Characteristic classes of fiber bundles

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In this paper, we construct new characteristic classes of fiber bundles via flat connections with values in infinite-dimensional Lie algebras of derivations. In fact, choosing a fiberwise metric, we construct a chain map to the de Rham complex on the base space, and show that the induced map on cohomology groups is independent of the choice of metric. Moreover, we show that, applied to a surface bundle, our construction gives Morita–Miller–Mumford classes.

1 Introduction

The purpose of this paper is to construct characteristic classes of fiber bundles which are not necessarily principal bundles whose structure group is a finite-dimensional Lie group. The difficulty is that a diffeomorphism group, which is considered as the structure group for a general fiber bundle, is huge compared to a finite-dimensional Lie group.

An idea to overcome the difficulty is a "linearization" which means to replace the diffeomorphism group $\mathrm{Diff}(X)$ for the fiber X with the automorphism group of the tensor algebra of the first homology group $H = H_1(X)$. A main tool is the Maurer–Cartan form of the space of expansions which was originally considered by Kawazumi [4; 5] in the case of free groups.

Diagrammatically, the construction is as follows:
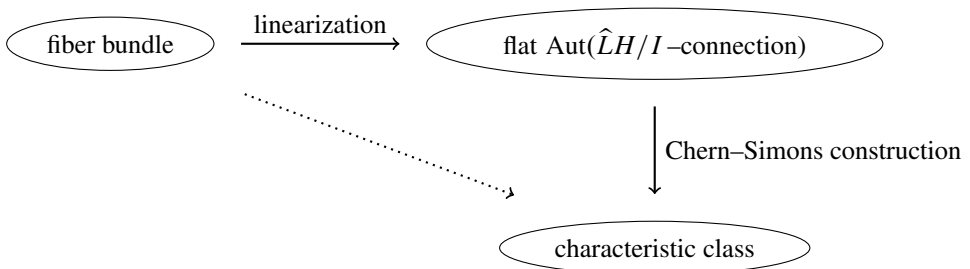
In this paper, for any fiber bundle $E \to B$ whose structure group satisfies a certain condition, choosing a fiberwise metric, we construct a chain map from the Chevalley–Eilenberg complex to the de Rham complex on the base space B via a flat $\mathrm{Aut}(\widehat{L}H/I)$–connection, and show that the induced map on cohomology groups is independent of the choice. We show that our construction gives Morita–Miller–Mumford classes if applied to a closed surface bundle. The similar construction of Morita–Miller–Mumford classes of 1–punctured surface bundles was previously given in [5].

Our construction is algebraic, and is closely related to the formal geometry of Kontsevich [7; 8] (see also Penkava and Schwarz [13]). For example, we have a description of our characteristic classes in terms of a graph complex with general valency when the Lie algebra is free. We intend to get such a graph complex in general cases, which is hopefully closely related to a combinatorial model of unstable homology of mapping class groups as in Godin [3].

It is interesting to compare our construction to a different approach on diffeomorphism groups with noncommutative geometry in Lott [9].

The paper is organized as follows: in Section 2, we introduce tools for construction of characteristic classes. The notions defined in Sections 2A and 2B are used in Section 3, and Johnson maps defined in Section 2C are used in Section 4. In Sections 3A and 3B, we construct characteristic classes under two different conditions of fiber bundles. In Section 3C, we clarify the relation between characteristic classes constructed in Sections 3A and 3B. In Section 3D, we describe these characteristic classes by Lie algebra cohomologies of the Lie algebra of derivations. In Section 4, we prove that our characteristic classes of a closed surface bundle give Morita–Miller–Mumford classes.

2 Preliminaries

For a finitely generated group π, we set $H = \pi^{\mathrm{ab}} \otimes \mathbb{R}$. We consider the completed tensor Hopf algebra $\widehat{T}H$ and the completed free Lie algebra $\widehat{L}H$ generated by H. The Lie algebra $\widehat{L}H$ can be regarded as the primitive part of the completed Hopf algebra $\widehat{T}H$. (For completed Hopf algebras, see Quillen [14].) Let J be the augmented

ideal of $\widehat{T}H$, which consists of all elements whose constant term is zero. The algebra $\widehat{T}H$ is described by the projective limit of finite-dimensional vector spaces

$$\widehat{T}H = \prod_{n=0}^{\infty} H^{\otimes n} = \varprojlim_{n} \widehat{T}H/J^n,$$

so it is endowed with the limit topology. We define $L_{\geq k} := \widehat{L}H \cap J^k$ and $L_k := \widehat{L}H \cap H^{\otimes k}$.

A closed two-sided Lie ideal I contained in $L_{\geq 2}$ is called a *decomposable ideal* of $\widehat{L}H$. We identify I with the closed two-sided ideal generated by I in $\widehat{T}H$. Then $\widehat{T}H/I$ is a completed Hopf algebra.

We denote the completion of the group Hopf algebra $\mathbb{R}\pi$ of π by $\widehat{\mathbb{R}}\pi$.

2A Automorphisms and derivations

In this section, we will define infinite-dimensional Lie groups satisfying the diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Inn}(\widehat{L}H/I) & \longrightarrow & \mathrm{IAut}(\widehat{L}H/I) & \longrightarrow & \mathrm{IOut}(\widehat{L}H/I) & \longrightarrow & 1 \\
& & {\scriptstyle \text{surj.}}\uparrow & & {\scriptstyle \text{surj.}}\uparrow & & {\scriptstyle \text{surj.}}\uparrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\widehat{L}H) & \longrightarrow & \mathrm{IAut}_I(\widehat{L}H) & \longrightarrow & \mathrm{IOut}_I(\widehat{L}H) & \longrightarrow & 1
\end{array}
$$

and corresponding Lie algebras

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{IDer}(\widehat{L}H/I) & \longrightarrow & \mathrm{Der}^+(\widehat{L}H/I) & \longrightarrow & \mathrm{ODer}^+(\widehat{L}H/I) & \longrightarrow & 0 \\
& & {\scriptstyle \text{surj.}}\uparrow & & {\scriptstyle \text{surj.}}\uparrow & & {\scriptstyle \text{surj.}}\uparrow & & \\
0 & \longrightarrow & \mathrm{IDer}(\widehat{L}H) & \longrightarrow & \mathrm{Der}_I^+(\widehat{L}H) & \longrightarrow & \mathrm{ODer}_I^+(\widehat{L}H) & \longrightarrow & 0
\end{array}
$$

for a decomposable ideal I. Here the rows are exact.

We define the group of *positive automorphisms* of $\widehat{L}H/I$ by

$$\mathrm{IAut}(\widehat{L}H/I) := \{ f \in \mathrm{Aut}(\widehat{L}H/I) : f(x) = x + L_{\geq 2}/I \},$$

where $\mathrm{Aut}(\widehat{L}H/I)$ is the group of completed Lie algebra automorphisms of $\widehat{L}H/I$. It is a projective limit of finite-dimensional Lie groups, and has the Lie algebra

$$\mathrm{Der}^+(\widehat{L}H/I) := \{ X \in \mathrm{Der}(\widehat{L}H/I) : X(H) \subset L_{\geq 2}/I \},$$

where $\mathrm{Der}(\widehat{L}H/I)$ is the Lie algebra consisting of continuous Lie algebra derivations on $\widehat{L}H/I$. We call an element of $\mathrm{Der}^+(\widehat{L}H/I)$ a *positive derivation* on $\widehat{L}H/I$.

Hopf algebra automorphisms and Hopf derivations of $\widehat{T}H/I$ are determined by their restrictions to the primitive part $\widehat{L}H/I$, so $\mathrm{IAut}(\widehat{L}H/I)$ and $\mathrm{Der}^{+}(\widehat{L}H/I)$ can be regarded as a subgroup of completed Hopf algebra automorphisms of $\widehat{T}H/I$ and a Lie subalgebra of continuous Hopf derivations on $\widehat{T}H/I$, respectively. Here Hopf derivation means derivation and coderivation.

An inner automorphism of $\widehat{L}H/I$ is a Hopf algebra automorphism $\iota_a\colon \widehat{L}H/I \to \widehat{L}H/I$ defined for all x in $\widehat{L}H/I$ by

$$\iota_a(x) := axa^{-1},$$

where a is a given fixed group-like element of $\widehat{T}H/I$. Then ι_a is a positive automorphism of $\widehat{L}H/I$. We denote the normal subgroup of $\mathrm{IAut}(\widehat{L}H/I)$ consisting of inner automorphisms by $\mathrm{Inn}(\widehat{L}H/I)$. It has the Lie algebra

$$\mathrm{IDer}(\widehat{L}H/I) := \{\mathrm{ad}(a) \in \mathrm{Der}^{+}(\widehat{L}H/I) : a \in \widehat{L}H/I\},$$

which is the Lie algebra of inner derivations on $\widehat{L}H/I$. We also define the quotient group

$$\mathrm{IOut}(\widehat{L}H/I) := \mathrm{IAut}(\widehat{L}H/I)/\mathrm{Inn}(\widehat{L}H/I),$$

and its Lie algebra,

$$\mathrm{ODer}^{+}(\widehat{L}H/I) := \mathrm{Der}^{+}(\widehat{L}H/I)/\mathrm{IDer}(\widehat{L}H/I).$$

We also use the groups of positive automorphisms and outer automorphisms of $\widehat{L}H$ which preserve a decomposable ideal I:

$$\mathrm{IAut}_I(\widehat{L}H) := \{f \in \mathrm{IAut}(\widehat{L}H) : f(I) = I\}, \quad \mathrm{IOut}_I(\widehat{L}H) := \mathrm{IAut}_I(\widehat{L}H)/\mathrm{Inn}(\widehat{L}H).$$

These Lie algebras are the Lie algebras of positive derivations and outer derivations on $\widehat{L}H$ which preserve I:

$$\mathrm{Der}_I^{+}(\widehat{L}H) := \{X \in \mathrm{Der}^{+}(\widehat{L}H) : X(I) \subset I\}, \quad \mathrm{ODer}_I^{+}(\widehat{L}H) := \mathrm{Der}_I^{+}(\widehat{L}H)/\mathrm{IDer}(\widehat{L}H),$$

respectively.

If I is a homogeneous ideal, the Lie algebra of derivations on $\widehat{L}H/I$ has the natural grading.

Definition Let H be a vector space and I a homogeneous decomposable ideal of $\widehat{L}H$. The degree k component of $\mathrm{Der}(\widehat{L}H/I)$ is defined by

$$\mathrm{Der}^k(LH/I) := \{X \in \mathrm{Der}^{+}(\widehat{L}H/I) : X(H) \subset L_{k+1}/(L_{k+1} \cap I)\}.$$

Then we have the decomposition

$$\mathrm{Der}^+(\hat{L}H/I) = \prod_{k=1}^{\infty} \mathrm{Der}^k(LH/I).$$

In the same way, $\mathrm{IDer}^k(LH/I)$ and $\mathrm{ODer}^k(LH/I)$ are defined.

2B Space of expansions

In this section, we define the spaces containing Chen expansions of a fundamental group. These spaces play an important role in our construction of characteristic classes in Section 3. It is a generalization of the spaces of Magnus expansions of free groups in [5].

Definition For a decomposable ideal I of $\hat{L}H$, an I–*expansion* of π is a completed Hopf algebra isomorphism $\theta\colon \hat{\mathbb{R}}\pi \to \hat{T}H/I$ satisfying

$$\theta(g) = 1 + [g] + J^2/I$$

for any $g \in \pi$. (For such a type of isomorphism, see [10].) The space of I–expansions is denoted by $\Theta(\pi, I)$.

The group $\mathrm{IAut}(\hat{L}H/I)$ acts on $\Theta(\pi, I)$ as follows:

$$f \cdot \theta := f \circ \theta$$

for $f \in \mathrm{IAut}(\hat{L}H/I)$ and $\theta \in \Theta(\pi, I)$. This action is free and transitive because $\theta \circ \theta_0^{-1} \in \mathrm{IAut}(\hat{L}H/I)$ and

$$(\theta \circ \theta_0^{-1}) \cdot \theta_0 = \theta$$

for all $\theta, \theta_0 \in \Theta(\pi, I)$. We describe the space of conjugacy classes of I–expansions by $\bar{\Theta}(\pi, I) := \mathrm{Inn}(\hat{L}H/I)\backslash\Theta(\pi, I)$. The group $\mathrm{IOut}(\hat{L}H/I)$ acts on $\bar{\Theta}(\pi, I)$ freely and transitively. (We denote the conjugacy class of $\theta \in \Theta(\pi, I)$ by $[\theta]$.)

We consider the space $\bar{\Theta}(\pi)$ of conjugacy classes of all possible expansions of π. It is parametrized by

$$\mathrm{IOut}(\hat{L}H) \times_{\mathrm{IOut}_{I_0}(\hat{L}H)} \bar{\Theta}(\pi, I_0)$$

fixing a decomposable ideal I_0 which satisfies $\Theta(\pi, I_0) \neq \varnothing$. We consider $\mathrm{IOut}_I(\hat{L}H)$ acting on $\bar{\Theta}(\pi, I)$ through the natural homomorphism $\mathrm{IOut}_I(\hat{L}H) \to \mathrm{IOut}(\hat{L}H/I)$. The outer automorphism group $\mathrm{Out}(\pi)$ of π acts on $\bar{\Theta}(\pi)$ by

$$\varphi \cdot [\theta] = [\,|\varphi| \circ \theta \circ \varphi^{-1}]$$

for $\varphi \in \mathrm{Out}(\pi)$, $\theta \in \bar{\Theta}(\pi)$. Here $|\varphi|$ means the Hopf algebra isomorphism between quotient Hopf algebras of the tensor Hopf algebra $\hat{T}H$ induced by $\varphi \in \mathrm{Out}(\pi)$.

The space $\mathcal{I}(\pi)$ of all ideals I satisfying $\bar{\Theta}(\pi, I) \neq \varnothing$ is parametrized by

$$\text{IOut}(\hat{L}H)/\text{IOut}_{I_0}(\hat{L}H) = \text{IAut}(\hat{L}H)/\text{IAut}_{I_0}(\hat{L}H)$$

fixing a decomposable ideal I_0 which satisfies $\bar{\Theta}(\pi, I_0) \neq \varnothing$. The group

$$\text{GL}(\pi) := \text{Im}(\text{Aut}(\pi) \to \text{GL}(H)) = \text{Im}(\text{Out}(\pi) \to \text{GL}(H))$$

acts on $\mathcal{I}(\pi)$ by

$$\varphi \cdot I = \varphi(I)$$

for $\varphi \in \text{GL}(\pi)$, $I \in \mathcal{I}(\pi)$. Through the natural homomorphism $\text{Out}(\pi) \to \text{GL}(\pi)$, the natural map $\bar{\Theta}(\pi) \to \mathcal{I}(\pi)$ is regarded as an $\text{Out}(\pi)$–equivariant $\bar{\Theta}(\pi, I_0)$–bundle.

2C Generalization of a Johnson map

We define a generalization of a Johnson map for free groups by Kawazumi [4]. The results in this section shall be used only in Section 4.

Proposition 2.1 *Let π be a finitely generated group. If π has a homogeneous decomposable ideal, ie there exists a homogeneous decomposable ideal I such that $\Theta(\pi, I) \neq \varnothing$, then such an ideal is unique. We denote it by I_π.*

Proof Suppose a homogeneous decomposable ideal I' also satisfies $\Theta(\pi, I') \neq \varnothing$. Take $\theta \in \Theta(\pi, I)$ and $\theta' \in \Theta(\pi, I')$ and set $f := \theta' \circ \theta^{-1}$. Then

$$f(x) = x + J^2/I'$$

for $x \in H$. From the equation

$$0 = f(y) = y + J^{k+1}/(J^{k+1} \cap I')$$

for $y \in H^{\otimes k} \cap I$, we get $y \in I'$ by assumption that I' is homogeneous. Since I is also homogeneous, we obtain $I \subset I'$. In the same way, we can prove $I \supset I'$. Therefore we have $I = I'$. $\qquad\square$

Definition Let π be a finitely generated group which has a homogeneous decomposable ideal I_π. For $\theta \in \Theta(\pi, I_\pi)$, we define the *Johnson map* $\tau^\theta \colon \text{Aut}(\pi) \to \text{IAut}(\hat{L}H/I_\pi)$ associated to θ by

$$\tau^\theta(\varphi) = \theta \circ \varphi \circ \theta^{-1} \circ |\varphi|^{-1}.$$

Note that uniqueness of I_π gives $|\varphi|(I_\pi) = I_\pi$. By the decomposition

$$\text{IAut}(\hat{L}H/I_\pi) \subset \prod_{p=1}^{\infty} \text{Hom}(H, L_{p+1}/(L_{p+1} \cap I_\pi)),$$

we denote the $\mathrm{Hom}(H, L_{p+1}/(L_{p+1} \cap I_\pi))$ component of τ^θ by τ_p^θ.

For a group-like element $a \in \hat{T}H/I_\pi$, the equation

$$\tau^{\iota_a \theta}(\varphi) = \iota_a \theta \varphi \theta^{-1} \iota_a^{-1} |\varphi|^{-1} = \iota_a \tau^\theta(\varphi) \iota_{|\varphi|(a)}^{-1}$$

holds. Then we can define $\tau^{[\theta]} \colon \mathrm{Out}(\pi) \to \mathrm{IOut}(\hat{L}H/I_\pi)$ for $[\theta] \in \bar{\Theta}(\pi, I_\pi)$. We can also obtain $\tau_1^{[\theta]}$ for $[\theta] \in \bar{\Theta}(\pi, I_\pi)$.

Definition Given a filtered algebra A with decreasing filtration $\{F_k(A)\}_{k=0}^\infty$,

$$\mathrm{gr}(A) := \bigoplus_{k=0}^\infty F_k(A)/F_{k+1}(A)$$

is a graded algebra by the multiplication induced by the multiplication of A. When A is the completion of a graded algebra V, then $\pi(\hat{V}) = V$ holds.

When G is a group with decreasing central filtration $\{F_k(G)\}_{k=0}^\infty$, a similar thing happens. Then

$$\mathrm{gr}(G) := \bigoplus_{k=0}^\infty F_k(G)/F_{k+1}(G)$$

is a graded Lie algebra by the Lie bracket induced by the commutator of G.

Propositions 2.2, 2.3, 2.4 and 2.5 are straightforward generalizations of those in [4].

Proposition 2.2 For an I_π–expansion $\theta \in \Theta(\pi, I_\pi)$, the map $\mathrm{gr}(\theta) \colon \mathrm{gr}(\pi) \otimes \mathbb{R} \to \mathrm{gr}(\hat{T}H/I_\pi) = TH/I_\pi$ induced by θ is the natural identification $\mathrm{gr}(\pi) \otimes \mathbb{R} \to LH/I_\pi$. Specifically, $\mathrm{gr}(\theta)$ does not depend on the choice of I_π–expansion θ. (Here the group π and the algebra $\hat{T}H$ are filtered by the lower central series $\{\Gamma_k \pi\}_{k=1}^\infty$ and the power series $\{J^k\}_{k=1}^\infty$ of the augmented ideal J, respectively.)

Proof Decompose θ into the sum

$$\theta = \sum_{k=0}^\infty \theta_k,$$

with respect to grading. For a positive integer k, $x \in \Gamma_{k-1}\pi$ and $y \in \pi$,

$$\theta([x, y]) = 1 + \theta_{k-1}(x)[y] - [y]\theta_{k-1}(x) + J^{k+1}/(J^{k+1} \cap I_\pi).$$

Therefore $\theta_k([x, y]) = [\theta_{k-1}(x), [y]]$ holds. Thus the result follows by induction. $\square$

Let π be a finitely generated group. The *Andreadakis filtration* of the automorphism group $\mathrm{Aut}(\pi)$ of π is defined by

$$\mathcal{A}_\pi(k) := \mathrm{Ker}(\mathrm{Aut}(\pi) \to \mathrm{Aut}(\pi/\Gamma_{k+1}\pi)).$$

Note $\mathcal{A}_\pi(1) = \mathrm{IAut}(\pi)$. The kth Johnson homomorphism

$$\tau_k \colon \mathcal{A}_\pi(k)/\mathcal{A}_\pi(k+1) \to \mathrm{Hom}(\pi^{\mathrm{ab}}, \mathrm{gr}^{k+1}(\pi))$$

is defined as follows: For $\varphi \in \mathcal{A}_\pi(k)$ and $x \in \pi$, we set $s_\varphi(x) := x^{-1}\varphi(x) \in \Gamma_{k+1}\pi$. If $x \in \Gamma_2\pi$, then $s_\varphi(x) \in \Gamma_{k+2}\pi$ holds. Thus, we can define

$$\tau_k(\varphi)([x]) = [s_\varphi(x)] \in \Gamma_{k+1}\pi/\Gamma_{k+2}\pi.$$

In addition, the fact $\tau_k(\varphi) = 0$ for $\varphi \in \mathcal{A}_\pi(k+1)$ induces the homomorphism

$$\tau_k \colon \mathcal{A}_\pi(k)/\mathcal{A}_\pi(k+1) \to \mathrm{Hom}(\pi^{\mathrm{ab}}, \mathrm{gr}^{k+1}(\pi)).$$

The direct sum of these maps with respect to k defines the Lie algebra homomorphism

$$\tau \colon \mathrm{gr}^+(\mathrm{Aut}(\pi)) \to \mathrm{Der}^+(\mathrm{gr}(\pi))$$

(see Satoh [15] for details). Here we denote the positive degree part of $\mathrm{gr}(G)$ by $\mathrm{gr}^+(G)$ for a group G with central filtration. By definition, an inner automorphism is mapped to an inner derivation by this map. Then the Lie algebra homomorphism

$$\bar{\tau} \colon \mathrm{gr}^+(\mathrm{Out}(\pi)) \to \mathrm{ODer}^+(\mathrm{gr}(\pi))$$

is induced by τ. Here $\mathrm{Out}(\pi)$ is filtered by

$$\mathcal{O}_\pi(k) := \mathrm{Ker}(\mathrm{Out}(\pi) \to \mathrm{Out}(\pi/\Gamma_{k+1}\pi)).$$

Proposition 2.3 *For $\theta \in \Theta(\pi, I_\pi)$ and $m \geq 1$, the map induced by τ^θ,*

$$\mathrm{gr}^m(\tau^\theta) \colon \mathrm{gr}^m(\mathrm{Aut}(\pi)) \to \mathrm{gr}^m(\mathrm{Aut}(\widehat{L}H/I_\pi)) = \mathrm{Der}^m(LH/I_\pi),$$

is equal to the mth Johnson homomorphism τ_m through the natural identification $\mathrm{gr}(\pi) \otimes \mathbb{R} \to LH/I_\pi$. Specifically, $\mathrm{gr}^+(\tau^\theta)$ does not depend on the choice of I_π–expansion θ. Here the group $\mathrm{Aut}(\pi)$ is filtered by $\{\mathcal{A}_\pi(k)\}_{k=0}^\infty$ and the group $\mathrm{Aut}(\widehat{L}H/I_\pi)$ is filtered by $\{\mathrm{Aut}^{\geq k}(\widehat{L}H/I_\pi)\}_{k=0}^\infty$, where

$$\mathrm{Aut}^{\geq k}(\widehat{L}H/I_\pi) = \{f \in \mathrm{Aut}(\widehat{L}H/I_\pi) : f(x) = x + L_{\geq k+1}/(L_{\geq k+1} \cap I_\pi) \text{ for } x \in H\}.$$

Proof Suppose $\tau_k^\theta(\varphi) = \tau_k(\varphi)$ for $\varphi \in \mathcal{A}_\pi(k)$ and $1 \leq k < m$. Then $\tau_k^\theta(\varphi) = 0$ holds for $\varphi \in \mathcal{A}_\pi(m)$ and $1 \leq k < m$. From

$$\tau^\theta(\varphi)([x]) = \theta\varphi\theta^{-1}([x]) = \theta(\varphi(x)) + \tau_m^\theta(\varphi)([x])$$

for $x \in \pi$, we obtain

$$\tau_m(\varphi)([x]) = \theta_{m+1}(s_\varphi(x)) \equiv \theta(s_\varphi(x)) - 1 = \theta(x)^{-1}\theta(\varphi(x)) - 1 \equiv \tau_m^\theta(\varphi)([x])$$

modulo $L_{\geq m+2}/(L_{\geq m+2} \cap I_\pi)$. Then we have $\tau_m^\theta(\varphi) = \tau_m(\varphi)$ for all $m \geq 1$ and $\varphi \in \mathcal{A}_\pi(m)$ inductively. $\square$

Proposition 2.4 *The first Johnson map τ_1^θ is a cocycle of* $\mathrm{Aut}(\pi)$ *with coefficient* $\mathrm{Der}^1(LH/I_\pi)$, *and the cohomology class* $[\tau_1^\theta] \in H^1(\mathrm{Aut}(\pi); \mathrm{Der}^1(LH/I_\pi))$ *does not depend on the choice of* I_π*–expansion* θ.

Proof The cocycle condition of τ_1^θ is written by

$$\tau_1^\theta(\varphi\psi) = \tau_1^\theta(\varphi) + |\varphi|\tau_1^\theta(\psi)$$

for every $\varphi, \psi \in \mathrm{Aut}(\pi)$. This follows from the formula

$$\tau^\theta(\varphi\psi) = \tau^\theta(\varphi)|\varphi|\tau^\theta(\psi)|\varphi|^{-1}.$$

Independence of its cohomology class is proved as follows. Because $\theta_2'(x) = \theta_2(x)$ for $x \in \Gamma_2\pi$ by [Proposition 2.2](), we can define $F \in \mathrm{Der}^1(LH/I_\pi)$ by

$$F([x]) := \theta_2'(x) - \theta_2(x)$$

for $[x] \in H$. By the formula

$$\tau_1^\theta(\varphi)([x]) = \theta_2(x) - |\varphi|\theta_2(\varphi^{-1}(x))$$

for $[x] \in H$ and $\varphi \in \mathrm{Aut}(\pi)$, we have

$$\tau_1^\theta - \tau_1^{\theta'} = dF.$$

Thus $[\tau_1^{\theta'}] = [\tau_1^\theta] \in H^1(\mathrm{Aut}(\pi); \mathrm{Der}^1(LH/I_\pi))$. $\square$

Similar results for the outer automorphism group also hold:

Proposition 2.5 *For $m \geq 1$, the map induced by τ^θ,*

$$\mathrm{gr}^m(\tau^\theta)\colon \mathrm{gr}^m(\mathrm{Out}(\pi)) \to \mathrm{gr}^m(\mathrm{Out}(\widehat{L}H/I_\pi)) = \mathrm{ODer}^m(LH/I_\pi)$$

for $\theta \in \bar{\Theta}(\pi, I_\pi)$, is identified with the mth Johnson homomorphism $\bar{\tau}_m$ of $\mathrm{Out}(\pi)$. Here the group $\mathrm{Out}(\pi)$ is filtered by $\{\mathcal{O}_\pi(k)\}_{k=0}^\infty$, and the group $\mathrm{Out}(\widehat{L}H/I_\pi)$ is filtered by $\{\mathrm{Out}^{\geq k}(\widehat{L}H/I_\pi)\}_{k=0}^\infty$, where

$$\mathrm{Out}^{\geq k}(\widehat{L}H/I_\pi) = \mathrm{Im}\big(\mathrm{Aut}^{\geq k}(\widehat{L}H/I_\pi) \to \mathrm{Out}(\widehat{L}H/I_\pi)\big).$$

Also, the first Johnson map τ_1^θ is a cocycle of $\mathrm{Out}(\pi)$ with coefficient $\mathrm{ODer}^1(LH/I_\pi)$, and the cohomology class

$$[\tau_1^\theta] \in H^1(\mathrm{Out}(\pi); \mathrm{ODer}^1(LH/I_\pi))$$

does not depend on the choice of $\theta \in \bar{\Theta}(\pi, I_\pi)$.

3 Characteristic classes

For an oriented closed manifold X, we set the fundamental group $\pi = \pi_1(X)$ and the first homology group $H = H_1(X; \mathbb{R})$. For the purpose, we recall a result of K-T Chen.

Definition [1; 2] Let X be a manifold. We denote the suspension of $H_+(X; \mathbb{R}) := \bigoplus_{p>0} H_p(X; \mathbb{R})$ by $H_+ := H_+(X; \mathbb{R})[1]$. The completed tensor algebra $T := \hat{T}H_+$ of the suspension H_+ is a (completion of a) graded algebra. A pair (ω, δ) satisfying the following conditions is a *formal homology connection*:

(i) A T–coefficient differential form $\omega \in A^*(X) \otimes T$ is described by

$$\omega = \sum_{k=1}^\infty \sum_{i_1,\dots,i_k} \omega_{i_1\cdots i_k} X_{i_1} \cdots X_{i_k},$$

where $X_1, \dots, X_n$ is a homogeneous basis of H_+ and the differential form $\omega_{i_1\cdots i_k} \in A^*(X)$ is a $(\deg X_{i_1} + \cdots + \deg X_{i_k} + 1)$–form and they satisfy

$$\int_{X_p} \omega_p = 1.$$

(ii) A linear map $\delta\colon T \to T$ is a differential with degree -1 of the graded algebra T such that

$$\delta(H_+) \subset \prod_{q=2}^\infty H_+^{\otimes q}.$$

(iii) The flatness condition $\delta\omega + d\omega = \epsilon(\omega) \wedge \omega$ holds, where $\epsilon\colon A^*(X) \to A^*(X)$ is defined by $\epsilon(\alpha) = (-1)^p\alpha$ for $\alpha \in A^p(X)$, and δ and ϵ are extended onto $A^*(X) \otimes T$.

Let (ω, δ) be a formal homology connection of X. Then we can obtain the chain map $C_*(\Omega X; \mathbb{R}) \to T$ from the cubical chain complex of the loop space ΩX of X to the algebra T defined by the iterated integral

$$\sigma \mapsto \sum_{n=0}^\infty \int_\sigma \underbrace{\omega \cdots \omega}_{n}.$$

Furthermore, by the result of Chen, the homomorphism $H_0(\Omega X; \mathbb{R}) \to H_0(T, \delta)$ induces a Hopf algebra isomorphism $\theta_\omega \colon \widehat{\mathbb{R}}\pi \simeq \widehat{T}H/I_\omega$ where $I_\omega := \delta(H_2(X; \mathbb{R})[1])$. This is called a Chen expansion. In our notation, I_ω is a decomposable ideal and θ_ω is an I_ω–expansion.

Theorem 3.1 [1; 2] *Let (X, g) be an oriented closed Riemannian manifold and*

$$A^*(X) = \mathcal{H}_g \oplus dA^*(X) \oplus d_g^* A^*(X)$$

be the Hodge decomposition of (X, g). Here $\mathcal{H}_g$ is the space of harmonic forms and $d_g^* \colon A^*(X) \to A^*(X)$ is the adjoint operator of d with respect to g. Then there exists a unique formal homology connection (ω_g, δ_g) such that

$$\omega_g = \sum_{i=1}^m \omega_i X_i + \sum_{p \geq 2} \sum_{i_1, \ldots, i_p} \omega_{i_1 \cdots i_p} X_{i_1} \cdots X_{i_p},$$

where $X_1, \ldots, X_m$ is a basis of H_+, $\omega_i \in \mathcal{H}_g$ and $\omega_{i_1 \cdots i_p} \in d_g^* A^*(X)$.

We denote the group of diffeomorphisms of X preserving the orientation by $\mathrm{Diff}_+(X)$. For a Riemannian metric g on X and $\varphi \in \mathrm{Diff}_+(X)$, we define the metric $\varphi_* g$ on X by

$$(\varphi_* g)(u, v) := g(\varphi^* u, \varphi^* v)$$

for cotangent vectors $u, v \in T_x^* X$ and $x \in X$. Then, since

$$\mathcal{H}_{\varphi_* g} = (\varphi^*)^{-1}(\mathcal{H}_g), \quad d_{\varphi_* g}^* A^*(X) = (\varphi^*)^{-1}(d_g^* A^*(X))$$

for $\varphi \in \mathrm{Diff}_+(X)$, we have

$$(\omega_{\varphi_* g}, \delta_{\varphi_* g}) = \big(((\varphi^*)^{-1} \otimes |\varphi|)(\omega_g), |\varphi| \circ \delta_g \circ |\varphi|^{-1}\big).$$

Here $|\varphi|$ means the algebra isomorphism $\widehat{T}H \to \widehat{T}H$ induced by $\varphi \in \mathrm{Diff}_+(X)$. Thus the corresponding Chen expansions and ideals satisfy

(3-1) $\quad \theta_{\omega_{\varphi_* g}} = |\varphi| \circ \theta_{\omega_g} \circ \varphi_*^{-1} \colon \widehat{\mathbb{R}}\pi_1(X, \varphi(*)) \to \widehat{T}H/|\varphi|(I_{\omega_g}), \quad I_{\omega_{\varphi_* g}} = |\varphi|(I_{\omega_g})$

for a diffeomorphism φ of X preserving the base point $*$. Here we denote the homomorphism $\widehat{T}H/I_{\omega_g} \to \widehat{T}H/|\varphi|(I_{\omega_g})$ induced by $|\varphi| \colon \widehat{T}H \to \widehat{T}H$ by the same symbol $|\varphi|$.

Based on these considerations, we describe characteristic classes of fiber bundles using expansion spaces as follows: let $E \to B$ be an oriented fiber bundle whose fiber X is an oriented closed manifold and set $\pi = \pi_1(X)$. We choose a fiberwise metric $g_{E/B}$ on $E \to B$. Take an open covering $\{U_i\}_i$ of B and local trivializations

$\varphi_i\colon \pi^{-1}(U_i) \simeq U_i \times X$ of E. Then $g_{E/B}$ defines fiberwise metrics g_i of trivial bundles $U_i \times X \to U_i$. The map $\theta_i\colon U_i \to \bar{\Theta}(\pi)$ can be defined by $x \mapsto [\theta_{\omega_{g_i(x)}}]$. For any differential form $\alpha \in A^*(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)}$,

$$\theta_i^*\alpha = \theta_j^*\alpha$$

holds on $U_i \cap U_j$ because the correspondence from metrics to Chen expansions is $\mathrm{Diff}_+(X)$–equivariant from the equations (3-1). (It also means $\theta_i^*\alpha$ is independent of the choice of local trivializations.) Thus the family $\{\theta_i^*\alpha\}_i$ of differential forms on the open covering defines the differential form $\theta^*\alpha$ on B by gluing. The correspondence $\alpha \mapsto \theta^*\alpha$ is a chain map $A^*(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \to A^*(B)$. Given two distinct open coverings, we can prove that the maps $A^*(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \to A^*(B)$ for two coverings are equal by taking a refinement of these open coverings.

Since any two fiberwise metrics g_0, g_1 of a fiber bundle can be connected by a segment $(1-t)g_0 + tg_1$, the chain maps $A^*(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \to A^*(B)$ for distinct metrics are chain homotopy equivalent. Therefore the induced homomorphism

$$\Phi_E\colon H^*_{\mathrm{DR}}(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \to H^*_{\mathrm{DR}}(B)$$

does not depend on the choice of fiberwise metric. So the following theorem holds:

Theorem 3.2 *For an oriented fiber bundle $E \to B$ whose fiber is X, the map $A^*(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \to A^*(B)$ constructed above is a chain map, and the induced map Φ_E on cohomology groups is independent of the choice of fiberwise metric.*

Remark 3.3 Let $\mathcal{G}$ be the structure group of $E \to B$ and set $\widetilde{S} := \mathrm{Im}(\mathcal{G} \to \mathrm{Out}(\pi))$. We obtain the map

$$H^*_{\mathrm{DR}}(\bar{\Theta}(\pi))^{\widetilde{S}} \to H^*_{\mathrm{DR}}(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \xrightarrow{\Phi_E} H^*_{\mathrm{DR}}(B).$$

We also denote this map by Φ_E.

3A Homologically trivial bundles

In this section we assume that a fiber bundle $E \to B$ with fiber X has the structure group

$$\mathcal{T}(X) := \mathrm{Ker}(\mathrm{Diff}_+(X) \to \mathrm{GL}(H)).$$

Fix a decomposable ideal I_0 which satisfies $\Theta(\pi, I_0) \neq \varnothing$. We choose a fiberwise metric g of $E \to B$. Since the structure group of $E \to B$ is $\mathcal{T}(X)$, the correspondence of ideals by g gives a map $q\colon B \to \mathcal{I}(\pi)$. Because the topological group $\mathrm{IOut}_{I_0}(\widehat{L}H)$ is contractible, the pullback $q^* \mathrm{IOut}(\widehat{L}H) \to B$ of the principal $\mathrm{IOut}_{I_0}(\widehat{L}H)$–bundle

$\mathrm{IOut}(\widehat{L}H) \to \mathcal{I}(\pi)$ is trivial. Taking a trivialization of the principal bundle, we get the $\mathrm{IOut}(\pi)$–equivariant map

$$s\colon q^*\overline{\Theta}(\pi) = q^* \mathrm{IOut}(\widehat{L}H) \times_{\mathrm{IOut}_{I_0}(\widehat{L}H)} \overline{\Theta}(\pi, I_0) \simeq B \times \overline{\Theta}(\pi, I_0) \to \overline{\Theta}(\pi, I_0).$$

Thus we can obtain the characteristic map

$$H^*_{\mathrm{DR}}(\overline{\Theta}(\pi, I_0))^{\mathrm{IOut}(\pi)} \xrightarrow{s^*} H^*_{\mathrm{DR}}(q^*\overline{\Theta}(\pi))^{\mathrm{IOut}(\pi)} \xrightarrow{\Phi_E} H^*_{\mathrm{DR}}(B),$$

where $\mathrm{IOut}(\pi) := \mathrm{Ker}(\mathrm{Out}(\pi) \to \mathrm{GL}(H))$. This map does not depend on the choice of fiberwise metric and trivialization. (Independence from the choice of trivialization of $q^* \mathrm{IOut}(\widehat{L}H) \to B$ comes from contractibility of $\mathrm{IOut}_{I_0}(\widehat{L}H)$.)

Remark 3.4 We remark that the map Φ_E of [Theorem 3.2](#) and [Remark 3.3](#) factors the natural maps

$$H^*_{\mathrm{DR}}(\overline{\Theta}(\pi))^{\mathrm{IOut}(\pi)} \to H^*_{\mathrm{DR}}(q^*\overline{\Theta}(\pi))^{\mathrm{IOut}(\pi)} \to H^*_{\mathrm{DR}}(B)$$

by construction. The map $H^*_{\mathrm{DR}}(q^*\overline{\Theta}(\pi))^{\mathrm{IOut}(\pi)} \to H^*_{\mathrm{DR}}(B)$ is also regarded as Φ_E.

3B The fiber which has a homogeneous ideal and a splitting

If the fundamental group $\pi = \pi_1(X)$ of fiber X satisfies some conditions, we can obtain stronger results. In this section we assume that the fundamental group $\pi = \pi_1(X)$ of fiber X has the homogeneous ideal I_π and there exists a splitting

$$\mathrm{Der}^+(\widehat{L}H) = V \oplus \mathrm{Der}^+_{I_\pi}(\widehat{L}H)$$

as graded S–vector spaces. Here a group S is the image of a structure group of $E \to B$ in $\mathrm{GL}(\pi)$.

Example 1 If π is a free group, we have $I_\pi = 0$. So π clearly satisfies the condition above for any $S \subset \mathrm{GL}(\pi)$. For example, the fundamental group of a manifold whose second Betti number is zero is a free group.

Example 2 We consider the case of $X = \Sigma_g$, an oriented surface of genus $g \geq 1$. Then $\pi = \pi_1(X)$ has the homogeneous ideal $I_\pi = (\omega)$. Here we denote the intersection form of Σ_g by

$$\omega = \sum_{i=1}^{g} [X_i, X_{i+g}] \in L_2,$$

where $X_1, \ldots, X_{2g}$ is a symplectic basis of H. We set

$$S = \mathrm{Im}\big(\mathrm{Diff}_+(\Sigma_g) \to \mathrm{GL}(\pi)\big) = \mathrm{Sp}\big(H_1(\Sigma_g; \mathbb{Z}), \omega\big) \subset \mathrm{Sp}(H, \omega).$$

Since all finite-dimensional representations of $\mathrm{Sp}(H, \omega)$ are completely reducible, π satisfies the splitting condition.

Lemma 3.5 *The* $\mathrm{GL}(\pi)$*–equivariant principal* $\mathrm{IAut}_{I_\pi}(\widehat{L}H)$*–bundle* $\mathrm{IAut}(\widehat{L}H) \to \mathcal{I}(\pi)$ *is* S*–equivariant trivial.*

Proof Let t be a real number. Then for any $X \in \mathrm{Der}^k(LH)$, we define the map $w_t \colon \mathrm{Der}^+(\widehat{L}H) \to \mathrm{Der}^+(\widehat{L}H)$ by $w_t(X) = t^k X$. If $X \in \mathrm{Der}^k(LH)$ and $Y \in \mathrm{Der}^l(LH)$, we have

$$[w_t(X), w_t(Y)] = [t^k X, t^l Y] = t^{k+l}[X, Y] = w_t([X, Y]).$$

So w_t is a Lie algebra homomorphism. We define $F_t \colon \mathrm{IAut}(\widehat{L}H) \to \mathrm{IAut}(\widehat{L}H)$ by

$$F_t(\exp X) = \exp w_t(X).$$

Since w_t is a Lie algebra homomorphism, F_t is a group homomorphism, and F_t is commutative with the action of $\mathrm{GL}(H)$ on $\mathrm{IAut}(\widehat{L}H)$ since w_t preserves the degree of derivations. Furthermore, since I_π is a homogeneous ideal, F_t can be restricted to the $\mathrm{GL}(\pi)$–equivariant map $\mathrm{IAut}_{I_\pi}(\widehat{L}H) \to \mathrm{IAut}_{I_\pi}(\widehat{L}H)$ and induces $\mathcal{I}(\pi) \to \mathcal{I}(\pi)$.

We set $\mathrm{IAut}^{\leq k}(\widehat{L}H) := \mathrm{IAut}(\widehat{L}H/L_{\geq k+1})$. We also write the corresponding Lie algebras in the same way. Since

$$\mathrm{IAut}(\widehat{L}H) = \varprojlim_k \mathrm{IAut}^{\leq k}(\widehat{L}H),$$

we have that

$$\mathcal{I}(\pi) = \varprojlim_k \mathrm{IAut}^{\leq k}(\widehat{L}H)/\mathrm{IAut}_{I_\pi}^{\leq k}(\widehat{L}H).$$

We set $\mathcal{I}(\pi)_k := \mathrm{IAut}^{\leq k}(\widehat{L}H)/\mathrm{IAut}_{I_\pi}^{\leq k}(\widehat{L}H)$.

By assumption of a splitting

$$\mathrm{Der}^+(\widehat{L}H) = V \oplus \mathrm{Der}_{I_\pi}^+(\widehat{L}H),$$

a splitting

$$\mathrm{Der}^{\leq k}(\widehat{L}H) = V^{\leq k} \oplus \mathrm{Der}_{I_\pi}^{\leq k}(\widehat{L}H)$$

is induced. Here the superscript $\leq k$ of graded vector spaces means the part with degree $\leq k$.

The map $\mathrm{Der}^{\leq k}(\widehat{L}H) \to \mathrm{IAut}^{\leq k}(\widehat{L}H)$ defined by

$$X + Y \mapsto \exp X \cdot \exp Y$$

is a local diffeomorphism at 0. Therefore so is the map $c\colon V^{\leq k} \to \mathcal{I}(\pi)_k$ induced by the map above. It gives a diffeomorphism $c\colon U \to O$ of a neighborhood U of 0 and a neighborhood O of $[\mathrm{id}]$. Then, for all $t > 0$, the following diagram commutes:

$$
\begin{array}{ccc}
U & \xrightarrow{\ c\ } & O \\
\downarrow{\scriptstyle w_t} & & \downarrow{\scriptstyle F_t} \\
w_t(U) & \xrightarrow{\ c\ } & F_t(O)
\end{array}
$$

Therefore, the restriction $c\colon w_t(U) \to F_t(U)$ is also a diffeomorphism. Thus,

$$
c\colon V^{\leq k} = \bigcup_t w_t(U) \to \bigcup_t F_t(O) = \mathcal{I}(\pi)_k
$$

is a global diffeomorphism. The inverse limit

$$
V \to \mathcal{I}(\pi)
$$

of the maps is also a diffeomorphism. We obtain a section

$$
\mathcal{I}(\pi) \simeq V \xrightarrow{\ \exp\ } \mathrm{IAut}(\hat{L}H)
$$

of the S–equivariant principal $\mathrm{IAut}_{I_\pi}(\hat{L}H)$–bundle $\mathrm{IAut}(\hat{L}H) \to \mathcal{I}(\pi)$. $\qquad\square$

Let $\widetilde{S}$ be the image of a structure group of $E \to B$ in $\mathrm{Out}(\pi)$. From [Lemma 3.5](#), the $\mathrm{GL}(\pi)$–equivariant principal $\mathrm{IOut}_{I_\pi}(\hat{L}H)$–bundle $\mathrm{IOut}(\hat{L}H) \to \mathcal{I}(\pi)$ is $\widetilde{S}$–equivariantly trivial. Then there exists an $\widetilde{S}$–equivariant diffeomorphism

$$
\begin{aligned}
\bar{\Theta}(\pi) &= \mathrm{IOut}(\hat{L}H) \times_{\mathrm{IOut}_{I_\pi}(\hat{L}H)} \bar{\Theta}(\pi, I_\pi) \\
&\simeq (\mathcal{I}(\pi) \times \mathrm{IOut}_{I_\pi}(\hat{L}H)) \times_{\mathrm{IOut}_{I_\pi}(\hat{L}H)} \bar{\Theta}(\pi, I_\pi) = \mathcal{I}(\pi) \times \bar{\Theta}(\pi, I_\pi).
\end{aligned}
$$

Since the map $F_t\colon \mathcal{I}(\pi) \to \mathcal{I}(\pi)$ defined in the proof of the theorem above is a $\mathrm{GL}(\pi)$–equivariant homotopy, $\mathcal{I}(\pi)$ is $\mathrm{GL}(\pi)$–contractible. Thus we have:

Theorem 3.6 *The space $\bar{\Theta}(\pi)$ is $\widetilde{S}$–equivariant homotopic to $\bar{\Theta}(\pi, I_\pi)$.*

Thus the characteristic map Φ_E of $E \to B$ is described by

$$
H^*_{\mathrm{DR}}(\bar{\Theta}(\pi, I_\pi))^{\widetilde{S}} = H^*_{\mathrm{DR}}(\bar{\Theta}(\pi))^{\widetilde{S}} \to H^*_{\mathrm{DR}}(B).
$$

3C Relation between two constructions

Let π be a finitely generated group which has a homogeneous ideal I_π. We consider the case of $S = 1$, $\widetilde{S} = \mathrm{IOut}(\pi)$, $I_0 = I_\pi$ in Sections 3A and 3B. We choose a trivialization of $q^* \mathrm{IOut}(\widehat{L}H) \to B$ which was induced by a trivialization of $\mathrm{IOut}(\widehat{L}H) \to \mathcal{I}(\pi)$ in Section 2B. Then the diagram

$$
\begin{array}{ccccc}
q^* \bar{\Theta}(\pi) & \xrightarrow{\ \sim\ } & B \times \bar{\Theta}(\pi, I_\pi) & \longrightarrow & \bar{\Theta}(\pi, I_\pi) \\
\downarrow & & \downarrow & \nearrow & \\
\bar{\Theta}(\pi) & \xrightarrow{\ \sim\ } & \mathcal{I}(\pi) \times \bar{\Theta}(\pi, I_\pi) & \text{h.e.} &
\end{array}
$$

commutes. So taking $\mathrm{IOut}(\pi)$–invariant de Rham cohomologies, we obtain the diagram

$$
\begin{array}{ccccc}
H^*_{\mathrm{DR}}(\bar{\Theta}(\pi, I_\pi))^{\mathrm{IOut}(\pi)} & \longrightarrow & H^*_{\mathrm{DR}}(q^* \bar{\Theta}(\pi))^{\mathrm{IOut}(\pi)} & \longrightarrow & H^*_{\mathrm{DR}}(B) \\
& \searrow & \uparrow & \nearrow & \\
& & H^*_{\mathrm{DR}}(\bar{\Theta}(\pi))^{\mathrm{IOut}(\pi)} & &
\end{array}
$$

Thus the characteristic maps obtained in two ways are equal in the common case.

Remark 3.7 Using the construction in this section, we can obtain characteristic maps of a fiber bundle whose structure group is a subgroup of $\mathrm{Diff}_+(X, *)$ by replacing Out with Aut, $\bar{\Theta}$ with Θ, ODer^+ with Der^+, and so on.

3D Lie algebra cohomology of derivations

We construct invariant differential forms on $\bar{\Theta}(\pi, I)$ from a flat connection on $\bar{\Theta}(\pi, I)$ and Chevalley–Eilenberg cochains.

Since $\mathrm{IOut}(\widehat{L}H/I)$ acts freely and transitively on $\bar{\Theta}(\pi, I)$, we get the $\mathrm{ODer}^+(\widehat{L}H/I)$–coefficient differential form $\eta \in A^1(\bar{\Theta}(\pi, I); \mathrm{ODer}^+(\widehat{L}H/I))$ by pulling back the right-invariant Maurer–Cartan form of $\mathrm{IOut}(\widehat{L}H/I)$. The form η satisfies the equation

$$d\eta + \tfrac{1}{2}[\eta, \eta] = 0.$$

Lemma 3.8 Let $\varphi \in \mathrm{Out}(\pi)$ be an automorphism of π which satisfies $|\varphi|(I) = I$. The Maurer–Cartan form η on $\bar{\Theta}(\pi, I)$ satisfies

$$\varphi^* \eta = \mathrm{Ad}(|\varphi|)\eta.$$

Proof For $\theta_0 \in \bar{\Theta}(\pi, I)$, we will denote $q_{\theta_0} \colon \bar{\Theta}(\pi, I) \to \mathrm{IOut}(\hat{L}H/I)$ by $\theta \mapsto \theta \circ \theta_0^{-1}$ and the Maurer–Cartan form on $\mathrm{IOut}(\hat{L}H/I)$ by

$$\tilde{\eta} \in A^*(\mathrm{IOut}(\hat{L}H/I); \mathrm{ODer}^+(\hat{L}H/I)).$$

Then $\eta = q_{\theta_0}^* \tilde{\eta}$ and it is independent of the choice of θ_0. Since $q_{\theta_0} \circ \varphi = \mathrm{Ad}(|\varphi|) \circ q_{\varphi \cdot \theta_0}$,

$$\varphi^* \eta = (q_{\theta_0} \circ \varphi)^* \tilde{\eta} = q_{\varphi \cdot \theta_0}^* \mathrm{Ad}(|\varphi|)^* \tilde{\eta} = \mathrm{Ad}(|\varphi|) q_{\varphi \cdot \theta_0}^* \tilde{\eta} = \mathrm{Ad}(|\varphi|) \eta.$$

Thus we complete the proof. $\square$

From Lemma 3.8, the map $C^*(\mathrm{ODer}^+(\hat{L}H/I)) \to A^*(\bar{\Theta}(\pi, I))^{\mathrm{IOut}(\pi)}$ can be defined by

$$c \mapsto c(\eta^m)$$

for any $c \in C^m(\mathrm{ODer}^+(\hat{L}H/I))^{\mathrm{GL}(\pi)}$. Here the power η^m of η is defined by the product

$$(A^*(X) \otimes \mathrm{ODer}^+(\hat{L}H/I))^{\otimes m}$$
$$= A^*(X)^{\otimes m} \otimes \mathrm{ODer}^+(\hat{L}H/I)^{\otimes m} \to A^*(X) \otimes \mathrm{ODer}^+(\hat{L}H/I)^{\otimes m}$$

which is the wedge product with respect to $A^*(X)$–components. We set

$$\eta = \sum \eta_\lambda D_\lambda,$$

using a (topological) basis $\{D_\lambda\}_{\lambda \in \Lambda}$ of $\mathrm{ODer}^+(\hat{L}H/I)$ and a well-ordered set Λ. Then we can write

$$\eta^m = \sum_{\lambda_1 < \cdots < \lambda_m} \eta_{\lambda_1} \wedge \cdots \wedge \eta_{\lambda_m} D_{\lambda_1} \wedge \cdots \wedge D_{\lambda_m} \in A^*(X) \otimes \overset{m}{\bigwedge} \mathrm{ODer}^+(\hat{L}H/I).$$

Thus we can define the above linear map $C^*(\mathrm{ODer}^+(\hat{L}H/I)) \to A^*(\bar{\Theta}(\pi, I))^{\mathrm{IOut}(\pi)}$. This map is a chain map. In fact, we have

$$d(c(\eta^m)) = \sum_{s=1}^m (-1)^{s-1} c(\underbrace{\eta \cdots \overset{s}{d\eta} \cdots \eta}_{m})$$
$$= \sum_{s=1}^m \frac{(-1)^s}{2} c(\underbrace{\eta \cdots \overset{s}{[\eta, \eta]} \cdots \eta}_{m})$$
$$= (dc)(\eta^{m+1}).$$

Combining it with the result in Section 3A, we obtain the following:

Theorem 3.9 Let $E \to B$ be a fiber bundle whose fiber is X and structure group is $\mathcal{T}(X)$. Fix a decomposable ideal satisfying $\bar{\Theta}(\pi, I_0) \neq \varnothing$. Then the map

$$H^*(\mathrm{ODer}^+(\widehat{L}H/I_0)) \to H^*_{\mathrm{DR}}(\bar{\Theta}(\pi, I_0))^{\mathrm{IOut}(\pi)} \to H^*_{\mathrm{DR}}(B)$$

is an invariant of an oriented fiber bundle $E \to B$.

If $I_0 = I_\pi$ is the homogeneous ideal, the group $\mathrm{GL}(\pi)$ acts on $\mathrm{ODer}^+(\widehat{L}H/I_\pi)$ by the adjoint action. So we define by $C^*(\mathrm{ODer}^+(\widehat{L}H/I_\pi))^{\mathrm{GL}(\pi)}$ the chain complex of $\mathrm{GL}(\pi)$–invariant Chevalley–Eilenberg cochains of $\mathrm{ODer}^+(\widehat{L}H/I_\pi)$ with respect to the action. In the same way as the construction above, we obtain the chain map

$$C^*(\mathrm{ODer}^+(\widehat{L}H/I_\pi))^{\mathrm{GL}(\pi)} \to A^*(\bar{\Theta}(\pi, I_\pi))^{\mathrm{Aut}(\pi)}.$$

We have the following theorem using the construction in [Section 3B](#):

Theorem 3.10 Let $E \to B$ be a fiber bundle whose fiber is X and structure group is a subgroup $\mathcal{G}$ of the diffeomorphism group $\mathrm{Diff}_+(X)$. We set $\pi = \pi_1(X)$, $\widetilde{S} = \mathrm{Im}(\mathcal{G} \to \mathrm{Out}(\pi))$, and $S = \mathrm{Im}(\mathcal{G} \to \mathrm{GL}(\pi))$. We suppose the group π has the homogeneous ideal I_π and there exists a splitting

$$\mathrm{Der}^+(\widehat{L}H) = V \oplus \mathrm{Der}^+_{I_\pi}(\widehat{L}H)$$

as graded S–vector spaces. Then the map

$$H^*(\mathrm{ODer}^+(\widehat{L}H/I_\pi))^S \to H^*_{\mathrm{DR}}(\bar{\Theta}(\pi, I_\pi))^{\widetilde{S}} \to H^*_{\mathrm{DR}}(B)$$

is an invariant of an oriented fiber bundle $E \to B$.

4 Closed surface bundles

Let $\mathcal{M}_g$ be the mapping class group of an oriented closed surface Σ_g of genus $g \geq 2$ and let $\mathcal{T}_g$ be the Teichmüller space of genus g, which is the space Met_g of metrics which have constant curvature -1 on Σ_g modulo $\mathrm{Diff}_0(\Sigma_g)$. The quotient orbifold $\mathbb{M}_g := \mathcal{T}_g/\mathcal{M}_g$ is called the moduli space of genus g. Since $\mathcal{T}_g$ is contractible, we have the canonical isomorphism

$$H^*(\mathbb{M}_g; \mathbb{R}) \simeq H^*(\mathcal{M}_g; \mathbb{R}).$$

We denote the first real homology group $H_1(\Sigma_g; \mathbb{R})$ of Σ_g by H and the intersection form on Σ_g by ω. The alternating form ω can be regarded as an element of L_2 by the identity

$$\omega = \sum_{i=1}^{g}[X_i, X_{i+g}],$$

where $X_1, \ldots, X_{2g}$ is any symplectic basis of H.

Lemma 4.1 [12] *There exist identifications*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{IDer}^1(LH/(\omega)) & \longrightarrow & \mathrm{Der}^1(LH/(\omega)) & \longrightarrow & \mathrm{ODer}^1(LH/(\omega)) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & H & \xrightarrow{\cdot \wedge \omega} & \Lambda^3 H & \longrightarrow & \Lambda^3 H/H & \longrightarrow & 0
\end{array}
$$

According to [6], all Morita–Miller–Mumford classes of $\mathcal{M}_g$ are constructed from the twisted cohomology class $[\tilde{k}] \in H^1(\mathcal{M}_g; \Lambda^3 H/H)$, which is the cohomology class of a crossed homomorphism $\tilde{k} \colon \mathcal{M}_g \to \Lambda^3 H/H$ defined in [11], through the map $\alpha^{\mathrm{KM}} \colon \mathrm{Hom}(\Lambda^*(\Lambda^3 H/H), \mathbb{R})^{\mathrm{Sp}(H)} \to H^*(\mathcal{M}_g; \mathbb{R})$ defined by

$$ c \mapsto c([\tilde{k}]^m) $$

for $c \in \mathrm{Hom}(\Lambda^m(\Lambda^3 H/H), \mathbb{R})^{\mathrm{Sp}(H)}$. Here the power is related to the cup product. We shall describe a relation between the map α_{KM} and our construction.

Since every hyperbolic metric on Σ_g admits a Kähler structure, the corresponding ideal of a hyperbolic metric is (ω). So we can define the map $\theta_C \colon \mathcal{T}_g \to \bar{\Theta}(\pi, (\omega))$ by giving Chen expansions corresponding to hyperbolic metrics. It can be constructed by the argument which is an analogue for the case of 1–punctured surfaces in [5]. Pulling back the flat connection $\eta \in A^1(\bar{\Theta}(\pi, (\omega)); \mathrm{ODer}^+(\hat{L}H/(\omega)))$ defined by the action of $\mathrm{IOut}(\hat{L}H/(\omega))$ on $\bar{\Theta}(\pi, (\omega))$, we obtain the flat connection

$$ \theta_C^* \eta \in A^1\big(\mathbb{M}_g; \mathcal{T}_g \times_{\mathcal{M}_g} \mathrm{ODer}^+(\hat{L}H/(\omega))\big). $$

Since

$$ \theta_C^* \colon A^*\big(\bar{\Theta}(\pi, (\omega))\big)^{\mathcal{M}_g} \to A^*(\mathcal{T}_g)^{\mathcal{M}_g} = A^*(\mathbb{M}_g), $$

we obtain

$$ \theta_C^*(c(\eta_1^m)) = c([\theta_C^* \eta_1]^m) = (-1)^m c([\tilde{k}]^m) = (-1)^m \alpha^{\mathrm{KM}}(c) =: \bar{\alpha}^{\mathrm{KM}}(c) $$

for $c \in \mathrm{Hom}(\Lambda^*(\Lambda^3 H/H), \mathbb{R})^{\mathrm{Sp}(H)} \subset Z^*(\mathrm{ODer}^+(\hat{L}H/(\omega)))^{\mathrm{Sp}(H)}$. Here we use

$$ -[\theta_C^* \eta_1] = [\tilde{k}] \in H^1(\mathcal{M}_g; \Lambda^3 H/H) \simeq H^1(\mathbb{M}_g; \mathcal{T}_g \times_{\mathcal{M}_g} \Lambda^3 H/H), $$

where η_1 is the $\mathrm{ODer}^1(\hat{L}H/(\omega))$–component of η. It is proved as follows: The crossed homomorphism τ_1^θ for any $\theta \in \bar{\Theta}(\pi, (\omega))$ is an extension of the first Johnson homomorphism $\tau_1 \colon \mathcal{I}_g \to \Lambda^3 H/H$ from Proposition 2.5. The equation

$$ [\tilde{k}] = [\tau_1^\theta] \in H^1(\mathcal{M}_g; \Lambda^3 H/H) $$

follows from the result of [11] (see [4]). The relation between the holonomy τ^θ and the iterated integral of the flat connection η gives

$$\tau^\theta(\varphi)^{-1} = \sum_{n=0}^{\infty} \int_{\theta}^{\varphi \cdot \theta} \underbrace{\eta \cdots \eta}_{n}$$

for $\varphi \in \mathcal{M}_g$ and $\theta \in \bar{\Theta}(\pi, (\omega))$. Therefore the correspondence follows from the equation

$$\tau_1^{\theta_C(x)}(\varphi) = -\int_{\theta_C(x)}^{\varphi \cdot \theta_C(x)} \eta_1 = -\int_{x}^{\varphi_* x} \theta_C^* \eta_1$$

for $\varphi \in \mathcal{M}_g$ and a fixed point $x \in \mathcal{T}_g$. So we obtain $[\tilde{k}] = [\tau_1^{\theta_C(x)}] = -[\theta_C^* \eta_1]$.

Thus we get the theorem:

Theorem 4.2 *The following diagram commutes:*

$$\text{Hom}(\Lambda^*(\Lambda^3 H/H), \mathbb{R})^{\text{Sp}(H)} \longrightarrow H^*(\text{ODer}^+(\hat{L}H/(\omega)))^{\text{Sp}(H)} \xrightarrow{\theta_C^*} H_{\text{DR}}^*(\mathbb{M}_g)$$

$$\bar{\alpha}^{KM} \searrow \quad \Big\| $$

$$H^*(\mathcal{M}_g; \mathbb{R})$$

The map θ_C^* can be interpreted from the viewpoint of our characteristic map as follows. We consider the oriented Σ_g–bundle

$$\text{Met}_g \times_{\text{Diff}_+(\Sigma_g)} \Sigma_g \to \mathbb{M}_g.$$

We note that $\text{Met}_g \times_{\text{Diff}_+(\Sigma_g)} \Sigma_g \to \mathbb{M}_g$ is not a fiber bundle exactly since the fiber on $[x] \in \mathbb{M}_g$, $x \in \text{Met}_g$, is isomorphic to the global quotient orbifold $\Sigma_g / \text{Isom}(\Sigma_g, x)$, where $\text{Isom}(\Sigma_g, x)$ is the isometry group of Riemann surface (Σ_g, x). However our construction of the characteristic map works as well as fiber bundles. We give the tautological metric μ to this bundle, ie the metric μ_c on the fiber of c represents the class c for all $c \in \mathbb{M}_g$.

The chain map constructed from the Chen expansion of μ in the manner of Section 3B is equal to $\theta_C^* : A^*(\bar{\Theta}(\pi, (\omega)))^{\mathcal{M}_g} \to A^*(\mathbb{M}_g)$.

Theorem 4.3 *The homomorphism* $\theta_C^* : H^*(\text{ODer}^+(\hat{L}H/(\omega)))^{\text{Sp}(H)} \to H_{\text{DR}}^*(\mathbb{M}_g)$ *is the characteristic map of the fiber bundle* $\text{Met}_g \times_{\text{Diff}_+(\Sigma_g)} \Sigma_g \to \mathbb{M}_g$ *constructed in Section 3B.*

The theorem above means our characteristic maps are nontrivial.

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