, 0 \rangle$ contains an element that is detected by $P^{2^{n-1}(2j+1)}v_n^2$. This Toda bracket consists entirely of multiples of ρ . □

Lemma 5.7 $d_3(\rho^{2^{n-1}-1}P^{2^{n-1}k}v_n) = P^{2^{n-3}+2^{n-1}k}v_{n-2}^2$ for $n \geq 4$.

Proof Lemma 5.6 shows that $P^{2^{n-3}+2^{n-1}k}v_{n-2}^2$ detects a class that is divisible by ρ . By inspection of Figure 2, there are no classes in lower filtration. Therefore, $P^{2^{n-3}+2^{n-1}k}v_{n-2}^2$ must detect zero, ie must be hit by a differential. It is apparent from Figure 2 that there is only one possible differential. □

Lemma 5.8 describes the higher Adams differentials.

Lemma 5.8 For $n \geq r + 1$ and $r \geq 3$,

$$d_r(\rho^{2^n-2^{n-r}+2-r+2}P^{2^{n-1}k}v_n) = P^{2^{n-1}k+2^{n-2}-2^{n-r}}v_{n-r+1}^2.$$

Proof The proof is essentially the same as the proof of Lemma 5.7. In the Milnor–Witt stem congruent to 2 modulo 4, Lemma 5.6 implies that every homotopy element is divisible by ρ . This implies that they must all be hit by differentials. Figure 2 indicates that there is just one possible pattern of differentials. □

From Lemma 5.8, it is straightforward to derive the h_1 -inverted Adams E_∞ -page, as shown in Figure 3.

Proposition 5.9 The h_1 -inverted Adams E_∞ -page is the \mathcal{R} -module on generators given in Table 4 for $n \geq 2$.

(mw, c)	generator	ρ -torsion
$(0, 0)$	1	∞
$(3, 1) + k(8, 8)$	$P^{2k}v_2$	3
$(7, 4) + k(16, 16)$	$\rho^3 P^{4k}v_3$	4
$(15, 11) + k(32, 32)$	$\rho^{10} P^{8k}v_4$	5
$(2^n - 1, 2^n - n - 1) + k(2^{n+1}, 2^{n+1})$	$\rho^{2^n-n-2} P^{2^{n-1}k}v_n$	$n + 1$

Table 4: \mathcal{R} -module generators for the Adams E_∞ -page

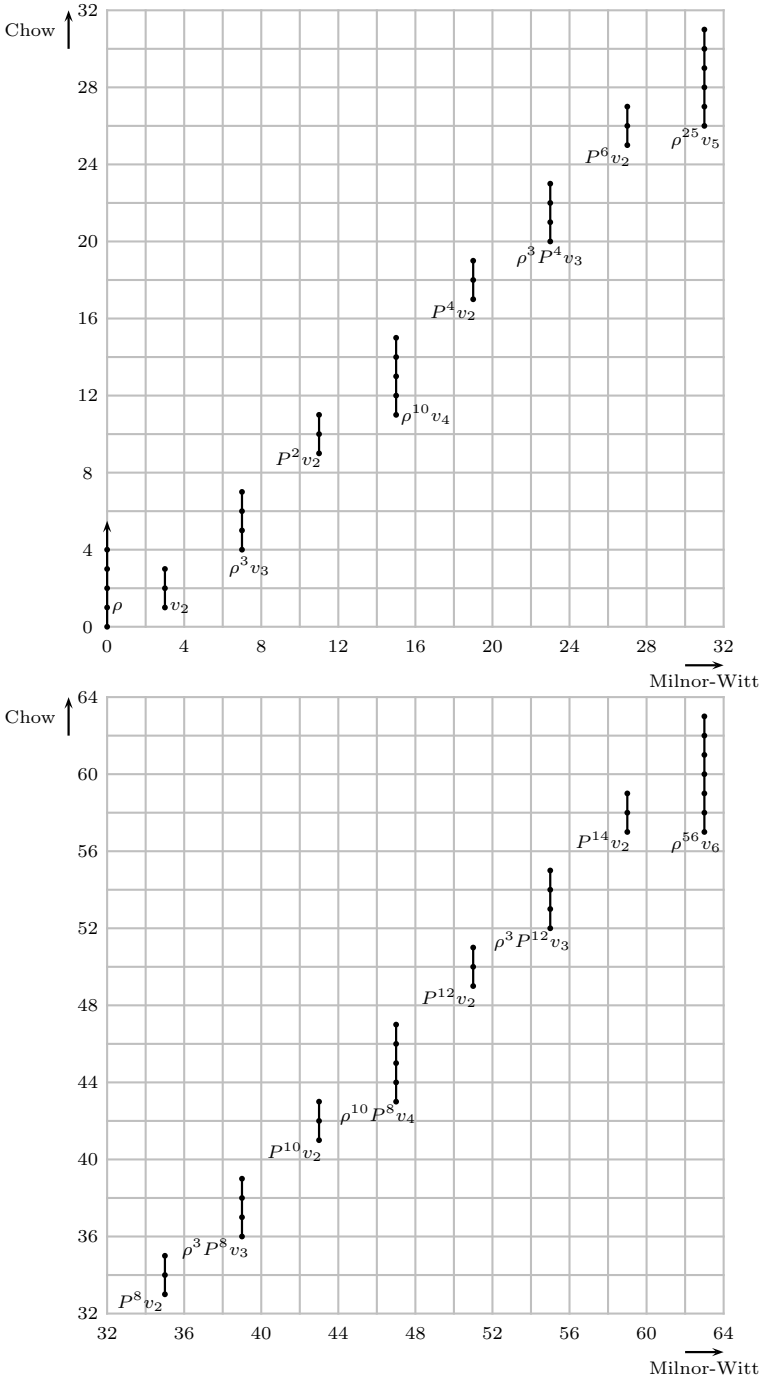


Figure 3: The η -inverted \mathbb{R} -motivic Adams E_∞ -page

6 η -inverted homotopy groups

From the h_1 -inverted Adams E_∞ -page, it is a short step to the η -inverted stable homotopy ring. First we must choose generators. Recall that Π_n is the Milnor–Witt n -stem $\bigoplus_p \hat{\pi}_{p+n,p}^{\mathbb{R}}$.

Definition 6.1 For k nonnegative and n at least 2, let $P^{2^{n-1}k}\lambda_n$ be an element of $\Pi_{2^{n+1}k+2^n-1}[\eta^{-1}]$ that is detected by $\rho^{2^n-n-2}P^{2^{n-1}k}v_n$.

There are choices in these definitions, which are measured by Adams E_∞ -page elements in higher filtration. For example, there are four possible choices for λ_2 because of the presence of ρv_2 and $\rho^2 v_2$ in higher filtration.

Theorem 6.2 *The η -inverted \mathbb{R} -motivic stable homotopy ring, as a $\mathbb{Z}_2[\eta^{\pm 1}]$ -module, is generated by 1 and $P^{2^{n-1}k}\lambda_n$ for $n \geq 2$ and $k \geq 0$. The generator $P^{2^{n-1}k}\lambda_n$ lies in $\Pi_{2^{n+1}k+2^n-1}[\eta^{-1}]$ and is annihilated by 2^{n+1} . All products are zero, except for those involving 2 or η .*

Proof In the η -inverted stable homotopy ring, ρ and 2 differ by a unit because $\rho\eta^2 = -2\eta$; see [9]. Therefore, the ρ -torsion information given in Proposition 5.9 translates to 2-torsion information in homotopy.

Except for 1, all $\mathbb{Z}_2[\eta^{\pm 1}]$ -module generators lie in Milnor–Witt stems that are congruent to 3 modulo 4. Therefore, such generators must multiply to zero. □

Table 5 lists all generators through the Milnor–Witt 63-stem. The table also identifies Toda brackets that contain each generator. These Toda brackets are computed in Section 7.

Table 5 also reveals a pattern that matches the classical image of J .

Corollary 6.3 *If $k > 1$, then $\Pi_{4k-1}[\eta^{-1}]$ is isomorphic to $\mathbb{Z}/2^{u+1}[\eta^{\pm 1}]$ as a module over $\mathbb{Z}_2[\eta^{\pm 1}]$, where u is the 2-adic valuation of $4k$.*

7 Toda brackets

Even though its primary multiplicative structure is uninteresting, the η -inverted \mathbb{R} -motivic stable homotopy ring has rich higher structure in the form of Toda brackets. We will explore some of the 3-fold Toda bracket structure. In particular, we will show that all of the generators can be inductively constructed via Toda brackets, starting

mw	E_∞	$\hat{\pi}_{*,*}^{\mathbb{R}}[\eta^{-1}]$	2^k -torsion	bracket	indeterminacy
0	1	1	∞		
3	v_2	λ_2	3		
7	$\rho^3 v_3$	λ_3	4	$\langle 2^3, \lambda_2, \lambda_2 \rangle$	$2^3 \lambda_3$
11	$P^2 v_2$	$P^2 \lambda_2$	3	$\langle 2^4, \lambda_3, \lambda_2 \rangle$	
15	$\rho^{10} v_4$	λ_4	5	$\langle 2^3, \lambda_2, P^2 \lambda_2 \rangle$	$2^3 \lambda_4$
19	$P^4 v_2$	$P^4 \lambda_2$	3	$\langle 2^5, \lambda_4, \lambda_2 \rangle$	
23	$\rho^3 P^4 v_3$	$P^4 \lambda_3$	4	$\langle 2^5, \lambda_4, \lambda_3 \rangle$	
27	$P^6 v_2$	$P^6 \lambda_2$	3	$\langle 2^5, \lambda_4, P^2 \lambda_2 \rangle$	
31	$\rho^{25} v_5$	λ_5	6	$\langle 2^3, \lambda_2, P^6 \lambda_2 \rangle$	$2^3 \lambda_5$
35	$P^8 v_2$	$P^8 \lambda_2$	3	$\langle 2^6, \lambda_5, \lambda_2 \rangle$	
39	$\rho^3 P^8 v_3$	$P^8 \lambda_3$	4	$\langle 2^6, \lambda_5, \lambda_3 \rangle$	
43	$P^{10} v_2$	$P^{10} \lambda_2$	3	$\langle 2^6, \lambda_5, P^2 \lambda_2 \rangle$	
47	$\rho^{10} P^8 v_4$	$P^8 \lambda_4$	5	$\langle 2^6, \lambda_5, \lambda_4 \rangle$	
51	$P^{12} v_2$	$P^{12} \lambda_2$	3	$\langle 2^6, \lambda_5, P^4 \lambda_2 \rangle$	
55	$\rho^3 P^{12} v_3$	$P^{12} \lambda_3$	4	$\langle 2^6, \lambda_5, P^4 \lambda_3 \rangle$	
59	$P^{14} v_2$	$P^{14} \lambda_2$	3	$\langle 2^6, \lambda_5, P^6 \lambda_2 \rangle$	
63	$\rho^{56} v_6$	λ_6	7	$\langle 2^3, \lambda_2, P^{14} \lambda_2 \rangle$	$2^3 \lambda_6$

Table 5: $\mathbb{Z}_2[\eta^{\pm 1}]$ -module generators for $\hat{\pi}_{*,*}^{\mathbb{R}}[\eta^{-1}]$

from just 2 and λ_2 . Table 5 lists one possible Toda bracket decomposition for each generator of Π_n for all n less than or equal to 63.

We observed in the proof of Theorem 6.2 that the element ρ of the Adams E_∞ -page detects the element 2 of the η -inverted stable homotopy ring. We will use this fact frequently in the following results.

Lemma 7.1 *The Toda bracket $\langle 2^3, \lambda_2, \lambda_2 \rangle$ contains an element detected by $\rho^3 v_3$ in Π_7 , and its indeterminacy is detected by $\rho^6 v_3$.*

Proof Moss’s convergence theorem [10, Theorem 1.2] and the differential $d_2(v_3) = v_2^2$ show that $\langle 2^3, \lambda_2, \lambda_2 \rangle$ is detected by $\rho^3 v_3$.

The indeterminacy follows from the facts that there are no multiples of λ_2 and that there is a unique multiple of 2^3 in Π_7 . □

Remark 7.2 The proof of Lemma 7.1 applies just as well to show that $\langle 2^4, \lambda_3, \lambda_3 \rangle$ is detected by $\rho^{10}v_4$ in Π_{15} . In higher stems, the analogous brackets do not produce generators. For example, the Massey product $\langle \rho^5, \rho^{10}v_4, \rho^{10}v_4 \rangle$ is already defined in Ext, which implies that the corresponding Toda bracket must be detected in filtration least 27. However, $\rho^{25}v_5$ detects the generator of Π_{31} , and it lies in filtration 26.

Lemma 7.3 For n at least 2, the Toda bracket $\langle 2^3, \lambda_2, P^{2^{n-1}-2}\lambda_2 \rangle$ is detected by the class $\rho^{2^{n+1}-n-3}v_{n+1}$. The indeterminacy in this Toda bracket is generated by $2^3\lambda_{n+1}$.

Proof This follows from Moss’s convergence theorem [10, Theorem 1.2], together with the Adams differential $d_n(\rho^{2^{n+1}-n-6}v_{n+1}) = P^{2^{n-1}-2}v_2^2$. □

Lemma 7.4 For n at least 2, the Toda bracket $\langle 2^{n+2}, \lambda_{n+1}, P^{2^{n-1}-2}\lambda_2 \rangle$ is detected by $P^{2^n-2}v_2$. The Toda bracket has no indeterminacy.

Proof Lemma 4.8 implies that there is a Massey product

$$P^{2^n-2}v_2 = \langle \rho^{n+2}, \rho^{2^{n+1}-n-3}v_{n+1}, P^{2^{n-1}-2}v_2 \rangle,$$

with no indeterminacy. Moss’s convergence theorem [10, Theorem 1.2] establishes the desired result. □

Lemma 7.5 If $m > n \geq 2$, then the Toda bracket $\langle 2^{m+1}, \lambda_m, P^{2^{n-1}k}\lambda_n \rangle$ is detected by $\rho^{2^n-n-2}P^{2^{m-2}+2^{n-1}k}v_n$. The Toda bracket has no indeterminacy.

Proof Lemma 4.8 implies that there is a Massey product

$$\rho^{2^n-n-2}P^{2^{m-2}+2^{n-1}k}v_n = \langle \rho^{m+1}, \rho^{2^m-m-2}v_m, \rho^{2^n-n-2}P^{2^{n-1}k}v_n \rangle.$$

Moss’s convergence theorem [10, Theorem 1.2] establishes the desired result. □

Proposition 7.6 Every generator $P^{2^{n-1}k}\lambda_n$ of the η -inverted \mathbb{R} -motivic stable homotopy ring can be constructed via iterated 3-fold Toda brackets starting from 2 and λ_2 .

Proof Lemmas 7.3 and 7.4 alternately show that the generators λ_n and the generators $P^{2^n-2}\lambda_2$ can be constructed via iterated 3-fold Toda brackets starting from 2 and λ_2 . Then Lemma 7.5 shows that any $P^{2^{n-1}k}\lambda_n$ can be constructed. □

Example 7.7 Suppose we wish to find a Toda bracket decomposition for $P^{40}\lambda_3$. Since $40 = 2^{7-2} + 2^{3-1} \cdot 4$, we can apply Lemma 7.5 with $m = 7$, $n = 3$ and $k = 4$ to conclude that $P^{40}\lambda_3$ is detected by the Toda bracket $\langle 2^8, \lambda_7, P^8\lambda_3 \rangle$.

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