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On the homotopy of $Q(3)$ and $Q(5)$ at the prime 2

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KYLE M ORMSBY

We study modular approximations $Q(\ell)$, $\ell = 3, 5$, of the $K(2)$–local sphere at the prime 2 that arise from ℓ–power degree isogenies of elliptic curves. We develop Hopf algebroid level tools for working with $Q(5)$ and record Hill, Hopkins and Ravenel's computation of the homotopy groups of $\mathrm{TMF}_0(5)$. Using these tools and formulas of Mahowald and Rezk for $Q(3)$, we determine the image of Shimomura's 2–primary divided β–family in the Adams–Novikov spectral sequences for $Q(3)$ and $Q(5)$. Finally, we use low-dimensional computations of the homotopy of $Q(3)$ and $Q(5)$ to explore the rôle of these spectra as approximations to $S_{K(2)}$.

55Q45, 55Q51

In [3], motivated by Goerss, Henn, Mahowald and Rezk [9], the p–local spectrum $Q(\ell)$ ($p \nmid \ell$) is defined as the totalization of an explicit semi-cosimplicial E_∞–ring spectrum of the form

$$Q(\ell)^\bullet = \big(\mathrm{TMF} \Rightarrow \mathrm{TMF}_0(\ell) \times \mathrm{TMF} \Rrightarrow \mathrm{TMF}_0(\ell)\big).$$

The spectrum $Q(\ell)$ serves as a kind of approximation to the $K(2)$–local sphere; see Section 3.1 for more details on its construction. In [4], it is proven that there is an equivalence

$$Q(\ell)_{K(2)} \simeq (E_2^{h\Gamma_\ell})^{h\,\mathrm{Gal}},$$

where Γ_ℓ is a certain subgroup of the Morava stabilizer group $\mathbb{S}_2$ coming from isogenies of elliptic curves. The subgroup Γ_ℓ is dense if p is odd and ℓ generates a dense subgroup of $\mathbb{Z}_p^\times$; see [6]. Based on this, it is conjectured that there are fiber sequences

$$D_{K(2)} Q(\ell) \to S_{K(2)} \xrightarrow{u} Q(\ell) \tag{0.0.1}$$

for such choices of ℓ (and the case of $\ell = 2$ and $p = 3$ is handled by explicit computation in [3], and is closely related to [9]). Density also is used in [5] to show that, for such ℓ, $Q(\ell)$ detects the exact divided β–family pattern for $p \geq 5$.

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However, in the case of $p=2$, $\mathbb{Z}_2^\times$ is not topologically cyclic, and the closure of Γ_ℓ in $\mathbb{S}_2$ is the inverse image of the closure of the subgroup $\ell^{\mathbb{Z}} < \mathbb{Z}_2^\times$ under the reduced norm

$$N\colon \mathbb{S}_2 \to \mathbb{Z}_2^\times.$$

It is not altogether clear in this case what the analog of the conjecture (0.0.1) should be, though one possibility is suggested in [6]. Although the 2-primary "duality resolution" of Bobkova, Goerss, Henn, Mahowald and Rezk (see Henn [10] and Bobkova and Goerss [7]) seems to take the form of a fiber sequence like (0.0.1), we will observe that the mod $(2, v_1)$–behavior of $Q(3)$ actually precludes $Q(3)$ from being half of the duality resolution (see Remark 4.5.2). The nondensity of Γ_3, together with the appearance of both $\mathrm{TMF}_0(3)$ and $\mathrm{TMF}_0(5)$ factors in $\mathrm{TMF} \wedge \mathrm{TMF}$ also suggests that, from a TMF–resolutions perspective, $Q(3)$ alone may not be seeing enough homotopy, and that a combined approach of $Q(3)$ and $Q(5)$ may be required at the prime 2.

The goal of this paper is to explore such an approach by extending the work of Mahowald and Rezk [20] on $Q(3)$, and initiating a similar study of $Q(5)$.

The first testing ground for the effectiveness of $Q(3)$ or $Q(5)$ at detecting v_2–periodic homotopy at the prime 2 is Shimomura's 2–primary divided beta family [22]. To the authors' surprise, $Q(3)$ was found to exactly detect Shimomura's divided beta patterns on the 2–lines of the E_2 term of its Adams–Novikov spectral sequence, as we shall explain in Section 4. Hence $Q(3)$ is all that is needed to detect the shape of the divided beta family. The authors were equally surprised to find no such phenomenon for $Q(5)$ — the beta family for $Q(5)$ has greater v_1–divisibility than that for the sphere. On the other hand, the $K(2)$–localization of $Q(5)$ is built out of homotopy fixed point spectra of groups with larger 2–torsion than $Q(3)$. This raises the possibility that while $Q(5)$ may be less effective when it comes to beta elements, it could detect exotic torsion in higher cohomological degrees that is invisible to $Q(3)$. This possibility is explored through some low-dimensional computations.

We now summarize the contents of this paper. In Section 1 we review and expand the theory of $\Gamma_0(5)$–structures on elliptic curves. A $\Gamma_0(5)$–structure is an elliptic curve equipped with a cyclic subgroup of order 5. We recall an explicit description of the scheme representing $\Gamma_1(5)$–structures (elliptic curves with a point of order 5) in terms of *Tate normal form* curves and use this description to present several Hopf algebroids that stackify to the moduli space of $\Gamma_0(5)$–structures. We then use these Hopf algebroids and the geometry of elliptic curves to determine the maps defining $Q(5)^\bullet$.

In Section 2 we compute the homotopy fixed point spectral sequence

$$H^*(\mathbb{F}_5^\times; \pi_* \mathrm{TMF}_1(5)) \implies \pi_* \mathrm{TMF}_0(5).$$

The ring $\pi_* \mathrm{TMF}_1(5)$ and the action of $\mathbb{F}_5^\times$ on it are determined by Tate normal form, allowing us to produce a detailed group cohomology computation. We then compute the differentials and hidden extensions in the spectral sequence by a number of methods: TMF–module structure, transfer-restriction arguments, and comparison with the homotopy orbit spectral sequence. Our use of the homotopy orbit spectral sequence to determine hidden extensions is somewhat novel and may find use in other contexts. Note that the computation of $\pi_* \mathrm{TMF}_0(5)$ was first due to Mahowald and Rezk (unpublished) using this descent spectral sequence. Hill, Hopkins and Ravenel [12] then rediscovered this computation using the slice spectral sequence.

Since $Q(\ell)$ is the totalization of a cosimplicial spectrum, we can compute the E_2–term of its Adams–Novikov spectral sequence as the cohomology of a double complex. The differentials in the double complex are either internal cobar differentials for the Weierstrass or $\Gamma_0(5)$ Hopf algebroids or external differentials determined by the cosimplicial structure of $Q(\ell)^\bullet$. In Section 3 we review formulas for the external differentials in the $\ell = 3$ and $\ell = 5$ cases. The $Q(3)$ formulas are due to Mahowald and Rezk [20] while those for $Q(5)$ are derived from Section 1.

In Section 4 we compute several chromatic spectral sequences related to $Q(3)$ and $Q(5)$. Definitions are stated in Section 4.1 and the technique we use is carefully laid out in Section 4.4. Stated precisely, we compute $H^{0,*}(M_0^2 C^*_{\mathrm{tot}}(Q(3)))$ and $H^{0,*}(M_1^1 C^*_{\mathrm{tot}}(Q(5)))$, both of which are related to the divided β–family in the $Q(\ell)$ spectra. We compare these groups to Shimomura's 2–primary divided β–family for the sphere spectrum (ie the groups $\mathrm{Ext}^{0,*}(M_0^2 \mathrm{BP}_*)$, reviewed in Theorem 4.2.1). In Theorem 4.2.2 we find that $\mathrm{Ext}^{0,*}(M_0^2 \mathrm{BP}_*)$ is isomorphic to $H^{0,*}(M_0^2 C^*_{\mathrm{tot}} Q(3))$, so $Q(3)$ precisely detects the divided β–family. In contradistinction, Theorem 4.2.4 and Corollary 4.9.4 show that the divided β–family for $Q(5)$ has extra v_1–divisibility.

Finally, in Section 5 we compute $\pi_n Q(3)$ and $\pi_n Q(5)$ for $0 \le n < 48$. More precisely, what we actually compute is the portion of these homotopy groups detected by connective versions of TMF.[1] These computations give evidence for some homotopy which $Q(5)$ detects which is not detected by $Q(3)$.

In this paper we assume the reader has some familiarity with the theory of elliptic curves, level structures, and the stacks which parametrize these objects. We also assume the reader is familiar with TMF, and its variants. To give extensive background on these subjects would take us outside of the scope of this paper. For the reader looking for outside resources, we recommend the 2007 Talbot conference proceedings [8]. The expository articles contained there should point the inquiring reader in the

[1] It is likely that what we are computing is a "connective" version of $Q(\ell)$ built out of the connective versions of $\mathrm{TMF}_0(\ell)$ recently constructed in Hill and Lawson [13], though we do not pursue this here.

right direction. This paper is itself extending computations of the first author [3] and Mahowald and Rezk [20]. The reader is encouraged to have some familiarity with these cases before jumping into the computations contained herein.

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1 Elliptic curves with level 5 structures

We consider the moduli problems of $\Gamma_1(5)$– and $\Gamma_0(5)$–structures on elliptic curves. An elliptic curve with a $\Gamma_1(5)$–structure over a commutative $\mathbb{Z}\left[\frac{1}{5}\right]$–algebra R is a pair (C, P) where C is an elliptic curve over R, and $P \in C$ is a point of exact order 5. An elliptic curve with a $\Gamma_0(5)$–structure is a pair (C, H) with C an elliptic curve over R and $H < C$ a subgroup of order 5. Let $\mathcal{M}_i(5)$ denote the moduli stack $\big($over $\operatorname{Spec}\mathbb{Z}\left[\frac{1}{5}\right]\big)$ of $\Gamma_i(5)$–structures.

Let $\mathcal{M}_i^1(5)$ denote the moduli stack of tuples (C, P, v) (respectively (C, H, v)) where v is a nonzero tangent vector at $0 \in C$. This is equivalent to the moduli problem in which a nontrivial invariant differential is recorded. Note that in the case where $i = 1$, we can use translation by P to equivalently specify this structure as a tuple (C, P, v') where v' is a nonzero tangent vector at P.

As we proceed, we will freely move between moduli problems of the form $\mathcal{M}_i(\ell)$ and $\mathcal{M}_i^1(\ell)$, so we will comment briefly here on the significance in topological modular forms of recording or not recording a tangent vector. As is customary in the subject, let ω denote the invertible sheaf of invariant differentials on the moduli stack of elliptic curves, $\mathcal{M}$, so that sections of $\omega^{\otimes t}$ correspond to modular forms of weight t which are meromorphic at the cusp. Recall that the elliptic spectral sequence takes the form

$$E_2^{s,t} = H^s(\mathcal{M}; \omega^{\otimes t}) \Longrightarrow \pi_{t-s}\,\mathrm{TMF}.$$

Now consider the stack $\mathcal{M}^1$ of elliptic curves with the data of a nonzero tangent vector at 0. This stack is equipped with a $\mathbb{G}_m$–action

$$\mathbb{G}_m \times \mathcal{M}^1 \to \mathcal{M}^1$$

that scales this vector, which on points is given by

$$(z, (C, v)) \mapsto (C, z \cdot v).$$

Let π denote the forgetful map

$$\pi\colon \mathcal{M}^1 \to \mathcal{M}$$

which on points is given by

$$(C, v) \mapsto C.$$

The stack $\mathcal{M}^1$ is a $\mathbb{G}_m$–torsor over $\mathcal{M}$, and ω is the associated line bundle over $\mathcal{M}$. We take a moment to spell this out in more concrete terms.

If $\mathcal{X}$ is any scheme or stack with a $\mathbb{G}_m$–action, the structure sheaf admits a decomposition

$$\mathcal{O}_{\mathcal{X}} \cong \bigoplus_{t \in \mathbb{Z}} \mathcal{O}_{\mathcal{X}}(t),$$

where the sections of $\mathcal{O}_{\mathcal{X}}(t)$ consist of those functions f on $\mathcal{X}$ satisfying

$$f(z \cdot x) = z^t \cdot f(x).$$

One source of $\mathbb{G}_m$–equivariant stacks arises from the stackification of commutative Hopf algebroids which are graded. Suppose that (T, Ξ) is a commutative Hopf algebroid with a grading, and let $\mathcal{X}$ be the associated stack:

$$\mathcal{X} = \operatorname{Spec} T /\!/ \operatorname{Spec} \Xi.$$

In this setting, the grading on T endows the scheme $\operatorname{Spec} T$ with a $\mathbb{G}_m$–action, and the grading on Ξ ensures that this $\mathbb{G}_m$ action descends to the quotient, endowing $\mathcal{X}$ with the structure of a $\mathbb{G}_m$–action.

In the case of the $\mathbb{G}_m$–action on $\mathcal{M}^1$, we have

$$\pi_* \mathcal{O}_{\mathcal{M}^1}(1) = \omega.$$

The cohomology ring $H^*(\mathcal{M}^1;\mathcal{O}_{\mathcal{M}^1})$ inherits an additional integer grading; we will write this bigraded ring as

$$H^{s,t}(\mathcal{M}^1;\mathcal{O}_{\mathcal{M}^1}) := H^s(\mathcal{M}^1;\mathcal{O}_{\mathcal{M}^1}(t)),$$

so there is an isomorphism

$$H^{s,t}(\mathcal{M}^1;\mathcal{O}_{\mathcal{M}^1}) \cong H^s(\mathcal{M};\omega^{\otimes t}).$$

Similar statements hold for $\mathcal{M}_i^1(\ell)$. As such, if our interest is in computing the E_2–term of the elliptic spectral sequence for $\mathrm{TMF}_i(\ell)$, then it suffices to study moduli problems in which we record a nonzero tangent vector (or equivalently a nontrivial invariant differential). For the remainder of this section, all presentations of $\mathbb{G}_m$–equivariant moduli stacks by Hopf algebroids shall be implicitly graded, with generators named "g_k" to implicitly lie in degree k.

The maps in the cosimplicial E_∞ ring $Q(5)^\bullet$ arise by evaluating the TMF–sheaf $\mathcal{O}^{\mathrm{top}}$ on maps $\mathcal{M}_0(5) \to \mathcal{M}$ and $\mathcal{M}_0(5) \to \mathcal{M}_0(5)$. Recall that the Weierstrass Hopf algebroid (A,Γ) stackifies to $\mathcal{M}^1$; we review the structure of (A,Γ) in Section 1.2. In this section we produce a Hopf algebroid (B^1,Λ^1) representing $\mathcal{M}_0^1(5)$ and produce Hopf algebroid formulas for the maps in (the cohomology of) the semi-simplicial stack associated to $Q(\ell)^\bullet$.

1.1 Representing $\mathcal{M}_1(5)$

In this section we will give explicit presentations of $\mathcal{M}_1(5)$ and $\mathcal{M}_1^1(5)$. Consider the rings

$$B := \mathbb{Z}\left[\tfrac{1}{5}, b, \Delta^{-1}\right],$$
$$B^1 := \mathbb{Z}\left[\tfrac{1}{5}, a_1, a_2, a_3, \Delta^{-1}\right]/(a_2^3 + a_3^2 - a_1a_2a_3),$$

where Δ is given respectively by

$$\Delta(b) = b^5(b^2 - 11b - 1),$$
$$\Delta(a_1,a_2,a_3) = -8a_1^2a_3^2a_2^2 + 20a_1a_3^3a_2 - a_1^4a_3^2a_2 - 11a_3^4 + a_1^3a_3^3.$$

We have the following theorem.

Theorem 1.1.1 *The stacks $\mathcal{M}_1(5)$ and $\mathcal{M}_1^1(5)$ are affine schemes, given by*

$$\mathcal{M}_1(5) = \operatorname{Spec} B,$$
$$\mathcal{M}_1^1(5) = \operatorname{Spec} B^1.$$

Proof We first use the techniques of [15, Section 4.4] (which is a recapitulation of a method from [19]) to produce an explicit model for $\mathcal{M}_1^1(5)$ as an affine scheme. The procedure is exhibited graphically in Figure 1.

Suppose (C, P) is a $\Gamma_1(5)$–structure over a commutative ring R in Weierstrass form with $P = (\alpha, \beta)$. For $r, s, t \in R$ and $\lambda \in R^\times$ let $\varphi_{r,s,t,\lambda}$ denote the coordinate change

$$x \mapsto \lambda^{-2}x + r, \quad y \mapsto \lambda^{-3}y + \lambda^{-2}sx + t.$$

Move P to $(0,0)$ via the coordinate change $\varphi_{-\alpha,0,-\beta,1}\colon (C, P) \to (C_{\underline{a}'}, (0,0))$, where $C_{\underline{a}'}$ has Weierstrass form

$$y^2 + a_1'xy + a_3'y = x^3 + a_2'x^2 + a_4'x.$$

(Note that $a_6' = 0$ because $(0,0)$ is on the curve.) Next eliminate a_4' by applying the transformation $\varphi_{0,-a_4'/a_3',0,1}$. The result is a smooth Weierstrass curve

$$T^1(a_1, a_2, a_3) : y^2 + a_1xy + a_3y = x^3 + a_2x^2 \tag{1.1.2}$$

with $\Gamma_1(5)$–structure $(0,0)$ which we call the *homogeneous Tate normal form* of (C, P).

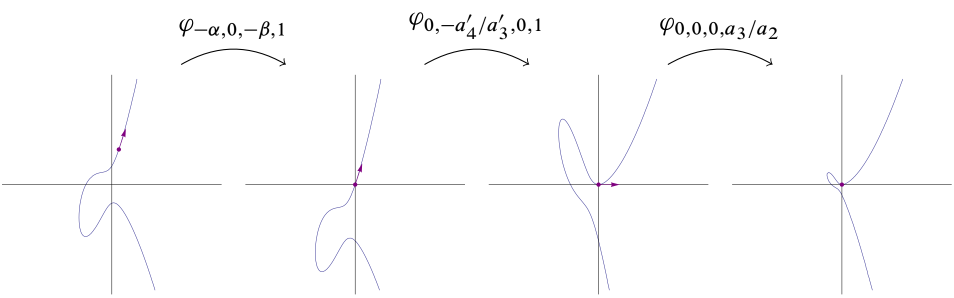

Figure 1: The procedure for putting a $\Gamma_1(5)$–structure in (homogeneous and nonhomogeneous) Tate normal form. From left to right: the curves C, $C_{\underline{a}'}$, T_1 and T.

Since $(0,0)$ has order 5 in $T^1(a_1, a_2, a_3)$ we must have

$$[3](0,0) = [-2](0,0), \tag{1.1.3}$$

where $[n]$ denotes the $\mathbb{Z}$–module structure of the elliptic curve group law. Explicitly expanding the left- and right-hand sides of this equation in projective coordinates, we find that

$$\big(a_2(-a_1a_2a_3 + a_3^2) : a_1a_2a_3^2 - a_2^3a_3 - a_3^3 : a_2^3\big) = (-a_2 : 0 : 1).$$

It follows that a_1, a_2, a_3 must satisfy

$$a_2^3 + a_3^2 = a_1 a_2 a_3 \tag{1.1.4}$$

in order for $(T^1(a_1, a_2, a_3), (0,0))$ to be a $\Gamma_1(5)$–structure. (The referee points out that one can also arrive at this condition by contemplating the geometric meaning of (1.1.3).)

We may compute the discriminant of $T^1(a_1, a_2, a_3)$ as

$$\Delta = -8a_1^2 a_3^2 a_2^2 + 20a_1 a_3^3 a_2 - a_1^4 a_3^2 a_2 - 11a_3^4 + a_1^3 a_3^3. \tag{1.1.5}$$

Let $f^1(a_1, a_2, a_3) := a_2^3 + a_3^2 - a_1 a_2 a_3$ and let

$$B^1 := \mathbb{Z}[a_1, a_2, a_3, \Delta^{-1}]/(f^1).$$

Then

$$\mathcal{M}_1^1(5) = \operatorname{Spec} B^1.$$

We now consider $\Gamma_1(5)$–structures without distinguished tangent vectors and produce a (nonhomogeneous) Tate normal form which is the universal elliptic curve for $\mathcal{M}_1(5)$. Begin with a $\Gamma_1(5)$–structure (C, P) and change coordinates to put it in homogeneous Tate normal form $T^1(a_1, a_2, a_3)$. Now apply the coordinate transformation $\varphi_{0,0,0,a_3/a_2}$. (This transformation is permissible because $(0,0)$ has order greater than 3.) After applying the transformation, the coefficients of y and x^2 are equal. Let

$$T(b,c) : y^2 + (1-c)xy - by = x^3 - bx^2 \tag{1.1.6}$$

denote the resulting smooth Weierstrass curve.

Since $(0,0)$ has order 5 we know (1.1.3) holds; it follows that

$$b = c \tag{1.1.7}$$

in (1.1.6). Abusing notation, let

$$T(b) : y^2 + (1-b)xy - by = x^3 - bx^2; \tag{1.1.8}$$

we call this the (*nonhomogeneous*) *Tate normal form* of (C, P). The discriminant of $T(b)$ is

$$\Delta = b^5(b^2 - 11b - 1). \tag{1.1.9}$$

Let

$$B := \mathbb{Z}\left[\tfrac{1}{5}, b, \Delta^{-1}\right].$$

The preceding two paragraphs show that

$$\mathcal{M}_1(5) = \operatorname{Spec} B. \qquad \square$$

Corollary 1.1.10 *The moduli space $\mathcal{M}_1^1(5)$ is represented by*

$$\operatorname{Spec} \mathbb{Z}\left[\tfrac{1}{5}, a_1, u^{\pm 1}, \Delta^{-1}\right],$$

where

$$\Delta = -11u^{12} + 64a_1u^{11} - 154a_1^2u^{10} + 195a_1^3u^9 - 135a_1^4u^8 + 46a_1^5u^7 - 4a_1^6u^6 - a_1^7u^5.$$

Proof The rings in question are isomorphic via the homomorphism

$$B^1 \to \mathbb{Z}\left[\tfrac{1}{5}, a_1, u^{\pm 1}, \Delta^{-1}\right]$$

determined by

$$a_1 \mapsto a_1, \quad a_2 \mapsto u(a_1 - u), \quad a_3 \mapsto u^2(a_1 - u).$$

(Note that u corresponds to a_3/a_2, and both a_2 and a_3 are invertible in B^1.) This takes $T^1(a_1, a_2, a_3)$ to the curve

$$y^2 + a_1xy + u^2(a_1 - u)y = x^3 + u(a_1 - u)x^2,$$

whose discriminant may be computed manually. □

The simple structure of $\mathcal{M}_1(5)$ has an immediate topological corollary that we record here.

Corollary 1.1.11 *The $K(2)$–localization of* $\mathrm{TMF}_1(5)$ *is a height-2 Lubin–Tate spectrum for the formal group law $\widehat{T(b)}$ defined over $\mathbb{F}_2$:*

$$\mathrm{TMF}_1(5)_{K(2)} \simeq E_2(\mathbb{F}_2, \widehat{T(b)}).$$

Proof The $K(2)$–localization of $\mathrm{TMF}_1(5)$ is controlled by the $\mathbb{F}_2$–supersingular locus $\mathcal{M}_1(5)^{\mathrm{ss}}_{\mathbb{F}_2}$ of $\mathcal{M}_1(5)$. The 2–series of the formal group law for $T = T(b)$ takes the form

$$[2]_{\widehat{T}}(z) = 2z + (b-1)z^2 + 2bz^3 + (b^2 - 2b)z^4 + \cdots.$$

(This is easily deduced from the standard formula for the formal group law of a Weierstrass curve found, for example, in [23, page 120].) Hence $\widehat{T}$ is supersingular over $\mathbb{F}_2$ if and only if $b = 1$. Note that $\Delta(T(1)) = -11$, a unit in $\mathbb{Z}_2$ and $\mathbb{F}_2$. It follows that

$$\mathcal{M}_1(5)^{\mathrm{ss}}_{\mathbb{F}_2} = \operatorname{Spec} \mathbb{F}_2.$$

Let $E_2 = E_2(\mathbb{F}_2, \widehat{T})$ with $\pi_0 E_2 = \mathbb{Z}_2[\![u_1]\!]$. The map

$$\operatorname{Spec} \pi_0 E_2 \to \mathcal{M}_1(5)$$

induced by

$$B \to \pi_0 E_2, \quad b \mapsto u_1 + 1$$

induces the $K(2)$–localization of $\mathrm{TMF}_1(5)$. □

1.2 Representing maps $\mathcal{M}_1^1(5) \to \mathcal{M}^1$

There are two important maps $\mathcal{M}_1^1(5) \to \mathcal{M}^1$ which we analyze. On the level of points, the first is the forgetful map

$$f\colon \mathcal{M}_1^1(5) \to \mathcal{M}^1, \quad (C, P) \mapsto C.$$

The second is the quotient map

$$q\colon \mathcal{M}_1^1(5) \to \mathcal{M}^1, \quad (C, P) \mapsto C/\langle P\rangle.$$

Let (A, Γ) denote the usual Weierstrass curve Hopf algebroid with

$$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}], \quad \Gamma = A[r, s, t]$$

that stackifies to $\mathcal{M}^1$. (Note that Γ does not have a polynomial generator λ precisely because the coordinate change $\varphi_{r,s,t,\lambda}$ preserves tangent vectors if and only if $\lambda = 1$.)

Theorem 1.2.1 *The morphisms f and q above are represented by*

$$f^*\colon A \to B^1, \quad a_i \mapsto \begin{cases} a_i & \text{if } i = 1, 2, 3, \\ 0 & \text{if } i = 4, 6, \end{cases}$$

and

$$q^*\colon A \to B^1, \quad a_i \mapsto \begin{cases} a_i & \text{if } i = 1, 2, 3, \\ 5a_1^2 a_2 - 10 a_1 a_3 - 10 a_2^2 & \text{if } i = 4, \\ a_1^4 a_2 - 2a_1^3 a_3 - 12 a_1^2 a_2^2 + 19 a_2^3 - a_3^2 & \text{if } i = 6. \end{cases}$$

The associated maps $\Gamma \to B^1$ take r, s, t to 0 since $\mathcal{M}_1^1(5)$ is a scheme.

Computing q requires that we find a Weierstrass curve representation of $C/\langle P\rangle$ in terms of the Weierstrass coefficients of C. This procedure is well-studied by number theorists under the name *Vélu's formulae* (see [24] and [17, Section 2.4]) and is implemented in the computer algebra system `Magma`. In fact, if ϕ is an isogeny on C in Weierstrass form with kernel H, then Vélu's formulae compute Weierstrass coefficients for the target of ϕ in terms of the Weierstrass coefficients of C and the defining equations of the subgroup scheme H. We briefly review the formulae here for reference.

Suppose $H < C$ is a finite subgroup with ideal sheaf generated by a monic polynomial $\psi(x)$, where C is a Weierstrass curve of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

For simplicity, assume that the isogeny $\phi\colon C \to C/H$ has odd degree. (The even degree case can be handled as a separate case, but we will not need it in this paper.) Write

$$\psi(x) = x^n - s_1x^{n-1} + \cdots + (-1)^n s_n.$$

Then C/H has Weierstrass equation

$$y_H^2 + a_1x_Hy_H + a_3y_H = x_H^3 + a_2x_H^2 + (a_4 - 5t)x_H + (a_6 - b_2t - 7w),$$

where

$$\begin{aligned} t &= 6(s_1^2 - 2s_2) + b_2s_1 + nb_4, \\ w &= 10(s_1^3 - 3s_1s_2 + 3s_3) + 2(b_2(s_1^2 - 2s_2) + 3b_4s_1 + nb_6, \end{aligned}$$

and

$$b_2 = a_1^2 + 4a_2, \quad b_4 = a_1a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6.$$

Vélu's formulae also give explicit equations for the isogeny $\phi\colon (x,y) \mapsto (x_H, y_H)$, but they are cumbersome to write down and we will not need them here.

Proof of Theorem 1.2.1 The representation of f is obvious: $T^1(a_1,a_2,a_3)$ is already in Weierstrass form with $a_4, a_6 = 0$.

Consider the case of $C = T(a_1,a_2,a_3)$ with $H = \langle P \rangle$ an order-5 subgroup. Using the elliptic curve addition law we see that H is the subgroup scheme of C cut out by the polynomial $x(x+a_2)$. Putting this data into Vélu's formulae, we find that C/H has Weierstrass form

$$\begin{aligned} \text{(1.2.2)} \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 &+ (5a_1^2a_2 - 10a_1a_3 - 10a_2^2)x \\ &+ (a_1^4a_2 - 2a_1^3a_3 - 12a_1^2a_2^2 + 19a_2^3 - a_3^2), \end{aligned}$$

from which our formula for q follows. □

1.3 Hopf algebroids for $\mathcal{M}_0^1(5)$

Recall that a $\Gamma_0(5)$–structure (C,H) consists of an elliptic curve C along with a subgroup $H < C$ of order 5. Unlike the moduli problem of $\Gamma_1(5)$–structures, $\mathcal{M}_0(5)$ is not representable by a scheme. Still, it is the case that $\mathcal{M}_1(5)$ admits a $C_4 = \mathbb{F}_5^\times$–action such that $\mathcal{M}_0(5)$ is the geometric quotient $\mathcal{M}_1(5)/\!/\mathbb{F}_5^\times$. An element

$g \in \mathbb{F}_5^\times$ takes (C, P) to $(C, [\tilde{g}]P)$ for $\tilde{g}$ any lift of g in $\mathbb{Z}$. Similarly, we can write $\mathcal{M}_0^1(5) = \mathcal{M}_1^1(5)/\!/\mathbb{F}_5^\times$.

While it is typically easier to use this quotient stack presentation of $\mathcal{M}_0(5)$ and $\mathcal{M}_0^1(5)$ (and this will be the perspective we will be taking in the computations later in this paper), we will note that there is also a presentation of these moduli stacks by "(r,s,t)" Hopf algebroids. Let B^1 be as before and define

$$\Lambda^1 := B^1[r,s,t]/\sim,$$

where $\sim$ consists of the relations

$$\begin{aligned}
3r^2 &= 2st + a_1rs + a_3s + a_1t - 2a_2r,\\
t^2 &= r^3 + a_2r^2 - a_1rt - a_3t,\\
s^6 &= -3a_1s^5 + 9rs^4 + 3a_2s^4 - 3a_1^2s^4 + 4ts^3\\
&\qquad + 20a_1rs^3 + 6a_1a_2s^3 + 2a_3s^3 - a_1^3s^3 + 6a_1ts^2\\
&\qquad - 27r^2s^2 - 18a_2rs^2 + 12a_1^2rs^2 - 3a_2^2s^2 + 3a_1^2a_2s^2\\
&\qquad + 3a_1a_3s^2 - 12rts - 4a_2ts + 2a_1^2ts - 33a_1r^2s\\
&\qquad - 20a_1a_2rs - 6a_3rs + a_1^3rs - 3a_1a_2^2s - 2a_3a_2s\\
&\qquad + a_1^2a_3s + 4t^2 - 2a_1rt - 2a_1a_2t + 4a_3t + 27r^3\\
&\qquad + 27a_2r^2 - 2a_1^2r^2 + 9a_2^2r - a_1^2a_2r - a_1a_3r.
\end{aligned}$$

Theorem 1.3.1 *The rings (B^1, Λ^1) form a Hopf algebroid stackifying to $\mathcal{M}_0^1(5)$. The structure maps are given by*

$$\begin{aligned}
\eta_R(a_1) &= a_1 + 2s, & \psi(r) &= r\otimes 1 + 1\otimes r,\\
\eta_R(a_2) &= a_2 + 3r - s^2 - a_1s, & \psi(s) &= s\otimes 1 + 1\otimes s,\\
\eta_R(a_3) &= a_3 + 2t + a_1r, & \psi(t) &= t\otimes 1 + s\otimes r + 1\otimes t.
\end{aligned}$$

Proof The reader will note that the structure maps are identical to those for the standard Weierstrass Hopf algebroid (A,Γ). The relations $\sim$ are precisely those required so that $\varphi_{r,s,t,1}$ transforms $T^1(a_1,a_2,a_3)$ (where $a_2^3 + a_3^2 = a_1a_2a_3$) into another homogeneous Tate normal curve. □

There are forgetful and quotient maps on $\mathcal{M}_0^1(5)$ that on points are given by

$$f\colon \mathcal{M}_0^1(5) \to \mathcal{M}^1, \quad (C,H) \mapsto C$$

and

$$q\colon \mathcal{M}_0^1(5) \to \mathcal{M}^1, \quad (C, H) \mapsto C/H.$$

(We elide the tangent vectors for concision.)

Corollary 1.3.2 *The maps f and q on $\mathcal{M}_0^1(5)$ are represented by*

$$f^*\colon (A, \Gamma) \to (B^1, \Lambda^1), \quad a_i \mapsto \begin{cases} a_i & \text{if } i = 1, 2, 3, \\ 0 & \text{if } i = 4, 6, \end{cases} \qquad r, s, t \mapsto r, s, t$$

and

$$q^*\colon (A, \Gamma)(B^1, \Lambda^1),$$

$$a_i \mapsto \begin{cases} a_i & \text{if } i = 1, 2, 3, \\ 5a_1^2 a_2 - 10 a_1 a_3 - 10 a_2^2 & \text{if } i = 4, \\ a_1^4 a_2 - 2a_1^3 a_3 - 12 a_1^2 a_2^2 + 19 a_2^3 - a_3^2 & \text{if } i = 6, \end{cases} \qquad r, s, t \mapsto r, s, t.$$

Proof This is a consequence of Theorems 1.2.1 and 1.3.1. □

1.4 The Atkin–Lehner dual

We will now compute the Atkin–Lehner dual

$$t\colon \mathcal{M}_0^1(5) \to \mathcal{M}_0^1(5).$$

Each $\Gamma_0(5)$–structure (C, H) can also be represented as a pair (C, ϕ) where $\phi\colon C \to C'$ has kernel H. On points, the Atkin–Lehner dual is given by

$$t\colon \mathcal{M}_0^1(5) \to \mathcal{M}_0^1(5), \quad (C, \phi) \mapsto (C/H, \widehat{\phi}),$$

where $\widehat{\phi}$ is the dual isogeny to ϕ.

We can lift t to stacks closely related to $\mathcal{M}_1^1(5)$. Recall [16, Section 2.8] that for each $\Gamma_0(5)$–structure (C, ϕ) there is an associated scheme-theoretic Weil pairing

$$\langle -, - \rangle_\phi\colon \ker \phi \times \ker \widehat{\phi} \to \mu_5.$$

Choose a primitive fifth root of unity ζ. For a $\Gamma_1(5)$–structure (C, P) let (C, ϕ_P) denote the associated $\Gamma_0(5)$–structure where $\phi_P\colon C \to C'$ is an isogeny with kernel $\langle P \rangle$. If we work in $\mathcal{M}_1^1(5)_\zeta$, ie $\mathcal{M}_1^1(5)$ considered as a $\mathbb{Z}[\frac{1}{5}, \zeta]$–scheme, then there is a unique element $Q \in \ker \widehat{\phi_P}$ such that $\langle P, Q \rangle_\phi = \zeta$. We define

$$t_\zeta\colon \mathcal{M}_1^1(5)_\zeta \to \mathcal{M}_1^1(5)_\zeta$$

in the obvious way so that $t_\zeta(C, P) = (C', Q)$.

The maps t and t_ζ fit in the commutative diagram

$$\begin{tikzcd} \mathcal{M}_1^1(5)_\zeta \arrow[r, "t_\zeta"] \arrow[d] & \mathcal{M}_1^1(5)_\zeta \arrow[d] \\ \mathcal{M}_0^1(5)_\zeta \arrow[r, "t"'] & \mathcal{M}_0^1(5)_\zeta \end{tikzcd}$$

where the vertical maps take (C, P) to (C, ϕ_P).

We can gain some computational control over t via the following method. First, recall from Corollary 1.1.10 that for each homogeneous Tate normal curve $T^1(a_1, a_2, a_3)$ there is a unit u such that $a_2 = u(a_1 - u)$ and $a_3 = u^2(a_1 - u)$. Abusing notation, denote the same curve by $T^1(a_1, u)$, and let H denote the canonical cyclic subgroup of order 5 generated by $(0, 0)$. The defining polynomial for H is $x(x + u(a_1 - u))$. Denote the isogeny with kernel H by ϕ. Note that the range of ϕ is the curve C/H given by Vélu's formulae in (1.2.2).

Using Kohel's formulas [17] (as implemented by the computer algebra system `Magma`), we can determine that the kernel of $\widehat{\phi}$ is the subgroup scheme determined by

$$f := x^2 + (a_1^2 - a_1 u + u^2)x + \tfrac{1}{5}(a_1^4 - 7a_1^3 u - 11a_1^2 u^2 + 47a_1 u^3 - 29u^4).$$

Then over the ring $R := \mathbb{Z}[\frac{1}{5}, \zeta][a_1, u^{\pm}]$ the polynomial f splits and we find that

$$(\ker \widehat{\phi})(R) = \{\infty, (x_0, y_{00}), (x_0, y_{01}), (x_1, y_{10}), (x_1, y_{11})\},$$

where

$$\begin{aligned}
x_0 &= \tfrac{1}{5}(\zeta^3 + \zeta^2 - 2)a_1^2 + \tfrac{1}{5}(9\zeta^3 + 9\zeta^2 + 7)a_1 u + \tfrac{1}{5}(-11\zeta^3 - 11\zeta^2 - 8)u^2, \\
x_1 &= \tfrac{1}{5}(-\zeta^3 - \zeta^2 - 3)a_1^2 + \tfrac{1}{5}(-9\zeta^3 - 9\zeta^2 - 2)a_1 u + \tfrac{1}{5}(11\zeta^3 + 11\zeta^2 + 3)u^2, \\
y_{00} &= \tfrac{1}{5}(\zeta^2 + 2\zeta + 2)a_1^3 + \tfrac{1}{5}(\zeta^3 + 7\zeta^2 + 17\zeta + 5)a_1^2 u \\
&\qquad + \tfrac{1}{5}(9\zeta^3 - 29\zeta^2 - 31\zeta - 14)a_1 u^2 + \tfrac{1}{5}(-11\zeta^3 + 22\zeta^2 + 11\zeta + 8)u^3, \\
y_{01} &= \tfrac{1}{5}(-\zeta^3 - 2\zeta^2 - 2\zeta)a_1^3 + \tfrac{1}{5}(-10\zeta^3 - 16\zeta^2 - 17\zeta - 12)a_1^2 u \\
&\qquad + \tfrac{1}{5}(2\zeta^3 + 40\zeta^2 + 31\zeta + 17)a_1 u^2 + \tfrac{1}{5}(11\zeta^3 - 22\zeta^2 - 11\zeta - 3)u^3, \\
y_{10} &= \tfrac{1}{5}(2\zeta^3 + \zeta + 2)a_1^3 + \tfrac{1}{5}(16\zeta^3 - \zeta^2 + 6\zeta + 4)a_1^2 u \\
&\qquad + \tfrac{1}{5}(-40\zeta^3 - 9\zeta^2 - 38\zeta - 23)a_1 u^2 + \tfrac{1}{5}(22\zeta^3 + 11\zeta^2 + 33\zeta + 19)u^3, \\
y_{11} &= \tfrac{1}{5}(-\zeta^3 + \zeta^2 - \zeta + 1)a_1^3 + \tfrac{1}{5}(-7\zeta^3 + 10\zeta^2 - 6\zeta - 2)a_1^2 u \\
&\qquad + \tfrac{1}{5}(29\zeta^3 - 2\zeta^2 + 38\zeta + 15)a_1 u^2 + \tfrac{1}{5}(-22\zeta^3 - 11\zeta^2 - 33\zeta - 14)u^3.
\end{aligned}$$

Choose (x_0, y_{00}) as a preferred generator of $\widehat{H}$. Let $\zeta' = \langle (0,0), (x_0, y_{00}) \rangle_\phi$. Then applying the method of Theorem 1.1.1 we can put $(C/H, (x_0, y_{00}))$ in homogeneous Tate normal form. What we find is a curve $T^1(t^*_{\zeta'}(a_1), t^*_{\zeta'}(u))$ with

$$\begin{aligned} t^*_{\zeta'}(a_1) &= \tfrac{1}{5}(-8\zeta^3 - 6\zeta^2 - 14\zeta - 7)a_1 + \tfrac{1}{5}(14\zeta^3 - 2\zeta^2 + 12\zeta + 6)u, \\ t^*_{\zeta'}(u) &= \tfrac{1}{5}(-\zeta^3 - 7\zeta^2 - 8\zeta - 4)a_1 + \tfrac{1}{5}(8\zeta^3 + 6\zeta^2 + 14\zeta + 7)u. \end{aligned} \tag{1.4.1}$$

Remark 1.4.2 We could produce similar formulas for any of the (x_i, y_{ij}) and these would correspond to different choices of ζ' for the Atkin–Lehner dual on $\Gamma_1(5)$–structures. The applications below will be independent of this choice.

The equations (1.4.1) permits a description of the Atkin–Lehner dual on the ring of $\Gamma_0(5)$–modular forms. We refer the reader to [11, Section 3.1.1] for a thorough description of modular forms as global sections. Recall briefly that for a congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, level Γ–modular forms are precisely the global sections of (the tensor powers of) the dualizing sheaf $\omega^{\otimes *}$ on the moduli stack $\mathcal{M}(\Gamma)$ of level Γ–structures,

$$\mathrm{MF}(\Gamma) = H^0(\mathcal{M}(\Gamma); \omega^{\otimes *}).$$

Let $\mathrm{MF}(\Gamma_1(5))_\zeta$ denote the ring of $\Gamma_1(5)$–modular forms over the ring $\mathbb{Z}\big[\frac{1}{5}, \zeta\big]$; it is isomorphic to $\mathbb{Z}\big[\frac{1}{5}, \zeta\big][a_1, u^\pm, \Delta^{-1}]$ since $\mathcal{M}_1(5)$ is a scheme. Then

$$\mathrm{MF}(\Gamma_0(5)) = (\mathrm{MF}(\Gamma_1(5))_\zeta^{\mathrm{Gal}})^{\mathbb{F}_5^\times},$$

where Gal denotes the copy of $\mathbb{F}_5^\times$ acting on coefficients.

Theorem 1.4.3 *The map $t^*\colon \mathrm{MF}(\Gamma_0(5)) \to \mathrm{MF}(\Gamma_0(5))$ induced by the Atkin–Lehner dual is the restriction of the unique map on $\mathrm{MF}(\Gamma_1(5)_\zeta)$ determined by* (1.4.1).

2 The homotopy groups of $\mathrm{TMF}_0(5)$

By étale descent along the cover

$$\mathcal{M}_1(5) \to \mathcal{M}_1(5) /\!\!/ \mathbb{F}_5^\times = \mathcal{M}_0(5),$$

we have $\mathrm{TMF}_0(5) \simeq \mathrm{TMF}_1(5)^{h\mathbb{F}_5^\times}$, and we may thus compute the associated homotopy point spectral sequence

$$E_2^{s,t} = H^s(\mathbb{F}_5^\times; \pi_t \, \mathrm{TMF}_1(5)) \Longrightarrow \pi_{t-s} \, \mathrm{TMF}_0(5).$$

The referee indicates that the first computation of this spectral sequence actually dates back to as early as 2003, with calculations of Mahowald and Rezk. Hill, Hopkins and

Ravenel computed $\pi_* \mathrm{TMF}_0(5)$ in [12]. As a self-contained homotopy fixed point spectral sequence computation of $\pi_* \mathrm{TMF}_0(5)$ is not yet available in the literature, we reproduce it in this section (though we note that the homotopy fixed point spectral sequence is actually a localization of the slice spectral sequence, and therefore the structure of this spectral sequence can actually be culled from [12]).

2.1 Computation of the E_2–term

Consider the representation of $\mathcal{M}_1^1(5)$ implicit in Corollary 1.1.10. In the context of spectral sequence computations, we will let $x = u$ and let $y = a_1 - u$. Let σ denote the reduction of 3 in $\mathbb{F}_5^\times$, a generator.

Lemma 2.1.1 *The action of $\mathbb{F}_5^\times$ on $\pi_* \mathrm{TMF}_1(5) = \mathbb{Z}\left[\frac{1}{5}, x, y, \Delta^{-1}\right]$ is determined by*

$$\sigma \cdot x = y, \quad \sigma \cdot y = -x. \tag{2.1.2}$$

Proof Consider the Tate normal curve T with $a_1 = x + y$, $a_2 = xy$ and $a_3 = x^2 y$. (This is the Tate normal curve of Corollary 1.1.10 under our coordinate change $x = u$, $y = a_1 - u$.) We can compute $[2](0,0) = (-xy, xy^2)$. The lemma then amounts to noting that the Tate normal curve associated with the $\Gamma_1(5)$–structure $((-xy, xy^2), T)$ has $a_1 = y - x$, $a_2 = -xy$, $a_3 = xy^2$. □

Note that we may manually compute the discriminant as

$$\Delta = x^5 y^5 (x^2 - 11xy - y^2),$$

so x and y are invertible elements of $\pi_* \mathrm{TMF}_1(5)$.

Theorem 2.1.3 *The E_2–term of the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$ is given by*

$$H^*(\mathbb{F}_5^\times; \pi_* \mathrm{TMF}_1(5)) = \mathbb{Z}\left[\tfrac{1}{5}\right][b_2, b_4, \delta, \eta, \nu, \gamma, \xi, \Delta^{-1}]/\sim,$$

where $\Delta = \delta^2(b_4 - 11\delta)$ and $\sim$ consists of the relations

$$\begin{aligned}
&2\eta = 0, && 4\xi = 0, && \eta\nu = 0,\\
&2\nu = 0, && \nu^2 = 2\xi, && \nu\gamma = 0,\\
&2\gamma = 0, && \gamma^2 = (b_2^2 + \delta)\eta^2, && b_2\xi = \delta\eta^2,\\
&b_2\nu = 0, && b_4^2 = b_2^2\delta - 4\delta^2, && b_4\xi = b_2^2\xi + 2\delta\xi + \delta\eta\gamma,\\
&b_4\nu = 0, && b_4\gamma = (b_4 + \delta)b_2\eta, && \gamma b_2 = \eta(b_2^2 + b_4).
\end{aligned}$$

The generators lie in bidegrees $(t-s, s)$:

$$|b_2| = (4,0), \qquad |\nu| = (3,1),$$
$$|b_4| = |\delta| = (8,0), \qquad |\gamma| = (5,1),$$
$$|\eta| = (1,1), \qquad |\xi| = (6,2).$$

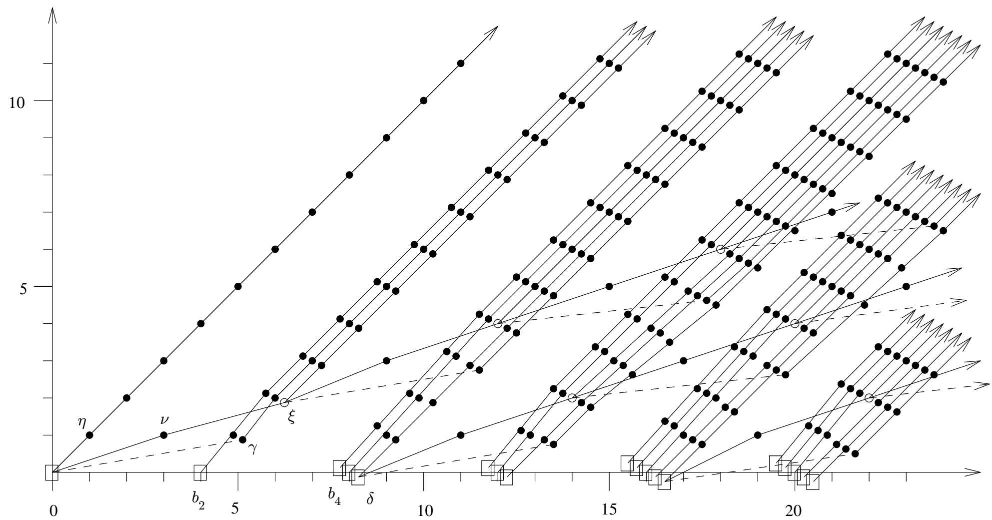

Figure 2: A delocalization of the E_2–term of the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$ (the actual E_2–term is obtained from this figure by inverting Δ)

Figure 2 shows a picture of the subring of the E_2–term of the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$ generated (as a $\mathbb{Z}[\frac{1}{5}]$–algebra) by

$$b_2, \quad b_4, \quad \delta, \quad \eta, \quad \nu, \quad \gamma, \quad \xi.$$

The full E_2–term is obtained after inverting Δ. Here and elsewhere in this paper, we use boxes $\square$ to represent $\mathbb{Z}$ (or $\mathbb{Z}[\frac{1}{5}]$ in this case), filled circles $\bullet$ to represent $\mathbb{Z}/2$, and open circles $\circ$ to represent $\mathbb{Z}/4$.

The proof of Theorem 2.1.3 is a routine but fairly involved calculation following from (2.1.2). We will establish this theorem with a series of lemmas. Let T_* denote the graded subring of $\pi_* \mathrm{TMF}_1(5)$ generated by x and y, so that

$$\pi_* \mathrm{TMF}_1(5) = T_*[\Delta^{-1}].$$

For a $\mathbb{F}_5^\times$–module M, we shall use $H^*(M)$ to denote $H^*(\mathbb{F}_5^\times; M)$.

The first step is to determine the structure of T_* as an $\mathbb{F}_5^\times$–module. We begin by setting some notation for $\mathbb{F}_5^\times$–modules. Let $\mathbb{Z}[\frac{1}{5}]$ denote the $\mathbb{F}_5^\times$–module with trivial action, let $\mathbb{Z}[\frac{1}{5}](-1)$ denote $\mathbb{Z}[\frac{1}{5}]$ with the sign action $\sigma \cdot n = -n$, let $\tau = \mathbb{Z}[\frac{1}{5}]^2$ with the twist action $\sigma \cdot (m,n) = (n,m)$, and let $\psi = \mathbb{Z}[\frac{1}{5}]^2$ with the cycle action $\sigma \cdot (m,n) = (n,-m)$.

Lemma 2.1.4 *The graded ring T_* admits the following additive decomposition as an $\mathbb{F}_5^\times$–module:*

$$\begin{aligned} T_{8n} &= \tau\{x^{4n}, x^{4n-1}y, \dots, x^{2n+1}y^{2n-1}\} \oplus \mathbb{Z}[\tfrac{1}{5}]\{x^{2n}y^{2n}\}, \\ T_{8n+4} &= \tau\{x^{4n+2}, x^{4n+1}y, \dots, x^{2n+2}y^{2n}\} \oplus \mathbb{Z}[\tfrac{1}{5}](-1)\{x^{2n+1}y^{2n+1}\}, \\ T_{4n+2} &= \psi\{x^{2n+1}, x^{2n}y, \dots, x^{n+1}y^{n}\}. \end{aligned}$$

Define the following $\mathbb{F}_5^\times$–invariants in T_*:

$$b_2 := x^2 + y^2, \quad b_4 := x^3y - xy^3, \quad \delta := x^2y^2.$$

(Warning: the b_2 and b_4 here are not related to the b_2 and b_4 mentioned in relation to Vélu's formulae, or the b_2 and b_4 traditionally used in the theory of elliptic curves.) Note that δ is almost a cube root of Δ: we have

$$\Delta = \delta^2(b_4 - 11\delta).$$

The following lemma is fairly easily checked.

Lemma 2.1.5 *The ring of $\mathbb{F}_5^\times$–invariants of T_* admits the presentation*

$$H^0(T_*) = \mathbb{Z}[\tfrac{1}{5}][b_2, b_4, \delta]/(b_4^2 - b_2^2\delta + 4\delta^2).$$

We now turn our attention to the higher cohomology of T_*. The following lemma gives an additive description of these cohomology groups, as a module over

$$H^*(\mathbb{Z}[\tfrac{1}{5}]) = \mathbb{Z}[\tfrac{1}{5}, \beta]/(4\beta)$$

(where β lies in H^2).

Lemma 2.1.6 *There is an* additive *isomorphism of $H^*(\mathbb{Z}[\frac{1}{5}])$–modules*

$$H^*(T_*) \cong \mathbb{Z}[\tfrac{1}{5}][b_2, b_4, \delta, \eta, \nu, \gamma, \beta]/\sim',$$

where $\sim'$ consists of the relations

$$\begin{aligned} &2\eta = 0, \quad &&2\gamma = 0, \quad &&2b_2\beta = 0, \quad &&b_4^2 = b_2^2\delta - 4\delta^2, \\ &2\nu = 0, &&4\beta = 0, &&2b_4\beta = 0 \end{aligned}$$

and

$$(*) \qquad \begin{cases} \nu^2 = 0, & \eta\nu = 0, \quad \eta^2 = 0, \quad \eta\gamma = 0, \quad b_4\gamma = 0, \\ \gamma^2 = 0, & b_2\nu = 0, \quad \nu\gamma = 0, \quad b_4\nu = 0, \quad \gamma b_2 = 0. \end{cases}$$

The relations marked $(*)$ in the preceding lemma are not actual multiplicative relations in $H^*(T_*)$, they just yield the correct additive answer. To properly compute the ring structure of $H^*(T_*)$, we need to replace these "fake" relations with true relations.

Proof The invariants introduced in the previous lemma allow for a more convenient additive description of T_* as an $\mathbb{F}_5^\times$–module:

$$\begin{aligned}
T_{8n} &= \tau\{x^2\}\{b_2^{2n-1}, b_4 b_2^{2n-3}, \delta b_2^{2n-3}, b_4\delta b_2^{2n-5}, \delta^2 b_2^{2n-5}, \dots\}\\
&\qquad \oplus \tau\{x^3 y\}\{\delta^{n-1}\} \oplus \mathbb{Z}\big[\tfrac{1}{5}\big]\{\delta^n\},\\
T_{8n+4} &= \tau\{x^2\}\{b_2^{2n}, b_4 b_2^{2n-2}, \delta b_2^{2n-2}, b_4\delta b_2^{2n-4}, \delta^2 b_2^{2n-4}, \dots\}\\
&\qquad \oplus \mathbb{Z}\big[\tfrac{1}{5}\big](-1)\{xy\}\{\delta^n\},\\
T_{8n+2} &= \psi\{x\}\{b_2^{2n}, b_4 b_2^{2n-2}, \delta b_2^{2n-2}, b_4\delta b_2^{2n-4}, \delta^2 b_2^{2n-4}, \dots\},\\
T_{8n+6} &= \psi\{x\}\{b_2^{2n+1}, b_4 b_2^{2n-1}, \delta b_2^{2n-1}, b_4\delta b_2^{2n-3}, \delta^2 b_2^{2n-3}, \dots\} \oplus \psi\{x^3\}\{\delta^n\}.
\end{aligned}$$

To compute the higher cohomology $H^*(T_*)$ we begin by noting that

$$\begin{aligned}
H^*\big(\mathbb{Z}\big[\tfrac{1}{5}\big]\big) &= \mathbb{Z}\big[\tfrac{1}{5}\big][\beta]/4\beta, &\qquad H^*\big(\mathbb{Z}\big[\tfrac{1}{5}\big](-1)\big) &= \mathbb{Z}\big[\tfrac{1}{5}\big][\beta]/2\beta[1],\\
H^*(\tau) &= \mathbb{Z}\big[\tfrac{1}{5}\big][\beta]/2\beta, & H^*(\psi) &= \mathbb{Z}\big[\tfrac{1}{5}\big][\beta]/2\beta[1],
\end{aligned}$$

where β has cohomological degree 2, $[1]$ denotes a cohomological degree shift by 1, and each cohomology ring has the obvious $H^*\big(\mathbb{Z}\big[\tfrac{1}{5}\big]\big)$–module structure. We define

$$\eta \in H^1(\psi\{x\}), \quad \nu \in H^1\big(\mathbb{Z}\big[\tfrac{1}{5}\big](-1)\{xy\}\big), \quad \gamma \in H^1(\psi\{x^3\})$$

to be the unique nontrivial elements in their respective cohomology groups. We then have the following additive presentation of $H^*(T_*)$:

$$\begin{aligned}
H^*(T_{8n}) &= \mathbb{Z}\big[\tfrac{1}{5}\big][\beta]/(2\beta)\{b_2^{2n}, b_4 b_2^{2n-2}, \delta b_2^{2n-2}, b_4\delta b_2^{2n-4}, \delta^2 b_2^{2n-4}, \dots, b_4\delta^{n-1}\}\\
&\qquad \oplus \mathbb{Z}\big[\tfrac{1}{5}\big][\beta]/(4\beta)\{\delta^n\},\\
H^*(T_{8n+4}) &= \mathbb{Z}\big[\tfrac{1}{5}\big][\beta]/(2\beta)\{b_2^{2n+1}, b_4 b_2^{2n-1}, \delta b_2^{2n-1}, b_4\delta b_2^{2n-3}, \delta^2 b_2^{2n-3}, \dots\}\\
&\qquad \oplus \mathbb{Z}\big[\tfrac{1}{5}\big][\beta]/(2\beta)\{\nu\delta^n\},\\
H^*(T_{8n+2}) &= \mathbb{Z}\big[\tfrac{1}{5}\big][\beta]/(2\beta)\{\eta b_2^{2n}, \eta b_4 b_2^{2n-2}, \eta\delta b_2^{2n-2}, \eta b_4\delta b_2^{2n-4}, \eta\delta^2 b_2^{2n-4}, \dots\},\\
H^*(T_{8n+6}) &= \mathbb{Z}\big[\tfrac{1}{5}\big][\beta]/(2\beta)\{\eta b_2^{2n+1}, \eta b_4 b_2^{2n-1}, \eta\delta b_2^{2n-1}, \eta b_4\delta b_2^{2n-3}, \eta\delta^2 b_2^{2n-3}, \dots\}\\
&\qquad \oplus \mathbb{Z}\big[\tfrac{1}{5}\big][\beta]/(2\beta)\{\gamma\delta^n\}.
\end{aligned}$$

The statement of the lemma follows. $\square$

The following proposition fills in the multiplicative structure missing from the previous lemma.

Proposition 2.1.7 *There is an isomorphism of rings*

$$H^*(T_*) \cong \mathbb{Z}\left[\tfrac{1}{5}\right][b_2, b_4, \delta, \eta, \nu, \gamma, \beta]/\sim,$$

where $\sim$ consists of the relations

$$\begin{array}{lll} 2\eta = 0, & \eta^2 = b_2\beta, & \nu\gamma = 0, \\ 2\nu = 0, & \nu^2 = 2\delta\beta, & \eta\gamma = (b_4 + b_2^2 + 2\delta)\beta, \\ 2\gamma = 0, & \gamma^2 = (b_2^2 + \delta)\eta^2, & \eta\nu = 0, \\ 4\beta = 0, & b_4\gamma = (b_4 + \delta)b_2\eta, & b_2\gamma = (b_2^2 + b_4)\eta, \\ b_2\nu = 0, & b_4\nu = 0, & b_4^2 = b_2^2\delta - 4\delta^2. \end{array}$$

Note that we are able to drop the relations

$$2b_2\beta = 0 \quad \text{and} \quad 2b_4\beta = 0$$

appearing in Lemma 2.1.6, as they follow from the relations

$$\eta^2 = b_2\beta \quad \text{and} \quad \eta\gamma = (b_4 + b_2^2 + 2\delta)\beta,$$

respectively.

Proof The following multiplicative relations are immediately deduced from dimensional considerations:

$$\eta\nu = 0, \quad b_2\nu = 0, \quad \nu\gamma = 0.$$

Moreover, the ring structure on T_* restricts to give a pairing

$$H^1\left(\mathbb{Z}\left[\tfrac{1}{5}\right](-1)\{xy\}\right) \otimes H^0(\tau\{x^2\}) \to H^1(\tau\{x^3y\}) = 0$$

which implies

$$\nu b_4 = 0.$$

In order to determine most of the remaining relations, we observe that

$$\begin{aligned} H^0(T_2/2) &= \mathbb{F}_2\{v_1\}, \\ H^0(T_4/2) &= \mathbb{F}_2\{v_1^2, \delta^{1/2}\}, \\ H^0(T_6/2) &= \mathbb{F}_2\{v_1^3, v_1\delta^{1/2}\}, \end{aligned}$$

with

$$v_1 := x + y, \quad \delta^{1/2} := xy.$$

Note that the mod 2 reductions of b_2, b_4, and δ are v_1^2, $v_1^2\delta^{1/2}$, and $(\delta^{1/2})^2$, respectively (and this explains the notation "$\delta^{1/2}$"). It follows easily from the long exact sequence

$$\cdots \to H^0(T_*) \xrightarrow{\cdot 2} H^0(T_*) \to H^0(T_*/2) \xrightarrow{\partial} H^1(T_*) \xrightarrow{\cdot 2} \cdots$$

that

$$\eta = \partial(v_1), \quad \nu = \partial(\delta^{1/2}), \quad \gamma = \partial(v_1^3 + \delta^{1/2}v_1).$$

We deduce that

$$\begin{aligned} b_4\gamma &= \partial((v_1^2\delta^{1/2})(v_1^3 + \delta^{1/2}v_1)) \\ &= \partial((v_1^2\delta^{1/2} + \delta)v_1^2 \cdot v_1) \\ &= (b_4 + \delta)b_2\eta \end{aligned}$$

and

$$\begin{aligned} b_2\gamma &= \partial(v_1^2(v_1^3 + v_1\delta^{1/2})) \\ &= \partial((v_1^4 + v_1^2\delta^{1/2})v_1) \\ &= (b_2^2 + b_4)\eta. \end{aligned}$$

To obtain the relation involving ν^2, we note from the exact sequence

$$\begin{tikzcd} H^1\big(\mathbb{Z}[\tfrac{1}{5}](-1)\{x^3y^3\}\big) \arrow[r, "\cdot 2"'] \arrow[d, equal] & H^1\big(\mathbb{Z}[\tfrac{1}{5}](-1)\{x^3y^3\}\big) \arrow[r] \arrow[d, equal] & H^1(\mathbb{F}_2\{x^3y^3\}) \\ \mathbb{F}_2\{\delta\nu\} & \mathbb{F}_2\{\delta\nu\} & \end{tikzcd}$$

that the mod 2 reduction of $\delta\nu$ is nontrivial in $H^1(T_*/2)$. From this it follows that $\delta^{1/2}\nu$ is nontrivial in $H^1(T_*/2)$, and in particular, it must generate

$$H^1(\mathbb{F}_2\{x^2y^2\}) \cong \mathbb{F}_2.$$

It then follows from the long exact sequence

$$\begin{tikzcd} H^1\big(\mathbb{Z}[\tfrac{1}{5}]\{x^2y^2\}\big) \arrow[r] \arrow[d, equal] & H^1(\mathbb{F}_2\{x^2y^2\}) \arrow[r, "\partial"'] \arrow[d, equal] & H^2\big(\mathbb{Z}[\tfrac{1}{5}]\{x^2y^2\}\big) \arrow[r, "\cdot 2"'] \arrow[d, equal] & H^2\big(\mathbb{Z}[\tfrac{1}{5}]\{x^2y^2\}\big) \arrow[d, equal] \\ 0 & \mathbb{F}_2\{\delta^{1/2}\nu\} & \mathbb{Z}/4\{\delta\beta\} & \mathbb{Z}/4\{\delta\beta\} \end{tikzcd}$$

that

$$2\delta\beta = \partial(\delta^{1/2}\nu) = \nu^2.$$

A similar argument handles the relation involving η^2. We note from the exact sequence

$$\begin{tikzcd} H^1(\psi\{b_2x\}) \arrow[r, "\cdot 2"'] \arrow[d, equal] & H^1(\psi\{b_2x\}) \arrow[r] \arrow[d, equal] & H^1(\psi/2\{b_2x\}) \\ \mathbb{F}_2\{b_2\eta\} & \mathbb{F}_2\{b_2\eta\} & \end{tikzcd}$$

that the mod 2 reduction of $b_2\eta$ is nontrivial in $H^1(T_*/2)$. From this it follows that $v_1\eta$ is nontrivial in $H^1(T_*/2)$, and in particular, it must generate

$$H^1(\tau/2\{x^2+xy\}) \cong \mathbb{F}_2.$$

It then follows from the long exact sequence

$$\begin{tikzcd} H^1(\tau\{x^2+xy\}) \arrow[r] \arrow[d, equal] & H^1(\tau/2\{x^2+xy\}) \arrow[r, "\partial"'] \arrow[d, equal] & H^2(\tau\{x^2+xy\}) \arrow[r, "\cdot 2"'] \arrow[d, equal] & H^2(\tau\{x^2+xy\}) \\ 0 & \mathbb{F}_2\{v_1\eta\} & \mathbb{F}_2\{\beta b_2\} & \end{tikzcd}$$

that

$$\beta b_2 = \partial(v_1\eta) = \eta^2.$$

The relation involving γ^2 now follows from the fact that multiplication by b_2^2 gives an injection

$$\cdot b_2^2\colon H^2(T_{12}) \hookrightarrow H^2(T_{20})$$

and we have

$$b_2^2\gamma^2 = \eta^2(b_2^2+b_4)^2 = b_2^2\eta^2(b_2^2+\delta).$$

The only relation left is the one involving $\eta\gamma$. To this end we have the following 1–cochain representatives, whose values on $\sigma^i \in \mathbb{F}_5^\times$ are given by:

g	1	σ	σ^2	σ^3
$\eta(g)$	0	x	$x+y$	y
$\gamma(g)$	0	x^3	x^3+y^3	y^3

Each of these 1–cochains $\phi(g)$ satisfies the 1–cocycle condition

$$(\delta\phi)(g_1,g_2) = g_1\phi(g_2) - \phi(g_1g_2) + \phi(g_2) = 0.$$

We also record a 2–cocycle $\beta(g_1, g_2)$ which represents β; its values on (g_1, g_2) are recorded in the following table:

$g_2 \downarrow\ g_1 \rightarrow$	1	σ	σ^2	σ^3
1	0	0	0	0
σ	0	0	0	1
σ^2	0	0	1	1
σ^3	0	1	1	1

Recall for 1–cocycles $\phi(g)$ and $\psi(g)$ the explicit chain-level formula for the 2–cocycle $\phi \cup \psi$ (see, for instance, [1]):

$$(\phi \cup \psi)(g_1, g_2) = (g_1 \phi(g_2))\psi(g_1).$$

Using our explicit cochain representatives, we compute that $\eta\gamma + \beta(b_4 + b_2^2 - 2\delta)$ is represented by the 2–cocycle $\psi(g_1, g_2)$ whose values are given by the following table:

$g_2 \downarrow\ g_1 \rightarrow$	1	σ	σ^2	σ^3
1	0	0	0	0
σ	0	x^3y	$-x^4 - xy^3$	$x^4 + x^3y - xy^3$
σ^2	0	$-x^4 + x^3y$	$-2xy^3$	$x^4 + x^3y$
σ^3	0	$x^3y - xy^3 + y^4$	$x^4 - xy^3$	$x^4 + x^3y + y^4$

This 2–cocycle is the coboundary of the 1–cochain ϕ given by the following table:

g	1	σ	σ^2	σ^3
$\phi(g)$	0	$-xy^3$	$-xy^3$	$x^4 - xy^3$

□

We can now deduce Theorem 2.1.3 from the preceding proposition by observing that

$$H^*(\pi_* \operatorname{TMF}_1(5)) = H^*(T_*)[\Delta^{-1}].$$

Since inverting Δ inverts δ, we can replace the generator β with the generator

$$\xi := \beta\delta.$$

The relations of Theorem 2.1.3 are then easily seen to be equivalent to those of the preceding proposition after inverting Δ. The authors find it easier to work with the generator ξ in the homotopy fixed point spectral sequence computations which follow, as it, as well as the other generators in the presentation of Theorem 2.1.3, lies in the first quadrant of the homotopy fixed point spectral sequence (with traditional Adams-style indexing).

As Δ is the product $\delta^2(b_4-11\delta)$, inverting Δ in Theorem 2.1.3 is a rather opaque procedure. Clearly it means that δ and $b_4-11\delta$ must be inverted. Inverting δ is relatively straightforward: the entire cohomology is then δ–periodic, and everything in H^0, as well as η multiples on these classes, is a polynomial algebra[2] over

$$\widetilde{j} := b_4/\delta \in H^0(\pi_* \mathrm{TMF}_1(5)).$$

This class seems to act like a kind of j–invariant in the theory of modular forms for $\Gamma_0(5)$. The relationship to the classical j–invariant is given by the equation

$$j = \frac{c_4^3}{\Delta} = \frac{(\widetilde{j}^2 - 12\widetilde{j} + 16)^3}{\widetilde{j} - 11}.$$

(We are grateful to the referee for suggesting the importance of this element.)

However, inverting $b_4-11\delta$ (or equivalently $\widetilde{j}-11$) is far more subtle, as there are many relations involving b_4 and hence $\widetilde{j}$. We propose two perspectives to help analyze the resulting localized cohomology groups.

Perspective 1 Work 2–locally. The only torsion in $\mathrm{TMF}_0(5)$ is 2–torsion, and arguably this spectrum is most interesting from the perspective of 2–local homotopy theory. We will argue that in this context, the effect of inverting $(\widetilde{j}-11)$ can be analyzed with a simple set of relations.

Perspective 2 (We thank the referee for pointing out this alternative perspective.) Instead of focusing on b_2, make b_4 (or equivalently $\widetilde{j} = b_4/\delta$) the more fundamental variable to express things in. This perspective has the advantage of making H^0 a free module over the ring

$$\mathbb{Z}\left[\tfrac{1}{5}, \widetilde{j}, (11-\widetilde{j})^{-1}, \delta^{\pm 1}\right]$$

at the expense of being able to easily identify b_2–periodic (ie 2–primary v_1–periodic) classes.

Perspective 1 is arguably the better perspective to take if the reader is interested in 2–local homotopy theory. Perspective 2 is arguably more appropriate for those readers interested in $\mathrm{TMF}_0(5)$ from a global perspective (ie with only 5 inverted).

Perspective 1 (2–local) We offer the following simple corollary to Theorem 2.1.3, which is easily deduced from the relations therein.

[2] By this, we mean that in each bidegree, the resulting localized E_2–term takes the form $A \otimes \mathbb{Z}[\widetilde{j}]$.

Corollary 2.1.8 *In $H^*(\pi_* \mathrm{TMF}_1(5))$, we have*

$$\begin{aligned} 11(\tilde{j}-11)^{-1}b_4 &= b_2^2(\tilde{j}-11)^{-1} - 4\delta(\tilde{j}-11)^{-1} - b_4, \\ (\tilde{j}-11)^{-1}\nu &= \nu, \\ (\tilde{j}-11)^{-1}\gamma &= \gamma + (\tilde{j}-11)^{-1}(b_4b_2\delta^{-1} + b_2)\eta, \\ (\tilde{j}-11)^{-1}\beta &= (\tilde{j}-11)^{-1}(\eta\gamma\delta^{-1} - b_2\eta^2\delta^{-1}) - \beta. \end{aligned}$$

The appearance of the factor of 11 in the first relation of the previous corollary complicates the situation, but this complication disappears after we invert 11. In particular, we deduce from this corollary that, at least additively, $H^*(\pi_* \mathrm{TMF}_1(5)_{(2)})$ can be visualized from Figure 2 by first inverting δ, and then formally adjoining a polynomial algebra on $(\tilde{j}-11)^{-1}$ on all classes of the form

$$\delta^i b_2^j \eta^k, \quad i \in \mathbb{Z},\ j \geq 0,\ k \geq 0.$$

Remark 2.1.9 If we complete at $(2, b_2)$ (as in the case of the E_2–term of the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)_{K(2)}$), then the situation becomes simpler: the class $(\tilde{j}-11)^{-1}$ is already invertible in $H^*((T_*)^\wedge_{(2,b_2)})[\delta^{-1}]$.

Perspective 2 (global) The referee, in addition to suggesting the previous far more streamlined and readable approach to Theorem 2.1.3, found an alternative set of generators for $H^*(\pi_* \mathrm{TMF}_1(5))$ which gives a cleaner presentation if the reader does not wish to work 2–locally. Replace the generator γ with the generator

$$\tilde{\gamma} := \gamma + b_2\eta.$$

We then have the following presentation of $H^*(\pi_* \mathrm{TMF}_1(5))$:

$$H^0(\pi_* \mathrm{TMF}_1(5)) = \mathbb{Z}\left[\tfrac{1}{5}, \tilde{j}, (11-\tilde{j})^{-1}, b_2, \delta^{\pm 1}\right]/(b_2^2 = (\tilde{j}^2+4)\delta)$$

and

$$H^*(\pi_* \mathrm{TMF}_1(5)) = H^0(\pi_* \mathrm{TMF}_1(5))[\beta, \eta, \nu, \tilde{\gamma}]/\sim,$$

where $\sim$ consists of the relations

$$\begin{array}{llll} 4\beta = 0, & b_2\tilde{\gamma} = \delta\tilde{j}\eta, & \eta^2 = b_2\beta, & \eta\nu = 0, \\ 2\eta = 0, & b_2\eta = \tilde{j}\tilde{\gamma}, & \tilde{\gamma}^2 = \delta b_2\beta, & \nu\tilde{\gamma} = 0, \\ 2\tilde{\gamma} = 0, & \tilde{j}\nu = 0, & \eta\tilde{\gamma} = (\tilde{j}+2)\delta\beta, & 2\tilde{j}\beta = 0, \\ 2\nu = 0, & b_2\nu = 0, & \nu^2 = 2\delta\beta, & 2b_2\beta = 0. \end{array}$$

2.2 The behavior of transfer and restriction in the homotopy fixed point spectral sequence

Our next task is to compute the differentials in the homotopy fixed point spectral sequence

$$H^s(\mathbb{F}_5^\times; \pi_t \,\mathrm{TMF}_1(5)) \Rightarrow \pi_{t-s} \,\mathrm{TMF}_0(5). \tag{2.2.1}$$

One might expect this could be accomplished by comparison with the well known descent spectral sequence for TMF. However, it will turn out that the images of many elements of π_* TMF in $\pi_* \,\mathrm{TMF}_0(5)$ will be detected on different lines of the respective spectral sequences. An analysis of transfer and restriction maps relating these two spectral sequences will remedy this complication.

Let $\mathcal{M}(5)$ denote the moduli space of elliptic curves with full level structure, and TMF(5) the corresponding spectrum of topological modular forms. Using the portion

$$\begin{tikzcd} \mathcal{M}(5) \arrow[d, dash, "B"] \arrow[dd, dash, bend right=60, "\mathrm{GL}_2(\mathbb{F}_5)"'] \\ \mathcal{M}_0(5) \arrow[d, dash] \\ \mathcal{M} \end{tikzcd}$$

of [16, Diagram 7.4.3] (where B is the Borel subgroup of upper triangular matrices), the spectrum TMF(5) has an action of $\mathrm{GL}_2(\mathbb{F}_5)$, and we have

$$\mathrm{TMF}\big[\tfrac{1}{5}\big] \simeq \mathrm{TMF}(5)^{h\,\mathrm{GL}_2(\mathbb{F}_5)} \quad \text{and} \quad \mathrm{TMF}_0(5) \simeq \mathrm{TMF}(5)^{hB}.$$

We finally note that the moduli space $\mathcal{M}(5)$ is representable by an affine scheme (see for example [16]). It follows (see for example [8, Chapter 5]) that the descent spectral sequences for TMF and $\mathrm{TMF}_0(5)$

$$H^s(\mathcal{M}; \omega^{\otimes t})\big[\tfrac{1}{5}\big] \;\Rightarrow\; \pi_{2t-s}\,\mathrm{TMF}\big[\tfrac{1}{5}\big] \quad \text{and} \quad H^s(\mathcal{M}_0(5); \omega^{\otimes t}) \;\Rightarrow\; \pi_{2t-s}\,\mathrm{TMF}_0(5)$$

are isomorphic to the Čech descent spectral sequences associated to the étale affine covers

$$\mathcal{M}(5) \to \mathcal{M} \quad \text{and} \quad \mathcal{M}(5) \to \mathcal{M}_0(5),$$

respectively. However, as these étale affine covers are in fact Galois, with Galois groups $\mathrm{GL}_2(\mathbb{F}_5)$ and B, respectively, the Čech descent spectral sequences are precisely the homotopy fixed point spectral sequences:

$$H^s(\mathrm{GL}_2(\mathbb{F}_5); \pi_{2t}\,\mathrm{TMF}(5)) \;\Rightarrow\; \pi_{2t-s}\,\mathrm{TMF}\big[\tfrac{1}{5}\big],$$
$$H^s(B; \pi_{2t}\,\mathrm{TMF}(5)) \;\Rightarrow\; \pi_{2t-s}\,\mathrm{TMF}_0(5).$$

We do not need to know anything about $\pi_*\,\mathrm{TMF}(5)$ to understand these spectral sequences; the E_2–terms are isomorphic to $H^*(\mathcal{M}, \omega^{\otimes *})\big[\tfrac{1}{5}\big]$ and $H^*(\mathcal{M}_0(5), \omega^{\otimes *})$, respectively.

The descent spectral sequence for TMF is computed in many places. For example, Bauer, in [2], and Hopkins and Mahowald, in [14] (in Part II of [8]), compute the Adams–Novikov spectral sequence for tmf. It is explained in [18] that the descent spectral sequence for TMF may be obtained from the Adams–Novikov spectral sequence for tmf by inverting Δ. Alternatively, it is also explained in [18] that the descent spectral sequence for TMF can be obtained from the descent spectral sequence for Tmf by inverting Δ, and the descent spectral sequence for Tmf is described in [18] and in [8, Chapter 13].

The homotopy fixed point spectral sequence

$$H^s(B; \pi_{2t}\,\mathrm{TMF}(5)) \;\Rightarrow\; \pi_{2t-s}\,\mathrm{TMF}_0(5)$$

is also isomorphic to the homotopy fixed point spectral sequence

$$H^s(\mathbb{F}_5^\times; \pi_{2t}\,\mathrm{TMF}_1(5)) \;\Rightarrow\; \pi_{2t-s}\,\mathrm{TMF}_0(5).$$

Indeed, the latter is also a Čech descent spectral sequence, but for the affine étale Galois cover

$$\mathcal{M}_1(5) \to \mathcal{M}_0(5).$$

Lemma 2.2.2 *The transfer-restriction composition*

$$\pi_*\,\mathrm{TMF}\big[\tfrac{1}{5}\big] \xrightarrow{\mathrm{Res}} \pi_*\,\mathrm{TMF}_0(5) \xrightarrow{\mathrm{Tr}} \pi_*\,\mathrm{TMF}\big[\tfrac{1}{5}\big]$$

is multiplication by $[\mathrm{GL}_2(\mathbb{F}_5) : B] = 6$.

Proof The theorem is true on the level of homotopy fixed point spectral sequence E_2–terms: the composite

$$H^s(\mathrm{GL}_2(\mathbb{F}_5); \pi_t\,\mathrm{TMF}(5)) \xrightarrow{\mathrm{Res}} H^s(B; \pi_t\,\mathrm{TMF}(5)) \xrightarrow{\mathrm{Tr}} H^s(\mathrm{GL}_2(\mathbb{F}_5); \pi_t\,\mathrm{TMF}(5))$$

is multiplication by $[\mathrm{GL}_2(\mathbb{F}_5) : B] = 6$. Since there are no nontrivial elements of $E_\infty^{s,t}$ with $t - s = 0$ and $s > 0$ (see for example [2]), it follows that the transfer-restriction on the unit $1_{\mathrm{TMF}} \in \pi_0\,\mathrm{TMF}\big[\tfrac{1}{5}\big]$ is given by

$$\mathrm{Tr}\,\mathrm{Res}(1_{\mathrm{TMF}}) = 6 \cdot 1_{\mathrm{TMF}}.$$

We compute, using the projection formula, that for $a \in \pi_* \mathrm{TMF}\left[\frac{1}{5}\right]$, we have

$$\operatorname{Tr}\operatorname{Res}(a) = \operatorname{Tr}\operatorname{Res}(a \cdot 1_{\mathrm{TMF}}) = \operatorname{Tr}((\operatorname{Res} a) \cdot 1_{\mathrm{TMF}_0(5)}) = a \cdot \operatorname{Tr}(1_{\mathrm{TMF}_0(5)}) = 6 \cdot a. \quad \square$$

We deduce the following corollary.

Corollary 2.2.3 *Suppose that $z \in \pi_* \mathrm{TMF}$ satisfies $2z \neq 0$. Then the element $\operatorname{Res}(z)$ in $\pi_* \mathrm{TMF}_0(5)$ is nonzero. Moreover, if z has Adams–Novikov filtration s_1, and $2z$ has Adams filtration s_2, then the Adams–Novikov filtration s of $\operatorname{Res}(z)$ satisfies $s_1 \leq s \leq s_2$.*

Finally, in order to properly utilize the previous corollary, we record the behavior of the restriction.

Lemma 2.2.4 *Using the notation of [2] for*

$$H^s(\mathrm{GL}_2(\mathbb{F}_5); \pi_{2t} \mathrm{TMF}(5)) \cong H^s(\mathcal{M}, \omega^{\otimes t})\left[\tfrac{1}{5}\right],$$

the restriction

$$H^s(\mathrm{GL}_2(\mathbb{F}_5); \pi_t \mathrm{TMF}(5)) \xrightarrow{\operatorname{Res}} H^s(B; \pi_t \mathrm{TMF}(5))$$

has the following behavior on selected elements:

$$\begin{aligned} h_1 &\mapsto \eta, & c_4 &\mapsto b_2^2 - 12b_4 + 12\delta, \\ h_2 &\mapsto \nu, & c_6 &\mapsto -b_2^3 + 18b_2b_4 - 72b_2\delta, \\ g &\mapsto \delta\xi^2 \mod (2, b_2, \gamma\eta), & \Delta &\mapsto \delta^2(b_4 - 11\delta). \end{aligned}$$

Proof Consider the element a_1 of the elliptic curve Hopf algebroid. It is primitive modulo (2), and hence gives an element

$$a_1 \in H^0(\mathcal{M}_{\mathbb{F}_2}, \omega) \cong H^0(\mathrm{GL}_2(\mathbb{F}_5); \pi_2 \mathrm{TMF}(5)/2).$$

However, in $\pi_2 \mathrm{TMF}_1(5)$, we have $a_1 = x + y$, and this gives rise in the proof of Proposition 2.1.7 to an element

$$v_1 \in H^0(\mathbb{F}_5^\times; \pi_2 \mathrm{TMF}_1(5)/2) \cong H^0(B; \pi_2 \mathrm{TMF}(5)/2).$$

We therefore have that $a_1 \mapsto v_1$ under the restriction map

$$H^0(\mathrm{GL}_2(\mathbb{F}_5); \pi_2 \mathrm{TMF}(5)/2) \to H^0(B; \pi_2 \mathrm{TMF}(5)/2).$$

Consider the following diagram:

$$\begin{tikzcd}
H^0(\mathrm{GL}_2(\mathbb{F}_5); \pi_2 \,\mathrm{TMF}(5)/2) \arrow[r, "\partial"] \arrow[d, "\mathrm{Res}"'] & H^1(\mathrm{GL}_2(\mathbb{F}_5); \pi_2 \,\mathrm{TMF}(5)) \arrow[d, "\mathrm{Res}"] \\
H^0(B; \pi_2 \,\mathrm{TMF}(5)/2) \arrow[r, "\partial"] & H^1(B; \pi_2 \,\mathrm{TMF}(5))
\end{tikzcd}$$

In the proof of Proposition 2.1.7 we showed that $\partial(v_1) = \eta$, and the Bockstein spectral sequence computations of [2] give $\partial(a_1) = h_1$. We deduce $\mathrm{Res}(h_1) = \eta$.

The restriction of h_2 must be nontrivial by Corollary 2.2.3. The element ν is the only nonzero element in the group $H^1(B; \pi_4 \,\mathrm{TMF}(5))$, so we must have $\mathrm{Res}(h_2) = \nu$.

The restriction of g is computed by computing the restriction modulo $(2, a_1)$ (where $a_1 \in \pi_2 \,\mathrm{TMF}(5)$ is the image of $a_1 = x + y \in \pi_2 \,\mathrm{TMF}_1(5)$):

$$\overline{\mathrm{Res}}\colon H^*(\mathrm{GL}_2(\mathbb{F}_5); \pi_* \,\mathrm{TMF}(5)/(2, a_1)) \to H^*(B; \pi_* \,\mathrm{TMF}(5)/(2, a_1)).$$

Since the mod 2 supersingular locus of $\mathcal{M}_1(5)$ is given by

$$\mathcal{M}_1(5)^{\mathrm{ss}}_{\mathbb{F}_2} = \operatorname{Spec} \pi_0(\mathrm{TMF}_1(5/(2, a_1)),$$

the mod 2 supersingular locus of $\mathcal{M}(5)$ is given by

$$\mathcal{M}(5)^{\mathrm{ss}}_{\mathbb{F}_2} = \operatorname{Spec} \pi_0(\mathrm{TMF}(5/(2, a_1)).$$

As such, there are isomorphisms

$$\begin{aligned} H^s(\mathrm{GL}_2(\mathbb{F}_5); \pi_{2t} \,\mathrm{TMF}(5)/(2, a_1)) &\cong H^s(\mathcal{M}^{\mathrm{ss}}_{\mathbb{F}_2}, \omega^{\otimes t}) \\ &\cong H^s(G_{24}; \pi_{2t} E_2/(2, a_1))^{\mathrm{Gal}} \end{aligned}$$

and

$$\begin{aligned} H^s(B; \pi_{2t} \,\mathrm{TMF}(5)/(2, a_1)) &\cong H^s(\mathcal{M}_0(5)^{\mathrm{ss}}_{\mathbb{F}_2}, \omega^{\otimes t}) \\ &\cong H^s(C_4; \pi_{2t} E_2/(2, a_1))^{\mathrm{Gal}}, \end{aligned}$$

where G_{24} is the automorphism group of the unique supersingular curve over $\mathbb{F}_4$ and $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Under these isomorphisms the mod $(2, a_1)$ restriction map above is equivalent to the restriction map

$$\overline{\mathrm{Res}}\colon H^*(G_{24}; \pi_* E_2/(2, u_1))^{\mathrm{Gal}} \to H^*(C_4; \pi_* E_2/(2, u_1))^{\mathrm{Gal}}.$$

Note that $\pi_{24} E_2/(2, u_1)$ is $\mathbb{F}_4$, with trivial action by G_{24}. We therefore have

$$H^4(G_{24}; \pi_{24} E_2/(2, u_1))^{\mathrm{Gal}} \cong H^4(Q_8; \mathbb{F}_2) \cong \mathbb{F}_2\{g\},$$

where g is the image of the element

$$g \in H^4(\mathcal{M};\omega^{12}) \cong H^4(\mathrm{GL}_2(\mathbb{F}_5);\pi_{24}\,\mathrm{TMF}(5))$$

of [2] under the reduction map

$$\begin{aligned} H^4(\mathrm{GL}_2(\mathbb{F}_5);\pi_{24}\,\mathrm{TMF}(5)) \to {} & H^4(\mathrm{GL}_2(\mathbb{F}_5);\pi_{24}\,\mathrm{TMF}(5)/(2,a_1)) \\ & \cong H^4(G_{24};\pi_{24}E_2/(2,u_1))^{\mathrm{Gal}}. \end{aligned}$$

(This follows from the construction of g in [2] using Bockstein spectral sequences.) We also have

$$H^4(C_4;\pi_{24}E_2/(2,u_1))^{\mathrm{Gal}} \cong H^4(C_4;\mathbb{F}_2) \cong \mathbb{F}_2\{\beta^2\},$$

and the restriction gives an isomorphism

$$\overline{\mathrm{Res}}\colon H^4(Q_8;\mathbb{F}_2) \xrightarrow{\cong} H^4(C_4;\mathbb{F}_2).$$

Now, consider the map

$$\mathrm{red}\colon H^*(B;\pi_*\,\mathrm{TMF}(5))/(2,b_2) \to H^*(B;\pi_*\,\mathrm{TMF}(5)/(2,a_1)).$$

Since $a_1 \equiv x+y$ and $b_4 \equiv xy(x^2+y^2)$, it follows that $\mathrm{red}(b_4)=0$. We therefore have (using Proposition 2.1.7)

$$\mathrm{red}(\gamma\eta) = \mathrm{red}(b_4\gamma) = \mathrm{red}((b_4+b_2^2+2\delta)\beta) = 0.$$

Therefore the map red descends to a map

$$\overline{\mathrm{red}}\colon H^*(B;\pi_*\,\mathrm{TMF}(5))/(2,b_2,\gamma\eta) \to H^*(B;\pi_*\,\mathrm{TMF}(5)/(2,a_1)).$$

Now

$$H^4(B;\pi_{24}\,\mathrm{TMF}(5))/(2,b_2,\gamma\eta) = \mathbb{F}_2\{\xi^2\delta\}$$

and $\overline{\mathrm{red}}(\xi^2\delta)$ is the generator of $H^4(C_4;\mathbb{F}_2)$. We therefore have

$$\overline{\mathrm{red}}\,\mathrm{Res}(g) = \overline{\mathrm{Res}}(g) = \overline{\mathrm{red}}(\delta\xi^2),$$

and the result concerning the restriction of g follows.

The restrictions of c_4, c_6, and Δ may be computed from the map of Hopf algebroids induced by the map f, computed in Theorem 1.2.1 (see Section 3.4). □

Corollary 2.2.5 *The elements η and ν are permanent cycles in the homotopy fixed point spectral sequence for $\pi_*\,\mathrm{TMF}_0(5)$.*

2.3 Computation of the differentials and hidden extensions

The following sequence of propositions specifies the behavior of the homotopy fixed point spectral sequence (2.2.1) culminating in Theorem 2.3.12, a complete description of $\pi_* \mathrm{TMF}_0(5)$.

Proposition 2.3.1 *In the homotopy fixed point spectral sequence (2.2.1), $E_2 = E_3$ and the d_3–differentials are determined by*

$$d_3 b_2 = \eta^3, \quad d_3\xi = \delta^{-1}\eta\xi^2, \quad d_3\gamma = \delta^{-1}\eta\gamma\xi,$$

and $d_3(b_4) = d_3(\delta) = 0$.

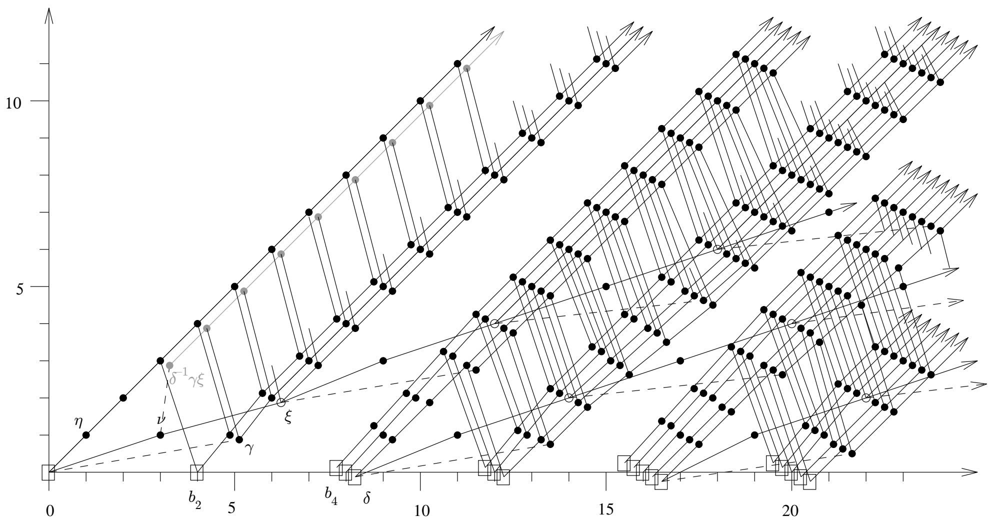

Figure 3: The d_3–differentials in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$

Figure 3 shows the d_3 differentials in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$. While most terms involving Δ^{-1} (and hence δ^{-1}) are excluded, those depicted are shown in gray.

Proof There is no room for d_2–differentials.

Note that $d_3 a_1^2 h_1 = h_1^4$ in the Adams–Novikov spectral sequence for TMF (we use the notation of [2]). Under the restriction map $\mathrm{TMF} \to \mathrm{TMF}_0(5)$, this differential maps to $d_3 b_2 \eta = \eta^4$, from which it follows that $d_3 b_2 = \eta^3$, and therefore $d_3(b_2^2) = 0$.

Note that since the possible targets of $d_3(b_4)$ and $d_3(\delta)$ are 2–torsion, we have $d_3(\delta^2) = d_3(b_4^2) = 0$. The element Δ is a d_3–cycle in the Adams–Novikov spectral sequence for TMF [2]. It follows that

$$0 = d_3(\delta^2(b_4 - 11\delta)) = \delta^2(d_3(b_4) + d_3(\delta))$$

and therefore

$$d_3(b_4) = d_3(\delta).$$

However, we have

$$0 = d_3(b_4^2) = d_3(b_2^2\delta - 4\delta^2) = b_2^2 d_3(\delta).$$

Since multiplication by b_2^2 is injective on the possible targets of $d_3(\delta)$, we conclude

$$d_3(b_4) = d_3(\delta) = 0.$$

By Corollary 2.2.3, 2ν must be detected in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$ in Adams–Novikov filtration between 1 and 3. Since $2\nu = 0$ in the E_2–page, it follows that in fact the filtration has to be between 2 and 3, and the only candidates live in filtration 3.

We claim that the filtration 3 class $\delta^{-1}\gamma\xi$ detects 2ν in $\mathrm{TMF}_0(5)$. To verify this claim, one can determine from Lemma 2.1.6 and Proposition 2.1.7 that $E_2^{3,6}$ is an $\mathbb{F}_2$–vector space. One subtlety to determining this $\mathbb{F}_2$–vector space is the fact that inverting Δ in $H^*(T_*)$ is equivalent to inverting δ and $b_4 - 11\delta$. However, Corollary 2.1.8, and the discussion that follows, makes it clear that we have

$$E_2^{3,6}/\eta^3 = \mathbb{F}_2\{\delta^{-1}\gamma\xi\}.$$

Finally, as the d_3 differentials determined up to this point completely determine the differentials supported by the 0–line, we can easily deduce that the image of d_3 in $E_2^{3,6}$ is precisely the image of η^3. We therefore deduce that $\delta^{-1}\gamma\xi$ is the only potential candidate to detect 2ν on the E_3–page of the spectral sequence.

Now observe that as a result of Corollary 2.1.8, and the discussion which follows, we have

$$d_3\gamma = a\delta^{-1}\eta\gamma\xi + \sum_{k,l\geq 0} a'_{k,l}\tilde{j}^k(\tilde{j}-11)^{-l}\eta^4 + \sum_{m\geq 0} a''_m\tilde{j}^m\delta^{-1}b_4\eta^4$$

for coefficients $a, a'_{k,l}, a''_m \in \mathbb{Z}/2$ with all but finitely many equal to zero. The class representing $2\eta\nu$, ie $\delta^{-1}\eta\gamma\xi$, must die in the spectral sequence. Since we have already established all of the terms involving η^4 are the targets of established d_3–differentials, this is only possible if $a = 1$.

We therefore have, using $b_2\xi = \delta\eta^2$,

$$\begin{aligned} d_3(b_2\gamma) &= d_3(b_2)\gamma + b_2 d_3(\gamma) \\ &= \sum_{k,l\geq 0} a'_{k,l}\tilde{j}^k(\tilde{j}-11)^{-l} b_2\eta^4 + \sum_{m\geq 0} a''_m \tilde{j}^m b_2\delta^{-1}b_4\eta^4. \end{aligned}$$

Turning this around, we have

$$\begin{aligned} \sum_{k,l\geq 0} a'_{k,l}\tilde{j}^k(\tilde{j}-11)^{-l} b_2\eta^4 + \sum_{m\geq 0} a''_m \tilde{j}^m b_2\delta^{-1}b_4\eta^4 &= d_3(b_2\gamma) \\ &= d_3(\eta(b_2^2+b_4)) \\ &= 0. \end{aligned}$$

We deduce that the coefficients $a'_{k,l}$ and a''_m are all zero.

Since $\delta^{-1}\gamma\xi$ is a permanent cycle, we have

$$0 = d_3\delta^{-1}\gamma\xi = (d_3\delta^{-1}\xi)\gamma - \delta^{-1}\xi(d_3\gamma).$$

Hence $d_3\xi = \delta^{-1}\eta\xi^2$. □

Corollary 2.3.2 *The E_4 term of the homotopy fixed point spectral sequence is given by*

$$E_4 = \mathbb{Z}\left[\tfrac{1}{5}\right][2b_2, b_2^2, b_4, \delta, \eta, \nu, \xi^2, \nu\xi, \gamma\xi, \delta^{-1}, (\tilde{j}-11)^{-1}]/\sim,$$

where $\sim$ consists of the relations

$$\begin{aligned}
b_4^2 &= b_2^2\delta - 4\delta^2, & \eta^3 &= 0, & \eta\gamma\xi &= 0, \\
2\eta &= 0, & \nu^3 &= 0, & b_2^2\nu &= 0, \\
2\nu &= 0, & (\gamma\xi)^2 &= 0, & b_4\nu &= 0, \\
2\gamma\xi &= 0, & \eta\nu &= 0, & \nu(\nu\xi) &= 2\xi^2, \\
4\xi^2 &= 0, & \eta\xi^2 &= 0, & \nu(\gamma\xi) &= 0, \\
b_2^2(\gamma\xi) &= 0, & 2b_2\xi^2 &= 0, & b_2^2\xi^2 &= 0, \\
b_4\xi^2 &= 2\delta\xi^2, & (\nu\xi)(\gamma\xi) &= 0, & b_4(\gamma\xi) &= \delta\eta^2(b_4+\delta).
\end{aligned}$$

Here we have omitted relations like $(2b_2)^2 = 4b_2^2$, $(2b_2)\nu = 0$ and $2(\nu\xi) = 0$, as they follow "from the notation". Everything is δ–periodic, and multiplication by $(\tilde{j}-1)^{-1} = (\delta^{-1}b_4 - 11)^{-1}$ satisfies the following relations (which follow from those *above):*

$$\begin{aligned}
11(\tilde{j}-11)^{-1}b_4 &= b_2^2(\tilde{j}-11)^{-1} - 4\delta(\tilde{j}-11)^{-1} - b_4,\\
(\tilde{j}-11)^{-1}\nu &= \nu,\\
(\tilde{j}-11)^{-1}\xi^2 &= -\xi^2,\\
(\tilde{j}-11)^{-1}\nu\xi &= \nu\xi,\\
(\tilde{j}-11)^{-1}\gamma\xi &= \gamma\xi.
\end{aligned}$$

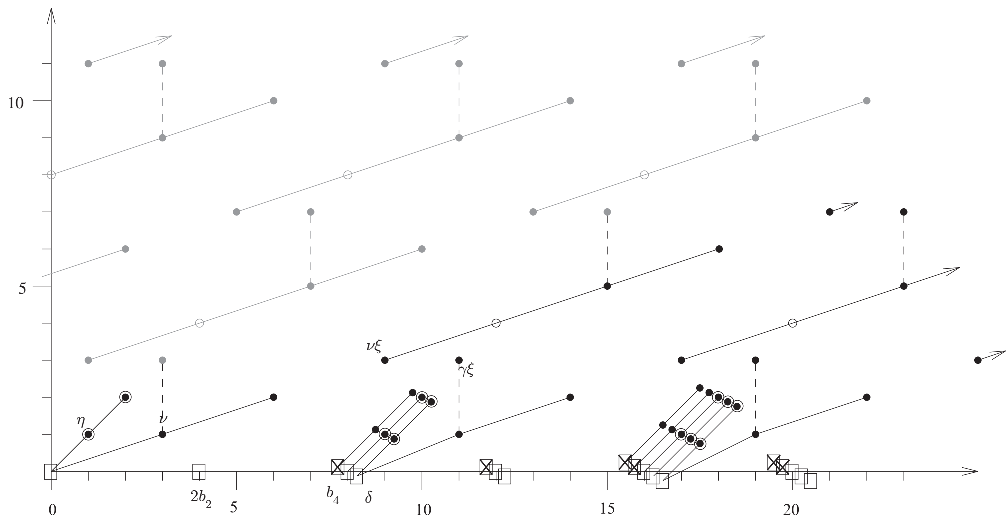

Figure 4: The $E_4 = E_5$ term in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)_{(2)}$

Figure 4 shows the resulting E_4–term in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)_{(2)}$. The authors find this easier to visualize (2)–locally (ie from Perspective 1 of Section 2.1). Terms involving δ^{-1} are excluded on the 0, 1 and 2–lines, and in lines greater than 2 are shown in gray. As in the other charts in this paper, solid dots denote $\mathbb{Z}/2$, and open circles denote $\mathbb{Z}/4$. If we localize at (2), the other symbols in the figure denote the following:

$$\square = \mathbb{Z}_{(2)}[(\tilde{j}-11)^{-1}], \quad \boxtimes = \mathbb{Z}_{(2)}, \quad \odot = \mathbb{Z}/2[(\tilde{j}-11)^{-1}].$$

In the following sequence of propositions, we will establish the rest of the differentials in the homotopy fixed point spectral sequence. Figure 5 displays these differentials. In this figure, the gray patterns represent the (infinite rank) *bo*–patterns.

We will need to observe the following to compute our d_5–differentials.

Lemma 2.3.3 *On the level of E_5–terms the restriction map (from the homotopy fixed point spectral sequence for* TMF *to the homotopy fixed point sequence for* $\mathrm{TMF}_0(5)$*) sends $\overline{\kappa}$ to $\delta\xi^2$.*

Proof In the homotopy fixed point spectral sequence for TMF, the element $\overline{\kappa}$ is detected by g. By Lemma 2.2.4, we have

$$\mathrm{Res}(g) = \delta\xi^2 \mod (2, b_2, \gamma\eta).$$

The lemma follows, as the elements of $H^4(\mathbb{F}_5^\times; \pi_{24}\,\mathrm{TMF}_1(5))$ which are divisible by 2, b_2, or $\gamma\eta$ are all killed by d_3–differentials. □

Corollary 2.3.4 *The element $\delta\xi^2$ is a permanent cycle in the homotopy fixed point spectral sequence for* $\mathrm{TMF}_0(5)$.

Proposition 2.3.5 *In the homotopy fixed point spectral sequence for $\pi_*\,\mathrm{TMF}_0(5)$ we have $E_4 = E_5$, and the d_5–differentials are determined by annihilating*

$$2b_2, \quad b_2^2, \quad b_4, \quad \eta, \quad \nu, \quad \gamma\xi$$

and by the equations

$$d_5(\delta) = \delta^{-1}\nu\xi^2, \quad d_5(\xi^2) = \delta^{-2}\nu\xi^4, \quad d_5(\nu\xi) = 2\delta^{-2}\xi^4.$$

Proof of Proposition 2.3.5, part 1 There is no room for d_4–differentials. We have already observed that η and ν are permanent cycles. Dimensional considerations also immediately show

$$d_5(2b_2) = d_5(\gamma\xi) = 0.$$

Note that the only possible target for a d_5–differential on b_2^2 or b_4 is $\nu\delta^{-1}\xi^2$. Since $\nu^2\delta^{-1}\xi^2$ is nontrivial in E_5, such nontrivial differentials would only be possible if νb_4 or νb_2^2 were nontrivial, but this is not the case. We deduce that

$$d_5(b_4) = d_5(b_2^2) = 0.$$

The element $\overline{\kappa} \in \pi_{20}S$ is in the Hurewicz image of TMF. In the Adams–Novikov spectral sequence for TMF, $d_5\Delta = \nu\overline{\kappa}$. We deduce that

$$\begin{aligned}
\nu\delta\xi^2 &= d_5(\delta^2(b_4 - 11\delta)) \\
&= 2\delta d_5(\delta)(b_4 - 11\delta) + \delta^2 d_5(b_4) - 11\delta^2 d_5(\delta) \\
&= 2\delta b_4 d_5(\delta) - 33\delta^2 d_5(\delta).
\end{aligned}$$

Since the only available class for $d_5(\delta)$ to hit is 2–torsion in the E_5–page, we deduce that

$$\delta^2 d_5(\delta) = \nu\delta\xi^2.$$

We have already observed that $\delta\xi^2$ is a permanent cycle since it detects $\overline{\kappa}$. We may therefore compute

$$0 = d_5(\delta\xi^2) = d_5(\delta)\xi^2 + \delta d_5(\xi^2) = \delta^{-1}\nu\xi^4 + \delta d_5(\xi^2).$$

We deduce that

$$d_5(\xi^2) = \delta^{-2}\nu\xi^4.$$

The only class left to handle is $\nu\xi$. We will defer the proof of this differential until after we establish the d_7–differentials. □

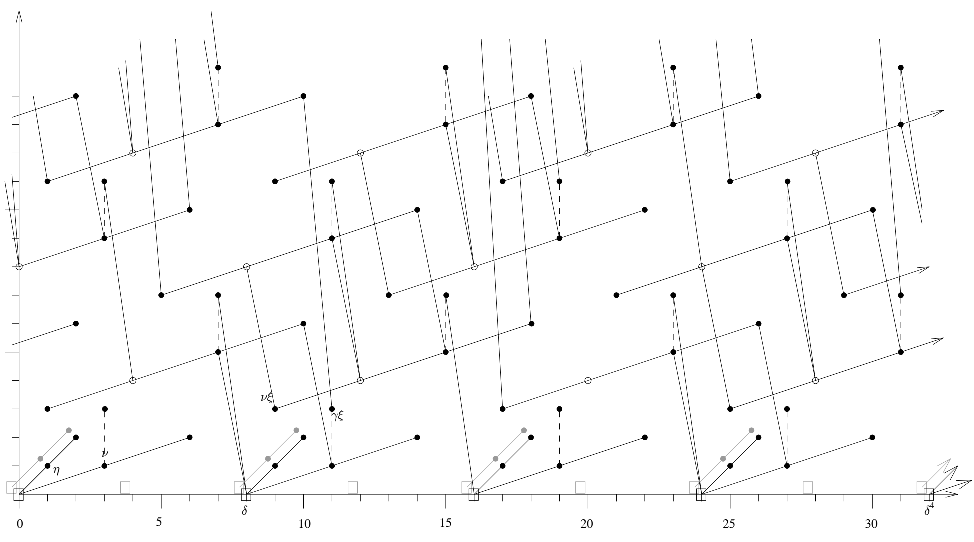

Figure 5: The E_4 term in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$ with d_r–differentials, $r \geq 4$

Proposition 2.3.6 *In the homotopy fixed point spectral sequence for $\pi_* \mathrm{TMF}_0(5)$ we have $E_6 = E_7$, and the d_7–differentials are determined by annihilating*

$$2b_2, \quad b_2^2, \quad \eta, \quad \nu, \quad \delta\xi^2, \quad \delta\nu\xi, \quad \gamma\xi, \quad \delta\gamma\xi, \quad \delta b_4$$

and by the equations

$$d_7(2\delta) = d_7(b_4) = \delta^{-2}\gamma\xi^3, \quad d_7(\delta^2) = \delta^{-1}\gamma\xi^3.$$

Proof There is no room for d_6–differentials. We have already observed that η, ν, and $\delta\xi^2$ are permanent cycles, since they are in the Hurewicz image. The elements $2b_2$, b_2^2, $\delta\nu\xi$, $\gamma\xi$, and $\delta\gamma\xi$ are d_7–cycles for dimensional reasons.

In order to establish the next round of differentials, we will first determine $d_7(2\delta^3)$ and $d_7(\delta^2 b_4)$ (of course, these differentials are determined by $d_7(2\delta)$, $d_7(b_4)$, and $d_7(\delta^2)$). Note that $2\nu\overline{\kappa}$ is 0 in $\pi_*\,\mathrm{TMF}$, from which we deduce that the class represented by $\delta^{-1}\gamma\xi\overline{\kappa}$ is 0 in $\pi_*\,\mathrm{TMF}_0(5)$ via the restriction map. The element $\gamma\xi^3$ detects this class, so it must be the target of a differential, and the only (not necessarily exclusive) possibilities at this point are:

Case 1 $d_7(2\delta^3) = \gamma\xi^3$.

Case 2 $d_7(\delta^{2-i} b_4 b_2^{2i}) = \gamma\xi^3$ for some $i \geq 0$.

Case 3 $d_7(\delta^{2-i} b_2^{2i+2}) = \gamma\xi^3$ for some $i \geq 0$.

Multiplying by the permanent cycle $\mathrm{Res}(\overline{\kappa}) = \delta\xi^2$, Case 2 yields

$$d_7(\delta^{5-i}\eta^4 b_2^{2i} + 2\delta^{4-i}\xi^2 b_2^{2i} + \delta^{4-i}\xi\gamma\eta) \neq 0.$$

If $i > 0$, this is a contradiction because

$$\delta^{5-i}\eta^4 b_2^{2i} = 2\delta^{4-i}\xi^2 b_2^{2i} = \delta^{4-i}\xi\gamma\eta = 0$$

in the E_7–page for $i > 0$. Therefore Case 2 for $i > 0$ cannot occur. Similarly, multiplying Case 3 by $\overline{\kappa}$ gives

$$d_7(\delta^{5-i} b_2^{2i}\eta^4) \neq 0,$$

again a contradiction. We conclude that either Case 1 or Case 2 with $i = 0$ must hold. Therefore

$$d_7(2\delta^3) = a\gamma\xi^3, \quad d_7(\delta^2 b_4) = b\gamma\xi^3,$$

with $a = 1$ or $b = 1$. Multiplying both of the above differentials by $\overline{\kappa}$ yields

$$d_7(2\delta^4\xi^2) = a\delta\gamma\xi^5, \quad d_7(2\delta^4\xi^2) = b\delta\gamma\xi^5.$$

We deduce that $a = b = 1$. Hence we deduce that

$$d_7(2\delta^3) = \gamma\xi^3, \quad d_7(\delta^2 b_4) = \gamma\xi^3.$$

We now turn our attention to $d_7(2\delta)$, $d_7(b_4)$ and $d_7(\delta^2)$. The only possible targets for these differentials are $\delta^{-2}\gamma\xi^3$ (for $d_7(2\delta)$ and $d_7(b_4)$) and $\delta^{-1}\gamma\xi^3$ (for $d_7(\delta^2)$). Write

$$d_7(2\delta) = c\delta^{-2}\gamma\xi^3, \quad d_7(b_4) = d\delta^{-2}\gamma\xi^3, \quad d_7(\delta^2) = e\delta^{-1}\gamma\xi^3.$$

Then we have

$$\gamma\xi^3 = d_7(\delta^2 b_4) = d_7(\delta^2)b_4 + \delta^2 d_7(b_4) = e\delta^{-2}\gamma\xi^3 b_4 + d\gamma\xi^3.$$

Using the relations we find that $\delta^{-2}\gamma\xi^3 b_4 = 0$, and we therefore deduce that $d = 1$. Similarly, we have

$$\gamma\xi^3 = d_7(2\delta^3) = d_7(\delta^2)2\delta + \delta^2 d_7(2\delta) = 2e\delta^{-1}\gamma\xi^3 + c\gamma\xi^3.$$

Since $2\delta^{-1}\gamma\xi^3 = 0$, we deduce that $c = 1$. We have shown

$$d_7(2\delta) = \delta^{-2}\gamma\xi^3, \quad d_7(b_4) = \delta^{-2}\gamma\xi^3.$$

To establish the final d_7 differential on δ^2, note that the restriction map $\mathrm{TMF} \to \mathrm{TMF}_0(5)$ takes $2\nu\Delta$ to $2\nu\Delta$ which is nonzero in $\pi_* \mathrm{TMF}_0(5)$. Since $2\nu\Delta\bar{\kappa} = 0 \in \pi_* \mathrm{TMF}$, we know $2\nu\Delta\bar{\kappa} = 0 \in \pi_* \mathrm{TMF}_0(5)$. The element $\gamma\xi^3\delta^3$ detects this class. It follows that $\delta^{-1}\gamma\xi^3$ must be the target of a differential. By the same argument used earlier, multiplication by $\bar{\kappa}$ shows that the only possible sources of a differential killing $\delta^{-1}\gamma\xi^3$ are δ^2 and δb_4. Write

$$d_7(\delta^2) = e\delta^{-1}\gamma\xi^3, \quad d_7(\delta b_4) = f\delta^{-1}\gamma\xi^3,$$

so that e or f equals $1 \bmod 2$. Multiplying both of these differentials by $\bar{\kappa}$ yields

$$d_7(\delta^3\xi^2) = e\gamma\xi^5, \quad d_7(2\delta^3\xi^2) = f\gamma\xi^5.$$

Thus we have $e \equiv 1 \bmod 2$, and $f \equiv 0 \bmod 2$, and

$$d_7(\delta^2) = \delta^{-1}\gamma\xi^3, \quad d_7(\delta b_4) = 0. \qquad \square$$

Proof of Proposition 2.3.5, part 2 We now return to the proof of Proposition 2.3.5 to establish the one remaining differential, $d_5(\nu\xi)$. We note that

$$d_5(\delta\nu\xi) = 0$$

since the only possible nontrivial target of such a differential would be $2\xi^2$, and this supports a nontrivial d_7–differential by Proposition 2.3.6. We therefore have

$$0 = d_5(\delta\nu\xi) = \delta^{-1}\nu^2\xi^3 + \delta d_5(\nu\xi) = \delta^{-1}2\xi^4 + \delta d_5(\nu\xi).$$

We conclude that we have

$$d_5(\nu\xi) = 2\delta^{-2}\xi^4. \qquad \square$$

To handle the next round of differentials we will need the following lemma.

Lemma 2.3.7 *The Hurewicz image of the element κ in $\pi_{14}\,\mathrm{TMF}$ restricts to a nontrivial class in $\pi_{14}\,\mathrm{TMF}_0(5)$, detected by $\nu^2\delta$ in the homotopy fixed point spectral sequence.*

Proof Applying Corollary 2.2.3 to the class $\Delta^4\kappa \in \pi_{110}\,\mathrm{TMF}$ of order 4, we find that $\mathrm{Res}(\Delta^4\kappa)$ is nontrivial, and detected in the homotopy fixed point spectral sequence by a class in filtration between 4 and 14. Given our d_5–differentials, the only candidate is $\nu^2\delta^{13}$. Since E_2 is δ–periodic, and since κ is detected in filtration 2 in TMF, it follows that on the level of E_2 pages κ restricts to $\nu^2\delta$. The lemma follows, since $\nu^2\delta$ is not the target of a differential. □

Proposition 2.3.8 *In the homotopy fixed point spectral sequence for $\pi_*\,\mathrm{TMF}_0(5)$, $E_8 = E_9 = E_{10}$ and the d_{11}–differentials are determined by*

$$d_{11}(\gamma\xi) = \delta^{-4}\xi^7.$$

Proof In $\pi_*\,\mathrm{TMF}$ we have $\bar{\kappa}^3\kappa = 0$. The restriction of this element in $\mathrm{TMF}_0(5)$ is detected in the homotopy fixed point spectral sequence by $\delta^4\xi^7$, so the latter must be the target of a differential. The only possibility is $d_{11}(\delta^8\gamma\xi) = \delta^4\xi^7$. Since δ^4 persists to the E_{11}–page, and there are no nontrivial targets for $d_{11}(\delta^4)$, it follows that E_{11} is δ^4–periodic, and the proposition follows. □

Proposition 2.3.9 *In the homotopy fixed point spectral sequence for $\pi_*\,\mathrm{TMF}_0(5)$, $E_{12} = E_{13}$ and the d_{13}–differentials are determined by*

$$d_{13}(\delta\nu\xi) = \delta^{-4}\xi^8, \quad d_{13}(\delta^3\nu^2) = \delta^{-2}\nu\xi^7.$$

Proof In $\pi_*\,\mathrm{TMF}$ we have $\bar{\kappa}^6 = 0$. Since $\mathrm{Res}(\bar{\kappa}^6)$ is detected by $\delta^6\xi^{12}$ in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$, the latter must be the target of a differential. Since $\bar{\kappa}\delta^6\xi^{12}$ is nontrivial in E_{13}, if $d_r(x) = \delta^6\xi^{12}$ it must be the case that $\bar{\kappa}\cdot x \neq 0$. The only such candidate is

$$d_{13}(\delta^{11}\nu\xi^5) = \delta^6\xi^{12}.$$

Dividing by $\bar{\kappa}^2$, it follows that we have

$$d_{13}(\delta^9\nu\xi) = \delta^4\xi^8.$$

Since δ^4 persists to E_{13} with no possible targets for a nontrivial $d_{13}(\delta^4)$, it follows that

$$d_{13}(\delta\nu\xi) = \delta^{-4}\xi^8.$$

The differential $d_{13}\delta^3\nu^2 = \delta^{-2}\nu\xi^7$ actually follows from the differential above, though perhaps not so obviously, so we will explain in more detail. The element $\xi^3\nu$ persists to the E_{13}–page, and there are no possibilities for it supporting a nontrivial d_{13}–differential. However, by the previous paragraph,

$$\overline{\kappa}^4\xi^3\nu = \delta^4\xi^{11}\nu = d_{13}(\delta^9\xi^4\nu^2) \neq 0 \in E_{13}.$$

Dividing by $\overline{\kappa}^2$, we get

$$d_{13}(\delta^7\nu^2) = \delta^2\xi^7\nu$$

and thus

$$d_{13}(\delta^3\nu^2) = \delta^{-2}\xi^7\nu. \qquad \square$$

This concludes the determination of the differentials in the homotopy fixed point spectral sequence; there are no further possibilities. We now turn to determining the hidden extensions in this spectral sequence. To do this, we will recompute $\pi_* \mathrm{TMF}_0(5)$ using a homotopy orbit spectral sequence. This different presentation will turn out to elucidate the multiplicative structure missed by the homotopy fixed point spectral sequence.

The Tate spectral sequence

$$\widehat{H}^s(\mathbb{F}_5^\times; \pi_t \mathrm{TMF}_1(5)) \Rightarrow \pi_{t-s} \mathrm{TMF}_1(5)^{t\mathbb{F}_5^\times}$$

can be easily computed from the homotopy fixed point spectral sequence — one simply has to invert ξ. A picture of the resulting spectral sequence (just from E_4 and beyond) is displayed in Figure 6.

Note that everything dies in this spectral sequence. Therefore, we have established the following lemma. (There may be other more conceptual ways of proving the following lemma — for instance, it is well known to hold $K(2)$–locally, and the unlocalized statement might follow from the fact that $\mathcal{M}_1(5) \to \mathcal{M}_0(5)$ is a Galois cover.)

Lemma 2.3.10 *The Tate spectrum* $\mathrm{TMF}_1(5)^{t\mathbb{F}_5^\times}$ *is trivial, and therefore the norm map*

$$N\colon \mathrm{TMF}_1(5)_{h\mathbb{F}_5^\times} \to \mathrm{TMF}_1(5)^{h\mathbb{F}_5^\times}$$

is an equivalence.

Thus the homotopy groups of $\pi_* \mathrm{TMF}_1(5)_{h\mathbb{F}_5^\times} = \pi_* \mathrm{TMF}_0(5)$ are isomorphic to $\pi_* \mathrm{TMF}_1(5)^{h\mathbb{F}_5^\times}$ as modules over $\pi_* \mathrm{TMF}$. However, these $\pi_* \mathrm{TMF}$–modules are computed in an entirely different way by the homotopy orbit spectral sequence

$$H_s(\mathbb{F}_5^\times; \pi_t \mathrm{TMF}_1(5)) \Rightarrow \pi_{s+t} \mathrm{TMF}_0(5).$$

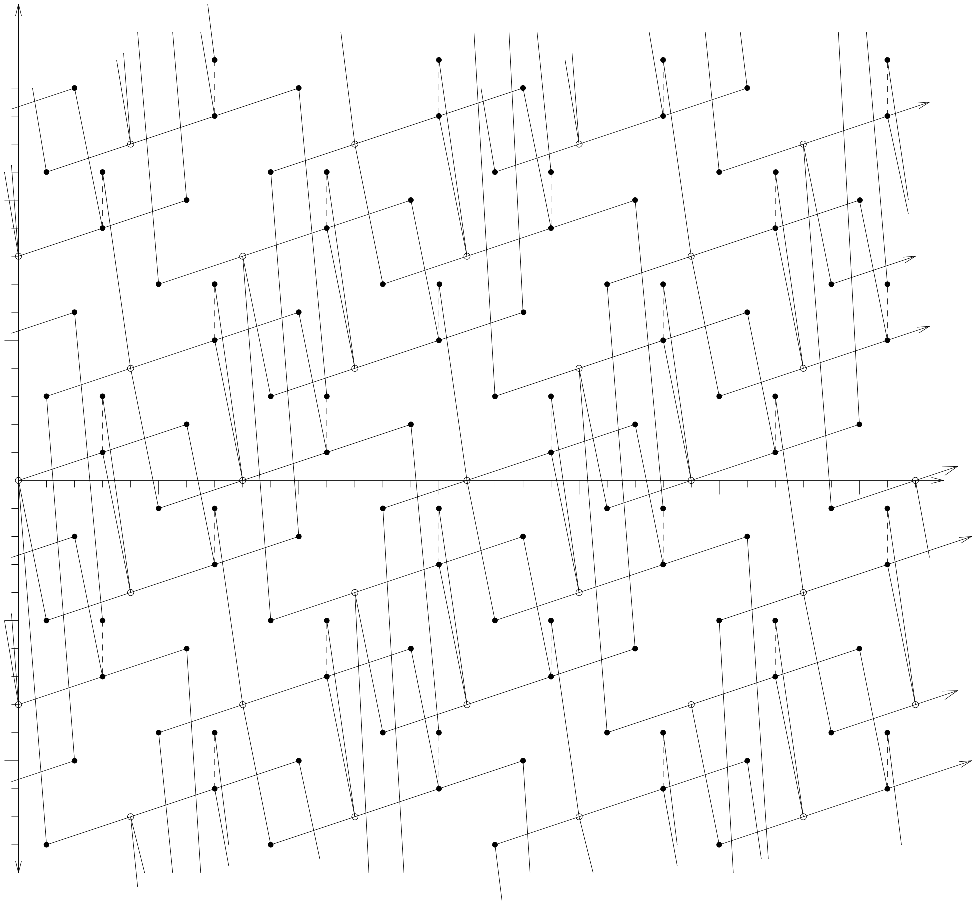

Figure 6: The E_4 term in the Tate spectral sequence for $\mathrm{TMF}_1(5)^{t\mathbb{F}_5^\times}$ with d_r–differentials, $r \geq 4$

Nevertheless, the homotopy orbit spectral sequence (with differentials) can be computed by simply truncating the Tate spectral sequence (and manually computing H_0 where appropriate). The resulting homotopy orbit spectral sequence is displayed in Figure 7. As with our other spectral sequences, we are displaying the E_4–page, with all remaining differentials. The (infinite rank) *bo* patterns are displayed in gray.

There are many hidden extensions (as $\pi_*\,\mathrm{TMF}$ modules) in the homotopy orbit spectral sequence (HOSS) which are not hidden in the homotopy fixed point spectral sequence (HFPSS). Since $\pi_0\,\mathrm{TMF}_0(5)$ is seen to be torsion free in the HFPSS, there must be additive extensions as indicated in Figure 7, and $1 \in \pi_0\,\mathrm{TMF}_0(5)$ must be detected on the $s = 12$ line. Since the HFPSS shows η, η^2 and ν are nontrivial in $\pi_*\,\mathrm{TMF}_0(5)$,

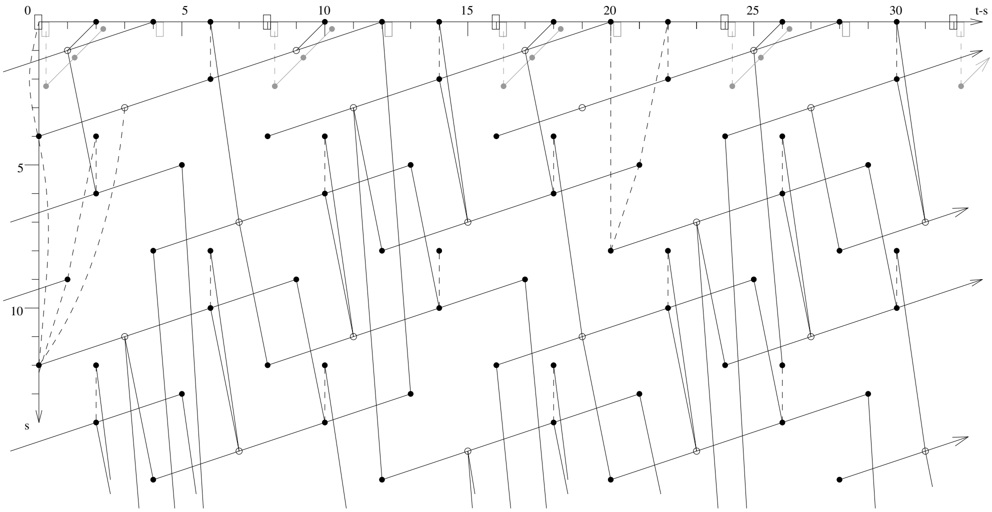

Figure 7: The E_4 term in the homotopy orbit spectral sequence for $\mathrm{TMF}_1(5)_{h\mathbb{F}_5^\times}$ with d_r–differentials, $r \geq 4$

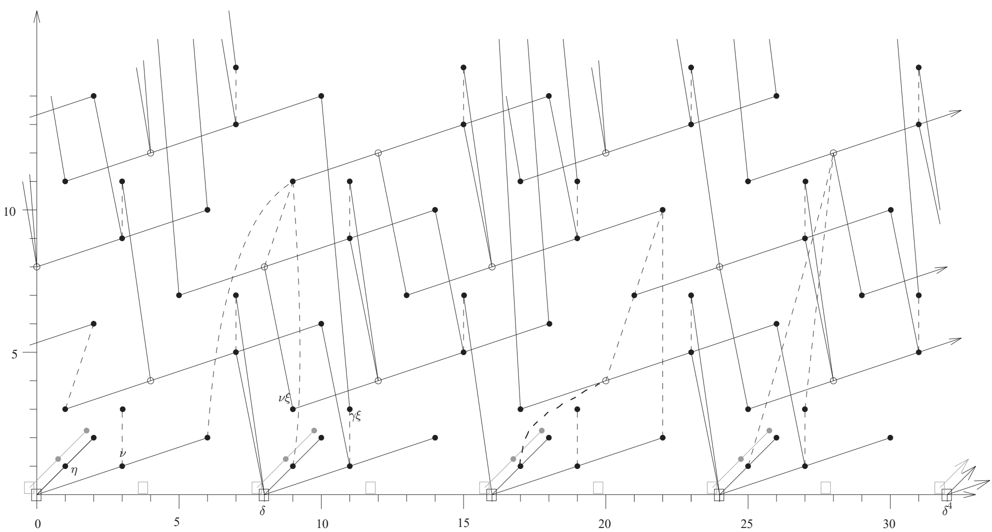

Figure 8: The hidden extensions in the homotopy fixed point spectral sequence for $\mathrm{TMF}_0(5)$

there must be corresponding hidden extensions in the HOSS. Multiplying these by $\overline{\kappa}$ in the HOSS yields hidden η and η^2 extensions supported by $\overline{\kappa}$.

We will now deduce the hidden extensions in the HFPSS from multiplicative structure in the HOSS. The resulting extensions are displayed in Figure 8.

Since $\eta\overline{\kappa}$ and $\eta^2\overline{\kappa}$ are seen to be nontrivial in $\pi_* \mathrm{TMF}_0(5)$ using hidden extensions in the HOSS, we obtain corresponding new hidden extensions in the HFPSS. With the one exception $\eta\cdot\delta^2\gamma\xi$, all of the other hidden extensions displayed in Figure 8 follow from nonhidden extensions in the HOSS. The remaining extension is addressed in the following lemma.

Lemma 2.3.11 *In the homotopy fixed point spectral sequence for* $\mathrm{TMF}_0(5)$*, there is a hidden extension*

$$\eta\cdot\delta^2\gamma\xi = \delta^{-1}\xi^6.$$

Proof Observe that since ν^3 is nontrivial in $\pi_* \mathrm{TMF}_0(5)$, and in $\pi_* \mathrm{TMF}$ we have $\nu^3 = \eta\epsilon$, it must follow that ϵ is detected by $\delta^{-2}\xi^4$ in the HFPSS. Thus $\overline{\kappa}\epsilon$ is detected by $\delta^{-1}\xi^6$. However, $\overline{\kappa}\epsilon$ is η–divisible in $\pi_* \mathrm{TMF}$. It follows that it must also be η–divisible in $\pi_* \mathrm{TMF}_0(5)$, and the hidden extension claimed is the only possibility to make this happen. □

Theorem 2.3.12 *The homotopy groups* $\pi_* \mathrm{TMF}_0(5)$ *are given by the* δ^4*–periodic pattern in Figure 9; the gray classes in the figure represent infinite direct sums of bo–patterns, generated ((2)–locally) by classes*

$$\begin{aligned}
&\delta^j b_2^{2k}\eta^a(\tilde{j}-11)^{-l}, && j\in\mathbb{Z},\ k>0,\ 0\le a\le 2,\ l\ge 0,\\
&\delta^j b_2^{2k} b_4\eta^a, && j\in\mathbb{Z},\ k\ge 0,\ 0\le a\le 2,\\
&2\delta^j b_2^{2k+1}(\tilde{j}-11)^{-l}, && j\in\mathbb{Z},\ k\ge 0,\ l\ge 0,\\
&2\delta^j b_2^{2k+1} b_4, && j\in\mathbb{Z},\ k\ge 0,\ l\ge 0.
\end{aligned}$$

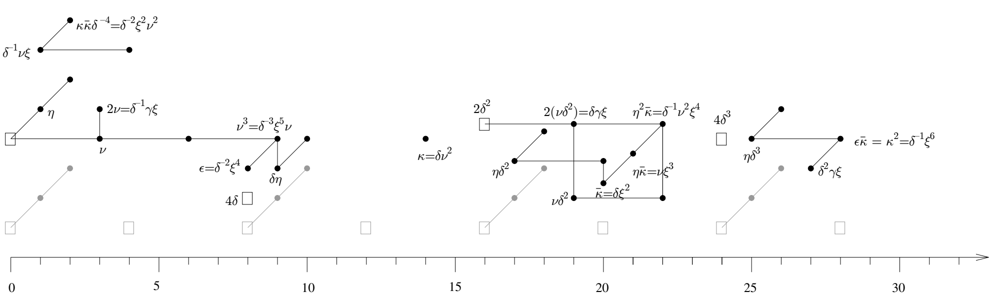

Figure 9

Remark 2.3.13 One easily sees from the calculation of the d_5 and d_7–differentials that $\tilde{j}$ and hence $(\tilde{j}-11)^{-1}$ are permanent cycles in the homotopy fixed point spectral sequence. It is reasonable to record how these act on the $\pi_* \operatorname{TMF}_0(5)$. Amongst the classes of the form

$$2^i \delta^j b_2^k b_4^\epsilon \eta^a (\tilde{j} - 11)^{-l}$$

(where we take $\epsilon \in \{0, 1\}$, and $l = 0$ if $\epsilon = 1$), multiplication by $\tilde{j}$ is easy to compute using $\tilde{j} = b_4 \delta^{-1}$ and the relation

$$b_4^2 = b_2^2 \delta - 4\delta^2.$$

Multiplication by $(\tilde{j} - 11)^{-1}$ is governed by the relation

$$11(\tilde{j} - 11)^{-1} b_4 = b_2^2 (\tilde{j} - 11)^{-1} - 4\delta(\tilde{j} - 11)^{-1} - b_4.$$

Amongst all other classes x in the chart not of the form above, we have

$$\tilde{j} x = 0 \quad \text{and} \quad (\tilde{j} - 11)^{-1} x = x.$$

Remark 2.3.14 The relation $\epsilon\bar{\kappa} = \kappa^2$ in the chart of Theorem 2.3.12 corresponds to the same relation in the stable homotopy groups of spheres. This relation represents a hidden ϵ–extension in the classical Adams spectral sequence for the sphere (in the ASS, $c_0 g = 0$ and d_0^2 detects the generator of π_{28}^s). In the homotopy fixed point spectral sequence above, the relation

$$(\delta^{-2}\xi^4)(\delta\xi^2) = \delta^{-1}\xi^6$$

implies that $\delta^{-1}\xi^6$ detects $\epsilon\bar{\kappa}$. Actually, this gives an amusing alternative proof of the relation $\epsilon\bar{\kappa} = \kappa^2$ in π_*^s: the fact that d_0^2 is a permanent cycle in the ASS implies that κ^2 is nontrivial, and we have just seen that $\epsilon\bar{\kappa}$ must be nontrivial, because it is detected in the Hurewicz image of $\operatorname{TMF}_0(5)$. Since $\pi_{28}^s = \mathbb{Z}/2$, the two classes must be equal. One could make a similar argument using TMF instead of $\operatorname{TMF}_0(5)$, as one sees $\epsilon\bar{\kappa}$ in a similar way as a nonhidden extension in the ANSS for TMF.

3 $Q(\ell)$–spectra

We now begin working with the $Q(\ell)$ spectra in earnest. We review the definition of $Q(\ell)$ in Section 3.1 and in Section 3.2 recall the double complex that computes the E_2–term of its Adams–Novikov spectral sequence.

In previous sections we have focused on data for $Q(5)$ but in Section 3.3 we review formulas of Mahowald and Rezk from [20] related to $Q(3)$. Finally in Section 3.4 we recall the formulas of Section 1 in forms that will be useful in subsequent calculations.

3.1 Definitions

In [3], the p–local spectrum $Q(\ell)$ ($p \nmid \ell$) is defined as the totalization of an explicit semi-cosimplicial E_∞–ring spectrum of the form

$$Q(\ell)^\bullet = \big(\mathrm{TMF} \rightrightarrows \mathrm{TMF}_0(\ell) \times \mathrm{TMF} \Rrightarrow \mathrm{TMF}_0(\ell)\big).$$

Here a semi-cosimplicial object is the same thing as a cosimplicial object, but without codegeneracy maps. The above expression is shorthand for a semi-cosimplicial spectrum $Q(\ell)^\bullet$ in which $Q(\ell)^k = *$ for $k > 2$. The coface maps from level 0 to level 1 are given by

$$d_0 = q^* \times \psi^\ell, \quad d_1 = f^* \times 1,$$

and the coface maps from level 1 to level 2 are given by

$$d_0 = t^* \circ \pi_2, \quad d_1 = f^* \circ \pi_1, \quad d_2 = \pi_2,$$

where π_i are the projections onto the components. These maps are induced by the maps of stacks

$$\begin{aligned}
\psi^\ell &\colon \mathcal{M}^1 \to \mathcal{M}^1, & (C, \vec{v}) &\mapsto (C, \ell \cdot \vec{v}),\\
f &\colon \mathcal{M}^1_0(\ell) \to \mathcal{M}^1, & (C, H, \vec{v}) &\mapsto (C, \vec{v}),\\
q &\colon \mathcal{M}^1_0(\ell) \to \mathcal{M}^1, & (C, H, \vec{v}) &\mapsto (C/H, (\phi_H)_*\vec{v}),\\
t &\colon \mathcal{M}^1_0(\ell) \to \mathcal{M}^1_0(\ell), & (C, H, \vec{v}) &\mapsto (C/H, \widehat{H}, (\phi_H)_*\vec{v}),
\end{aligned}$$

where $\phi_H\colon (C, H) \to C/H$ is the quotient isogeny. (Note that our t is ψ_d in [3, page 349], and we have corrected a small typo in its presentation here.) The map $\psi^\ell\colon \mathrm{MF}_k \to \mathrm{MF}_k$ is analogous to an Adams operation, and acts by multiplication by ℓ^k. Formulas for f^*, q^* and t^*, on the level of modular forms are typically computed differently for different choices of ℓ, and are more complicated.

3.2 The double complex

As done in the special case of $\ell = 2$ and $p = 3$ in [3], one can form a total cochain complex to compute the E_2–term for the Adams–Novikov spectral sequence for $Q(\ell)$. Let (A, Γ) denote the usual elliptic curve Hopf algebroid, and let $(B^1(\ell), \Lambda^1(\ell))$ denote a Hopf algebroid which stackifies to give $\mathcal{M}^1_0(\ell)$. Let $C^*_\Gamma(A)$, $C^*_{\Lambda^1(\ell)}(B^1)$ denote the corresponding cobar complexes, so the corresponding Adams–Novikov spectral sequences take the form

$$E_2^{s,2t} = H^s(\mathcal{M}, \omega^{\otimes t}) = H^s(C^*_\Gamma(A)_{2t}) \Rightarrow \pi_{2t-s}\,\mathrm{TMF},$$
$$E_2^{s,2t} = H^s(\mathcal{M}_0(\ell), \omega^{\otimes t}) = H^s(C^*_{\Lambda^1(\ell)}(B^1(\ell))_{2t}) \Rightarrow \pi_{2t-s}\,\mathrm{TMF}_0(\ell).$$

Corresponding to the cosimplicial decomposition of $Q(\ell)$ we can form a double complex $C^{*,*}(Q(\ell))$:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^1_\Gamma(A) \arrow[r] \arrow[u] & C^1_{\Lambda^1(\ell)}(B^1(\ell)) \oplus C^1_\Gamma(A) \arrow[r] \arrow[u] & C^1_{\Lambda^1(\ell)}(B^1(\ell)) \arrow[r] \arrow[u] & \cdots \\
C^0_\Gamma(A) \arrow[r] \arrow[u] & C^0_{\Lambda^1(\ell)}(B^1(\ell)) \oplus C^0_\Gamma(A) \arrow[r] \arrow[u] & C^0_{\Lambda^1(\ell)}(B^1(\ell)) \arrow[r] \arrow[u] & \cdots
\end{tikzcd} \tag{3.2.1}$$

Let $C^*_{\mathrm{tot}}(Q(\ell))$ denote the corresponding total complex. Then the Adams–Novikov spectral sequence for $Q(\ell)$ takes the form

$$E_2^{s,2t} = H^s(C^*_{\mathrm{tot}}(Q(\ell))_{2t}) \Rightarrow \pi_{2t-s}Q(\ell).$$

3.3 Recollections about $Q(3)$

Mahowald and Rezk [20] performed a study of the explicit formulas for $Q(3)$ similar to our current treatment of $Q(5)$. We summarize some of their results here for the reader's convenience.

The moduli space $\mathcal{M}_1^1(3)$ is represented by the affine scheme Spec $B^1(3)$ with

$$B^1(3) = \mathbb{Z}\left[\tfrac{1}{3}, a_1, a_3, \Delta^{-1}\right]$$

with

$$\Delta = a_3^3(a_1^3 - 27a_3).$$

The corresponding universal $\Gamma_1(3)$ structure is carried by the Weierstrass curve

$$y^2 + a_1xy + a_3y = x^3$$

with point $P = (0,0)$ of order 3. The $\mathbb{G}_m$–action on $\mathcal{M}_1^1(3)$ induces a grading on $B^1(3)$, for which a_i has weight i. It follows that

$$\pi_* \mathrm{TMF}_1(3) = \mathbb{Z}\left[\tfrac{1}{3}, a_1, a_3, \Delta^{-1}\right]$$

with topological degrees $|a_i| = 2i$. The spectrum $\mathrm{TMF}_1(3)$ admits a complex orientation with $v_1 = a_1$ and $v_2 = a_3$.

The group $\mathbb{F}_3^\times = \{\pm 1\}$ acts on $\mathcal{M}_1^1(5)$ by sending an R–point (C, P) (where P is a point of exact order 3 on C) to the R–point $(C, [-1](P))$. This induced action of $\mathbb{F}_3^\times$

on the ring $B^1(3)$ is given by

$$[-1](a_1) = -a_1, \quad [-1](a_3) = -a_3.$$

We have

$$\mathcal{M}_0^1(3) = \mathcal{M}_1^1(3) /\!/ \mathbb{F}_3^\times$$

and hence an equivalence

$$\mathrm{TMF}_0(3) \simeq \mathrm{TMF}_1(3)^{h\mathbb{F}_3^\times}.$$

The resulting homotopy fixed point spectral sequence takes the form

$$H^s(\mathbb{F}_3^\times; \pi_t \, \mathrm{TMF}_1(3)) \;\Rightarrow\; \pi_{t-s} \, \mathrm{TMF}_0(3).$$

In particular, the ring of modular forms (meromorphic at the cusps) for $\Gamma_0(3)$ is the subring

$$\mathrm{MF}(\Gamma_0(3)) = H^0(\mathbb{F}_3^\times; \mathrm{MF}(\Gamma_1(3))) = \mathbb{Z}\big[\tfrac{1}{3}, a_1^2, a_1 a_3, a_3^2, \Delta^{-1}\big] \subset B^1(3).$$

Mahowald and Rezk also compute the effects of the maps

$$f^*, q^*\colon A \to B^1(3) \quad \text{and} \quad t^*\colon B^1(3) \to B^1(3)$$

as

$$\begin{aligned}
f^*(a_1) &= a_1, & q^*(a_1) &= a_1,\\
f^*(a_2) &= 0, & q^*(a_2) &= 0,\\
f^*(a_3) &= a_3, & q^*(a_3) &= 3a_3,\\
f^*(a_4) &= 0, & q^*(a_4) &= -6a_1a_3,\\
f^*(a_6) &= 0, & q^*(a_6) &= -(9a_3^2 + a_1^3 a_3)
\end{aligned}$$

and

$$\begin{aligned}
t^*(a_1^2) &= -3a_1^2,\\
t^*(a_1a_3) &= \tfrac{1}{3}a_1^4 - 9a_1a_3,\\
t^*(a_3^2) &= -\tfrac{1}{27}a_1^6 + 2a_1^3a_3 - 27a_3^2.
\end{aligned}$$

3.4 The formulas for $Q(5)$

The moduli space $\mathcal{M}_1^1(5)$ is represented by the affine scheme $\operatorname{Spec} B^1(5)$ with

$$B^1(5) = \mathbb{Z}\big[\tfrac{1}{5}, a_1, u, \Delta^{-1}\big]$$

with

$$\Delta = -11u^{12} + 64a_1u^{11} - 154a_1^2u^{10} + 195a_1^3u^9 - 135a_1^4u^8 + 46a_1^5u^7 - 4a_1^6u^6 - a_1^7u^5.$$

The corresponding universal $\Gamma_1(5)$ structure is carried by the Weierstrass curve

$$y^2 + a_1xy + (a_1u^2 - u^3)y = x^3 + (a_1u - u^2)x^2$$

with point $P = (0,0)$ of order 5. The $\mathbb{G}_m$–action on $\mathcal{M}_1^1(5)$ induces a grading on $B^1(5)$, for which a_1 and u both have weight 1. It follows that

$$\pi_* \mathrm{TMF}_1(5) = \mathbb{Z}\big[\tfrac{1}{5}, a_1, u, \Delta^{-1}\big]$$

with topological degrees $|a_1| = |u| = 2$. The spectrum $\mathrm{TMF}_1(5)$ admits a complex orientation with $v_1 = a_1$ and $v_2 \equiv u^3 \mod (2, v_1)$.

The group $\mathbb{F}_5^\times \cong C_4$ acts on $\mathcal{M}_1^1(5)$: for $5 \nmid n$, the mod 5 reduction $[n] \in \mathbb{F}_5^\times$ acts by sending an R–point (C, P) (where P is a point of exact order 5 on C) to the R–point $(C, [n](P))$. This induced action of the generator $[2]$ of $\mathbb{F}_5^\times$ on the ring $B^1(5)$ is given by

$$[2](a_1) = a_1 - 2u, \quad [2](u) = a_1 - u.$$

These have the more convenient expressions

$$[2](u) = b_1, \quad [2](b_1) = -u,$$

where $b_1 := a_1 - u$. We have

$$\mathcal{M}_0^1(5) = \mathcal{M}_1^1(5)/\!/\mathbb{F}_5^\times$$

and hence an equivalence

$$\mathrm{TMF}_0(5) \simeq \mathrm{TMF}_1(5)^{h\mathbb{F}_5^\times}.$$

The resulting homotopy fixed point spectral sequence takes the form

$$H^s(\mathbb{F}_5^\times; \pi_t \mathrm{TMF}_1(5)) \Rightarrow \pi_{t-s} \mathrm{TMF}_0(5).$$

In particular, the ring of modular forms (meromorphic at the cusps) for $\Gamma_0(5)$ is the subring

$$\mathrm{MF}(\Gamma_0(5)) = H^0(\mathbb{F}_5^\times; \mathrm{MF}(\Gamma_1(5)) = \frac{\mathbb{Z}\big[\tfrac{1}{5}, b_2, b_4, \delta\big][\Delta^{-1}]}{(b_4^2 = b_2^2\delta - 4\delta^2)} \subset B^1(5)$$

where

$$b_2 := u^2 + b_1^2, \quad b_4 := u^3b_1 - ub_1^3, \quad \delta := u^2b_1^2.$$

Note that δ is almost a cube root of Δ: we have

$$\Delta = \delta^2 b_4 - 11\delta^3.$$

The effects of the maps

$$f^*, q^*\colon A \to B^1(5), \quad \text{and} \quad t^*\colon B^1(5) \to B^1(5)$$

are

$$\begin{aligned}
f^*(a_1) &= a_1, & q^*(a_1) &= a_1,\\
f^*(a_2) &= a_1u - u^2, & q^*(a_2) &= -u^2 + a_1u,\\
f^*(a_3) &= a_1u^2 - u^3, & q^*(a_3) &= -u^3 + a_1u^2,\\
f^*(a_4) &= 0, & q^*(a_4) &= -10u^4 + 30a_1u^3 - 25a_1^2u^2 + 5a_1^3u,\\
f^*(a_6) &= 0, & q^*(a_6) &= -20u^6 + 59a_1u^5 - 70a_1^2u^4 + 45a_1^3u^3 - 15a_1^4u^2 + a_1^5u
\end{aligned}$$

and

$$\begin{aligned}
t^*(a_1) &= \tfrac{1}{5}(-8\zeta^3 - 6\zeta^2 - 14\zeta - 7)a_1 + \tfrac{1}{5}(14\zeta^3 - 2\zeta^2 + 12\zeta + 6)u,\\
t^*(u) &= \tfrac{1}{5}(-\zeta^3 - 7\zeta^2 - 8\zeta - 4)a_1 + \tfrac{1}{5}(8\zeta^3 + 6\zeta^2 + 14\zeta + 7)u.
\end{aligned}$$

In the formulas for t^*, we use ζ to denote a 5^{th} root of unity. This results in the following formulas for f^*, q^* and t^* on rings of modular forms:

$$\begin{aligned}
f^*(c_4) &= b_2^2 - 12b_4 + 12\delta, & q^*(c_4) &= b_2^2 + 228b_4 + 492\delta,\\
f^*(c_6) &= -b_2^3 + 18b_2b_4 - 72b_2\delta, & q^*(c_6) &= -b_2^3 + 522b_2b_4 + 10,008b_2\delta
\end{aligned}$$

and

$$\begin{aligned}
t^*(b_2) &= -5b_2,\\
t^*(b_4) &= \tfrac{1}{5}(11b_2^2 - 117b_4 - 88\delta),\\
t^*(\delta) &= \tfrac{1}{5}(b_2^2 - 22b_4 + 117\delta).
\end{aligned}$$

4 Detection of the β–family by $Q(3)$ and $Q(5)$

The Miller–Ravenel–Wilson divided β–family [21] is an important algebraic approximation of the $K(2)$–local sphere at the prime 2. It was computed for the prime 2 by Shimomura in [22]. Here we use the standard chain of Bockstein spectral sequences and the formulas of Section 3.3 and Section 3.4 to compute algebraic chromatic data in the $Q(3)$ and $Q(5)$ spectra. These are compared to Shimomura's calculations, resulting in

Theorems 4.2.2 and 4.2.4. The surprising observation is that $Q(3)$ precisely detects the divided β–family, while the analogous family in $Q(5)$ has extra v_1–divisibility.

4.1 The chromatic spectral sequence

Following [21], given a BP_*–module N, we will let

$$M_i^{n-i}N := N/(p,\dots,v_{i-1},v_i^\infty,\dots,v_{n-1}^\infty)[v_n^{-1}].$$

If N is a $\mathrm{BP}_*\mathrm{BP}$–comodule, then so is $M_i^{n-i}N$. Letting $\mathrm{Ext}^{*,*}(N)$ denote the groups

$$\mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*,N),$$

there is a chromatic spectral sequence

$$E_1^{n,s,t} = \mathrm{Ext}^{s,t}(M_0^nN) \Rightarrow \mathrm{Ext}^{s+n,t}(N).$$

The groups $\mathrm{Ext}^{0,*}(M_0^n\mathrm{BP}_*)$ detect the n^{th} Greek letter elements in $\mathrm{Ext}^{*,*}(\mathrm{BP}_*)$.

The E_1–term of this spectral sequence may be computed by first computing the groups $\mathrm{Ext}^{*,*}(M_n^0)$ and then using the v_i–Bockstein spectral sequences (BSS) of the form

$$\mathrm{Ext}^{*,*}(M_{i+1}^{n-i-1}N)\otimes\mathbb{F}_p[v_i]/(v_i^\infty) \Rightarrow \mathrm{Ext}^{*,*}(M_i^{n-i}N).$$

4.2 Statement of results

For the remainder of this section we work exclusively at the prime 2. Shimomura used these spectral sequences to make the following computation.

Theorem 4.2.1 [22] *The groups $\mathrm{Ext}^0(M_0^2\mathrm{BP}_*)$ are spanned by the elements*

$$\frac{1}{2^kv_1^j},\qquad j\ge 1 \text{ and } k\le k(j);$$
$$\frac{v_2^{m2^n}}{2^kv_1^j},\qquad 2\nmid m,\ k\le k(j),\ j\le\begin{cases}a(1), & k=3, n=2,\\ a(n-k+1), & \text{otherwise},\end{cases}$$

where

$$k(j) := \begin{cases}1, & j\not\equiv 0 \mod 2,\\ \nu_2(j)+2, & j\equiv 0 \mod 2\end{cases}\quad\text{and}\quad a(i) := \begin{cases}1, & i=0,\\ 2, & i=1,\\ 3\cdot 2^{i-1}, & i\ge 2.\end{cases}$$

The "names" $v_2^i/2^k v_1^j$ are not the exact names of $\mathrm{BP}_*\mathrm{BP}$–primitives in $M_0^2\,\mathrm{BP}_*$, but rather the names of the elements detecting them in the sequence of BSSs:

$$\mathrm{Ext}^{*,*}(M_2^0\,\mathrm{BP}_*)\otimes\frac{\mathbb{F}_2[v_0,v_1]}{(v_0^\infty,v_1^\infty)}\Rightarrow\mathrm{Ext}^{*,*}(M_1^1\,\mathrm{BP}_*)\otimes\frac{\mathbb{F}_2[v_0]}{(v_0^\infty)}\Rightarrow\mathrm{Ext}^{*,*}(M_0^2\,\mathrm{BP}_*).$$

Put a linear order on the monomials $v_0^k v_1^j$ in $\mathbb{F}_2[v_0^k,v_1^j]$ by left lexicographical ordering on the sequence of exponents (k,j). With respect to this ordering, the actual primitives correspond to elements

$$\frac{v_2^i}{2^k v_1^j}+\text{terms with smaller denominators.}$$

The main theorem of this section is the following.

Theorem 4.2.2 *The map*

$$\mathrm{Ext}^0(M_0^2\,\mathrm{BP}_*)\to H^0(M_0^2C^*_{\mathrm{tot}}(Q(3)))$$

is an isomorphism.

Remark 4.2.3 It was observed by Mahowald and Rezk [20] that the map

$$\mathrm{Ext}^0(M_1^1\,\mathrm{BP}_*)\to H^0(M_1^1C^*_{\mathrm{tot}}(Q(3)))$$

is an isomorphism.

However, the same cannot hold for $Q(5)$. Indeed, the following theorem implies it does not even hold on the level of M_1^1.

Theorem 4.2.4 *The map*

$$\mathrm{Ext}^0(M_1^1\,\mathrm{BP}_*)\to H^0(M_1^1C^*_{\mathrm{tot}}(Q(5)))$$

is not *an isomorphism.*

4.3 Leibniz and doubling formulas

The group $H^0(M_0^2C^*_{\mathrm{tot}}(Q(\ell)))$ is the kernel of the map

$$M_0^2C^0_{\mathrm{tot}}(Q(\ell))\xrightarrow{d_0-d_1}M_0^2C^1_{\mathrm{tot}}(Q(\ell)),$$

where d_0 and d_1 are the cosimplicial coface maps of the total complex. Explicitly, we are applying M_0^2 to the map

$$D_{\mathrm{tot}}\colon A_{(2)} \xrightarrow{(\eta_R-\eta_L)\oplus(q^*-f^*)\oplus(\psi^\ell-1)} \Gamma_{(2)} \oplus B^1(\ell)_{(2)} \oplus A_{(2)}.$$

The projection of D_{tot} onto the last component is very easy to understand; it is given by

$$\psi^\ell - 1\colon A \to A.$$

As long as ℓ generates $\mathbb{Z}_2^\times/\{\pm 1\}$, in degree $2t$ the map $\psi^\ell - 1$, up to a unit in $\mathbb{Z}_{(2)}^\times$, corresponds to multiplication by a factor of $2^{k(t)}$. It therefore suffices to understand the composite D of D_{tot} with the projection onto the first two components:

$$D\colon A_{(2)} \xrightarrow{(\eta_R-\eta_L)\oplus(q^*-f^*)} \Gamma_{(2)} \oplus B^1(\ell)_{(2)}.$$

We shall make repeated use of the following lemma about this map D.

Lemma 4.3.1 *The map D satisfies the following two identities:*

$$D(xy) = D(x)\eta_R(y) + xD(y), \tag{4.3.2}$$

$$D(x^2) = 2xD(x) + D(x)^2. \tag{4.3.3}$$

Here, Γ is given the A–module structure induced by the map η_L, and $B^1(3)$ is given the A–module structure induced from the map f^*. Consequently, we have

$$D(xy) \equiv xD(y) \mod (D(x)). \tag{4.3.4}$$

Proof These identities hold for any map $D = d_0 - d_1\colon R^0 \to R^1$, the difference of two ring maps:

$$\begin{aligned} D(xy) &= d_0(x)d_0(y) - d_1(x)d_1(y) \\ &= d_1(x)(d_0(y) - d_1(y)) + (d_0(x) - d_1(x))d_0(y) \\ &= d_1(x)D(y) + D(x)d_0(y); \\ D(x^2) &= d_0(x)^2 - d_1(x)^2 \\ &= (d_0(x) - d_1(x))^2 + 2d_0(x)d_1(x) - 2d_1(x)^2 \\ &= D(x)^2 + 2d_1(x)D(x). \end{aligned}$$

□

Observe that using the fact that $a_1 = v_1$, there are isomorphisms

$$\begin{aligned}
\Gamma_{(2)} &\cong \mathbb{Z}_{(2)}[v_1][a_2, a_3, a_4, a_6, r, s, t][\Delta^{-1}], \\
B^1(3)_{(2)} &\cong \mathbb{Z}_{(2)}[v_1][a_3][\Delta^{-1}], \\
B^1(5)_{(2)} &\cong \mathbb{Z}_{(2)}[v_1][u][\Delta^{-1}].
\end{aligned}$$

Express elements of $\Gamma_{(2)}$ (respectively, $B^1(3)_{(2)}$, $B^1(5)_2$) "$(2, v_1)$–adically" so that every element is expressed as a power of the discriminant times a sum of terms

$$\Delta^\ell \sum_{k \geq 0} \sum_{j \geq 0} 2^k v_1^j c_{k,j}$$

for $\ell \in \mathbb{Z}$ and $c_{k,j} \in \mathbb{F}_2[a_2, a_3, a_4, a_6, r, s, t]$ (respectively $\mathbb{F}_2[a_3]$, $\mathbb{F}_2[u]$). We shall compare terms by saying that

$$2^k v_1^j c_{j,k} \quad \textit{is larger than} \quad 2^{k'} v_1^{j'} c_{j',k'}$$

if (k, j) is larger than (k', j') with respect to left lexicographical ordering. We shall be concerned with ordered sums of monomials of the form

$$\begin{aligned}
& v_1^{i_0} c_{0,i_0} + \text{terms of the form } v_1^j c_{0,j} \text{ with } j > i_0 \\
& \quad + 2v_1^{i_1} c_{1,i_1} + \text{terms of the form } 2v_1^j c_{1,j} \text{ with } j > i_1 \\
& \qquad + 4v_1^{i_2} c_{2,i_2} + \text{terms of the form } 4v_1^j c_{2,j} \text{ with } j > i_2 \\
& \qquad\quad + \cdots \\
& \qquad\qquad + 2^n v_1^{i_n} c_{n,i_n} + \text{larger terms}
\end{aligned}$$

for $i_0 > i_1 > \cdots > i_n$ and $n \geq 1$. Note that we permit the coefficients c_{k,i_k} to be zero. We shall abbreviate such expressions as

$$\begin{aligned}
(\dagger) \qquad & v_1^{i_0} c_{0,i_0} + \cdots \\
& \quad + 2v_1^{i_1} c_{1,i_1} + \cdots \\
& \qquad + 4v_1^{i_2} c_{2,i_2} + \cdots \\
& \qquad\quad + \cdots \\
& \qquad\qquad + 2^n v_1^{i_n} c_{n,i_n} + \cdots .
\end{aligned}$$

The following observation justifies considering such representations.

Lemma 4.3.5 *Suppose $x \in A_{(2)}$ is such that $D(x)$ is of the form $(\dagger)$. Then we have*

$$
\begin{aligned}
D(x^2) = v_1^{2i_0} c_{0,i_0}^2 + \cdots \\
+ 2v_1^{i_0} c_{0,i_0} x + \cdots \\
+ 4v_1^{i_1} c_{1,i_1} x + \cdots \\
+ 8v_1^{i_2} c_{2,i_2} x + \cdots \\
+ \cdots \\
+ 2^{n+1} v_1^{i_n} c_{n,i_n} x + \cdots ,
\end{aligned} \tag{4.3.6}
$$

and for m odd we have

$$
\begin{aligned}
D(x^m) = v_1^{i_0} c_{0,i_0} x^{m-1} + \cdots \\
+ 2v_1^{i_1} c_{1,i_1} x^{m-1} + \cdots \\
+ 4v_1^{i_2} c_{2,i_2} x^{m-1} + \cdots \\
+ \cdots \\
+ 2^n v_1^{i_n} c_{n,i_n} x^{m-1} + \cdots .
\end{aligned} \tag{4.3.7}
$$

Proof The identity (4.3.6) follows immediately from (4.3.3). We prove (4.3.7) by induction on $m = 2j+1$. Suppose that we know (4.3.7) for all odd $m' < m$. Write $j = 2^t s$ for s odd. Then by the inductive hypothesis, and repeated applications of (4.3.6), we deduce that

$$
\begin{aligned}
D(x^j) = v_1^{i_0} c'_{0,i_0} + \cdots \\
+ 2v_1^{i_1} c'_{1,i_1} + \cdots \\
+ 4v_1^{i_2} c'_{2,i_2} + \cdots \\
+ \cdots \\
+ 2^n v_1^{i_n} c'_{n,i_n} + \cdots .
\end{aligned}
$$

Applying (4.3.6), we have

$$
\begin{aligned}
D(x^{2j}) = v_1^{2i_0} (c'_{0,i_0})^2 + \cdots \\
+ 2v_1^{i_0} c'_{0,i_0} x^j + \cdots \\
+ 4v_1^{i_1} c'_{1,i_1} x^j + \cdots \\
+ 8v_1^{i_2} c'_{2,i_2} x^j + \cdots \\
+ \cdots \\
+ 2^{n+1} v_1^{i_n} c'_{n,i_n} x^j + \cdots .
\end{aligned}
$$

It follows from (4.3.4) that we have

$$\begin{aligned} D(x^{2j+1}) = D(x^{2j}x) = v_1^{i_0}c_{0,i_0}x^{2j} &+ \cdots \\ &+ 2v_1^{i_1}c_{1,i_1}x^{2j} + \cdots \\ &+ 4v_1^{i_2}c_{2,i_2}x^{2j} + \cdots \\ &+ \cdots \\ &+ 2^n v_1^{i_n}c_{n,i_n}x^{2j} + \cdots . \end{aligned}$$

□

4.4 Overview of the technique

The technique for the proof of Theorem 4.2.2 is as follows (following [21] and [22]):

Step 1 Compute the differentials from the $s=0$ to the $s=1$–lines in the v_1–BSS

(4.4.1) $$H^{s,*}(M_2^0 C^*_{\mathrm{tot}}(Q(3))) \otimes \mathbb{F}_2[v_1]/(v_1^\infty) \Rightarrow H^{s,*}(M_1^1 C^*_{\mathrm{tot}}(Q(3))).$$

This establishes the existence and v_1–divisibility of v_2^i/v_1^j in $H^{0,*}(C^*_{\mathrm{tot}}(Q(3)))$.

Step 2 For i, j as above, demonstrate that $v_2^i/2^k v_1^j$ exists in $H^{0,*}(M_0^2 C^*_{\mathrm{tot}}(Q(3)))$ by writing down an element

$$x_{i/j,k} = \frac{a_3^i}{2^k v_1^j} + \text{terms with smaller denominators} \in M_0^2 A$$

with $D_{\mathrm{tot}}(x) = 0$.

Step 3 Given j, find the maximal k such that $x_{i/j,k}$ exists by using the exact sequence

$$H^{0,*}(M_0^2 C^*_{\mathrm{tot}}(Q(3))) \xrightarrow{\cdot 2} H^{0,*}(M_0^2 C^*_{\mathrm{tot}}(Q(3))) \xrightarrow{\partial} H^{1,*}(M_1^1 C^*_{\mathrm{tot}}(Q(3))).$$

Specifically, the maximality of k is established by showing that $\partial(x_{i/j,k}) \neq 0$. The nontriviality of $\partial(x_{i/j,k})$ can be demonstrated by considering its image under the inclusion

$$H^{1,*}(M_1^1 C^*_{\mathrm{tot}}(Q(3))) \hookrightarrow \operatorname{Coker} M_1^1(D_{\mathrm{tot}}),$$

where $M_1^1(D_{\mathrm{tot}})$ is the map

$$M_1^1(D_{\mathrm{tot}})\colon M_1^1 A \to M_1^1 \Gamma \oplus M_1^1 B^1(3) \oplus M_1^1 A$$

essentially computed in Step 1 by the computation of the differentials from $s=0$ to $s=1$ in the spectral sequence (4.4.1).

4.5 Computation of $H^{*,*}(M_2^0 C^*_{\mathrm{tot}}(Q(3)))$

We have [2, Section 7]

$$H^{*,*}(M_2^0 C^*_\Gamma(A)) = \mathbb{F}_2[a_3^{\pm 1}, h_1, h_2, g]/(h_2^3 = a_3 h_1^3),$$
$$H^{*,*}(M_2^0 C^*_{\Lambda^1(3)}(B^1)) = \mathbb{F}_2[a_3^{\pm}, h_{2,1}]$$

with (s,t)–bidegrees

$$|a_3| = (0,6), \quad |h_1| = (1,2), \quad |h_2| = (1,4), \quad |g| = (4,24), \quad |h_{2,1}| = (1,6),$$

and $h_{2,1}^4 = g$. Moreover, the spectral sequence of the double complex gives

$$\begin{aligned} (4.5.1)\quad & H^{s,t}(M_2^0 C^*_\Gamma(A)) \\ & \oplus H^{s-1,t}(M_2^0 C^*_\Gamma(A)) \oplus H^{s-1,t}(M_2^0 C^*_{\Lambda^1(3)}(B^1)) \\ & \oplus H^{s-2,t}(M_2^0 C^*_{\Lambda^1(3)}(B^1)) \\ & \qquad\qquad\qquad \Rightarrow H^{s,t}(M_2^0 C^*_{\mathrm{tot}}(Q(3))). \end{aligned}$$

In order to differentiate the terms x with the same name (such as a_3) occurring in the different groups in the E_1–term of spectral sequence (4.5.1), we shall employ the following notational convention:

$$\begin{array}{ll} x \in C^*_\Gamma(A) & \text{on the 0–line,} \\ \bar{x} \in C^*_\Gamma(A) & \text{on the 1–line,} \\ x' \in C^*_{\Lambda^1(\ell)}(B^1) & \text{on the 1–line,} \\ \bar{x}' \in C^*_{\Lambda^1(\ell)}(B^1) & \text{on the 2–line.} \end{array}$$

The formulas of Section 3.3 show that the only nontrivial d_1 differentials in spectral sequence (4.5.1) are

$$d_1(g^i(\bar{a}_3)^j) = h_{2,1}^{4i}(\bar{a}_3')^j.$$

Since g is the image of the element $g \in \mathrm{Ext}^{4,24}(\mathrm{BP}_*)$ (the element that detects $\bar{\kappa}$ in the ANSS for the sphere), and the spectral sequence (4.5.1) is a spectral sequence of modules over $\mathrm{Ext}^{*,*}(\mathrm{BP}_*)$, we deduce that there are no possible d_r–differentials for

$r > 1$. We deduce that we have

$$\begin{aligned} H^{*,*}(M_2^0 C^*_{\mathrm{tot}}(Q(3))) = {} & \mathbb{F}_2[a_3^{\pm 1}, h_1, h_2, g]/(h_2^3 = a_3 h_1^3) \\ & \oplus \mathbb{F}_2[\overline{a}_3^{\pm 1}, \overline{g}]\{\overline{h}_1, \overline{h}_2, \overline{h}_1^2, \overline{h}_2^2, \overline{h}_2^3 = \overline{a}_3 \overline{h}_1^3\} \\ & \oplus \mathbb{F}_2[(a_3')^{\pm 1}, h_{2,1}'] \\ & \oplus \mathbb{F}_2[(\overline{a}_3')^{\pm 1}, \overline{g}']\{\overline{h}_{2,1}', (\overline{h}_{2,1}')^2, (\overline{h}_{2,1}')^3\}. \end{aligned}$$

Remark 4.5.2 Note that $H^{*,*}(M_2^0 C^*_{\mathrm{tot}}(Q(3)))$ is less than half of $\mathrm{Ext}^{*,*}(M_2^0\,\mathrm{BP}_*)$. This indicates that $Q(3)$ cannot agree with "half" of the proposed duality resolution of Goerss, Henn, Mahowald and Rezk at $p = 2$ [10], despite the fact that it is built from the same spectra. In particular, the fiber of the map

$$S_{K(2)} \to Q(3)_{K(2)}$$

cannot be the dual of $Q(3)_{K(2)}$.

4.6 Computation of $H^{0,*}(M_1^1 C^*_{\mathrm{tot}}(Q(3)))$

We now compute the differentials in the v_1–BSS

$$H^{s,*}(M_2^0 C^*_{\mathrm{tot}}(Q(3))) \otimes \mathbb{F}_2[v_1]/(v_1^\infty) \ \Rightarrow\ H^{s,*}(M_1^1 C^*_{\mathrm{tot}}(Q(3))) \tag{4.6.1}$$

from the $s = 0$–line to the $s = 1$–line. This computation was originally done by Mahowald and Rezk [20], but we redo it here to establish notation, and to motivate the rationale behind some of the computations to follow.

One computes using the formulas of Section 3.3:

$$\begin{aligned} D(x_0) &\equiv a_1 s^2 \mod (2, v_1^2), \\ D(x_1) &\equiv a_1^2 a_3 s \mod (2, v_1^3), \\ D(x_2) &\equiv (a_1')^6 (a_3')^2 \mod (2, v_1^7) \end{aligned} \tag{4.6.2}$$

for

$$\begin{aligned} x_0 &:= a_3 + a_1 a_2 \equiv a_3 \mod (2, v_1), \\ x_1 &:= x_0^2 + a_1^2 a_4 + a_1^2 a_2^2 \equiv a_3^2 \mod (2, v_1), \\ x_2 &:= \Delta \equiv a_3^4 \mod (2, v_1). \end{aligned}$$

Remark 4.6.3 The above formulas for x_i were obtained by the following method. In the complex $M_2^0 C_\Gamma^*(A)$, we have

$$\begin{aligned} d(a_2) &= r + \cdots, \\ d(a_4 + a_2^2) &= s^4 + \cdots, \\ d(a_6) &= t^2 + \cdots. \end{aligned}$$

These are used in [2, Section 6] to produce a complex which is closely related to the cobar complex on the double of $A(1)_*$. To arrive at x_0 we calculate

$$D(a_3) = a_1 r + \cdots,$$

which means that we need to add the correction term $a_1 a_2$ to arrive at x_0. The expression for x_1 was similarly produced. The definition Δ is a natural candidate for x_2, as it is an element of the form $a_3^4 + \cdots$ which is already known to be a cocycle in $C_\Gamma^0(A)$.

It follows from inductively applying (4.3.6) that we have

$$D(x_2^{2^{n-2}}) \equiv (a_1')^{3\cdot 2^{n-1}} (a_3')^{2^{n-1}} \mod (2, v_1^{3\cdot 2^{n-1}+1}).$$

It follows from (4.3.7) that for m odd we have

$$\begin{aligned} D(x_0^m) &\equiv a_1 s^2 a_3^{m-1} \mod (2, v_1^2), \\ D(x_1^m) &\equiv a_1^2 a_3^{2m-1} s \mod (2, v_1^3), \\ D(x_2^{m2^{n-2}}) &\equiv (a_1')^{3\cdot 2^{n-1}} (a_3')^{m2^n - 2^{n-1}} \mod (2, v_1^{3\cdot 2^{n-1}+1}). \end{aligned}$$

We deduce the following.

Lemma 4.6.4 *The v_1–BSS differentials in (4.6.1) from the $(s=0)$–line to the $(s=1)$–line are given by*

$$\begin{aligned} d_1\left(\frac{a_3^m}{v_1^j}\right) &= \frac{a_3^{m-1} h_2}{v_1^{j-1}}, \\ d_2\left(\frac{a_3^{2m}}{v_1^j}\right) &= \frac{a_3^{2m-1} h_1}{v_1^{j-2}}, \\ d_{3\cdot 2^{n-1}}\left(\frac{a_3^{m2^n}}{v_1^j}\right) &= \frac{(a_3')^{m2^n - 2^{n-1}}}{v_1^{j-3\cdot 2^{n-1}}}, \end{aligned}$$

where m is odd.

Corollary 4.6.5 *The groups $H^{0,*}(M_1^1 C_{\mathrm{tot}}^*(Q(3)))$ are generated by the elements*

$$\frac{a_3^{m2^n}}{v_1^j}$$

for m odd and $j \le a(n)$.

4.7 Computation of $H^{0,*}(M_0^2 C_{\mathrm{tot}}^*(Q(3)))$

We now prove Theorem 4.2.2, which is more specifically stated below.

Theorem 4.7.1 *The groups $H^{0,*}(M_0^2 C_{\mathrm{tot}}^*(Q(3)))$ are spanned by elements*

$$\frac{1}{2^k v_1^j}, \qquad j \ge 1 \text{ and } k \le k(j);$$

$$\frac{a_3^{mp^n}}{2^k v_1^j}, \qquad 2 \nmid m,\ k \le k(j),\ \text{and } j \le \begin{cases} a(1), & k = 3,\ n = 2, \\ a(n-k+1), & \text{otherwise.} \end{cases}$$

In many cases, the bounds on 2–divisibility will follow from the following simple observation.

Lemma 4.7.2 *Suppose the element*

$$\frac{a_3^i}{2^k v_1^j} \in H^{0,2t}(M_0^2 C_{\mathrm{tot}}^*(Q(3)))$$

exists. Then $k \le k(t)$.

Proof The formula

$$(\psi^3 - 1)\frac{a_3^i}{2^k v_1^j} = (3^t - 1)\frac{\overline{a}_3^i}{2^k v_1^j}$$

implies that in order for

$$0 \ne D_{\mathrm{tot}}\left(\frac{a_3^i}{2^k v_1^j}\right) \in M_0^2 C_{\mathrm{tot}}^1(Q(3))$$

we must have $k \le \nu_2(3^t - 1)$. □

Proof of Theorem 4.7.1 Lemma 4.6.4 established that for m odd, $a_3^{m2^n}/2v_1^j$ exists for $1 \le j \le a(n)$. In order to prove the required 2–divisibility of these elements, we need to prove that

$$D\left(\frac{a_3^{4m}}{8v_1^2}+\cdots\right)=0, \tag{4.7.3}$$

$$D\left(\frac{a_3^{m2^n}}{4v_1^{2j}}+\cdots\right)=0, \qquad 2j \le a(n-1), \tag{4.7.4}$$

$$D\left(\frac{a_3^{m2^n}}{2^k v_1^{j2^{k-2}}}+\cdots\right)=0, \qquad k \ge 3,\ j2^{k-2} \le a(n-k+1). \tag{4.7.5}$$

In light of Lemma 4.7.2, to establish that these are the maximal 2–divisibilities of these elements, we need only check that

$$\partial\left(\frac{a_3^{m}}{2v_1}+\cdots\right)\not\equiv 0 \mod D(M_1^1 A), \tag{4.7.6}$$

$$\partial\left(\frac{a_3^{m2^n}}{2v_1^{2j}}+\cdots\right)\not\equiv 0 \mod D(M_1^1 A), \quad a(n-1) < 2j \le a(n), \tag{4.7.7}$$

$$\partial\left(\frac{a_3^{m2^n}}{2^{k-1} v_1^{j2^{k-2}}}+\cdots\right)\not\equiv 0 \mod D(M_1^1 A), \quad k \ge 2, \quad a(n-k+1) < j2^{k-1} \le a(n-k+2). \tag{4.7.8}$$

Proof of (4.7.6) Using the formulas of Section 3.3, we have

$$\begin{aligned} D(x_0) = a_1 s^2 &+ \cdots \\ &+ 2(t+rs+s^3+a_2 s)+\cdots \\ &+ 2a_3' + \cdots . \end{aligned} \tag{4.7.9}$$

It follows from (4.3.7) that we have for m odd

$$\begin{aligned} D(x_0^m) = a_1 a_3^{m-1} s^2 &+ \cdots \\ &+ 2a_3^{m-1}(t+rs+s^3+a_2 s)+\cdots \\ &+ 2(a_3')^m + \cdots . \end{aligned} \tag{4.7.10}$$

Since we have

$$\eta_R(a_1) = a_1 + 2s \tag{4.7.11}$$

we deduce from (4.7.10), using (4.3.2),

$$\begin{aligned}D(x_0^m a_1) = {} & a_1^2 a_3^{m-1} s^2 + \cdots \\ & + 2a_3^m s + 2a_1 a_3^{m-1}(t+rs) + \cdots \\ & + 2a_1'(a_3')^m + \cdots .\end{aligned} \tag{4.7.12}$$

Reducing modulo the invariant ideal $(4, v_1^2)$ we deduce

$$\partial\left(\frac{a_3^m}{2v_1} + \cdots\right) = \frac{a_3^m h_1}{v_1^2} + \cdots .$$

Lemma 4.6.4 implies that this expression is not in $D(M_1^1 A)$ if $m \equiv 3 \bmod 4$. However, if $m \equiv 1 \bmod 4$, then Lemma 4.6.4 implies that $a_3^m h_1/v_1^2$ is killed in the v_1–BSS (4.6.1) by $d_2(a_3^{m+1}/v_1^4)$. We compute, using the formulas of Section 3.3,

$$\begin{aligned}D(x_1) = {} & a_1^2 a_3 s + a_1^3(t+rs) + \cdots \\ & + 2a_1 a_3 s^2 + \cdots \\ & + 2(a_1')^3 a_3' + \cdots .\end{aligned} \tag{4.7.13}$$

We deduce using (4.3.7) that for m odd we have

$$\begin{aligned}D(x_1^m) = {} & a_1^2 a_3^{2m-1} s + a_1^3 a_3^{2m-2}(t+rs) + \cdots \\ & + 2a_1 a_3^{2m-1} s^2 + \cdots \\ & + 2(a_1')^3 (a_3')^{2m-1} + \cdots .\end{aligned} \tag{4.7.14}$$

We deduce that for $m \equiv 1 \bmod 4$ we have

$$\begin{aligned}D(a_1^3 x_0^m + 2x_1^{\frac{m+1}{2}}) = {} & a_1^4 a_3^{m-1} s^2 + \cdots \\ & + 2(a_1')^3 (a_3')^m + \cdots .\end{aligned}$$

Thus we have for $m \equiv 1 \bmod 4$

$$\partial\left(\frac{x_0^m}{2v_1} + \cdots\right) = \frac{(a_3')^m}{v_1} + \cdots$$

and Lemma 4.6.4 implies that this expression is not in $D(M_1^1 A)$. This establishes (4.7.6). □

Proof of (4.7.7) for $n = 1$ Equation (4.7.14) implies that

$$\partial\left(\frac{a_3^{2m}}{v_1^2} + \cdots\right) = \frac{a_3^{2m-1} h_2}{v_1} + \cdots ,$$

which, by Lemma 4.6.4, is not in $D(M_1^1 A)$. This establishes (4.7.7) for $n = 1$. □

Proof of (4.7.7) for $n = 2$ We compute using the formulas of Section 3.3

$$D(x_2) = (a_1')^6(a_3')^2 + \cdots + 2(a_1')^3(a_3')^3 + \cdots . \tag{4.7.15}$$

Applying (4.3.7), we get for m odd

$$D(x_2^m) = (a_1')^6(a_3')^{4m-2} + \cdots + 2(a_1')^3(a_3')^{4m-1} + \cdots . \tag{4.7.16}$$

It follows that

$$\partial\left(\frac{x_2^m}{2v_1^{2j}}\right) = \frac{(a_3')^{4m-1}}{v_1^{2j-3}} + \cdots$$

for $a(1) < 2j \le a(2)$, which is not in $D(M_1^1 A)$ by Lemma 4.6.4. This establishes (4.7.7) for $n = 2$. □

Proof of (4.7.3) We deduce from (4.7.16) that $a_3^{4m}/4v_1^2$ exists. In order to understand its 2–divisibility, we compute $\partial(a_3^{4m}/4v_1^2)$, which is the obstruction to divisibility. To do this we need to compute $D(x_2^m/8v_1^2)$. Since $(8, v_1^4)$ is an invariant ideal, we compute this from $D(a_1^2 x_2^m)$. Since

$$D(a_1^2) = 4s^2 + 4sa_1 \tag{4.7.17}$$

and

$$x_2 \equiv a_3^4 + 2a_1^2a_3^2a_4 + a_3^3a_1^3 \mod (4, v_1^4), \tag{4.7.18}$$

we deduce from (4.3.2) that

$$D(a_1^2x_2^m) = (a_1')^8(a_3')^{4m-2} + \cdots + 2(a_1')^5(a_3')^{4m-1} + \cdots + 4a_3^{4m}s^2 + 4a_1a_3^{4m}s + 4a_1^3a_3^{4m-1}s^2 + \cdots , \tag{4.7.19}$$

which gives

$$D\left(\frac{x_2^m}{8v_1^2}\right) = \frac{a_3^{4m}s^2}{2v_1^4} + \frac{a_3^{4m}s}{2v_1^3} + \frac{a_3^{4m-1}s^2}{2v_1}.$$

Lemma 4.6.4 tells us that $a_3^{4m}h_2/v_1^4$ is killed by x_0^{4m+1}/v_1^5. We compute

$$D(x_0) \equiv a_1s^2 + sa_1^2 \mod 2$$

and thus

$$D(x_0^4) \equiv a_1^4s^8 \mod (2, v_1^5).$$

Using the fact that

$$x_0^{4m} \equiv a_3^{4m} \mod (2, v_1^4)$$

we have

$$D(x_0^{4m+1}) \equiv a_1 a_3^{4m} s^2 + a_1^2 a_3^{4m} s + a_1^4 a_3^{4m-3} s^8 \mod (2, v_1^5).$$

and thus

$$D\left(\frac{x_2^m}{8v_1^2} + \frac{x_0^{4m+1}}{2v_1^5}\right) = \frac{a_3^{4m-3}s^8}{2v_1} + \frac{a_3^{4m-1}s^2}{2v_1}.$$

Since $a_4 + a_2^2$ kills s^4 (see Remark 4.6.3), $(a_4 + a_2^2)^2$ kills s^8, and we compute

$$D((a_4 + a_2^2)^2) \equiv s^8 + a_3^2 s^2 \mod (2, v_1).$$

Therefore we have

$$D\left(\frac{x_2^m}{8v_1^2} + \frac{x_0^{4m+1}}{2v_1^5} + \frac{a_3^{4m-3}(a_4 + a_2^2)^2}{2v_1}\right) = 0. \tag{4.7.20}$$

This establishes (4.7.3). □

Proof of (4.7.4) Iterated application of (4.3.3) to (4.7.16) yields

$$\begin{aligned} D(x_2^{m2^{n-2}}) = (a_1')^{3\cdot 2^{n-1}}(a_3')^{m2^n - 2^{n-1}} &+ \cdots \\ + 2(a_1')^{3\cdot 2^{n-2}}(a_3')^{m2^n - 2^{n-2}} &+ \cdots \\ + 4(a_1')^{3\cdot 2^{n-3}}(a_3')^{m2^n - 2^{n-3}} &+ \cdots \\ &+ \cdots \\ + 2^{n-1}(a_1')^3(a_3')^{m2^n - 1} &+ \cdots . \end{aligned} \tag{4.7.21}$$

It follows that

$$D\left(\frac{x_2^{m2^{n-2}}}{4v_1^{2j}}\right) = 0, \quad 2j \le a(n-1).$$

This establishes (4.7.4). □

Proof of (4.7.5) Suppose that j is even. Then the ideal $(2^k, v_1^{j2^{k-2}})$ is invariant, and reducing (4.7.21) modulo this invariant ideal gives

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}}\right) = 0, \quad j2^{k-2} \le a(n-k+1).$$

This establishes (4.7.5) for j even.

Suppose now that j is odd. Then the ideal $(2^k, v_1^{j2^{k-2}+2^{k-2}})$ is invariant, and in order to compute $D(x_2^{m2^{n-2}}/2^k v_1^{j2^{k-2}})$ we must compute $D(a_1^{2^{k-2}} x_2^{m2^{n-2}})$ modulo $(2^k, v_1^{j2^{k-2}+2^{k-2}})$. Repeated application of (4.3.3) to (4.7.17) yields

$$D(a_1^{2^{k-2}}) \equiv 2^{k-1} a_1^{2^{k-2}-2} s^2 + 2^{k-1} a_1^{2^{k-2}-1} s \mod 2^k. \tag{4.7.22}$$

We also note that since

$$x_2 \equiv a_3^4 + a_1^3 a_3^3 + \cdots \mod 2$$

we have

$$\begin{aligned} x_2^{m2^{n-2}} &\equiv a_3^{m2^n} + a_1^{3\cdot 2^{n-2}} a_3^{3\cdot 2^{n-2}+(m-1)2^{n-2}} + \cdots \mod 2 \\ &\equiv a_3^{m2^n} + a_1^{3\cdot 2^{n-2}} a_3^{2^{n-1}+m2^{n-2}} + \cdots \mod 2. \end{aligned} \tag{4.7.23}$$

Applying (4.3.2) to (4.7.21)–(4.7.23), we get

$$\begin{aligned} D(a_1^{2^{k-2}} x_2^{m2^{n-2}}) = (a_1')^{3\cdot 2^{n-1}+2^{k-2}} (a_3')^{m2^n - 2^{n-1}} &+ \cdots \\ + 2(a_1')^{3\cdot 2^{n-2}+2^{k-2}} (a_3')^{m2^n-2^{n-2}} &+ \cdots \\ + 4(a_1')^{3\cdot 2^{n-3}+2^{k-2}} (a_3')^{m2^n-2^{n-3}} &+ \cdots \\ &+ \cdots \\ + 2^{k-1}(a_1')^{3\cdot 2^{n-k}+2^{k-2}} (a_3')^{m2^n-2^{n-k}} &+ \cdots \\ + 2^{k-1} a_1^{2^{k-2}-2} a_3^{m2^n} s^2 + 2^{k-1} a_1^{2^{k-2}-1} a_3^{m2^n} s & \\ + 2^{k-1} a_1^{3\cdot 2^{n-2}} a_3^{2^{n-1}+m2^{n-2}} s^2 &+ \cdots . \end{aligned} \tag{4.7.24}$$

We deduce that for j odd and $j2^{k-2} \le a(n-k+1)$ we have

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}}\right) = \frac{a_3^{m2^n} s^2}{2v_1^{j2^{k-2}+2}} + \frac{a_3^{m2^n} s}{2v_1^{j2^{k-2}+1}}.$$

However, Lemma 4.6.4 implies that $a_3^{m2^n} h_2/v_1^{j2^{k-2}+2}$ is killed by $a_3^{m2^n+1}/v_1^{j2^{k-2}+3}$. It follows from (4.7.9) that we have

$$D(x_0^{m2^n}) \equiv a_1^{m2^n} s^{m2^{n+1}} + \cdots \mod 2$$

and hence

$$D(x_0^{m2^n+1}) \equiv a_1 a_3^{m2^n} s^2 + a_1^2 a_3^{m2^n} s + a_1^{m2^n} a_3 s^{m2^{n+1}} + \cdots \mod 2.$$

This implies that we have

$$D\left(\frac{x_0^{m2^n+1}}{2v_1^{j2^{k-2}+3}}\right) = \frac{a_3^{m2^n}s^2}{2v_1^{j2^{k-2}+2}} + \frac{a_3^{m2^n}s}{2v_1^{j2^{k-2}+1}} \tag{4.7.25}$$

and therefore

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}} + \frac{x_0^{m2^n+1}}{2v_1^{j2^{k-2}+3}}\right) = 0.$$

This establishes (4.7.5). □

Proof of (4.7.7) for $n \geq 3$ It follows from (4.7.21) that for $a(n-1) < 2j \leq a(n)$, we have

$$D\left(\frac{x_2^{m2^{n-2}}}{4v_1^{2j}}\right) = \frac{(a_3')^{m2^n-2^{n-2}}}{2v_1^{2j-a(n-1)}} + \cdots$$

and hence

$$\partial\left(\frac{x_2^{m2^{n-2}}}{2v_1^{2j}}\right) = \frac{(a_3')^{m2^n-2^{n-2}}}{v_1^{2j-a(n-1)}} + \cdots.$$

This element is not in $D(M_1^1 A)$ by Lemma 4.6.4. This establishes (4.7.7). □

Proof of (4.7.8) Suppose that j is even. Then the ideal $(2^k, v_1^{j2^{k-2}})$ is invariant, and reducing (4.7.21) modulo this invariant ideal gives

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}}\right) = \frac{(a_3')^{m2^n-2^{n-k}}}{2v_1^{j2^{k-2}-a(n-k+1)}} + \cdots, \quad a(n-k+1) < j2^{k-2} \leq a(n-k+2)$$

and therefore

$$\partial\left(\frac{x_2^{m2^{n-2}}}{2^{k-1} v_1^{j2^{k-2}}}\right) = \frac{(a_3')^{m2^n-2^{n-k}}}{v_1^{j2^{k-2}-a(n-k+1)}} + \cdots, \quad a(n-k+1) < j2^{k-2} \leq a(n-k+2).$$

Since $k \geq 3$, this is not in $D(M_1^1 A)$ by Lemma 4.6.4. This establishes (4.7.5) for j even.

Suppose now that j is odd. Then the ideal $(2^k, v_1^{j2^{k-2}+2^{k-2}})$ is invariant, and in order to compute $D(x_2^{m2^{n-2}}/2^k v_1^{j2^{k-2}})$ we must compute $D(a_1^{2^{k-2}} x_2^{m2^{n-2}})$ modulo $(2^k, v_1^{j2^{k-2}+2^{k-2}})$. It follows from (4.7.24) that for j odd and $a(n-k+1) < j2^{k-2} \leq a(n-k+2)$ we have

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}}\right) = \frac{a_3^{m2^n}s^2}{2v_1^{j2^{k-2}+2}} + \frac{a_3^{m2^n}s}{2v_1^{j2^{k-2}+1}} + \frac{(a_3')^{m2^n-2^{n-k}}}{2v_1^{j2^{k-2}-a(n-k+1)}} + \cdots.$$

Using (4.7.25), we have

$$D\left(\frac{x_2^{m2^{n-2}}}{2^k v_1^{j2^{k-2}}} + \frac{x_0^{m2^n+1}}{2v_1^{j2^{k-2}+3}}\right) = \frac{(a_3')^{m2^n-2^{n-k}}}{2v_1^{j2^{k-2}-a(n-k+1)}} + \cdots$$

and therefore

$$\partial\left(\frac{x_2^{m2^{n-2}}}{2^{k-1} v_1^{j2^{k-2}}}\right) = \frac{(a_3')^{m2^n-2^{n-k}}}{v_1^{j2^{k-2}-a(n-k+1)}} + \cdots .$$

Since $k \geq 3$, this is not in $D(M_1^1 A)$ by Lemma 4.6.4. This establishes (4.7.5) for j odd. □

This completes the proof of Theorem 4.7.1. □

4.8 Computation of $H^{*,*}(M_2^0 C^*_{\mathrm{tot}}(Q(5)))$

We have (as before)

$$H^{*,*}(M_2^0 C^*_\Gamma(A)) = \mathbb{F}_2[a_3^{\pm 1}, h_1, h_2, g]/(h_2^3 = a_3 h_1^3),$$
$$H^{*,*}(M_2^0 C^*_{\Lambda^1(5)}(B^1)) = \mathbb{F}_2[u^{\pm}, h_{2,1}],$$

with (s,t)–bidegrees

$$|a_3| = (0,6), \qquad |h_1| = (1,2), \qquad |h_2| = (1,4),$$
$$|g| = (4,24), \qquad |u| = (0,2), \qquad |h_{2,1}| = (1,6),$$

and $h_{2,1}^4 = g$. Moreover, the spectral sequence of the double complex gives

$$\begin{aligned}(4.8.1)\quad & H^{s,t}(M_2^0 C^*_\Gamma(A)) \\ & \oplus H^{s-1,t}(M_2^0 C^*_\Gamma(A)) \oplus H^{s-1,t}(M_2^0 C^*_{\Lambda^1(5)}(B^1)) \\ & \oplus H^{s-2,t}(M_2^0 C^*_{\Lambda^1(5)}(B^1)) \\ & \qquad\qquad \Rightarrow H^{s,t}(M_2^0 C^*_{\mathrm{tot}}(Q(5))).\end{aligned}$$

As before, we will differentiate the terms x with the same name occurring in the different groups in the E_1–term of spectral sequence (4.8.1). We shall employ the following notational convention:

$$\begin{aligned} &x \in C^*_\Gamma(A) && \text{on the 0–line,}\\ &\overline{x} \in C^*_\Gamma(A) && \text{on the 1–line,}\\ &y \in C^*_{\Lambda^1(\ell)}(B^1) && \text{on the 1–line,}\\ &\overline{y} \in C^*_{\Lambda^1(\ell)}(B^1) && \text{on the 2–line.} \end{aligned}$$

The formulas of Section 3.4 show that the only nontrivial d_1 differentials in spectral sequence (4.5.1) are

$$d_1(g^i\overline{a}_3^j) = h_{2,1}^{4i}\overline{u}^{3j}.$$

Since the spectral sequence (4.8.1) is a spectral sequence of modules over $\mathrm{Ext}^{*,*}(\mathrm{BP}_*)$, we deduce that there are no possible d_r–differentials for $r > 1$. We deduce that we have

$$\begin{aligned} H^{*,*}(M_2^0C^*_{\mathrm{tot}}(Q(5))) = \mathbb{F}_2[a_3^{\pm 1}, h_1, h_2, g]/(h_2^3 = a_3h_1^3)&\\ \oplus \mathbb{F}_2[\overline{a}_3^{\pm 1}, \overline{g}]\{\overline{h}_1, \overline{h}_2, \overline{h}_1^2, \overline{h}_2^2, \overline{h}_2^3 = \overline{a}_3\overline{h}_1^3\}&\\ \oplus \mathbb{F}_2[u^{\pm 1}, h_{2,1}]&\\ \oplus \mathbb{F}_2[\overline{u}^{\pm 3}, g]\{\overline{h}_{2,1}, (\overline{h}_{2,1})^2, (\overline{h}_{2,1})^3\}&\\ \oplus \mathbb{F}_2[\overline{u}^{\pm 3}, \overline{h}_{2,1}]\{\overline{u}, \overline{u}^2\}.& \end{aligned}$$

4.9 Computation of $H^{0,*}(M_1^1C^*_{\mathrm{tot}}(Q(5)))$

We now compute the differentials in the v_1–BSS

$$H^{s,*}(M_2^0C^*_{\mathrm{tot}}(Q(5))) \otimes \mathbb{F}_2[v_1]/(v_1^\infty) \Rightarrow H^{s,*}(M_1^1C^*_{\mathrm{tot}}(Q(5))) \tag{4.9.1}$$

from the $s = 0$–line to the $s = 1$–line.

One computes using the formulas of Section 3.4:

$$\begin{aligned} D(x_0) &\equiv a_1s^2 \mod (2, v_1^2),\\ D(x_1) &\equiv a_1^2a_3s \mod (2, v_1^3),\\ D(x_2) &\equiv a_1^8u^4 \mod (2, v_1^9), \end{aligned} \tag{4.9.2}$$

for x_i as in Section 4.6. The formula for $D(x_2)$ already informs us that the v_1–BSS for $Q(5)$ differs from the v_1–BSS for $Q(3)$.

It follows from inductively applying (4.3.6) that we have

$$D(x_2^{2^{n-2}}) \equiv a_1^{2^{n+1}}u^{2^n} \mod (2, v_1^{2^{n+1}+1}).$$

It follows from (4.3.7) that for m odd, we have

$$\begin{aligned} D(x_0^m) &\equiv a_1 s^2 a_3^{m-1} \quad \mod (2, v_1^2), \\ D(x_1^m) &\equiv a_1^2 a_3^{2m-1} s \quad \mod (2, v_1^3), \\ D(x_2^{m2^{n-2}}) &\equiv a_1^{2^{n+1}} u^{3m2^n - 2^{n+1}} \quad \mod (2, v_1^{2^{n+1}+1}). \end{aligned}$$

We deduce the following.

Lemma 4.9.3 *The v_1–BSS differentials in (4.6.1) from the $(s=0)$–line to the $(s=1)$–line are given by*

$$\begin{aligned} d_1\left(\frac{a_3^m}{v_1^j}\right) &= \frac{a_3^{m-1}h_2}{v_1^{j-1}}, \\ d_2\left(\frac{a_3^{2m}}{v_1^j}\right) &= \frac{a_3^{2m-1}h_1}{v_1^{j-2}}, \\ d_{2^{n+1}}\left(\frac{a_3^{m2^n}}{v_1^j}\right) &= \frac{u^{3m2^n-2^{n+1}}}{v_1^{j-2^{n+1}}}, \end{aligned}$$

where m is odd.

Corollary 4.9.4 *The groups $H^{0,*}(M_1^1 C^*_{\mathrm{tot}}(Q(5)))$ are generated by the elements*

$$\begin{aligned} &1/v_1^j, && j \geq 1, \\ &\frac{a_3^{m2^n}}{v_1^j}, && m \text{ odd and } j \leq \begin{cases} 1, & n = 0, \\ 2, & n = 1, \\ 2^{n+1}, & n \geq 2. \end{cases} \end{aligned}$$

In particular, the map

$$\mathrm{Ext}^{0,*}(M_1^1 \mathrm{BP}_*) \to H^{0,*}(M_1^1 C^*_{\mathrm{tot}}(Q(5)))$$

is not *an isomorphism.*

5 Low-dimensional computations

In this section we explore the 2–primary homotopy $\pi_* Q(3)$ and $\pi_* Q(5)$ for $0 \leq * < 48$ (everything is implicitly 2–localized). In the case of $Q(3)$, Mark Mahowald has done similar computations, over a much vaster range, for the closely related Goerss–Henn–Mahowald–Rezk conjectural resolution of the 2–primary $K(2)$–local sphere — there is definitely some overlap here. In the case of $Q(5)$ the computations represent some genuinely unexplored territory, and give evidence that $Q(5)$ may detect more non-β–family v_2–periodic homotopy than $Q(3)$.

We do these low-dimensional computations in the most simple-minded manner, by computing the Bousfield–Kan spectral sequence

$$E_1^{s,t}(Q(\ell)) \Rightarrow \pi_{t-s}Q(\ell)$$

with

$$E_1^{s,t} = \begin{cases} \pi_t \,\mathrm{TMF}, & s=0, \\ \pi_t \,\mathrm{TMF}_0(\ell) \oplus \pi_t \,\mathrm{TMF}, & s=1, \\ \pi_t \,\mathrm{TMF}_0(\ell), & s=2. \end{cases}$$

Actually, as the periodic versions of TMF typically have π_t of infinite rank, we only compute a certain "connective cover" of the spectral sequence — we only include holomorphic modular forms in this low-dimensional computation (ie we do not invert Δ). Thus we are only computing a portion of the spectral sequence, which we shall refer to as the *holomorphic summand*. Note that the authors are not claiming that there exists a bounded-below version of $Q(\ell)$ whose homotopy groups the holomorphic summand converges to (it remains an interesting open question how such connective versions of $Q(\ell)$ could be obtained by extending the semi-cosimplicial complex over the cusps). Indeed, recent advances by Hill and Lawson [13] may produce such a bounded-below $Q(\ell)$–spectrum, but we do not pursue this possibility here.

In the following calculations, we employ a leading term algorithm, which basically amounts to only computing the leading terms of the differentials in row echelon form. Similarly to the previous section, we write everything 2–adically, and employ a lexicographical ordering on monomials

$$2^i v_1^j x.$$

Namely, we say that $2^i v_1^j x$ is *lower* than $2^{i'} v_1^{j'} x'$ if $i < i'$, or if $i = i'$ and $j < j'$. We will write "leading term" differentials: the expression

$$x \mapsto y$$

indicates that

$$d_r(x + \text{higher terms}) = y + \text{higher terms}.$$

5.1 The case of $Q(3)$

In the case of $\mathrm{TMF}_0(3)$, recall that the modular forms for $\Gamma_0(3)$ are spanned by those monomials $a_1^i a_3^j$ in $\mathbb{Z}[\frac{1}{3}, a_1, a_3]$ with $i+j$ even. In this section we will refer to a_1 as v_1 and a_3 as v_2, because that is what they correspond to under the complex orientation.

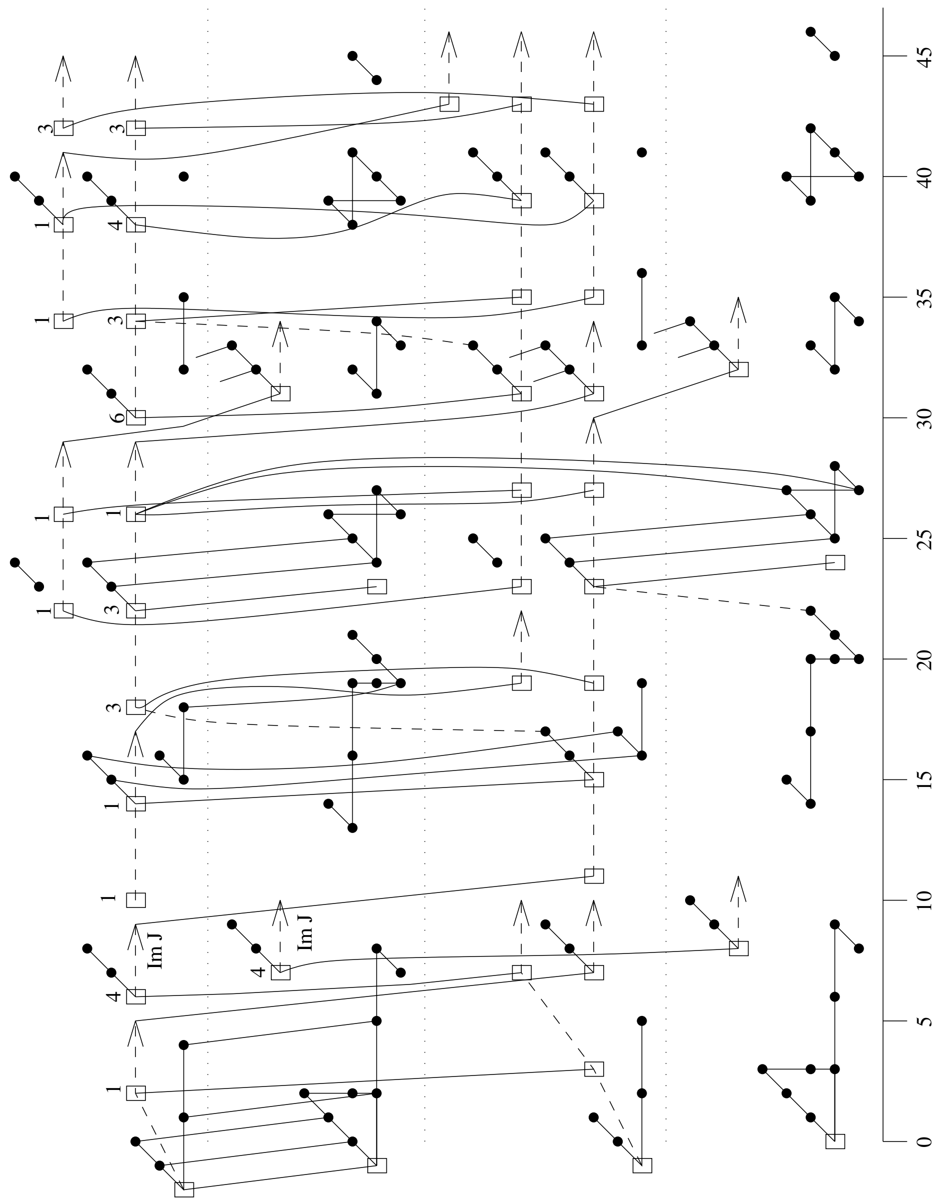

Figure 10: The holomorphic summand of the spectral sequence $E_r^{s,t}(Q(3))$ in low degrees

Figure 10 shows a low-dimensional portion of the holomorphic summand of the spectral sequence $E_r^{s,t}(Q(3))$. There are many aspects of this chart that deserve explanation/remark.

- The copies of $\pi_*\operatorname{TMF}$ and $\pi_*\operatorname{TMF}_0(3)$ are separated by dotted lines. The bottom pattern is the $s=0$ line of the spectral sequence ($\pi_*\operatorname{TMF}$). The next pattern up is the $\pi_*\operatorname{TMF}_0(3)$ summand of the $s=1$ line, followed by the $\pi_*\operatorname{TMF}$ summand of the $s=1$ line. The top pattern is the $s=2$ line of the spectral sequence ($\pi_*\operatorname{TMF}_0(3)$). The spectral sequence is Adams-indexed, with the x–axis corresponding to the coordinate $t-s$.

- Dots indicate $\mathbb{Z}/2$. Boxes indicate $\mathbb{Z}_{(2)}$. The solid lines between the dots indicate 2–extensions, and η and ν multiplication.

- Horizontal dashed lines denote *bo*–patterns. Arrows indicate the *bo* patterns continue.

- There are two *bo*–patterns which are denoted "Im J". These *bo*–patterns (together with the *bo*–patterns which hit them with differentials) combine to form Im J patterns.

- Differentials are indicated with vertical curvy lines. All differentials displayed only indicate the leading terms of the differentials, as explained in the beginning of this section. For example, the d_1 differential from the 1–line to the 2–line showing
$$v_1^2v_2^2 \mapsto 2v_2^2v_1^2$$
actually corresponds to a differential
$$d_1(v_1^2v_2^2+v_1^5v_2)=2v_2^2v_1^2+\text{higher terms.}$$
The differentials on the torsion-free portions spanned by the modular forms are computed using the Mahowald–Rezk formulas.

- Differentials on the torsion summand can often be computed by noting that the maps f, t, q and ψ^3 that define the coface maps of the semi-cosimplicial spectrum $Q(3)^\bullet$ are all maps of ring spectra, and in particular all have the same effect on elements in the Hurewicz image. There are a few notable exceptions, which we explain below.

- Dashed lines between layers indicate hidden extensions. These (probably) do not represent all hidden extensions: there are several possible hidden extensions which we have not resolved.

- The differentials supported by the non-Hurewicz classes x and ηx in $\pi_{17}\operatorname{TMF}_0(3)$ and $\pi_{18}\operatorname{TMF}_0(3)$ are deduced because they kill the Hurewicz image of $\beta_{4/4}\eta$ and $\beta_{4/4}\eta^2$, which are zero in π_*S.

- The d_2–differentials are computed by observing that there is a (zero) hidden extension $\eta^3v_1^6v_2^2[1]=4v_1^5v_2^3[2]$ (where $[s]$ means s–line).

$v_1^2 \mapsto 2v_1^2$		
$v_1 v_2 \mapsto v_1^4$	$v_1^4 \mapsto 16 v_1 v_2$	
$v_2^2 \mapsto 2v_2^2$		
$v_1^2 v_2^2 \mapsto 2v_2^2 v_1^2$		
$v_1 v_2^3 \mapsto v_2^2 v_1^4$	$v_1^4 v_2^2 \mapsto 8 v_2^3 v_1$	
$8\Delta \mapsto 8 v_2^3 v_1^3$	$v_2^4 \mapsto 2v_2^4$	$v_1^4 v_2^2 \mapsto 8 v_2^3 v_1$
$v_2^4 \mapsto 2v_2^4$		
$v_2^4 v_1^2 \mapsto 2v_2^4 v_1^2$	$v_1^8 v_2^2 \mapsto 8 v_2^3 v_1^5$	
$c_4 \Delta \mapsto v_2^4 v_1^4$	$v_2^5 v_1 \mapsto v_2^3 v_1^7$	$v_1^4 v_2^4 \mapsto 64 v_2^5 v_1$
$v_2^4 v_1^6 \mapsto 8 v_2^5 v_1^3$	$v_2^6 \mapsto 2v_2^6$	
$v_2^4 v_1^8 \mapsto 16 v_2^5 v_1^5$	$v_2^6 v_1^2 \mapsto 2 v_2^6 v_1^2$	
$v_2^7 v_1 \mapsto v_2^6 v_1^4$	$v_2^6 v_1^4 \mapsto 8 v_2^7 v_1$	$v_1^{10} v_2^4 \mapsto 8 v_2^5 v_1^7$

Table 1: Leading terms of d_1 differentials between torsion-free classes on the 1– and 2–lines of the spectral sequence

- Up to the natural deviations introduced by computing with the Bousfield–Kan spectral sequence, and not the Adams–Novikov spectral sequence, the divided β–family is faithfully reproduced on the 2–line with the exception of the additional copy of Im J (there in fact should be infinitely many copies of such Im J summands) and one peculiar abnormality: the element $\beta_{8/8}$, detected by $32 v_1 v_2^5$, is 32–divisible. This extra divisibility does not contradict the results of Section 4 — the results there pertain to the monochromatic layer $M_2 Q(3)$, and not $Q(3)$ directly.

- Boxes which are targets of differentials are labeled with numbers. A number n above a box indicates that after all differentials are run, you are left with a $\mathbb{Z}/2^n$.

- It is interesting to note that the permanent cycles on the zero line in this range are exactly the image of the TMF–Hurewicz homomorphism.

We did not label the modular forms generating the boxes in the spectral sequence. In the case of π_* TMF, the dimensions resolve this ambiguity. The remaining ambiguity is resolved by Table 1, which indicates all of the leading terms of d_1 differentials between torsion-free classes on the 1– and 2–lines of the spectral sequence.

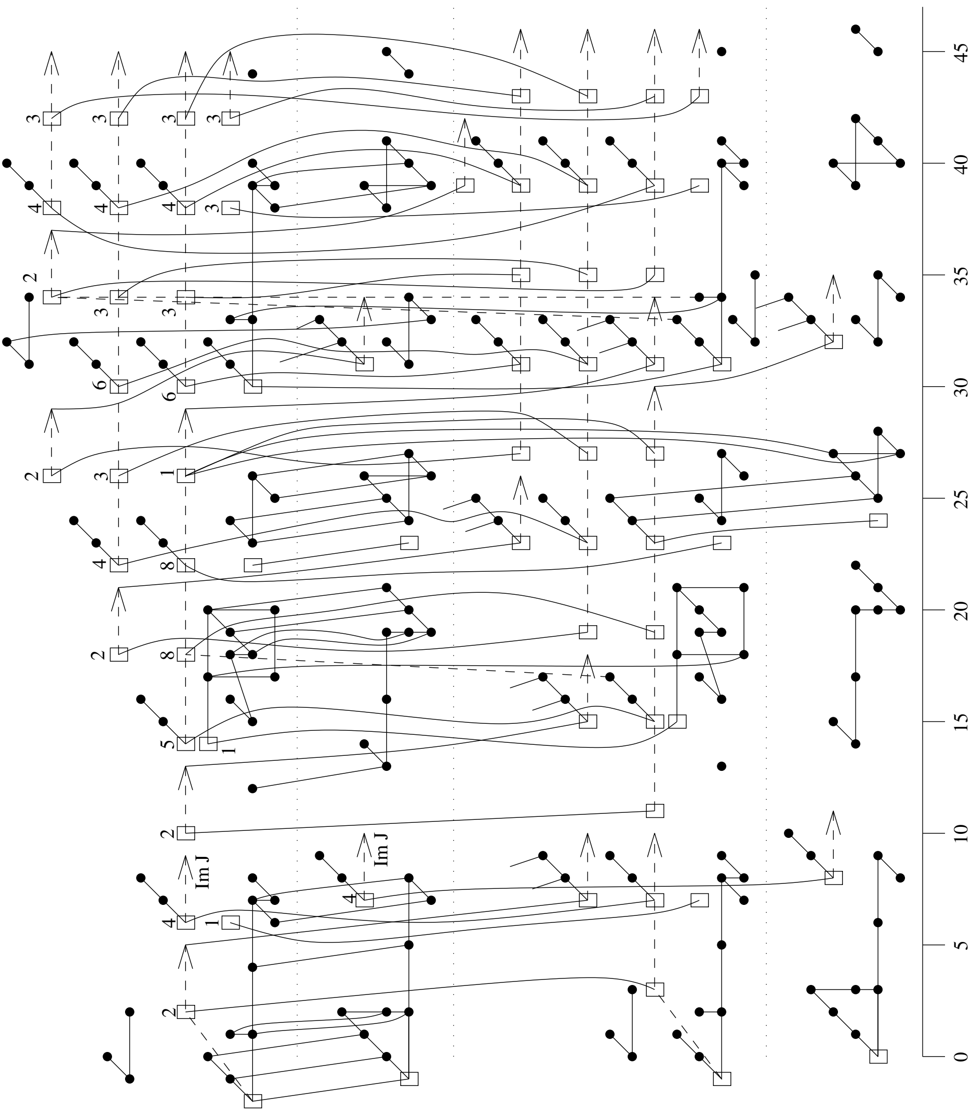

Figure 11: The holomorphic summand of the spectral sequence $E_r^{s,t}(Q(5))$ in low degrees

5.2 The case of $Q(5)$

Figure 11 displays the spectral sequence for $Q(5)$. Essentially all of the conventions and remarks for the $Q(3)$ computation above extend to the $Q(5)$ computation. Table 2 contains the leading terms of differentials from the torsion-free elements in the 1–line to those in the 2–line.

$b_2 \mapsto 4b_2$				
$b_4 \mapsto b_2^2$	$\delta \mapsto 2\delta$	$b_2^2 \mapsto 16b_4$		
$b_2\delta \mapsto 4b_2\delta$				
$b_4\delta \mapsto b_2^2\delta$	$\delta^2 \mapsto 2\delta^2$	$b_2^2\delta \mapsto 32b_4\delta$		
$b_2\delta^2 \mapsto 4\delta^2 b_2$	$b_2^3\delta \mapsto 8b_2b_4\delta$			
$8\Delta \mapsto 8\delta^3$	$b_4\delta^2 \mapsto b_2^2\delta^2$	$4\delta^3 \mapsto 8b_2^2b_4\delta$	$b_2^2\delta^2 \mapsto 16b_4\delta^2$	
$b_2^5\delta \mapsto 8b_4\delta b_2^3$	$b_2\delta^3 \mapsto 4b_2\delta^3$	$b_2^3\delta^2 \mapsto 8b_4\delta^2 b_2$		
$c_4\Delta \mapsto b_2^2\delta^3$	$\delta^4 \mapsto 2\delta^4$	$b_4\delta^3 \mapsto b_2^4 b_4\delta$	$b_2^2\delta^3 \mapsto 64b_4\delta^3$	$b_2^4\delta^2 \mapsto 64b_2^2b_4\delta^2$
$b_2\delta^4 \mapsto 4b_2\delta^4$	$b_2^3\delta^3 \mapsto 8b_2b_4\delta^3$	$b_2^5\delta^2 \mapsto 8b_2^3b_4\delta^2$		
$b_4\delta^4 \mapsto b_2^2\delta^4$	$4\delta^5 \mapsto 8\delta^5$	$b_2^4\delta^3 \mapsto 16b_2^2b_4\delta^3$	$b_2^2\delta^4 \mapsto 16b_4\delta^4$	$b_2^6\delta^2 \mapsto 16b_4\delta^2 b_2^4$
$b_2\delta^5 \mapsto 4b_2b_4\delta^4$	$b_2^3\delta^4 \mapsto 8b_2\delta^5$	$b_2^5\delta^3 \mapsto 8\delta^3 b_4 b_2^6$	$b_2^7\delta^2 \mapsto 8b_2^5b_4\delta^2$	

Table 2: Leading terms of differentials from the torsion-free elements in the 1–line to those in the 2–line

We make the following remarks:

- The 2–line now bears little resemblance to the divided β–family. This is in sharp contrast with the situation with $Q(3)$. This fits well with our premise that while $Q(3)$ reproduces the divided β–family almost flawlessly, $Q(5)$ does not.

- The much more robust torsion in $\pi_* \mathrm{TMF}_0(5)$ gives a significant source of homotopy in $\pi_* Q(5)$ which does not appear in $\pi_* Q(3)$. In particular, the elements

$$\nu\delta^4, \quad \nu^2\delta^4, \quad \epsilon\delta^4$$

seem like candidates to detect the elements in $\pi_* S$ with Adams spectral sequence names

$$h_5h_2^2, \quad h_5h_2^3, \quad h_5h_3h_1,$$

though the ambiguity resulting from the leading term algorithm makes it difficult to resolve this in the affirmative. These classes are *not* seen by $Q(3)$.

- Just as in the case of $Q(3)$, the permanent cycles on the zero line in this range are exactly the image of the TMF–Hurewicz homomorphism.

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The number of strings on essential tangle decompositions of a knot can be unbounded

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We construct an infinite collection of knots with the property that any knot in this family has n–string essential tangle decompositions for arbitrarily high n.

57M25, 57N10

1 Introduction

An *n–string tangle* $(B, \mathcal{T})$ is a ball B together with collection of n disjoint arcs $\mathcal{T}$ properly embedded in B, for $n \in \mathbb{N}$. We say that $(B, \mathcal{T})$ is *essential* if n is 1 and its arc is knotted,[1] or if n is bigger than 1 and there is no properly embedded disk in B disjoint from $\mathcal{T}$ and separating the components of $\mathcal{T}$ in B. Otherwise, we say that the tangle is *inessential*. (See Figure 1 for examples.)

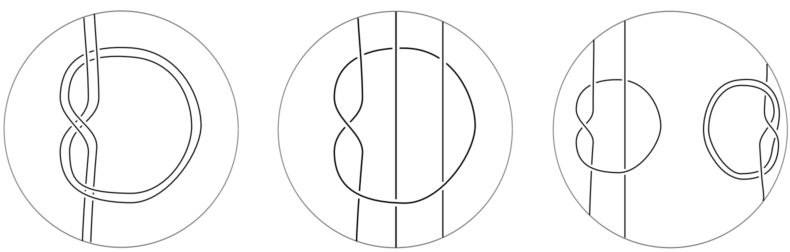

Figure 1: Examples of essential tangles (left and middle), and an inessential tangle (right)

Let K be a knot in S^3 and S a 2–sphere in general position with K. Each ball bounded by S in S^3 intersects K in the same number n of arcs. So these balls together with the arcs of intersection with K are n–string tangles. In this case, we say that S defines a *n–string tangle decomposition* of K, and if both tangles are essential we say that the tangle decomposition of K defined by S is *essential*. A knot is composite if and only if it has a 1–string essential tangle decomposition; otherwise

[1] An arc of $\mathcal{T}$ is *unknotted* if it cobounds a disk embedded in B together with an arc in ∂B; otherwise, it is said to be *knotted*.

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the knot is prime. Note also that S defines an essential tangle decomposition for K if and only if the intersection of S with the exterior of K, $E(K)$,[2] is an essential surface in $E(K)$; see Definition 3.

A tangle decomposition of a knot is natural and has been relevant for knot theory and its applications. The concept of a "tangle" was first used in the work of Conway [3], where he defines and classifies (2–string) rational tangles and uses it as an instrument to list knots. The concept of an essential tangle was first used in [8], where Kirby and Lickorish prove that any knot is concordant to a prime knot. They actually define *prime tangle*, that is an essential tangle with no local knots.[3] Another example is the work of Lickorish in [9], where he proves, for instance, that if a knot has a 2–string prime tangle decomposition, then the knot is prime. Tangles are also used in applied mathematics to study the DNA topology. The paper [2] by Buck surveys the subject concisely and also explains how tangles are useful to the study of the topological properties of DNA, an application pioneered by Ernst and Sumners in [5].

This paper addresses the question of if the number of strings on essential tangle decompositions of a fixed knot is bounded. There are results showing some evidence for this to be true. For instance, knots with no closed essential surfaces (see Culler, Gordon, Luecke and Shalen [4]), tunnel number one knots (see Gordon and Reid [6]) and free genus-one knots (see Matsuda and Ozawa [10]) have no essential tangle decompositions. There also are knots with a unique essential tangle decomposition; see Ozama [12]. Furthermore, in Proposition 2.1 of [11], Mizuma and Tsutsumi proved that, for a given knot, the number of strings in essential tangle decompositions, without parallel strings,[4] is bounded. The proof of this result allows a more general statement. That is, the number of strings that are not parallel to other strings in an essential tangle decomposition of a fixed knot is bounded. So, from this flow of results and intuition on essential tangle decompositions, the following theorem and its corollary are surprising.

Theorem 1 *There is an infinite collection of prime knots such that, for all $n \geq 2$, each knot has a n–string essential tangle decomposition.*

Corollary 2 *There is an infinite collection of knots such that, for all $n \geq 1$, each knot has a n–string essential tangle decomposition.*

[2] We denote by $E(K)$ the *exterior* of a knot K, that is, $S^3 - \operatorname{int} N(K)$, where $N(K)$ is a regular neighborhood of K.

[3] A tangle $(B, \mathcal{T})$ has no local knots if any 2–sphere intersecting $\mathcal{T}$ transversely in two points bounds a ball in B meeting $\mathcal{T}$ in an unknotted arc.

[4] Two strings of a tangle in a ball B are parallel if there is an embedded disk in B cobounded by these strings and two arcs in ∂B.

Essential surfaces are very important in the study of 3–manifold topology. And as observed above, to each n–string essential tangle decomposition of a knot corresponds a meridional essential surface in the exterior of the knot, with $2n$ boundary components. Therefore, from the results in this paper, there are knots with meridional planar essential surfaces in their exteriors with all possible numbers of boundary components. Furthermore, from Lemma 1.2 in Bleiler [1], the double cover of S^3 along these knots contains genus-g closed incompressible surfaces, meeting the fixed point set of the covering action in $2(g+1)$ points, and separating the double cover in irreducible and ∂–irreducible components, for all $g \geq 1$.

The reference used for standard definitions and results of knot theory is Rolfsen's book [13], and throughout this paper we work in the piecewise-linear category.

In Section 2, we show the existence of handlebody-knots (see Definition 4) with incompressible planar surfaces in their exteriors with b boundary components for all $b \geq 2$. In Section 3, we use these handlebody-knots to prove Theorem 1 and its corollary. The main techniques used are standard in 3–manifold topology. Throughout the paper, the number of connected components of a topological space X is denoted by $|X|$.

2 Meridional incompressible planar surfaces in handlebody-knots complements

To prove Theorem 1, we use the correspondence between n–string essential tangle decompositions of a knot and meridional planar essential surfaces in the knot exterior. We start by defining these surfaces.

Definition 3 A *planar surface* is a surface obtained from a 2–sphere by removing the interior of a finite number of disks.

Let H be a handlebody embedded in S^3.

A surface P properly embedded in $E(H) = S^3 - \operatorname{int} H$ is *meridional* if each boundary component of P bounds a disk in H.

An embedded disk D in $E(H)$ is a *compressing disk* for P if $D \cap P = \partial D$ and ∂D does not bound a disk in P. We say that P is *incompressible* if there is no compressing disk for P in $E(H)$.

An embedded disk D in $E(H)$ is a *boundary compressing disk* for P if $\partial D \cap P = \alpha$, with α a connected arc not cutting a disk from P, and $\partial D - \alpha = \beta$ a connected arc in ∂H. We say that P is *boundary incompressible* if there is no boundary compressing disk for P in $E(H)$.

The surface P is *essential* if it is incompressible and boundary incompressible.

In this section, we present handlebody-knots whose exteriors contain meridional incompressible planar surfaces with n boundary components for any $n \geq 2$. This embedding will later be used in the proof of Theorem 1. We consider next the definition of handlebody-knot.

Definition 4 A *handlebody-knot* of genus g in S^3 is an embedded handlebody of genus g in S^3. A *spine* γ of a handlebody-knot Γ is an embedded graph in S^3 with Γ as a regular neighborhood.

Let Γ be the genus-two handlebody-knot 4_1 from the list of [7], with spine γ, as in Figure 2. Consider also a collection of distinct knots C_i, for $i \in \mathbb{N}$, and C some other nontrivial knot. We work with γ as if defined by two vertices, two loops e_1 and e_2 (one for each vertex), and an edge e between the two vertices.

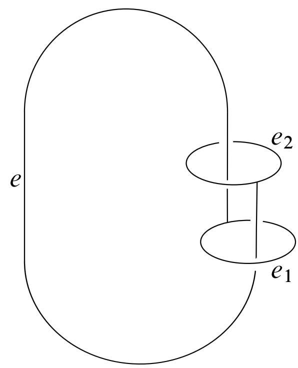

Figure 2: The spine γ of the handlebody-knot Γ, with labels of the two loops e_1 and e_2, and of the edge e

Consider two disjoint closed arcs a_1 and a_2 in e, as in Figure 3 (left). In this figure we also have represented an embedded 2–sphere S_2 in S^3 that intersects γ in e at two points, p_1 and p_2, and separates the arcs a_1 and a_2. Denote the ball bounded by S_2 containing a single component of e by $B_{2,1}$ and the other by $B_{2,2}$. Denote by l_1 and l_2 the components of $B_{2,2} \cap \gamma$ that contain e_1 and e_2, respectively, and note that l_j intersects S_2 at p_j, for $j = 1, 2$.

We perform an unusual connected sum operation between γ and the knots C and C_i along the arcs a_1 and a_2. That is, we take a ball in S^3 intersecting γ in a_1, and a ball in S^3 intersecting C_i at a single unknotted arc. A connected sum operation is obtained by removing both balls and gluing their boundaries through a homeomorphism in a way that the boundary points of a_1 are mapped to the boundary points of the chosen arc in C_i. A similar operation is obtained from the arc a_2 and C. From these operations

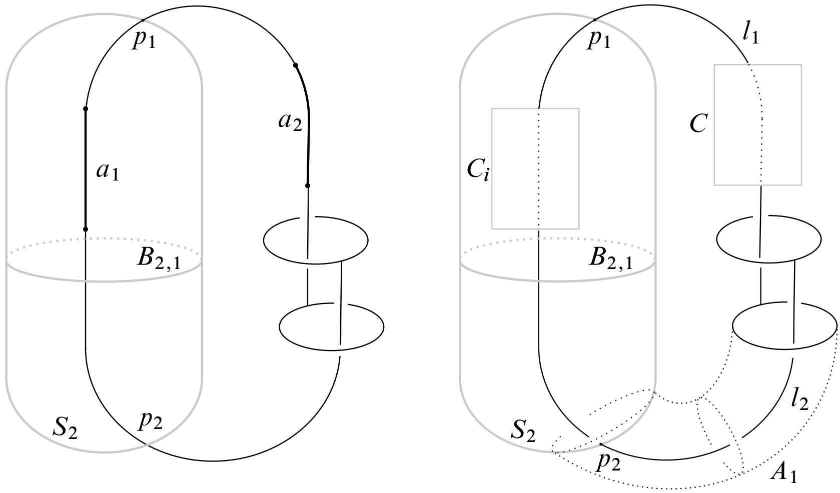

Figure 3: The arcs a_1 and a_2 in γ and the sphere S_2 (left); the spines γ_i of the handlebody-knots Γ_i and the annulus A_1 (right). Note that C_i and C label the pattern of the respective knots.

we get the handlebody-knots as represented schematically in Figure 3 (right), which we denote by Γ_i with a respective spine γ_i. For each handlebody-knot Γ_i we consider the swallow-follow torus X_i defined by the connected sum of C with C_i. A minimal JSJ–decomposition for the complement of Γ_i is defined by the torus X_i, cutting from $E(\Gamma_i)$ the exterior of $C_i \# C$, and a JSJ–decomposition of $E(C_i \# C)$. Also, the torus X_i cuts from $E(\Gamma_i)$ the only component obtained from the JSJ–decomposition containing the boundary of $E(\Gamma_i)$. Hence, from the unicity of minimal JSJ–decomposition of compact 3–manifolds, for any other minimal JSJ–decomposition of $E(\Gamma_i)$, the torus cutting the component with the boundary of $E(\Gamma_i)$ is isotopic to X_i. Consequently, if Γ_i is ambient isotopic to Γ_j for $i \neq j$, the torus X_i is isotopic to X_j, which means that $E(C_i \# C)$ is ambient isotopic to $E(C_j \# C)$. This is a contradiction with the torus $C_i \# C$ and $C_j \# C$ being distinct. Then, the handlebody-knots Γ_i are not ambient isotopic.

Both loops e_1 and e_2 cobound an embedded annulus in $B_{2,2}$, parallel to the component of e in $B_{2,2}$ each encircles, with interior disjoint from γ_i and intersecting S_2 in the other boundary component. Consider such an annulus with a boundary component in e_1, denoted A_1, as it is illustrated in Figure 3 (right). We proceed with an isotopy of γ_i along A_1, taking l_1 passing through S_2, and we obtain γ_i as in Figure 4 (left). We refer to this isotopy as an *annulus isotopy* of γ_i. After this isotopy we denote S_2 by S_3, considering its relative position with Γ_i, and the respective balls it bounds by $B_{3,1}$ and $B_{3,2}$. We assume that l_1 intersects S_3 at p_1. Note that all intersections

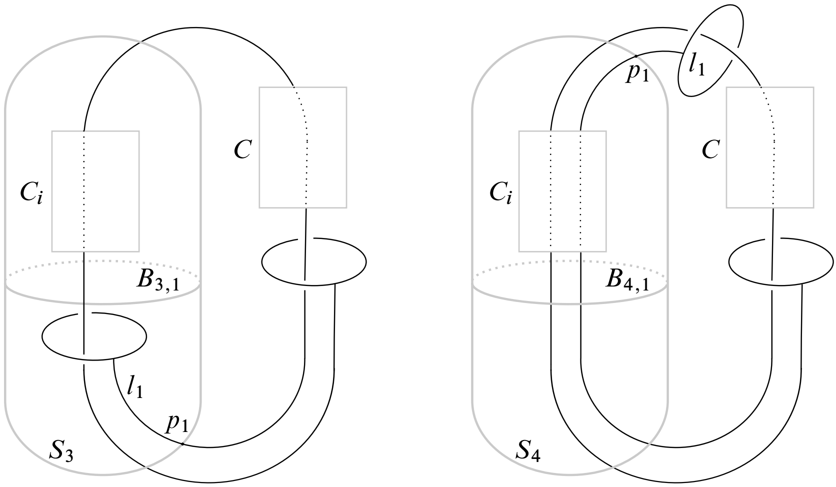

Figure 4: The spine γ_i after one (left), and two (right), annulus isotopies, and the spheres S_3 and S_4

of γ_i and S_3 are in the arc of e between p_1 and p_2. Again, we consider an embedded annulus A_2 in $B_{3,1}$, cobounded by e_1 and its intersection with S_3, parallel to the component of $e \cap B_{3,1}$ disjoint from e_1 and in the direction of the local knot C_i, following its pattern. By an annulus isotopy of γ_i along A_2 taking l_1 passing through S_3, we obtain γ_i as in Figure 4 (right). After this isotopy, we denote S_3 by S_4, considering its relative position with Γ_i, and the respective balls it bounds by $B_{4,1}$ and $B_{4,2}$. The ball $B_{4,1}$ intersects γ_i in two parallel arcs, and we still assume that $l_1 \cap S_4$ is p_1. Note again that all intersections of γ_i and S_4 are in the arc of e between p_1 and p_2.

For a canonical position, we isotope e_1 along the component of $e \cap B_{4,2}$, disjoint from e_1 and e_2, encircling l_2; see Figure 5 (left). We can now continue the previous process. Consider again an annulus A_3 in $B_{4,2}$, cobounded by e_1 and its intersection with S_4, parallel to the components of $e \cap B_{4,2}$ other than l_1, and in the opposite direction of the local knot C. By an annulus isotopy of γ_i along A_3, taking l_1 passing through S_4, we obtain γ_i as in Figure 5 (right). After this isotopy, we denote S_4 by S_5, considering its relative position with Γ_i, and we denote the balls it bounds by $B_{5,1}$ and $B_{5,2}$. Again, l_1 intersects S_5 at p_1, and all intersections of S_5 with γ_i are in the arc of e between p_1 and p_2. For the next step, proceed with an annulus isotopy along an annulus A_4 in $B_{5,1}$ cobounded by e_1, parallel to the components of $e \cap B_{5,1}$ disjoint from e_1, in the direction of the local knot C_i, following its pattern.

After $2(k-1)$ (for $k = 1, 2, \dots$) annulus isotopies as the ones explained above, we get γ_i as in Figure 6 (left). From S_2, we obtain S_{2k} and the balls it bounds, $B_{2k,1}$ and $B_{2k,2}$. The ball $B_{2k,1}$ intersects γ_i in k parallel arcs with the pattern of C_i, and

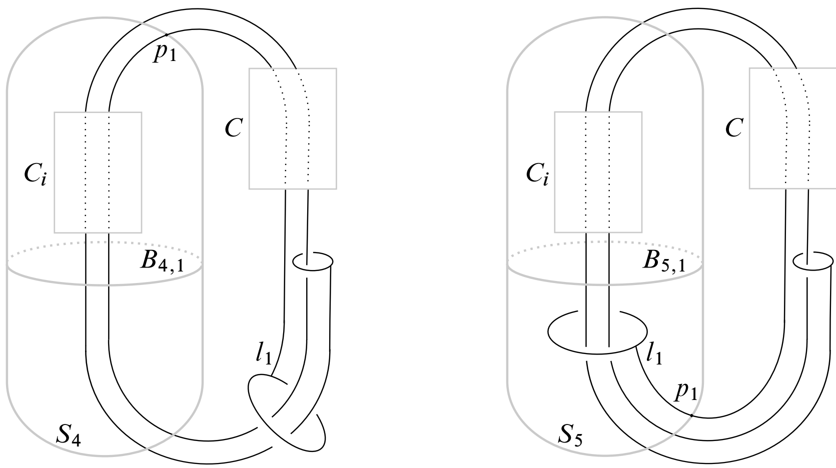

Figure 5: The spine γ_i of Figure 4 (left) in a canonical position (left), and γ_i after another annulus isotopy (right)

the ball $B_{2k,2}$ intersects γ_i in $k-2$ parallel arcs with the pattern of C, another arc with the pattern of C encircled by l_2, and l_1 that encircles all these other components.

After $2k-1$ (for $k=1,2,\dots$) annulus isotopies, we obtain γ_i as in Figure 6 (right). From S_2, we obtain S_{2k+1} and the balls it bounds, $B_{2k+1,1}$ and $B_{2k+1,2}$. The ball $B_{2k+1,1}$ intersects γ_i in k parallel arcs with the pattern of C_i and l_1 encircling these arcs, and the ball $B_{2k+1,2}$ intersects γ_i in $k-1$ parallel arcs with the pattern of C, together with another arc with the pattern of C and l_2 which encircles this arc.

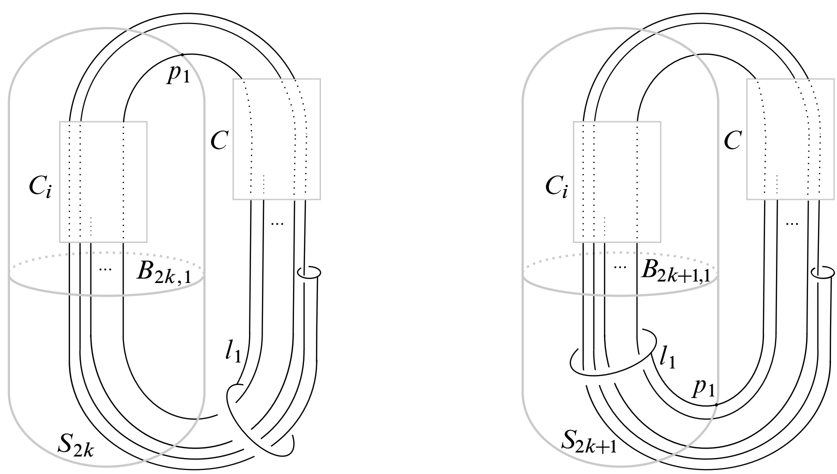

Figure 6: The spine γ_i after an even number (left), and an odd number (right), of annulus isotopies, and the corresponding spheres S_{2k} and S_{2k+1}, $k \in \mathbb{N}$

Note after each isotopy we assume that l_j intersects S_n, for $n = 2, 3, \dots,$ in p_j and that all points of $S_n \cap \gamma_i$ are in the arc between p_1 and p_2 in e.

We denote $S^3 - \operatorname{int} \Gamma_i$ by $E(\Gamma_i)$, and $S^3 - \gamma_i$ by $E(\gamma_i)$. Let Q_n, for $n = 2, 3, \dots,$ be the intersection of S_n with $E(\Gamma_i)$ in S^3.

Lemma 5 *The surface Q_n is incompressible in $E(\Gamma_i)$.*

Proof As Γ_i is a regular neighborhood of γ_i, if Q_n is compressible in $E(\Gamma_i)$, then S_n is compressible in $E(\gamma_i)$. Hence it suffices to prove that S_n is incompressible in $E(\gamma_i)$.

Case 1 Suppose n is even. Then S_n is as in Figure 6 (left).

(i) In this case, the ball $B_{n,1}$ intersects γ_i in a collection of $k = n/2$ parallel knotted arcs. Then $(B_{n,1}, B_{n,1} \cap \gamma_i)$ is an essential tangle. In fact, suppose there is a compressing disk D for S_n in $B_{n,1} - (B_{n,1} \cap \gamma_i)$. Then D separates the arcs $B_{n,1} \cap \gamma_i$ into two collections. Let s_1 and s_2 be two arcs in $B_{n,1}$ which are separated by D. As s_1 and s_2 are parallel, there is a disk E with boundary $s_1 \cup s_2$ and two arcs, α_1 and α_2, in S_n, each with one end in s_1 and the other in s_2. Consider D and E in general position and suppose that $|D \cap E|$ is minimal. If D intersects E in simple closed curves or in arcs with both ends in α_1 or both in α_2, we can reduce $|D \cap E|$ by an innermost arc type of argument, which is a contradiction. Therefore, all arcs of $D \cap E$ have one end in α_1 and the other end in α_2. Hence both s_1 and s_2 are parallel to outermost arcs of $D \cap E$ in D, which implies that s_1 and s_2 are parallel to S_n. This is a contradiction because the arcs s_1 and s_2 are knotted by construction.

(ii) If $n \leq 4$, then the ball $B_{n,2}$ intersects γ_i in l_1 and l_2, and when $n = 4$, also in an arc encircled by both l_1 and l_2. In this case, if there is a compressing disk for S_n in $B_{n,2} - (B_{n,2} \cap \gamma_i)$ it separates a component l_1 or l_2 from the other components. This implies that e_1 or e_2 bound a disk in the complement of γ_i, which is a contradiction with Γ_i being a knotted handlebody-knot. Otherwise, suppose that $n > 4$. Thus $B_{n,2}$ intersects γ_i in $(n/2) - 2$ parallel arcs with the pattern of C, another arc with the pattern of C encircled by l_2, and the component l_1 that encircles the arc encircled by l_1 and the $(n/2) - 2$ parallel arcs. With exception to l_1 and l_2, all other arcs are parallel as properly embedded arcs in $B_{n,2}$. Thus if a compressing disk for S_n in $B_{n,2} - (B_{n,2} \cap \gamma_i)$ separates these arcs, following an argument as in Case 1(i) we have a contradiction with these arcs being knotted. Therefore, a compressing disk for S_n in $B_{n,2} - (B_{n,2} \cap \gamma_i)$ separates a single component l_1 or l_2 from all the other components, or it separates both components l_1 and l_2 from the other parallel arcs. As e_1 bounds a disk disjoint from l_2, in both cases e_1 bounds a disk in the complement of γ_i, which is a contradiction with Γ_i being a knotted handlebody-knot.

Case 2 Suppose now that n is odd. Then S_n is as in Figure 6 (right).

(i) The ball $B_{n,1}$ intersects γ_i in a collection of $(n-1)/2$ parallel arcs and l_1 which encircles these arcs. If there is a compressing disk D of S_n in $B_{n,1} - (B_{n,1} \cap \gamma_i)$ separating the parallel arcs, following an argument as in Case 1(i) we have a contradiction with these arcs being knotted. If D separates the component l_1 from the other components, following an argument as in Case 1(ii) we have a contradiction with Γ_i being a knotted handlebody-knot.

(ii) If $n=3$, the ball $B_{n,2}$ intersects γ_i in an arc with pattern C and l_2 which encircles the arc. If there is a compressing disk for S_n in $B_{n,2} - (B_{n,2} \cap \gamma_i)$ in this case, then it separates the component l_2 from the arc with pattern C. From the same argument used in Case 1(ii), we have a contradiction with Γ_i being a knotted handlebody-knot. If $n > 3$, then the ball $B_{n,2}$ intersects γ_i in $(n-1)/2$ parallel arcs and l_2 which encircles one of the previous arcs. Without considering l_2, if a compressing disk for S_n in $B_{n,2} - (B_{n,2} \cap \gamma_i)$ separates the parallel arcs, then following an argument as in Case 1(i) we have a contradiction with the arcs being knotted. If S_n has a compressing disk in $B_{n,2} - (B_{n,2} \cap \gamma_i)$, then this disk isolates the component l_2 from the other components, and following the argument as in Case 1(ii) we have a contradiction with Γ_i being a knotted handlebody-knot. □

The surface Q_n is boundary compressible in $E(\Gamma_i)$ as there are boundary compressing disks over the regular neighborhoods of l_1 and l_2. However, our construction of the handlebody-knots Γ_i could have been made in such a way that the surfaces Q_n are incompressible and boundary incompressible in their complements. For that purpose, we could do a connected sum of γ_i with two knots along two arcs in e_1 and e_2. After this operation, there won't be boundary compressing disks of Q_n over the regular neighborhoods of l_1 and l_2 in $E(\Gamma_i)$. And as these are the only possible boundary compressing disks, because all other components $\gamma_i - \gamma_i \cap S_n$ correspond to knotted arcs in their respective balls, after these connected sums the surfaces Q_n would also be boundary incompressible in the complement of the handlebody-knots. But for the purpose of this paper, we will use the handlebody-knots Γ_i.

3 Knots with essential tangle decompositions with an arbitrarily high number of strings

In this section, we use the handlebody-knots Γ_i to construct infinitely many examples of knots with essential tangle decompositions for all numbers of strings.

Let N_1 and N_2 be torus knots in the boundary of the solid tori T_1 and T_2 (that we assume to be in different copies of S^3). Consider a regular neighborhood B_i of an arc of N_i intersecting T_i at a ball, for $i=1,2$. We isotope B_i and $B_i \cap N_i$ away from the interior of T_i such that B_i intersects T_i at a disk, for $i=1,2$. We proceed with a connected sum of N_1 and N_2 by removing the interior of B_1 and attaching the exterior of B_2 in such a way that the disks $B_1 \cap T_1$ and $B_2 \cap T_2$ are identified. Hence the knot $N_1 \# N_2$, denoted by K, is in the boundary of a genus-two handlebody H, obtained by gluing T_1 and T_2 along a disk in their boundaries. We denote the identification disk of T_1 and T_2 in H by D. In Figure 7, we have the example of this connected sum with two trefoils, that we will use as reference for the remainder of the paper.

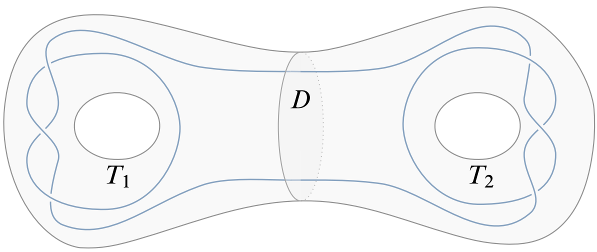

Figure 7: The handlebody H with the connected sum of two trefoil knots

Consider disks D_1 and D_2 parallel to D in H, such that the cylinder $C_{1,2}$ cut by $D_1 \cup D_2$ from H intersects K in two parallel arcs, each with one end in D_1 and the other in D_2. We also keep denoting by T_1 and T_2 the solid tori cut from H by D_1 and D_2, respectively; see Figure 8. Let s be a spine of H that intersects $C_{1,2}$ in a single arc. We denote by d_i the point $D_i \cap s$, and by t_i the intersection of s with T_i, for $i=1,2$.

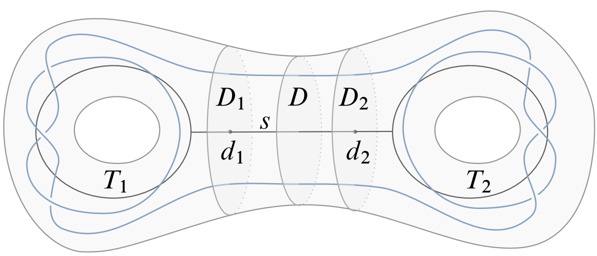

Figure 8: The handlebody H and the spine s with the connected sum of two trefoil knots

We now embed the knot K in Γ_i as follows. Consider an embedding h_i of H in S^3 taking H homeomorphically to Γ_i, such that $h_i(s)=\gamma_i$, $h_i(d_j)=p_j$, $h_i(t_j)=l_j$ and also that $h_i(T_j)=L_j$, for $j=1,2$.

Proof of Theorem 1 Denote by K_i the knots $h_i(K)$, $i \in \mathbb{N}$, for a fixed knot K. To prove that the handlebody-knots Γ_i are distinct, let X_i be the torus cutting from

$E(K_i)$ the exterior of $C_i \# C$. The component cut by X_i from $E(K_i)$ containing the boundary torus is the same for every knot K_i. Hence, from the unicity of minimal JSJ–decomposition of compact 3–manifolds, if two knots K_i and K_j are ambient isotopic, the tori X_i and X_j are also ambient isotopic, contradicting $C_i \# C$ and $C_j \# C$ being distinct. Thus the knots K_i define a collection of distinct knots.

To prove the statement of the theorem, we will show that the spheres S_n, for $n \geq 2$, define n–string essential tangle decomposition for the knots K_i, and that these knots are prime.

We start by proving that S_n defines an n–string essential tangle decomposition of K_i. Let $E(K_i)$ be the exterior of K_i in S^3; that is, $E(K_i) = S^3 - \operatorname{int} N(K_i)$. Let P_n be the intersection of S_n with $E(K_i)$ for a fixed n. To prove that S_n defines an essential tangle decomposition for K_i, we need to prove that P_n is essential in $E(K_i)$, ie that P_n is incompressible and boundary incompressible.

First, we observe that P_n is boundary incompressible. In fact, as the strings of $K \cap B_{n,i}$ in $B_{n,i}$ are knotted for $i = 1, 2$, there is no boundary compressing disk for P_n in $E(K_i)$.

Now we prove that P_n is incompressible in $E(K_i)$. Let Δ_j, for $j = 1, \dots, n$, be the disks of intersection between Γ_i and S_n with $\Delta_1 = L_1 \cap S_n$ and $\Delta_n = L_2 \cap S_n$. Denote by $C_{j,j+1}$ the cylinder cut by $\Delta_j \cup \Delta_{j+1}$ from Γ_i. Denote also by $\partial^* C_{j,j+1}$ the annulus $C_{j,j+1} \cap \partial \Gamma_i$; that is, $\partial^* C_{j,j+1} = \partial C_{j,j+1} - (\Delta_j \cup \Delta_{j+1})$. Note that $C_{j,j+1} \cap K$ is a collection of two arcs parallel to $\partial^* C_{j,j+1}$, each with one end in Δ_j and the other in Δ_{j+1}. We also let $\partial^* L_1$ and $\partial^* L_2$ denote $\partial L_1 - \Delta_1$ and $\partial L_2 - \Delta_n$. Furthermore, we denote by s_j the string component of the tangle decomposition of K_i defined by S_n, in L_j, for $j = 1, 2$. Note that s_j is parallel to $\partial^* L_j$. We isotope s_j into $\partial^* L_j$ and denote the annulus $\partial^* L_j \cap E(K_i)$ by Λ_j.

Suppose that P_n is compressible in $E(K_i)$ with D a compressing disk, properly embedded in $B_{n,1}$ or $B_{n,2}$, in general position with Γ_i. If D is disjoint from Γ_i, we have a contradiction with Lemma 5. In this way, we assume that D intersects Γ_i and that $|D \cap \partial \Gamma_i|$ is minimal over all isotopy classes of compressing disks of P_n in $E(K_i)$.

In particular, assume that D intersects an annulus $\partial^* C_{j,j+1}$. If $D \cap \bigcup_{j=1}^{n-1} \partial^* C_{j,j+1}$ contains a simple closed curve or an arc with both ends in the same disk of $\Gamma_i \cap S_n$, by considering an outermost one between such curves and arcs in $\partial^* C_{j,j+1}$, and by cutting and pasting along the disk it bounds or cobounds, we get a contradiction with the minimality of $|D \cap \partial \Gamma_i|$. Thus $D \cap \bigcup_{j=1}^{n-1} \partial^* C_{j,j+1}$ is a collection of arcs with ends in distinct disks of $\Gamma_i \cap S_n$. Consider an outermost arc of $D \cap \bigcup_{j=1}^{n-1} \partial^* C_{j,j+1}$ in D, say a, and without loss of generality, suppose it belongs to $\partial^* C_{j,j+1}$. The arc a

is parallel to a string of the tangle defined by S_n that is in $C_{j,j+1}$, which contradicts the fact that all strings of the tangle decomposition of K_i defined by S_n are knotted. Consequently, we can assume that $D \cap \bigcup_{j=1}^{n-1} \partial^* C_{j,j+1}$ is empty.

Then we are assuming that D intersects $\partial\Gamma_i$ at $\partial^* L_1$ or $\partial^* L_2$, or more precisely, at Λ_1 or Λ_2. We denote by a_j and a'_j the arcs of $\partial\Lambda_j$ parallel to s_j in $\partial^* L_j$, and by b_j and b'_j the arcs cut by ∂a_j and $\partial a'_j$, respectively, in the boundary of $\partial^* L_j$. The boundary components of Λ_j are $a_j \cup b_j$ and $a'_j \cup b'_j$. Note that, as $D \cap s_j$ is empty, the disk D is disjoint from a_j and a'_j. Note also that $a_j \cup b_j$ is a torus knot in the torus $\partial^* L_j \cup (S_n - L_j \cap S_n)$, denoted T'_j. If D intersects Λ_j in inessential simple closed curves or arcs with both ends in b_j or both ends in b'_j, then by cutting and pasting along a disk cut by such curve or arc, we have a contradiction with the minimality of $|D \cap \partial\Gamma_i|$. If D intersects Λ_j in an essential simple closed curve, then $a_j \cup b_j$ is parallel to a simple closed curve in D, which contradicts $a_j \cup b_j$ being knotted. Consequently, D intersects Λ_j in a collection of arcs, each with one end in b_j and the other in b'_j. Let O be an outermost disk in D cut by the arcs of $D \cap \Lambda_j$. Then O is a disk in a solid torus bounded by T'_j and intersects the torus knot $a_j \cup b_j$ in T'_j at a single point. As we are working in S^3, either O is parallel to T'_j or it is a meridian to a solid torus bounded by T_j. In either case, O intersects any torus knot in T'_j at least in two points, which contradicts O intersecting $a_j \cup b_j$ once.

Therefore, we have that P_n is essential in the complement of K_i, which ends the proof that S_n defines an n–string essential tangle decomposition of K_i.

Now we prove that the knots K_i are prime. From Theorem 1 of [1], if a knot has a 2–string prime tangle decomposition, that is, if the tangles are essential and with no local knots, then the knot is prime. We have that the knot K_i has a 2–string essential tangle decomposition defined by S_2. So to prove that it is prime, we just need to show that the tangle decomposition defined by S_2 has no local knots. The ball $B_{2,1}$ intersects K_i in two parallel arcs. Hence if there is a 2–sphere intersecting only one of the arcs at a single component, this component has to be unknotted. The ball $B_{2,2}$ intersects γ_i in l_1 and l_2; thus it intersects K_i at two strings each with the pattern of a torus knot. Note that even though the pattern of the knot C is in l_2, it does not affect the topological type of the string in L_2. Suppose the tangle in $B_{2,2}$ contains a local knot. That is, there is a ball Q intersecting only one of the strings, and at a knotted arc. As the torus knots are prime, this knotted arc contains the whole pattern of the string; that is, the intersections of Q and $B_{2,2}$ with this string are topologically the same. Therefore, as the strings in $B_{2,2}$ are parallel to the boundary of L_1 and L_2, and Q intersects only one of them, we have that Q contains either e_1 or e_2, or we can isotope e_1 and e_2 in such a way that Q contains either e_1 or e_2. But then, either e_1 or e_2 bound a disk in the complement of γ_i and, as in Case 1(ii) from the proof of Lemma 5, we have a contradiction with Γ_i being a knotted handlebody-knot. Consequently, the tangle decomposition defined by S_2 contains no local knots, and the knots K_i are prime. □

Corollary 2 is now an immediate consequence.

Proof of Corollary 2 In Theorem 1, we proved that the spheres S_n, for $n \geq 2$, define an n–string essential tangle decomposition for the knots K_i. Hence, considering the knots K_i connected sum with some other knot, we have infinitely many knots with n–string essential tangle decompositions for all $n \in \mathbb{N}$, as in the statement of this corollary. □

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The L^2–(co)homology of groups with hierarchies

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We study group actions on manifolds that admit hierarchies, which generalizes the idea of Haken n–manifolds introduced by Foozwell and Rubinstein. We show that these manifolds satisfy the Singer conjecture in dimensions $n \leq 4$. Our main application is to Coxeter groups whose Davis complexes are manifolds; we show that the natural action of these groups on the Davis complex has a hierarchy. Our second result is that the Singer conjecture is equivalent to the cocompact action dimension conjecture, which is a statement about all groups, not just fundamental groups of closed aspherical manifolds.

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Introduction

In his PhD thesis [10], Foozwell introduced Haken n–manifolds as a higher dimensional analogue of Haken 3–manifolds. Loosely speaking, these are closed n–manifolds that can be cut inductively along codimension-1 submanifolds to a disjoint union of n–balls. The exact definition is somewhat technical. The resulting sequence of manifolds is called a hierarchy. Foozwell and Rubinstein have explored many properties of these manifolds, in particular, they have shown [11; 12] that their universal covers are homeomorphic to $\mathbb{R}^n$ and their fundamental groups have solvable word problem. Both of these properties show that Haken n–manifolds are a special class of aspherical manifolds; see Davis [4] and Mess [18].

The classical Euler characteristic conjecture, attributed to Hopf, predicts the sign of the Euler characteristic of a closed aspherical $2n$–dimensional manifold M^{2n}: $(-1)^n\chi(M^{2n}) \geq 0$. In a special case of right-angled Coxeter group manifolds, this conjecture becomes a purely combinatorial statement about flag simplicial triangulations of $(2n-1)$–spheres, known as the Charney–Davis conjecture [3].

Another classical conjecture about aspherical manifolds, the Singer conjecture, predicts that the reduced L^2–homology of the universal cover vanishes except possibly in the middle dimension. Since one can use L^2–Betti numbers to compute χ, the Singer conjecture immediately implies the Euler characteristic conjecture.

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Edmonds [9] proved the Euler characteristic conjecture for closed Haken 4–manifolds by showing that it was equivalent to the Charney–Davis conjecture for 3–spheres, which holds true by a result of Davis and the first author [8], where the Singer conjecture for 4–dimensional right-angled Coxeter group manifolds is proved. The equivalence of the two conjectures was extended by Davis and Edmonds [6] to all even dimensions. In fact, they showed this equivalence for *generalized Haken $2n$–manifolds*, where they allow the hierarchy to end in any compact contractible manifold.

The starting point of this paper was a question of Edmonds whether the Singer conjecture holds for Haken 4–manifolds.

One advantage of studying homological properties of Haken n–manifolds is that we can ignore most of the technicalities and study a more general class of manifolds that is closer to the loose definition above. Since we are interested in group actions that are not free, and because we think it is simpler, we build the hierarchies out of contractible manifolds with a proper and cocompact group action.

We say a group G *admits a hierarchy* if it acts on a contractible manifold M that can be cut inductively along codimension-1 contractible G–invariant submanifolds to a disjoint union of compact contractible manifolds. An example to keep in mind is $\mathbb{Z}^n$ acting on $\mathbb{R}^n$ with quotient the n–torus T^n. Cutting T^n along T^{n-1} corresponds to cutting along $\mathbb{Z}^n$–translates of $\mathbb{R}^{n-1}$ inside $\mathbb{R}^n$. In a similar way, hierarchies for Haken n–manifolds lift to our hierarchies on the universal covers.

The paper is organized as follows. We develop a general theory of group actions with hierarchies in Section 1. In Section 2 we prove that Coxeter group manifolds admit hierarchies. Section 3 recalls the necessary background material on L^2–(co)homology. Finally, in Section 4 we study various vanishing conjectures about L^2–homology.

Our first result is that the Singer conjecture holds for all groups that admit a hierarchy in dimension 4. Our main application of this result is to Coxeter groups: Theorem 4.16 generalizes the result in [8] for right-angled Coxeter groups and a later result of Schroeder [24] for even Coxeter groups.

We also introduce the notion of the *cocompact action dimension* of a group: the minimal dimension of a contractible manifold, possibly with boundary, which admits a proper cocompact action by the group. Our second result is that the Singer conjecture is actually a statement about all groups, not just about fundamental groups of closed aspherical manifolds. Namely, we show in the smooth or PL categories that the Singer conjecture is equivalent to the cocompact action dimension conjecture: the L^2–cohomology of a group vanishes above half of its cocompact action dimension. We also show that for type VF groups, the cocompact action dimension conjecture is equivalent to the action dimension conjecture [8, Conjecture 8.9.1].

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1 Hierarchies for group actions

Definition Let G be a discrete group. A *G–space M* is a topological space with a G–action. We say a G–space is *proper* or *cocompact* if the action of G is proper or cocompact respectively. If N is a G–invariant subspace of M then (M,N) is a *pair of G–spaces.*

Definition A *convex polyhedral cone C* in $\mathbb{R}^n$ is the intersection of a finite collection $\{B_i^+\}$ of linear half-spaces in $\mathbb{R}^n$ (a half-space is linear if its bounding hyperplane B_i is a linear subspace). C is *nondegenerate* if it has nonempty interior. A *hyperplane arrangement in a nondegenerate cone C* is a finite collection $\{A_i\}$ of linear hyperplanes such that each A_i intersects the interior of C.

We assume that our manifolds are topological and mention explicitly when we require a smooth or PL structure.

Definition Let M be a proper, cocompact G–manifold, and $\mathcal{E}=\{E_i\}_{i=0}^r$ a collection of codimension-1 G–submanifolds. $(M,\mathcal{E})$ is *tidy* if:

- The components of M are contractible.
- The components of any intersection of the E_i are either contractible, or contained in ∂M.
- $(M,\partial M,\mathcal{E})$ locally looks like a hyperplane arrangement in a nondegenerate cone in $\mathbb{R}^n$: every point in M has a chart which maps M into a nondegenerate cone in $\mathbb{R}^n$, the point to the origin, ∂M into the boundary of the cone, and the E_i into a hyperplane arrangement in the cone.

This local structure implies that each component L of any intersection of the E_i is a manifold and either $L\subset\partial M$ or $L\cap\partial M=\partial L$. In the first case we call L a boundary component, and in the second an interior component. Moreover, the condition that hyperplanes intersect the interior of the cone implies that each E_i has only interior components. Finally, note that if $x\in M-\partial M$, the local picture is that of a linear hyperplane arrangement in $\mathbb{R}^n$.

In the case where $\mathcal{E}$ consists of just one submanifold F, this definition is equivalent to requiring that F is locally flat as a submanifold with boundary (sometimes called a neat submanifold), and the components of both M and F are contractible. We will call such a pair (M, F) a *tidy pair*.

In this case, since the components of F are contractible, it admits a collar neighborhood, and since the components of M are contractible, F is separating. By *cutting* M along F we mean taking the disjoint union N of the closures in M of components of $M - F$. We say that N is M *cut-open* along F. The action of G on $M - F$ extends by continuity to a proper cocompact action on N. So $N = M \ominus F$ is a G–manifold with boundary $\partial M - \partial F$ union two copies of F.

Note that $N - \partial N$ is naturally identified with $M - \partial M - F$. Using this identification, we can cut any closed G–subspace L of M along F by taking the closure of $L - F$ in N: $L \ominus F := Cl_N(L - F)$. Note that we still have a natural identification $(L \ominus F) - \partial N = L - \partial M - F$.

Associated to the cut there is an exact sequence of a triple $(M, F \cup \partial M, \partial M)$:

$$\cdots \to H_c^{k-1}(F \cup \partial M, \partial M) \to H_c^k(M, F \cup \partial M) \\ \to H_c^k(M, \partial M) \to H_c^k(F \cup \partial M, \partial M) \to \cdots .$$

By excision, we have $H_c^k(F \cup \partial M, \partial M) \cong H_c^k(F, \partial F)$ and $H_c^k(M, F \cup \partial M) \cong H_c^k(N, \partial N)$, so the above sequence becomes

$$\cdots \to H_c^{k-1}(F, \partial F) \to H_c^k(N, \partial N) \to H_c^k(M, \partial M) \to H_c^k(F, \partial F) \to \cdots . \tag{1}$$

Finally, applying Poincaré duality and reindexing, we obtain a sequence

$$\cdots \to H_k(F) \to H_k(N) \to H_k(M) \to H_{k-1}(F) \to \cdots . \tag{2}$$

Lemma 1.1 *If (M, F) is a tidy pair and N is M cut-open along F, then the components of N are contractible manifolds.*

Notice that N may or may not have more G–orbits of components than does M.

Proof The van Kampen theorem implies that components of N are simply connected, and sequence (2) shows that N is acyclic. □

Lemma 1.2 *Suppose that $(M, \mathcal{E})$ is tidy, and let N be M cut-open along E_0. Then $(N, \{E_i \ominus E_0\}_{i=1}^r)$ is also tidy.*

Proof We check the conditions of tidiness. Contractibility of the components of N follows immediately from Lemma 1.1 since (M, E_0) is a tidy pair. After cutting, the local picture is mostly preserved, we just have to check near E_0. If $x \in E_0$, the new charts come from restricting the old chart to one of the two halfcones bounded by the hyperplane corresponding to E_0, and taking hyperplanes which pass through the interior of that halfcone. Note that this description of cutting an arrangement in a nondegenerate cone along one of the hyperplanes into two nondegenerate cones with arrangements agrees with the procedure of cutting M and the E_i by E_0 described above in terms of closures.

Next, we show that the interior components of an intersection $\bigcap(E_{i_\alpha} \ominus E_0)$ are contractible. Let L be the union of these interior components. It follows from the local structure that L is a manifold with $\partial L = L \cap \partial N$. Therefore, it's enough to show that $L - \partial L = L - \partial N$ has contractible components. These components come from cutting the components of $\bigcap(E_{i_\alpha} - \partial M)$ which are not contained in E_0. The local picture of $\bigcap(E_{i_\alpha} - \partial M)$ intersecting $E_0 - \partial M$ is of a hyperplane intersecting a subspace, and we ignore the case when the subspace is contained in the hyperplane, as this would produce a boundary component. Thus the intersections we are interested are transverse.

So, let D be the union of the interior components of $\bigcap E_{i_\alpha} - \partial M$ which are not contained in E_0. Since $E_0 \cap D$ has contractible components by hypothesis, it follows from the above that the pair $(D, E_0 \cap D)$ satisfies all conditions of tidiness except cocompactness and $L - \partial L = (D - \partial D) \ominus (E_0 \cap D)$. Therefore $L - \partial L$ has contractible components by Lemma 1.1. (The cutting procedure and the proof of the lemma did not use cocompactness.) □

Definition An *n–hierarchy* for an action of a discrete group G on a manifold M is a sequence

$$(M_0, F_0), (M_1, F_1), \dots, (M_m, F_m), (M_{m+1}, \varnothing),$$

such that

- $M_0 = M$,
- M_{m+1} is a disjoint union of compact contractible n–manifolds,
- (M_i, F_i) is a tidy pair for each i,
- M_{i+1} is M_i cut-open along F_i.

More generally, if (M, N) is a proper, cocompact G–pair of manifolds, we can define a *hierarchy ending in N* in the same way, with the one difference being that $M_{m+1} = N$.

Definition G *admits an n–hierarchy* if there exists a contractible, n–dimensional G–manifold M and a hierarchy for the action.

Lemma 1.3 *Let G act on M with a hierarchy, and let M_1^0 be a component of M_1. Then there is an induced hierarchy for the action of $\mathrm{St}_G(M_1^0)$ on M_1^0, where $\mathrm{St}_G(M_1^0)$ is the stabilizer of M_1^0.*

Proof We claim the following sequence is a hierarchy for M_1^0:

$$(M_1^0, F_1 \cap M_1^0), (M_2 \cap M_1^0, F_2 \cap M_1^0), \dots, (M_{m+1} \cap M_1^0, \varnothing).$$

We have that M_1^0 is a contractible $\mathrm{St}_G(M_1^0)$–manifold by Lemma 1.1. Since each F_i is G–invariant, $F_i \cap M_1^0$ is $\mathrm{St}_G(M_1^0)$–invariant, and the other conditions of our hierarchy follow immediately. □

Theorem 1.4 *Let M be a proper, cocompact G–manifold, and $\mathcal{E} = \{E_i\}_{i=0}^r$ a collection of submanifolds such that $(M, \mathcal{E})$ is tidy. If the components of the complement $M - \cup_i E_i$ have compact closure in M, then the action of G on M admits a hierarchy.*

Proof The proof is to apply Lemma 1.2 repeatedly, as this implies that if we cut along each E_i, we get a hierarchy ending in $M - \cup_i E_i$. To be precise, let $F_j = E_j$ cut-along by $E_0, E_1, \dots E_{j-1}$, and let $M_0 = M$ and $M_{j+1} = M_j$ cut along by F_j. Since each E_i is G–invariant, (M_j, F_j) is a tidy pair for all j. □

2 Coxeter groups

Recall that a Coxeter group W has generators s_i with relations $s_i^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$ for some $m_{ij} \in \mathbb{N} \cup \infty$. In other words, W is generated by reflections and each pair of reflections generates a dihedral subgroup (possibly D_∞). The *nerve* of a Coxeter group is a simplicial complex with vertices corresponding to generators s_i, and $s_{i_1}, \dots, s_{i_n}$ a simplex if and only if the subgroup generated by $s_{i_1}, \dots, s_{i_n}$ is finite. A Coxeter group is *right-angled* if $m_{ij} = 2$ or ∞ for all i, j.

Definition A *mirror structure* on a space X is an index set S and a collection of subspaces $\{X_s\}_{s \in S}$. For each $x \in X$, let

$$S(x) := \{s \in S \mid x \in X_s\}.$$

An example to keep in mind is a convex polytope in $\mathbb{E}^n$ or $\mathbb{H}^n$ with mirrors the codimension-1 faces. We will assume that our index set S is finite.

Definition Let X have a mirror structure, and let W be a Coxeter group with generators $s \in S$. Let W_T denote the subgroup generated by $s \in T \subset S$. Let $\sim$ denote the following equivalence relation on $W \times X$: $(w_1, x) \sim (w_2, y)$ if and only if $x = y$ and $w_1 w_2^{-1} \in W_{S(x)}$. The *basic construction* is the space

$$\mathcal{U}(W, X) := W \times X / \sim .$$

Therefore, $\mathcal{U}(W, X)$ is constructed by gluing together copies of X along its mirrors, with the exact gluing dictated by the Coxeter group. A standard example is where X is a right-angled pentagon in $\mathbb{H}^2$ with mirrors the edges of X, and W is the right-angled Coxeter group generated by reflections in these edges. Then $\mathcal{U}(W, X) \cong \mathbb{H}^2$.

Let W be a Coxeter group with nerve L. Again, L is the simplicial complex with vertex set corresponding to S and simplices corresponding to subsets of S that generate finite subgroups of W. Let K be the cone on the barycentric subdivision of L. K admits a natural mirror structure with K_s the closed star of the vertex corresponding to s in the barycentric subdivision of L. The *Davis complex* $\Sigma(W, S)$ is defined to be the simplicial complex $\mathcal{U}(W, K)$.

Lemma 2.1 [5] *$\Sigma(W, S)$ has the following properties:*

- *W acts properly and cocompactly on $\Sigma(W, S)$ with fundamental domain K.*
- *Σ admits a cellulation such that the link of every vertex can be identified with L. Therefore, if L is a triangulation of S^{n-1}, then $\Sigma(W, S)$ is an n–manifold.*
- *$\Sigma(W, S)$ admits a piecewise Euclidean metric that is* CAT(0).

We assume from now on that W is a Coxeter group with nerve a PL triangulation of S^{n-1}. If $w \in W$ acts as a reflection on $\Sigma(W, S)$, we call the fixed point set a wall, and denote it Σ^w.

Lemma 2.2 *Walls in $\Sigma(W, S)$ have the following properties:*

- *The stabilizer of each wall acts properly and cocompactly on the wall.*
- *Each wall and each half-space is a geodesically convex subset of $\Sigma(W, S)$.*
- *The collection of walls separates $\Sigma(W, S)$ into disjoint copies of the fundamental domain K.*
- *The stabilizer of each point in $\Sigma(W, S)$ is a finite Coxeter group, and the walls containing that point can be locally identified with the fixed hyperplanes of the standard action of this Coxeter group on $\mathbb{R}^n$.*

Though each wall of Σ is a contractible submanifold, a W–orbit of a wall has in general quite complicated topology. Even in the simple case where W is generated by reflections in a equilateral triangle in $\mathbb{R}^2$ the W–orbit of a wall is not contractible, as W–translates of a wall can intersect nontrivially. However, passing to a suitable subgroup fixes this problem.

Theorem 2.3 *W has a finite index torsion-free normal subgroup Γ, and the action of Γ on $\Sigma(W,S)$ admits a hierarchy.*

Proof The existence of such a subgroup Γ is well-known. The cutting submanifolds that we choose will be Γ–orbits of walls in $\Sigma(W,S)$.

A lemma of Millson and Jaffee [19] shows that any torsion-free normal subgroup of W has the trivial intersection property: for all $\gamma \in \Gamma$, either $\gamma\Sigma^s = \Sigma^s$ or $\gamma\Sigma^s \cap \Sigma^s = \varnothing$. Therefore, each Γ–orbit is a disjoint union of walls and has contractible components.

Once we have removed all the walls, we are left with disjoint copies of the fundamental domain K, and since Γ is of finite index in W, there are only finitely many orbits of walls to remove, so by Lemma 2.2 this is a tidy collection. Therefore, we are done by Theorem 1.4. □

Remark If W is a Coxeter group with nerve a PL triangulation of D^{n-1}, then $\Sigma(W,S)$ is an n–manifold with boundary, and these groups also virtually admit hierarchies.

3 L^2–homology

Let X be a proper, cocompact G–CW–complex, and let $C_*(X)$ denote the usual cellular chains of X, which we regard as left $\mathbb{Z}G$–modules. The *square-summable chains of X* are the tensor product

$$C_*^{(2)}(X) = L^2(G) \otimes_{\mathbb{Z}G} C_*(X),$$

where $L^2(G)$ is the Hilbert space of real-valued square-summable functions on G.

The usual boundary homomorphism $\partial\colon C_*(X) \to C_{*-1}(X)$ extends to a boundary operator $\partial\colon C_*^{(2)}(X) \to C_{*-1}^{(2)}(X)$ whose adjoint is the coboundary operator $\delta\colon C_*^{(2)}(X) \to C_{*+1}^{(2)}(X)$.

The (reduced) L^2–(co)homology groups can be defined as the kernel of the Laplacian operator:

$$L^2H_*(X;G) \cong L^2H^*(X;G) \cong \ker(\partial\delta + \delta\partial)\colon C_*^{(2)}(X) \to C_*^{(2)}(X).$$

These are Hilbert G–modules, and one defines L^2–Betti numbers as their von Neumann dimension. These definitions can be extended to arbitrary topological spaces with G–action using, for example, singular (co)chains, as follows; see [17, Chapter 6].

Let $\mathcal{N}(G)$ denote the von Neumann algebra of bounded G–equivariant operators on $L^2(G)$. As explained in [17], given an algebraic $\mathcal{N}(G)$–module A there is a well-behaved notion of dimension $\dim_{\mathcal{N}(G)}(A)$. The key feature of $\dim_{\mathcal{N}(G)}$ is *additivity* under short exact sequences.

Consider equivariant singular (co)homology with $\mathcal{N}(G)$ coefficients: $H_*^G(X,\mathcal{N}(G)) := H_*(\mathcal{N}(G)\otimes_{\mathbb{Z}G} C_*^{\mathrm{sing}}(X))$ and $H_G^*(X,\mathcal{N}(G)) := H^*(\mathrm{Hom}_{\mathbb{Z}G}(C_*^{\mathrm{sing}}(X),\mathcal{N}(G)))$. The i^{th} *L^2–Betti number* $b_i^{(2)}(X;G)$ is defined to be $\dim_{\mathcal{N}(G)}(H_i^G(X;\mathcal{N}(G)))$. We will also consider the cohomological version $b_{(2)}^i(X;G) := \dim_{\mathcal{N}(G)}(H_G^i(X;\mathcal{N}(G)))$.

Since the category of finitely generated projective $\mathcal{N}(G)$–modules is equivalent to the category of Hilbert G–modules (via completion), the resulting theory is equivalent to the combinatorial version for G–CW–complexes.

We record as a lemma some of the basic algebraic properties of L^2–homology that we will need. In the next section we will often use the fact that in the exact sequences a term between two zero-dimensional terms has to be zero-dimensional itself.

Lemma 3.1

- **Functoriality** *A G–equivariant map $f\colon (X_1,Y_1)\to(X_2,Y_2)$ between pairs of G–spaces induces a map $f_*\colon H_k^G(X_1,Y_1;\mathcal{N}(G))\to H_k^G(X_2,Y_2;\mathcal{N}(G))$. If f is a weak G–equivariant homotopy equivalence, then f_* is an isomorphism.*
- **Exact sequence of a pair** *Let (X,Y) be a pair of G–spaces, then the sequence*

  $$\cdots\to H_i^G(Y;\mathcal{N}(G))\to H_i^G(X;\mathcal{N}(G))\to H_i^G(X,Y;\mathcal{N}(G))\to\cdots$$

  is exact.
- **Multiplicativity** *Let $H<G$ be a subgroup of finite index. If X is a G–space then $b_i^{(2)}(X;H)=[G:H]b_i^{(2)}(X;G)$ and $b_{(2)}^i(X;H)=[G:H]b_{(2)}^i(X;G)$*
- **Excision** *Suppose (X,A,B) is a triple of G–spaces such that $Cl_X(B)\subset Int(A)$, then the map $H_*^G(X-B,A-B;\mathcal{N}(G))\to H_*^G(X,A;\mathcal{N}(G))$ is a isomorphism.*
- **Poincaré duality** *If G acts properly, cocompactly, and preserving orientation on an orientable n–manifold $(M,\partial M)$, then $H_G^i(M;\mathcal{N}(G))\cong H_{n-i}^G(M,\partial M;\mathcal{N}(G))$ and $H_i^G(M;\mathcal{N}(G))\cong H_G^{n-i}(M,\partial M;\mathcal{N}(G))$.*

- **Induction principle** *The L^2–homology of a G–space X is induced from the L^2–homology of its components:*

$$H_i^G(X;\mathcal{N}(G)) = \bigoplus_{[X^0]\in\pi_0(X)/G} \mathcal{N}(G)\otimes_{\mathcal{N}(\mathrm{St}_G X^0)} H_i^{\mathrm{St}_G X^0}(X^0;\mathcal{N}(\mathrm{St}_G X^0)),$$
$$b_i^{(2)}(X;G) = \sum_{[X^0]\in\pi_0(X)/G} b_i^{(2)}(X^0,\mathrm{St}_G X^0),$$

where the sums are over representatives of the orbits of the components of X.

- **Künneth formula** *If M is a G–space and Y is an H–space, then*

$$b_n^{(2)}(X\times Y) = \sum_{i+j=n} b_i^{(2)}(X)b_j^{(2)}(Y).$$

Remark The first four statements are quite standard and have analogous versions for L^2–cohomology. The proofs of the last two statements in [17] strongly depend on nice properties of $\dim_{\mathcal{N}(G)}$ with respect to tensor products and colimits. It's unclear whether their cohomological versions hold in this generality.

We will need the following version of Poincaré duality.

Lemma 3.2 *If $(M,\partial M)$ is an n–dimensional proper cocompact G–manifold with orientable components, then*

$$b_i^{(2)}(M,\partial M;G) = \sum_{[M^0]\in\pi_0(M)/G} b^{n-i}_{(2)}(M^0;\mathrm{St}_G M^0),$$

where the sums are over representatives of the orbits of the components of M.

Proof By the induction principle,

$$b_i^{(2)}(M,\partial M;G) = \sum_{[M^0]\in\pi_0(M)/G} b_i^{(2)}(M^0,\partial M;\mathrm{St}_G M^0).$$

Since each M^0 is contractible, it is orientable, and by taking, if necessary, an index 2 subgroup of $\mathrm{St}_G M^0$ we get a cocompact orientation preserving action on M^0. Thus we can apply Poincaré duality and multiplicativity to each M^0 to finish the proof. □

Definition For a discrete group G, define

$$b_k^{(2)}(G) := b_k^{(2)}(EG;G),$$
$$b^k_{(2)}(G) := b^k_{(2)}(EG;G).$$

By the functoriality property, these are well-defined. In fact, one can use L^2–(co)homology of any proper contractible G–space, since the chain complex of such a space still gives a projective resolution of $\mathbb{Q}$ over the group ring $\mathbb{Q}G$, and we are using $\mathcal{N}(G)$ coefficients anyway.

Note that in general the relation between the homological and cohomological versions of L^2–Betti numbers is unclear, however if X is a proper and cocompact G–CW–complex, a cellular version of the Hodge decomposition shows $b^k_{(2)}(X;G) = b_k^{(2)}(X;G)$. In the next two lemmas, we establish some partial results in this direction.

Lemma 3.3 *If X is a countable proper G–CW–complex, then $b^k_{(2)}(X;G) \geq b_k^{(2)}(X;G)$. In particular for any group G, $b^k_{(2)}(G) \geq b_k^{(2)}(G)$.*

Proof X is the colimit of a directed sequence of proper, cocompact G–complexes $\{X_i \mid i \in \mathbb{N}\}$. By [17, Theorems 6.13 and 6.18], we have

$$b_k^{(2)}(X;G) = \sup_i \inf_{j\geq i} \dim_{\mathcal{N}(G)}(\operatorname{im} i_{i,j}\colon H_k^G(X_i) \to H_k^G(X_j)),$$
$$\dim_{\mathcal{N}(G)} \varprojlim H^k_G(X_i;G) = \sup_i \inf_{j\geq i} \dim_{\mathcal{N}(G)}(\operatorname{im} i^{i,j}\colon H^k_G(X_j) \to H^k_G(X_i)).$$

Since X_i and X_j are cocompact proper G–complexes, the terms on the right-hand side are equal, and because $H^k_G(X)$ surjects onto $\varprojlim H^k_G(X_i)$, we have that $b^k_{(2)}(X;G) \geq b_k^{(2)}(X;G)$.

The last sentence follows, since the standard bar construction gives a countable model for EG. □

Lemma 3.4 *If G acts properly and cocompactly on an n–dimensional contractible manifold without boundary, then $b_k^{(2)}(G) = b^{n-k}_{(2)}(G) = b_{n-k}^{(2)}(G) = b^k_{(2)}(G)$.*

Proof Applying Lemma 3.3 and 3.2 twice we get

$$b_k^{(2)}(G) = b^{n-k}_{(2)}(G) \geq b_{n-k}^{(2)}(G) = b^k_{(2)}(G) \geq b_k^{(2)}(G).$$

Thus the inequalities above are equalities, and the result is proved. □

Remark In general, it is not true that a proper G–action on a manifold M is weak G–homotopy equivalent to a G–action on a countable CW–complex, even for finite groups. For example, Ancel and Guilbault [1] proved that doubling an open manifold along a $\mathcal{Z}$–boundary results in a closed manifold, and therefore any such $\mathcal{Z}$–boundary is the fixed point set of an involution acting on a closed manifold. The $\mathcal{Z}$–boundaries may have uncountable fundamental group.

We will also need the following version of excision.

Lemma 3.5 *Let (X, A, B) be a G–triple of spaces. Suppose that for every open U in X there is an excision isomorphism $H_*(U, U \cap A, \mathbb{R}) \cong H_*(U - B, (U \cap A) - B, \mathbb{R})$. Then we have an isomorphism $H_*^G(X, A, \mathcal{N}(G)) \cong H_*^G(X - B, A - B, \mathcal{N}(G))$.*

Proof For this proof, we will say a G–subspace Y of X "satisfies (A, B)–excision" if $H_*^G(Y, Y \cap A, \mathcal{N}(G)) \cong H_*^G(Y - B, (Y \cap A) - B, \mathcal{N}(G))$. If G is finite, we have $H_*^G(X, A, \mathcal{N}(G)) \cong H_*(X, A, \mathbb{R})$, so by assumption we get the desired isomorphism. Now, suppose G is infinite.

Consider the collection of open G–invariant subsets V satisfying (A, B)–excision, partially ordered by inclusion. Since singular homology commutes with direct limits, it follows by the Zorn lemma that there is a maximal such V. We claim that $V = X$.

Indeed, otherwise by properness there is a open set $U \not\subset V$ with finite stabilizer G_U, and for which $gU \cap U = \varnothing$ for $g \notin G_U$. We have the relative Mayer–Vietoris sequence with $\mathcal{N}(G)$ coefficients:

$$H_*^G(GU \cap V, A) \to H_*^G(GU, A) \oplus H_*^G(V, A) \to H_*^G(GU \cup V, A).$$

We have a corresponding Mayer–Vietoris sequences where we have excised B. Note that $GU \cong G \times_{G_U} U$ and $GU \cap V \cong G \times_{G_U} (U \cap V)$, since each element of G not in G_U moves U off of itself. Therefore, we have induction isomorphisms:

$$H_*^G(GU, \mathcal{N}(G)) = \mathcal{N}(G) \otimes_{\mathcal{N}(G_U)} H_*^{G_U}(U, \mathcal{N}(G_U))$$

and

$$H_*^G(GU \cap V, \mathcal{N}(G)) = \mathcal{N}(G) \otimes_{\mathcal{N}(G_U)} H_*^{G_U}(U \cap V, \mathcal{N}(G_U)).$$

By the finite group case, we have (A, B)–excision for U and $U \cap V$, and therefore, we have (A, B)–excision for GU and $GU \cap V$. Since we have (A, B)–excision for V, by the 5 lemma, this implies excision for $GU \cup V$, contradiction. □

4 Vanishing conjectures and results

Conjecture (Singer conjecture) *If G acts properly and cocompactly on a contractible n–manifold without boundary, then $b_i^{(2)}(G) = 0$ for $i \neq n/2$.*

In general, it seems this conjecture is stronger than the original Singer conjecture, which assumed G to be torsion-free. By the multiplicativity of L^2–Betti numbers, the two versions are equivalent for type VF groups (groups which are virtually finite type).

The conjecture holds for trivial reasons in dimensions ≤ 2. In dimension 3, Lott and Lück [16] proved the conjecture for all fundamental groups of manifolds that satisfy the geometrization conjecture, therefore by Perelman's work [21; 22; 23] it holds for all type VF groups acting properly and cocompactly on contractible 3–manifolds. We record this as a theorem.

Theorem 4.1 *The Singer conjecture is true for type VF groups in dimensions $n \le 3$.*

Definition The *L^2–dimension* of a group G, $L^2\dim(G)$ is the largest degree n, such that $b^n_{(2)}(G) \neq 0$.

Definition The *action dimension* $\mathrm{actdim}(G)$ of a group G is the least dimension of a contractible manifold which admits a proper G–action.

Action dimension was introduced and studied by Bestvina, Kapovich, and Kleiner in [2]. One consequence of their work is that an n–fold product of nonabelian free groups does not act properly discontinuously on a contractible $(2n-1)$–manifold. Since nonabelian free groups have $b^1_{(2)}(F_n) \neq 0$, it follows from the Künneth formula that the n–fold products have nontrivial $b^n_{(2)}$. Therefore, as noted in [8], their result is implied by the following conjecture.

Conjecture (actdim conjecture) $\mathrm{actdim}(G) \ge 2L^2\dim(G)$.

Remark In [8], the conjecture is stated in terms of homology. Lemma 3.3 implies that the above version is potentially stronger than the original.

We note the following bounds, which are well-known for cellular actions.

Lemma 4.2 *We have $\mathrm{actdim}(G) \ge \mathrm{cd}_{\mathbb{Q}}(G)$. If G is virtually torsion free, then $\mathrm{actdim}(G) \ge \mathrm{vcd}(G)$.*

Proof We only prove the first inequality, the proof of the second one is entirely similar. Suppose G acts properly on a contractible n–manifold M^n, and let A be a $\mathbb{Q}[G]$–module. We need to show that $H^i(G; A) = 0$ for $i > n$. We will use equivariant sheaf cohomology; see [13, Chapter V].

Denote by $\mathcal{A}$ the constant sheaf on M with stalk A, and by $\mathcal{A}^G$ the sheaf on M/G whose sections over an open set in M/G are G–invariant sections of $\mathcal{A}$ over its preimage in M. Since the action is proper and the coefficients are rational, [13, Corollaire to Théorème 5.3.1, page 204] applies and we obtain a spectral sequence with E_2–term $H^i(G; H^j(M, \mathcal{A}))$ which converges to $H^{i+j}(M/G, \mathcal{A}^G)$. Since M is contractible, the spectral sequence collapses, and we get $H^i(G, A) = H^i(M/G, \mathcal{A}^G)$.

Since M is a manifold, it is a separable metrizable space, and [20, Theorem 4.3.4] implies that M/G is also metrizable, in particular paracompact, and $\dim M/G \le n$. Finally, paracompactness of M/G allows us to use Čech cohomology to conclude that $H^i(M/G, \mathcal{A}^G) = 0$ for $i > n$. □

Definition The *cocompact action dimension* $\mathrm{cadim}(G)$ of a group G is the least dimension of a proper cocompact contractible G–manifold.

Obviously, $\mathrm{actdim}(G) \le \mathrm{cadim}(G)$. We do not know of a type VF group G with $\mathrm{actdim}(G) < \mathrm{cadim}(G)$.

Conjecture (cadim conjecture) $\mathrm{cadim}(G) \ge 2L^2\mathrm{dim}(G)$.

Note that these conjectures all have smooth and PL versions.

Lemma 4.3 *The actdim conjecture implies the cadim conjecture, which in turn implies the Singer conjecture.*

Proof The first implication is trivial. By considering cohomology with compact support, we see that a group acting properly and cocompactly on a contractible n–manifold without boundary has $\mathrm{actdim} = \mathrm{cadim} = \mathrm{cd}_{\mathbb{Q}} = n$, so the second implication follows from Lemma 3.4. □

Shmuel Weinberger pointed out that a recent theorem of Craig Guilbault [14] implies that for type F groups the difference between cadim and actdim is at most 1, at least in high dimensions.

Theorem 4.4 [14] *For an open manifold M^n $(n \ge 5)$, $M^n \times \mathbb{R}$ is homeomorphic to the interior of a compact $(n+1)$–manifold with boundary if and only if M^n has the homotopy type of a finite complex.*

If G acts freely and properly on a contractible manifold M, we can apply Guilbault's theorem to the interior of M/G to get the following.

Corollary 4.5 *If G is type F and $\mathrm{actdim}(G) \ge 5$, then $\mathrm{cadim}(G) \le \mathrm{actdim}(G) + 1$.*

Although the precise relationship between actdim and cadim is unclear, we can still show the two conjectures are equivalent, at least for type VF groups.

Theorem 4.6 *The cadim and actdim conjectures are equivalent for type VF groups.*

Proof We need to show that the cadim conjecture implies the actdim conjecture. So, let G acting properly on a contractible n–manifold M be a counterexample to the actdim conjecture, ie $2L^2\mathrm{dim}(G) - n > 0$. By removing the boundary, we can assume that M is open. If H is finite index in G and type F, then M/H is an open aspherical manifold of finite homotopy type, and $2L^2\dim(H)-n>0$. Note that by the Künneth formula, by taking direct products of H with itself, we can assume that H acts freely on a contractible n–manifold M with $2L^2\dim(H)-n$ arbitrarily large. (The Künneth formula applies here since H is type F and therefore homological and cohomological L^2–Betti numbers are the same.) By Theorem 4.4, $M/H\times\mathbb{R}$ is the interior of a compact manifold. Therefore, the action of H on the universal cover of this compact manifold is a counterexample to the cadim conjecture. □

Since Guilbault's result holds in the PL and smooth categories, the smooth and PL versions of these conjectures are also equivalent to each other.

Remark These conjectures put restrictions on the embedding dimension of a $K(G,1)$–space. For example, if $b^i_{(2)}(G)\neq 0$, the cadim conjecture implies that no $K(G,1)$ space can embed in $\mathbb{R}^{2i-1}$.

Since any contractible proper G–manifold can be used to compute $b^i_{(2)}(G)$, using Lemma 3.2 we obtain an equivalent series of conjectures in terms of manifolds.

Conjecture (cadim conjecture in dimension n) *Suppose $(M,\partial M)$ is an n–manifold with contractible components which admits a proper cocompact G–action. Then $b_i^{(2)}(M,\partial M;G)=0$ for $i<n/2$.*

We now consider these conjectures in the context of manifolds with hierarchies. Excision (Lemma 3.5) allows us to apply the argument used to derive sequence (1) to L^2–homology to obtain the following.

Lemma 4.7 *If (M,F) is a tidy pair and N is M cut-open along F, there is an exact sequence with $\mathcal{N}(G)$ coefficients*

$$(3)\qquad \cdots\to H_k^G(F,\partial F)\xrightarrow{i_*}H_k^G(M,\partial M)\to H_k^G(N,\partial N)\to H_{k-1}^G(F,\partial F)\to\cdots$$

Thus we have the following apparently weaker version of the cadim conjecture.

Conjecture (weak cadim conjecture) *If (M^{2k+1},F^{2k}) is a tidy pair, then the map induced by inclusion $i_*\colon H_k^G(F,\partial;\mathcal{N}(G))\to H_k^G(M,\partial;\mathcal{N}(G))$ has zero-dimensional image.*

Lemma 4.8 *Suppose that (M^n,F) is a tidy G–pair, N is M cut-open by F, and the cadim conjecture in dimension $(n-1)$ holds for F. Then the cadim conjecture in dimension n holds for M if and only if it holds for N and, if $n=2k+1$ is odd, the weak cadim conjecture holds for (M,F).*

Proof First, suppose that the cadim conjecture holds for M. We have $b_i^{(2)}(M,\partial M)=0$ for $i<n/2$, and $b_i^{(2)}(F,\partial F)=0$ for $i<(n-1)/2$. Then the cadim conjecture holds for N by sequence (3).

Next, suppose the cadim conjecture holds for N, so that $b_i^{(2)}(N,\partial N)=0$ for $i<n/2$, and $b_i^{(2)}(F,\partial F)=0$ for $i<(n-1)/2$. By the same argument as above, we can say that $b_i^{(2)}(M,\partial M)=0$ for $i<(n+1)/2$.

Now, we only have to consider the case where $n=2k+1$ and $i=k+1$. The weak cadim conjecture says that the map $H_k^G(F,\partial F;\mathcal{N}(G))\to H_k^G(M,\partial M;\mathcal{N}(G))$ in sequence (3) has zero-dimensional image. The result follows. □

Theorem 4.9 *The cadim conjecture in dimension $2k-1$ implies the cadim conjecture in dimension $2k$ for manifolds with hierarchies. The cadim conjecture in dimension $2k$ and the weak cadim conjecture in dimension $2k+1$ imply the cadim conjecture in dimension $2k+1$ for manifolds with hierarchies.*

Proof This is immediate by induction on the length of the hierarchy, using Lemmas 1.3 and 4.8, and noting that the cadim conjecture holds for manifolds with compact components. □

A somewhat surprising result is a converse to the second implication in Lemma 4.3, at least if we somewhat restrict the category. By *the cadim conjecture with PL boundary*, we mean the version of the cadim conjecture where we only allow manifolds whose boundaries admit PL structures and actions restricted to the boundaries are PL.

Theorem 4.10 *The Singer conjecture and the cadim conjecture with PL boundary are equivalent.*

The result follows immediately from the following key lemma and induction.

Lemma 4.11 *The Singer conjecture in dimension n and the cadim conjecture with PL boundary in dimension $(n-1)$ imply the cadim conjecture with PL boundary in dimension n.*

In the proof of this lemma we will need the following result, which is probably well-known to the experts, but we could not find any reference for it.

Proposition 4.12 *Suppose a group G acts PL properly and cocompactly on a polyhedron M. Then M has a G–equivariant PL triangulation.*

Proof Using cocompactness choose a finite subpolyhedron $P \subset M$, so that $GP = M$, and set $F := \{f \in G \mid fP \cap P \neq \varnothing\}$. By properness, F is finite. Set $Q := \bigcup\{fP \mid f \in F\}$. Let $\mathcal{T}$ be a triangulation of Q such that each fP is a subcomplex of Q. The pullback $f^*\mathcal{T}$ is a triangulation of P such that $f\colon f^*\mathcal{T} \to \mathcal{T}$ is simplicial. Let $\mathcal{S}$ be a common subdivision of $\{f^*\mathcal{T} \mid f \in F\}$. Then each $f \in F$ is a linear map $\mathcal{S} \to \mathcal{T}$. For each $x \in M$ set

$$C_x = \bigcap\{g\sigma \mid g \in G,\ \sigma \in \mathcal{S},\ g\sigma \ni x\}.$$

Then the collection $\mathcal{C} = \{C_x \mid x \in M\}$ is a G–equivariant cover of M, which restricts to a cellulation of P, since for $x \in P$ the last two conditions imply that $g \in F$, and thus C_x is a finite intersection of linear simplices: a closed convex cell. Therefore, $\mathcal{C}$ is a cellulation of M, and taking the barycentric subdivision gives an equivariant triangulation of M. □

Proof of Lemma 4.11 We use the equivariant Davis reflection group trick as in [7; 5]. The idea is that the trick turns the input of the cadim conjecture (a contractible manifold with boundary and proper cocompact group action) into the input of the Singer conjecture (a contractible manifold without boundary and proper cocompact group action). In addition, the newly constructed manifold action admits a hierarchy ending at a disjoint union of copies of the original. Once this has been established, the proof is more or less the same as that of Theorem 4.9.

Suppose that G acts properly and cocompactly on a contractible n–manifold with boundary $(M, \partial M)$ (if $\partial M = \varnothing$, we are done since we are assuming the Singer conjecture holds). Let L be a flag PL triangulation of M that is equivariant with respect to the G–action. Suppose that the stabilizer of any simplex fixes the simplex pointwise and that $g.v \cap \mathrm{Lk}(v) = \varnothing$ for all $g \in G$ and and $v \in L^0$ (by subdividing, these triangulations can always be constructed). We can now apply the equivariant reflection group trick. Indeed, L determines a right-angled Coxeter group W, and we can form the basic construction $\mathcal{U} = \mathcal{U}(W, M)$. By the conditions imposed on L, there is an action of G on W which determines a semidirect product $W \rtimes G$. Since $\mathcal{U}/W \rtimes G \cong M/G$, $W \rtimes G$ acts cocompactly on $\mathcal{U}$. Here are some key properties of the reflection group trick:

- Each wall is a codimension-1 contractible submanifold of N.
- There are a finite number of $W \rtimes G$–orbits of walls, and each orbit is a disjoint union of walls.

- Any component of a nonempty intersection of orbits of walls is itself a Davis complex and is therefore contractible.
- The stabilizer of each wall acts properly and cocompactly on the wall.
- The collection of walls looks locally like a right-angled hyperplane arrangement in $\mathbb{R}^n$ (this is where we need the triangulation to be PL).

It follows, similarly to Theorem 1.4, that the $W \rtimes G$ action on $\mathcal{U}$ admits a hierarchy that ends in disjoint copies of M, where the cutting submanifolds are $W \rtimes G$–orbits of walls. Since $\mathcal{U}$ has no boundary, and we are assuming that the Singer conjecture holds for $\mathcal{U}$, the cadim conjecture in dimension n holds for $\mathcal{U}$. Since we are also assuming the cadim conjecture in dimension $n-1$, it follows by applying Lemma 4.8 inductively that the cadim conjecture holds for the original M. □

If M is a PL manifold, and the action of G is PL, then the basic construction $\mathcal{U}$ and the action of $W \rtimes G$ in the above argument are also PL.

If M is a smooth manifold, and the action of G is smooth, the existence of a smooth equivariant triangulation is part of the main result of [15]. Moreover, in this case the reflection trick produces a smooth manifold with the smooth action.

Thus we have the following.

Corollary 4.13 *The Singer conjecture and the cadim conjecture are equivalent in the smooth and the PL categories.*

Since for a type VF group the reflection trick produces another type VF group, Theorem 4.6 gives us a corollary.

Corollary 4.14 *For type VF groups the Singer conjecture and the action dimension conjecture are equivalent in the smooth and the PL categories.*

Since TOP = PL in dimension 2, Theorem 4.1 and Theorem 4.10 imply another.

Corollary 4.15 *The cadim conjecture holds for all type VF groups in dimensions less than or equal to* 3.

Now, Corollary 4.15, Theorem 4.9 and Lemma 4.3 imply our main theorem.

Theorem 4.16 *The Singer conjecture holds for all type VF groups that admit a hierarchy in dimensions less than or equal to* 4.

Theorem 4.16 and Theorem 2.3 now imply our main applications.

Theorem 4.17 *If W is a Coxeter group with nerve a triangulation of S^3, then the Singer conjecture holds for W acting on $\Sigma(W,S)$.*

Theorem 4.18 *If W is a Coxeter group with nerve a triangulation of D^3, then $b_i^{(2)}(W)=0$ for $i>2$.*

Remark The hierarchies for Coxeter groups have more structure in the following sense: the hierarchy for $\Sigma(W,S)$ induces a hierarchy on each wall. This means that if we restrict our attention to Coxeter groups, we can relax many of the assumptions. For instance, Theorem 4.9 restricted to Coxeter groups need only assume the cadim conjecture in dimension $2k-1$ for manifolds with hierarchies.

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Combinatorial proofs in bordered Heegaard Floer homology

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Using bordered Floer theory, we give a combinatorial construction and proof of invariance for the hat version of Heegaard Floer homology. As part of the proof, we also establish combinatorially the invariance of the linear-categorical representation of the strongly based mapping class groupoid given by the same theory.

57R58; 57R56

1 Introduction

Heegaard Floer homology, introduced by Ozsváth and Szabó [14; 15], gives several kinds of invariants for closed 3–manifolds. The invariants are defined using holomorphic curves, so in general they are not directly computable from their definitions. However, for the hat version of the invariant, denoted $\widehat{\mathrm{HF}}$, there are ways to give combinatorial definitions. There are two steps in this process. First, we want to give to a particular kind of description of a 3–manifold (such as a Heegaard splitting) a description of $\widehat{\mathrm{HF}}$ associated to that 3–manifold. This means that, at least in principle, the invariant can be computed algorithmically for any 3–manifold. Second, we want to give combinatorial proofs for the main properties of $\widehat{\mathrm{HF}}$, beginning with the statement that it depends only on the diffeomorphism class of the 3–manifold, rather than on a particular description of it.

Bordered Floer theory gives a way to extend the hat version of Heegaard Floer homology to 3–manifolds with one or two boundary components. The theory is also defined using holomorphic curves. However, some of the invariants associated to certain simple types of 3–manifolds with boundary have been computed. By breaking an arbitrary closed 3–manifold into simpler pieces, the theory gives a combinatorial description of $\widehat{\mathrm{HF}}$ (see Lipshitz, Ozsváth and Thurston [11]) achieving the first step in the process described above.

In this paper, we give the second step of the process; namely, we prove combinatorially that the construction of $\widehat{\mathrm{HF}}$ given by bordered Floer theory in fact produces an invariant of the 3–manifold. One main result we use is an alternative description of $\widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})$ given by Zhan [21]. This allows us to use a combinatorial construction that is easier to reason about.

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An intermediate statement in the proof, which may be of independent interest, is that bordered Floer theory gives a linear-categorical representation of the strongly based mapping class groupoid, which contains the strongly based mapping class group. By a linear-categorical representation of a group or groupoid, we mean assigning homotopy equivalence classes of bimodules to each element of the group (resp. groupoid), in such a way that composition in the group (resp. groupoid) corresponds to taking an appropriate tensor product of bimodules.

Sarkar and Wang [16] gave the first combinatorial description of $\widehat{\mathrm{HF}}$, giving a systematic way to convert any Heegaard diagram into a nice diagram, in which counting holomorphic curves is combinatorial. Ozsváth, Stipsicz, and Szabó [13] gave the first combinatorial proof of invariance for $\widehat{\mathrm{HF}}$, using another way to convert general Heegaard diagrams into *convenient* diagrams — a more restricted kind of nice diagram — and by studying Heegaard moves on convenient diagrams.

Linear-categorical representations of important groups in topology have also been investigated before. Bordered Floer theory actually gives a family of representations of the strongly based mapping class groupoid. For a given genus g, the representations are indexed by an integer w, called the weight, between $-g$ and g. The representation that is relevant for 3–manifold invariants, and that we will focus on in this paper, corresponds to $w = 0$. The cases $w = \pm g$ are trivial. The cases $w = \pm(g-1)$ are described combinatorially by Lipshitz, Ozsváth and Thurston [9], and a combinatorial proof of invariance is given by Siegel [19]. Linear-categorical representations of other groups occurring in topology have also been studied; see the introduction by Khovanov and Thomas [7] for a review and a list of references. One major example is linear-categorical representations of the braid group, studied by, for example, Khovanov and Seidel [6], Seidel and Thomas [18], Cautis and Kamnitzer [3], Seidel and Smith [17], and Khovanov [5].

In Section 2, we review the structure of bordered Floer theory, and describe its combinatorial construction as considered here. In Section 3, we prove some preliminary results on type *DA* bimodules and our construction of the type *DA* invariants. Using these results, we prove in Section 4 the intermediate statement on the linear-categorical representation of the strongly based mapping class groupoid. Finally, we complete the proof of invariance for closed 3–manifolds in Section 5.

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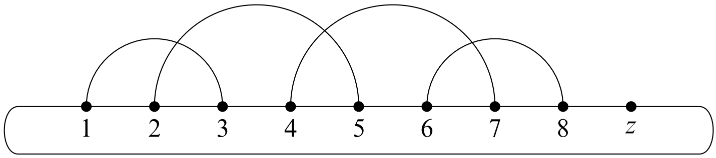


Figure 1: Linear pointed matched circle with $k=2$

2 Overview of the construction

In the first part of this section, we briefly review the structure of bordered Floer theory, as is defined analytically in [10; 12]. In the second part, we describe some of the existing combinatorial constructions given in [11], and then the construction that will be studied in this paper.

2A Pointed matched circles and strand algebras

In bordered Floer theory, the connected, compact, orientable surfaces that serve as boundary components of 3–manifolds are specified using pointed matched circles. A pointed matched circle is a quadruple $\mathcal{Z}=(Z,z,\boldsymbol{a},M)$, consisting of a circle Z, a point $z\in Z$, a set of $4k$ points $\boldsymbol{a}\subset Z\setminus\{z\}$, and a two-to-one map M from $\boldsymbol{a}$ to $\{1,2,\dots,2k\}$, pairing the points in $\boldsymbol{a}$, that satisfies the following condition: if we thicken the circle Z to an annulus $Z\times[0,1]$ and attach a 1–handle to the outside boundary $Z\times\{1\}$ of the annulus joining each pair of points in $\boldsymbol{a}$, then the new outside boundary must be a single circle. Given this requirement, we may glue a disk onto that boundary, obtaining a genus k surface $F^\circ(\mathcal{Z})$ with one boundary component $Z\times\{0\}$ and a basepoint z on the boundary. We say that the pointed matched circle $\mathcal{Z}$ *parametrizes* $F^\circ(\mathcal{Z})$. Let $-\mathcal{Z}$ be the pointed matched circle obtained by reversing orientation on $\mathcal{Z}$. Then $F^\circ(-\mathcal{Z})$ is the orientation reversal of $F^\circ(\mathcal{Z})$.

Let $F(\mathcal{Z})$ be the result of filling the boundary of $F^\circ(\mathcal{Z})$ with a disk. Then $F(\mathcal{Z})$ is a closed surface of genus k, marked with a homotopically trivial circle Z and a basepoint $z\in Z$. We will also say $F(\mathcal{Z})$ is parametrized by $\mathcal{Z}$.

An example of a pointed matched circle for $k=2$ is shown in Figure 1.

To each pointed matched circle $\mathcal{Z}$, bordered Floer theory associates a combinatorially defined DG algebra $\mathcal{A}(\mathcal{Z})$. We refer to the original papers for the description of $\mathcal{A}(\mathcal{Z})$. Here we just fix some notations and terminologies used in this paper. For any generator $a\in\mathcal{A}(\mathcal{Z})$, the *multiplicity* of a, denoted $\operatorname{mult}(a)$, is an element in $H_1(Z\setminus\{z\},\boldsymbol{a})$ recording how many times the strands in a cover each nonbasepoint interval on Z. The *length* of a is the sum of coefficients in $\operatorname{mult}(a)$. Equivalently, it is the sum of lengths of the strands in a. It is clear from the definitions that the algebra $\mathcal{A}(-\mathcal{Z})$ is the opposite algebra of $\mathcal{A}(\mathcal{Z})$. In particular, there is a canonical identification of their generators. For any $a \in \mathcal{A}(\mathcal{Z})$, let $\bar{a}$ denote the corresponding element in $\mathcal{A}(-\mathcal{Z})$. If $i \in \mathcal{A}(\mathcal{Z})$ is an idempotent, let $o(i) \in \mathcal{A}(\mathcal{Z})$ denote the idempotent complementary to i. A *chord* is a single strand on Z. For any given chord ξ, we define $a(\xi) \in \mathcal{A}(\mathcal{Z})$ to be the sum of all generators that result from adding horizontal strands to ξ.

Given a 3–manifold Y with one boundary component ∂Y, a parametrization of ∂Y by a pointed matched circle $\mathcal{Z} = (Z, z, \boldsymbol{a}, M)$ is a diffeomorphism $\phi\colon F(\mathcal{Z}) \to \partial Y$. This marks ∂Y with a circle and a basepoint on the circle, which by abuse of notation we will also call Z and z. Bordered Floer theory associates two invariants to a 3–manifold Y with boundary ∂Y parametrized by $\mathcal{Z}$:

- A type A invariant $\widehat{\mathrm{CFA}}(Y)_{\mathcal{A}(\mathcal{Z})}$ is a right A_∞–module over $\mathcal{A}(\mathcal{Z})$.
- A type D invariant ${}^{\mathcal{A}(-\mathcal{Z})}\widehat{\mathrm{CFD}}(Y)$ is a left type D module over $\mathcal{A}(-\mathcal{Z})$.

They are invariants of Y up to homotopy equivalence of A_∞–modules or type D modules. We use the following standard convention in expressing types of actions on the module: each algebra is written on the side it acts on, subscripts indicate A_∞–actions, and superscripts indicate type D actions. These may be omitted when there is no danger of confusion.

These invariants satisfy the following pairing theorem: let Y_1 and Y_2 be two 3–manifolds with boundaries parametrized by $\mathcal{Z}$ and $-\mathcal{Z}$, respectively. Let $Y = Y_1 \cup_\partial Y_2$ be the closed 3–manifold obtained by gluing them along their boundaries (with the gluing map induced by the parametrizations). Then the chain complex $\widehat{\mathrm{CF}}(Y)$, whose homology is $\widehat{\mathrm{HF}}(Y)$, is given by

$$\widehat{\mathrm{CF}}(Y) \simeq \widehat{\mathrm{CFA}}(Y_1)_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{\mathrm{CFD}}(Y_2) \tag{1}$$

[10, Theorem 1.3].

The theory extends to 3–manifolds with two boundary components as follows: given Y with two boundary components $\partial_L Y$ and $\partial_R Y$, fix parametrizations $\phi_1\colon F(\mathcal{Z}_1) \to \partial_L Y$ and $\phi_2\colon F(\mathcal{Z}_2) \to \partial_R Y$. This induces circles Z_1 and Z_2 on $\partial_L Y$ and $\partial_R Y$, and basepoints $z_1 \in Z_1, z_2 \in Z_2$. We further fix a map γ from the framed cylinder $(S^1, z) \times [0,1]$ into Y, so that $(S^1, z) \times \{0\}$ and $(S^1, z) \times \{1\}$ map to (Z_1, z_1) and (Z_2, z_2), respectively. We call the totality of the data $(Y, \partial_L Y, \partial_R Y, \phi_1, \phi_2, \gamma)$ a *strongly bordered* 3*–manifold with two boundary components*. From now on whenever we mention a 3–manifold Y with two boundary components, we mean a strongly bordered 3–manifold, omitting the other data when they are clear from context. To a 3–manifold Y with two boundary components, bordered Floer theory associates the following invariants:

- A type *AA* invariant $\widehat{\mathrm{CFAA}}(Y)_{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}$ is a right A_∞–bimodule over $\mathcal{A}(\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_2)$.
- A type *DD* invariant ${}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFDD}}(Y)$ is a left type D bimodule over $\mathcal{A}(-\mathcal{Z}_1)$ and $\mathcal{A}(-\mathcal{Z}_2)$.
- A type *DA* invariant ${}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(Y)_{\mathcal{A}(\mathcal{Z}_2)}$ is a left type D, right A_∞–bimodule over $\mathcal{A}(-\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_2)$.
- A type *AD* invariant ${}^{\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFAD}}(Y)_{\mathcal{A}(\mathcal{Z}_1)}$ is a right A_∞, left type D bimodule over $\mathcal{A}(\mathcal{Z}_1)$ and $\mathcal{A}(-\mathcal{Z}_2)$.

These bimodules satisfy similar pairing theorems, as described in [12, Section 7.1]. The general rule is that box tensor products can be taken between a right A_∞–action and a left type D action over the same algebra $\mathcal{A}(\mathcal{Z})$. Taking this box tensor product corresponds to gluing two boundaries parametrized by $\mathcal{Z}$ and $-\mathcal{Z}$.

Following the convention in [21], we will write actions on the various kinds of modules and bimodules as sums of *arrows*. For example, if the coefficient of $\boldsymbol{y}$ is 1 in $m_{1,i,j}(\boldsymbol{x};a_1,\dots,a_i;b_1,\dots,b_j)$, where each $a_k, 1\le k\le i$ and $b_l, 1\le l\le j$ is a generator of the appropriate algebra, we say there is an arrow

$$m_{1,i,j}\colon (\boldsymbol{x};a_1,\dots,a_i;b_1,\dots,b_j)\to \boldsymbol{y}.$$

Likewise, an arrow in the type *DA* action is of the form

$$\delta^1_{1+i}\colon (\boldsymbol{x},a_1,\dots,a_i)\to b\otimes \boldsymbol{y},$$

and an arrow in the type *DD* action is of the form

$$\delta^1\colon \boldsymbol{x}\to a\otimes b\otimes \boldsymbol{y}.$$

We will also need the concept of duality on bimodules, called opposite structures in [12, Definition 2.2.31, 2.2.53]. For a left type *DD* bimodule ${}^{A,B}M$, its *dual* $\overline{M}^{A,B}$ is the type *DD* bimodule over the same generators, where each arrow $\delta^1_M\colon \boldsymbol{x}\to a_1\otimes a_2\otimes \boldsymbol{y}$ in the type *DD* action of ${}^{A,B}M$ corresponds to an arrow $\delta^1_{\overline{M}}\colon \boldsymbol{y}\to a_1\otimes a_2\otimes \boldsymbol{x}$ in the type *DD* action of $\overline{M}^{A,B}$. In this way the left actions by A and B (equivalently left actions by A^{opp} and B^{opp}) become right actions. Thus, we will also write the dual as ${}^{A^{\mathrm{opp}},B^{\mathrm{opp}}}\overline{M}$. Similarly, we can define duals on type *DA* and type *AA* bimodules. The dual commutes with box tensor product, that is,

$$\overline{M_A\boxtimes {}^AN}=\overline{N}^A\boxtimes {}_A\overline{M}=\overline{M}_{A^{\mathrm{opp}}}\boxtimes {}^{A^{\mathrm{opp}}}\overline{N},$$

where M and N may have additional actions.

2B Gradings on bordered invariants

In this section we give a brief overview of gradings on the bordered invariants. For details, see [10, Chapter 10] and [12, Section 6.5].

We begin with gradings on the DG algebra $\mathcal{A}(\mathcal{Z})$. There are two kinds of gradings: one by a larger group $G'(\mathcal{Z})$, and a refined grading by a smaller group $G(\mathcal{Z})$. Both $G(\mathcal{Z})$ and $G'(\mathcal{Z})$ are noncommutative, equipped with a distinguished central element λ.

An element of $G'(\mathcal{Z})$ is specified by a pair (k,α), where $k \in \frac{1}{2}\mathbb{Z}$ and $\alpha \in H_1(Z', \boldsymbol{a})$. With points of $\boldsymbol{a}$ labeled $1,\dots,4k$, we can write α as a sequence of integers α_i, $1 \le i \le 4k-1$, where α_i is the multiplicity of α at the interval $[i, i+1]$. Then multiplication on $G'(\mathcal{Z})$ is defined by

$$(k,\alpha)\cdot(l,\beta) = (k+l+L(\alpha,\beta), \alpha+\beta),$$

where

$$L(\alpha,\beta) = \sum_{i=1}^{4k-2} \tfrac{1}{2}(\alpha_i\beta_{i+1} - \alpha_{i+1}\beta_i).$$

Actually, the grading lies in an index 2 subgroup of $G'(\mathcal{Z})$, but we will not make use of this here.

For later use, we define an antihomomorphism

$$R\colon G'(\mathcal{Z}) \to G'(-\mathcal{Z})$$

given by

$$R(k,\alpha_1,\dots,\alpha_{4k-1}) = (k, -\alpha_{4k-1},\dots,-\alpha_1).$$

To define the grading of a generator of $\mathcal{A}(\mathcal{Z})$, we first define a map

$$m\colon H_1(Z',\boldsymbol{a}) \times H_0(\boldsymbol{a}) \to \tfrac{1}{2}\mathbb{Z}.$$

For an interval α (with orientation from Z) and a point p, let $m(\alpha, p)$ equal 1 if p is in the interior of α, $\frac{1}{2}$ if p is on the boundary, and 0 otherwise. This is then extended bilinearly to all of $H_1(Z',\boldsymbol{a}) \times H_0(\boldsymbol{a})$ to define m.

Given a generator $a \in \mathcal{A}(\mathcal{Z})$, let $\boldsymbol{\rho}$ be the nonhorizontal strands of a. Let $\mathrm{inv}(\boldsymbol{\rho})$ be the number of inversions in $\boldsymbol{\rho}$, $S \in H_0(\boldsymbol{a})$ be the starting points of $\boldsymbol{\rho}$, and $[a] \in H_1(Z',\boldsymbol{a})$ be the multiplicity of a. Then

$$\mathrm{gr}'(a) = (\mathrm{inv}(\boldsymbol{\rho}) - m([a], S), [a]).$$

Next, we consider relative gradings on the type D invariant. Fix a bordered Heegaard diagram $\mathcal{H}$. Let $\boldsymbol{x}, \boldsymbol{y}$ be generators and $B \in \pi_2(\boldsymbol{x},\boldsymbol{y})$, define $g'(B) \in G'(\mathcal{Z})$ as

$$g'(B) = (-e(B) - n_{\boldsymbol{x}}(B) - n_{\boldsymbol{y}}(B), \partial^{\partial}(B)).$$

Here $e(B)$ is the Euler measure of B, and $n_{\boldsymbol{x}}(B), n_{\boldsymbol{y}}(B)$ are multiplicities of B at $\boldsymbol{x}, \boldsymbol{y}$ (each corner around $\boldsymbol{x}$ or $\boldsymbol{y}$ counts as multiplicity $\frac{1}{4}$), and $\partial^{\partial}(B)$ is the boundary of B on $H_1(Z', \boldsymbol{a})$.

There is a grading set for each spin^c class on $\mathcal{H}$ (in most bordered cases we consider here, there is just one spin^c class). The grading set $S'_D(\mathcal{H}, \mathfrak{s})$ for the Heegaard diagram $\mathcal{H}$ and spin^c class $\mathfrak{s}$ is defined as follows: choose a base generator $\boldsymbol{x}_0$ with spin^c class $\mathfrak{s}$. Let $P'(\boldsymbol{x}_0)$ be the set of $g'(P)$ for all $P \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x}_0)$ (the domains in $\pi_2(\boldsymbol{x}_0, \boldsymbol{x}_0)$ are called *periodic domains*). Then

$$S'_D(\mathcal{H}, \mathfrak{s}) = G'(-\mathcal{Z})/R(P'(\boldsymbol{x}_0)).$$

This grading set has an obvious left action by $G'(\mathcal{Z})$. For another generator $\boldsymbol{x}$ in the same spin^c class, choose a domain $B_0 \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x})$, and set

$$\mathrm{gr}'(\boldsymbol{x}) = R(g'(B_0)) \cdot R(P'(\boldsymbol{x}_0)).$$

The type D action respects this relative grading in the sense that, for each arrow $\delta^1\colon \boldsymbol{x} \to a \otimes \boldsymbol{y}$ in the action, we have

$$\lambda^{-1}\,\mathrm{gr}'(\boldsymbol{x}) = \mathrm{gr}'(a)\,\mathrm{gr}'(\boldsymbol{y}).$$

Relative gradings on type A invariants are similar. The grading set is

$$S'_A(\mathcal{H}, \mathfrak{s}) = P'(\boldsymbol{x}_0)\backslash G'(\mathcal{Z}).$$

This carries a natural right action by $G'(\mathcal{Z})$. For any generator $\boldsymbol{x}$ in the spin^c class $\mathfrak{s}$, choose a domain $B_0 \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x})$ and set

$$\mathrm{gr}'(\boldsymbol{x}) = P'(\boldsymbol{x}_0) \cdot g'(B_0).$$

The A_∞–action respects the relative grading in the sense that, for each arrow

$$m_{1,k}\colon (\boldsymbol{x}; a_1, \dots, a_k) \to \boldsymbol{y},$$

we have

$$\lambda^{k-1}\,\mathrm{gr}'(\boldsymbol{x})\,\mathrm{gr}'(a_1)\cdots\mathrm{gr}'(a_k) = \mathrm{gr}'(\boldsymbol{y}).$$

Gradings on bimodules are defined similarly. In particular, a domain in a Heegaard diagram with two boundary components parametrized by $\mathcal{Z}_1$ and $\mathcal{Z}_2$ gives rise to an element of

$$G'(\mathcal{Z}_1) \times_\lambda G'(\mathcal{Z}_2) = G'(\mathcal{Z}_1) \times G'(\mathcal{Z}_2)/(\lambda_1 = \lambda_2).$$

The grading set is a certain coset of $G'(\mathcal{Z}_1) \times_\lambda G'(\mathcal{Z}_2)$.

Now we briefly discuss refined gradings, which contain essentially the same information, but are cleaner to work with theoretically.

The group $G(\mathcal{Z})$ can be considered as a subgroup of $G'(\mathcal{Z})$, generated by λ and elements of the form $(0,[p,q])$, where p,q is a pair of matched points, and $[p,q]$ denotes the interval in $H_1(Z',\boldsymbol{a})$ between p and q. An element (k,α) of $G'(\mathcal{Z})$ is in $G(\mathcal{Z})$ if and only if $M_*(\partial\alpha)=0$, where $\partial\colon H_1(Z',\boldsymbol{a})\to H_0(\boldsymbol{a})$ is the boundary operator and $M_*\colon H_0(\boldsymbol{a})\to\mathbb{Z}^{2k}$ is a map sending each matched pair of points to the same basis element of $\mathbb{Z}^{2k}$.

To construct the refined grading on $\mathcal{A}(\mathcal{Z})$, we first choose a base idempotent s_0 in $\mathcal{A}(\mathcal{Z})$. Then for every idempotent s, choose a grading element $\psi(s)=(k,\alpha)\in G'(\mathcal{Z})$ such that $M_*(\partial\alpha)=s-s_0$. For an algebra element a with left idempotent s and right idempotent t, we set

$$\mathrm{gr}(a)=\psi(s)\,\mathrm{gr}'(a)\psi(t)^{-1}.$$

It is easy to check that this element lies in $G(\mathcal{Z})$ and that the two conditions on the grading are satisfied.

Similarly, we can refine the grading on the bordered invariants to use $G(\mathcal{Z})$ rather than $G'(\mathcal{Z})$. We will omit the details here.

We will not perform any detailed grading computations in this paper, but will simply note that all such computations can be done combinatorially from the Heegaard diagram. For a module (or bimodule) M of any type, grading imposes a constraint on what kind of arrows can appear in the A_∞ or type D action on M. One such constraint is this: if a domain in a Heegaard diagram with two boundary components touches the two boundaries at intervals i and i', respectively, then for each arrow in the algebra action of a bimodule corresponding to that Heegaard diagram, its multiplicities at i and at i' must be the same. Such constraints are crucial in establishing uniqueness properties of bimodule invariants, to be discussed in the following sections.

2C The strongly based mapping class groupoid

An important class of 3–manifolds with two boundary components is the mapping cylinders of surface diffeomorphisms. Gluing with these 3–manifolds can be considered as changing the parametrization on the boundary of a bordered 3–manifold.

The strongly based mapping class groupoid of genus g is a category whose objects are pointed matched circles with $4g$ points. Each object $\mathcal{Z}$ corresponds to a surface $F^\circ(\mathcal{Z})$ of genus g, with standard parametrization by $\mathcal{Z}$. The morphisms from $\mathcal{Z}_1$ to $\mathcal{Z}_2$ in the category are isotopy classes of diffeomorphisms $\phi\colon F^\circ(\mathcal{Z}_1)\to F^\circ(\mathcal{Z}_2)$, sending the basepoint $z_1\in F^\circ(\mathcal{Z}_1)$ to the basepoint $z_2\in F^\circ(\mathcal{Z}_2)$. Identity and composition in the category correspond to the identity diffeomorphism and composition of diffeomorphisms, respectively.

If we fix a pointed matched circle $\mathcal{Z}$ and only consider morphisms from $\mathcal{Z}$ to itself, we obtain the strongly based mapping class group of $F_{g,1}$ (where $F_{g,1}$ denotes a genus g surface with one circle boundary). This is simply the group of isotopy classes of boundary-preserving self-diffeomorphisms of $F_{g,1}$.

Given a diffeomorphism $\phi\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, we can construct its mapping cylinder $Y(\phi) = F(\mathcal{Z}_2) \times [0,1]$ as a strongly bordered 3–manifold with two boundary components, parametrized by $-\mathcal{Z}_1$ and $\mathcal{Z}_2$. The left boundary $\partial_L Y(\phi) = F(\mathcal{Z}_2) \times \{0\}$ is parametrized by the induced map $\phi_*\colon -F(\mathcal{Z}_1) \to -F(\mathcal{Z}_2)$ (reverse orientation and extend over the disk filling the boundary), while the right boundary is parametrized by the identity map on $F(\mathcal{Z}_2)$. The map $\gamma\colon (S^1, z) \times [0,1] \to Y(\phi)$ is simply the inclusion $(Z, z) \times [0,1] \to F(\mathcal{Z}_2) \times [0,1]$.

This establishes a one-to-one correspondence between strongly bordered 3–manifolds that are topologically $F_g \times [0,1]$, and morphisms in the strongly based mapping class groupoid with genus g. For a morphism $\phi\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, we write $\widehat{\mathrm{CFAA}}(\phi)_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}$ to denote the type *AA* invariant $\widehat{\mathrm{CFAA}}(Y(\phi))_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}$ associated to the mapping cylinder of ϕ. Likewise, we use notations ${}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(\phi)_{\mathcal{A}(\mathcal{Z}_2)}$ and ${}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFDD}}(\phi)$ for the other invariants corresponding to $Y(\phi)$.

For future reference, we write down the pairing theorems involving *DA* invariants. For morphisms $\phi_1\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ and $\phi_2\colon F^\circ(\mathcal{Z}_2) \to F^\circ(\mathcal{Z}_3)$, the *DA* invariant for $\phi_2 \circ \phi_1\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_3)$ is given by

$$ {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(\phi_2 \circ \phi_1)_{\mathcal{A}(\mathcal{Z}_3)} = {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(\phi_1)_{\mathcal{A}(\mathcal{Z}_2)} \boxtimes {}^{\mathcal{A}(\mathcal{Z}_2)}\widehat{\mathrm{CFDA}}(\phi_2)_{\mathcal{A}(\mathcal{Z}_3)}. \tag{2} $$

For a morphism $\phi\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ and a 3–manifold Y with boundary parametrized by $\psi\colon F(-\mathcal{Z}_2) \to \partial Y$, let Y' be the same manifold with boundary parametrized by $\psi \circ \phi_*\colon F(-\mathcal{Z}_1) \to \partial Y$, then

$$ {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathrm{CFD}}(Y') = {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(\phi)_{\mathcal{A}(\mathcal{Z}_2)} \boxtimes {}^{\mathcal{A}(\mathcal{Z}_2)}\widehat{\mathrm{CFD}}(Y). \tag{3} $$

2D Invariants of the identity diffeomorphism

Let $\mathbb{I}_{\mathcal{Z}}$ be the identity morphism $F^\circ(\mathcal{Z}) \to F^\circ(\mathcal{Z})$. All bimodule invariants associated to $\mathbb{I}_{\mathcal{Z}}$ have special significance in the theory. First, it can be shown [12, Section 8.1] that

$$ \widehat{\mathrm{CFDA}}(\mathbb{I}_{\mathcal{Z}}) \simeq {}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}, $$

where the latter denotes the identity type *DA* bimodule over $\mathcal{A}(\mathcal{Z})$. This is the bimodule generated over $\mathbb{F}_2$ by idempotents of $\mathcal{A}(\mathcal{Z})$, and with the algebra action given by

$$ \delta^1_2(i, a) = a \otimes j, $$

for any generator $a \in \mathcal{A}(\mathcal{Z})$, where i and j are the left and right idempotents of a.

The type *DD* invariant ${}^{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z})}\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}})$ and *AA* invariant $\widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z})}$ relate the type A and type D invariants through taking the tensor product. For any 3–manifold Y with one boundary component parametrized by $\mathcal{Z}$, the relations are

$$\widehat{\mathrm{CFA}}(Y)_{\mathcal{A}(\mathcal{Z})} = \widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(-\mathcal{Z})}\widehat{\mathrm{CFD}}(Y), \tag{4}$$

$${}^{\mathcal{A}(-\mathcal{Z})}\widehat{\mathrm{CFD}}(Y) = \widehat{\mathrm{CFA}}(Y)_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z})}\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}}). \tag{5}$$

One implication is that $\widehat{\mathrm{CFD}}(Y)$ and $\widehat{\mathrm{CFA}}(Y)$ contain the same information about Y. Likewise, there are relations among the bimodule invariants, showing that all bimodule invariants also contain the same information. For any 3–manifold Y with two boundary components parametrized by $\mathcal{Z}_1$ and $\mathcal{Z}_2$, we have

$$\begin{aligned}&{}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(Y)_{\mathcal{A}(\mathcal{Z}_2)}\\ &\qquad = \widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}_2})_{\mathcal{A}(-\mathcal{Z}_2),\mathcal{A}(\mathcal{Z}_2)} \boxtimes_{\mathcal{A}(-\mathcal{Z}_2)} {}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFDD}}(Y),\end{aligned} \tag{6}$$

$$\begin{aligned}&\widehat{\mathrm{CFAA}}(Y)_{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}\\ &\qquad = \widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}_1})_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_1)} \boxtimes_{\mathcal{A}(-\mathcal{Z}_1)} {}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(Y)_{\mathcal{A}(\mathcal{Z}_2)},\end{aligned} \tag{7}$$

$$\begin{aligned}&{}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(Y)_{\mathcal{A}(\mathcal{Z}_2)}\\ &\qquad = \widehat{\mathrm{CFAA}}(Y)_{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)} \boxtimes_{\mathcal{A}(\mathcal{Z}_1)} {}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_1)}\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}_1}),\end{aligned} \tag{8}$$

$$\begin{aligned}&{}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFDD}}(Y)\\ &\qquad = {}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(Y)_{\mathcal{A}(\mathcal{Z}_2)} \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} {}^{\mathcal{A}(\mathcal{Z}_2),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}_2}).\end{aligned} \tag{9}$$

The above equations are special cases of the pairing theorems, where one of the bordered 3–manifolds is a cylinder with trivial parametrization. They indicate the importance of finding combinatorial descriptions of $\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}})$ and $\widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})$, which we now consider.

First, we describe the combinatorial model $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ of ${}^{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z})}\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}})$, given in [11, Theorem 1]. It is generated over $\mathbb{F}_2$ by the set of pairs of complementary idempotents $i \otimes i'$, with $i \in \mathcal{A}(\mathcal{Z})$ and $i' = \overline{o(i)} \in \mathcal{A}(-\mathcal{Z})$. The type *DD* action is given by

$$\delta^1(i \otimes i') = \sum_{\substack{\xi \in \mathcal{C} \\ i a(\xi) = a(\xi) j \\ i' \overline{a(\xi)} = \overline{a(\xi)} j'}} (a(\xi) \otimes \overline{a(\xi)}) \otimes (j \otimes j'), \tag{10}$$

where $\mathcal{C}$ is the set of chords on $\mathcal{Z}$ whose two endpoints are not matched. Intuitively, the arrows in the type *DD* action are exactly those whose two algebra outputs both contain exactly one chord connecting two unpaired points, and covering corresponding intervals in $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$.

Next, we consider the invariant $\widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z})}$. A formula for it is given in [12, Proposition 9.2] as follows (here we simplify $\mathcal{A}(\mathcal{Z})$ to $\mathcal{A}$ and $\mathcal{A}(-\mathcal{Z})$ to $\mathcal{A}'$):

$$
\begin{aligned}
\widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}',\mathcal{A}} &= \mathrm{Mor}^{\mathcal{A}}\big({}_{\mathcal{A}'}\mathcal{A}'_{\mathcal{A}'} \boxtimes_{\mathcal{A}'} {}^{\mathcal{A},\mathcal{A}'}\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}}), {}^{\mathcal{A}}\mathbb{I}_{\mathcal{A}}\big) \\
&= \big(\overline{\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}})}^{\mathcal{A}',\mathcal{A}} \boxtimes_{\mathcal{A}'} {}_{\mathcal{A}'}\overline{\mathcal{A}'}_{\mathcal{A}'}\big) \boxtimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}.
\end{aligned} \tag{11}
$$

This bimodule has a large number of generators, making it difficult to use for the computations needed in this paper. The main result of [21] is to describe a bimodule $\widehat{\mathcal{A}\mathcal{A}}(\mathbb{I}_{\mathcal{Z}})$ homotopy equivalent to this (and hence is also a combinatorial model of $\widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})$), but with a minimal number of generators. The bimodule $\widehat{\mathcal{A}\mathcal{A}}(\mathbb{I}_{\mathcal{Z}})$ is generated over $\mathbb{F}_2$ by the set of pairs of complementary idempotents, but with much more complex A_∞–bimodule actions. We will briefly review this construction in Section 3C.

One of the pairing theorems imply the following relation among the combinatorial models for $\mathbb{I}_{\mathcal{Z}}$:

$$
{}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})} \simeq \widehat{\mathcal{A}\mathcal{A}}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z})} \boxtimes_{\mathcal{A}(-\mathcal{Z})} {}^{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z})}\widehat{\mathcal{D}\mathcal{D}}(\mathbb{I}_{\mathcal{Z}}). \tag{12}
$$

The two sides are not equal but only homotopy equivalent. This homotopy equivalence is proven combinatorially as Corollary 3.10 in Section 3D.

2E Invariants of arcslides

The strongly based mapping class groupoid is generated by a particularly simple class of morphisms called *arcslides*. We will now review their definitions and the invariants associated to them. The relations among arcslides will be described at the beginning of Section 3.

Given a pointed matched circle $\mathcal{Z}_1$, and two matched pairs of points $B = (b_1, b_2)$ and $C = (c_1, c_2)$ in $\boldsymbol{a} \subset Z_1$, such that b_1 and c_1 are adjacent in $\boldsymbol{a}$, an arcslide of b_1 over c_1 moves b_1 to be adjacent to c_2, on the side opposite to its original position with respect to c_1. This results in a new pointed matched circle $\mathcal{Z}_2$. Such a move corresponds to a certain diffeomorphism $F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, which we will also call an arcslide. See Figure 2 for two examples of arcslides. The first example is an *overslide* meaning b_1 is outside the interval $[c_1, c_2]$. The second example is an *underslide* meaning b_1 is inside that interval.

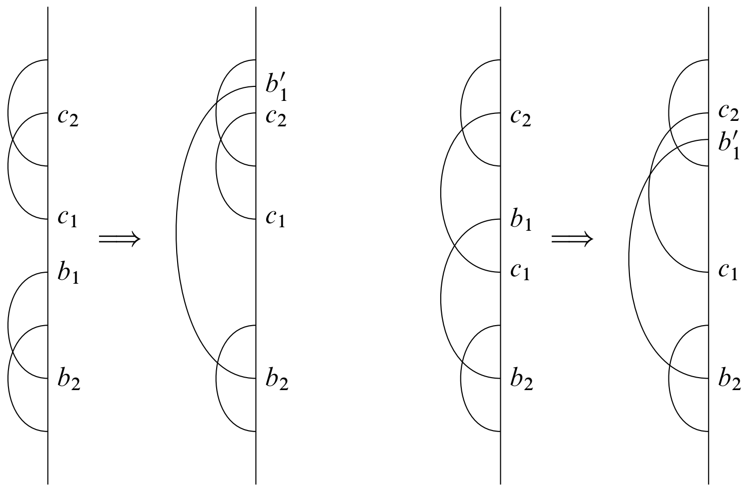


Figure 2: Two examples of arcslides

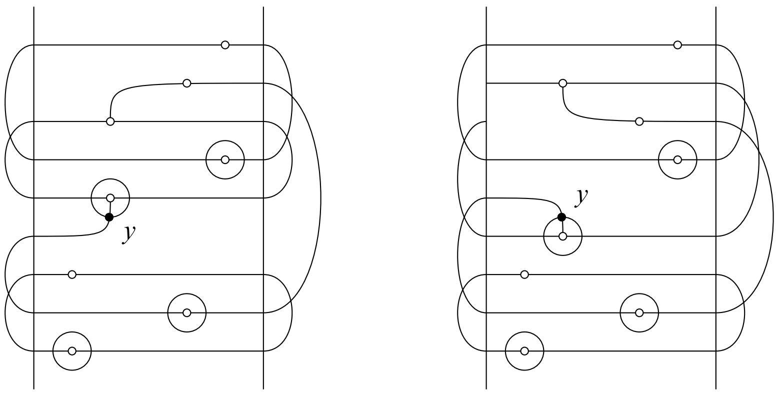


Figure 3: Examples of Heegaard diagrams of arcslides

Given an arcslide $\tau\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, the invariant $\widehat{\mathrm{CFDD}}(\tau)$ is a left type *DD* bimodule over $\mathcal{A}(\mathcal{Z}_1)$ and $\mathcal{A}(-\mathcal{Z}_2)$. Constructing a combinatorial model of this bimodule, denoted $\widehat{\mathcal{DD}}(\tau)$, is the main subject of [11]. This model is computed from a standard Heegaard diagram for the mapping cylinder $Y(\tau)$. For the two arcslides in Figure 2, these standard Heegaard diagrams are shown in Figure 3. The tiny circles in the diagrams are 1–handle attachment points, paired according to their vertical positions. The larger circles are β circles, and all other arcs inside the boundary are α–arcs. Later on, we will draw more schematic versions of these diagrams, omitting some of the β circles and attaching points of 1–handles.

Each generator of $\widehat{\mathcal{DD}}(\tau)$ corresponds to a g–tuple of intersection points between α and β curves, where each β circle contains exactly one point, and each α–arc contains at most one point. The (type D) idempotent of the generator specifies which pairs of α–arcs are *not* occupied by the generator. For the standard Heegaard diagram of arcslides, a generator is uniquely specified by its idempotents on the two sides.

There is an obvious identification between pairs of points on the two sides, using which we can talk about two idempotents on different sides being complementary, etc. There are two types of generators in $\widehat{\mathcal{DD}}(\tau)$. A generator of type X has complementary idempotents, and a generator of type Y has idempotents that are complementary except for both containing the C pair and neither containing the B pair. The type X generators are those that do not occupy the intersection point y in Figure 3, while the type Y generators do.

The type *DD* action on the bimodule can be described as follows: given a pointed matched circle $\mathcal{Z}$, let $C(\mathcal{Z})$ denote the collection of sets of chords in $\mathcal{Z}$. For some $\xi \in C(\mathcal{Z})$, let $a(\xi) \in \mathcal{A}(\mathcal{Z})$ denote the sum of all generators of $\mathcal{A}(\mathcal{Z})$ produced by adding horizontal strands to ξ (this definition extends the case where ξ is a chord). For any arcslide $\tau\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$, there is a collection of pairs $C_\tau \subseteq \{(\xi_i, \eta_i) \mid \xi_i \in C(\mathcal{Z}_1), \eta_i \in C(\mathcal{Z}_2)\}$ such that

$$\delta^1(i \otimes i') = \sum_{\substack{(\xi_k,\eta_k)\in C_\tau \\ i a(\xi_k) = a(\xi_k) j \\ i' \overline{a(\eta_k)} = \overline{a(\eta_k)} j' \\ j \otimes j' \text{ is a generator}}} (a(\xi_k) \otimes \overline{a(\eta_k)}) \otimes (j \otimes j'), \tag{14}$$

where generators are represented by their type D idempotents.

Intuitively, there is a term in the type *DD* action whenever the idempotent agrees, and the moving strands part of the algebra coefficients match one of the fixed patterns. Depending on whether the arcslide is an underslide or an overslide, there are six or eight types of elements in C_τ. See [11, Figures 21 and 28] for diagrams of these patterns. Note that not all pairs in Figure 28 are actually in C_τ: there is an additional choice involved. In the following computations we will only use some of the simpler pairs, involving algebra elements that have small total length. In particular we will not need to consider any pair where a choice is necessary.

The following properties of $\widehat{\mathcal{DD}}(\tau)$ can be directly verified for the above description:

Relation with Heegaard diagram Every arrow in the type *DD* action comes from a domain in the Heegaard diagram. For a Heegaard diagram $\mathcal{H}$, write $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to denote the union of α and β curves, respectively. A *domain* in $\mathcal{H}$ is a nonnegative integral linear combination of connected components of $\mathcal{H}\setminus\{\boldsymbol{\alpha},\boldsymbol{\beta}\}$. Each arrow $\boldsymbol{x}\to a_1\otimes a_2\otimes \boldsymbol{y}$ in the type *DD* action comes from a domain B, such that a_1 and a_2 have multiplicities equal to the multiplicities of B at the corresponding boundaries. Moreover, let $\partial^\alpha B$ be the part of the boundary of B on the α curves, and let $\partial(\partial^\alpha B)$ be the part of the boundary of $\partial^\alpha B$ in the interior of the diagram, as a signed sum of intersection points, then $\partial(\partial^\alpha B)=\boldsymbol{x}-\boldsymbol{y}$. Intuitively, the α–boundaries of B start at points of $\boldsymbol{x}$ and end at points of $\boldsymbol{y}$, and vice versa for the β–boundary.

Grading There is a refined grading on the generators of $\widehat{\mathcal{DD}}(\tau)$ to a particular grading set S_τ, which has left-right actions by $G(\mathcal{Z}_1)$ and $G(\mathcal{Z}_2)$. Both actions are free and transitive, which means S_τ induces a group isomorphism $G(\mathcal{Z}_1)\to G(\mathcal{Z}_2)$. This group isomorphism is an invariant of τ, up to composing by inner automorphisms of $G(\mathcal{Z}_1)$ and $G(\mathcal{Z}_2)$. In other words, τ induces an element in the set of outer isomorphisms $\mathrm{Out}(G(\mathcal{Z}_1),G(\mathcal{Z}_2))$. In fact, this outer isomorphism corresponds to the actions of τ on the homology of the surface; see [11, Section 6.2] for details.

Stabilization Given arcslide $\tau\colon F^\circ(\mathcal{Z}_1)\to F^\circ(\mathcal{Z}_2)$, let $\mathring{\mathcal{Z}}_1=\mathcal{Z}_1\#\mathcal{Z}^1$ and $\mathring{\mathcal{Z}}_2=\mathcal{Z}_2\#\mathcal{Z}^1$, where $\mathcal{Z}^1$ is the genus 1 pointed matched circle, and # denotes connect sum on pointed matched circles. Let $\mathring{\tau}\colon F(\mathring{\mathcal{Z}}_1)\to F(\mathring{\mathcal{Z}}_2)$ be the arcslide acting as identity on the new handle, and as τ elsewhere. This is called the *stabilization* of τ. Then $\widehat{\mathcal{DD}}(\tau)$ and $\widehat{\mathcal{DD}}(\mathring{\tau})$ are related as follows: fix any idempotent i_o on $\mathcal{Z}^1$ (occupying one of the two possible pairs), then there is an injection from generators of $\widehat{\mathcal{DD}}(\tau)$ into generators of $\widehat{\mathcal{DD}}(\mathring{\tau})$, sending $i\otimes i'$ in $\widehat{\mathcal{DD}}(\tau)$ to $(i\#i_o)\otimes(i'\#\overline{o(i_o)})$ in $\widehat{\mathcal{DD}}(\mathring{\tau})$. For any generator $\boldsymbol{x}$ in $\widehat{\mathcal{DD}}(\tau)$, let $\mathring{\boldsymbol{x}}$ be the corresponding generator in $\widehat{\mathcal{DD}}(\mathring{\tau})$. Then for any two generators $\boldsymbol{x}$, $\boldsymbol{y}$ in $\widehat{\mathcal{DD}}(\tau)$, there is a one-to-one correspondence between arrows from $\boldsymbol{x}$ to $\boldsymbol{y}$ in the *DD* action of $\widehat{\mathcal{DD}}(\tau)$ and arrows from $\mathring{\boldsymbol{x}}$ to $\mathring{\boldsymbol{y}}$ in the *DD* action of $\widehat{\mathcal{DD}}(\mathring{\tau})$ that do not cover any region around the adjoined $\mathcal{Z}^1$, with $\boldsymbol{x}\to a\otimes b\otimes\boldsymbol{y}$ corresponding to $\mathring{\boldsymbol{x}}\to\mathring{a}\otimes\mathring{b}\otimes\mathring{\boldsymbol{y}}$, where $\mathring{a}$ and $\mathring{b}$ are obtained from a and b by adjoining the appropriate idempotents.

Duality For any arcslide $\tau\colon F^\circ(\mathcal{Z}_1)\to F^\circ(\mathcal{Z}_2)$, let $\bar{\tau}\colon F^\circ(-\mathcal{Z}_1)\to F^\circ(-\mathcal{Z}_2)$ be the arcslide with reversed orientation. Now we have that ${}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathcal{DD}}(\tau)$ and ${}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}\widehat{\mathcal{DD}}(\bar{\tau})$ are dual to each other (using definition of dual at the end of Section 2A); that is,

$$\overline{{}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathcal{DD}}(\tau)}\simeq {}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}\widehat{\mathcal{DD}}(\bar{\tau}). \tag{15}$$

Furthermore, the invariant ${}^{\mathcal{A}(\mathcal{Z}_2),\mathcal{A}(-\mathcal{Z}_1)}\widehat{\mathcal{DD}}(\tau^{-1})$ is homotopy equivalent to the right side of (15), after switching the two algebra actions. This comes from the fact that the mapping cylinder of τ^{-1} is the mirror image of the mapping cylinder of τ.

2F Main constructions

We now summarize the combinatorial constructions that will be studied in this paper. From here on, we will no longer use the analytical definitions of invariants, but define everything combinatorially from scratch. We will use notations such as $\widehat{\mathrm{CFAA}}$ to denote (combinatorially defined) homotopy equivalence classes of bimodules, and notations such as $\widehat{\mathcal{DD}}$, $\widehat{\mathcal{DA}}$ to denote particular combinatorial models in the equivalence classes of bimodules. For modules with one algebra action, we will use $\widehat{\mathrm{CFA}}$ and $\widehat{\mathrm{CFD}}$ for both models and equivalence classes, as no confusion will arise there. Since all combinatorial definitions below use either constructions derived from the analytical definition, or the appropriate box tensor product, it is clear that the entire construction agrees with the analytical definitions.

First, we will use models $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$, ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$, and $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ to define $\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}})$, $\widehat{\mathrm{CFDA}}(\mathbb{I}_{\mathcal{Z}})$, and $\widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})$, respectively. Then Corollary 3.10 shows (12) holds for our combinatorial construction. This means box tensoring with $\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}})$ and $\widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})$ are inverse operations on equivalence classes of bimodules.

Next, we define $\widehat{\mathcal{DA}}(\tau)$ as the box tensor product

$$
{}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathcal{DA}}(\tau)_{\mathcal{A}(\mathcal{Z}_2)} = \widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}_2})_{\mathcal{A}(-\mathcal{Z}_2),\mathcal{A}(\mathcal{Z}_2)} \boxtimes_{\mathcal{A}(-\mathcal{Z}_2)} {}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathcal{DD}}(\tau) \tag{16}
$$

Given this, we can define $\widehat{\mathcal{DA}}(\phi)$ for an arbitrary element ϕ of the strongly based mapping class groupoid, by factoring ϕ into arcslides. The precise statement is the following.

Construction 2.1 Given an element $\phi\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_{n+1})$ of the strongly based mapping class groupoid, with factorization $\phi = \tau_n \circ \cdots \circ \tau_1$, where $\tau_i\colon F^\circ(\mathcal{Z}_i) \to F^\circ(\mathcal{Z}_{i+1})$. Write $\boldsymbol{\tau}$ for the sequence $\tau_1, \dots, \tau_n$. Define

$$
{}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathcal{DA}}(\phi,\boldsymbol{\tau})_{\mathcal{A}(\mathcal{Z}_{n+1})} = {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathcal{DA}}(\tau_1)_{\mathcal{A}(\mathcal{Z}_2)} \boxtimes \cdots \boxtimes {}^{\mathcal{A}(\mathcal{Z}_n)}\widehat{\mathcal{DA}}(\tau_n)_{\mathcal{A}(\mathcal{Z}_{n+1})}.
$$

Theorem 2.2 *The homotopy type of $\widehat{\mathcal{DA}}(\phi,\boldsymbol{\tau})$ does not depend on the choice of factorization $\boldsymbol{\tau}$. Hence, $\widehat{\mathcal{DA}}(\phi,\boldsymbol{\tau})$ is an invariant of ϕ up to homotopy equivalence.*

This theorem is proven in Section 4. Given this, we can define $\widehat{\mathrm{CFDA}}(\phi)$ to be the equivalence class of $\widehat{\mathcal{DA}}(\phi,\boldsymbol{\tau})$, for any choice of factorization $\boldsymbol{\tau}$.

The other bimodule invariants $\widehat{\mathrm{CFDD}}(\phi)$, $\widehat{\mathrm{CFAA}}(\phi)$, and $\widehat{\mathrm{CFAD}}(\phi)$ for a general morphism $\phi\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ can be defined as follows:

$$
{}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFDD}}(\phi) = {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(\phi)_{\mathcal{A}(\mathcal{Z}_2)} \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} {}^{\mathcal{A}(\mathcal{Z}_2),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}_2}),
$$

$$
\widehat{\mathrm{CFAA}}(\phi)_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)} = \widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}_1})_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_1)} \boxtimes_{\mathcal{A}(\mathcal{Z}_1)} {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(\phi)_{\mathcal{A}(\mathcal{Z}_2)},
$$

$$
{}^{\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFAD}}(\phi)_{\mathcal{A}(-\mathcal{Z}_1)} = \widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}_1})_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_1)} \boxtimes_{\mathcal{A}(\mathcal{Z}_1)} {}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFDD}}(\phi).
$$

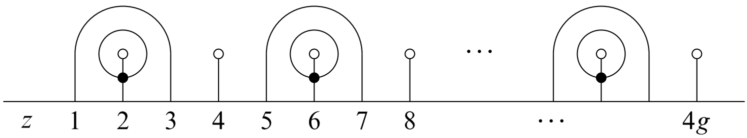


Figure 4: Heegaard diagram for the 0–framed handlebody. The numbers at bottom label points in $\mathcal{Z}^g$. Points in $-\mathcal{Z}^g$ are labeled in the reverse order.

Since $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ is the quasi-inverse of $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ (that is, inverse up to homotopy equivalence), we know $\widehat{\mathrm{CFDD}}(\tau)$ for an arcslide τ can also be represented by $\widehat{\mathcal{DD}}(\tau)$. Also, expanding out the definitions, we see $\widehat{\mathrm{CFAD}}(\phi) \simeq \widehat{\mathrm{CFAA}}(\phi) \boxtimes \widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}_2})$.

This concludes our construction of bimodule invariants (we will not need bimodule invariants other than those for mapping classes of surface diffeomorphisms). To construct invariants of closed 3–manifolds, we need one more building block: $\widehat{\mathrm{CFD}}$ of the 0*–framed handlebody* $\boldsymbol{H}^g$. Here $\boldsymbol{H}^g$ is the 3–manifold with one parametrized boundary given by the Heegaard diagram in Figure 4.

In this diagram, the small circles are 1–handle attachment points, paired consecutively. The larger circles are β circles, and all other arcs inside the boundary are α arcs. From the way the α arcs meet the boundary, we see that the boundary of $\boldsymbol{H}^g$ is parametrized by the split pointed matched circle of genus g, denoted $\mathcal{Z}^g$. This is the pointed matched circle with matching

$$(1,3),(2,4),(5,7),(6,8),\ldots,(4g-3,4g-1),(4g-2,4g).$$

While it is true that $-\mathcal{Z}^g = \mathcal{Z}^g$, we will usually distinguish them in order to emphasize orientation changes.

The orientation reversal $-\boldsymbol{H}^g$ is called the ∞*–framed handlebody*. Its boundary is parametrized by $-\mathcal{Z}^g$. The Heegaard diagram for $-\boldsymbol{H}^g$ is the reflection of that for $\boldsymbol{H}^g$.

The invariant $\widehat{\mathrm{CFD}}(\boldsymbol{H}^g)$ has left type D action by $\mathcal{A}(-\mathcal{Z}^g)$. It can be defined using the following model: there is a single generator $\boldsymbol{x}$, corresponding to the set of intersection points indicated in Figure 4. The idempotent of $\boldsymbol{x}$ contains pairs $(2,4),(6,8),\ldots$ in $-\mathcal{Z}^g$ (pairs corresponding to α–arcs not occupied by $\boldsymbol{x}$; note the labeling of points in $-\mathcal{Z}^g$ is reversed). The type D action is

$$\delta^1(\boldsymbol{x}) = \sum_{\xi \in \mathcal{D}} a(\xi) \cdot \boldsymbol{x},$$

where $\mathcal{D}$ is the set of chords $\{2 \to 4, 6 \to 8, \ldots\}$. The invariant $\widehat{\mathrm{CFD}}(-\boldsymbol{H}^g)$ can be defined to be the dual of $\widehat{\mathrm{CFD}}(\boldsymbol{H}^g)$.

We now give the combinatorial construction of $\widehat{\mathrm{HF}}(Y)$ for a closed 3–manifold Y, following the spirit of the construction in [11].

Construction 2.3 Let Y be a closed 3–manifold. Choose a Heegaard splitting $Y_1 \cup_u Y_2$ of Y, where $u\colon \partial Y_1 \to -\partial Y_2$ is the gluing map. Fix circle and basepoint (Z, z) on the gluing boundary $Y_1 \cap Y_2$, and diffeomorphisms $f_1\colon \boldsymbol{H}^g \to Y_1$ and $f_2\colon -\boldsymbol{H}^g \to Y_2$, preserving (Z, z), from the standard handlebodies to Y_1 and Y_2. Let $f_{1*}\colon F^\circ(\mathcal{Z}^g) \to \partial Y_1$ and $f_{2*}\colon F^\circ(-\mathcal{Z}^g) \to \partial Y_2$ be the restrictions of f_1 and f_2 to the boundary. Let $\psi = \overline{f}_{2*}^{-1} \circ u \circ f_{1*}$ be the induced gluing map. This is an element of the strongly based mapping class group on $F^\circ(\mathcal{Z}^g)$. Define

$$\widehat{\mathrm{HF}}(Y, Y_1, Y_2, u, f_1, f_2) \\ = \big(\widehat{\mathrm{CFAA}}(\psi)_{\mathcal{A}(-\mathcal{Z}^g),\mathcal{A}(\mathcal{Z}^g)} \boxtimes {}^{\mathcal{A}(-\mathcal{Z}^g)}\widehat{\mathrm{CFD}}(\boldsymbol{H}^g)\big) \boxtimes {}^{\mathcal{A}(\mathcal{Z}^g)}\widehat{\mathrm{CFD}}(-\boldsymbol{H}^g).$$

Theorem 2.4 *The homotopy type of $\widehat{\mathrm{HF}}(Y, Y_1, Y_2, u, f_1, f_2)$ does not depend on the choice of Heegaard splitting $Y = Y_1 \cup_u Y_2$ or the parametrizations f_1, f_2. Therefore it gives an invariant of Y up to homotopy equivalence.*

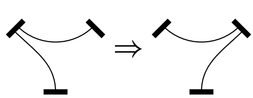
We will prove Theorem 2.4 combinatorially in Section 5. Given this theorem, we can write $\widehat{\mathrm{HF}}(Y)$ for $\widehat{\mathrm{HF}}(Y, Y_1, Y_2, u, f_1, f_2)$, for some choice of Heegaard splitting and parametrizations. From the construction, it is clear that this is equivalent to the definition of $\widehat{\mathrm{HF}}(Y)$ using holomorphic curves.

3 Computations on *DA* invariants

In this section, we prove some preliminary results on type *DA* bimodules, and perform some computations on the type *DA* invariants of arcslides, in preparation for the proof of Theorem 2.2 in Section 4.

First, we give an outline for the proof of Theorem 2.2. We want to show that the combinatorial construction of $\widehat{\mathcal{DA}}(\phi, \boldsymbol{\tau})$ does not depend on the choice of factorization of ϕ into arcslides $\boldsymbol{\tau}$. For this purpose, it is necessary to understand relations among arcslides. This is studied in detail in [1; 2]. The notions of pointed matched circles and arcslides correspond to linear chord diagrams and chord slides in these papers. We now give a summary of the results.

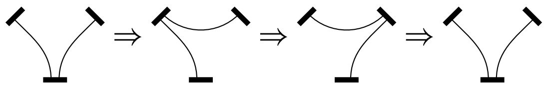
Locally, an arcslide can be viewed as one end of the B pair sliding along the C pair:



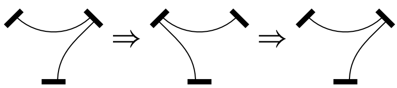
In this diagram, the three short segments denote portions of the straight line in the pointed matched circle. The upper, stationary arc denotes the C pair; and the lower, moving arc denotes the B pair.

There are five types of relations on arcslides, and together they generate all relations. The local diagrams for the five types of relations are as follows (see [2, Theorem 6.2, Figure 6.1]):

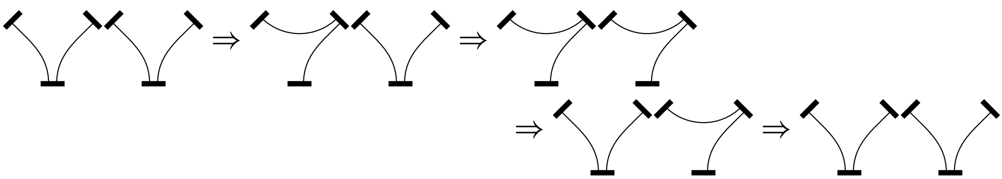
- Triangle



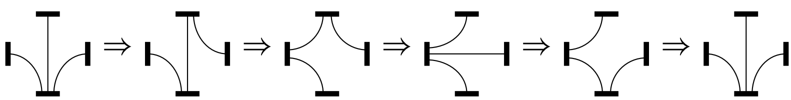
- Involution



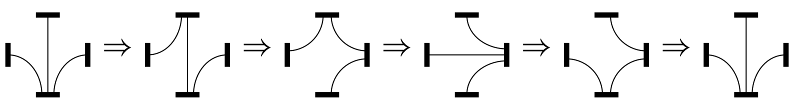
- Commutativity



- Left pentagon



- Right pentagon



Each relation gives one way to factor the identity morphism $\mathbb{I}_{\mathcal{Z}}$ starting and ending at some pointed matched circle $\mathcal{Z}$. For proving Theorem 2.2, it suffices to check that for each such factorization

$$\mathbb{I}_{\mathcal{Z}} = \tau_n \circ \cdots \circ \tau_1,$$

the corresponding homotopy equivalence

$$^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})} \simeq \widehat{\mathcal{DA}}(\tau_1) \boxtimes \cdots \boxtimes \widehat{\mathcal{DA}}(\tau_n) \tag{17}$$

holds. Note that in general, the starting and ending pointed matched circles of each τ_i may be different from $\mathcal{Z}$. This is the main reason why we need to consider strongly based mapping class groupoids, even if we are only interested in statements about strongly based mapping class groups.

The overall strategy for verifying (17) is as follows: from the description of $\widehat{\mathcal{DA}}(\tau_i)$, we can readily enumerate the set of generators on the right side of the equation. There are, however, more generators on the right side than on the left side. The *cancellation lemma* for type *DA* bimodules describes conditions under which we can prove that a bimodule is homotopy equivalent to one with two fewer generators. Using it, we can remove generators from the right side in pairs, so that the set of remaining generators matches that on the left side. The cancellation lemma is stated and proven in Section 3A.

It turns out that a type *DA* bimodule with the same set of generators as ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$, and with a few more properties in common with ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$, must be homotopy equivalent to ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$. We prove two lemmas of this kind, which we call *rigidity lemmas*, in Section 3B. The first lemma will be used to prove the involution relation, and the second lemma will be used for all other relations. The idea here is that once the involution relation is proven, we can show that $\widehat{\mathcal{DA}}(\tau)$ is quasi-invertible for any arcslide τ, which implies that any box tensor product of such bimodules is also quasi-invertible (recall that a type *DA* bimodule ${}^{A}M_{B}$ is *quasi-invertible* if there exists ${}^{B}N_{A}$ such that ${}^{A}M_{B} \boxtimes {}^{B}N_{A} \simeq {}^{A}\mathbb{I}_{A}$). This means checking the quasi-invertibility condition in the second lemma becomes trivial, and we can avoid checking the more involved condition in the first lemma that it replaces. We note here that the rigidity lemmas depend on specific properties of $\mathcal{A}(\mathcal{Z})$, and is not applicable to DG algebras in general.

To apply the cancellation lemma, and in the case of the involution relation, the rigidity lemma, we need to compute certain arrows in the type *DA* action of the bimodule on the right side. To prepare for this, we review the construction of $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ in Section 3C, and compute in Section 3D some arrows in the type *DA* action of $\widehat{\mathcal{DA}}(\tau)$ for arcslides τ (the components in the tensor product).

3A Cancellation lemmas

In this section we state cancellation lemmas for type D modules and type *DA* bimodules over DG algebras. Both are generalizations of the cancellation lemma in the case of chain complexes. These results are well known; see, for example, [8, Section 2.6].

Let A be a DG algebra over a ground ring $\boldsymbol{k}$, where $\boldsymbol{k}$ is a direct sum of copies of $\mathbb{F}_2$ (in our application, $A = \mathcal{A}(\mathcal{Z})$ and $\boldsymbol{k}$ is generated by the indecomposable idempotents). Let M be a left type D module over A with a fixed set of generators $\mathcal{G}$. We can describe the action δ^1 on M in terms of coefficients as follows: for any $x \in \mathcal{G}$, expand $\delta^1(x)$ as

$$\delta^1(x) = \sum_{\boldsymbol{y} \in \mathcal{G}} c_{xy} \otimes \boldsymbol{y}$$

$$\begin{tikzcd} x \arrow[dr] & a \arrow[d] \arrow[dr] & \\ & b & y \end{tikzcd}$$

Figure 5: Standard example of a zigzag in M. This becomes $x \to c_{xb}c_{ab}^{-1}c_{ay} \otimes y$ in M'.

for some choice of c_{xy}.

Here the tensor product is implicitly taken over $\boldsymbol{k}$, and as a result there is some flexibility in the choice of c_{xy}. We generally choose c_{xy} to consist of as few generators of A as possible, except when choosing c_{xy} to be invertible whenever possible.

Now suppose that for some $a, b \in \mathcal{G}$, the coefficient c_{ab} is invertible in A, and $d(c_{ab}) = 0$. Then there is a new type D module M', generated by $\mathcal{G}' = \mathcal{G} \setminus \{a, b\}$ and with type D action

$$\delta^{1'}(x) = \sum_{y \in \mathcal{G}'} (c_{xy} + c_{xb}c_{ab}^{-1}c_{ay}) \otimes y \tag{18}$$

for any $x \in \mathcal{G}'$. The first part of each term in the sum is simply the original δ^1 (excluding terms involving a and b). The second part is as follows: for each *zigzag* in M, as shown in Figure 5, the term $c_{xb}c_{ab}^{-1}c_{ay} \otimes y$ is added to $\delta^{1'}x$. The coefficient can be read out by following the arrows from x to y, treating a reversed arrow as taking inverse.

Theorem 3.1 (cancellation lemma for type D modules) *With the above definitions, the action $\delta^{1'}$ on M' satisfies the type D structure equation, and the resulting type D module M' is homotopy equivalent to M.*

Proof We prove this by giving explicit type D morphisms and homotopies, in terms of coefficients as we did for the type D action. The necessary data are morphisms $f\colon M \to M'$ and $g\colon M' \to M$, and homotopy $h\colon M \to M$, satisfying the identities $f \circ g = \mathbb{I}_{M'}$ and $g \circ f = \mathbb{I}_M + h \circ \delta^1 + \delta^1 \circ h$. The morphisms f and g are given by

$$f(a) = 0, \quad f(b) = \sum_{y \in \mathcal{G}'} c_{ab}^{-1}c_{ay} \otimes y, \quad \text{and} \quad f(x) = 1 \otimes x \;\text{ for } x \in \mathcal{G}'; \tag{19}$$

$$g(x) = 1 \otimes x + c_{xb}c_{ab}^{-1} \otimes a. \tag{20}$$

Part of f can be visualized using the zigzag by following arrows from b to y. Likewise, part of g can be visualized by following arrows from x to a. The homotopy $h\colon M \to M$ is given by $h(b) = c_{ab}^{-1} \otimes a$ and $h(x) = 0$ for any $x \neq b$.

It remains to verify that $\delta^{1'}$ satisfies the type D structure equations, and the maps f, g, and h satisfy the required identities. This can be done by converting the equations into their coefficient form. For example, the type D structure equation

$$(\mu_2 \otimes \mathbb{I}_M) \circ (\mathbb{I}_A \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \mathbb{I}_M) \circ \delta^1 = 0,$$

when written in terms of coefficients, becomes

$$d(c_{xy}) + \sum_{z \in \mathcal{G}} c_{xz} c_{zy} = 0 \tag{21}$$

for any $x, y \in \mathcal{G}$.

The structure equation for a type D morphism ϕ: $M \to N$ is

$$(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta^1_N) \circ \phi^1 + (\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \phi^1) \circ \delta^1_M + (\mu_1 \otimes \mathbb{I}_N) \circ \phi^1 = 0.$$

For any $x \in \mathcal{G}(M)$ and $y \in \mathcal{G}(N)$, let ϕ_{xy} be the coefficient of y in $\phi^1(x)$. Then, applying the above equation to an arbitrary generator x of M, we see that the structure equation is equivalent to

$$d(\phi_{xy}) + \sum_{z' \in \mathcal{G}(N)} \phi_{xz'} c_{z'y,N} + \sum_{z \in \mathcal{G}(M)} c_{xz,M} \phi_{zy} = 0 \tag{22}$$

for any $y \in \mathcal{G}(N)$.

The composition of two morphisms ϕ: $M \to N$ and ψ: $N \to P$ is given by

$$(\psi \circ \phi)^1 = (\mu_2 \otimes \mathbb{I}_P) \circ (\mathbb{I}_A \otimes \psi^1) \circ \phi^1.$$

In terms of coefficients, this is

$$(\psi \circ \phi)_{xy} = \sum_{z \in \mathcal{G}(N)} \phi_{xz} \psi_{zy}, \tag{23}$$

for any $x \in \mathcal{G}(M)$ and $y \in \mathcal{G}(P)$.

It is then routine to verify these equations, using the assumption that c_{ab} is invertible and $d(c_{ab}) = 0$. □

The cancellation lemma for type *DA* bimodules follows from that for type D modules, by viewing type *DA* bimodules over A' and A as type D modules over $\widehat{\mathrm{Cob}}(A) \otimes A'$; see [12, Remark 2.2.35].

Definition 3.2 Given a strand algebra A, let A_+ be the DG subalgebra of A generated by the nonidempotent generators. The cobar resolution $\mathrm{Cob}(A)$ is defined as $T^*(A_+[1]^*)$, the tensor algebra of the dual of A_+. This can be given the structure of a DG algebra, whose product is that of tensor algebra, and whose differential consists of the following arrows:

- $a_1^* \otimes \cdots \otimes b^* \otimes \cdots \otimes a_k^* \to a_1^* \otimes \cdots \otimes a_i^* \otimes \cdots \otimes a_k^*$ for each i and term b in da_i,
- $a_1^* \otimes \cdots \otimes a_i^* \otimes \cdots \otimes a_k^* \to a_1^* \otimes \cdots \otimes b^* \otimes b'^* \otimes \cdots \otimes a_k^*$ for each i and generators b, b' such that $bb' = a_i$.

Furthermore, we write $\widehat{\mathrm{Cob}}(A)$ to denote the completion of $\mathrm{Cob}(A)$ with respect to the length filtration, that is, an element of $\widehat{\mathrm{Cob}}(A)$ is a formal sum of elements in $(A_+[1]^*)^{\otimes i}$ for possibly infinitely many i.

The category of type *DA* bimodules over A' on the D–side and A on the A–side is equivalent to the category of type D modules over $\widehat{\mathrm{Cob}}(A) \otimes A'$, where the arrow

$$\delta^1_{1+i}\colon (x; a_1, \ldots, a_i) \to a' \otimes y$$

in the action of a type *DA* bimodule M corresponds to the arrow

$$\delta^1\colon x \to (a_1^* \otimes \cdots \otimes a_i^*) \otimes a' \otimes y$$

in the action of the type D module corresponding to M.

Using this correspondence, we can define coefficients on a type *DA* bimodule.

Definition 3.3 Given two generators x, y of M, define the coefficient C_{xy} to be the formal sum, in $\widehat{\mathrm{Cob}}(A') \otimes A$, of $(a_1^* \otimes \cdots \otimes a_i^*) \otimes a'$ over all arrows of the form $\delta^1_{1+i}\colon (x; a_1, \ldots, a_i) \to a' \otimes y$. As in the type D case, we choose a' to be invertible whenever possible when writing the action in terms of arrows.

This allows us to state the cancellation lemma for type *DA* bimodules, following immediately from the cancellation lemma in the type D case, and the equivalence of categories.

Theorem 3.4 (cancellation lemma for type *DA* bimodules) *Let ${}^{A'}M_A$ be a type DA bimodule, with a fixed set $\mathcal{G}$ of generators. Suppose there are $x, y \in \mathcal{G}$ such that $C_{xy} = 1 \otimes a$ with $a \in A$ invertible and $da = 0$. Then $C_{xy}^{-1} = 1 \otimes a^{-1}$, and the type DA bimodule M' generated by $\mathcal{G}' = \mathcal{G} \setminus \{x, y\}$ and with coefficients $C'_{ab} = C_{ab} + C_{ay} C_{xy}^{-1} C_{xb}$ is homotopy equivalent to M.*

We end with a remark on grading. If M is graded by a grading set S_M, and if every generator being cancelled is homogeneous in grading, then M' is also graded by S_M, with the grading of each generator in M' equal to the grading of the corresponding generator in M. The arrows that are added to M' satisfy the grading constraints, because they come from traversing a zigzag as in Figure 5, where each of the three arrows in the zigzag satisfy the grading constraints. The homogeneity condition of the cancelled generators will be automatically satisfied in our case.

3B Characterization of the identity bimodule

In this section, we prove two lemmas describing conditions under which we can assert a type *DA* bimodule ${}^{\mathcal{A}(\mathcal{Z})}M_{\mathcal{A}(\mathcal{Z})}$ is homotopy equivalent to the identity bimodule ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$. The main result we use is the characterization of $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ given in [11]. We will start by reviewing that result here.

Definition 3.5 [11, Definition 3.1] The *diagonal subalgebra* of $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(-\mathcal{Z})$ is the algebra generated by $a \otimes b$, where a and b satisfy the following conditions: $\mathrm{mult}(a) = \mathrm{mult}(\bar{b})$, the left idempotents of a and b are complementary, and the right idempotents of a and b are complementary.

Proposition 3.6 [11, Proposition 3.8, proof of Theorem 1] *Let M be a left type DD bimodule over $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$, where $\mathcal{Z}$ has genus greater than one. Suppose M satisfies the following conditions, then M is isomorphic to $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$:*

(1) *The generators of M are in one-to-one correspondence with the idempotents of $\mathcal{A}(\mathcal{Z})$, so that the generator corresponding to idempotent i has (type D) idempotents i and $\overline{o(i)}$.*

(2) *For any arrow $\boldsymbol{x} \to a \otimes b \otimes \boldsymbol{y}$ in the differential of M, the element $a \otimes b$ lies in the diagonal subalgebra.*

(3) *M is graded by a λ–free grading set S, with a left-right $G(\mathcal{Z})$-$G(\mathcal{Z})$ action.*

(4) *The differential in M contains all arrows of the form*
$$\boldsymbol{x} \to a(\xi) \otimes \overline{a(\xi)} \otimes \boldsymbol{y},$$
where ξ is a length-1 chord.

In the case where $\mathcal{Z}$ has genus one, if M satisfies an additional stability condition, in the sense of [11, Definition 1.8], then we can still conclude that $M = \widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$.

The following result will be used in the proof of the second lemma.

Proposition 3.7 *Suppose a type DA bimodule ${}^{\mathcal{A}(\mathcal{Z})}M_{\mathcal{A}(\mathcal{Z})}$ satisfies the following two conditions:*

(1) *M is homotopy equivalent to the identity bimodule ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$.*

(2) *The generators of M are in one-to-one correspondence with the idempotents of $\mathcal{A}(\mathcal{Z})$, so that the generator corresponding to idempotent i has both left (type D) and right (type A) idempotent equal to i.*

Then the type DA action on M contains all arrows of the form

$$\delta_2^1\colon (\boldsymbol{x}, a(\xi)) \to a(\xi) \otimes \boldsymbol{y}, \tag{24}$$

where ξ is a length-1 chord.

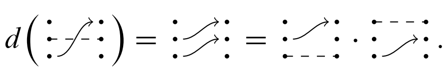
Proof Consider generators $\boldsymbol{x}, \boldsymbol{y}$ corresponding to idempotents $i, j \in \mathcal{A}(\mathcal{Z})$, and ξ a length-1 chord, such that the idempotent matches in the arrow (24). We want to show that (24) does exist as an arrow.

Let T_D be a type D module over $\mathcal{A}(\mathcal{Z})$ with two generators $\boldsymbol{x}_D$ and $\boldsymbol{y}_D$, whose idempotents are i and j, such that $\delta^1(\boldsymbol{x}_D) = a(\xi) \otimes \boldsymbol{y}_D$ and $\delta^1(\boldsymbol{y}_D) = 0$. Since $d(a(\xi)) = 0$, it is clear that δ^1 satisfies the type D structure equation.

Likewise, let T_A be the A_∞–module over $\mathcal{A}(\mathcal{Z})$ with two generators $\boldsymbol{x}_A$ and $\boldsymbol{y}_A$ whose idempotents are i and j, and $m_{1,1}\colon (\boldsymbol{x}_A, a(\xi)) \to \boldsymbol{y}_A$ is the only arrow in the A_∞–action.

Consider the tensor product $T_A \boxtimes N \boxtimes T_D$, with $N = M$ or $N = \mathbb{I}$. This is a chain complex with two generators $\boldsymbol{x}_A \otimes \boldsymbol{x} \otimes \boldsymbol{x}_D$ and $\boldsymbol{y}_A \otimes \boldsymbol{y} \otimes \boldsymbol{y}_D$, and there is an arrow between these two if and only if the arrow (24) exists in N for the given $\boldsymbol{x}, \boldsymbol{y}$ and $a(\xi)$. In particular, $T_A \boxtimes \mathbb{I} \boxtimes T_D$ has zero homology. By assumption, $M \simeq \mathbb{I}$, so $T_A \boxtimes M \boxtimes T_D$ must also have zero homology. This shows the arrow (24) exists in M. $\square$

Remark The argument in the above proof only works when ξ has length 1. If otherwise, we may have $d(a(\xi)) \neq 0$, and δ^1 on T_D no longer satisfies the type D structure equation. Indeed, in the case where ξ has length 2, we may have:



Hence, if there are generators $\boldsymbol{x}_D$ and $\boldsymbol{y}_D$ in T_D with arrow $\boldsymbol{x}_D \to a(\xi) \otimes \boldsymbol{y}_D$, where the idempotents of $\boldsymbol{x}_D$ and $\boldsymbol{y}_D$ contain the middle point, then there must be an additional generator $\boldsymbol{z}_D$ with appropriate arrows from $\boldsymbol{x}_D$ to $\boldsymbol{z}_D$ and from $\boldsymbol{z}_D$ to $\boldsymbol{y}_D$, so that the type D structure equation remains satisfied. This is why we may have, for example, arrow (DA4) instead of (DA2) in ${}^{\mathcal{A}(\mathcal{Z})}M_{\mathcal{A}(\mathcal{Z})}$, according the computations in Section 3D1.

We now state and prove the lemmas on the characterization of ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$.

Lemma 3.8 *Let $M = {}^{\mathcal{A}(\mathcal{Z})}M_{\mathcal{A}(\mathcal{Z})}$ be a left-right type DA bimodule over $\mathcal{A}(\mathcal{Z})$-$\mathcal{A}(\mathcal{Z})$. Suppose M satisfies the following properties, then M is homotopy equivalent to the identity bimodule ${}^{\mathcal{A}(\mathcal{Z})}\mathbb{I}_{\mathcal{A}(\mathcal{Z})}$.*

- (ID-1) *The generators of M are in one-to-one correspondence with the idempotents of $\mathcal{A}(\mathcal{Z})$, so that the generator corresponding to idempotent i has both left (type D) and right (type A) idempotent equal to i.*
- (ID-2) *M can be graded by a principal left-right $G(\mathcal{Z})$-$G(\mathcal{Z})$ set, such that the induced map $\phi \in \mathrm{Out}(G(\mathcal{Z}), G(\mathcal{Z}))$ (as in [11, Lemma 6.4]) is the identity map, and there is a choice of refined relative grading with every generator having grading zero. (The choice of grading refinement for $G(\mathcal{Z})$ is arbitrary but must be the same on both sides).*
- (ID-3) *The type DA action on M contains all arrows of the form*

  $$\delta_2^1\colon (\boldsymbol{x}, a(\xi)) \to a(\xi) \otimes \boldsymbol{y},$$

  where ξ is a length-1 chord.
- (ID-4) *M is stable in the sense of [11, Definition 1.8] (this condition is only necessary when $\mathcal{Z}$ is the unique genus 1 pointed matched circle).*

Proof Consider the type *DD* bimodule $M_{DD} = M \boxtimes \widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$. We check that M_{DD} satisfies all the conditions of Proposition 3.6, which will show that M_{DD} is isomorphic to $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$. Since $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ is quasi-invertible, this implies $M \simeq \mathbb{I}$.

Using the fact that relative grading can be chosen on $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ so that every generator has grading zero, condition (ID-2) on the grading of M implies a similar condition on the grading of M_{DD}. The constraint that the type *DD* action must respect the grading implies that for each arrow

$$\boldsymbol{x} \to a \otimes b \otimes \boldsymbol{y},$$

the multiplicities of a and $\bar{b}$ must be the same. The idempotent conditions on the diagonal subalgebra follow from the constraints on idempotents on each arrow, and the fact that both $\boldsymbol{x}$ and $\boldsymbol{y}$ have complementary idempotents. This verifies condition (2) of Proposition 3.6.

The other deductions are trivial. (ID-1), (ID-2) and (ID-3) imply conditions (1), (3) and (4), respectively. Condition (ID-4) implies the stability of M_{DD}, needed for the genus 1 case. □

The condition (ID-3) in the previous lemma can still be difficult to verify in actual computations. It is possible to replace it as follows.

Lemma 3.9 *With the same notation as in Lemma 3.8, if M satisfies the conditions (ID-1), (ID-2), (ID-4), and the following condition, then it is homotopy equivalent to $\mathbb{I}$.*

- (ID-3') *M is invertible, with a quasi-inverse M' that satisfies y conditions (ID-1) and (ID-2).*

Proof It suffices to show that (ID-3'), together with the other conditions, implies (ID-3). We first show that $\delta_1^1 = 0$ on both M and M', that is, there are no arrows of the form

$$\delta_1^1\colon \boldsymbol{x} \to a \otimes \boldsymbol{y}.$$

By the grading constraints on any arrow, the algebra generator a must have multiplicity zero. That is, it must be an idempotent in $\mathcal{A}(\mathcal{Z})$. However, this would mean that the grading of $\boldsymbol{x}$ and $\boldsymbol{y}$ differ by λ, contradicting the assumption that all generators in M (or M') have grading zero.

Both M and its quasi-inverse M' also satisfy (ID-1), so by [12, Lemma 2.2.50] they can be represented as ${}^{\mathcal{A}(\mathcal{Z})}[\phi]_{\mathcal{A}(\mathcal{Z})}$ and ${}^{\mathcal{A}(\mathcal{Z})}[\phi']_{\mathcal{A}(\mathcal{Z})}$ respectively, for A_∞–algebra morphisms $\phi, \phi'\colon \mathcal{A}(\mathcal{Z}) \to \mathcal{A}(\mathcal{Z})$. Then $M' \boxtimes M$ is represented by ${}^{\mathcal{A}(\mathcal{Z})}[\phi' \circ \phi]_{\mathcal{A}(\mathcal{Z})}$.

Since $M' \boxtimes M$ satisfies the grading condition (ID-2), the map $\phi' \circ \phi$ must preserve gradings. This means that for $a = a(\xi)$ where ξ is a length-1 chord, the only possible term in $(\phi' \circ \phi)(a)$ is a. Since $M \boxtimes M'$ is homotopy equivalent to identity, by Proposition 3.7, we have $(\phi' \circ \phi)(a) = a$, which implies $\phi(a) \neq 0$. By the same grading argument, either $\phi(a) = 0$ or $\phi(a) = a$. So we must have $\phi(a) = a$. This shows ϕ is the identity map on length-1 chords, which implies condition (ID-3). □

3C Combinatorial model of $\widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})$

In this section, we review the construction of the combinatorial model $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ of $\widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})$ given in [21], in preparation for computing some arrows in $\widehat{\mathcal{DA}}(\tau)$ for arcslides τ in the next section.

The construction begins with (11). After expanding the definitions, this gives a model of $\widehat{\mathrm{CFAA}}(\mathbb{I}_{\mathcal{Z}})$ generated by the set of pairs $[a_1, a_2]$, where a_1 and a_2 are generators of $\mathcal{A}(\mathcal{Z})$, such that the initial idempotents of a_1 and a_2 are complementary. The differential and type *AA* action on these generators are given as [21, Proposition 1]. The smaller model $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ is obtained from this using homological perturbation theory. This involves finding the homology C' of C, the chain complex underlying the larger model, and giving chain maps $f\colon C \to C'$, $g\colon C' \to C$, and homotopy $H\colon C \to C$ verifying the homotopy equivalence between C and C'. The homology C' is generated by those $[a_1, a_2]$ where both a_1 and a_2 are idempotents (which are then complementary). The chain maps f and g are the obvious projection and inclusion maps. The homotopy H is summarized in [21, Figures 6 and 9].

From homological perturbation theory, we obtain the following description of the smaller model: the A_∞–bimodule $\widehat{\mathcal{A}\mathcal{A}}(\mathbb{I}_{\mathcal{Z}})_{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z})}$ is generated by pairs of complementary idempotents $i'\otimes i$, where $i\in\mathcal{A}(\mathcal{Z})$ and $i'=\overline{o(i)}\in\mathcal{A}(-\mathcal{Z})$. The generator $i'\otimes i$ has type A idempotents i' and i. Each arrow in the type *AA* action of $\widehat{\mathcal{A}\mathcal{A}}(\mathbb{I}_{\mathcal{Z}})$ comes from a sequence of moves between generators of C. There are three types of moves, the first two of which carry a coefficient.

- **Move A_1** If $c\bar{b}'\neq 0$, with $b'\in\mathcal{A}(-\mathcal{Z})$, move from $[c\bar{b}',a_2]$ to $[c,a_2]$ with coefficient b'.
- **Move A_2** If $a_2b\neq 0$, with $b\in\mathcal{A}(\mathcal{Z})$, move from $[a_1,a_2]$ to $[a_1,a_2b]$ with coefficient b.
- **Move H** Apply one of the arrows in the homotopy map H.

Each arrow then corresponds to a sequence $[a_{1,1},a_{1,2}],\dots,[a_{2n,1},a_{2n,2}]$ of generators of C, satisfying the following conditions:

- $[a_{1,1},a_{1,2}]=[o(i),i]$ and $[a_{2n,1},a_{2n,2}]=[o(j),j]$ for some idempotents $i,j\in\mathcal{A}(\mathcal{Z})$.
- Each $[a_{2k,1},a_{2k,2}]$ is obtained from $[a_{2k-1,1},a_{2k-1,2}]$ by applying either move A_1 or A_2.
- Each $[a_{2k+1,1},a_{2k+1,2}]$ is obtained from $[a_{2k,1},a_{2k,2}]$ by applying move H.

Let $b'_1,\dots,b'_p$ be the ordered sequence of coefficients for moves of type A_1, and $b_1,\dots,b_q$ be the ordered sequence of coefficients for moves of type A_2, then such a sequence of generators of C gives rise to an arrow

$$m_{1,p,q}\colon (i'\otimes i;b'_1,\dots,b'_p;b_1,\dots,b_q)\to j'\otimes j,$$

where $i'=\overline{o(i)}$ and $j'=\overline{o(j)}$.

An important property of $\widehat{\mathcal{A}\mathcal{A}}(\mathbb{I}_{\mathcal{Z}})$, which follows directly from this construction, is that for any arrow in the type *AA* action, the total multiplicity of the $\mathcal{A}(-\mathcal{Z})$ inputs (that is, the sum of multiplicities of $b'_1,\dots,b'_p$) equals that of the $\mathcal{A}(\mathcal{Z})$ inputs (the sum of multiplicities of $b_1,\dots,b_q$). From the definition using holomorphic curves, this is clear since each arrow comes from a domain in the standard Heegaard diagram of the identity diffeomorphism. We also note that $\widehat{\mathcal{A}\mathcal{A}}(\mathbb{I}_{\mathcal{Z}})$ can be given a refined relative grading where all generators have grading zero.

The definition of the homotopy map H involves first defining a specific ordering $<_{\mathcal{Z}}$ on the $4g-1$ intervals of the pointed matched circle $\mathcal{Z}$. This means the determination of arrows is not local, in the sense that if we restrict to a certain interval of $\mathcal{Z}$, containing points paired outside the interval, then the type *AA* arrows restricted to that interval may depend on how $\mathcal{Z}$ is configured outside the interval. However, we note that if all points are paired within the interval, then the ordering $<_{\mathcal{Z}}$ on these points (and therefore the type *AA* arrows) is independent of outside configurations (this follows directly from how the ordering $<_{\mathcal{Z}}$ is defined). In particular, $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ behaves well with respect to stabilization. That is, if $\mathring{\mathcal{Z}} = \mathcal{Z}\#\mathcal{Z}^1$, then $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ is isomorphic to the appropriate restriction of $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathring{\mathcal{Z}}})$.

3D Certain arrows in $\widehat{\mathcal{DA}}$ of arcslides

In this section we compute some of the arrows in $\widehat{\mathcal{DA}}(\tau)$ for a general arcslide τ, using (16). From its description in the previous section, one can expect arrows in $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ to be extremely complicated in general. The same would then be true for arrows in $\widehat{\mathcal{DA}}(\tau)$. We manage this complexity by focusing only on arrows whose algebra coefficients have a small total length (say length 1 or 2 on each side). It turns out that these are sufficient to prove the necessary properties of the box tensor products of $\widehat{\mathcal{DA}}(\tau_i)$ that we will need to consider.

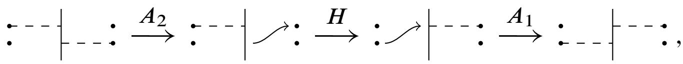
Since the algebra coefficients have small total length, the domain corresponding to the arrow is supported in a small part of the Heegaard diagram. For arcslides, the parts of the Heegaard diagram that we are particularly interested in are the differences with the Heegaard diagram for the identity diffeomorphism, that is, around the points b_1, c_1, c_2 and b_1'.

One source of complexity comes from the fact that the definition of the homotopy map H in the construction of $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ depends on the ordering $<_{\mathcal{Z}}$ on the intervals of the pointed matched circle. In a local situation, if we cannot tell which interval comes first in the ordering, we will need to cover all possible cases. Note that only the restriction of $<_{\mathcal{Z}}$ to the intervals covered by the algebra coefficients matter for determining the arrows.

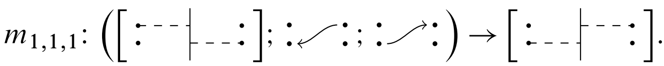
When we show a set of local arrows in a given local situation and restriction of the ordering $<_{\mathcal{Z}}$, we intend to make the following assertions:

- There is an arrow for every way of extending the local arrow by completing the pointed matched circle and adding the appropriate number of horizontal lines to the algebra coefficients.
- Every arrow in the bimodule action whose algebra coefficients lie within the area shown can be obtained by extending one of the local arrows.

We now begin with the simplest case: arrows in the type AA action on $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ where the algebra coefficients have length 1 on either side. The coefficients must then cover the same interval. The sequence of pairs $[a_{i,1}, a_{i,2}]$ is



where the middle H is [21, Case 3] of the homotopy map in the multiplicity-one case given there. This gives the arrow

(AA1) 

From (AA1), we obtain a simple method of deriving arrows in $\widehat{\mathcal{DA}}(\tau)$ from arrows in $\widehat{\mathcal{DD}}(\tau)$, in cases where the second coefficient of the type DD arrow has length 1 (the second algebra action is the one that is involved in the box tensor product). For each type DD arrow $\delta^1\colon \boldsymbol{x} \to a_1 \otimes a_2 \otimes \boldsymbol{y}$, where a_2 has length 1, there corresponds a type DA arrow $\delta^1_2\colon (\boldsymbol{x}, \bar{a}_2) \to a_1 \otimes \boldsymbol{y}$, where by abuse of notation we use the same symbol to denote corresponding generators of $\widehat{\mathcal{DA}}(\tau)$ and $\widehat{\mathcal{DD}}(\tau)$.

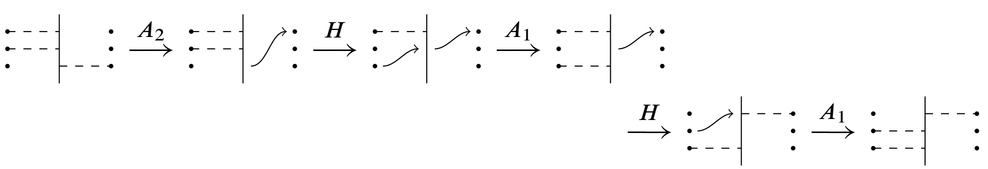
As an application, we give a combinatorial proof of the following corollary.

Corollary 3.10 *The tensor product $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}}) \boxtimes \widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ is homotopy equivalent to $\mathbb{I}$.*

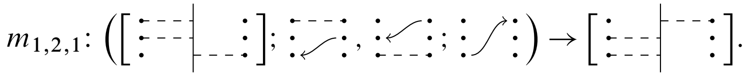
Proof Directly check each of the conditions in Lemma 3.8. For condition (ID-2), we use the refined relative grading on $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ with all generators having grading zero. For condition (ID-3), use the type AA arrows computed here. For (ID-4), use the stabilization property of $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ discussed at the end of Section 3C. $\square$

3D1 Type *AA* on a size-2 interval: the disjoint pairs case The next simplest case for type AA arrows is the size-2 interval. First, we assume that no two of the three points are paired with each other. There are four subcases, depending on whether the middle idempotent is occupied on the left or on the right, and whether the lower or the upper interval comes first in the ordering $<_{\mathcal{Z}}$.

Case 1 The middle idempotent is on the left, the lower interval comes first in ordering. The only sequence covering the size-2 interval is

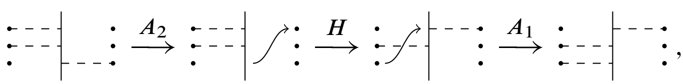


giving the arrow

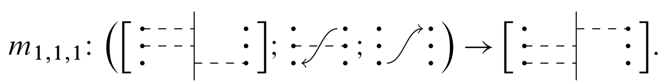
(AA2) 

Note that in the first H–move, we shift only the lower part of the strand to the left, since the lower interval comes first in the ordering.

Case 2 The middle is idempotent on the left, the upper interval comes first in ordering. The only sequence covering the size-2 interval is

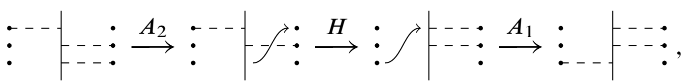


giving the arrow

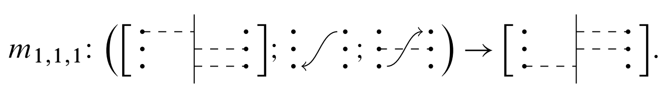
(AA3) 

Here the upper interval comes first, so we shift the entire strand to the left in the first H–move.

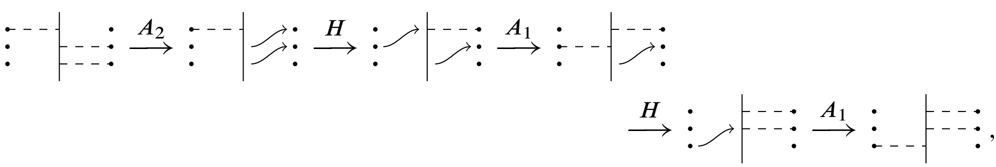
Case 3 The middle idempotent is on the right, the upper interval comes first in ordering. In this case there are two possible sequences covering the size-2 interval. The first one is



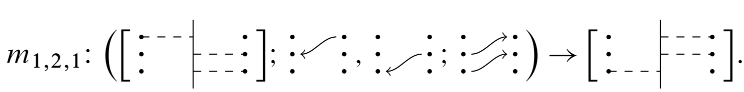
giving the arrow

(AA4) 

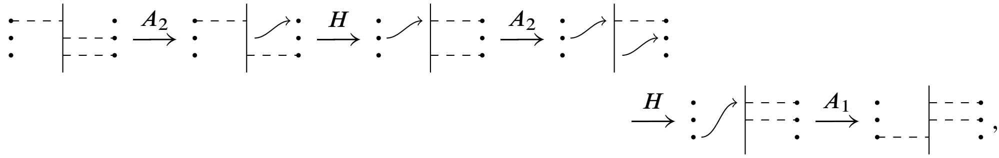
The second one is



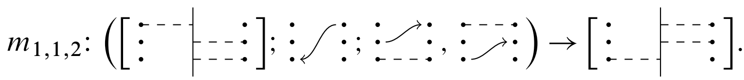
giving the arrow

(AA5) 

Case 4 The middle idempotent is on the right, the lower interval comes first in ordering. The only sequence covering the size-2 interval is



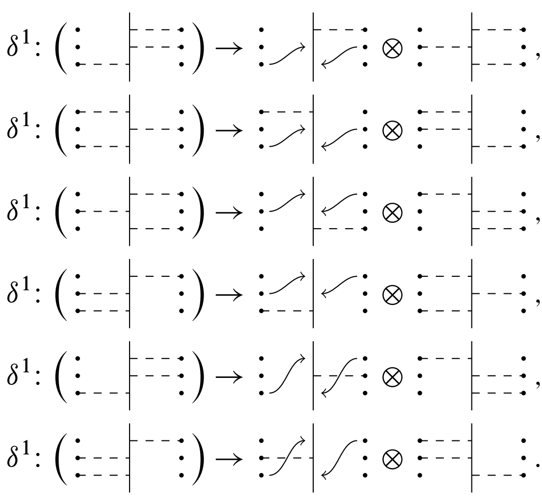
giving the arrow

(AA6) 

As examples, we show the computation of type *DA* arrows in $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}}) \boxtimes \widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ that cover a size-2 interval. While the results in the remainder of this section will not be used directly in what follows, it serves as a model for the calculations of similar arrows in $\widehat{\mathcal{DA}}(\tau)$ for an arcslide τ.

To compute the type *DA* arrows, we combine the previous results with what is known about type *DD* arrows in $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$. On the size-2 interval, the possibilities are given below (on each line, $\delta^1\colon \boldsymbol{x} \to (a, a') \otimes \boldsymbol{y}$ represents the arrow $\delta^1\colon \boldsymbol{x} \to a \otimes a' \otimes \boldsymbol{y}$, where $a \in \mathcal{A}(\mathcal{Z})$ and $a' \in \mathcal{A}(-\mathcal{Z})$):

(DD1)
(DD2)
(DD3)
(DD4)
(DD5)
(DD6)



It is now a matter of combining these following the rules of the box tensor product. In the figures below, for both type *DD* and type *AA* bimodules, we will show the first algebra action on the left and the second algebra action on the right. This is purely for ease of visualization, and does not indicate which side the algebras act on. Indeed, both actions on the type *DD* bimodule are on the left, and both actions on the type *AA* bimodule are on the right. Nevertheless, we will often talk about left action or left

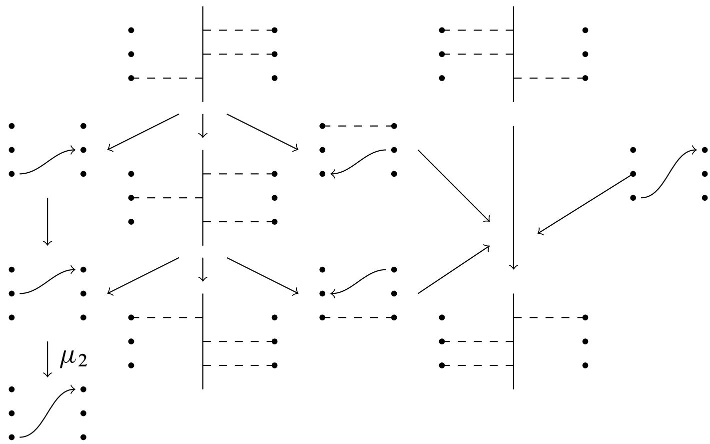


Figure 6: Formation of type DA operation, case 1

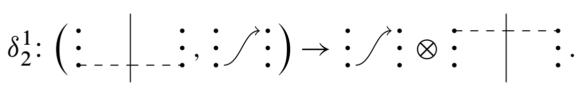
idempotents to match how the figures are drawn. Moreover, we will put the DD arrows on the left, and AA arrows on the right, since we are tensoring the second action in $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ with the first action in $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$.

Each type DA arrow comes from a single type AA arrow and zero or more type DD arrows. The left outputs (in $\mathcal{A}(\mathcal{Z})$) of the type DD arrows are multiplied together to give the overall type D output, while the right outputs (in $\mathcal{A}(-\mathcal{Z})$) are given as the left inputs to the type AA arrow. The overall type A inputs in $\mathcal{A}(\mathcal{Z})$ are given as the right inputs to the type AA arrow.

The right idempotent of the DD generator must agree with the left idempotent of the AA generator. The left idempotent of the DD generator and the right idempotent of the AA generator then combine to form the idempotent of the resulting DA generator.

We now look at each of the four cases.

Case 1 The middle idempotent is on the left, the lower interval comes first in ordering. This combination is shown in Figure 6. We use (DD1), (DD3), and (AA2). The resulting arrow is

(DA1) 

Case 2 The middle idempotent is on the left, the upper interval comes first in ordering; see Figure 7 We use (DD5) and (AA3). The resulting arrow is the same as in (DA1), so in this case the order of the two intervals already does not matter at the DA level.

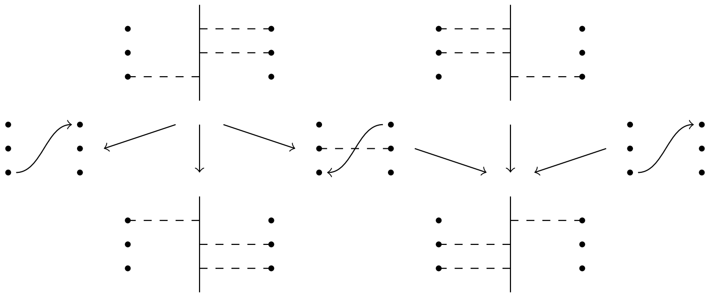


Figure 7: Formation of type *DA* operation, case 2

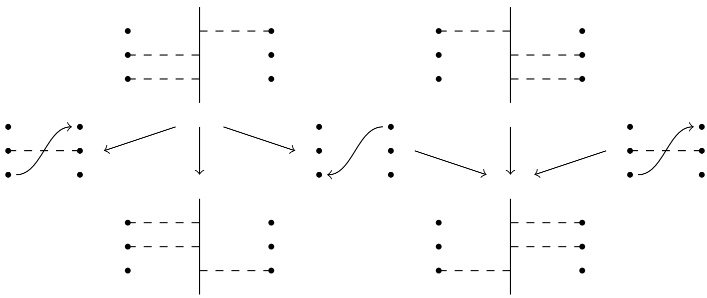


Figure 8: Formation of type *DA* operation, case 3

Case 3 The middle idempotent is on the right, the upper interval comes first in ordering; see Figure 8. We use (DD6) and (AA4), and the resulting arrow is

(DA2) $$\delta^1_2\colon (\ ,\ ) \to \ \otimes\ .$$

Another combination, using (DD4), (DD2), and (AA5), gives the arrow

(DA3) $$\delta^1_2\colon (\ ,\ ) \to \ \otimes\ .$$

Case 4 The middle idempotent is on the right, the lower interval comes first in ordering, shown in Figure 9. We use (DD6) and (AA6), and the resulting arrow is

(DA4) $$\delta^1_3\colon (\ ,\ ,\ ) \to \ \otimes\ .$$

This arrow shows that the model $\widehat{\mathcal{A}\mathcal{A}}(\mathbb{I}_{\mathcal{Z}}) \boxtimes \widehat{\mathcal{D}\mathcal{D}}(\mathbb{I}_{\mathcal{Z}})$ of $\widehat{\mathrm{CFDA}}(\mathbb{I}_{\mathcal{Z}})$ is not exactly the same, but only homotopy equivalent to $\mathbb{I}$.

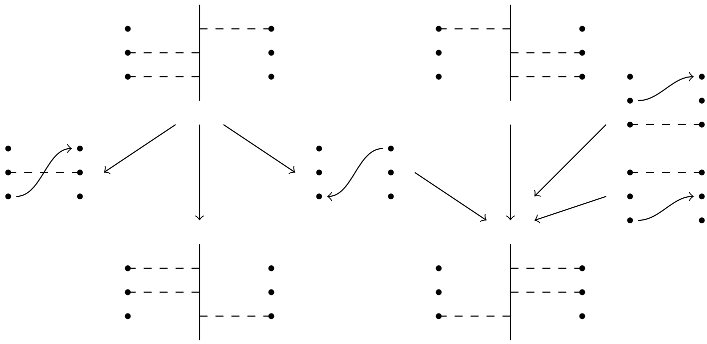
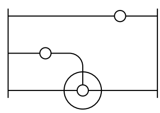


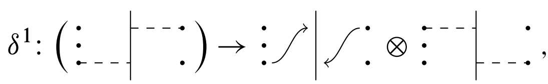
Figure 9: Formation of type *DA* operation, case 4

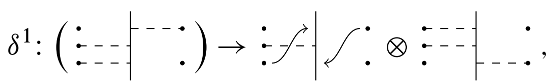
3D2 Pieces of arcslide We now compute some simple arrows in $\widehat{\mathcal{DA}}(\tau)$ for an arcslide τ. Since the method used here is similar to that in the previous section, we will show only the results.

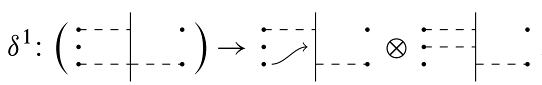
First, we consider the case where b_1 is directly above c_1, and compute the arrows in $\widehat{\mathcal{DA}}(\tau)$ corresponding to the region of the Heegaard diagram around b_1. The Heegaard diagram around b_1 is this:

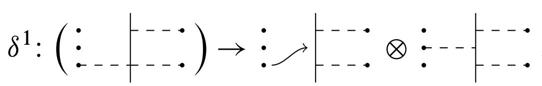


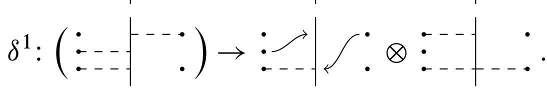
The possible type *DD* arrows are the following:

(DD7) 

(DD8) 

(DD9) 

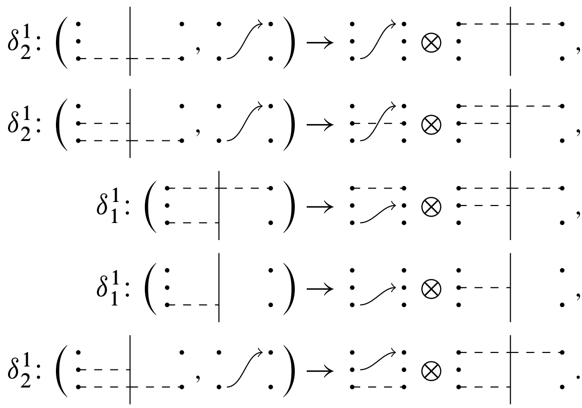
(DD10) 

(DD11) 

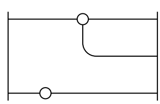
This comes directly from [11]. The only potentially tricky part is figuring out the possible locations of idempotents. For example, in the arrow $\delta^1\colon \boldsymbol{x} \to (a(\sigma), 1) \otimes \boldsymbol{y}$ (third and fourth arrows above; σ is the chord $c_1 \to b_1$), the left idempotent of $\boldsymbol{y}$ must be occupied at the B pair and unoccupied at the C pair. Since generators of $\widehat{\mathcal{DD}}(\tau)$ either have complementary idempotents or idempotents that are both occupied at C, the right idempotent of $\boldsymbol{y}$ must be occupied at C (so $\boldsymbol{y}$ is of type X). From the idempotent of $\boldsymbol{y}$, we can deduce that of $\boldsymbol{x}$, and see that $\boldsymbol{x}$ is of type Y. Similar arguments are used to list possible idempotents in the other cases.

Computing the type *DA* arrows in this case is relatively straightforward, as we are combining with the arrow (AA1) on a size-1 interval. The results are as follows, where (DA5)–(DA9) follow respectively from (DD7)–(DD11):

(DA5)
(DA6)
(DA7)
(DA8)
(DA9)

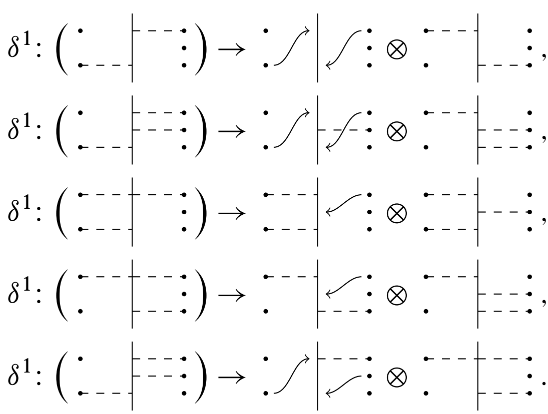


Now we consider other side of the same case, computing arrows in $\widehat{\mathcal{DA}}(\tau)$ corresponding to the region around b_1'. Since b_1 is directly above c_1, we have b_1' directly below c_2, and the Heegaard diagram around b_1' is this:

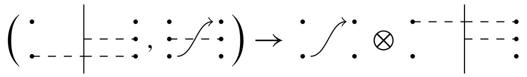


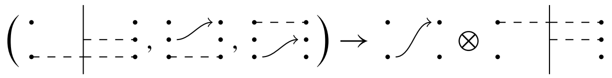
The type *DD* operations are these:

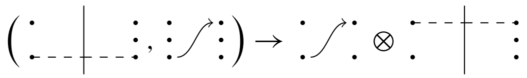
(DD12)
(DD13)
(DD14)
(DD15)
(DD16)

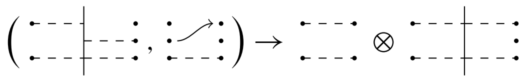


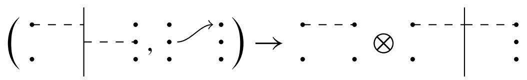
This time we will need to combine with type AA arrows on a size-2 interval, emulating the method in Section 3D1. The results are the following:

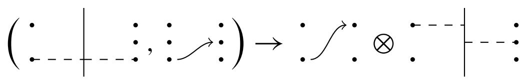
(DA10) (upper first) δ^1_2: 

(DA11) (lower first) δ^1_3: 

(DA12) δ^1_2: 

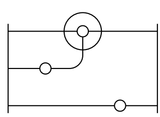
(DA13) δ^1_2: 

(DA14) δ^1_2: 

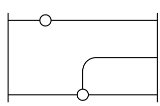
(DA15) δ^1_2: 

The first arrow follows from (DD12) and (AA4) only if the upper interval comes first in the ordering $<_{\mathcal{Z}'}$ for the right pointed matched circle. The second arrow follows from (DD12) and (AA6) only if the lower interval comes first in the ordering. The third arrow does not depend on ordering. However, it is formed in different ways for the two orderings: if upper interval comes first, it follows from (DD13) and (AA3); otherwise it follows (DD16), (DD15), and (AA2). The last three arrows are independent of ordering. They follow from (AA1) and respectively (DD14)–(DD16).

The cases where b_1 is directly below c_1 (and therefore b'_1 is directly above c_2) are very similar. Here is the Heegaard diagram around b_1:

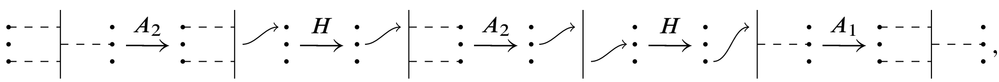


The first two DA arrows are the same as (DA5) and (DA6), and the last three are modified appropriately from (DA7)–(DA9). The Heegaard diagram around b'_1 is this:

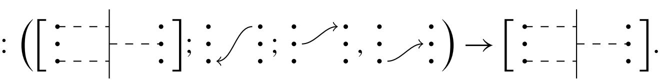


The first three DA arrows are the same as arrows are modified appropriately from (DA13)–(DA15).

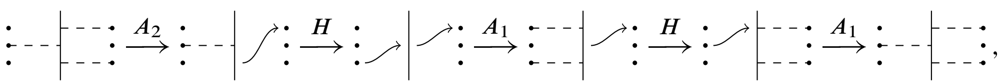
3D3 Type AA on a size-2 interval: the paired case We now consider the case of a size-2 interval, where the top and bottom points are paired with each other. Here the lower interval immediately precedes the upper interval in the ordering $<_{\mathcal{Z}}$. There are no arrows starting at generators where the middle idempotent is on the same side as the idempotent containing the top and bottom points. Starting at generators where the middle idempotent is to the right, there is a sequence



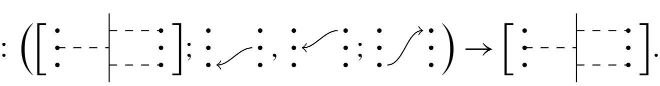
giving the arrow

(AA7) $m_{1,1,2}$: 

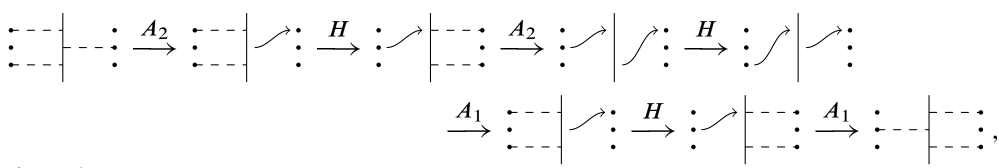
Starting at generators where the middle idempotent is to the left, there is a sequence



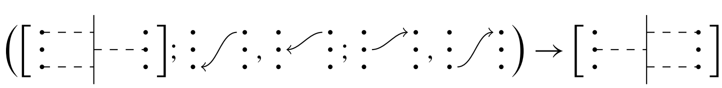
giving the arrow

(AA8) $m_{1,2,1}$: 

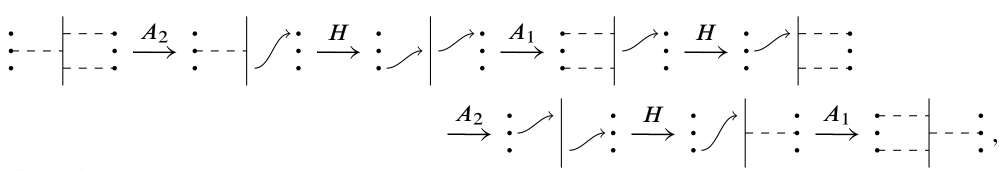
Furthermore, there are several infinite series of arrows formed by repeating the moves used above. We list the two arrows that will be used later in the paper:



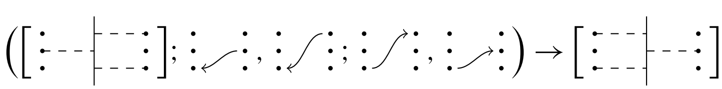
gives the arrow

(AA9) $m_{1,2,2}$: 

and

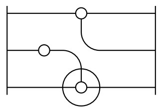


gives the arrow

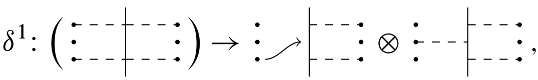
(AA10) $m_{1,2,2}$: 

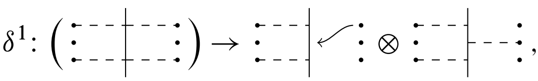
Note that in the derivation of the first arrow, we used [21, Case 3] of the homotopy map H in the multiplicity greater than one case. From this we see that the type *AA* bimodule can have infinitely many arrows. However, in our examples, only a finite number of them will be used when constructing the action on type *DA* invariants.

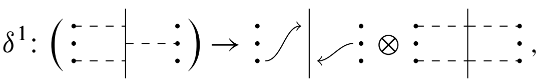
3D4 Short underslide Using results from the previous section, we compute type *DA* arrows for the short underslide. These are underslides where b_1 is the only point between c_1 and c_2. Hence b_1 and b_1' are located in the same region of the Heegaard diagram, which is the only region of interest. Here is the diagram for the case where b_1 is directly above c_1:

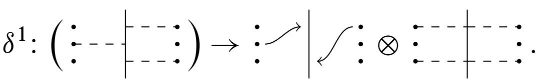


The possible type *DD* arrows are the following:

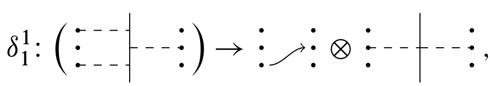
(DD17) 

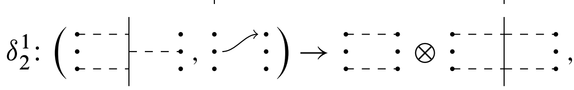
(DD18) 

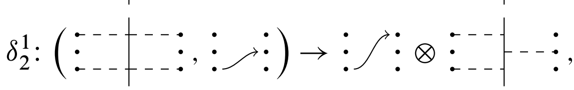
(DD19) 

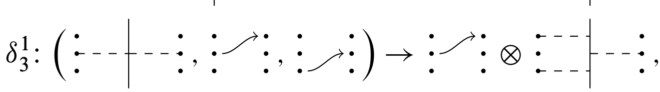
(DD20) 

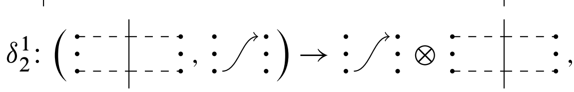
These give rise to type *DA* arrows

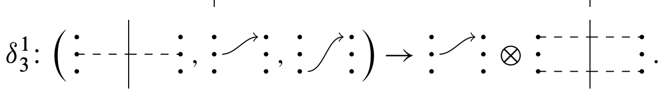
(DA16) 

(DA17) 

(DA18) 

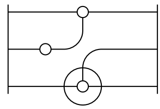
(DA19) 

(DA20) 

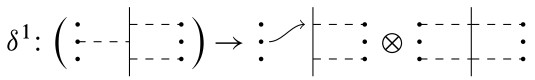
(DA21) 

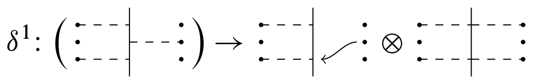
Here arrows (DA16)–(DA18) follow respectively from (DD17)–(DD19). Arrow (DA19) follows from (DD20) and (AA7). Arrow (DA20) follows from (DD19), (DD18), and (AA8). Arrow (DA21) follows from (DD20), (DD18), and (AA9).

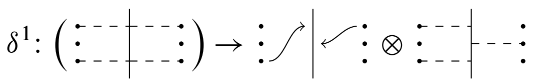
The diagram for the case where b_1 is directly below c_1 is this:

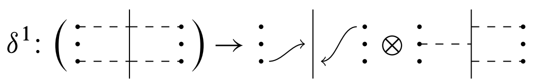


The possible type *DD* arrows are the following:

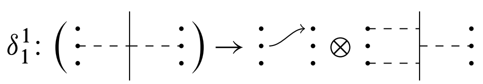
(DD21) 

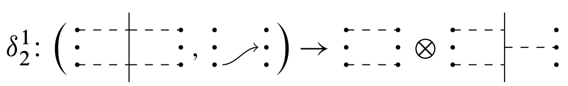
(DD22) 

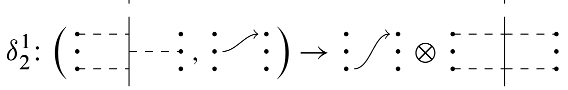
(DD23) 

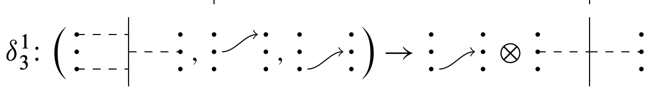
(DD24) 

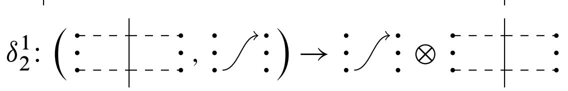
These give rise to type *DA* arrows:

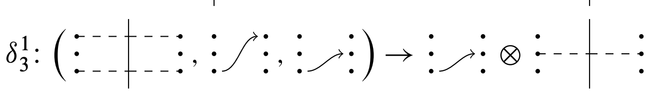
(DA22) 

(DA23) 

(DA24) 

(DA25) 

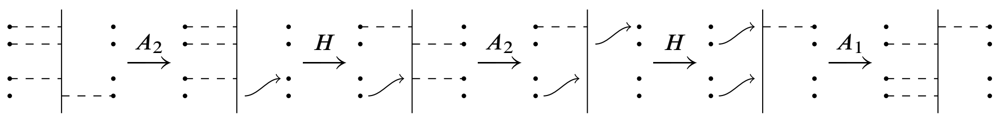
(DA26) 

(DA27) 

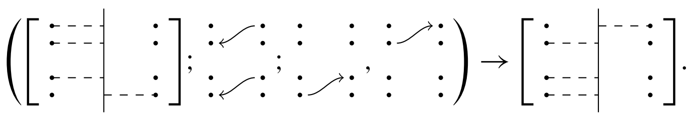
Here arrows (DA22)–(DA24) follow respectively from (DD21)–(DD23). Arrow (DA25) follows from (DD24) and (AA7). Arrow (DA26) follows from (DD22), (DD23), and (AA8). Arrow (DA27) follows from (DD22), (DD24), and (AA10).

3D5 Type *AA* on two separated intervals with pairing To consider more local situations for the arcslide, we will need type *AA* arrows on two separated intervals, such that either the two inner positions or the two outer positions are paired. These two cases are very similar, so we will only write out the first case here.

In this case, the upper interval immediately precedes the lower interval in the ordering $<_{\mathcal{Z}}$. If the middle idempotent (consisting of the two paired inner points) is occupied on the left, then it is not possible to multiply both intervals to the right as the first step. So the only possible sequence of $[a_{i,1}, a_{i,2}]$ is the following:

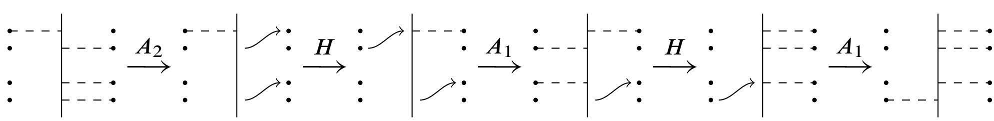


giving the arrow

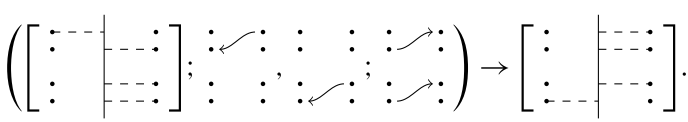
(AA11) $m_{1,1,2}$: 

In these diagrams, the two middle positions are paired, and there can be an arbitrary number of points between them in the full pointed matched circle. Since no arrow in $\widehat{\mathcal{DD}}(\mathbb{I}_{\mathcal{Z}})$ gives off an algebra element with two separate strands, this cannot be used to form a type *DA* arrow for the identity.

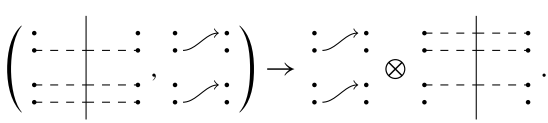
If the middle idempotent is occupied on the right, it is possible to multiply both intervals to the right as the first step, but not possible to multiply only the lower interval. So the only sequence is



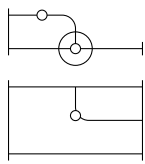
giving the arrow

(AA12) $m_{1,2,1}$: 

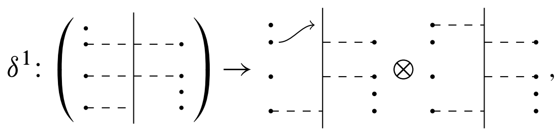
This leads to the following type *DA* arrow for identity:

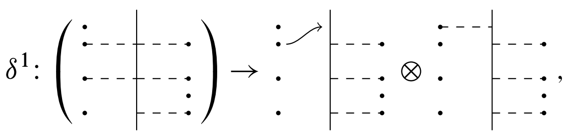
(DA28) δ_2^1: 

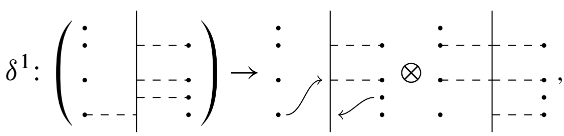
3D6 More pieces of arcslide We now compute some arrows whose corresponding domains touch both c_1 and c_2. We focus on the overslide cases; the underslide cases are similar. First, if b_1 is directly above c_1, the local Heegaard diagram is as follows. We focus on arrows whose domain is restricted inside this diagram:

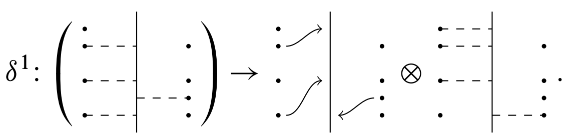


The two horizontal lines where the 1–handle is attached contain the α–arcs for the C pair. Immediately above and below are the points b_1 on the left and b_1' on the right. For clarity, we list all type *DD* and *DA* arrows in this region, even though some may already have been covered in previous cases. These are the type *DD* arrows:

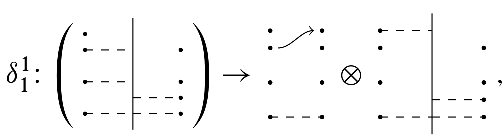
(DD25) 

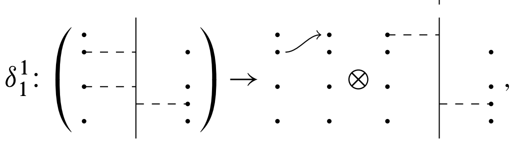
(DD26) 

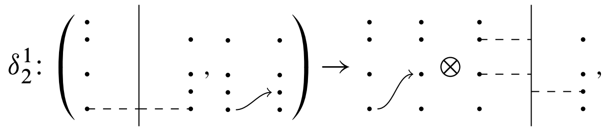
(DD27) 

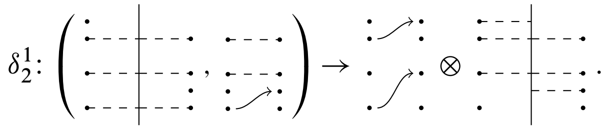
(DD28) 

These give rise to the following type *DA* arrows. Here (DA29)–(DA32) follow respectively from (DD25)–(DD28):

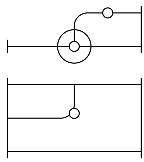
(DA29) 

(DA30) 

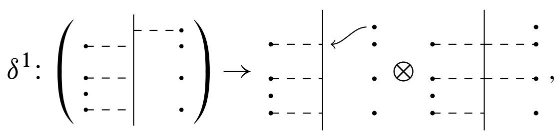
(DA31) 

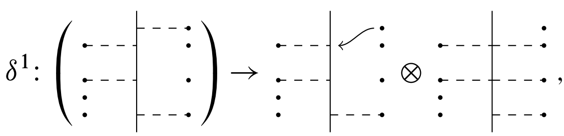
(DA32) 

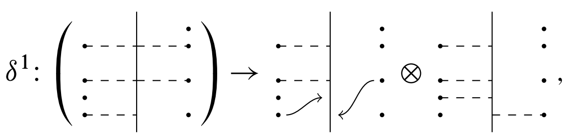
The case where b_1 is directly below c_1 is again more complicated. Here is the local Heegaard diagram:

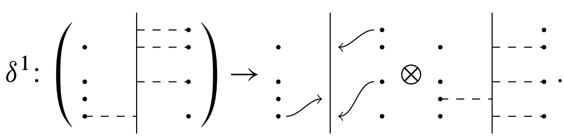


The type *DD* arrows are these:

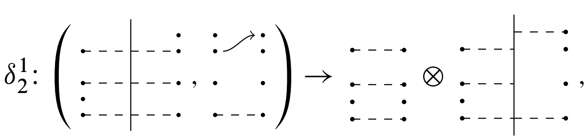
(DD29) 

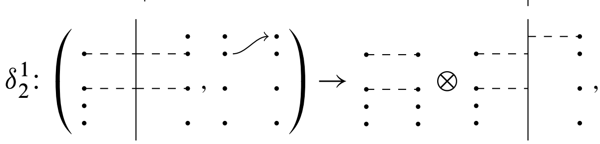
(DD30) 

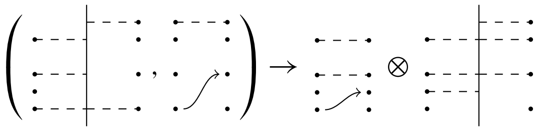
(DD31) 

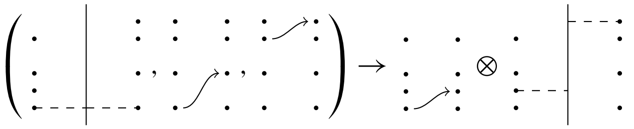
(DD32) 

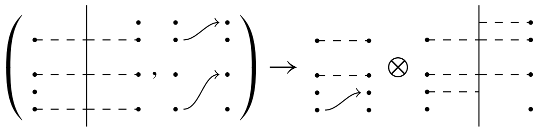
The resulting type *DA* arrows are the following:

(DA33) 

(DA34) 

(DA35) δ^1_2: 

(DA36) δ^1_3: 

(DA37) δ^1_2: 

Here, arrows (DA33)–(DA35) follow respectively from (DD29)–(DD31). Arrow (DA36) follows from (DD32) and (AA11). Arrow (DA37) follows from (DD29), (DD31), and (AA12).

4 Relations on the mapping class groupoid

In this section, we conclude the proof of Theorem 2.2. In Section 4A, we describe how to enumerate the set of generators of a box tensor product of $\widehat{\mathcal{DA}}(\tau_i)$, where τ_i are arcslides, and how properties of $\widehat{\mathcal{DD}}(\tau_i)$ carry over to properties of $\widehat{\mathcal{DA}}(\tau_i)$ and their box tensor products. With all these preparations in place, we prove (17) for the involution relation in Section 4B, and for the other relations in Section 4C.

4A Compositions of arcslides

Given an arcslide τ, the description of the set of generators of $\widehat{\mathcal{DA}}(\tau)$ follows from that of $\widehat{\mathcal{DD}}(\tau)$ and $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$. The generators are classified by their idempotents on the two sides (type D idempotent on the left and type A idempotent on the right). As before, we use the canonical identification of pairs of points between the pointed matched circles on the two sides. There are two types of generators in $\widehat{\mathcal{DA}}(\tau)$:

- Type X The idempotents on the two sides contain the same pairs.
- Type Y The idempotents on the two sides differ at exactly one pair, with the C pair occupied on the left and B pair occupied on the right.

Using this, and the definition of box tensor product, we can enumerate the set of generators of

$$\widehat{\mathcal{DA}}(\tau_1) \boxtimes \cdots \boxtimes \widehat{\mathcal{DA}}(\tau_n)$$

for a sequence of arcslides $\tau_1, \ldots, \tau_n$. We now describe the procedure in detail.

First, we combine the identification between pairs of points on the starting and ending pointed matched circles of a single arcslide to obtain an identification of pairs on all pointed matched circles appearing in the sequence. Note that even if the starting and ending pointed matched circle of a sequence is the same, the identification of pairs between the two, induced by the sequence of arcslides, may not be the identity. See the triangle relation for an example.

With this identification of pairs throughout a sequence, we can talk about a *pair of points* in the sequence. These are pairs of points, one for each pointed matched circle in the sequence, that are identified to be the same. We assign a number from 1 to d to each pair of points in the sequence that served as either the B pair or the C pair of some arcslide, where d is the total number of such pairs.

Each generator of the box tensor product $\widehat{\mathcal{DA}}(\tau_1) \boxtimes \cdots \boxtimes \widehat{\mathcal{DA}}(\tau_n)$ is of the form $\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_n$, where $\boldsymbol{x}_i$ is a generator of $\widehat{\mathcal{DA}}(\tau_i)$ for each $1 \le i \le n$, and the right idempotent of $\boldsymbol{x}_i$ agrees with the left idempotent of $\boldsymbol{x}_{i+1}$ for each $1 \le i < n$. A generator $\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_n$ is determined by the set of occupied pairs at the starting and ending pointed matched circles, and at each pointed matched circles in the middle. It is clear that each unnumbered pair must be either occupied throughout or unoccupied throughout. For the numbered pairs, the only possible changes are as follows: suppose that for a certain arcslide τ_i in the sequence, the B pair is numbered b_i and the C pair is numbered c_i; then it is possible to have c_i, but not b_i, occupied in the left idempotent of $\boldsymbol{x}_i$, and b_i, but not c_i, occupied in the right idempotent, with all other pairs staying the same. This corresponds to choosing $\boldsymbol{x}_i$ to have type Y.

We can therefore specify a *type* of generators by specifying which of the numbered pairs are occupied at each pointed matched circle. At each arcslide, the generator is either type X or type Y. In the first case, the occupied pairs must be the same before and after, and in the second case, the C pair occurs before and is replaced by the B pair. To choose a specific generator of a given type, it remains to choose which unnumbered pairs to occupy throughout, so that the total number of occupied pairs is g (half of the total $2g$ pairs).

We now study the involution relation as an example. This is the simplest case, which nevertheless illustrates most of the reasoning required. One possible Heegaard diagram for the involution relation is shown in Figure 10. There are two numbered pairs, that is, $d = 2$. Pair 1 served as the C pair and pair 2 served as the B pair for both arcslides. The possible types of generators are

$$_{()}X_{()}X_{()},\ _{(1)}X_{(1)}X_{(1)},\ _{(2)}X_{(2)}X_{(2)},\ _{(12)}X_{(12)}X_{(12)},\ _{(1)}X_{(1)}Y_{(2)},\ _{(1)}Y_{(2)}X_{(2)}.$$

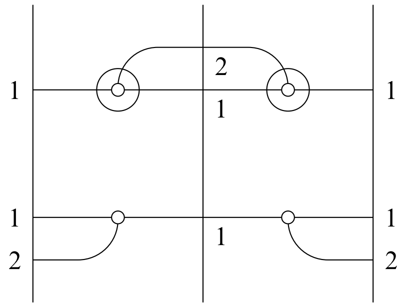


Figure 10: Heegaard diagram for the involution relation

For generators with X at both positions, any combination of occupying pairs is possible. For generators with X at first position and Y at second position, pair 1 and not pair 2 are occupied in the middle, so 1 can be replaced by 2 at the end. This implies pair 1 and not pair 2 are occupied at the beginning, as well. The same reasoning can be used for type YX, and to show that type YY is not possible.

Later, we may use $(*)$ to denote an arbitrary subset of the numbered pairs, that stays the same for a given generator. So for example, we may collect the first four types above into ${}_{(*)}X_{(*)}X_{(*)}$.

We now state several general facts about $\widehat{\mathcal{DA}}(\tau)$ and the box tensor products of such bimodules. These follow from the corresponding facts about $\widehat{\mathcal{DD}}(\tau)$ in Section 2E, and the definition of $\widehat{\mathcal{DA}}(\tau)$ as $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}}) \boxtimes \widehat{\mathcal{DD}}(\tau)$.

Remark 4.1 (relation with Heegaard diagram) Just as in the type *DD* case, each generator of $\widehat{\mathcal{DA}}(\tau)$ corresponds to a tuple of points in the standard Heegaard diagram for the arcslide, with its left (type D) idempotent the set of unoccupied α–arcs on the left, and its right (type A) idempotent the set of occupied α–arcs on the right. When Heegaard diagrams of arcslides are glued side-by-side along their boundaries, the result is a larger Heegaard diagram that now contains α–circles. Note that the boundaries that are glued along are removed from the resulting diagram. Each generator $\boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_n$ of the box tensor product corresponds to a tuple of points, with each α and β circles containing exactly one point, and each α–arc containing at most one point.

As in the type *DD* case, each arrow in $\widehat{\mathcal{DA}}(\tau)$ corresponds to a domain away from the basepoint in the Heegaard diagram of the arcslide. Likewise, each arrow in the box tensor product $\widehat{\mathcal{DA}}(\tau_1) \boxtimes \cdots \boxtimes \widehat{\mathcal{DA}}(\tau_n)$ corresponds to a domain in the Heegaard diagram obtained by gluing the diagrams for $\tau_1, \ldots, \tau_n$ in sequence. The multiplicity of the domain on the left (resp. right) boundary equals the total multiplicity of the algebra coefficients on the left (resp. right) of the arrow. The relation $\partial(\partial^\alpha B) = \boldsymbol{y} - \boldsymbol{x}$, when domain B represents an arrow from $\boldsymbol{x}$ to $\boldsymbol{y}$, still holds.

Remark 4.2 (grading) The remarks on the grading set of $\widehat{\mathcal{DD}}(\tau)$ for an arcslide τ extends to a similar statement for $\widehat{\mathcal{DA}}(\tau)$, and by taking box tensor products, extends to $\widehat{\mathcal{DA}}(\phi, \boldsymbol{\tau})$. Given ϕ: $F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ and a factorization $\boldsymbol{\tau}$ of ϕ, the bimodule $\widehat{\mathcal{DA}}(\phi, \boldsymbol{\tau})$ is graded by a set S with free and transitive left-right actions by $G(\mathcal{Z}_1)$ and $G(\mathcal{Z}_2)$. The grading set induces an element of $\mathrm{Out}(G(\mathcal{Z}_1), G(\mathcal{Z}_2))$. If ϕ begins and ends at the same pointed matched circle $\mathcal{Z}$, then it induces an element of the outer automorphism group $\mathrm{Out}(G(\mathcal{Z}), G(\mathcal{Z}))$. That element can be found from the action of ϕ on the homology of the surface. In particular, the identity morphism on $F^\circ(\mathcal{Z})$ induces the identity outer isomorphism on $G(\mathcal{Z})$.

Remark 4.3 (stabilization) Given τ: $F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ and its stabilization $\mathring{\tau}$: $F(\mathring{\mathcal{Z}}_1) \to F(\mathring{\mathcal{Z}}_2)$, the bimodule $\widehat{\mathcal{DA}}(\tau)$ is again an appropriate restriction of $\widehat{\mathcal{DA}}(\mathring{\tau})$. This follows from the corresponding relations between $\widehat{\mathcal{AA}}(\mathcal{Z})$ and $\widehat{\mathcal{AA}}(\mathring{\mathcal{Z}})$, for any pointed matched circle $\mathcal{Z}$. Taking box tensor products, the stabilization property extends to a relation between $\widehat{\mathcal{DA}}(\phi, \boldsymbol{\tau})$ and $\widehat{\mathcal{DA}}(\mathring{\phi}, \mathring{\boldsymbol{\tau}})$, where $\mathring{\phi}$ is the element of the mapping class groupoid that acts as identity on the adjoined $\mathcal{Z}^1$, and as ϕ elsewhere, and where $\mathring{\boldsymbol{\tau}}$ is the extension of the factorization $\boldsymbol{\tau}$.

The corresponding duality statements will be left to the end of Section 4. By then we will have defined all the other bimodule invariants for surface diffeomorphisms.

4B The involution relation

In this section we will verify the involution relation. Figure 10 shows one of the possible cases: overslide in the upward direction. The computations for overslide in the downward direction, and for underslides over a pair of points at distance greater than 2 from each other, are similar.

Recall that the box tensor product is generated by three types of generators:

$$_{(*)}X_{(*)}X_{(*)}, \quad _{(1)}X_{(1)}Y_{(2)}, \quad _{(1)}Y_{(2)}X_{(2)}.$$

For each type XY generator, there is a corresponding type YX generator that occupies the same unnumbered pairs. The plan is to cancel out pairs of XY and YX generators using this correspondence, and show that the resulting bimodule satisfies the four conditions in Lemma 3.8.

There are five domains that contribute type *DA* arrows of interest. They are shown in Figure 11. Domain C contributes an arrow with no A–side inputs and idempotent D–side output from any XY generator to the corresponding YX generator. This allows us to cancel all XY and YX generators using the cancellation lemma. We now focus on the resulting bimodule, with the type XX generators.

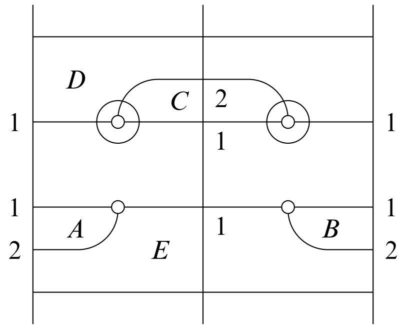


Figure 11: Domains A–E are connected components of $\mathcal{H}\setminus\{\boldsymbol{\alpha},\boldsymbol{\beta}\}$ containing the respective letters. The domains that contribute the type *DA* arrows of interest are A, B, C, $D+C$, and $E+C$. If the upper point in pair 1 (resp. the visible point in pair 2) is the topmost (resp. bottommost) point in the pointed matched circle, then the domain $D+C$ (resp. $E+C$) does not exist.

This bimodule clearly satisfies (ID-1). Condition (ID-2) can be checked by explicit grading computations, which use only the combinatorial features of the Heegaard diagram. In particular, the fact that the induced map $\phi\in\mathrm{Out}(G(\mathcal{Z}),G(\mathcal{Z}))$ is the identity is equivalent to the fact that the action of this composition of arcslides on $H_1(F(\mathcal{Z}))$ is the identity. The stability condition (ID-4) follows from Remark 4.3.

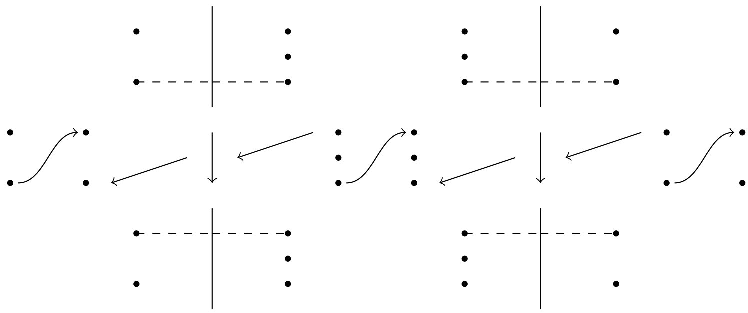
It remains to verify (ID-3). For this, we need to classify all arrows whose coefficients have length at most one on both boundaries. Such an arrow either exists before applying the cancellation lemma, or is produced via a zigzag. In the first case, they correspond to one connected domain between type XX generators. They include trivial horizontal strips in regions away from the slide, the domain $D+C$, and the domain $E+C$. In the second case, the zigzag must be of the form

$$\begin{tikzcd} XX \arrow[dr, "c_1"'] & XY \arrow[d] \arrow[dr, "c_2"] & \\ & YX & XX. \end{tikzcd}$$

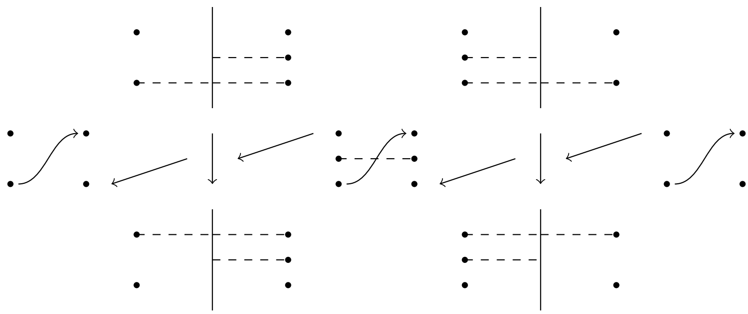
The coefficient c_1 must have length one on the left boundary and length zero on the right, and c_2 must have length zero on the left and length one on the right, or vice-versa. Looking at the Heegaard diagram, the only possibility is that c_1 is produced by domain A and c_2 by domain B.

In what follows, we show that for each of the domains $A, B, D+C$, and $E+C$, and any starting and ending generators with matching idempotents, there is exactly one arrow. These arrows, together with the ones coming from simple horizontal strips, cover each length-1 interval exactly once, which verifies (ID-3).

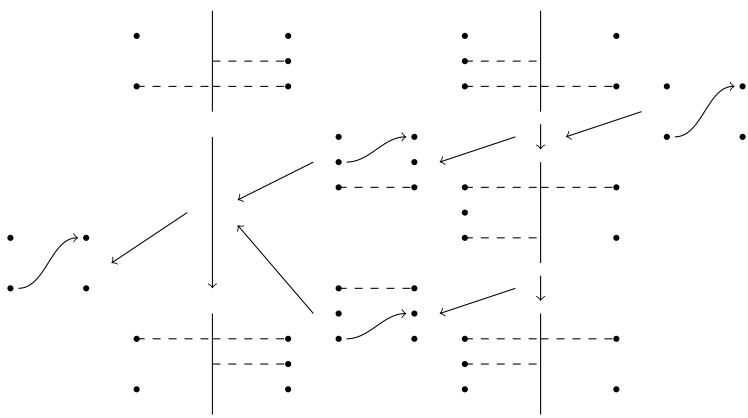
Domains A and B are straightforward since they involve only length-1 coefficients. The next case is the domain $D+C$. If pair 2 is not occupied, the arrow follows from (DA5) and (DA12):



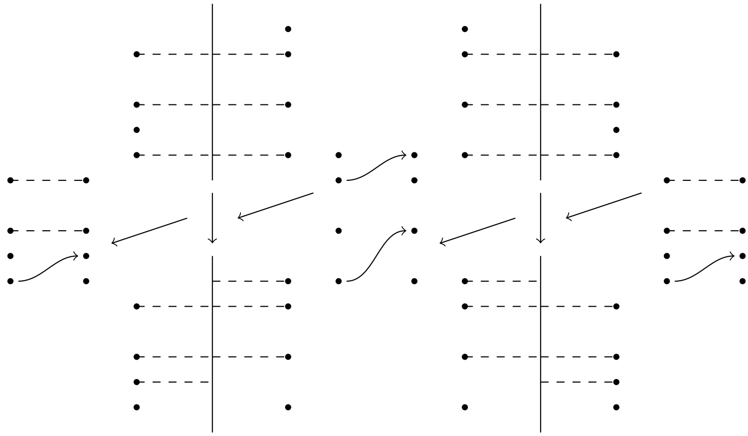
If pair 2 is occupied, then the type DA arrows on the left side depends on the ordering. However, in either case we get the same arrow after box tensoring. If the upper interval comes first, it follows from (DA6) and (DA10):



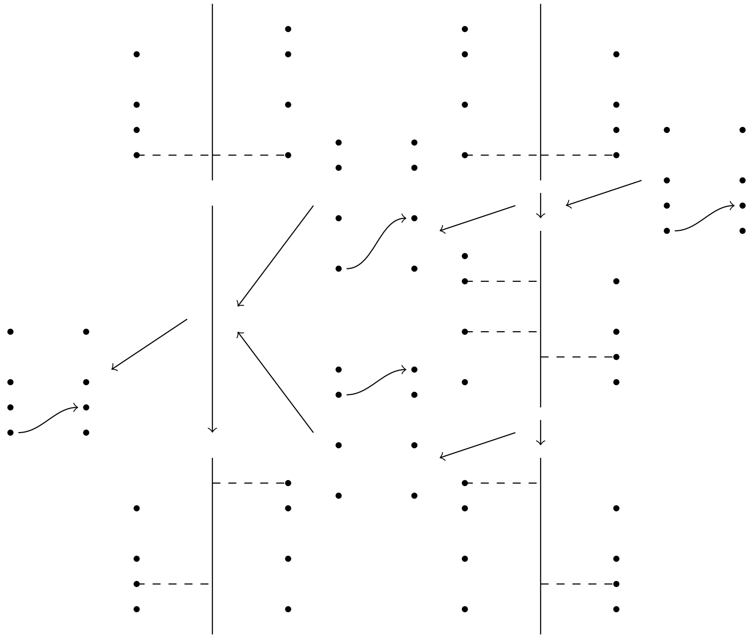
Otherwise, it follows from (DA9), (DA7), and (DA11):



Now for the domain $E + C$. If pair 1 is occupied, the arrow follows from (DA32) and (DA37):



If pair 1 is unoccupied, the arrow follows from (DA31), (DA30), and (DA36):



This finishes the verification of the involution relation, except for the case of a short underslide. The computations in that case involve size-2 intervals where the top and bottom points are paired, so we consider them separately. The diagram is shown in Figure 12.

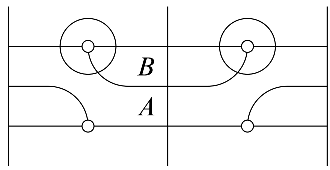
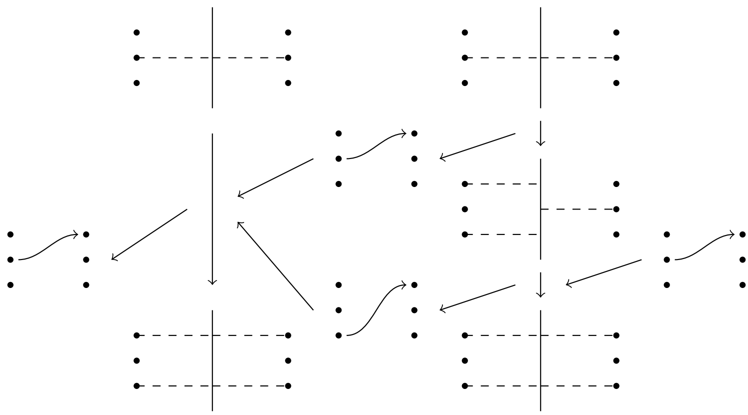


Figure 12: Diagram for involution in the short underslide case. Domains A and B are connected components of $\mathcal{H} \setminus \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ containing the respective letters.

The only difference in the verification is computing the arrow covering the upper length-1 interval. This arrow comes from the domain $A + 2B$, and is produced by (DA22), (DA24), and (DA21):



This concludes all cases of the involution relation. Two results follow immediately from this relation.

Corollary 4.4 *The bimodule $\widehat{\mathcal{DA}}(\tau)$ is quasi-invertible, and the same is true for any box tensor product of such bimodules.*

Proof For a single arcslide, the quasi-inverse is given by $\widehat{\mathcal{DA}}(\tau^{-1})$. It is clear that box tensor products of quasi-invertible bimodules are also quasi-invertible. □

The computations here allow us to prove a uniqueness statement on $\widehat{\mathcal{DD}}(\tau)$. A similar statement is proven in [11].

Corollary 4.5 *Let $\tau\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ be an arcslide. If a bimodule ${}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}M$ is stable, has the same generators and gradings as $\widehat{\mathcal{DD}}(\tau)$, and its type DD action matches that of $\widehat{\mathcal{DD}}(\tau)$ on all arrows with total lengths of coefficients at most 3, then M is homotopy equivalent to $\widehat{\mathcal{DD}}(\tau)$.*

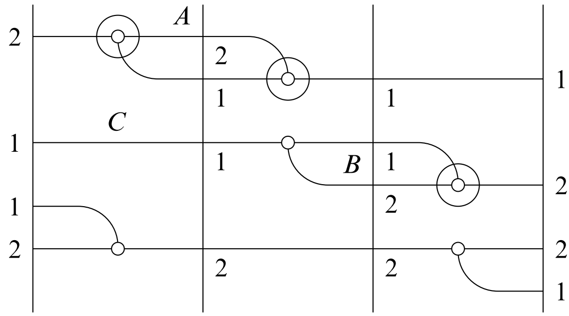


Figure 13: Heegaard diagram for the triangle relation. Domains A, B and C are connected components of $\mathcal{H} \setminus \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ containing the respective letters.

Proof Let $M_{DA} = \widehat{\mathcal{A}\mathcal{A}}(\mathbb{I}) \boxtimes M$. Since we only used type DD arrows whose coefficients have total length at most 3 in this section, we can perform the same computations on M_{DA} as on $\widehat{\mathcal{D}\mathcal{A}}(\tau)$, showing that

$$M_{DA} \boxtimes \widehat{\mathcal{D}\mathcal{A}}(\tau^{-1}) \simeq \mathbb{I} \simeq \widehat{\mathcal{D}\mathcal{A}}(\tau) \boxtimes \widehat{\mathcal{D}\mathcal{A}}(\tau^{-1}).$$

Since $\widehat{\mathcal{D}\mathcal{A}}(\tau^{-1})$ is quasi-invertible, we see M_{DA} is homotopy equivalent to $\widehat{\mathcal{D}\mathcal{A}}(\tau)$. Since $\widehat{\mathcal{A}\mathcal{A}}(\mathbb{I})$ is also quasi-invertible, we see M is homotopy equivalent to $\widehat{\mathcal{D}\mathcal{D}}(\tau)$. □

4C Other relations on arcslides

For each of the other relations on arcslides, we check the conditions in Lemma 3.9. Condition (ID-2) is checked by grading computations as before. (ID-3) follows from Corollary 4.4, with the quasi-inverse M' being the box tensor product of the inverse arcslides in the opposite order. (ID-4) follows from Remark 4.3. It remains to verify (ID-1); with the same technique given here we can show (ID-1) for the inverse M'.

4C1 Triangle For the triangle relation, the Heegaard diagram for one of the possible cases is shown in Figure 13. The other cases differ from this one only by switching the ordering of the points and underslides with overslides. The enumeration of generators, and which pairs of generators can be cancelled, are essentially similar.

The roles of the numbered pairs are as follows:

- Arcslide 1 $C = 2$, $B = 1$.
- Arcslide 2 $C = 1$, $B = 2$.
- Arcslide 3 $C = 2$, $B = 1$.

Only the sequence YXY is forbidden. For that sequence, pair 1 must be occupied after the first arcslide, and therefore after the second arcslide, so type Y is not possible at the third arcslide.

The possible types are these:

$$ {}_{(*)}X_{(*)}X_{(*)}X_{(*)}, \quad {}_{(2)}Y_{(1)}X_{(1)}X_{(1)}, \quad {}_{(1)}X_{(1)}Y_{(2)}X_{(2)}, \quad {}_{(2)}X_{(2)}X_{(2)}Y_{(1)}, $$
$$ {}_{(2)}Y_{(1)}Y_{(2)}X_{(2)}, \quad {}_{(1)}X_{(1)}Y_{(2)}Y_{(1)}, \quad {}_{(2)}Y_{(1)}Y_{(2)}Y_{(1)}, \quad {}_{(1)}X_{(1)}Y_{(2)}X_{(2)}. $$

There are two domains that give rise to cancellable arrows: domain A and B as shown in the figure. Domain A gives rise to arrows from $YY*$ to $XX*$, and domain B gives rise to arrows from $*YY$ to $*XX$. So the cancellable arrows are these:

$$ YYY \to XXY, \quad YYY \to YXX, $$
$$ {}_{(1)}XYY_{(1)} \to {}_{(1)}XXX_{(1)}, \quad {}_{(2)}YYX_{(2)} \to {}_{(2)}XXX_{(2)}. $$

We choose to cancel everything except the second set $YYY \to YXX$ (cancelling the first set of arrows eliminates the option of cancelling the second). These are the remaining generators:

$$ {}_{(2)}YXX_{(1)}, \quad {}_{(1)}XYX_{(2)}, \quad {}_{()}XXX_{()}, \quad {}_{(12)}XXX_{(12)}. $$

Since each type of idempotents at the two ends occurs exactly once, we have verified (ID-1). Note that pair 1 at the left becomes pair 2 at the right, and vice versa, under the bijection of pairs coming from the equality of pointed matched circles at the two ends.

For the triangle relation, it is not immediately clear that there exists a refined relative grading where all generators have grading zero, so we give more details on verifying this condition. Choose a generator in class ${}_{(2)}YXX_{(1)}$ as the base generator (with refined grading zero). To verify that any generator of class ${}_{(1)}XYX_{(2)}$ has grading zero, it suffices to check that any potential domain connecting them has the expected grading. The domain $B+C$ is such a domain. Its grading can be computed to be the same as that of a simple horizontal strip in the Heegaard diagram for identity, with the same boundaries at the two sides. If the genus is greater than 2, then generators of type ${}_{()}XXX_{()}$ and ${}_{(12)}XXX_{(12)}$ exist. They are connected to ${}_{(2)}YXX_{(1)}$ or ${}_{(1)}XYX_{(2)}$ by horizontal strips above either A or B. These domains also have the same gradings as the simple horizontal strips in the diagram for identity with the same boundaries, so the latter two types of generators must also have grading zero.

4C2 Commutativity The Heegaard diagram for one of the cases of the commutativity relation is shown in Figure 14; as in the triangle case, the other possibilities are similar. The role of the numbered pairs are as follows:

- Arcslide 1 $C = 1, B = 2$.
- Arcslide 2 $C = 3, B = 4$.
- Arcslide 3 $C = 1, B = 2$.
- Arcslide 4 $C = 3, B = 4$.

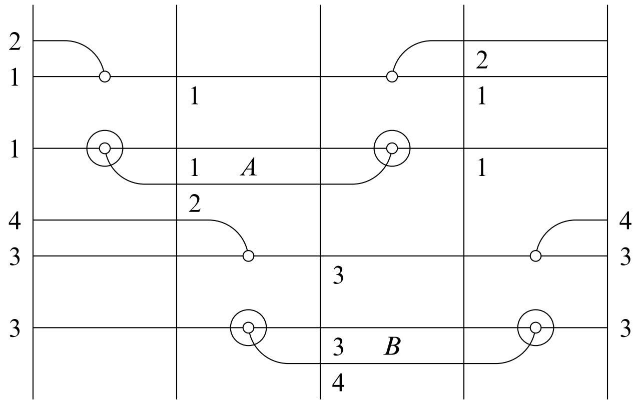


Figure 14: Heegaard diagram for the commutativity relation. Domains A and B are connected components of $\mathcal{H} \setminus \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ containing the respective letters.

The restriction on the types is that at most one of the types at arcslides 1 and 3 can be Y, and at most one at arcslides 2 and 4 can be Y. The possibilities are:

$$\begin{gathered} {}_{(*)}X_{(*)}X_{(*)}X_{(*)}X_{(*)}, \quad {}_{(1)}Y_{(2)}X_{(2)}X_{(2)}X_{(2)} \text{ (3 or 4)}, \\ {}_{(1)}X_{(1)}X_{(1)}Y_{(2)}X_{(2)} \text{ (3 or 4)}, \quad {}_{(3)}X_{(3)}Y_{(4)}X_{(4)}X_{(4)} \text{ (1 or 2)}, \\ {}_{(3)}X_{(3)}X_{(3)}X_{(3)}Y_{(4)} \text{ (1 or 2)}, \quad {}_{(13)}Y_{(23)}Y_{(24)}X_{(24)}X_{(24)}, \\ {}_{(13)}Y_{(23)}X_{(23)}X_{(23)}Y_{(24)}, \quad {}_{(13)}X_{(13)}Y_{(14)}Y_{(24)}X_{(24)}, \quad {}_{(13)}X_{(13)}X_{(13)}Y_{(23)}Y_{(24)}, \end{gathered}$$

where (3 or 4) means "with pairs 3 and/or 4 possibly added to each idempotent".

The two domains giving rise to cancellable arrows are labelled A and B in the Figure 14. Domain A gives rise to arrows from $YaXb$ to $XaYb$, for any valid choice of $a, b \in \{X, Y\}$. Likewise, domain B gives rise to arrows from $aYbX$ to $aXbY$. So the cancellable arrows are:

$$ {}_{(1)}YXXX_{(2)} \to {}_{(1)}XXYX_{(2)} \text{ (3 or 4)}, \quad {}_{(3)}XYXX_{(4)} \to {}_{(3)}XXXY_{(4)} \text{ (1 or 2)}, $$

$$\begin{tikzcd} {}_{(13)}YYXX_{(24)} \arrow[r] \arrow[d] & {}_{(13)}YXXY_{(24)} \arrow[d] \\ {}_{(13)}XYYX_{(24)} \arrow[r] & {}_{(13)}XXYY_{(24)}. \end{tikzcd}$$

The first two arrows cancel all generators with one Y. For generators with two Y's, we can either cancel both horizontal arrows or both vertical arrows in the square above. In the end, only generators of type ${}_{(*)}XXXX_{(*)}$ remain, which verifies (ID-1).

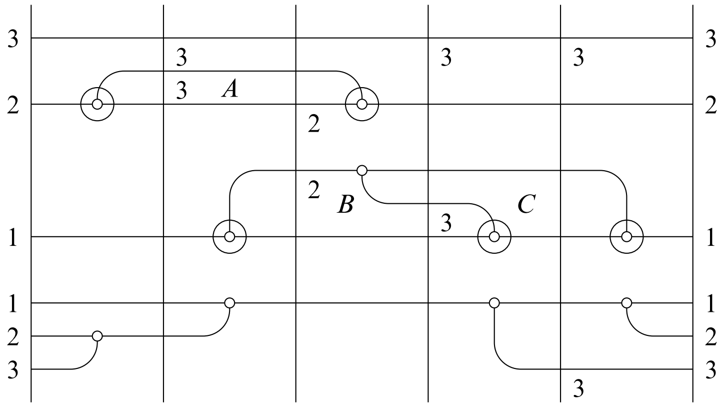


Figure 15: Heegaard diagram for the pentagon relation. Domains A, B, and C are connected components of $\mathcal{H} \setminus \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ containing the respective letters.

4C3 Left and right pentagon The Heegaard diagram for one of the cases of the left pentagon relation is shown in Figure 15. Other cases of the left and right pentagon relation are similar.

The role of the numbered pairs are as follows:

- Arcslide 1 $C = 2, B = 3$.
- Arcslide 2 $C = 1, B = 2$.
- Arcslide 3 $C = 2, B = 3$.
- Arcslide 4 $C = 1, B = 3$.
- Arcslide 5 $C = 1, B = 2$.

The possible types are:

- ${}_{(*)}X_{(*)}X_{(*)}X_{(*)}X_{(*)}X_{(*)}$,
- ${}_{(12)}Y_{(13)}Y_{(23)}X_{(23)}X_{(23)}X_{(23)}$,
- ${}_{(12)}X_{(12)}X_{(12)}Y_{(13)}X_{(13)}Y_{(23)}$,
- ${}_{(12)}Y_{(13)}X_{(13)}X_{(13)}X_{(13)}Y_{(23)}$,
- ${}_{(1)}X_{(1)}Y_{(2)}Y_{(3)}X_{(3)}X_{(3)}$,
- ${}_{(2)}Y_{(3)}X_{(3)}X_{(3)}X_{(3)}X_{(3)}$, ${}_{(12)}Y_{(13)}X_{(13)}X_{(13)}X_{(13)}X_{(13)}$,
- ${}_{(1)}X_{(1)}Y_{(2)}X_{(2)}X_{(2)}X_{(2)}$, ${}_{(13)}X_{(13)}Y_{(23)}X_{(23)}X_{(23)}X_{(23)}$,
- ${}_{(2)}X_{(2)}X_{(2)}Y_{(3)}X_{(3)}X_{(3)}$, ${}_{(12)}X_{(12)}X_{(12)}Y_{(13)}X_{(13)}X_{(13)}$,
- ${}_{(1)}X_{(1)}X_{(1)}X_{(1)}Y_{(3)}X_{(3)}$, ${}_{(12)}X_{(12)}X_{(12)}X_{(12)}Y_{(23)}X_{(23)}$,
- ${}_{(1)}X_{(1)}X_{(1)}X_{(1)}X_{(1)}Y_{(2)}$, ${}_{(13)}X_{(13)}X_{(13)}X_{(13)}X_{(13)}Y_{(23)}$.

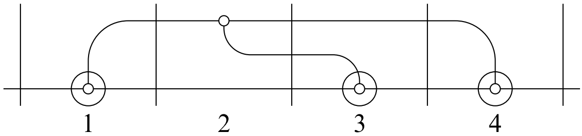


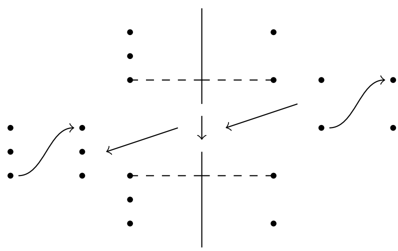
Figure 16: Domain $B+C$

Domains A, B, and C, respectively, give rise to the following arrows:

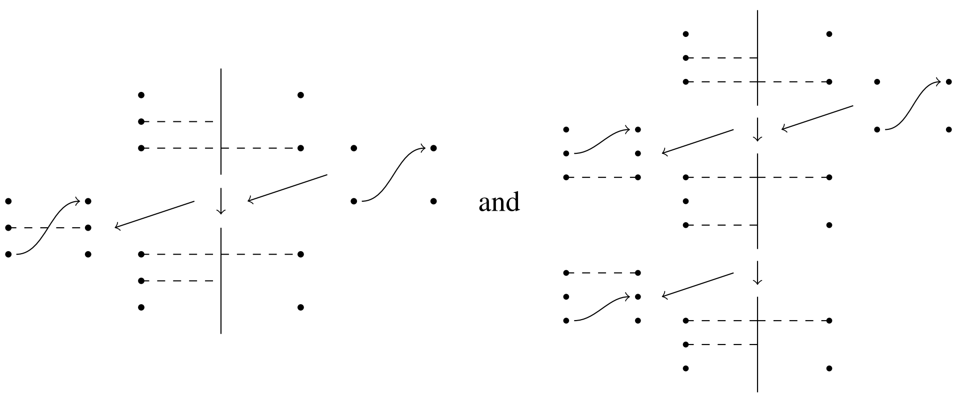
$$XXY**\to YXX**, \quad *XXY*\to *YYX*, \quad **YXY\to **XYX.$$

Other domains that may give arrows are $B+C$ and $A+B$. We first analyze $B+C$, showing that it will always contribute an arrow whenever idempotent matches. The calculation involves box tensoring the four type DA bimodules as shown in Figure 16.

The arrow needed in the fourth piece is simple. For the third, there are several ways to cover the domain. First, if pair 3 is unoccupied in the middle pieces, use (DA5):



If pair 3 is occupied, there are two different ways: (DA6), or (DA9) and (DA7):



Now looking at the possible arrows in the second piece, we see there is always exactly one way to continue forming the arrow in the box tensor product to the second piece (and then trivially to the first piece). If pair 3 is unoccupied, we use (DA12). If pair 3 is occupied, we use either (DA10) or (DA11), depending on the ordering $<_{\mathcal{Z}}$. This shows that the domain $B+C$ gives rise to the following arrows:

$$*_{(1)}X_{(1)}X_{(1)}X_{(1)}Y_{(2)} \to *_{(1)}Y_{(2)}X_{(2)}X_{(2)}X_{(2)},$$
$$*_{(13)}X_{(13)}X_{(13)}X_{(13)}Y_{(23)} \to *_{(13)}Y_{(23)}X_{(23)}X_{(23)}X_{(23)}.$$

Finally, we consider the domain $A+B$. This domain potentially contributes arrows of the form $XXXX* \to YYXX*$. The only possible choice of idempotents is

$${}_{(12)}X_{(12)}X_{(12)}X_{(12)}Y_{(23)}X_{(23)} \to {}_{(12)}Y_{(13)}Y_{(23)}X_{(23)}X_{(23)}X_{(23)}.$$

Rather than computing the type DA arrows for this domain like in the previous case, we note that the sequence

$${}_{(12)}XXYXY_{(23)} \to {}_{(12)}YXXXY_{(23)} \to {}_{(12)}YYXXX_{(23)}$$

must cancel against something in the type DA structure equation. This is possible only if the domain $A+B$ contributes an arrow.

In summary, the cancellable arrows are these:

$$\begin{gathered}
{}_{(1)}XXXXY_{(2)} \to {}_{(1)}XYXXX_{(2)}, \quad {}_{(13)}XXXXY_{(23)} \to {}_{(13)}XYXXX_{(23)}, \\
{}_{(2)}XXYXX_{(3)} \to {}_{(2)}YXXXX_{(3)}, \quad {}_{(12)}XXYXX_{(13)} \to {}_{(12)}YXXXX_{(13)}, \\
{}_{(1)}XXXYX_{(3)} \to {}_{(1)}XYYXX_{(3)},
\end{gathered}$$

$$\begin{tikzcd}
{}_{(12)}XXYXY_{(23)} \arrow[r] \arrow[d] & {}_{(12)}YXXXY_{(23)} \arrow[d] \\
{}_{(12)}XXXYX_{(23)} \arrow[r] & {}_{(12)}YYXXX_{(23)}.
\end{tikzcd}$$

The four types of generators starting with (12) and ending with (23) form the square above, and are cancelled using either the horizontal or vertical arrows. The other ten types of generators containing at least one Y are cancelled using the first five arrows. So only generators of type ${}_{(*)}XXXXX_{(*)}$ remain, which verifies (ID-1).

This concludes the proof of Theorem 2.2, showing that the bimodule $\widehat{\mathcal{DA}}(\phi,\boldsymbol{\tau})$ is independent of the choice of factorization $\boldsymbol{\tau}$ up to homotopy equivalence. This allows us to write $\widehat{\mathrm{CFDA}}(\phi)$ for the homotopy equivalence class of $\widehat{\mathcal{DA}}(\phi,\boldsymbol{\tau})$, and define the other invariants $\widehat{\mathrm{CFDD}}(\phi)$, $\widehat{\mathrm{CFAA}}(\phi)$, and $\widehat{\mathrm{CFAD}}(\phi)$ combinatorially by box tensoring with appropriate identity bimodules.

We finish with a discussion of how duality on $\widehat{\mathcal{DD}}(\tau)$ extends to the other bimodule invariants.

Lemma 4.6 *For any element $\phi\colon F^\circ(\mathcal{Z}_1)\to F^\circ(\mathcal{Z}_2)$ of the strongly based mapping class groupoid, we have*

$$ {}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{\mathrm{CFAD}}(\phi^{-1})_{\mathcal{A}(-\mathcal{Z}_2)} \simeq \overline{{}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(\phi)_{\mathcal{A}(\mathcal{Z}_2)}}. \tag{25}$$

Proof First, we consider the case of an arcslide τ. Using the definition of $\widehat{\mathrm{CFAD}}$ and the fact that $\widehat{\mathrm{CFAA}}(\mathbb{I})$ and $\widehat{\mathrm{CFDD}}(\mathbb{I})$ are quasi-inverses, we have

$$ {}^{\mathcal{A}(\mathcal{Z}_2),\mathcal{A}(-\mathcal{Z}_1)}\widehat{\mathrm{CFDD}}(\tau^{-1}) \simeq {}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{\mathrm{CFAD}}(\tau^{-1})_{\mathcal{A}(-\mathcal{Z}_2)} \boxtimes {}^{\mathcal{A}(\mathcal{Z}_2),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}_2}).$$

On the other hand,

$$\begin{aligned} {}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}\overline{\widehat{\mathrm{CFDD}}(\tau)} &\simeq \overline{{}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{\mathrm{CFDA}}(\tau)_{\mathcal{A}(\mathcal{Z}_2)} \boxtimes {}^{\mathcal{A}(\mathcal{Z}_2),\mathcal{A}(-\mathcal{Z}_2)}\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}_2})} \\ &\simeq {}^{\mathcal{A}(-\mathcal{Z}_1)}\overline{\widehat{\mathrm{CFDA}}(\tau)}_{\mathcal{A}(-\mathcal{Z}_2)} \boxtimes {}^{\mathcal{A}(-\mathcal{Z}_2),\mathcal{A}(\mathcal{Z}_2)}\overline{\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}_2})}. \end{aligned}$$

By the remarks on duality at the end of Section 2E, we see $\widehat{\mathrm{CFDD}}(\tau^{-1})$ and $\overline{\widehat{\mathrm{CFDD}}(\tau)}$ are homotopy equivalent after switching the two algebra actions. It is also clear from the construction of $\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}_2})$ that it is isomorphic to $\overline{\widehat{\mathrm{CFDD}}(\mathbb{I}_{\mathcal{Z}_2})}$ after switching the algebra actions. This implies (25) for arcslides τ.

For a general surface diffeomorphism ϕ, factor it into arcslides τ_i. The statement then follows from the case of arcslides, and the fact that taking duals distributes over the box tensor product. □

5 The 3–manifold invariant

In this section, we prove Theorem 2.4, showing that the homotopy type of the chain complex $\widehat{\mathrm{HF}}$ given in Construction 2.3 does not depend on the choices made. There are two main components of the proof, given by the two lemmas below.

Let $\mathrm{MCG}_0(\mathcal{Z}^g)$ denote the strongly based mapping class group on $F_{g,1}$, parametrized by the genus g split pointed matched circle $\mathcal{Z}^g$. Recall that $\boldsymbol{H}^g$ denotes the 0–framed handlebody, and its orientation reversal $-\boldsymbol{H}^g$ is the ∞–framed handlebody.

Lemma 5.1 (stabilization) *Let ψ be an element of $\mathrm{MCG}_0(\mathcal{Z}^g)$. Consider $F_{g+1,1}$, parametrized by $\mathcal{Z}^{g+1}$ as the surface obtained from $F_{g,1}$ by adding a handle in a neighborhood of the basepoint. Let $\mathring{\psi}$ be the element of $\mathrm{MCG}_0(\mathcal{Z}^{g+1})$ that fixes the new handle and acts as ψ elsewhere. Then*

$$\begin{aligned} &\big(\widehat{\mathrm{CFAA}}(\psi)\boxtimes\widehat{\mathrm{CFD}}(\boldsymbol{H}^g)\big)\boxtimes\widehat{\mathrm{CFD}}(-\boldsymbol{H}^g) \\ &\qquad\simeq \big(\widehat{\mathrm{CFAA}}(\mathring{\psi})\boxtimes\widehat{\mathrm{CFD}}(\boldsymbol{H}^{g+1})\big)\boxtimes\widehat{\mathrm{CFD}}(-\boldsymbol{H}^{g+1}). \end{aligned}\tag{26}$$

Definition 5.2 Define $\mathrm{MCG}_0^{\beta}(\mathcal{Z}^g)$ to be the subgroup of $\mathrm{MCG}_0(\mathcal{Z}^g)$ consisting of maps that extend to automorphisms of $\boldsymbol{H}^g$. Likewise, define $\mathrm{MCG}_0^{\alpha}(\mathcal{Z}^g)$ to be the subgroup of $\mathrm{MCG}_0(\mathcal{Z}^g)$ consisting of maps that extend to automorphisms of $-\boldsymbol{H}^g$ (using identification $\mathcal{Z}^g = -\mathcal{Z}^g$ to consider $-\boldsymbol{H}^g$ as parametrized by $\mathcal{Z}^g$).

Lemma 5.3 (reparametrization of the 0–framed handlebody) *For each element $\phi \in \mathrm{MCG}_0^{\beta}(\mathcal{Z}^g)$, we have*

$$^{\mathcal{A}(-\mathcal{Z}^g)}\widehat{\mathrm{CFAD}}(\phi)_{\mathcal{A}(-\mathcal{Z}^g)} \boxtimes {}^{\mathcal{A}(-\mathcal{Z}^g)}\widehat{\mathrm{CFD}}(\boldsymbol{H}^g) \simeq {}^{\mathcal{A}(-\mathcal{Z}^g)}\widehat{\mathrm{CFD}}(\boldsymbol{H}^g). \tag{27}$$

We first show that these two lemmas imply Theorem 2.4.

Proof of Theorem 2.4 There are two choices made in Construction 2.3: the choice of Heegaard splitting $Y = Y_1 \cup Y_2$, and choice of parametrizations of Y_1 and Y_2 by standard handlebodies. It is well known that any two Heegaard splittings become isotopic after a finite number of stabilizations. Also, any stabilization can be isotopied to the standard one, adding a handle in a neighborhood of the basepoint. If ψ is a valid choice of element in $\mathrm{MCG}_0(\mathcal{Z}^g)$ in the second stage of the construction, then $\mathring{\psi}$ is a valid choice of element in $\mathrm{MCG}_0(\mathcal{Z}^{g+1})$ after a standard stabilization. So Lemma 5.1 implies that Construction 2.3 is invariant under stabilizations.

Now we consider choice of parametrizations of Y_1 and Y_2. Recall $\psi = \overline{f}_{2*}^{-1} \circ u \circ f_{1*}$, where $u\colon \partial Y_1 \to -\partial Y_2$ is the gluing map, $f_1\colon \boldsymbol{H}^g \to Y_1$ is the parametrization of Y_1 by $\boldsymbol{H}^g$, and $f_2\colon -\boldsymbol{H}^g \to Y_2$ is the parametrization of Y_2 by $-\boldsymbol{H}^g$. Hence, changing parametrization of Y_1 changes ψ to $\psi' = \psi \circ \phi_1$, where $\phi_1 \in \mathrm{MCG}_0^{\beta}(\mathcal{Z}^g)$, and changing parametrization of Y_2 changes ψ to $\psi' = \overline{\phi}_2^{-1} \circ \psi$, where $\phi_2 \in \mathrm{MCG}_0^{\alpha}(\mathcal{Z}^g)$.

It remains to show the following:

$$\widehat{\mathrm{CFAA}}(\psi \circ \phi_1) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^g) \simeq \widehat{\mathrm{CFAA}}(\psi) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^g),$$
$$\widehat{\mathrm{CFAA}}(\overline{\phi}_2^{-1} \circ \psi) \boxtimes \widehat{\mathrm{CFD}}(-\boldsymbol{H}^g) \simeq \widehat{\mathrm{CFAA}}(\psi) \boxtimes \widehat{\mathrm{CFD}}(-\boldsymbol{H}^g),$$

for $\phi_1 \in \mathrm{MCG}_0^{\beta}(\mathcal{Z}^g)$ and $\phi_2 \in \mathrm{MCG}_0^{\alpha}(\mathcal{Z}^g)$.

The first equation follows directly from Lemma 5.3:

$$\begin{aligned}\widehat{\mathrm{CFAA}}(\psi \circ \phi_1) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^g) &\simeq \widehat{\mathrm{CFAA}}(\psi) \boxtimes \widehat{\mathrm{CFAD}}(\phi_1) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^g)\\ &\simeq \widehat{\mathrm{CFAA}}(\psi) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^g).\end{aligned}$$

For the second equation, by taking the dual of (27), and using Lemma 4.6, we get

$$^{\mathcal{A}(\mathcal{Z}^g)}\widehat{\mathrm{CFDA}}(\phi^{-1})_{\mathcal{A}(\mathcal{Z}^g)} \boxtimes {}^{\mathcal{A}(\mathcal{Z}^g)}\widehat{\mathrm{CFD}}(-\boldsymbol{H}^g) \simeq {}^{\mathcal{A}(\mathcal{Z}^g)}\widehat{\mathrm{CFD}}(-\boldsymbol{H}^g) \tag{28}$$

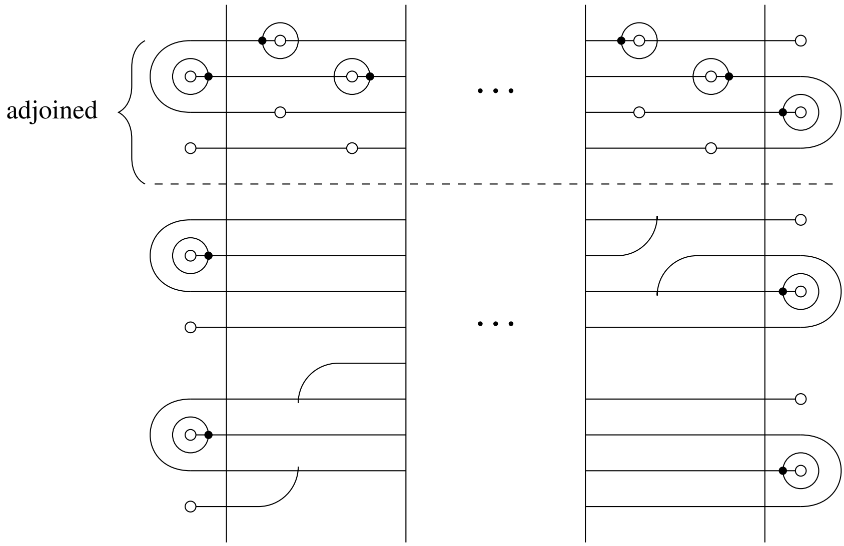


Figure 17: Proof of invariance under stabilization

for any $\phi \in \mathrm{MCG}_0^{\beta}(\mathcal{Z}^g)$. Then the second equation follows, as

$$\begin{aligned}\widehat{\mathrm{CFAA}}(\bar{\phi}_2^{-1} \circ \psi) \boxtimes \widehat{\mathrm{CFD}}(-\boldsymbol{H}^g) &\simeq \widehat{\mathrm{CFAA}}(\psi) \boxtimes \widehat{\mathrm{CFDA}}(\bar{\phi}_2^{-1}) \boxtimes \widehat{\mathrm{CFD}}(-\boldsymbol{H}^g) \\ &\simeq \widehat{\mathrm{CFAA}}(\psi) \boxtimes \widehat{\mathrm{CFD}}(-\boldsymbol{H}^g),\end{aligned}$$

since $\phi_2 \in \mathrm{MCG}_0^{\alpha}(\mathcal{Z}^g)$ implies $\bar{\phi}_2 \in \mathrm{MCG}_0^{\beta}(\mathcal{Z}^g)$. $\square$

Proof of Lemma 5.1 Choose factorization $\boldsymbol{\tau}$ for ψ, then $\mathring{\boldsymbol{\tau}}$ is a factorization for $\mathring{\psi}$. Choose $\widehat{\mathcal{DA}}(\psi, \boldsymbol{\tau})$ and $\widehat{\mathcal{DA}}(\mathring{\psi}, \mathring{\boldsymbol{\tau}})$ as models for the $\widehat{\mathrm{CFDA}}$ invariants behind the $\widehat{\mathrm{CFAA}}$ invariants. The lemma then follows from the stabilization property for $\widehat{\mathcal{DA}}(\psi, \boldsymbol{\tau})$. We can see this by comparing the Heegaard diagrams underlying the two sides of (26). First, the Heegaard diagram for $\widehat{\mathcal{DA}}(\mathring{\psi}, \mathring{\boldsymbol{\tau}})$ is constructed from that for $\widehat{\mathcal{DA}}(\psi, \boldsymbol{\tau})$ by adjoining a horizontal strip of diagrams for the identity diffeomorphism of the genus 1 surface at the top. Likewise, the Heegaard diagrams of $\boldsymbol{H}^{g+1}$ and $-\boldsymbol{H}^{g+1}$ are obtained from that of $\boldsymbol{H}^g$ and $-\boldsymbol{H}^g$ by adjoining diagrams of $\boldsymbol{H}^1$ and $-\boldsymbol{H}^1$ to the top. These constructions are combined in Figure 17.

By Remark 4.1, generators in the chain complex

$$\big(\widehat{\mathrm{CFAA}}(\psi) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^g)\big) \boxtimes \widehat{\mathrm{CFD}}(-\boldsymbol{H}^g) \tag{29}$$

correspond to certain tuples of intersection points in the part of the diagram below the dashed line in Figure 17, while generators in the chain complex

$$\big(\widehat{\mathrm{CFAA}}(\mathring{\psi}) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^{g+1})\big) \boxtimes \widehat{\mathrm{CFD}}(-\boldsymbol{H}^{g+1}) \tag{30}$$

correspond to certain tuples of intersection points in the full diagram. Likewise, there is a correspondence between arrows in the type *DA* action on the two sides, and domains in appropriate parts of the diagram.

The choice of intersection points in the adjoined portion of the diagram is forced (as marked in the figure), which means that it is the same for all generators in (30). So there is a one-to-one correspondence between generators in (29) and (30). Moreover, since there are no closed domains above the dashed line, all arrows in (30) automatically have domains restricted below the dashed line. By Remark 4.3, there is a one-to-one correspondence between these arrows and the arrows in (29). This shows the chain complexes (29) and (30) are isomorphic, proving Lemma 5.1. □

For Lemma 5.3, we need to show

$$\widehat{\mathrm{CFAD}}(\phi) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^g) \simeq \widehat{\mathrm{CFD}}(\boldsymbol{H}^g)$$

for any $\phi \in \mathrm{MCG}_0^\beta(\mathcal{Z}_g)$. It suffices to verify the equation for a set of generators of $\mathrm{MCG}_0^\beta(\mathcal{Z}_g)$.

We find generators for the strongly based mapping class group by appealing to results on the usual mapping class group. Let F_g be the genus g surface with a basepoint. Let $\mathrm{MCG}(F_g)$ be the group of isotopy classes of diffeomorphisms on F_g that fixes the basepoint, with isotopies also required to fix the basepoint. It is related to $\mathrm{MCG}_0(\mathcal{Z}_g)$ by a short exact sequence (see [4, Section 4.2.5]):

$$0 \to \mathbb{Z} \xrightarrow{\tau_\partial} \mathrm{MCG}_0(\mathcal{Z}_g) \to \mathrm{MCG}(F_g) \to 0.$$

Here τ_∂ maps the generator of $\mathbb{Z}$ to the *boundary Dehn twist* in $\mathrm{MCG}_0(\mathcal{Z}_g)$. This is the element that performs a Dehn twist along a loop parallel to the boundary of $F_{g,1}$.

There is, likewise, a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\tau_\partial} \mathrm{MCG}_0^\beta(\mathcal{Z}_g) \to \mathrm{MCG}^\beta(F_g) \to 0,$$

where $\mathrm{MCG}^\beta(F_g)$ is the subgroup of $\mathrm{MCG}(F_g)$ consisting of restrictions of automorphisms of the 0–framed handlebody $\boldsymbol{H}^g$. This exact sequence shows that a generating set of $\mathrm{MCG}_0^\beta(\mathcal{Z}_g)$ can be obtained by adding the boundary Dehn twist to the lifting of a generating set of $\mathrm{MCG}^\beta(F_g)$.

A generating set of $\mathrm{MCG}^{\beta}(F_g)$ is given in [20] (the corresponding notation in that paper is $\mathrm{MCG}^*(F_g)$). We reproduce the list of generators, together with the action of each generator on $\pi_1(F_g)$ here. For an element $\psi \in \mathrm{MCG}^{\beta}(F_g)$, let $\psi_\sharp\colon \pi_1(F_g) \to \pi_1(F_g)$ be its action on $\pi_1(F_g)$. We let $a_1, b_1, \dots, a_g, b_g$ be a set of standard generators of $\pi_1(F_g)$, with each b_i contractible in the handlebody, and each a_i intersecting b_i once. Let $s_i = a_i^{-1} b_i^{-1} a_i b_i$, so that $s_n \cdots s_2 s_1 = 1$ is a relation in $\pi_1(F_g)$. In [20], a genus g surface is considered as a sphere with g handles attached. Each handle, together with its immediate base, is called a *knob*. We refer to that paper for diagrams and geometric description of these generators.

Theorem 5.4 (Suzuki [20]) *The group $\mathrm{MCG}^{\beta}(F_g)$ is generated by $\rho, \omega_1, \tau_1, \rho_{12}, \theta_{12}$ and ξ_{12}, whose actions on $\pi_1(F_g)$ are the following:*

- **Cyclic translation of handles** $\rho_\sharp\colon a_i \to a_{i+1}, b_i \to b_{i+1}$, *where indices are taken modulo g.*
- **Twisting a knob** $\omega_{1\sharp}\colon a_1 \to a_1^{-1} s_1^{-1}, b_1 \to a_1^{-1} b_1^{-1} a_1$, $a_j \to a_j$, $b_j \to b_j$ *for* $2 \le j \le n$.
- **Twisting a handle, or Dehn twist** $\tau_{1\sharp}\colon a_1 \to a_1 b_1^{-1}$, $b_1 \to b_1$, $a_j \to a_j$, $b_j \to b_j$ *for* $2 \le j \le n$.
- **Interchanging two knobs** $\rho_{12\sharp}\colon a_1 \to s_1^{-1} a_2 s_1, a_2 \to a_1$, $b_1 \to s_1^{-1} b_2 s_1$, $b_2 \to b_1$, $a_j \to a_j$, $b_j \to b_j$ *for* $3 \le j \le n$.
- **Sliding along $\boldsymbol{a_2}$** $\theta_{12\sharp}\colon a_1 \to a_1(b_2^{-1} a_2^{-1} b_2)$, $a_j \to a_j$ *for* $j \ne 1$, $b_2 \to a_2 b_2 (a_1^{-1} b_1 a_1)(b_2^{-1} a_2^{-1} b_2)$, $b_j \to b_j$ *for* $j \ne 2$.
- **Sliding along $\boldsymbol{b_2}$** $\xi_{12\sharp}\colon a_1 \to b_1 a_1 b_2^{-1} s_2 (a_1^{-1} b_1^{-1} a_1), a_2 \to a_2 b_2 (a_1^{-1} b_1^{-1} a_1) b_2^{-1}$, $a_j \to a_j$ *for* $j \ne 1, 2$, $b_i \to b_i$ *for* $1 \le i \le g$.

Of these, only ρ is nonlocal in the sense that it is not restricted to a part of the surface with fixed genus. All other generators are restricted to a genus 1 or 2 part of the surface. We can remove ρ in favor of other local generators by writing

$$\rho^{-1} = \rho_{12} \circ \rho_{23} \circ \cdots \circ \rho_{g-1,g} \circ (\omega_{g\sharp})^{-2},$$

where $\omega_{g\sharp}$ is similar to $\omega_{1\sharp}$, except acting on the g^{th} handle, and $\rho_{i,i+1}$ interchanges the i^{th} and $(i+1)^{\text{th}}$ knobs. The equation can be verified by comparing the actions of two sides on $\pi_1(F_g)$: the initial $(\omega_{g\sharp})^{-2}$ has the effect of conjugating a_g and b_g by s_g^{-1}. After interchanging the knobs in succession, the action of the right side on $\pi_1(F_g)$ is $a_1 \to s_1^{-1} s_2^{-1} \cdots s_g^{-1} a_g s_g \cdots s_2 s_1$, $a_2 \to a_1$, $a_3 \to a_2$, and so on, and similarly for the b_i's. We then apply the relation $s_g \cdots s_2 s_1 = 1$.

Since the boundary Dehn twist equals ρ^g, the g^{th} power of the cyclic translation of handles, the same generators also generate the group $\mathrm{MCG}_0^{\beta}(\mathcal{Z}_g)$. Thus, we have proven the following.

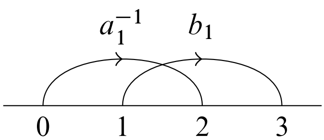
Corollary 5.5 *The group* $\mathrm{MCG}_0^{\beta}(F_g)$ *is generated by* $\omega_1, \omega_g, \tau_1, \theta_{12}, \xi_{12}$ *and* $\rho_{i,i+1}$ *for* $1 \le i \le g-1$.

Each of the generators in Corollary 5.5 is confined to one or two knobs on the surface. Our strategy will be to check (27) on a surface of the corresponding genus, that is, 1 or 2, then extend the result to the general case. First, we compute a decomposition of these generators into arcslides. An arcslide with B pair (b_1, b_2) and C pair (c_1, c_2), with b_1 sliding over c_1, is denoted $b_1 \to c_1$. The points are always labeled 0 to $4g-1$ from left to right. The results are

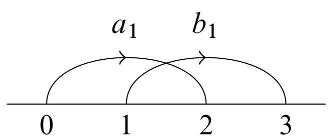
$$\begin{aligned}
\rho_{12}\colon & 3 \to 4, 6 \to 7, 5 \to 6, 4 \to 5, 2 \to 3, 5 \to 6, 4 \to 5, 3 \to 4, \\
& 1 \to 2, 4 \to 5, 3 \to 4, 2 \to 3, 0 \to 1, 3 \to 4, 2 \to 3, 1 \to 2, \\
\theta_{12}\colon & 4 \to 3, 1 \to 0, 1 \to 2, 5 \to 4, 6 \to 5, \\
\xi_{12}\colon & 0 \to 1, 3 \to 4, 6 \to 7, 6 \to 5, 2 \to 3, 1 \to 2, 3 \to 2, \\
\omega_1\colon & 2 \to 3, 1 \to 2, 2 \to 3, 1 \to 2, 2 \to 3, 1 \to 2, \\
\tau_1\colon & 2 \to 3.
\end{aligned}$$

To verify these decompositions, we compute their actions on $\pi_1(F^\circ(\mathcal{Z}^g))$. For any pointed matched circle $\mathcal{Z}$, recall that the surface with circle boundary $F^\circ(\mathcal{Z})$ is formed by attaching 1–handles to Z along the matched pairs of points in $\boldsymbol{a} \subset Z$, then gluing in a solid disk on the other side. Choosing $z \in Z$ as the basepoint, the fundamental group of $F^\circ(\mathcal{Z})$ is generated freely by paths through the 1–handles. We choose the following orientation for the generators of the fundamental group. For the genus 1 cases, we have

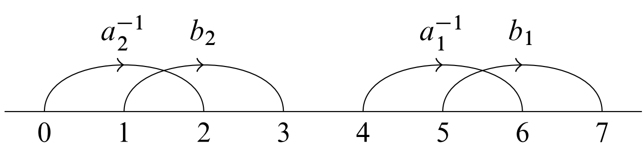
ω_1:



τ_1:



and for all genus 2 cases,



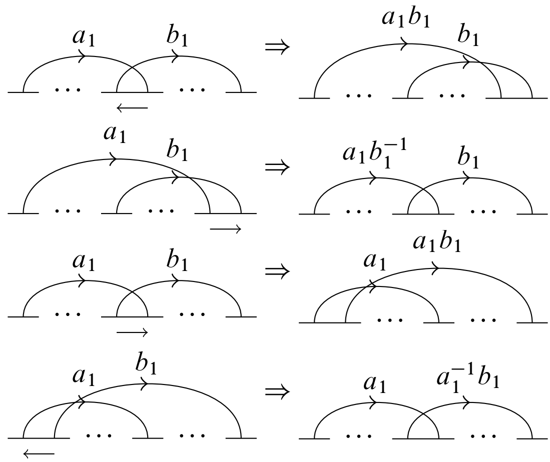


Figure 18: Actions of arcslides on the fundamental group (underslide)

An arcslide $\tau\colon \mathcal{Z}_1 \to \mathcal{Z}_2$ induces an action

$$\tau_*\colon \pi_1(F^\circ(\mathcal{Z}_1)) \to \pi_1(F^\circ(\mathcal{Z}_2))$$

on the fundamental groups. We describe this action by expressing each generator of $\pi_1(F^\circ(\mathcal{Z}_2))$ (corresponding to a pair of matched points in $\mathcal{Z}_2$) in terms of the images under τ_* of generators of $\pi_1(F^\circ(\mathcal{Z}_1))$. This can be computed from the definition of arcslides; see, for example, [11, Figure 3]. The results are shown in Figures 18 and 19. For example, the first diagram means that if the two displayed handles in the starting pointed matched circle correspond to generators a_1 and b_1, then the two displayed handles in the ending pointed matched circle correspond to $\tau_*(a_1 b_1)$ and $\tau_*(b_1)$ (the relation for handles unaffected by the arcslide is clear).

As an example, we verify the decomposition of θ_{12} into arcslides. Only the two middle pairs, corresponding to generators b_2 and a_1^{-1}, are moved during this sequence of arcslides. We follow what happens to these two pairs in Table 1. We identify pairs of points in a sequence of arcslides as before. Each line in the table writes the generator corresponding to pairs identified with the initial b_2 and a_1^{-1} pairs in terms of $\tau_*(\cdot)$ of the initial generators.

After this sequence of arcslides, the two middle pairs have switched positions. So the action is $a_1^{-1} \to b_2^{-1} a_2 b_2 a_1^{-1}$ and $b_2 \to a_2 b_2 a_1^{-1} b_1 a_1 b_2^{-1} a_2^{-1} b_2$. The first equation can be rewritten as $a_1 \to a_1 b_2^{-1} a_2^{-1} b_2$. This agrees with the fundamental group action given in Theorem 5.4.

Proof of Lemma 5.3 The same argument as in the proof of Lemma 5.1 shows that if $\widehat{\mathrm{CFAD}}(\phi) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^g) \simeq \widehat{\mathrm{CFD}}(\boldsymbol{H}^g)$, then $\widehat{\mathrm{CFAD}}(\mathring{\phi}) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^{g+1}) \simeq \widehat{\mathrm{CFD}}(\boldsymbol{H}^{g+1})$,

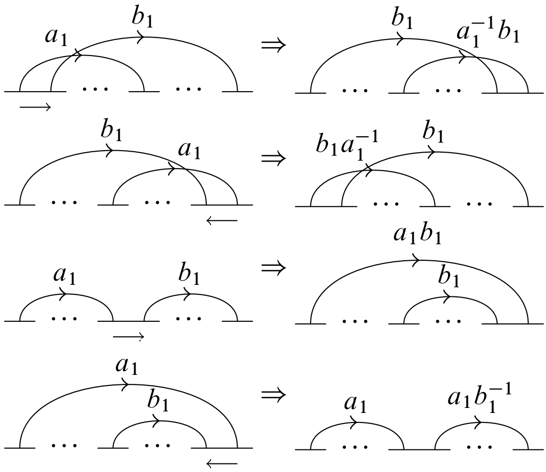


Figure 19: Actions of arcslides on the fundamental group (overslide)

Arcslide	b_2	a_1^{-1}
$4 \to 3$	b_2	$b_2a_1^{-1}$
$1 \to 0$	b_2	$a_2b_2a_1^{-1}$
$1 \to 2$	$b_2^{-1}a_2b_2a_1^{-1}$	$a_2b_2a_1^{-1}$
$5 \to 4$	$b_2^{-1}a_2b_2a_1^{-1}$	$a_2b_2a_1^{-1}b_1$
$6 \to 5$	$b_2^{-1}a_2b_2a_1^{-1}$	$a_2b_2a_1^{-1}b_1a_1b_2^{-1}a_2^{-1}b_2$

Table 1: Trajectories of generators b_2 and a_1^{-1} under various arcslides.

where $\mathring{\phi}$ is the element of $\mathrm{MCG}_0(\mathcal{Z}^{g+1})$ that fixes the new handle and acts as ϕ elsewhere. Here there is again an one-to-one correspondence on the generators between $\widehat{\mathrm{CFAD}}(\phi) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^g)$ and $\widehat{\mathrm{CFAD}}(\mathring{\phi}) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^{g+1})$. There is exactly one domain in the adjoined portion that can (and does) contribute an arrow. The evaluation there is equivalent to the evaluation of $\widehat{\mathrm{CFAD}}(\mathbb{I}_{\mathcal{Z}}) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^1) \simeq \widehat{\mathrm{CFD}}(\boldsymbol{H}^1)$ on the genus 1 pointed matched circle, giving the arrow in $\widehat{\mathrm{CFD}}(\boldsymbol{H}^{g+1})$ that is inside the adjoined pointed matched circle. The remaining domains must be outside the adjoined region, showing a one-to-one correspondence between arrows in $\widehat{\mathrm{CFAD}}(\phi) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^g)$, and the remaining arrows in $\widehat{\mathrm{CFAD}}(\mathring{\phi}) \boxtimes \widehat{\mathrm{CFD}}(\boldsymbol{H}^{g+1})$. This argument works whether $\mathcal{Z}^{g+1}$ is formed as $\mathcal{Z}^1 \# \mathcal{Z}^g$ or as $\mathcal{Z}^g \# \mathcal{Z}^1$.

From this, we see that it is sufficient to verify (27) for each of the generators of $\mathrm{MCG}_0^{\beta}(F_g)$ in its respective minimum genus (1 or 2) case.

To do so, we decompose each generator ϕ of $\mathrm{MCG}_0^{\beta}(F_g)$ (for $g=1$ or 2 depending on ϕ) into arcslides $\tau_n \circ \cdots \circ \tau_1$, as given above. Then we directly compute the left side of (27) using the constructions for $\widehat{\mathcal{DA}}(\tau_i)$. This reduces to a finite computation, which we performed on a computer using a Python program (which implements the description of $\widehat{\mathcal{AA}}(\mathbb{I}_{\mathcal{Z}})$ and the box tensor product). The code for the computation can be found at `https://github.com/bzhan/auto2`. The entire computation took less than 20 seconds.

This concludes the proof of Lemma 5.3, and therefore Theorem 2.4. □

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Quadratic-linear duality and rational homotopy theory of chordal arrangements

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To any graph and smooth algebraic curve C, one may associate a "hypercurve" arrangement, and one can study the rational homotopy theory of the complement X. In the rational case ($C = \mathbb{C}$), there is considerable literature on the rational homotopy theory of X, and the trigonometric case ($C = \mathbb{C}^\times$) is similar in flavor. The case when C is a smooth projective curve of positive genus is more complicated due to the lack of formality of the complement. When the graph is chordal, we use quadratic-linear duality to compute the Malcev Lie algebra and the minimal model of X, and we prove that X is rationally $K(\pi, 1)$.

16S37, 52C35, 55P62

1 Introduction

This paper explores the topology of the rational, trigonometric, and projective (in particular, elliptic) analogues of hyperplane arrangements. The rational case consists of linear arrangements, which are finite sets of codimension-one linear subspaces of a complex vector space. The trigonometric case consists of toric arrangements, which are finite sets of codimension-one subtori in a complex torus. The elliptic case consists of abelian arrangements, which are finite sets of codimension-one abelian subvarieties of a product of elliptic curves. We focus our attention on unimodular and supersolvable arrangements, which are classified by chordal graphs and are therefore called chordal arrangements. Chordal arrangements can be defined without reference to abelian group structure and hence make sense for curves of arbitrary genus. When we discuss the projective case, we will only consider curves of *positive* genus, as our method does not apply to $\mathbb{P}^1$ (whose cohomology ring is not Koszul).

In each case, we study a differential graded algebra (DGA) that is a model (in the sense of rational homotopy theory) for the complement of the arrangement.

- For linear arrangements, the complement is formal, which means that the cohomology algebra with trivial differential is itself a model. A combinatorial presentation for this algebra is given by Orlik and Solomon [12].

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- For toric arrangements, the complement is also formal, and in the unimodular case, a combinatorial presentation of the cohomology ring is given by De Concini and Procesi [6].
- In the projective case, the complement is not necessarily formal, but combinatorially presented models (with nontrivial differential) are given by the first author [2] in the elliptic case and by Dupont [7] in general.

When the matroid associated to the arrangement is supersolvable, the above model is Koszul; this is due to Shelton and Yuzvinsky for linear arrangements [18], and we prove it in the toric and projective cases (Theorems 3.3.3, 3.4.3, and 3.5.3). By studying the quadratic dual of the model, one can obtain a combinatorial presentation for a Lie algebra and use it to compute the $\mathbb{Q}$–nilpotent completion of the fundamental group and the minimal model. This is done by Papadima and Yuzvinsky in the linear case [14], and the toric case is completely analogous. In the projective case, the lack of formality makes this computation more subtle: we need to use nonhomogeneous quadratic duality, where the dual to a Koszul differential graded algebra is a quadratic-linear algebra. With this tool, we extend Papadima and Yuzvinsky's results to the projective setting (Theorem 5.2.1).

We also prove that complements of chordal arrangements are rational $K(\pi, 1)$ spaces. In the rational and toric cases, this follows from formality and Koszulity [14]. In the projective case (where we lack formality) it is not automatic, but we obtain it from our concrete description of the minimal model (Corollary 5.2.3).

In the projective case, our results were inspired by [1], where Bezrukavnikov constructed a model of the ordered configuration space of an arbitrary smooth, projective, complex curve of positive genus; he showed that his model was Koszul, gave a presentation for the dual Lie algebra, and described the minimal model. In fact, our results generalize his since the ordered configuration space is the complement of the braid arrangement (which is chordal).

In Section 2, we review known results on the cohomology of arrangements in each of our cases, giving explicit presentations for the algebras we will consider. In Section 3, we review the proof that the cohomology ring of the complement to a chordal linear arrangement is Koszul, and then we prove the analogous results in the toric and projective cases. In Section 4, we review definitions and results from rational homotopy theory and quadratic-linear duality. The reader can skip ahead to Section 5 and refer back to Section 4 as needed. In Section 5, we review some known results on the rational homotopy theory of linear arrangements, which also apply to the toric case, and then we prove the analogous results for the projective case.

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2 Cohomology

In this section, we collect known results about the cohomology of the complement of a chordal arrangement in each of our cases. Since we will only be considering graphic arrangements in Sections 3 and 5 (see Remark 3.1.1), we will state all of the results in graphical language here, as well. Since our goal will be to study the rational homotopy theory of these spaces, we will also restrict our attention to cohomology with rational coefficients throughout this paper.

2.1 Definitions

An *ordered graph* is a graph $\Gamma = (\mathcal{V}, \mathcal{E})$ with an ordering on the vertices $\mathcal{V}$. We will assume throughout that our graphs are simple (that is, they have no loops or multiple edges). An ordered graph can be considered as a directed graph in the following way: For each edge $e \in \mathcal{E}$, label its larger vertex by $h(e)$ (for "head" of an arrow) and its smaller vertex by $t(e)$ (for "tail" of an arrow). An order on the vertices of Γ induces an order on the edges by setting $e < e'$ if $h(e) < h(e')$ or if $h(e) = h(e')$ and $t(e) < t(e')$.

Remark 2.1.1 None of the structures in this section will depend on the ordering of the vertices, but it will simplify the notation. The order chosen will also be necessary for the proofs in Section 3.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be an ordered graph. Let C be $\mathbb{C}$, $\mathbb{C}^\times$, or a complex projective curve, and let $C^{\mathcal{V}}$ be the complex vector space (respectively torus or projective variety) whose coordinates are indexed by the vertices $\mathcal{V}$. For each edge $e \in \mathcal{E}$, let

$$H_e = \{x_{\mathcal{V}} \in C^{\mathcal{V}} \mid x_{h(e)} = x_{t(e)}\}.$$

The collection $\mathcal{A}(\Gamma, C) = \{H_e \mid e \in \mathcal{E}\}$ is a *graphic* arrangement in $C^{\mathcal{V}}$. In each case, denote the complement of an arrangement $\mathcal{A}$ in V by $X_{\mathcal{A}} := V \setminus \bigcup_{H \in \mathcal{A}} H$. In the case that C is $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve, $\mathcal{A}(\Gamma, C)$ is a *linear*, *toric*, or *abelian* arrangement, respectively.

Example 2.1.2 Let $C = \mathbb{C}$, $\mathbb{C}^\times$, or a complex projective curve. If Γ is the complete graph on n vertices, then $\mathcal{A} = \mathcal{A}(\Gamma, C)$ is the braid arrangement, and its complement $X_{\mathcal{A}}$ is the ordered configuration space of n points on C.

2.2 Linear arrangements

For linear arrangements, a combinatorial presentation for the cohomology ring was first given by Orlik and Solomon [12]. The fact that the complement of a linear arrangement is formal (that is, its cohomology ring is a model for the space; see Section 4.1) is originally due to Brieskorn [5]. Here, we state these results for graphic arrangements.

Theorem 2.2.1 [13, Theorems 3.126 and 5.89] *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be an ordered graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C})$. Then $X_{\mathcal{A}}$ is formal and $H^*(X_{\mathcal{A}}, \mathbb{Q})$ is isomorphic to the exterior algebra on the $\mathbb{Q}$–vector space spanned by*

$$\{g_e \mid e \in \mathcal{E}\}$$

modulo the ideal generated by

$$\sum\nolimits_j (-1)^j g_{e_1} \cdots \hat{g}_{e_j} \cdots g_{e_k} \quad \text{whenever } \{e_1 < \cdots < e_k\} \text{ is a cycle.}$$

2.3 Toric arrangements

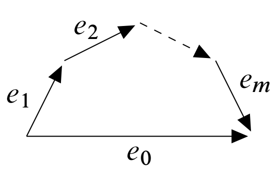
De Concini and Procesi studied the cohomology of the complement of a toric arrangement. If $\mathcal{A}$ is a unimodular toric arrangement (that is, all multiple intersections of subtori in $\mathcal{A}$ are connected), they show that the complement $X_{\mathcal{A}}$ is formal and give a presentation for the cohomology ring. Here, we state the result for graphic arrangements (which are always unimodular).

Theorem 2.3.1 [6, Theorem 5.2] *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be an ordered graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C}^\times)$. Then $X_{\mathcal{A}}$ is formal and $H^*(X_{\mathcal{A}}, \mathbb{Q})$ is isomorphic to the exterior algebra on the $\mathbb{Q}$–vector space spanned by*

$$\{x_v, g_e \mid v \in \mathcal{V}, e \in \mathcal{E}\}$$

modulo the ideal generated by the following:

(i) (a) *Whenever $e_0, e_1, \ldots, e_m$ is a cycle such that $t(e_0) = t(e_1)$, $h(e_0) = h(e_m)$, and $h(e_i) = t(e_{i+1})$ for $i = 1, \ldots, m-1$ (as pictured below),*



we have

$$g_{e_1} g_{e_2} \cdots g_{e_m} - \sum (-1)^{|I|+m+s_I} g_{e_{i_1}} \cdots g_{e_{i_k}} \psi_{e_{j_1}} \cdots \psi_{e_{j_{m-k-1}}} g_{e_0},$$

where the sum is taken over all $I = \{i_1 < \cdots < i_k\} \subsetneq \{1, \ldots, m\}$ *with complement* $\{j_1 < \cdots < j_{m-k}\}$, $\psi_{e_\ell} = x_{h(e_\ell)} - x_{t(e_\ell)}$, *and* s_I *is the parity of the permutation* $(i_1, \ldots, i_k, j_1, \ldots, j_{m-k})$.

(b) *If we again have a cycle, but have some arrows reversed, relabel the arrows so that* $e_1 < \cdots < e_s < e_0$, *then take the relation from* (i-a) *and replace each* ψ_{e_i} *with* $-\psi_{e_i}$ *and each* g_{e_i} *with* $-g_{e_i} - \psi_{e_i}$ *whenever* e_i *points in the opposite direction of* e_0.

(ii) $(x_{h(e)} - x_{t(e)})g_e$ *for* $e \in \mathcal{E}$.

The presentation encodes both the combinatorics of the arrangement and the geometry of the ambient space. The generators x_v come from the cohomology of $\mathbb{C}^\times$, while the generators g_e are similar to that of the Orlik–Solomon algebra for its rational counterpart. However, the toric analogue of the Orlik–Solomon relation is much more complicated.

2.4 Abelian arrangements

The elliptic analogue has a very different flavor, since the complement to an arrangement is not formal. If $\mathcal{A}$ is a unimodular abelian arrangement (that is, all multiple intersections of subvarieties in $\mathcal{A}$ are connected), the first author gave a presentation for a model for $X_{\mathcal{A}}$ [2, Theorem 4.1]. The presentation for graphic abelian arrangements is also a special case of one given by Dupont and Bloch [7], which we state in the next subsection.

Theorem 2.4.1 *Let* E *be a complex elliptic curve. Let* $\Gamma = (\mathcal{V}, \mathcal{E})$ *be an ordered graph, and let* $\mathcal{A} = \mathcal{A}(\Gamma, E)$. *Define the differential graded algebra* $A(\mathcal{A})$ *as the exterior algebra on the* $\mathbb{Q}$*–vector space spanned by*

$$\{x_v, y_v, g_e \mid v \in \mathcal{V}, e \in \mathcal{E}\}$$

modulo the ideal generated by the following relations:

(i) $\sum_j (-1)^j g_{e_1} \cdots \widehat{g}_{e_j} \cdots g_{e_k}$ *whenever* $\{e_1 < \cdots < e_k\}$ *is a cycle and*

(ii) $(x_{h(e)} - x_{t(e)})g_e$ *and* $(y_{h(e)} - y_{t(e)})g_e$ *for each* $e \in \mathcal{E}$.

The differential is defined by putting $dx_v = dy_v = 0$ *and*

$$dg_e = (x_{h(e)} - x_{t(e)})(y_{h(e)} - y_{t(e)}).$$

The DGA $(A(\mathcal{A}), d)$ *is a model for* $X_{\mathcal{A}}$.

In a similar way to toric arrangements, this algebra encodes both the combinatorics of the arrangement (with the Orlik–Solomon relation) and the geometry of the ambient space. The generators x_v, y_v come from the cohomology of E, while the g_e are from the Orlik–Solomon algebra of its rational counterpart.

2.5 Higher-genus curves

By the work of Dupont and Bloch, we have the following presentation for graphic arrangements in the case that C is a complex projective curve of positive genus [7], which we state here.

Theorem 2.5.1 *Let C be a complex projective curve with genus $g \geq 1$. Define the differential graded algebra $A(\mathcal{A})$ as the exterior algebra on the $\mathbb{Q}$–vector space spanned by*

$$\{x_v^i, y_v^i, g_e \mid v \in \mathcal{V}, e \in \mathcal{E}, i = 1, \dots, g\}$$

modulo the ideal generated by the following relations:

(i) $\sum_j (-1)^j g_{e_1} \cdots \widehat{g}_{e_j} \cdots g_{e_k}$ *whenever* $\{e_1 < \cdots < e_k\}$ *is a cycle,*

(ii) $(x^i_{h(e)} - x^i_{t(e)})g_e$ *and* $(y^i_{h(e)} - y^i_{t(e)})g_e$ *for each* $e \in \mathcal{E}$*,*

(iii) (a) $x_v^i y_v^j$, $x_v^i x_v^j$ *and* $y_v^i y_v^j$ *for* $i \neq j$*, and*

(b) $x_v^i y_v^i - x_v^j y_v^j$.

The differential is defined by putting $dx_v^i = dy_v^i = 0$ *and*

$$dg_e = x^1_{h(e)} y^1_{h(e)} + x^1_{t(e)} y^1_{t(e)} - \sum_{i=1}^{g} (x^i_{h(e)} y^i_{t(e)} + x^i_{t(e)} y^i_{h(e)}).$$

The DGA $(A(\mathcal{A}), d)$ is a model for $X_{\mathcal{A}}$.

Just as before, this algebra encodes both the combinatorics of the arrangement and the geometry of the ambient space. The generators x_v^i, y_v^i come from the cohomology of $C^{\mathcal{V}}$, and we write these generators and relations here explicitly since we will use this presentation to show that the algebra is Koszul in Section 3.5. A more elegant way of writing the differential is to say that the generator g_e maps to $[\Delta_e] \in H^2(C^{\mathcal{V}})$, where Δ_e is the diagonal corresponding to the coordinates indexed by $h(e)$ and $t(e)$ in $C^{\mathcal{V}}$.

3 Koszulity

In this section, we will show that for chordal arrangements, the algebras presented in Theorems 2.2.1, 2.3.1, 2.4.1, and 2.5.1 are Koszul. The cohomology of the complement of a chordal linear arrangement was first shown to be Koszul by Shelton and Yuzvinsky [18]. In Section 3.2, we outline the proof presented by Yuzvinsky in [20]. The analogous results for toric and abelian arrangements, as well as for higher-genus curves, are new and presented in Sections 3.3, 3.4, and 3.5, respectively.

3.1 Chordal arrangements

Let C be $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve, and let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a simple graph. If Γ is chordal (that is, every cycle with more than three vertices has a chord), then the graphic arrangement $\mathcal{A}(\Gamma, C)$ is said to be *chordal*.

A *perfect elimination ordering* is an order on the vertices so that for all $v \in \mathcal{V}$, v is a simplicial vertex (a vertex whose neighbors form a clique) in the graph

$$\Gamma_v := \Gamma - \{v' \in \mathcal{V} \mid v' > v\}.$$

Such an ordering exists if and only if Γ is chordal [9, page 851]. From now on, we will use such an order when discussing chordal graphs.

We say a set $S = \{e_1 < \cdots < e_k\}$ is a *broken circuit* if there is some edge e with $e < e_1$ such that $S \cup \{e\}$ is a cycle. A set $S \subseteq \mathcal{E}$ is *nbc* (nonbroken circuit) if no subset of it is a broken circuit. Let $F \subseteq \mathcal{E}$ be a flat of the matroid of Γ, and consider the subgraph $\Gamma[F]$ of Γ, which has edges F and vertices adjacent to edges in F. We say that an nbc set S is associated to F if $S \subseteq F$ and S spans $\Gamma[F]$.

Remark 3.1.1 In the case of linear, toric, or abelian arrangements, the essential property that we need for our results is that the arrangement is unimodular (for Theorems 2.3.1 and 2.4.1) and supersolvable (for Theorems 3.3.3 and 3.4.3). We could state all of our results in the language of unimodular and supersolvable arrangements; however, this isn't any more general than the language of chordal graphs. This is because Ziegler showed that a matroid is unimodular and supersolvable if and only if it is chordal graphic (Proposition 2.6 and Theorem 2.7 of [21]). In fact, since the edge set of $\Gamma - v$ is a modular hyperplane when v is a simplicial vertex [21, Proposition 4.4], the maximal chain of modular flats in the matroid corresponds exactly to our ordering on the vertices.

3.2 Linear arrangements

Yuzvinsky proved that the Orlik–Solomon ideal has a quadratic Gröbner basis when $\mathcal{A}$ is supersolvable (eg chordal), which implies that $H^*(X_{\mathcal{A}})$ is Koszul [20, Corollary 6.21]. We outline his technique as we will use similar techniques in the toric and abelian cases.

For ease of notation, whenever $C = \{e_1 < \cdots < e_k\}$ we will use $g_C := g_{e_1} \cdots g_{e_k}$ and $\partial g_C := \sum_j (-1)^j g_{e_1} \cdots \widehat{g}_{e_j} \cdots g_{e_k}$. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph with a perfect elimination ordering on the vertices (and edges).

First, Yuzvinsky showed that the set $G = \{\partial g_C \mid C \text{ is a circuit}\}$ is a Gröbner basis for the ideal $I = \langle G \rangle$ in the exterior algebra $\Lambda(g_e \mid e \in \mathcal{E})$, with the degree-lexicographic order such that $g_e < g_{e'}$ whenever $e < e'$. The leading (or initial) term of ∂g_C is

$\mathrm{In}(\partial g_C) = g_{C'}$ where $C' \subseteq C$ is the broken circuit associated to C. Recall that a subset G of an ideal I is a Gröbner basis if $\mathrm{In}(I) = \langle \mathrm{In}(G) \rangle$. To prove that this is a Gröbner basis, Yuzvinsky used the fact that the set of monomials not in $\mathrm{In}(I)$ form a basis for $H^*(X_{\mathcal{A}}) = \Lambda(g_e)/I$. The set of monomials not in $\mathrm{In}(I)$ is the basis $\{g_C \mid C \text{ is nbc}\}$.

Moreover, since Γ is chordal, this Gröbner basis can be reduced to a quadratic Gröbner basis. This is because we have the following property (which follows immediately from Proposition 6.19 of [20]): $S \subseteq \mathcal{E}$ is an nbc set if and only if for all distinct $e, e' \in S$ we have $h(e) \neq h(e')$. A circuit C is not nbc, hence there exist distinct edges $e, e' \in C$ such that $h(e) = h(e')$. But then $\{e, e'\}$ contains (and hence is) a broken circuit, and so it is contained in some circuit T with $|T| = 3$. Thus $\mathrm{In}(\partial g_T) = g_e g_{e'}$ divides $\mathrm{In}(\partial g_C)$, and we can reduce our Gröbner basis to a quadratic one.

3.3 Toric arrangements

Since the complement to a chordal toric arrangement is formal (as in the linear case), we want to show that its cohomology ring is Koszul. Our argument will be similar to (but slightly more complicated than) the linear case. We will provide a $\mathbb{Q}$–basis for the cohomology ring, use it to show that our generating set of the ideal is a Gröbner basis, and then reduce the Gröbner basis to a quadratic one.

Lemma 3.3.1 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph. Let F be a flat of the arrangement $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C}^\times)$, and let S be a nonbroken circuit associated to F. Define I_F to be the ideal generated by*

$$\{x_{h(e)} - x_{t(e)} \mid e \in F\}$$

in $\Lambda(x_v \mid v \in \mathcal{V}) \cong H^*((\mathbb{C}^\times)^{\mathcal{V}})$, and let $H_F = \bigcap_{e \in F} H_e \subseteq (\mathbb{C}^\times)^{\mathcal{V}}$.

(1) *With the degree-lexicographic order and $x_v < x_{v'}$ whenever $v < v'$, the set*

$$G_S := \{x_{h(e)} - x_{t(e)} \mid e \in S\}$$

is a Gröbner basis for I_F.

(2) *The set $\{x_{i_1} \cdots x_{i_r} \mid h(e) \notin \{i_1, \ldots, i_r\} \text{ for each } e \in S\}$ is a basis for*

$$H^*(H_F) \cong \Lambda(x_v \mid v \in \mathcal{V})/I_F,$$

and this basis does not depend on the choice of nbc *set S.*

Proof For linear relations, finding a Gröbner basis is equivalent to Gaussian elimination, and so consider the matrix M_F whose rows are indexed by edges $e \in F$, whose columns are indexed by the vertices in decreasing order, and whose entries are zero except $(M_F)_{e,h(e)} = 1$ and $(M_F)_{e,t(e)} = -1$ (so that row e corresponds to the element $x_{h(e)} - x_{t(e)}$). Note that since $|S| = \mathrm{rk}(S) = \mathrm{rk}(M_F)$, we may use row operations so that the rows corresponding to elements of S remain unchanged while all other rows are zero. Moreover, since $|\{h(e) \mid H_e \in S\}| = |S|$, the matrix is in row echelon form. Thus, G_S is a Gröbner basis for I_F.

For part (2), by Gröbner basis theory, the set of monomials not in $\mathrm{In}(I_F)$ form a basis for $H^*(H_F)$. Since the ideal $\mathrm{In}(I_F)$ is generated by $\mathrm{In}(G_S) = \{x_{h(e)} | e \in S\}$, the monomials not in $\mathrm{In}(I_F)$ are precisely those stated. Since S is an nbc set associated to F, it spans the subgraph $\Gamma[F]$. Thus $\{h(e) \mid e \in F\} = \{h(e) \mid e \in S\}$ and the basis given does not depend on S. □

For ease of notation, we will use

$$x_A g_C := x_{a_1} \cdots x_{a_r} g_{c_1} \cdots g_{c_k},$$

where $A = \{a_1 < \cdots < a_r\}$ and $C = \{c_1 < \cdots < c_k\}$. We will also denote relations (i-a) and (i-b) from Theorem 2.3.1 by r_C for a cycle C.

Lemma 3.3.2 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C}^\times)$. Define P to be the set of all monomials $x_A g_C$ such that C is a nonbroken circuit and $h(e) \notin A$ for all $e \in C$. Then P is a basis for $H^*(X_{\mathcal{A}})$.*

Proof There is a decomposition into the flats of $\mathcal{A}$ [6, Remark 4.3(2)] (see also [2, Lemma 3.1]), which is given as follows: For a flat F, let $H_F = \bigcap_{e \in F} H_e \subseteq (\mathbb{C}^\times)^{\mathcal{V}}$, and let V_F be the vector space spanned by g_C for all nbc sets C associated to F. Then

$$H^*(X_{\mathcal{A}}) = \bigoplus_F H^*(H_F) \otimes V_F.$$

Denote $H^*(H_F) \otimes V_F$ by A_F. To show that P is a basis for $H^*(X_{\mathcal{A}})$, it suffices to show that

$$P \cap A_F = \{x_A g_C \mid h(c) \notin A \text{ for } c \in C, \text{ where } C \text{ is an nbc set associated to } F\}$$

is a basis for A_F. But this follows from Lemma 3.3.1. □

Theorem 3.3.3 *Let $\mathcal{A}$ be a chordal toric arrangement. Then $H^*(X_{\mathcal{A}})$ is Koszul.*

Proof Fix a degree-lexicographic order on $H^*(X_{\mathcal{A}})$ that is induced by our order on $\mathcal{V}$. That is, $g_e < g_{e'}$ if $e < e'$, and $x_{h(e)} < g_e < x_{h(e)+1}$. We will show that

$$G = \{(x_{h(e)} - x_{t(e)})g_e, r_S \mid e \in \mathcal{E} \text{ and } S \text{ is a circuit}\}$$

is a Gröbner basis with this order which can be reduced to a quadratic Gröbner basis.

We have
$$\mathrm{In}(G) = \{x_{h(e)} g_e,\ g_C \mid e \in \mathcal{E} \text{ and } C \text{ is a broken circuit}\}.$$

Then P is the set of monomials that are not in $\langle \mathrm{In}(G)\rangle$. Since $\langle \mathrm{In}(G)\rangle \subseteq \mathrm{In}(I)$, the monomials that are not in $\mathrm{In}(I)$ are contained in P. Since the set of monomials not in $\mathrm{In}(I)$ is a basis for $H^*(X_{\mathcal{A}})$ contained in the basis P, and $H^*(X_{\mathcal{A}})$ is finite dimensional, we must have equality throughout. That means that the monomials in $\langle \mathrm{In}(G)\rangle$ are exactly the monomials in $\mathrm{In}(I)$. Since these ideals are generated by monomials, they must be equal. Note that the relations of type (ii) are already quadratic. In a similar way as in the linear case, we can reduce our relations r_C to quadratic ones as well. □

3.4 Abelian arrangements

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let E be a complex elliptic curve. For the chordal abelian arrangement $\mathcal{A} = \mathcal{A}(\Gamma, E)$, consider the algebra $A(\mathcal{A})$ from Theorem 2.4.1 (ignoring the differential). In this subsection, we will prove that $A(\mathcal{A})$ is Koszul. The proof is very similar to (but slightly more complicated than) the toric case.

Lemma 3.4.1 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph. Let $F \subseteq \mathcal{E}$ be a flat of the arrangement $\mathcal{A} = \mathcal{A}(\Gamma, E)$, and let S be a nonbroken circuit associated to F. Define I_F to be the ideal generated by*
$$\{x_{h(e)} - x_{t(e)},\ y_{h(e)} - y_{t(e)} \mid e \in F\}$$
in $\Lambda(x_v, y_v \mid v \in \mathcal{V}) \cong H^*(E^{\mathcal{V}})$, and let $H_F = \bigcap_{e\in F} H_e \subseteq E^{\mathcal{V}}$.

(1) *With the degree-lexicographic order and $x_v < y_v < x_{v'} < y_{v'}$ whenever $v < v'$, the set $G_S := \{x_{h(e)} - x_{t(e)},\ y_{h(e)} - y_{t(e)} \mid e \in S\}$ is a Gröbner basis for I_F.*

(2) *The set $\{x_{i_1} \cdots x_{i_r} y_{j_1} \cdots y_{j_t} \mid h(e) \notin \{i_1, \dots, i_r, j_1, \dots, j_t\} \text{ for each } e \in S\}$ is a basis for*
$$H^*(H_F) \cong \Lambda(x_v, y_v \mid v \in \mathcal{V})/I_F,$$
and this basis does not depend on the choice of nbc *set S.*

Proof Consider the matrix M_F from the proof of Lemma 3.3.1. Build a 2×2 block matrix, where the upper left and lower right blocks are copies of M_F and the other blocks are zero. In the upper half of the matrix, row e corresponds to $x_{h(e)} - x_{t(e)}$, and in the lower half of the matrix, row e corresponds to $y_{h(e)} - y_{t(e)}$. By a similar argument as before, we can eliminate rows that don't correspond to elements of S and we're left with a matrix in row echelon form. Thus, we have a Gröbner basis.

The proof of the second statement mimics the proof in the toric case, with
$$\mathrm{In}(G_S) = \{x_{h(e)}, y_{h(e)} \mid e \in S\}. \qquad □$$

Lemma 3.4.2 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, E)$. Define P to be the set of all monomials $x_A y_B g_C$ such that C is a nonbroken circuit and $h(e) \notin (A \cup B)$ for all $e \in C$. Then P is a basis for $A(\mathcal{A})$.*

Proof By Lemma 3.1 in [2], there is a decomposition into the flats of $\mathcal{A}$, given by the following: For a flat F, let $H_F = \bigcap_{e \in F} H_e \subseteq E^{\mathcal{V}}$, and let V_F be the vector space spanned by g_C for all nbc sets C associated to F. Then

$$A(\mathcal{A}) = \bigoplus_F H^*(H_F) \otimes V_F.$$

Denote $H^*(H_F) \otimes V_F$ by A_F. To show that P is a basis for $A(\mathcal{A})$, it suffices to show that

$$P \cap A_F = \{x_A y_B g_C \mid h(c) \notin (A \cup B) \text{ for } c \in C, \text{ where } C \text{ is an nbc set associated to } F\}$$

is a basis for A_F. But this follows from Lemma 3.4.1. □

Theorem 3.4.3 *Let $\mathcal{A}$ be a chordal abelian arrangement. Then $A(\mathcal{A})$ is Koszul.*

Proof Fix a degree-lexicographic order on $A(\mathcal{A})$ that is induced by our order on $\mathcal{V}$. That is, $g_e < g_{e'}$ if $e < e'$, and $x_{h(e)} < y_{h(e)} < g_e < x_{h(e)+1} < y_{h(e)+1}$. We claim that

$$G = \{(x_{h(e)} - x_{t(e)})g_e,\ (y_{h(e)} - y_{t(e)})g_e,\ \partial g_S \mid e \in \mathcal{E} \text{ and } S \text{ is a circuit}\}$$

is a Gröbner basis with this order. Here,

$$\mathrm{In}(G) = \{x_{h(e)} g_e,\ y_{h(e)} g_e,\ g_C \mid e \in \mathcal{E} \text{ and } C \text{ is a broken circuit}\},$$

and P from Lemma 3.4.2 is the set of monomials not in $\langle \mathrm{In}(G) \rangle$. By an argument similar to that in the toric case, we can conclude that G is a Gröbner basis. Moreover, using the fact that we have a chordal graph, we can again reduce this (in the same way) to a quadratic Gröbner basis, thus proving Koszulity. □

3.5 Higher-genus curves

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let C be a complex projective curve of genus $g > 1$. For the chordal arrangement $\mathcal{A} = \mathcal{A}(\Gamma, C)$, consider the algebra $A(\mathcal{A})$ from Theorem 2.5.1 (ignoring the differential). In this subsection, we will prove that $A(\mathcal{A})$ is Koszul. The proof is very similar to that of the abelian case in Section 3.4.

Lemma 3.5.1 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph. Let $F \subseteq \mathcal{E}$ be a flat of the arrangement $\mathcal{A} = \mathcal{A}(\Gamma, C)$, and let S be a nonbroken circuit associated to F. Denote $H_F = \bigcap_{e \in F} H_e$ in $C^{\mathcal{V}}$. Then $H^*(H_F) \cong \Lambda(x_v^i, y_v^i \mid v \in \mathcal{V}, i = 1, \dots, g)/I_F$ where I_F is the ideal generated by the relations*

(i) $x^i_{h(e)} - x^i_{t(e)}$ and $y^i_{h(e)} - y^i_{t(e)}$ for $e \in F$,

(ii) $x_v^i x_v^j$, $y_v^i y_v^j$ and $x_v^i y_v^j$ for $i \neq j$, and

(iii) $x_v^i y_v^i - x_v^j y_v^j$.

This algebra has basis

$$\big\{x^1_{A_1} \cdots x^g_{A_g} y^1_{B_1} \cdots y^g_{B_g} \mid A_i \cap B_i = \varnothing \text{ for } i > 1;\ \{h(e) \mid e \in S\} \cap (A_i \cup B_i) = \varnothing \text{ for } i = 1, \dots, g\big\},$$

and this basis does not depend on the choice of S.

Proof Consider the exterior algebra modulo the first relation, which we can write as the exterior algebra

$$\Lambda(x_v^i,\ y_v^i \mid v \notin \{h(e) \mid e \in S\},\ i = 1, \dots, g)$$

by a similar argument as in the proof of Lemma 3.4.1. Note that, as before,

$$\{h(e) \mid e \in S\} = \{h(e) \mid e \in F\},$$

and so this does not depend on the choice of S. Now consider relations (ii) and (iii) in this algebra. This is a Gröbner basis G with

$$\mathrm{In}(G) = \big\{x_v^i x_v^j,\ y_v^i y_v^j,\ x_v^i y_v^j\ (i \neq j),\ x_v^i y_v^i\ (i > 1),\ v \notin \{h(e) \mid e \in S\}\big\}.$$

The set of monomials in our proposed basis are exactly those not divisible by $\mathrm{In}(G)$ and are hence a basis. □

Lemma 3.5.2 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, C)$. Define P to be the set of all monomials $x^1_{A_1} \cdots x^g_{A_g} y^1_{A_1} \cdots y^g_{A_g} g_S$ such that S is a nonbroken circuit, $h(e) \notin (A_i \cup B_i)$ for all $e \in S$ and all i, and $A_i \cap B_i = \varnothing$ for $i > 1$. Then P is a basis for $A(\mathcal{A})$.*

Proof There is again a decomposition into the flats of $\mathcal{A}$, given by the following: For a flat F, let $H_F = \bigcap_{e \in F} H_e \subseteq C^{\mathcal{V}}$, and let V_F be the vector space spanned by g_S for all nbc sets S associated to F. Then

$$A(\mathcal{A}) = \bigoplus_F H^*(H_F) \otimes V_F.$$

Denote $H^*(H_F) \otimes V_F$ by A_F. To show that P is a basis for $A(\mathcal{A})$, it suffices to show that $P \cap A_F$ is a basis of A_F. But this follows from Lemma 3.5.1. □

Theorem 3.5.3 *Let C be a complex projective curve of genus $g > 1$, and let $\mathcal{A}$ be a chordal arrangement in $C^{\mathcal{V}}$. Then $A(\mathcal{A})$ is Koszul.*

Proof Fix a degree-lexicographic order on $A(\mathcal{A})$ that is induced by our order on $\mathcal{V}$. We claim that

$$G = \{(x^i_{h(e)} - x^i_{t(e)})g_e, (y^i_{h(e)} - y^i_{t(e)})g_e, \partial g_S, R \mid e \in \mathcal{E}, \text{ and } S \text{ is a circuit}\}$$

is a Gröbner basis with this order, where R denotes the set of relations (iii-a) and (iii-b) in $A(\mathcal{A})$. Here,

$$\mathrm{In}(G) = \{x^1_v y^1_v,\ x^i_{h(e)} g_e,\ y^i_{h(e)} g_e,\ g_B \mid B \text{ is a broken circuit}\},$$

and P from Lemma 3.5.2 is the set of monomials not in $\langle \mathrm{In}(G) \rangle$. By an argument similar to the previous cases, we can conclude that G is a Gröbner basis. Moreover, using the fact that we have a chordal graph, we can again reduce this (in the same way) to a quadratic Gröbner basis, thus proving Koszulity. □

4 Rational homotopy theory and quadratic duality

In this section, we collect definitions and results from rational homotopy theory, quadratic duality, and the relationship between these two subjects. This section is meant to provide a background on the necessary theory; the reader can skip ahead and refer back as needed. Throughout this section, all DGAs will be assumed to be connective (that is, their cohomology has a nonnegative grading). Except in Section 4.3, all DGAs will be commutative (that is, graded-commutative).

4.1 Rational homotopy theory

The fundamental problem of rational homotopy theory is to understand the topology of the $\mathbb{Q}$–completion $X \to \mathbb{Q}_\infty(X)$ of a topological space X as defined in [4, Chapter I.4]. When X is a simply connected CW-complex, we have $\pi_i \mathbb{Q}_\infty(X) \cong (\pi_i X) \otimes \mathbb{Q}$ and $H^*(X,\mathbb{Q}) \cong H^*(\mathbb{Q}_\infty(X),\mathbb{Q})$, but in general the relationship between X and $\mathbb{Q}_\infty(X)$ is more complicated. Still, the homotopy type of $\mathbb{Q}_\infty(X)$ is substantially simpler than that of X as the results of [17; 19; 3] show that the rational homotopy theory of connected $\mathbb{Q}$–finite spaces is determined by the quasi-isomorphism type of a particular DGA $(A_{PL}(X), d)$ with $H^*(A_{PL}(X), d) \cong H^*(X,\mathbb{Q})$. A DGA $(A(X), d)$ is a *model* for X if it is quasi-isomorphic to $(A_{PL}(X), d)$. The space X is *formal* if $(H^*(X,\mathbb{Q}), 0)$ is a model for X.

Let (B, d) be a DGA and for $n \geq 0$ let $(B(n), d)$ be the DG-subalgebra of B generated by B^i for $i \leq n$. Define $(B(-1), d)$ to be the DG-subalgebra generated by $1 \in B^0$. For $n \geq 0$, there is an increasing filtration $(B(n,q), d)$ on $(B(n), d)$ defined inductively as follows: Let $(B(n,0), d) = (B(n-1), d)$ and let $B(n, q+1)$ be the DG-subalgebra of B generated by $B(n-1)$ and $\{b \in B^n | db \in B(n,q)\}$. A commutative DGA (B, d) is *minimal* [3, Section 7.1] if it is connected, B is a free commutative graded algebra, and $B(n) = \bigcup_{q \geq 0} B(n,q)$ for all n.

Sullivan [19] showed that any homologically connected DGA has a minimal model that is unique up to unnatural isomorphism. Write $(M(X), d)$ for the minimal model of $(A_{PL}(X), d)$, which is called the minimal model of X. Every minimal DGA (M, d) has a canonical augmentation determined by the augmentation ideal M^+. This lets us define the homotopy groups $\pi^q M = H^q(M^+/(M^+ \cdot M^+), d)$ of (M, d). The following theorem relates the homotopy groups of $(M(X), d)$ to those of $\mathbb{Q}_\infty(X)$.

Theorem 4.1.1 [3, Theorem 12.8] *There are natural bijections*

$$\pi_q \mathbb{Q}_\infty(X) \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^q M(X), \mathbb{Q}).$$

They are group isomorphisms for $q \geq 2$.

In the next subsection we develop the technology to get more refined information about $\pi_1 X$ from the minimal model. This will be important because the spaces we are interested in are rational $K(\pi, 1)$ spaces.

4.2 Complete Lie algebras and nilpotent completion of groups

When X is not simply connected, we don't necessarily have the isomorphism

$$\pi_1 \mathbb{Q}_\infty(X) \cong (\pi_1 X) \otimes \mathbb{Q}.$$

To even make sense of the right hand side when $\pi_1 X$ isn't abelian, we need to review the *Malcev completion* (or $\mathbb{Q}$–nilpotent completion) $\widehat{G} \otimes \mathbb{Q}$ of a finitely presented group G. Then we will survey some results that show that $\widehat{G} \otimes \mathbb{Q}$ is entirely determined by a complete Lie algebra $L(G)$, which we call the *Malcev Lie algebra* of G.

Recall that the lower central series of G is defined by setting $\Gamma_1 G = G$ and then $\Gamma_{q+1} G = [G, \Gamma_q G]$. As in [10, Section 13.2], we will use a recursive procedure to define $N_i G \otimes \mathbb{Q}$ for each nilpotent group $N_i G = G / \Gamma_i G$ and then define

$$\widehat{G} \otimes \mathbb{Q} := \varprojlim (N_i G) \otimes \mathbb{Q}.$$

First we see that $N_1G = 0$ so we can define $(N_1G)\otimes\mathbb{Q} = 0$. Now assume that we have defined $(N_{i-1}G)\otimes\mathbb{Q}$. The N_iG fit into a series of exact sequences

$$0 \to \Gamma_{i-1}/\Gamma_i G \to N_iG \to N_{i-1}G \to 0,$$

which determine classes $\sigma_i \in H^2(N_{i-1}G, \Gamma_{i-1}/\Gamma_i G)$ (where $\Gamma_{i-1}/\Gamma_i G$ is given a trivial $N_{i-1}G$–module structure). It can be shown that

$$H^2(N_{i-1}, \Gamma_{i-1}/\Gamma_i G)\otimes\mathbb{Q} \cong H^2\big((N_{i-1}G)\otimes\mathbb{Q}, (\Gamma_{i-1}/\Gamma_i G)\otimes\mathbb{Q}\big),$$

so the class $\sigma_i\otimes 1$ determines an extension of $(N_{i-1}G)\otimes\mathbb{Q}$ by $(\Gamma_{i-1}/\Gamma_i G)\otimes\mathbb{Q}$. We then define this extension to be $N_iG\otimes\mathbb{Q}$.

For a minimal DGA (M,d) the analogue of the lower central series of $\pi_1 X$ is an increasing filtration on $\pi^1 M$ defined by $\Gamma^i\pi^1 = \mathrm{Im}(\pi^1 M(1,i-1) \to \pi^1 M)$.

Theorem 4.2.1 [3, Theorem 12.8] *Let $(M(X),d)$ be the minimal model of X. Then*

$$\mathrm{Hom}_{\mathbb{Q}}\big(\pi^1 M(X)/\Gamma^i\pi^1 M(X), \mathbb{Q}\big) \cong (N_i\pi_1 X)\otimes\mathbb{Q}.$$

There is a second construction of $\widehat{G}\otimes\mathbb{Q}$ that proceeds through the theory of complete Hopf algebras [17, Appendix A]. In particular, $\widehat{G}\otimes\mathbb{Q}$ is isomorphic to the group of group-like elements $\{x \in \overline{\mathbb{Q}[G]} \mid \Delta x = x\widehat{\otimes} x\}$ in the completion of the group algebra $\mathbb{Q}[G]$ with respect to the augmentation ideal. We can also define a complete Lie algebra by taking primitive elements

$$L(G) := \{x \in \overline{\mathbb{Q}[G]} \mid \Delta x = 1\widehat{\otimes} x + x\,\widehat{\otimes} 1\}$$

in $\overline{\mathbb{Q}[G]}$. Recall that there is a lower central series for Lie algebras defined by $\Gamma_1 L = L$ and $\Gamma_{i+1}L = [L, \Gamma_i L]$.

The following proposition tells us that understanding the Malcev Lie algebra $L(G)$ is enough to understand the quotients in the lower central series of G and also the completion $\overline{\mathbb{Q}[G]}$ with respect to the augmentation ideal.

Proposition 4.2.2 [1, Section 4]

(1) $(\Gamma_i/\Gamma_{i+1}G)\otimes\mathbb{Q} \cong \Gamma_i/\Gamma_{i+1}(\widehat{G}\otimes\mathbb{Q}) \cong \Gamma_i/\Gamma_{i+1}L(G)$.

(2) $\overline{U(L(G))} \cong \overline{\mathbb{Q}[G]}$ *as complete Hopf algebras.*

In fact, even more is true. The power series defining $\log(x)$ and e^x converge in any complete Hopf algebra and give an isomorphism between the Lie algebra of primitive elements and the group of group-like elements [17, Appendix A.2]. Thus, $L(G)$ completely determines $\widehat{G}\otimes\mathbb{Q}$ and vice versa.

4.3 Nonhomogeneous quadratic duality

In this subsection, we describe and outline definitions and results on the nonhomogeneous quadratic duality [16; 15; 1] between quadratic differential graded algebras and weak quadratic-linear algebras. This will give a tractable method for computing the minimal model $(M(X), d)$ and the Malcev Lie algebra $L(\pi_1(X))$ when we have a quadratic model $(A(X), d)$ for a space X.

First we need to establish some conventions on graded and filtered algebras. All graded and filtered algebras will be locally finite dimensional. All gradings will be concentrated in nonnegative degree and will be notated with superscripts. All $\mathbb{N}$–filtrations will be increasing, exhaustive, and indexed by subscripts. The tensor algebra on a k–vector space V will be denoted by $T(V)$. It is graded by putting V in degree 1, equipped with the increasing filtration induced by the grading, and augmented by the map $\epsilon\colon T(V) \to k$ that sends V to 0.

A *WQLA* (weak quadratic-linear algebra) is an augmented algebra $\epsilon\colon B \to k$, together with a choice of k–subspace W, satisfying the following:

(i) $1 \in W$.

(ii) B is generated multiplicatively by W.

(iii) Let $V = \ker(\epsilon|_W)$ and $J = \ker(T(V) \to B)$. The ideal J is generated by J_2.

A *QLA* (quadratic-linear algebra) is an augmented algebra $\epsilon\colon B \to k$ equipped with an exhaustive $\mathbb{N}$–filtration such that $\operatorname{gr} B$ is quadratic. In particular this implies that the choice $W = B_1$ makes B into a WQLA. A morphism of WQLAs $f\colon (B, \epsilon, W) \to (B', \epsilon', W')$ is a homomorphism of augmented algebras such that $f(W) \subset W'$. Morphisms of QLAs coincide with homomorphisms of augmented filtered algebras.

A WQLA B has an associated quadratic algebra $B^{(0)}$ which is defined by generators $V \cong W/k\cdot 1$ subject to the relations $I = J_2/J_1$. For QLAs, $\operatorname{gr} B \cong B^{(0)}$. We say that a WQLA B is *Koszul* if the underlying quadratic algebra $B^{(0)}$ is Koszul. Every Koszul WQLA is in fact a QLA [15, Section 3.3]. Denote the category of weak quadratic-linear algebras by `WQLA`, and denote its subcategory consisting of Koszul quadratic-linear algebras by `KLA`.

A DGA A has an underlying graded algebra given by forgetting the differential. We say that a DGA is *quadratic* (respectively *Koszul*) if its underlying algebra is quadratic (respectively Koszul). Let `QDGA` be the category of quadratic differential graded algebras, and denote its subcategory consisting of Koszul DGAs by `KDGA`.

There is a fully faithful contravariant functor $D_{\mathrm{WQLA}}\colon \mathtt{WQLA} \to \mathtt{QDGA}$ that is defined on objects as follows: Let (B, ϵ, W) be a WQLA. As a graded algebra $D(B, \epsilon, W) = B^{(0)!}$ is the quadratic dual to the quadratic algebra associated to B. Note that $J_2 \cap T_1(V) = 0$ and $I = J_2/J_1$. Thus we can represent J_2 as the graph of a linear map

$$(-h, -\phi)\colon I \to T_1(V) = k \oplus V.$$

It is easy to see that $J_2 \subset \ker(\epsilon)$ implies that $h = 0$. The map

$$d_1 = \phi^*\colon D(B, \epsilon, W)^1 \cong V^* \to I^* \cong D(B, \epsilon, W)^2$$

can be extended to a differential on $D(B, \epsilon, W)$.

On the other hand, there is a contravariant functor $D_{\mathrm{QDGA}}\colon \mathtt{QDGA} \to \mathtt{WQLA}$ defined on objects as follows: Let (A, d) be a quadratic DGA and let $V = A^1$. We can write $A \cong T(V)/J$. The map $d|_{A^1}\colon V = A^1 \to A^2 = (V \otimes V)/J$ has a dual map $\phi\colon J^\perp \to V^*$, where V^* is the dual vector space to V and $J^\perp \subseteq V^* \otimes V^*$ is the annihilator of J. Then $D(A, d) = (T(V^*)/I, \overline{\epsilon}, W_A)$, where I is the ideal generated by $\{x - \phi(x) \mid x \in J^\perp\}$, $\overline{\epsilon}$ is the augmentation induced by $\epsilon\colon T(V^*) \to k$, and $W_A = k \oplus V^*$.

Proposition 4.3.1 [15, Section 2.5] *The functors D_{WQLA} and D_{QDGA} restrict to a contravariant equivalence of categories between* KLA *and* KDGA*.*

$$\begin{tikzcd} \mathtt{WQLA} \arrow[r, shift right] & \mathtt{QDGA} \arrow[l, shift right] \\ \mathtt{KLA} \arrow[u, hook] \arrow[r, leftrightarrow, "\sim"] & \mathtt{KDGA} \arrow[u, hook] \end{tikzcd}$$

If (A, d) is a commutative QDGA, then $S^2(V) \subseteq J$ and hence $J^\perp \subseteq \Lambda^2(V^*)$. This implies that there is a Lie algebra $L = L(A)$ such that

$$D(A, d) \cong (U(L), \epsilon, W_A).$$

We call this Lie algebra the *Lie algebra dual to A*. Note that in general, $W_A \neq k \oplus L$, so the induced filtration is not the order filtration.

Example 4.3.2 We start with a very simple example of the duality we consider in Section 5. A model for the punctured elliptic curve is given by

$$A = \Lambda(x, y, g)/(xg, yg)$$

with differential defined by $dx = dy = 0$ and $dg = xy$. The QL-algebra dual to A is

$$B = T(a, b, c)/(ab - ba - c).$$

As augmented algebras $B \cong U(L)$, where L is the free Lie algebra on the generators a and b, but the filtration on B does not coincide with the order filtration on $U(L)$.

Let L be a finite-dimensional Lie algebra, and consider its universal enveloping algebra $U(L)$ equipped with the order filtration. It is an easy exercise to see that the QDGA dual to $U(L)$ is the graded algebra $\Lambda(L^*)$ equipped with differential dual to the Lie bracket. This is often called the standard (or Chevalley–Eilenberg) complex of L and is denoted by $(\Omega(L), d)$. We can extend this to the situation when the Lie algebra $L^\bullet$ is $\mathbb{N}$–graded with each graded piece finite dimensional by defining the standard complex of $L^\bullet$ to be the restricted dual subalgebra $\Omega(L^\bullet) := \bigoplus_{i,j}([\Lambda^i L]^j)^*$ of $\Lambda(L^*)$ with differential dual to the Lie bracket.

Given a minimal model $(M(X), d)$ for a space X, we can reconstruct the Lie algebra $L(\pi_1 X)$ as follows: The commutative DGAs $(M(1,i), d)$ are quadratic and hence dual to Lie algebras L_i. Moreover the inclusions $M(1,i) \to M(1,i+1)$ induce maps $L_{i+1} \to L_i$.

Theorem 4.3.3 [10, Theorem 13.2] *There are natural isomorphisms*

$$L_i \cong L(\pi_1 X)/\,\Gamma_i L(\pi_1 X) \quad \textit{and} \quad L(\pi_1 X) \cong \varprojlim L_i.$$

We can also recover $\overline{\mathbb{Q}[\pi_1 X]}$ from $(M(X), d)$. Let $(C^{\bullet,\bullet}(A,d), d_1, d_2)$ be the cobar bicomplex of (A,d) as defined in [1, Section 3] (also called the dual bar bicomplex in [15, Section 3]) and let $H_b^*(A,d)$ be the cohomology of its totalization.

Lemma 4.3.4 [1, Lemma 3.1] *Let (A,d) be a QDGA. Then $H_b^0(A,d)$ is naturally isomorphic to $D(A,d)$. The increasing columns filtration on the cobar complex induces the QLA structure on $D(A,d)$. The decreasing rows filtration on the cobar complex induces the filtration by powers of the augmentation ideal of $D(A,d)$.*

Proposition 4.3.5 [1, Proposition 4.0] *Let $(M(X), d)$ be the minimal model of X. Then*

$$\overline{\mathbb{Q}[\pi_1 X]} \cong \overline{H_b^0(M(X), d)},$$

where the completion on the left is with respect to the augmentation ideal and the completion on the right is with respect to the decreasing rows filtration.

Suppose that (A,d) is a quadratic model for a space X. Let L be the Lie algebra dual to A, and let $L(\pi_1 X)$ be the Malcev Lie algebra of X. The following theorem tells us how to obtain $L(\pi_1 X)$ from L, and also how to compute the minimal model of X when A is Koszul.

Theorem 4.3.6 *Let X be a space with a quadratic model $(A(X), d)$, and let $L = L(A(X))$ be the Lie algebra dual to $A(X)$.*

(1) $\overline{U(L)} \cong \overline{\mathbb{Q}[\pi_1 X]}$, *where the completions are each with respect to the augmentation ideal. This isomorphism respects the Hopf algebra structures.*

(2) $\overline{L} \cong L(\pi_1 X)$, *where the completion of L is with respect to the filtration by bracket length.*

(3) *If $A(X)$ is Koszul, and L is graded by bracket length with $L_i := L/\Gamma_i L$ finite dimensional for all i, then $(\Omega(L^\bullet), d)$ is the minimal model of X.*

(4) *Under the hypotheses of* (3), $\mathbb{Q}_\infty(X)$ *is a $K(\pi, 1)$ space.*

Proof Since $(A(X), d)$ is a model for X, there is a quasi-isomorphism from the minimal model $(M(X), d)$ to $(A(X), d)$. This gives a map on the degree-zero cohomology of their cobar complexes, $H_b^0(C_{A(X)}) \to H_b^0(C_{M(X)})$, which induces an isomorphism on the associated graded quotients with respect to the rows' filtration on the complexes [1, Lemma 3.3a]. Thus, there is an isomorphism on the completions with respect to the row filtration, $\overline{H_b^0(C_{A(X)})} \cong \overline{H_b^0(C_{M(X)})}$. Also by Lemma 4.3.4, $H_b^0(C_{A(X)})$ is the dual to $A(X)$, $U(L)$, and the rows' filtration is the filtration by the augmentation ideal. Hence $\overline{H_b^0(C_{A(X)})} \cong \overline{U(L)}$. Moreover, Proposition 4.3.5 says that $\overline{H_b^0(C_{M(X)})} \cong \overline{\mathbb{Q}[\pi_1 X]}$, completing the proof of (1).

Since the isomorphism in (1) respects the Hopf algebra structures, taking the primitive elements on each side yields the isomorphism in (2).

The projection $f\colon (U(L), \epsilon, W_{A(X)}) \to (U(L_i), \epsilon, \mathbb{Q} \oplus L_i)$ is a map of QLAs, and so it induces a map on the dual DGAs $g\colon (\Omega(L_i), d_i) \to (A(X), d)$. The grading of L by bracket length induces another grading on $U(L)$ and on $U(L_i)$ which we call weight, and the map f is an isomorphism for weight $j < i$. The weight gradings on $U(L)$ and $U(L_i)$ also induce weight gradings on $\mathrm{Ext}^*_{U(L)}(\mathbb{Q}, \mathbb{Q})$, $\mathrm{Ext}^*_{U(L_i)}(\mathbb{Q}, \mathbb{Q})$, $(A(X), d)$, and $(\Omega(L_i), d_i)$. The differentials on $(A(X), d)$ and $(\Omega(L_i), d_i)$ preserve weight and hence we have a weight grading on $H^*(A(X), d)$ and $H^*(\Omega(L_i), d_i)$. Consider the following diagram for weight $j < i$:

$$\begin{tikzcd}
\mathrm{Ext}^*_{U(L),j}(\mathbb{Q}, \mathbb{Q}) \arrow[d, leftrightarrow] & \mathrm{Ext}^*_{U(L_i),j}(\mathbb{Q}, \mathbb{Q}) \arrow[l, "f^*"'] \arrow[d, leftrightarrow] \\
H^*_j(A, d) & H^*_j(\Omega(L_i), d_i) \arrow[l, "g^*"']
\end{tikzcd}$$

Since $A(X)$ and $U(L_i)$ are Koszul, the maps on the right and left are both isomorphisms [1, Lemma 3.2]. Since $U(L)$ and $U(L_i)$ agree for weight $j < i$, one can see that f^*

is an isomorphism for weight $j < i$ by comparing the minimal graded free resolutions of $\mathbb{Q}$ considered as a $U(L)$–module and as a $U(L_i)$–module. Thus, the map g is a quasi-isomorphism for weight $j < i$. Since $\Omega(L^\bullet) = \varprojlim_i \Omega(L_i)$, we have a quasi-isomorphism from the standard complex of L to $A(X)$. Moreover, $\Omega(L^\bullet)$ is minimal and is hence the minimal model of $A(X)$.

Finally, note that the minimal model $\Omega(L^\bullet)$ is generated in degree 1. By Theorem 4.1.1, this happens exactly when the space is rationally $K(\pi, 1)$. □

5 Topology of $X_{\mathcal{A}}$

In this section, we will first review known results on the rational homotopy theory of linear arrangements. These results will apply to the toric case as well, and so we focus on proving the analogous results for abelian arrangements. In the projective case (with curves of positive genus), we compute the quadratic dual to the QDGA $(A(X_{\mathcal{A}}), d)$ for a chordal arrangement $\mathcal{A}$ and give a combinatorial presentation for the Lie algebra dual to $A(X_{\mathcal{A}})$. This then gives us a combinatorial description of $\overline{\mathbb{Q}[\pi_1 X_{\mathcal{A}}]}$, the Malcev Lie algebra $L(\pi_1 X_{\mathcal{A}})$, and the minimal model $(M(X_{\mathcal{A}}), d)$. Finally, we will show that $X_{\mathcal{A}}$ is a rational $K(\pi, 1)$ space.

5.1 Linear and toric arrangements

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C})$. Papadima and Yuzvinsky [14] describe the *holonomy Lie algebra*, L, of $X_{\mathcal{A}}$ and show that it is the Lie algebra dual to the cohomology ring $H^*(X_{\mathcal{A}})$. They also show that the standard complex of L is the minimal model of $X_{\mathcal{A}}$ [14, Propositions 3.1 and 4.4]. Moreover, Kohno [11] shows that the holonomy Lie algebra is isomorphic to the Malcev Lie algebra $L(\pi_1 X_{\mathcal{A}})$.

This Lie algebra L can be described as the free Lie algebra generated by c_e for $e \in \mathcal{E}$, modulo the relations

(i) $[c_e, c_{e'}] = 0$ if e and e' are not part of a cycle of size 3, and

(ii) $[c_{e_1}, c_{e_2} + c_{e_3}] = 0$ if $\{e_1, e_2, e_3\}$ is a cycle.

If X is a formal space, then $H^*(X)$ is Koszul if and only if X is rationally $K(\pi, 1)$ [14, Theorem 5.1]. In particular, $X_{\mathcal{A}}$ is a rational $K(\pi, 1)$ space. Falk first showed that $X_{\mathcal{A}}$ is a rational $K(\pi, 1)$ space when studying the minimal model [8, Proposition 4.6], but the generality of Papadima and Yuzvinsky's arguments allows us to directly apply them to toric arrangements, giving the following result:

Theorem 5.1.1 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph and $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C}^\times)$.*

(1) *The holonomy Lie algebra of $X_\mathcal{A}$ is the Lie algebra dual to $H^*(X_\mathcal{A})$, denoted by $L = L(H^*(X_\mathcal{A}))$.*

(2) *The minimal model of $X_\mathcal{A}$ is $(\Omega(L^\bullet), d)$, the standard complex of L.*

(3) *$X_\mathcal{A}$ is a rational $K(\pi, 1)$ space.*

However, the presentation for the Lie algebra is much more complicated.

5.2 Abelian arrangements and higher genus

For this subsection, fix a projective curve C of genus $g > 0$ and a chordal graph $\Gamma = (\mathcal{V}, \mathcal{E})$, and consider the chordal abelian arrangement $\mathcal{A} = \mathcal{A}(\Gamma, C)$. We will use quadratic-linear duality to study the rational homotopy theory of $X_\mathcal{A}$.

Let L be the free Lie algebra generated by a_v^i, b_v^i and c_e for $v \in \mathcal{V}$, $e \in \mathcal{E}$ and $i = 1, \dots, g$, subject to the following relations:

(i) $[a_v^i, a_w^j] = [b_v^i, b_w^j] = 0$ for $v, w \in \mathcal{V}$ with $v \neq w$,

(ii) (a) $[b_{h(e)}^i, a_{t(e)}^i] = [b_{t(e)}^i, a_{h(e)}^i] = c_e$ for $e \in \mathcal{E}$,

(b) $[a_v^i, b_w^j] = 0$ if $v \neq w$ and there is no edge connecting v and w, or if $i \neq j$,

(c) $\sum_{i=1}^g [a_v^i, b_v^i] = \sum_{v \in \{h(e), t(e)\}} c_e$ for $v \in \mathcal{V}$,

(iii) (a) $[a_v^i, c_e] = [b_v^i, c_e] = 0$ for $e \in \mathcal{E}$ and $h(e) \neq v \neq t(e)$,

(b) $[a_{h(e)}^i + a_{t(e)}^i, c_e] = [b_{h(e)}^i + b_{t(e)}^i, c_e] = 0$ for $e \in \mathcal{E}$,

(iv) (a) $[c_e, c_{e'}] = 0$ whenever e and e' are not part of a 3–cycle, and

(b) $[c_{e_1}, c_{e_2} + c_{e_3}] = 0$ whenever $\{e_1, e_2, e_3\}$ is a cycle.

The following theorem generalizes the main theorem of [1]. Using the Lie algebra dual to $A(\mathcal{A})$, this theorem gives a description of the Malcev Lie algebra of $X_\mathcal{A}$ when $\mathcal{A}$ is chordal.

Theorem 5.2.1 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, $\mathcal{A} = \mathcal{A}(\Gamma, C)$, and L be the Lie algebra described above. Then we have the following:*

(1) *Consider the universal enveloping algebra $U(L)$ as a QLA whose first filtered piece is spanned by a_v^i, b_v^i, c_e for $v \in \mathcal{V}$ and $e \in \mathcal{E}$. Then $U(L)$ is a Koszul QLA which is the nonhomogeneous quadratic dual to the Koszul DGA $A(\mathcal{A})$.*

(2) *$\overline{U(L)} \cong \overline{\mathbb{Q}[\pi_1(X_\mathcal{A})]}$, where the completions are each with respect to the augmentation ideal. This isomorphism respects the Hopf algebra structures.*

(3) *$\bar{L} \cong L(\pi_1(X_\mathcal{A}))$, where the completion of L is with respect to the filtration by bracket length.*

Proof (1) We can identify the QLA dual to $A(\mathcal{A})$ with L as follows. Let a_v^i, b_v^i, c_e be the dual basis to x_v^i, y_v^i, g_e. The relations in the quadratic dual correspond to quadratic elements of the basis from Lemma 3.5.2 since there is a natural isomorphism $\phi\colon I^{\perp} \cong ((V \otimes V)/I)^*$. The four types of relations (i)–(iv) in the presentation for L come from four types of basis elements for $(V \otimes V)/I$:

(i) $x_v^i x_w^j$ or $y_v^i y_w^j$ for $v \neq w$,

(ii) $x_v^i y_w^j$ for $v \neq w$, or $v = w$ and either $i \neq j$ or $i = j = 1$,

(iii) $x_v^i g_e$ or $y_v^i g_e$ for $v \neq h(e)$,

(iv) $g_{e_1} g_{e_2}$ for $\{e_1, e_2\}$ not a broken circuit.

The further subtypes in the relations for L arise when computing ϕ^{-1}. Since $A(\mathcal{A})$ is Koszul by Theorem 3.4.3 or 3.5.3, $U(L)$ is also Koszul.

Statements (2) and (3) follow from Theorem 4.3.6. □

Since $A(\mathcal{A})$ is a Koszul model for $X_{\mathcal{A}}$, Theorem 4.3.6 gives us the following proposition and corollary, which describes the minimal model of $X_{\mathcal{A}}$ and shows that $X_{\mathcal{A}}$ is rationally $K(\pi, 1)$.

Proposition 5.2.2 *Let C be a complex projective curve of genus $g \geq 1$, $\Gamma = (\mathcal{V}, \mathcal{E})$ a chordal graph, $\mathcal{A} = \mathcal{A}(\Gamma, C)$, and L be the Lie algebra described above. Consider $L^\bullet$ with the grading by bracket length. Then the standard complex $(\Omega(L^\bullet), d)$ is the minimal model for $X_{\mathcal{A}}$.*

Corollary 5.2.3 *Let C be a complex projective curve of genus $g \geq 1$ and $\mathcal{A} = \mathcal{A}(\Gamma, C)$ a chordal arrangement. Then its complement $X_{\mathcal{A}}$ is a rational $K(\pi, 1)$ space.*

Remark 5.2.4 Not only is $X_{\mathcal{A}}$ rationally $K(\pi, 1)$, but it is not hard to show that $X_{\mathcal{A}}$ is also $K(\pi, 1)$. As an easy case, a punctured projective curve is homotopic to a wedge of circles and hence is $K(\pi, 1)$. Then by induction on $|\mathcal{V}|$ and using the long exact sequence in homotopy of a fibration, one can show that if $\Gamma = (\mathcal{V}, \mathcal{E})$ is chordal, then the complement to $\mathcal{A}(\Gamma - v, C)$ is $K(\pi, 1)$. The fibration arises as the restriction of the projection $C^{\mathcal{V}} \to C^{\mathcal{V}-v}$ to $X_{\mathcal{A}(\Gamma,C)} \to X_{\mathcal{A}(\Gamma-v,C)}$, where $v \in V$ is the maximum vertex in our perfect elimination ordering. The fiber of this fiber bundle is homeomorphic to $C \setminus \{k \text{ points}\}$ where $k = |\mathcal{E} \setminus (\mathcal{E} - v)|$.

Remark 5.2.5 The fact that chordal arrangements are rationally $K(\pi, 1)$ gives us a class of examples of abelian arrangements which are not formal. If we did have formality, then Theorem 5.1 of [14] would imply that the cohomology ring is Koszul. However, if the arrangement is chordal and has at least one cycle, the cohomology ring is not even generated in degree one and hence cannot be Koszul.

We end with an example computation of the first few terms $\Omega(L/\Gamma_i L)$ of the minimal model $\varprojlim_i \Omega(L/\Gamma_i L)$ for the complement of the elliptic braid arrangement of type A_2.

Example 5.2.6 Consider the case of an elliptic curve. The braid arrangement of type A_2 corresponds to the complete graph Γ on three vertices $\mathcal{V} = \{1 < 2 < 3\}$ with edges labeled $\{12, 13, 23\}$.

The DGA $A(\mathcal{A})$ is the quotient of the exterior algebra $\Lambda(x_v, y_v, g_e)$ by the ideal generated by

(i) $(x_i - x_j)g_{ij}$, $(y_i - y_j)g_{ij}$, and

(ii) $g_{12}g_{13} - g_{12}g_{23} + g_{13}g_{23}$,

with differential $dg_{ij} = (x_i - x_j)(y_i - y_j)$.

Recall from the proof of Theorem 4.3.6 that the bracket length defines another grading on $U(L)$, which also gives another grading on $A(\mathcal{A})$ (by assigning the "weight" of a generator of $A(\mathcal{A})$ to be the bracket length of its dual in $U(L)$). Notice that there is a quasi-isomorphism up to weight less than i between $\Omega(L/\Gamma_i L)$ and $A(\mathcal{A})$, for each i, which in the limit induces the quasi-isomorphism between $\Omega(L)$ and $A(\mathcal{A})$.

(1) $L/\Gamma_1 L = 0$ and hence $\Omega_1 = \mathbb{Q}$, which is isomorphic to the weight-0 part of $A(\mathcal{A})$.

(2) $L/\Gamma_2 L$ is the vector space generated by a_v and b_v so that $\Omega_1 = \Lambda(x_v, y_v)$ with differential $d_2 = 0$. This is isomorphic to the weight ≤ 1 part of $A(\mathcal{A})$.

(3) $\Omega(L/\Gamma_3 L) = \Lambda(x_v, y_v, g_e)$ with differential $d_3\colon g_{ij} \mapsto (x_i - x_j)(y_i - y_j)$, which matches $A(\mathcal{A})$ up to weight 2.

(4) $\Omega(L/\Gamma_4 L) = \Lambda(x_v, y_v, g_e, k_{e,a}, k_{e,b})$, where
$$k_{e_a} := [a_{h(e)}, c_e]^* = -[a_{t(e)}, c_e]^*$$
and
$$k_{e,b} := [b_{h(e)}, c_e]^* = -[b_{t(e)}, c_e]^*.$$
The differential d_4 restricts to d_3 on the subalgebra $\Omega(L/L^{(3)})$, and we also have $d_4 k_{e,a} = (x_{h(e)} - x_{t(e)})g_e$ and $d_4 k_{e,b} = (y_{h(e)} - y_{t(e)})g_e$.

(5) $\Omega(L/\Gamma_5 L) = \Lambda(x_v, y_v, g_e, k_{e,a}, k_{e,b}, k_{e,aa}, k_{e,bb}, k_{e,ab}, k_C)$ where
$$k_{e,aa} := [a_{h(e)}, [a_{h(e)}, c_e]]^* = -[a_{t(e)}, [a_{h(e)}, c_e]]^*,$$
$$k_{e,bb} := [b_{h(e)}, [b_{h(e)}, c_e]]^* = -[b_{t(e)}, [b_{h(e)}, c_e]]^*,$$
$$k_{e,ab} := [a_{h(e)}, [b_{h(e)}, c_e]]^*$$

and

$$k_C = [c_{e_1}, c_{e_2}]^* = -[c_{e_1}, c_{e_3}]^* = [c_{e_2}, c_{e_3}]^*$$

whenever $\{e_1, e_2, e_3\}$ is a cycle. The differential is defined by

$$\begin{aligned} d_5 k_{e,aa} &= (x_{h(e)} - x_{t(e)}) k_{e,a}, \\ d_5 k_{e,bb} &= (y_{h(e)} - y_{t(e)}) k_{e,b}, \\ d_5 k_{e,ab} &= (x_{h(e)} - x_{t(e)}) k_{e,b} + (y_{h(e)} - y_{t(e)}) k_{e,a}, \end{aligned}$$

and

$$d_5 k_C = g_{e_1} g_{e_2} - g_{e_1} g_{e_3} + g_{e_2} g_{e_3}.$$

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Quasiflats in CAT(0) 2–complexes

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We show that if X is a piecewise Euclidean 2–complex with a cocompact isometry group, then every 2–quasiflat in X is at finite Hausdorff distance from a subset Q which is locally flat outside a compact set, and asymptotically conical.

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1 Introduction

In a number of rigidity theorems for quasi-isometries, an important step is to determine the structure of individual quasiflats; this is then used to restrict the behavior of quasi-isometries, often by exploiting the pattern of asymptotic incidence of the quasiflats. See Kleiner and Leeb [10; 9], Kapovich and Leeb [7], Eskin and Farb [5], Eskin [4], and Behrstock, Kleiner, Minsky and Mosher [1]. In this paper, we study 2–quasiflats in CAT(0) 2–complexes, and show that they have a very simple asymptotic structure.

Theorem 1.1 *Let X be a proper, piecewise Euclidean,* CAT(0) *2–complex with a cocompact isometry group. Then every 2–quasiflat $Q \subset X$ lies at finite Hausdorff distance from a subset $Q' \subset X$ which is locally flat, ie locally isometric to $\mathbb{R}^2$, outside a compact set.*

This result, and more refined statements appearing in later sections, are applied to 2–dimensional right-angled Artin groups by the present authors [2]. The main application is to show that if X, X' are the standard CAT(0) complexes of 2–dimensional right-angled Artin groups, then any quasi-isometry $X \to X'$ between them must map flats to within finite Hausdorff distance of flats.

The strategy for proving Theorem 1.1 is to replace the quasiflat Q with a canonical object that has more rigid structure. To that end, we first associate an element $[Q]$ of the locally finite homology group $H_2^{\mathrm{lf}}(X)$, and then show that the support set $\mathrm{supp}([Q])$ of $[Q]$ — the set of points $x \in X$ such that the induced homomorphism $H_2^{\mathrm{lf}}(X) \to H_2(X, X \setminus \{x\})$ is nontrivial on $[Q]$ — is at bounded Hausdorff distance

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from Q. The support set $Q' := \mathrm{supp}([Q])$ behaves much like a minimizing locally finite cycle, and this leads to asymptotically rigid behavior, in particular asymptotic flatness.

Remark 1.2 (1) Support sets were used implicitly in Kleiner and Leeb [9; 11].

(2) The paper Kleiner and Lang [8], which may be viewed as a more sophisticated version of the results presented here, exploits similar geometric ideas in asymptotic cones, to study k–quasiflats in CAT(0) spaces which have no $(k+1)$–quasiflats.

(3) Many of the results of this paper (though not Theorem 1.1 itself) can be adapted to n–quasiflats in n–dimensional CAT(0) complexes.

(4) One may use the results in this paper to give a new proof that quasi-isometries between Euclidean buildings map flats to within uniform Hausdorff distance of flats [9]. This then leads to a (partly) different proof of rigidity of quasi-isometries between Euclidean buildings.

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2 Preliminaries

CAT(κ) spaces

We recall some standard facts, and fix notation. We refer the reader to [3; 9] for more detail. Our notation and conventions are consistent with [9].

Let X be a CAT(0) space.

If x and y are in X, then $\overline{xy} \subset X$ denotes the geodesic segment with endpoints x, y. If p is in X, we let $\angle_p(x, y)$ denote the angle between x and y at p. This induces a pseudodistance on $X \setminus \{p\}$. By collapsing subsets of zero diameter and completing, we obtain the space of directions $\Sigma_p X$, which is a CAT(1) space. The quotient map yields the logarithm $\log_p\colon X \setminus \{p\} \to \Sigma_p X$; it associates to $x \in X \setminus \{p\}$ the direction at p of the geodesic segment $\overline{px}$. The tangent cone at p, denoted $C_p X$, is a CAT(0) space isometric to the cone over $\Sigma_p X$.

Given two constant (not necessarily unit) speed rays $\gamma_1, \gamma_2\colon [0, \infty) \to X$, their distance is defined to be

$$\lim_{t\to\infty} \frac{d(\gamma_1(t), \gamma_2(t))}{t}.$$

This defines a pseudodistance on the set of constant speed rays in X; the metric space obtained by collapsing zero diameter subsets is the *Tits cone* of X, denoted $C_T X$. The Tits cone is isometric to the Euclidean cone over the Tits boundary $\partial_T X$. For every $p \in X$, there are natural logarithm maps

$$\log_p\colon X \to C_p X, \qquad \log_p\colon C_T X \to X,$$
$$\log_p\colon X \setminus \{p\} \to \Sigma_p X, \quad \log_p\colon \partial_T X \to \Sigma_p X.$$

Definition 2.1 If Z is a CAT(1) space, $Y \subset Z$ and $z \in Z$, then the *antipodal set of z in Y* is

$$\mathrm{Ant}(z, Y) := \{y \in Y \mid d(z, y) = \pi\}.$$

Recall that by our definition, every CAT(1) space has diameter at most π.

If X is a CAT(0) complex and $p, x \in X$ are distinct points, $Y \subset \Sigma_x X$, then the antipodal set $\mathrm{Ant}(\log_x p, Y)$ is the set of directions in Y which are tangent to extensions of the geodesic segment $\overline{px}$ beyond x.

Locally finite homology

Let Z be a topological space. We recall that the k^{th} locally finite (singular) chain group $C_k^{\mathrm{lf}}(Z)$ is the collection of (possibly infinite) formal sums of singular k–simplices, such that for every compact subset $Y \subset Z$, only finitely many nonzero terms are contributed by singular simplices whose image intersects Y. The usual boundary operator yields a well-defined chain complex $C_*^{\mathrm{lf}}(Z)$; its homology is the *locally finite homology of Z*.

Suppose K is a simplicial complex. Then there is a simplicial version of the locally finite chain complex — the locally finite simplicial chain complex — defined by taking (possibly infinite) formal linear combinations of oriented simplices of K, where every simplex σ of K touches only finitely many simplices with nonzero coefficients. The usual proof that simplicial homology is isomorphic to singular homology gives an isomorphism between the locally finite simplicial homology of K, and the locally finite homology of its geometric realization $|K|$, when K is locally finite [6, 3.H, Exercise 6].

The *support set* of $\sigma \in H_k^{\mathrm{lf}}(Z)$ is the subset $\mathrm{supp}(\sigma) \subset Z$ consisting of the points $z \in Z$ for which the inclusion homomorphism

$$H_k^{\mathrm{lf}}(Z) \to H_k(Z, Z \setminus \{z\})$$

is nonzero on σ. This is a closed subset when Z is Hausdorff.

Now suppose K is an n–dimensional locally finite simplicial complex, with polyhedron Z. Then the simplicial chain groups $C_k^{\mathrm{lf}}(K)$ vanish for $k > n$, and hence $H_n^{\mathrm{lf}}(Z)$ is isomorphic to the group of locally finite simplicial n–cycles $Z_n^{\mathrm{lf}}(K)$. The support set of a locally finite simplicial n–cycle $\sigma \in Z_n^{\mathrm{lf}}(Z)$ is the union of the closed n–simplices with nonzero coefficient in σ, as follows from excision.

3 Locally finite homology and support sets

The key results in this section are the geodesic extension property of Lemma 3.1, and the asymptotic conicality result for support sets with quadratic area growth, in Theorem 3.11. We remark that most of the statements (and proofs) in this section extend with minor modifications to supports of n–dimensional locally finite homology classes in n–dimensional CAT(0) complexes.

In this section X will be a proper, piecewise Euclidean, CAT(0) 2–complex.

The geodesic extension property and metric monotonicity

The fundamental property of support sets is the extendability of geodesics.

Lemma 3.1 *Suppose $\sigma \in H_2^{\mathrm{lf}}(X)$, and let $S := \mathrm{supp}(\sigma) \subset X$ be the support of σ. If p is in X and x is in S, the geodesic segment $\overline{px}$ may be prolonged to a ray in S: there is a ray $\overline{x\xi} \subset S$ which fits together with $\overline{px}$ to form a ray $\overline{p\xi}$.*

Proof Let $\gamma\colon [0, L] \to X$ be the unit speed parametrization of $\overline{px}$, and let $\hat\gamma\colon I \to X$ be a maximal extension of γ such that $\hat\gamma(I \setminus [0, L]) \subset S$, where I is an interval contained in $[0, \infty)$. Since S is a closed subset of the complete space X, either $I = [0, R]$ for some $R < \infty$, or $I = [0, \infty)$.

Suppose $I = [0, R]$ for $R < \infty$, and let $y := \hat\gamma(R)$. Consider the closed ball $B := \overline{B(y, r)}$, where r is small enough that B is isometric to the r–ball in the tangent cone $C_y X$. Note that this implies that $S \cap B$ is also a cone. Let $\sigma = [\sigma_B + \tau]$, where $\sigma_B \in C_2^{\mathrm{lf}}(X)$ is carried by B (and is therefore a finite 2–chain), $\tau \in C_2^{\mathrm{lf}}(X)$ is carried by $X \setminus B(y, r)$, and $\partial\sigma_B = -\partial\tau$ is carried by $\partial B \cap S$. Consider the singular chain μ obtained by coning off $\partial\sigma_B$ at p. Then $\partial\mu = \partial\sigma_B$, so the contractibility of X implies that μ is homologous to σ_B relative to $\partial\mu$. Thus $\mu + \tau$ belongs to the homology class of σ. Therefore y lies in the carrier of μ, for otherwise $\mu + \tau$ would be carried by $X \setminus \{y\}$, contradicting the fact that $y \in \mathrm{supp}(\sigma)$. Thus there is a point $z \in \partial B \cap S$ such that the segment $\overline{pz}$ passes through y. Since $B \cap S$ is a cone, we have $\overline{yz} \subset S$. This implies that $\hat\gamma$ is not a maximal extension, which is a contradiction.

Another way to argue the last part of the proof is to observe that σ_B projects under $\log_y\colon X\setminus\{y\}\to\Sigma_y X$ to a nontrivial 1–cycle η in $\Sigma_y X$. Therefore, there must be a direction $v\in\Sigma_y S$ making an angle π with $\log_y p$, since otherwise η would lie in the open ball of radius π centered at $\log_y p$, which is contractible. Then $\hat\gamma$ may be extended in the direction v, which contradicts the maximality of $\hat\gamma$. □

Remark 3.2 The geodesic extension property has a flavor similar to convexity, but note that support sets need not be convex. To obtain an example, let Z be the union of two disjoint circles Y_1, Y_2 of length 2π with a geodesic segment of length less than π (so Z is a "pair of glasses"), and let X be the Euclidean cone over Z. Then the cone over $Y_1\cup Y_2$ is a support set, but is not convex.

Corollary 3.3 (monotonicity and lower density bound) *Suppose* $\sigma\in H_2^{\mathrm{lf}}(X)$ *and* $S:=\operatorname{supp}(\sigma)$. *We have the following properties:*

(1) **Metric monotonicity** *For all* $0<r\le R$, $p\in X$, *if* $\Phi\colon X\to X$ *is the map which contracts points toward* p *by the factor* r/R, *then*

$$B(p,r)\cap S\subset\Phi(B(p,R)\cap S). \tag{3.4}$$

(2) **Monotonicity of density** *For all* $0\le r\le R$,

$$\frac{\operatorname{Area}(B(p,r)\cap S)}{r^2}\le\frac{\operatorname{Area}(B(p,R)\cap S)}{R^2}. \tag{3.5}$$

(3) **Lower density bound** *For all* $p\in S$, $r>0$,

$$\operatorname{Area}(B(p,r)\cap S)\ge\pi r^2, \tag{3.6}$$

with equality only if $B(p,r)\cap S$ *is isometric to an* r*–ball in* $\mathbb{R}^2$.

Here $\operatorname{Area}(Y)$ *refers to* 2*–dimensional Hausdorff measure, which is the same as Lebesgue measure (computed by summing over the intersections with* 2*–dimensional faces).*

Remark 3.7 Since the map Φ in Corollary 3.3(1) has Lipschitz constant r/R, the inclusion (3.4) can be viewed as a much stronger version of the usual monotonicity formula for minimal submanifolds in nonpositively curved spaces, which corresponds to (3.5).

Proof of Corollary 3.3 Equation (3.4) follows from Lemma 3.1.

Assertion (2) follows from assertion (1) and the fact that Φ has Lipschitz constant r/R.

If $p\in S$, then σ determines a nonzero class $\Sigma_p\sigma\in H_1(\Sigma_p X)$, by the composition

$$H_2(X,X\setminus\{p\})\xrightarrow{\partial}H_1(X\setminus\{p\})\xrightarrow{\log_{\Sigma_p X}}H_1(\Sigma_p X).$$

Since $\Sigma_p X$ is a CAT(1) graph, $\operatorname{supp}(\Sigma_p\sigma)$ contains a cycle of length at least 2π. If $r>0$ is small, then $B(p,r)\cap S$ is isometric to a cone of radius r over $\operatorname{supp}(\Sigma_p\sigma)$, and therefore has area at least πr^2. Now (3.5) implies (3.6). Equality in (3.6) implies that $\operatorname{supp}(\Sigma_p\sigma)$ is a circle of length 2π, $B(p,r_0)\cap S$ is isometric to an r_0–ball in $\mathbb{R}^2$ for small $r_0>0$, and that the contraction map Φ is similarity. This implies assertion (3). □

The corollary implies that the ratio

$$\frac{\operatorname{Area}(B(p,r)\cap S)}{r^2}$$

has a (possibly infinite) limit $\bar{A}$ as $r\to\infty$, which is clearly independent of the basepoint. When it is finite we say that σ has *quadratic growth*. In this case, Corollary 3.3 implies that, for all $p\in X$ and $r>0$,

$$\frac{\operatorname{Area}(B(p,r)\cap S)}{r^2}\le \bar{A}. \tag{3.8}$$

Asymptotic conicality

We will use Lemma 3.1 and Corollary 3.3 to see that quadratic growth support sets are asymptotically conical, provided the CAT(0) 2–complex X satisfies a mild additional condition. To see why an additional assumption is needed, consider a piecewise Euclidean CAT(0) 2–complex X homeomorphic to $\mathbb{R}^2$, whose singular set consists of a sequence of cone points $\{p_i\}$ tending to infinity, where $\Sigma_{p_i}X$ is a circle of length $2\pi+\theta_i$, and $\sum_i\theta_i<\infty$. Then X is the support set of the locally finite fundamental class $[X]$ of the 2–manifold X, but is not locally flat outside any compact subset of X.

To exclude this kind of behavior, one would like to know, for instance, that the cone angle 2π is isolated among the set of cone angles of points in X. When dealing with general CAT(0) 2–complexes, one needs to know that if $p\in X$ and $v\in\Sigma_p X$ is a direction whose antipodal set $\operatorname{Ant}(v,\operatorname{supp}(\tau))$ in a 1–cycle $\tau\in Z_1(\Sigma_p X)$ has small diameter, then v is close to a suspension point of τ. This condition will hold automatically if X admits a cocompact group of isometries. The precise condition we need is the following.

Definition 3.9 A family $\mathcal{F}$ of CAT(1) graphs has *isolated suspensions* if for every $\alpha>0$ there is a $\beta>0$ such that if Γ is in $\mathcal{F}$, $\tau\in Z_1(\Gamma)$ is a 1–cycle, v is in Γ, and

$$\operatorname{diam}(\operatorname{Ant}(v,\operatorname{supp}(\tau))<\beta,$$

then $\operatorname{supp}(\tau)$ is a metric suspension and v lies at distance less than α from a pole (ie suspension point) of $\operatorname{supp}(\tau)$). A CAT(0) 2–complex X has *isolated suspensions* if the collection of spaces of directions $\{\Sigma_x X\}_{x\in X}$ has isolated suspensions.

Remark 3.10 It follows from a compactness argument that any finite collection of CAT(1) graphs has the isolated suspensions property. In particular, any CAT(0) 2–complex with a cocompact isometry group has the isolated suspension property.

For the remainder of this section X will be a piecewise Euclidean, proper CAT(0) 2–complex with isolated suspensions.

Theorem 3.11 *Suppose $\sigma \in H_2^{\mathrm{lf}}(X)$ has quadratic area growth, and $S := \mathrm{supp}(\sigma)$. Then for all $p \in X$ there is an $r_0 < \infty$ such that:*

(1) *If x is in $S \setminus B(p, r_0)$, then S is locally isometric to a product of the form $\mathbb{R} \times W$ near x, where W is an i–pod (ie a cone over a finite set). In particular, S is locally convex near x.*

(2) *The map $S \setminus B(p, r_0) \to [r_0, \infty)$ given by the distance function from p is a fibration with fiber homeomorphic to a finite graph with all vertices of valence at least 2.*

(3) *S is asymptotically conical in the following sense. For every $p \in X$ and every $\epsilon > 0$, there is an $r < \infty$ such that if $x \in S \setminus B(p, r)$, then the angle (at x) between the geodesic segment $\overline{xp}$ and the $\mathbb{R}$–factor of some local product splitting of S is less than ϵ.*

(4) *If the area growth of S is Euclidean, ie*
$$\frac{\mathrm{Area}(B(p,r) \cap S)}{\pi r^2} \to 1 \quad \textit{as} \quad r \to \infty,$$
then S is a 2–flat.

Before entering into the proof of this theorem, we point out that the proof is driven by the following observation. The locally finite cycle σ is an area minimizing object in the strongest possible sense: any compact piece τ solves the Plateau problem with boundary condition $\partial\tau$ (ie filling $\partial\tau$ with a least area chain); in fact, because of the dimension assumption, there is only one way to fill $\partial\tau$ with a chain. Then we adapt the standard monotonicity formula from minimal surface theory to see that the support set is asymptotically conical. Roughly speaking the idea is that the ratio
$$\frac{\mathrm{Area}(B(p,r) \cap \mathrm{supp}(\sigma))}{r^2}$$
is nondecreasing and bounded above, and hence has limit as $r \to \infty$. For large r, one concludes that the monotonicity inequality is nearly an equality, which leads to Theorem 3.11(2).

Proof of Theorem 3.11 We begin with a packing estimate.

Lemma 3.12 *For all $\epsilon > 0$ there is an N such that for all $r \geq 0$, the intersection $B(p,r) \cap S$ does not contain an ϵr–separated subset of cardinality greater than N.*

Proof Take $\epsilon < 1$, and suppose the points

$$x_1, \dots, x_k \in B(p,r) \cap S$$

are ϵr–separated. Then the collection

$$\left\{B\left(x_i, \tfrac{1}{2}\epsilon r\right) \cap S\right\}_{1 \leq i \leq k}$$

is disjoint, is contained in $B(p,2r) \cap S$, and by Corollary 3.3(2) it has area at least $k\pi(\frac{1}{2}\epsilon r)^2$. Thus (3.8) implies the lemma. □

Lemma 3.13 *For all $\beta > 0$ there is an $r < \infty$ such that if $x \in S \setminus B(p,r)$, then*

$$\operatorname{diam}(\operatorname{Ant}(\log_x p, \Sigma_x S)) < \beta. \tag{3.14}$$

Proof The idea is that quadratic area growth bounds the complexity of the support set from above, which implies that on sufficiently large scales, it looks very much like a metric cone. On the other hand, failure of (3.14) implies that there is a pair of rays leaving p which coincide until x, and then branch apart with an angle at least β; when x is far enough from p, this will contradict the approximately conical structure of S at large scales.

Pick $\delta, \mu > 0$, to be determined later.

By Lemma 3.12 there is finite upper bound on the cardinality of a δr–separated subset sitting in $B(p,r) \cap S$, where r ranges over $[1,\infty)$. Let N be the maximal such cardinality, which will be attained by some δr_0–separated subset $\{x_1, \dots, x_N\} \subset B(p,r_0) \cap S$, for some r_0. Applying Lemma 3.1, let $\gamma_1, \dots, \gamma_N \colon [0,\infty) \to X$ be constant speed geodesics emanating from p, such that $\gamma_i(r_0) = x_i$, and $\gamma_i(t) \in S$ for all $t \in [r_0,\infty)$, $1 \leq i \leq N$. The functions

$$t \mapsto \frac{d(\gamma_i(t), \gamma_j(t))}{t} \tag{3.15}$$

are nondecreasing, and hence for all $r \in [r_0,\infty)$ the collection

$$\gamma_1(r), \dots, \gamma_N(r)$$

is δr–separated, and by maximality, it is therefore a δr–net in $B(p,r) \cap S$ as well.

Using the monotonicity (3.15) again, we may find $r_1 \in [r_0, \infty)$ such that for all $1 \le i, j \le N$, and every $r \in [r_1, \infty)$,

$$\frac{d(\gamma_i(r), \gamma_j(r))}{r} + \mu > \lim_{t \to \infty} \frac{d(\gamma_i(t), \gamma_j(t))}{t}. \tag{3.16}$$

Now suppose $x \in S \setminus B(p, r_1)$, and $v_1, v_2 \in \mathrm{Ant}(\log_x p, \Sigma_x S)$ satisfy $\angle_x(v_1, v_2) \ge \beta$. The idea of the rest of the proof is to invoke Lemma 3.1 to produce two rays emanating from p which agree until they reach x, but then diverge at angle at least β; since both rays will be well-approximated by one of the γ_i, their separation behavior will contradict (3.16).

Let $r_2 := d(p, x)$. By Lemma 3.1 we may prolong the segment $\overline{px}$ into two rays $\overline{p\xi_1}, \overline{p\xi_2}$, such that $\log_{\Sigma_x} \xi_i = v_i$, and $\overline{p\xi_i} \setminus B(p, r_2) \subset S$. Let η_1, η_2 be the unit speed parametrizations of $\overline{p\xi_1}$ and $\overline{p\xi_2}$ respectively. Applying triangle comparison, we may choose an $r_3 \ge r_2$ such that

$$d(\eta_1(r_3), \eta_2(r_3)) > r_3 \cos \tfrac{1}{2}\beta. \tag{3.17}$$

Pick i, j such that

$$d(\gamma_i(r_3), \eta_1(r_3)) < \delta r_3 \quad \text{and} \quad d(\gamma_j(r_3), \eta_2(r_3)) < \delta r_3.$$

By triangle comparison, we have

$$d(\gamma_i(r_3), \gamma_j(r_3)) \ge d(\eta_1(r_3), \eta_2(r_3)) - 2\delta r_3 > r_3 \cos \tfrac{1}{2}\beta - 2\delta r_3$$

while

$$\begin{aligned} d(\gamma_i(r_2), \gamma_j(r_2)) &\le d(\gamma_i(r_2), \eta_1(r_2)) + d(\eta_1(r_2), \eta_2(r_2)) + d(\eta_2(r_2), \gamma_j(r_2)) \\ &\le 2\delta r_2, \end{aligned}$$

since $d(\eta_1(r_2), \eta_2(r_2)) = 0$. On the other hand, by (3.16)

$$\mu > \frac{d(\gamma_i(r_3), \gamma_j(r_3))}{r_3} - \frac{d(\gamma_i(r_2), \gamma_j(r_2))}{r_2} \ge \cos \tfrac{1}{2}\beta - 4\delta.$$

When $\mu + 4\delta < \cos \frac{1}{2}\beta$ this gives a contradiction. □

The lemma together with the definition of isolated suspensions implies (1) and (3) of Theorem 3.11. Part (4) follows from Corollary 3.3.

To prove Theorem 3.11(2), we apply the definition of isolated suspensions with $\alpha_0 = \frac{\pi}{4}$ and let $\beta_0 > 0$ be the corresponding constant; then we apply Lemma 3.13 with $\beta = \beta_0$, and let r_0 be the resulting radius. For each $x \in X \setminus B(p, r_0)$, the space of directions $\Sigma_x S$ is a metric suspension, and the direction $\log_x p \in \Sigma_x X$ makes an angle at most $\frac{\pi}{4}$ from a pole of $\Sigma_x S$.

We call a point $x \in S \setminus B(p, r_0)$ *singular* if its tangent cone is not isometric to $\mathbb{R}^2$; thus singular points in $S \setminus B(p, r_0)$ have tangent cones of the form $\mathbb{R} \times W$, where W is an i–pod with $i > 2$, and the set of regular points forms an open subset which carries the structure of a flat Riemannian manifold. Using a partition of unity, we may construct a smooth vector field ξ on the regular part of $S \setminus B(p, r_0)$ such that:

- $\xi(x)$ makes an angle at least $\frac{3\pi}{4}$ with $\log_x p$ at every regular point x.
- For each singular point $x \in S \setminus B(p, r_0)$ whose space of directions is the metric suspension of an i–pod, if we decompose a small neighborhood $B(x, \rho) \cap S$ into a union
$$C_1 \cup \cdots \cup C_i,$$
where the C_j are Euclidean half-disks of radius ρ which intersect along a segment η of length 2ρ, then the restriction of ξ to C_j extends to a smooth vector field ξ_j on the manifold with boundary C_j, and $\xi_j(y)$ is a unit vector tangent to $\eta = \partial C_j$ for every $y \in \eta$.

Now a standard Morse theory argument using a reparametrization of the flow of ξ implies that
$$d_p\colon S \setminus B(p, r_0) \to [r_0, \infty)$$
is a fibration, and that the fiber is locally homeomorphic to an i–pod near each point $x \in S \setminus B(p, r_0)$ whose space of directions is the metric suspension of an i–pod. Here $i \geq 2$. □

Asymptotic branch points

The next result will be used when we consider support sets associated with quasiflats.

Lemma 3.18 *Let $\sigma \in H_2^{\mathrm{lf}}(X)$ be a quadratic growth class with support S, pick $p \in X$, and let*
$$d_p\colon S \setminus B(p, r_0) \to [r_0, \infty)$$
be the fibration as in Theorem 3.11(2). If the fiber has a branch point, then for all $R < \infty$, the support set S contains an isometrically embedded copy of an R–ball

$$B_R := B(z, R) \subset \mathbb{R} \times W, \tag{3.19}$$

where W is an infinite tripod, and $z \in \mathbb{R} \times W$ lies on the singular line.

Proof Let $\pi\colon Y \to S \setminus B(p, r_0)$ be the universal covering map. Since $S \setminus B(p, r_0)$ is homeomorphic to $\mathcal{G} \times [0, \infty)$, the covering map π is equivalent to the product of the universal covering $\widetilde{\mathcal{G}} \to \mathcal{G}$ with the identity map $[0, \infty) \to [0, \infty)$. Since $\mathcal{G}$ contains a branch point, we may find a proper embedding $\phi\colon V \to \widetilde{\mathcal{G}}$ of a tripod V into $\widetilde{\mathcal{G}}$.

Consider the map ψ given by the composition

$$V \times [0,\infty) \to \widetilde{\mathcal{G}} \times [0,\infty) \to \mathcal{G} \times [0,\infty) \simeq S \setminus B(p, r_0).$$

We may put a locally CAT(0) metric on $V \times (0,\infty)$ by pulling back the metric from $S \setminus B(p, r_0)$. For each of the three rays $\gamma_i \subset V$ whose union is V, the metric on $\gamma_i \times (0,\infty)$ is locally isometric to a flat metric with geodesic boundary. It follows from a standard argument that if $y \in V \times (0,\infty)$ lies on the singular locus and $\psi(y)$ lies outside $B(p, r_0 + R)$, then the R–ball in $V \times (0,\infty)$ is isometric to B_R as in (3.19). Since ψ is a locally isometric map of a CAT(0) space into a CAT(0) space, it is an isometric embedding. □

4 Quasiflats in 2–complexes

In this section, X is a piecewise flat proper CAT(0) 2–complex with isolated suspensions.

Theorem 4.1 *Let $Q \subset X$ be an (L, A)–quasiflat. Then there is a nontrivial quadratic growth, locally finite homology class $\sigma \in H_2^{\mathrm{lf}}(X)$ whose support set $S \subset X$ is at Hausdorff distance at most $D = D(L, A)$ from Q, with the following properties:*

(1) *For every $p \in X$, there is an $r_0 \in [0,\infty)$ such that $S \setminus \overline{B(p, r_0)}$ is locally isometric to $\mathbb{R}^2$.*

(2) *S is asymptotically conical, in the following sense. For every $p \in X$ and every $\epsilon > 0$, there is an $r_1 \in [r_0,\infty)$ such that if $x \in S \setminus B(p, r_1)$, then the angle at x between the geodesic segment $\overline{xp}$ and S is less than ϵ, and the map $S \setminus B(p, r_1) \to [r_0,\infty)$ given by the distance function from p is a fibration with circle fiber.*

(3) *If the area growth of S is Euclidean, ie*

$$\frac{\mathrm{Area}(B(p,r) \cap S)}{\pi r^2} \to 1 \quad \textit{as} \quad r \to \infty,$$

then S is a 2–flat.

Proof Using a standard argument, we may assume without loss of generality (and at the cost of some deterioration in quasi-isometry constants which will be suppressed), that Q is the image of a C–Lipschitz (L, A)–quasi-isometric embedding $f\colon \mathbb{R}^2 \to X$, where $C = C(L, A)$. The mapping f is proper, and hence induces a homomorphism $f_*\colon H_2^{\mathrm{lf}}(\mathbb{R}^2) \to H_2^{\mathrm{lf}}(X)$ of locally finite homology groups. We define S to be the support set of the image of the fundamental class of $\mathbb{R}^2$ under f_*:

$$S := \mathrm{supp}(f_*([\mathbb{R}^2])) \subset \mathrm{Im}(f) = Q. \tag{4.2}$$

Lemma 4.3 *There are constants $D = D(L, A)$ and $a = a(L, A)$ such that:*

(1) *The Hausdorff distance between S and Q is at most D.*

(2) *For every $p \in X$, the area of $B(p, r) \cap S$ is at most $a(1+r)^2$.*

Proof Using the uniform contractibility of $\mathbb{R}^2$, one may construct a proper map $g\colon Q \to \mathbb{R}^2$ such that $d(g \circ f, \mathrm{id}_{\mathbb{R}^2})$ is bounded by a function of (L, A). In particular, the composition of proper maps

$$\mathbb{R}^2 \xrightarrow{f} Q \xrightarrow{g} \mathbb{R}^2$$

is properly homotopic to $\mathrm{id}_{\mathbb{R}^2}$. Therefore $(g \circ f)_*([\mathbb{R}^2]) = [\mathbb{R}^2]$, so

$$\mathrm{supp}((g \circ f)_*([\mathbb{R}^2])) = \mathbb{R}^2.$$

On the other hand

$$\mathrm{supp}((g \circ f)_*([\mathbb{R}^2])) \subset g(S),$$

which implies that $Q = \mathrm{Im}(f)$ is contained in a controlled neighborhood of S.

The last assertion follows from the fact that $S \subset Q$ and Q has quadratic area growth, being the image of a Lipschitz quasi-isometric embedding. □

Therefore Theorem 3.11 applies to S, and by part (2), we get a fibration

$$d_p\colon S \setminus B(p, r_0) \to [r_0, \infty)$$

whose fiber is homeomorphic to a finite graph $\mathcal{G}$ all of whose vertices have valence at least 2. If $\mathcal{G}$ had a branch point, we could apply Lemma 3.18, contradicting the fact that S is a quasiflat. Thus S is locally isometric to $\mathbb{R}^2$ outside $B(p, r_0)$. □

5 Square complexes

In this section, X is a locally finite CAT(0) square complex with isolated suspensions.

Remark 5.1 It is not difficult to show that if $\mathcal{F}$ is the collection of CAT(1) graphs Γ all of whose edges have length $\frac{\pi}{2}$, then $\mathcal{F}$ has isolated suspensions. In particular, any CAT(0) square complex has isolated suspensions. However, we will not need this fact for our primary applications, so we omit the proof.

Theorem 5.2 *Let $\sigma \in H_2^{\mathrm{lf}}(X)$ be a quadratic growth locally finite homology class whose support set S is a quasiflat. Then there is a finite collection $\{H_1, \dots, H_k\}$ of half-plane subcomplexes contained in S, and a finite subcomplex $W \subset S$ such that*

$$S = W \cup \left(\bigcup_i H_i\right).$$

Proof Pick $p \in X$ and $\epsilon \in \left(0, \frac{\pi}{2}\right)$. Let r_1 be as in Theorem 4.1, and set $Y_1 := S \setminus B(p, r_1)$. Then Y_1 is a complete flat Riemann surface with concave boundary $\partial Y_1 = S(p, r_0) \cap Y_1$. Now pick $\alpha \in \left(0, \frac{\pi}{8}\right)$, $r_2 \in [r_1, \infty)$, and let $Y_2 := S \setminus B(p, r_2)$.

Lemma 5.3 *Provided r_2 is sufficiently large (depending on α), for every $x \in Y_2$, and every semicircle $\tau \subset \Sigma_x S$ such that*

$$d(\tau, \log_x p) > \alpha,$$

there is a subset $Z \subset S$ isometric to a Euclidean half-plane, such that $\Sigma_x Z = \tau$.

Proof Let y be in Y_2 and $v \in \Sigma_y S$ be a tangent vector such that $\angle_y(v, \log_y p) > \alpha$. Provided $r_2 \sin\alpha > r_1$, there will be a unique geodesic ray $\gamma_v \subset S$ starting at y with direction v; this follows from a continuity argument, since triangle comparison implies that any geodesic segment with initial direction v remains outside $B(p, r_1)$.

If $\tau \subset \Sigma_x S$ is a semicircle (ie a geodesic segment of length π) and $\angle_x(\tau, \log_x p)$ is less than α, then the union of the rays γ_v, for $v \in \tau$, will form a subset of S isometric to a Euclidean half-plane. □

Continuing the proof of Theorem 5.2, we now assume that r_2 is large enough that Lemma 5.3 applies.

Our next step is to construct a finite collection of half-planes in S. Consider the boundary ∂Y_2. This is the frontier of the set $K := S \cap \overline{B(p, r_2)}$ in S. Since K is locally convex near $\partial K = \partial Y_2$, it follows that for each $x \in \partial Y_2$, there is a well-defined space of directions $\Sigma_x K$, which consists of the directions $v \in \Sigma_x S$ such that $\angle_x(v, \log_x p) \le \frac{\pi}{2}$. Also, there is a normal space $\nu_x K \subset \Sigma_x S$ consisting of the directions $v \in \Sigma_x S$ making an angle at least $\frac{\pi}{2}$ with $\Sigma_x K$. When ϵ is small, the angle $\angle_x(\log_x p, \Sigma_x S)$ is small, and hence $\pi - \angle_x(v, \log_x p)$ will be small for every $v \in \nu_x K$. In particular, when ϵ is small, for every $v \in \nu_v K$ there will be a semicircle $\tau_v \subset \Sigma_x S$ such that:

1. τ_v makes an angle at least $\frac{\pi}{8}$ with $\log_x p$.
2. If $Z_v \subset S$ is the subset obtained by applying Lemma 5.3 to τ_v, then the boundary of Z_v is parallel to one of the sides of a square $P \subset S$ which contains x.
3. The angle between ∂Z_v and v is at least $\frac{\pi}{8}$.

We let $H_v \subset Z_v$ be the largest half-plane subcomplex of Z_v. It follows from (2) that H_v may be obtained from Z_v by removing a strip of thickness less than 1 around ∂Z_v.

Now let $\mathcal{H}$ be the collection of all half-planes obtained this way, where x ranges over ∂Y_2, and $v \in \nu_x K$. Observe that this is a finite collection, since each $H \in \mathcal{H}$ has a boundary square lying in $B(p, 1 + r_2)$, and two half-planes $H, H' \in \mathcal{H}$ sharing a boundary square must be the same.

We now claim that $S\setminus\bigcup_{H\in\mathcal{H}} H$ is contained in $\overline{B\left(p, r_2+\sec\frac{\pi}{8}\right)}$. To see this note that if $y\in Y_2$, then there is a shortest path in S from y to K. Since S is locally convex, this path will be a geodesic segment $\overline{yx}$ in X, where $x\in\partial Y_2$. Let $v:=\log_x y\in\Sigma_x S$. Then $\overline{yx}$ is contained in Z_v, and in view of condition (3) above, all but an initial segment of length at most $\sec\frac{\pi}{8}$ will be contained in $H_v\subset Z_v$. The claim follows. $\square$

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String homology, and closed geodesics on manifolds which are elliptic spaces

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Let M be a closed, simply connected, smooth manifold. Let $\mathbb{F}_p$ be the finite field with p elements, where $p > 0$ is a prime integer. Suppose that M is an $\mathbb{F}_p$–elliptic space in the sense of Félix, Halperin and Thomas (1991). We prove that if the cohomology algebra $H^*(M, \mathbb{F}_p)$ cannot be generated (as an algebra) by one element, then any Riemannian metric on M has an infinite number of geometrically distinct closed geodesics. The starting point is a classical theorem of Gromoll and Meyer (1969). The proof uses string homology, in particular the spectral sequence of Cohen, Jones and Yan (2004), the main theorem of McCleary (1987), and the structure theorem for elliptic Hopf algebras over $\mathbb{F}_p$ from Félix, Halperin and Thomas (1991).

55P50; 55P35, 55T05, 58E10

1 Introduction

We work over a ground field $\mathbb{F}$ and use $\mathbb{F}$ as the coefficients of homology and cohomology. Our main applications are in the case where this ground field is the finite field $\mathbb{F}_p$ with p elements, where $p > 0$ is a prime integer.

Let $\mathrm{HL}_*(M)$ denote the string homology algebra of a closed, simply connected manifold M. String homology is a graded commutative $\mathbb{F}$–algebra defined as follows. Let LM be the free loop space of M. In [4], Chas and Sullivan define the *string product*

$$H_p(LM) \otimes H_q(LM) \to H_{p+q-n}(LM),$$

where n is the dimension of M. This product is studied from the point of view of homotopy theory in Cohen and Jones [5]. The *string homology algebra* is defined by setting $\mathrm{HL}_s(M) = H_{s+n}(LM)$ and using the string product to define the product. It is proved that this string product makes $\mathrm{HL}_*(M)$ into a graded commutative $\mathbb{F}$–algebra in both [4] and [5].

Our main result about string homology is the following theorem. In the statement, ΩX refers to the based loop space of X.

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Theorem 1.1 *Let M be a simply connected, closed manifold. Suppose there is a constant C and an integer K such that*

$$\sum_{i \le n} \dim H_i(\Omega M; \mathbb{F}_p) \le C n^K.$$

Let K_0 be the minimal exponent which can occur in this bound. Then the string homology algebra $\mathrm{HL}_*(M; \mathbb{F}_p)$ contains a polynomial algebra P over $\mathbb{F}_p$ on K_0 generators and $\mathrm{HL}_*(M; \mathbb{F}_p)$ is a finitely generated free module over P.

If $H_*(\Omega M; \mathbb{F}_p)$ satisfies the growth hypotheses in the statement of this theorem, then we say that $H_*(\Omega M; \mathbb{F}_p)$ has *polynomial growth*. The main application of this theorem is the following result.

Theorem 1.2 *Let M be a simply connected, closed manifold. Suppose $H_*(\Omega M; \mathbb{F}_p)$ has polynomial growth and the algebra $H^*(M; \mathbb{F}_p)$ cannot be generated by one element. Then for any metric on M, there is an infinite number of geometrically distinct closed geodesics on M.*

To obtain this result from Theorem 1.1 we use the Gromoll–Meyer theorem relating closed geodesics and the topology of the free loop space. A metric on M defines a function, the *energy function*, on LM given by

$$\gamma \mapsto \int_{S^1} \langle \gamma'(t), \gamma'(t) \rangle \, dt.$$

If $\gamma : S^1 \to M$ is a closed geodesic parametrised by arc length then γ is a critical point of the energy function, as is the loop γ_n defined by $\gamma_n(z) = \gamma(z^n)$. Furthermore every critical point of the energy function is of the form γ_n, where γ is a closed geodesic parametrised by arc length; see Bott [3].

The circle S^1 acts on LM by rotating loops and the energy function is S^1–invariant. It follows that any closed geodesic γ parametrised by arc length generates an infinite number of critical S^1 orbits of the energy function. In general these orbits will not be isolated, but if there are only a finite number of geometrically distinct closed geodesics these orbits will be isolated.

We use the following terminology for graded vector spaces V. If each V_i is finite-dimensional we say V has *finite type*. If V has finite type then we say it has *finite dimension* if $\dim V_i$ is zero for all but a finite number of i, *infinite dimension* if $\dim V_i$ is non-zero for an infinite number of i, and *doubly infinite dimension* if the sequence of numbers $\dim V_i$ is unbounded. Note that polynomial growth with exponent at least 2 is the same as doubly infinite dimension. Using Morse–Bott theory, Gromoll and Meyer showed in [12] that the relation between critical points of the energy function and closed geodesics leads to the following theorem.

Theorem 1.3 *Let M be a simply connected closed manifold. If $H_*(LM;\mathbb{F})$ has doubly infinite dimension for some field $\mathbb{F}$, then for any metric on M there is an infinite number of geometrically distinct closed geodesics on M.*

If $\pi_1(M)$ is finite, then we can apply this theorem to the universal cover $\widetilde{M}$ of M. If $\pi_1(M)$ is infinite and $\pi_1(M)$ has an infinite number of conjugacy classes, then LM has an infinite number of components. Given a metric on M we can choose a minimiser of the energy function in each component of LM and it follows that this metric has an infinite number of geodesics; see Ballmann, Thorbergsson and Ziller [1]. This leaves the case where $\pi_1(M)$ is infinite but only has a finite number of conjugacy classes. Very little is known about this case; see Bangert and Hingston [2].

In [20] Sullivan and Vigué-Poirrier took up the case where $\mathbb{F}=\mathbb{Q}$ and, as an application of the theory of minimal models in rational homotopy, proved the following theorem.

Theorem 1.4 *Suppose M is a closed, simply connected manifold and the algebra $H^*(M,\mathbb{Q})$ is not generated by one element. Then $H_*(LM,\mathbb{Q})$ is doubly infinite.*

There are other interesting applications of the Gromoll–Meyer theorem in Halperin and Vigué-Poirrier [13] and Ndombol and Thomas [18]. Both these papers assume connectivity hypotheses of the following type: if M is a simply connected closed manifold of dimension n, then there are explicitly given constants $a\neq 0$ and b for which $H_i(M;\mathbb{F})=0$ for $2\leq i\leq r$, where $r\geq an+b$.

A very important ingredient in the proof of Theorem 1.2 is the following theorem from McCleary [14].

Theorem 1.5 *Let X be a simply connected space such that the algebra $H^*(X;\mathbb{F}_p)$ cannot be generated by one element. Then $H_*(\Omega X;\mathbb{F}_p)$ is doubly infinite.*

Indeed the main idea which led to this paper is to use string homology with coefficients in $\mathbb{F}_p$ to convert this theorem into a result about string homology. The first step in this process is to use the spectral sequence of Cohen, Jones and Yan [6] to relate string homology and the homology of the based loop space. The second is to use the structure theorems for elliptic Hopf algebras over $\mathbb{F}_p$ from Félix, Halperin and Thomas [9] to obtain the information about the E_2–term of this spectral sequence required for the proof.

This paper is set out as follows. In Section 2 we deal with those aspects of string homology our main results require. The primary objective in Section 2 is to prove Theorem 2.1. In Section 3 we give applications of Theorem 2.1. For example we explain how this theorem applies to the main examples of McCleary and Ziller [15; 16]. In Section 4 we summarise the results from [9] we need and complete the proofs of the main theorems. Finally in Section 5 we give applications of the main theorem to homogeneous spaces.

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2 String homology

In [6, Theorem 1], it is shown that there is a multiplicative second quadrant spectral sequence $(E_r^{s,t}, d_r^{s,t})$ with

$$d_r^{s,t}\colon E_r^{s,t} \to E_r^{s-r,t+r-1}, \quad E_2^{s,t} = H^{-s}(M) \otimes H_t(\Omega M),$$

and converging to $\mathrm{HL}_*(M)$. We will refer to it as the *CJY spectral sequence*.

Here second quadrant means that $E_r^{s,t}$ is zero if $s > 0$ or $t < 0$. Multiplicative means that each term $E_r^{*,*}$ is a bigraded algebra, d_r is a bigraded derivation of the product, and the E_∞ term of the spectral sequence is the bigraded algebra associated to a filtration of $\mathrm{HL}_*(M)$. The edge homomorphism $h\colon \mathrm{HL}_*(M) \to E_\infty^{0,*} \subseteq H_*(\Omega M)$ is the natural algebra homomorphism $h\colon \mathrm{HL}_*(M) \to H_*(\Omega M)$. This gives us a method of relating the algebras $H_*(\Omega M)$ and $\mathrm{HL}_*(M)$.

The simplest way to construct this spectral sequence is to use the string topology spectrum $\mathcal{S}(M) = LM^{-TM}$ introduced in [6]. The skeletal filtration of M induces a filtration of LM using the evaluation map $LM \to M$, and this in turn induces a filtration of $\mathcal{S}(M)$. The spectral sequence is the spectral sequence obtained from this filtration of $\mathcal{S}(M)$.

Our main application of this spectral sequence is the following theorem.

Theorem 2.1 *Let M be a closed oriented manifold. Then $\mathrm{HL}_*(M;\mathbb{F}_p)$ contains a polynomial algebra over $\mathbb{F}_p$ on k generators if and only if the centre of $H_*(\Omega M;\mathbb{F}_p)$ contains a polynomial algebra over $\mathbb{F}_p$ on k generators.*

The first step is to prove the following lemma.

Lemma 2.2 *Let M be a closed manifold. The kernel of the ring homomorphism $h\colon \mathrm{HL}_*(M) \to H_*(\Omega M)$ is a nilpotent ideal.*

Proof Let

$$0 = F^{-n-1} \subseteq F^{-n} \subseteq \cdots \subseteq F^0 = \mathrm{HL}_*(M)$$

be the (negatively indexed) filtration of $\mathrm{HL}_*(M)$ coming from the CJY spectral sequence. Here n is the dimension of the manifold M. Then

$$F^{-i} F^{-j} \subseteq F^{-i-j}$$

and so $(F^{-1})^{n+1} = 0$. The proposition follows since F^{-1} is exactly the kernel of the edge homomorphism of this spectral sequence. □

Next we give a simple but very useful lemma.

Lemma 2.3 *Suppose M is a closed, simply connected manifold of dimension n. Let C be the centre of the algebra $H_*(\Omega M;\mathbb{F}_p)$. Then for any $x \in C$,*

$$x^{p^{n-2}} \in \mathrm{im}\big(h\colon \mathrm{HL}_*(M;\mathbb{F}_p) \to H_*(\Omega M;\mathbb{F}_p)\big).$$

Proof Since h is the edge homomorphism in the CJY spectral sequence we know that an element $y \in H_*(\Omega M;\mathbb{F}_p) = E_2^{0,*}$ is in the image of h if and only if it is an infinite cycle in this spectral sequence. Let $x \in H_*(\Omega M;\mathbb{F}_p) = E_2^{0,*}$ be a central element. Now x may or may not be a cycle for d_2 in the CJY spectral sequence. But d_2 is a derivation and x is central so it follows that

$$d_2 x^p = p x^{p-1} d_2 x.$$

Since the ground field is $\mathbb{F}_p$ it follows that $d_2 x^p = 0$. It may or may not be the case that x^p is a cycle for d_3 but the same argument shows that $x^{p^2} = (x^p)^p$ is a cycle for d_3. Because M has dimension n, it follows that $d_r = 0$ for $r \geq n+1$. Since M is simply connected $H^1(M;\mathbb{F}_p) = H^{n-1}(M;\mathbb{F}_p) = 0$. It follows that there are at most $n-2$ differentials on $E_2^{0,*}$ which could be non-zero, starting with d_2. Repeating this argument at most $n-2$ times shows that $x^{p^{n-2}} \in E_2^{0,*}$ is an infinite cycle and it follows that $x^{p^{n-2}}$ is in the image of h. □

We will also need the following result of [11].

Theorem 2.4 *The image of $h\colon \mathrm{HL}_*(M;\mathbb{F}_p) \to H_*(\Omega M;\mathbb{F}_p)$ is contained in the centre of $H_*(\Omega M,\mathbb{F}_p)$.*

To prove Theorem 2.1 we simply combine the previous three results.

Proof of Theorem 2.1 The kernel of $h\colon \mathrm{HL}_*(M;\mathbb{F}_p)\to H_*(\Omega M;\mathbb{F}_p)$ is a nilpotent ideal, and the image of h is contained in the centre of $H_*(\Omega M;\mathbb{F}_p)$. So if $\mathrm{HL}_*(M;\mathbb{F}_p)$ contains a polynomial algebra on k generators, then so does the centre of $H_*(\Omega M;\mathbb{F}_p)$. On the other hand, if the centre of $H_*(\Omega M;\mathbb{F}_p)$ contains the polynomial algebra $\mathbb{F}_p[x_1,\dots,x_k]$, then Lemma 2.3 shows that every element of the subalgebra of the E_2–term of the CJY spectral sequence

$$\mathbb{F}_p[(x_1)^{p^{n-2}},\dots,(x_k)^{p^{n-2}}]\subset H_*(\Omega M;\mathbb{F}_p)=E_2^{0,*}$$

is an infinite cycle. It follows that $\mathrm{HL}_*(M;\mathbb{F}_p)$ contains a polynomial algebra on k generators. □

3 Applications of Theorem 2.1

3.1 Sphere bundles over spheres

Let M be a k–sphere bundle over S^l. If l is odd then M has the same cohomology ring as the product of spheres $S^k\times S^l$ and the theorem of Sullivan and Vigué-Poirrier, Theorem 1.4, shows that any metric on M has an infinite number of closed geodesics. If l is even and $k\neq l-1$ the same argument applies. We are left with the case of a $2n-1$ sphere bundle over S^{2n}. So let $Q=Q_{2n,e}$ denote the sphere bundle

$$S^{2n-1}\to Q\to S^{2n}$$

with Euler class $e\in\mathbb{Z}$. We choose an orientation of S^{2n} to identify the Euler class with an integer. If $e\neq 0$ then the rational cohomology ring of $Q_{2n,e}$ is generated by one element and so we will not be able to use the theorem of Sullivan and Vigué-Poirrier.

There are three special cases to deal with, $2n=2,4,8$. In these dimensions there is a $2n-1$ sphere bundle over S^{2n} with Euler class ±1 but the non-existence of elements with Hopf invariant one shows that these are the only dimensions in which this can happen. In these special cases $Q_{2n,\pm1}$ is a homotopy sphere and we cannot use the Gromoll–Meyer theorem for any coefficient field $\mathbb{F}$. The remaining cases are dealt with by the following theorem.

Proposition 3.1 *If $e\neq 0,\pm1$, for any metric on $Q=Q_{2n,e}$, there is an infinite number of closed geodesics on Q.*

Proof Choose a prime p such that p divides e. Standard basic calculations in algebraic topology show that

$$H^*(Q;\mathbb{F}_p)=E[a_{2n-1},b_{2n}]\quad\text{and}\quad H_*(\Omega Q;\mathbb{F}_p)=P[u_{2n-2},v_{2n-1}].$$

Here E denotes the exterior algebra over $\mathbb{F}_p$ and P denotes the polynomial algebra over $\mathbb{F}_p$. The subscripts are the degrees of the elements. If $p=2$, then the algebra $P[u_{2n-2}, v_{2n-1}]$ is not graded commutative since $v_{2n-1}^2 \neq 0$. However the centre of $H_*(\Omega Q;\mathbb{F}_p)$ is precisely $P[u_{2n-2}, v_{2n-1}^2]$. Theorem 2.1 shows that $\mathrm{HL}_*(Q)$ contains a polynomial algebra on two generators and so $H_*(LQ;\mathbb{F}_p)$ has doubly infinite dimension. The Gromoll–Meyer theorem shows that for any metric on Q, there is an infinite number of distinct closed geodesics. □

3.2 The Grassmannian of oriented two planes in $\mathbb{R}^{2n+1}$

Let $G_2^+(\mathbb{R}^{2n+1})$ denote the Grassmannian of oriented 2–planes in $\mathbb{R}^{2n+1}$. Recall the following two calculations from the theory of characteristic classes.

Suppose 2 is a unit in the coefficient field $\mathbb{F}$. Then

(1) $H^*(G_2^+(\mathbb{R}^{2n+1});\mathbb{F}) = P[x_2]/(x_2^{2n})$,

(2) $H^*(G_2^+(\mathbb{R}^{2n+1});\mathbb{F}_2) = P[x_2]/(x_2^{n}) \otimes E(y_{2n})$.

So the algebra $H^*(G_2^+(\mathbb{R}^{2n+1});\mathbb{F}_p)$ can be generated by a single generator for $p\neq 2$, but in the case $p=2$ it requires at least two generators. Another standard calculation in algebraic topology shows that

$$H_*(\Omega G_2^+(\mathbb{R}^{2n+1});\mathbb{F}_2) = E(u_1)\otimes P[v_{2n-2}]\otimes P[w_{2n-1}] \cong H_*(\Omega(\mathbb{CP}^n\times S^{2n});\mathbb{F}_2).$$

Evidently this contains a central polynomial algebra generated by two elements. The following theorem follows from the Gromoll–Meyer theorem in the case of $\mathbb{F}_2$ coefficients.

Theorem 3.2 *Any metric on $G_2^+(\mathbb{R}^{2n+1})$ has an infinite number of closed geodesics.*

3.3 The list of examples from McCleary–Ziller

There is a list in McCleary and Ziller [15], based on the work of [19], consisting of one representative from each diffeomorphism class of homogeneous spaces G/K, where G is a compact connected Lie group and K is a connected closed subgroup, with two properties:

- G/K is not diffeomorphic to a sphere, a real, complex, or quaternionic projective space, nor is it diffeomorphic to the Cayley projective plane.
- The algebra $H^*(G/K;\mathbb{Q})$ is generated by one element.

In other words it is the list of examples of homogeneous spaces to which we would like to apply the theorem of Gromoll–Meyer, but cannot do so over the ground field $\mathbb{Q}$. This list contains two infinite families:

- The Stiefel manifold $V_2(\mathbb{R}^{2n+1})$ of two frames in $\mathbb{R}^{2n+1}$. This is a $2n-1$ sphere bundle over S^{2n} with Euler class 2, and Proposition 3.1 shows that any metric on $V_2(\mathbb{R}^{2n+1})$ has an infinite number of geometrically distinct closed geodesics.
- The Grassmannian of oriented 2–planes in $\mathbb{R}^{2n+1}$. Theorem 3.2 shows that any metric on this manifold has an infinite number of geometrically distinct closed geodesics.

There are another seven homogeneous spaces on this list. The first two are $\mathrm{SU}(2)/\mathrm{SO}(3)$ and $\mathrm{Sp}(2)/\mathrm{SU}(2)$, and the other five are homogeneous spaces for G_2. It is possible to go through these seven examples by direct calculations with loop spaces. However, we will deal with them in Section 5 as examples of our main theorem.

4 The proof of Theorem 1.1 and Theorem 1.2

We next need results contained in a series of interrelated papers by Félix, Halperin, Lemaire and Thomas on the homology of based loop spaces. We give a brief summary of the results we need.

4.1 Elliptic Hopf algebras

Let Γ be a graded Hopf algebra over the ground field $\mathbb{F}$. The *lower central series* of Γ is the sequence

$$\Gamma = \Gamma^{(0)} \supset \Gamma^{(1)} \supset \Gamma^{(2)} \supset \cdots \supset \Gamma^{(n)} \supset \cdots,$$

where $\Gamma^{(i+1)} = [\Gamma, \Gamma^{(i)}]$. By definition Γ is *nilpotent* if $\Gamma^{(s)} = \mathbb{F}$ for some s. Although the definition of the $\Gamma^{(i)}$ depends only on the algebra structure of Γ, it is straightforward to check that the $\Gamma^{(i)}$ are normal Hopf subalgebras of Γ.

We say that Γ is *connected* if $\Gamma_i = 0$ when $i < 0$ and $\Gamma_0 = \mathbb{F}$, and that Γ is *finitely generated* if it is finitely generated as an algebra. From [9] we have the following definition.

Definition 4.1 Fix a ground field $\mathbb{F}$. A Hopf algebra Γ over $\mathbb{F}$ is *elliptic* if it is connected, co-commutative, finitely generated, and nilpotent.

Note that the only part of the definition of an elliptic Hopf algebra which refers to the coproduct is the condition that it is co-commutative.

Here are some examples. In these examples we assume that the Hopf algebras in question are connected and co-commutative over a fixed ground field $\mathbb{F}$.

(1) If Γ is a finite-dimensional Hopf algebra, then Γ is elliptic. To prove this first note that since Γ is connected $\Gamma^{(i)}$ is $(i+1)$–connected. Since Γ is finite-dimensional it follows that $\Gamma^{(i)} = \mathbb{F}$ for sufficiently large i. So Γ is nilpotent. Since Γ is finite, it is finitely generated.

(2) If Γ is commutative, then Γ is elliptic if and only if Γ is finitely generated.

(3) Let L be a Lie algebra. Let $U(L)$ be the universal enveloping algebra of L. This becomes a Hopf algebra by defining the coproduct to be the unique coproduct which makes the elements of L primitive. Then $U(L)$ is an elliptic Hopf algebra if and only if L is a finitely generated nilpotent Lie algebra.

The structure theorem for elliptic Hopf algebras proved in [9] tells us that essentially these examples generate the class of all elliptic Hopf algebras by taking extensions.

Theorem 4.2 *Let $\mathbb{F}$ be a field and let Γ be a connected, finitely generated, co-commutative Hopf algebra over $\mathbb{F}$.*

- *If $\mathbb{F}$ has characteristic zero, then Γ is elliptic if and only if $\Gamma = U(L)$, where L is a finitely generated, nilpotent Lie algebra over $\mathbb{F}$.*
- *If $\mathbb{F}$ has characteristic $p \neq 0$, then Γ is elliptic if and only if it contains a finitely generated, central Hopf subalgebra C, such that $\Gamma/\!/C$ is finite.*

The statement of the second clause of the theorem is not quite the same as the statement of [9, Theorem B(ii)] but it is easily seen to be equivalent to it. From [17] we know that Γ is isomorphic to $C \otimes \Gamma/\!/C$ as a C algebra. Since C is finitely generated and commutative it follows from a theorem of Borel [17] that as an algebra C is isomorphic to $P \otimes A$, where P is a polynomial algebra over $\mathbb{F}$ in a finite number of variables and A is a finite-dimensional algebra. It follows that Γ is isomorphic to $P \otimes A \otimes \Gamma/\!/C$ as a P module. Since both A and $\Gamma/\!/C$ are finite-dimensional it follows that Γ is a finitely generated free module over P. This is the condition given in [9].

4.2 Depth and the Gorenstein condition

Let A be a graded augmented algebra over the ground field $\mathbb{F}$. We will assume that A is connected. We can form the vector spaces

$$\operatorname{Ext}_A^{i,j}(\mathbb{F}, A).$$

The *depth* of A, depth A, is defined as follows:

$$\operatorname{depth} A = \inf\{s \mid \operatorname{Ext}_A^{s,*}(\mathbb{F}, A) \neq 0\}.$$

If $n = \operatorname{depth} A$, then $\operatorname{Ext}_A^{s,t}(\mathbb{F}, A) = 0$ for $s < n$ and there is an integer t such that $\operatorname{Ext}_A^{n,t}(\mathbb{F}, A) \neq 0$. In particular the depth of A could be infinite, that is, $\operatorname{Ext}_A^{s,t}(\mathbb{F}, A) = 0$ for all (s, t).

The graded algebra A is *Gorenstein* if there is a pair of integers (n, m) such that

- $\operatorname{Ext}_A^{s,t}(\mathbb{F}, A) = 0$ if $(s, t) \neq (n, m)$,
- $\operatorname{Ext}_A^{n,m}(\mathbb{F}, A) = \mathbb{F}$.

The definitions of the Gorenstein condition and depth first appeared in classical commutative ring theory. Gorenstein rings generalise complete intersection rings.

It is straightforward to check that

- $\operatorname{depth} A \otimes B = \operatorname{depth} A + \operatorname{depth} B$,
- $A \otimes B$ is Gorenstein if and only if both A and B are Gorenstein.

In the case of a polynomial algebra $\mathbb{F}[x]$ with one generator of degree k,

$$\operatorname{Ext}_{\mathbb{F}[x]}^{1,k}(\mathbb{F}, \mathbb{F}[x]) = \mathbb{F}, \quad \operatorname{Ext}_{\mathbb{F}[x]}^{s,t}(\mathbb{F}, \mathbb{F}[x]) = 0 \quad (s, t) \neq (1, k).$$

In the case where $A = \mathbb{F}[x]/(x^n)$ is a truncated polynomial with generator of degree k,

$$\operatorname{Ext}_A^{0,-k(n-1)}(\mathbb{F}, A) = \mathbb{F}, \quad \operatorname{Ext}_A^{s,t}(\mathbb{F}, A) = 0 \quad (s, t) \neq (0, -k(n-1)).$$

The most elementary method for doing these calculations is to use the minimal resolution of $\mathbb{F}$ over $\mathbb{F}[x]$ and the minimal resolution of $\mathbb{F}$ over $\mathbb{F}[x]/(x^n)$. It follows that both the algebras $\mathbb{F}[x]$ and $\mathbb{F}[x]/(x^n)$ are Gorenstein, and

$$\operatorname{depth} \mathbb{F}[x] = 1, \quad \operatorname{depth} \mathbb{F}[x]/(x^n) = 0.$$

The following lemma is [8, Proposition 1.7].

Lemma 4.3 *Suppose A is an infinite tensor product of algebras. Then the depth of A is infinite.*

Suppose Γ is a connected Hopf algebra that is commutative as an algebra. By a theorem of Borel [17, Theorem 7.11] it follows that Γ is isomorphic as an algebra to a tensor product of polynomial algebras and truncated polynomial algebras. If Γ is not finitely generated then Lemma 4.3 shows that Γ has infinite depth. If Γ is finitely generated, then it has finite depth and it is isomorphic to $P \otimes A$, where P is a polynomial algebra with $m = \operatorname{depth} \Gamma$ variables and A is a finite tensor product of truncated polynomial algebras. This proves Theorem 4.2 in the case where Γ is commutative. One way to think of the proof of Theorem 4.2 is that it works by reducing the general case to the commutative case by using the condition that Γ is nilpotent.

The results of [7] and [8] show the relevance of the Gorenstein condition to topology. We summarise these results as the following theorem.

Theorem 4.4 *Let X be a simply connected finite complex.*

(1) *The Hopf algebra $H_*(\Omega X;\mathbb{F})$ has finite depth. In fact,* $\operatorname{depth} X \le \operatorname{LSCat} X$*, where* $\operatorname{LSCat} X$ *denotes the Lyusternik–Schnirelman category of X.*

(2) *If the Hopf algebra $H_*(\Omega X;\mathbb{F})$ is Gorenstein, then X is a Poincaré duality space.*

In [8] Félix, Halperin and Thomas extend the Gorenstein condition to differential graded algebras and show that a finite complex X is a Poincaré duality space if and only if the cochain algebra $S^*(X;\mathbb{F})$ is a Gorenstein differential graded algebra. While it is true that if $H^*(X;\mathbb{F})$ is Gorenstein then so is $S^*(X;\mathbb{F})$, the reverse implication is not true; see [8, Examples 3.3].

If X is a finite complex, then we know that $H_*(\Omega X;\mathbb{F})$ has finite type and finite depth. The following theorem gives some useful practical ways to deduce, in addition, that $H_*(\Omega X;\mathbb{F})$ is elliptic. For the proof see [9, Theorem C].

Theorem 4.5 *Suppose Γ is a connected, co-commutative Hopf algebra over $\mathbb{F}$ of finite type and that Γ has finite depth. Then the following are equivalent:*

(1) *Γ is elliptic.*

(2) *Γ is nilpotent.*

(3) *Γ has polynomial growth.*

(4) *Γ is Gorenstein.*

4.3 The proof of Theorem 1.1

If M is a closed, connected, oriented manifold of finite dimension, then $H_*(\Omega M;\mathbb{F}_p)$ is connected and co-commutative, and it has finite type and finite depth. We are assuming it has polynomial growth. It follows from Theorem 4.5 that $H_*(\Omega M;\mathbb{F}_p)$ is elliptic. Therefore, from Theorem 4.2, it is a finitely generated free module over a central subalgebra P that is a polynomial algebra on a finite number, say l, of variables. It follows that $H_*(\Omega M;\mathbb{F}_p)$ has polynomial growth with exponent l and indeed l is the minimal exponent which can occur in the inequality for polynomial growth. In the notation of Theorem 1.1, $l = K_0$. This proves Theorem 1.1.

4.4 The proof of Theorem 1.2

It follows from Theorem 4.2 that if Γ is an elliptic Hopf algebra over $\mathbb{F}_p$, then Γ is doubly infinite if and only if the centre of Γ contains a polynomial algebra on two generators. Now let M be a simply connected closed manifold satisfying the hypotheses of Theorem 1.2. Then, as in the proof of Theorem 1.1, it follows that $H_*(\Omega M;\mathbb{F}_p)$ is an elliptic Hopf algebra. Suppose in addition that the algebra $H^*(M;\mathbb{F}_p)$ cannot be generated by one element. From Theorem 1.5, it follows that $H_*(\Omega M;\mathbb{F}_p)$ is doubly infinite and so the centre of $H_*(\Omega M;\mathbb{F}_p)$ contains a polynomial algebra on two generators. By Theorem 2.1 it follows that $\mathrm{HL}_*(M;\mathbb{F}_p)$ contains a polynomial algebra on two generators and therefore $H_*(LM;\mathbb{F}_p)$ is doubly infinite. The Gromoll–Meyer theorem, Theorem 1.3, completes the proof.

5 Application to homogeneous spaces

The following theorem is [10, Example 3.2].

Theorem 5.1 *Let G be a simply connected, compact Lie group and K a connected, closed subgroup of G. Then the homogeneous space G/K is $\mathbb{F}_p$ elliptic for any prime p.*

The proof uses the fibration

$$\Omega G \to \Omega(G/K) \to K$$

for which the fundamental group $\pi_1(K)$ acts trivially on the groups $H_*(\Omega G; F_p)$. Then a Leray–Serre spectral sequence argument may be applied because K and ΩG are both elliptic and hence have polynomial growth.

Now return to the list from [15]. The seven examples of homogeneous spaces in this list not covered by Proposition 3.1 and Theorem 3.2 are $\mathbb{F}_p$ elliptic spaces for any prime p by Theorem 5.1. Furthermore, in each case, there is a prime p such that the cohomology algebra of the homogeneous space cannot be generated by a single element. Therefore by Theorem 1.2 any metric has an infinite number of geometrically distinct closed geodesics.

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Relative left properness of colored operads

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The category of $\mathfrak{C}$–colored symmetric operads admits a cofibrantly generated model category structure. In this paper, we show that this model structure satisfies a relative left properness condition, ie that the class of weak equivalences between Σ–cofibrant operads is closed under cobase change along cofibrations. We also provide an example of Dwyer which shows that the model structure on $\mathfrak{C}$–colored symmetric operads is not left proper.

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1 Introduction

Operads are combinatorial devices that encode families of algebras defined by multilinear operations and relations. Common examples are the operads $\mathbb{A}$, $\mathbb{C}$ and $\mathbb{L}$ whose algebras are associative, associative and commutative, and Lie algebras, respectively. Colored operads are a bit more exotic, with what is likely the most famous example being Voronov's "Swiss-cheese operad", which models the genus-zero moduli spaces that appear in open-closed string theory. Other examples of colored operads[1] encode complicated algebraic structures such as operadic modules, enriched categories, and even categories of operads themselves. The study of model category structures on categories of colored operads has found many recent applications, including the rectification of diagrams of operads by Berger and Moerdijk, [4] and the construction of simplicial models for ∞–operads by Cisinski and Moerdijk [7].

Our goal in this paper is to further the study of the Quillen model category structure of colored operads initiated by the second author [26], Cisinski and Moerdijk [7], and Caviglia [6]. Specifically, we are interested in understanding if the category of colored, symmetric operads is *left proper*; ie we wish to know if weak equivalences between *all* colored, symmetric operads are closed under cobase change along cofibrations. The main result of this paper is to say that this is not the case, but we give sufficient

[1] Colored operads are also sometimes called (symmetric) multicategories in the literature.

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conditions on a monoidal model category $\mathcal{M}$ in order for the model category structure of $\mathcal{M}$–enriched, colored, symmetric operads to be *relatively* left proper, ie for the class of weak equivalences between Σ–cofibrant operads to be closed under cobase change along cofibrations (Theorem 3.1.10). Recall that in any model category, the class of weak equivalences between cofibrant objects is closed under cobase change along cofibrations. The class of Σ–cofibrant operads is much larger than the class of cofibrant operads; in particular, this class includes small examples such as the associative operad $\mathbb{A}$. If one is instead willing to consider the category of *reduced* (or *constant-free*) operads (those satisfying $\mathsf{P}\binom{c}{\varnothing} = \varnothing$), then Batanin and Berger [1] prove a strict left properness result.

The question of (relative) left properness for categories of symmetric operads has many immediate applications. As an example, left properness makes it easier to identify homotopy pushouts since, in a left proper model category, any pushout along a cofibration is a homotopy pushout. Relative left properness allows us to make similar statements.

Furthermore, understanding when left properness holds allows us to describe the rectification of homotopy coherent diagrams and weak maps between homotopy O–algebras, as first proposed by Berger and Moerdijk in [4, Section 6]. More explicitly, it is well known that the structure of a model category on the category of $\mathcal{M}$–enriched operads is important for the study of up-to-homotopy algebras over an operad such as $\mathbb{A}_\infty$–algebras and $\mathbb{E}_\infty$–algebras which are respectively associative and commutative "up to homotopy." The deformations of algebraic structures and morphisms between algebraic structures are controlled by up-to-homotopy resolutions of (colored) operads. These resolutions include the W-construction of Boardman and Vogt [5], the cobar-bar resolutions of Ginzburg and Kapranov [12] and Kontsevich and Soibelman [19], and the Koszul resolutions of Fresse [10]. In their paper [4], Berger and Moerdijk show that a coherent theory of up-to-homotopy resolutions of operadic algebras is provided by a Quillen model category structure on $\mathfrak{C}$–colored operads in a general monoidal model category $\mathcal{M}$. (Relative) left properness is one way to establish when these resolutions can be rectified, in the sense of being weakly homotopy equivalent to strict O–algebras.

Related work To the knowledge of the authors, the idea of relative left properness, and much of the inspiration for this paper, was first established in the thesis of Spitzweck [28] where he considers semi-model structures of categories of operads in general monoidal model categories. Similarly, Dwyer and Hess [8] and Muro [24] established a left properness result which is identical to that of Theorem 3.1.10 for nonsymmetric, monochromatic operads enriched in simplicial sets and monoidal model categories, respectively. Of particular note, Muro's proof requires that his monoidal model categories satisfy weaker conditions than those imposed on the monoidal model categories in this work. The stronger conditions in Theorem 3.1.10 are due to both the extra complexity introduced by the addition of the symmetric group actions and the authors' desire to exhibit the most direct proof of this result which still applies in many situations.

It must also be noted that one could obtain similar results using the techniques of the recent paper of Batanin and Berger [1]; see Remark 3.1.11. The actual definition of relative left properness in [1] is slightly different, though morally the same, as that used in Spitzweck [28], Muro [24], and this paper, and we have made note of similarities in their results and our own throughout this paper. Again, the authors of this work have made stronger assumptions on our enriching monoidal model category, as it is our belief that these assumptions allowed for greater clarity in the arguments while still being applicable in most cases of interest. These assumptions also allow for generalizations to more complicated cases such as relative left properness of dioperads and wheeled properads (see the authors' [13]), the latter of which is inaccessible to the Batanin–Berger machinery; see [1, Proposition 10.8]. These generalizations will serve as key components of the authors' larger body of work constructing models for ∞–wheeled properads.

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2 Colored operads and algebras

In this section, we briefly recall the definitions of colored operads and algebras over colored operads.

2.1 Colors and profiles

Throughout, let $(\mathcal{M}, \otimes, \mathbb{I})$ be a closed, symmetric monoidal category with all small colimits. Let $\varnothing$ denote the initial object of $\mathcal{M}$ and $\mathrm{Hom}(X, Y) \in \mathcal{M}$ the internal hom object. We will briefly give the necessary definitions and notations regarding colored objects in $\mathcal{M}$. A more complete discussion of the following definitions can be found in [31].

Definition 2.1.1 (colored objects) Fix a nonempty set of *colors*, $\mathfrak{C}$.

(1) A $\mathfrak{C}$–*profile* is a finite sequence of elements in $\mathfrak{C}$,

$$\underline{c} = (c_1, \ldots, c_m) = c_{[1,m]},$$

with each $c_i \in \mathfrak{C}$. If $\mathfrak{C}$ is clear from the context, then we simply say *profile*. The empty $\mathfrak{C}$–profile is denoted $\varnothing$, which is not to be confused with the initial object in $\mathcal{M}$. Write $|\underline{c}| = m$ for the *length* of a profile $\underline{c}$.

(2) An object in the product category $\prod_{\mathfrak{C}} \mathcal{M} = \mathcal{M}^{\mathfrak{C}}$ is called a *$\mathfrak{C}$–colored object in $\mathcal{M}$*; similarly, a map of $\mathfrak{C}$–colored objects is a map in $\prod_{\mathfrak{C}} \mathcal{M}$. A typical $\mathfrak{C}$–colored object X is also written as $\{X_a\}$ with $X_a \in \mathcal{M}$ for each color $a \in \mathfrak{C}$.

(3) Fix $c \in \mathfrak{C}$. An $X \in \mathcal{M}^{\mathfrak{C}}$ is said to be *concentrated in the color c* if $X_d = \varnothing$ for all $c \neq d \in \mathfrak{C}$.

(4) Similarly, fix $c \in \mathfrak{C}$. For $f\colon X \to Y \in \mathcal{M}$, we say that f is said to be *concentrated in the color c* if both X and Y are concentrated in the color c.

Now we are ready to define the colored version of Σ–objects underlying the category of colored operads. These objects are also sometimes called symmetric sequences, Σ–modules, or collections in the literature.

Definition 2.1.2 (colored symmetric sequences) Fix a nonempty set $\mathfrak{C}$.

(1) If $\underline{a}$ and $\underline{b}$ are $\mathfrak{C}$–profiles, then a *map* (or *left permutation*) $\sigma\colon \underline{a} \to \underline{b}$ is a permutation $\sigma \in \Sigma_{|\underline{a}|}$ such that

$$\sigma\underline{a} = (a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(m)}) = \underline{b}.$$

This necessarily implies $|\underline{a}| = |\underline{b}| = m$.

(2) The *groupoid of $\mathfrak{C}$–profiles*, which has $\mathfrak{C}$–profiles as the objects and left permutations as the isomorphisms, is denoted by $\Sigma_{\mathfrak{C}}$. The opposite groupoid, $\Sigma_{\mathfrak{C}}^{\mathrm{op}}$, is the groupoid of $\mathfrak{C}$–profiles with *right permutations*

$$\underline{a}\sigma = (a_{\sigma(1)}, \dots, a_{\sigma(m)})$$

as isomorphisms.

(3) The *orbit* of a profile $\underline{a}$ is denoted by $[\underline{a}]$. The maximal connected subgroupoid of $\Sigma_{\mathfrak{C}}$ containing $\underline{a}$ is written as $\Sigma_{[\underline{a}]}$. Its objects are the left permutations of $\underline{a}$. There is an *orbit decomposition* of $\Sigma_{\mathfrak{C}}$:

(2.1.2.1) $$\Sigma_{\mathfrak{C}} \cong \coprod_{[\underline{a}] \in \Sigma_{\mathfrak{C}}} \Sigma_{[\underline{a}]},$$

where there is one coproduct summand for each orbit $[\underline{a}]$ of a $\mathfrak{C}$–profile.

(4) Define the diagram category

(2.1.2.2) $$\mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M}) = \mathcal{M}^{\Sigma_{\mathfrak{C}}^{\mathrm{op}} \times \mathfrak{C}},$$

whose objects are called $\mathfrak{C}$–*colored symmetric sequences* or just *symmetric sequences* when $\mathfrak{C}$ is understood. The decomposition (2.1.2.1) implies that there is a decomposition

$$\mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M}) \cong \prod_{([\underline{c}];d)\in\Sigma_{\mathfrak{C}}^{\mathrm{op}}\times\mathfrak{C}} \mathcal{M}^{\Sigma_{[\underline{c}]}^{\mathrm{op}}\times\{d\}}, \tag{2.1.2.3}$$

where $\Sigma_{[\underline{c}]}^{\mathrm{op}} \times \{d\} \cong \Sigma_{[\underline{c}]}^{\mathrm{op}}$.

(5) For $X \in \mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M})$, we write

$$X\binom{d}{[\underline{c}]} \in \mathcal{M}^{\Sigma_{[\underline{c}]}^{\mathrm{op}}\times\{d\}} \cong \mathcal{M}^{\Sigma_{[\underline{c}]}^{\mathrm{op}}} \tag{2.1.2.4}$$

for its $([\underline{c}];d)$–component. For $(\underline{c};d) \in \Sigma_{\mathfrak{C}}^{\mathrm{op}} \times \mathfrak{C}$ (ie $\underline{c}$ is a $\mathfrak{C}$–profile and $d \in \mathfrak{C}$), we write

$$X\binom{d}{\underline{c}} \in \mathcal{M} \tag{2.1.2.5}$$

for the value of X at $(\underline{c};d)$.

(6) Write $\mathbb{N}(\mathfrak{C})$ for the set $\mathrm{Ob}(\Sigma_{\mathfrak{C}}^{\mathrm{op}} \times \mathfrak{C})$; ie an element in $\mathbb{N}(\mathfrak{C})$ is a pair $(\underline{c};d) \in \Sigma_{\mathfrak{C}}^{\mathrm{op}} \times \mathfrak{C}$.

Remark 2.1.3 In the case where $\mathfrak{C} = \{*\}$, for each integer $n \geq 0$, there is a unique $\mathfrak{C}$–profile of length n, usually denoted by $[n]$. We have $\Sigma_{[n]} = \Sigma_n$, which is just the symmetric group Σ_n regarded as a one-object groupoid. So we have

$$\mathbb{N}(\mathfrak{C}) = \mathbb{N}, \quad \Sigma_{\mathfrak{C}} = \coprod_{n\geq 0} \Sigma_n = \Sigma \quad \text{and} \quad \mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M}) = \mathcal{M}^{\Sigma_{\mathfrak{C}}^{\mathrm{op}}\times\mathfrak{C}} = \mathcal{M}^{\Sigma^{\mathrm{op}}}.$$

So one-colored symmetric sequences are symmetric sequences (also known as Σ–objects and collections) in the usual sense.

Unless otherwise specified, we will assume that $\mathfrak{C}$ is a *fixed*, nonempty set of colors.

2.2 Colored circle product

We define $\mathfrak{C}$–colored operads to be monoids in $\mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M})$ with respect to the $\mathfrak{C}$–colored circle product. In order to define the latter, we need the following definition.

Definition 2.2.1 (tensored over a groupoid) Suppose $\mathcal{D}$ is a small groupoid, $X \in \mathcal{M}^{\mathcal{D}^{\mathrm{op}}}$, and $Y \in \mathcal{M}^{\mathcal{D}}$. Define the object $X \otimes_{\mathcal{D}} Y \in \mathcal{M}$ as the colimit of the composite

$$\mathcal{D} \xrightarrow{\cong\Delta} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{(X,Y)} \mathcal{M} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M},$$

where the first map is the composite of the diagonal map and the isomorphism $\mathcal{D}\times\mathcal{D} \cong \mathcal{D}^{\mathrm{op}} \times \mathcal{D}$.

We mainly use the construction $\otimes_{\mathcal{D}}$ when $\mathcal{D}$ is the finite connected groupoid $\Sigma_{[\underline{c}]}$ for some orbit $[\underline{c}] \in \Sigma_{\mathfrak{C}}$.

Convention 2.2.2 For an object $A \in \mathcal{M}$, we take $A^{\otimes 0}$ to mean $\mathbb{I}$, the $\otimes$–unit in $\mathcal{M}$.

Definition 2.2.3 (colored circle product) Suppose $X, Y \in \mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M})$, $d \in \mathfrak{C}$, $\underline{c} = (c_1, \dots, c_m) \in \Sigma_{\mathfrak{C}}$, and $[\underline{b}] = [(\underline{b}_1, \dots, \underline{b}_m)] \in \Sigma_{\mathfrak{C}}$ is an orbit.

(1) Define the object $Y^{\underline{c}} \in \mathcal{M}^{\Sigma_{\mathfrak{C}}^{\mathrm{op}}} \cong \prod_{[\underline{b}] \in \Sigma_{\mathfrak{C}}} \mathcal{M}^{\Sigma_{[\underline{b}]}^{\mathrm{op}}}$ as having the $[\underline{b}]$–component

$$Y^{\underline{c}}([\underline{b}]) = \coprod_{\{[\underline{b}_j] \in \Sigma_{\mathfrak{C}}\}_{1 \le j \le m}} \mathsf{Lan}^{\Sigma_{[\underline{b}]}^{\mathrm{op}}} \left[\bigotimes_{j=1}^{m} Y\binom{c_j}{[\underline{b}_j]} \right] \in \mathcal{M}^{\Sigma_{[\underline{b}]}^{\mathrm{op}}}. \tag{2.2.3.1}$$

The Kan extension in (2.2.3.1) is defined as shown:

$$\begin{tikzcd} \prod_{j=1}^{m} \Sigma_{[\underline{b}_j]}^{\mathrm{op}} \arrow[r, "\prod Y\binom{c_j}{-}"] \arrow[d, "\text{concatenation}"'] & \mathcal{M}^{\times m} \arrow[d, "\otimes"] \\ \Sigma_{[\underline{b}]}^{\mathrm{op}} \arrow[r, "\text{left Kan extension}", "\mathsf{Lan}^{\Sigma_{[\underline{b}]}^{\mathrm{op}}}[\bigotimes Y(:)]"'] & \mathcal{M} \end{tikzcd}$$

(2) Considering left permutations of $\underline{c}$ in (2.2.3.1), we obtain $Y^{[\underline{c}]} \in \mathcal{M}^{\Sigma_{\mathfrak{C}}^{\mathrm{op}} \times \Sigma_{[\underline{c}]}} \cong \prod_{[\underline{b}] \in \Sigma_{\mathfrak{C}}} \mathcal{M}^{\Sigma_{[\underline{b}]}^{\mathrm{op}} \times \Sigma_{[\underline{c}]}}$ with components

$$Y^{[\underline{c}]}([\underline{b}]) \in \mathcal{M}^{\Sigma_{[\underline{b}]}^{\mathrm{op}} \times \Sigma_{[\underline{c}]}}. \tag{2.2.3.2}$$

(3) Using the product decomposition (2.1.2.3) of $\mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M})$, the *$\mathfrak{C}$–colored circle product* $X \circ Y \in \mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M})$ is defined to have components

$$(X \circ Y)\binom{d}{[\underline{b}]} = \coprod_{[\underline{c}] \in \Sigma_{\mathfrak{C}}} X\binom{d}{[\underline{c}]} \otimes_{\Sigma_{[\underline{c}]}} Y^{[\underline{c}]}([\underline{b}]) \in \mathcal{M}^{\Sigma_{[\underline{b}]}^{\mathrm{op}} \times \{d\}}, \tag{2.2.3.3}$$

where the coproduct is indexed by all the orbits in $\Sigma_{\mathfrak{C}}$, as d runs through $\mathfrak{C}$ and $[\underline{b}]$ runs through all the orbits in $\Sigma_{\mathfrak{C}}$. The construction $\otimes_{\Sigma_{[\underline{c}]}}$ is as defined in Definition 2.2.1.

Remark 2.2.4 In the one-colored case (ie $\mathfrak{C} = \{*\}$), the $\mathfrak{C}$–colored circle product is equivalent to the circle product of Σ–objects in [25, Section 2.2.3]. An anonymous referee made the authors aware that the idea to first define the circle product through Day's convolution belongs to G M Kelly [18].

The following observation is the colored version of [14, Proposition 4.13].

Proposition 2.2.5 *With respect to* $\circ$, $\mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M})$ *is a monoidal category.*

Remark 2.2.6 We consider $\mathcal{M}^{\mathfrak{C}}$ as a subcategory of $\mathcal{M}^{\mathbb{N}(\mathfrak{C})}$ via the inclusion

$$\mathfrak{C} \longrightarrow \mathbb{N}(\mathfrak{C}), \quad c \longmapsto \binom{c}{\varnothing}$$

We use this to consider $\mathsf{O} \circ -$ as a functor with domain $\mathcal{M}^{\mathfrak{C}}$ in Example 2.3.5.

2.3 Colored operads as monoids

In the previous section we show that the category of $\mathfrak{C}$–colored operads is a category of monoids "with many objects". We make this explicit below.

Definition 2.3.1 For a nonempty set $\mathfrak{C}$ of colors, denote by $\mathsf{Operad}^{\mathfrak{C}}(\mathcal{M})$, or $\mathsf{Operad}^{\mathfrak{C}}$ when $\mathcal{M}$ is understood, the category of monoids [20, Section VII.3] in the monoidal category $(\mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M}), \circ)$. An object in $\mathsf{Operad}^{\mathfrak{C}}$ is called a $\mathfrak{C}$*–colored operad* in $\mathcal{M}$. We write $\varnothing_{\mathfrak{C}}$ for the initial object in $\mathsf{Operad}^{\mathfrak{C}}$.

Remark 2.3.2 Unpacking Definition 2.3.1, a $\mathfrak{C}$–colored operad is equivalent to a triple (O, γ, u) consisting of

- $\mathsf{O} \in \mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M})$,
- a $\mathfrak{C}$*–colored unit* map

  $$\mathbb{I} \xrightarrow{u_c} \mathsf{O}\binom{c}{c} \in \mathcal{M}$$

  for each color $c \in \mathfrak{C}$, and
- *operadic composition*

$$\mathsf{O}\binom{d}{\underline{c}} \otimes \bigotimes_{i=1}^{m} \mathsf{O}\binom{c_i}{\underline{b}_i} \xrightarrow{\gamma} \mathsf{O}\binom{d}{\underline{b}} \in \mathcal{M} \tag{2.3.2.1}$$

  for all $d \in \mathfrak{C}$, $\underline{c} = (c_1, \ldots, c_m) \in \Sigma_{\mathfrak{C}}$ with $m \geq 1$, and $\underline{b}_i \in \Sigma_{\mathfrak{C}}$, where $\underline{b} = (\underline{b}_1, \ldots, \underline{b}_m)$.

The triple (O, γ, u) is required to satisfy the obvious associativity, unity, and equivariance axioms, the details of which can be found in [31, Definition 11.14] . The detailed axioms in the one-colored case can also be found in [23]. This way of expressing a $\mathfrak{C}$–colored operad is close to the way an operad was defined in [22].

Remark 2.3.3 In the case $\mathfrak{C} = \{*\}$, write Operad for $\mathsf{Operad}^{\mathfrak{C}}$. Objects of this category are called 1*–colored operads* or *monochromatic operads*. In this case, we write $\mathsf{O}(n)$ for the $([n];*)$–component of $\mathsf{O} \in \mathsf{Operad}$, where $[n]$ is the orbit of the $\{*\}$–profile consisting of n copies of $*$ (this orbit has only one object). Our notion of a 1–colored operad agrees with the notion of an operad in, eg [23] and [14]. Note that even for 1–colored operads, our definition is slightly more general than the one in [21, Section II.1.2] because, in our definition, the 0–component $\mathsf{O}(0)$ corresponds to the empty profile, $\{*\}$. In general, the purpose of the 0–component (whether in the one-colored or the general colored cases) is to encode units in O–algebras. Also note that in [22], where an operad was first defined in the topological setting, the 0–component was required to be a point.

Definition 2.3.4 Suppose $n \geq 0$. A $\mathfrak{C}$–colored symmetric sequence X is said to be *concentrated in arity n* if

$$|\underline{c}| \neq n \quad \Longrightarrow \quad X\binom{d}{\underline{c}} = \varnothing \text{ for all } d \in \mathfrak{C}.$$

Example 2.3.5 (1) A $\mathfrak{C}$–colored symmetric sequence concentrated in arity 0 is precisely a $\mathfrak{C}$–colored object. In the $\mathfrak{C}$–colored circle product $X \circ Y$ (2.2.3.3), if Y is concentrated in arity 0, then so is $X \circ Y$ because, by (2.2.3.1),

$$\underline{b} \neq \varnothing \quad \Longrightarrow \quad Y^{\underline{c}}([\underline{b}]) = \varnothing$$

for all $\underline{c}$. In other words, there is a lift:

$$\begin{tikzcd} \mathcal{M}^{\mathfrak{C}} \arrow[r, dotted] \arrow[d] & \mathcal{M}^{\mathfrak{C}} \arrow[d] \\ \mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M}) \arrow[r, "\mathsf{O}\circ -"] & \mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M}) \end{tikzcd}$$

So if O is a $\mathfrak{C}$–colored operad, then the functor

(2.3.5.1) $$\mathsf{O} \circ -\colon \mathcal{M}^{\mathfrak{C}} \longrightarrow \mathcal{M}^{\mathfrak{C}}$$

defines a monad [20, Section VI.1] whose monadic multiplication and unit are induced by the multiplication $\mathsf{O} \circ \mathsf{O} \to \mathsf{O}$ and the unit $\varnothing_{\mathfrak{C}} \to \mathsf{O}$, respectively.

(2) A $\mathfrak{C}$–colored operad O concentrated in arity 1 is exactly an $\mathcal{M}$–enriched category with object set $\mathfrak{C}$. In this case, the nontrivial operadic compositions correspond to the categorical compositions. Restricting further to the 1–colored case ($\mathfrak{C} = \{*\}$), a 1–colored operad concentrated in arity 1 is precisely a monoid in $\mathcal{M}$.

2.4 Algebras over colored operads

The category of representations over an operad O is referred to, for classical reasons, as the category of *algebras over an operad.*

Definition 2.4.1 Suppose O is a $\mathfrak{C}$–colored operad. The category of algebras over the monad [20, Section VI.2]

$$\mathsf{O} \circ -\colon \mathcal{M}^{\mathfrak{C}} \longrightarrow \mathcal{M}^{\mathfrak{C}}$$

in (2.3.5.1) is denoted by $\mathsf{Alg}(\mathsf{O}; \mathcal{M})$ or simply $\mathsf{Alg}(\mathsf{O})$ when $\mathcal{M}$ is understood. Objects of $\mathsf{Alg}(\mathsf{O})$ are called O*–algebras* (in $\mathcal{M}$).

Definition 2.4.2 Suppose $A = \{A_c\}_{c \in \mathfrak{C}} \in \mathcal{M}^{\mathfrak{C}}$ is a $\mathfrak{C}$–colored object. For $\underline{c} = (c_1, \dots, c_n) \in \Sigma_{\mathfrak{C}}$ with associated orbit $[\underline{c}]$, define the object

$$A_{\underline{c}} = \bigotimes_{i=1}^{n} A_{c_i} = A_{c_1} \otimes \cdots \otimes A_{c_n} \in \mathcal{M} \tag{2.4.2.1}$$

and the diagram $A_{[\underline{c}]} \in \mathcal{M}^{\Sigma_{[\underline{c}]}}$ with values

$$A_{[\underline{c}]}(\underline{c}') = A_{\underline{c}'} \tag{2.4.2.2}$$

for each $\underline{c}' \in [\underline{c}]$. All the structure maps in the diagram $A_{[\underline{c}]}$ are given by permuting the factors in $A_{\underline{c}}$.

Remark 2.4.3 (unwrapping O–algebras) From the definition of the monad $\mathsf{O} \circ -$, an O–algebra A has a structure map $\mu\colon \mathsf{O} \circ A \to A \in \mathcal{M}^{\mathfrak{C}}$. For each color $d \in \mathfrak{C}$, the d–colored entry of $\mathsf{O} \circ A$ is

$$(\mathsf{O} \circ A)_d = \coprod_{[\underline{c}] \in \Sigma_{\mathfrak{C}}} \mathsf{O}\binom{d}{[\underline{c}]} \otimes_{\Sigma_{[\underline{c}]}} A_{[\underline{c}]}. \tag{2.4.3.1}$$

So the d–colored entry of the structure map μ consists of maps

$$\mathsf{O}\binom{d}{[\underline{c}]} \otimes_{\Sigma_{[\underline{c}]}} A_{[\underline{c}]} \xrightarrow{\mu} A_d \in \mathcal{M}$$

for all orbits $[\underline{c}] \in \Sigma_{\mathfrak{C}}$. The $\otimes_{\Sigma_{[\underline{c}]}}$ here means that we can unpack μ further into maps

$$\mathsf{O}\binom{d}{\underline{c}} \otimes A_{\underline{c}} \xrightarrow{\mu} A_d \in \mathcal{M} \tag{2.4.3.2}$$

for all $d \in \mathfrak{C}$ and all objects $\underline{c} \in \Sigma_{\mathfrak{C}}$. Then an O–algebra is equivalent to a $\mathfrak{C}$–colored object A together with structure maps (2.4.3.2) that are associative, unital, and equivariant in an appropriate sense, the details of which can be found in [31, Corollary 13.37] . The detailed axioms in the 1–colored case can also be found in [23].

Note that when $\underline{c} = \varnothing$, the map (2.4.3.2) takes the form

$$\mathsf{O}\binom{d}{\varnothing} \xrightarrow{\mu} A_d \tag{2.4.3.3}$$

for $d \in \mathfrak{C}$. In practice, this 0–component of the structure map gives A the structure of d–colored units. For example, in a unital associative algebra, the unit arises from the 0–component of the structure map.

Remark 2.4.4 The $\mathfrak{C}$*–colored endomorphism operad*, $\mathrm{End}(A)$, is defined by

$$\mathrm{End}\binom{d}{\underline{c}} = \mathrm{Hom}_{\mathcal{M}}(A_{\underline{c}}, A_d).$$

It is an elementary exercise to check that, for an $\mathfrak{C}$–colored operad O, an O–algebra A is equivalent to a map of $\mathfrak{C}$–colored operads

$$\mathsf{O} \xrightarrow{\mu} \mathrm{End}(A).$$

Some important examples of colored operads and their algebras follow.

Example 2.4.5 (free operadic algebras) Fix a $\mathfrak{C}$–colored operad O. There is an adjoint pair

$$\mathcal{M}^{\mathfrak{C}} \overset{\mathsf{O}\circ -}{\underset{}{\rightleftarrows}} \mathsf{Alg}(\mathsf{O}) \tag{2.4.5.1}$$

in which the right adjoint is the forgetful functor. The left adjoint takes a $\mathfrak{C}$–colored object A to the object $\mathsf{O} \circ A$ which has the canonical structure of an O–algebra, called the *free* O*–algebra of* A. In particular, free O–algebras always exist.

Example 2.4.6 If O is an $\mathcal{M}$–enriched category, then the category of O–algebras is the $\mathcal{M}$–enriched functor category $[\mathsf{O}, \mathcal{M}]$.

Example 2.4.7 ($\mathfrak{C}$–colored operads as operadic algebras) First, recall that $\mathbb{N}(\mathfrak{C}) = \mathrm{Ob}(\Sigma_{\mathfrak{C}}^{\mathrm{op}} \times \mathfrak{C})$. For each nonempty set of colors $\mathfrak{C}$, there exists an $\mathbb{N}(\mathfrak{C})$–colored operad $\mathsf{Op}^{\mathfrak{C}}$ and an isomorphism

$$\mathsf{Operad}^{\mathfrak{C}} \cong \mathsf{Alg}(\mathsf{Op}^{\mathfrak{C}}). \tag{2.4.7.1}$$

So $\mathfrak{C}$–colored operads are equivalent to algebras over the $\mathbb{N}(\mathfrak{C})$–colored operad $\mathsf{Op}^{\mathfrak{C}}$. This is a special case of [31, Lemma 14.4], which describes any category of generalized props (of which $\mathsf{Operad}^{\mathfrak{C}}$ is an example) as a category of algebras over some colored operad; in the case $\mathfrak{C} = \{*\}$, this construction appears in [4, Example 1.5.6]. As mentioned in Example 2.4.5, it follows that *free* $\mathfrak{C}$*–colored operads* (= free $\mathsf{Op}^{\mathfrak{C}}$–algebras) always exist. The construction of $\mathsf{Op}^{\mathfrak{C}}$ begins with an $\mathbb{N}(\mathfrak{C})$–colored operad

$\mathsf{Op}^{\mathfrak{C}}_{\mathsf{Set}}$ in the symmetric monoidal category of sets and Cartesian products. There is a strong symmetric monoidal functor

$$\mathsf{Set} \longrightarrow \mathcal{M}, \quad S \longmapsto \coprod_S \mathbb{I}. \tag{2.4.7.2}$$

The colored operad $\mathsf{Op}^{\mathfrak{C}}$ is the entrywise image of $\mathsf{Op}^{\mathfrak{C}}_{\mathsf{Set}}$ under this strong symmetric monoidal functor. Therefore, if $\mathcal{M}$ has a model structure in which $\mathbb{I}$ is cofibrant, then $\mathsf{Op}^{\mathfrak{C}}$ is entrywise cofibrant. In fact, when $\mathbb{I}$ is cofibrant, a careful inspection of $\mathsf{Op}^{\mathfrak{C}}$ shows that its underlying symmetric sequence is cofibrant in $\mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M})$. This is a key example for us, and we will elaborate on it more later.

2.5 Limits and colimits of colored operadic algebras

Limits of $\mathsf{Alg}(\mathsf{O})$ are taken in the underlying category of colored objects $\mathcal{M}^{\mathfrak{C}}$ via the free-forgetful adjoint pair

$$\mathcal{M}^{\mathfrak{C}} \underset{}{\overset{\mathsf{O}\circ -}{\rightleftarrows}} \mathsf{Alg}(\mathsf{O})$$

in (2.4.5.1) for a $\mathfrak{C}$–colored operad. The following observation is the colored version of a well known result (see, for example [25, Proposition 2.3.5], [14, Proposition 5.15], or the closely related [9, Proposition II.7.2]).

Proposition 2.5.1 *Suppose O is a $\mathfrak{C}$–colored operad. Then the category $\mathsf{Alg}(\mathsf{O})$ has all small limits and colimits, with reflexive coequalizers and filtered colimits preserved and created by the forgetful functor $\mathsf{Alg}(\mathsf{O}) \to \mathcal{M}^{\mathfrak{C}}$.*

2.6 Model structure on colored operadic algebras

In this section, we will assume that our cocomplete, closed, symmetric monoidal category $\mathcal{M}$ comes with a compatible cofibrantly generated Quillen model category structure; ie we assume that $\mathcal{M}$ is a *monoidal model category* [27, Definition 3.1] with cofibrant tensor unit.

The category of $\mathfrak{C}$–colored objects, $\mathcal{M}^{\mathfrak{C}}$, admits a cofibrantly generated model category structure where weak equivalences, fibrations, and cofibrations are defined entrywise, as described in [15, Proposition 11.1.10]. In this model category, a generating cofibration in $\mathcal{M}^{\mathfrak{C}} = \prod_{\mathfrak{C}} \mathcal{M}$ (ie a map in I) is a generating cofibration of $\mathcal{M}$, concentrated in one entry. Similarly, the set of generating acyclic cofibrations is $\mathsf{J} \times \mathfrak{C}$. In addition, the properties of being simplicial, or proper, are inherited from $\mathcal{M}$.

A functor F between two symmetric monoidal categories is called *symmetric monoidal* if there is a unit $\mathbb{I} \to F(\mathbb{I})$ and a binatural transformation

$$F(-) \otimes F(-) \Rightarrow F(- \otimes -)$$

satisfying unit, associativity, and symmetry conditions [20].

Definition 2.6.1 We say that $\mathcal{M}$ admits *functorial path data* if there exist a symmetric monoidal functor Path on $\mathcal{M}$ and monoidal natural transformations

$$s\colon \mathrm{Id} \Rightarrow \mathrm{Path} \quad \text{and} \quad d_0, d_1\colon \mathrm{Path} \Rightarrow \mathrm{Id}$$

such that for any fibrant X in $\mathcal{M}$,

$$X \xrightarrow{s} \mathrm{Path}(X) \xrightarrow{d_0 \times d_1} X \times X$$

is a path object (ie s is a weak equivalence and $d_0 \times d_1$ is a fibration).

Remark 2.6.2 The definition of functorial path data is adapted from Fresse [11, Fact 5.3]. As a particular example, Fresse showed that functorial path data exists if $\mathcal{M}$ is the category of chain complexes over a ring of characteristic 0 or the category of simplicial modules.

One way to check if $\mathcal{M}$ admits functorial path data is to check if $\mathcal{M}$ admits an interval object defined as follows.

Definition 2.6.3 We say that $\mathcal{M}$ admits a *cocommutative, coassociative coalgebra interval* J if the fold map $\mathbb{I} \sqcup \mathbb{I} \to \mathbb{I}$ can be factored as

$$\mathbb{I} \sqcup \mathbb{I} \xrightarrow{\alpha} J \xrightarrow{\beta} \mathbb{I},$$

in which α is a cofibration, β is a weak equivalence, J is a coassociative cocommutative comonoid in $\mathcal{M}$, and α and β are both maps of comonoids.

For example, the categories of compactly generated spaces and simplicial sets admit such cocommutative coalgebra intervals. The category of unbounded chain complexes over a ring which is *not* characteristic 0 admits an interval which is coassociative, but not cocommutative.

Lemma 2.6.4 [17, Proposition 3.10] *If $\mathcal{M}$ admits a coassociative, cocommutative coalgebra interval and $\mathbb{I}$ is cofibrant, then $\mathcal{M}$ admits functorial path data.*

Definition 2.6.5 A *symmetric monoidal fibrant replacement functor* is a functor $f\colon \mathcal{M} \to \mathcal{M}$ together with a natural transformation $r\colon \mathrm{Id} \Rightarrow f$ such that

- $r_X\colon X \to f(X)$ is a fibrant replacement for each object X,
- f is a symmetric monoidal functor, and

- for every X and Y in $\mathcal{M}$, the following diagram commutes:

$$\begin{tikzcd} X \otimes Y \arrow[r, "r_{X\otimes Y}"] \arrow[d, "r_X \otimes r_Y"'] & f(X\otimes Y) \\ fX \otimes fY \arrow[ur] & \end{tikzcd}$$

Throughout this paper, we will want our monoidal model category $\mathcal{M}$ to satisfy a number of conditions, as we want $\mathcal{M}$ to have a symmetric monoidal fibrant replacement functor. To simplify the listing of these conditions, we make the following definition.

Definition 2.6.6 A monoidal model category $\mathcal{M}$ is called *nice* if

- $\mathcal{M}$ is strongly cofibrantly generated, ie the domain of each generating (acyclic) cofibration is small with respect to the entire category;
- there is a symmetric monoidal fibrant replacement functor;
- there is functorial path data;
- every object is cofibrant;
- weak equivalences are closed under filtered colimits.

Examples of nice monoidal model categories are sSet, $\mathbb{Z}$–graded chain complexes in characteristic zero, and simplicial presheaves.

Remark 2.6.7 The definition of a nice monoidal model category automatically implies that our monoidal model categories are what are called "strongly h-monoidal" in Batanin and Berger [1, Propositions 1.8, 2.5], and that our monoidal model categories satisfy the monoid axiom of Schwede and Shipley [27, Definition 3.3], which also makes an appearance in the work of Muro [24].

The following is a restricted version of [4, Theorem 2.1] and is a colored operad analogue of [17, Theorem 3.11] which dealt with the more complicated case of colored props.

Theorem 2.6.8 *Suppose $\mathcal{M}$ is a nice monoidal model category and that O is a $\mathfrak{C}$–colored operad in $\mathcal{M}$. Then $\mathsf{Alg}(\mathsf{O})$ admits a strongly cofibrantly generated model category structure, in which:*

- *fibrations and weak equivalences are created in $\mathcal{M}^{\mathfrak{C}}$,*
- *the set of generating cofibrations is $\mathsf{O} \circ \mathsf{I}$, where I is the set of generating cofibrations in $\mathcal{M}^{\mathfrak{C}}$, and*
- *the set of generating acyclic cofibrations is $\mathsf{O} \circ \mathsf{J}$, where J is the set of generating acyclic cofibrations in $\mathcal{M}^{\mathfrak{C}}$.*

Example 2.6.9 The category of simplicial sets, sSet, is a Cartesian closed, cofibrantly generated, monoidal model category that admits a coassociative, cocommutative interval. As a symmetric monoidal fibrant replacement functor, we can choose either the Ex^{∞} functor or the singular chain complex of the geometric realization functor, since both are product-preserving. Similarly, the category of $\mathbb{Z}$–graded chain complexes over a field $\mathbb{K}$ with the projective model structure [16, Chapter 2] satisfies the conditions of Theorem 2.6.8.

Corollary 2.6.10 *If $\mathcal{M}$ is a nice monoidal model category, then $\mathsf{Alg}(\mathsf{Op}^{\mathfrak{C}}) \cong \mathsf{Operad}^{\mathfrak{C}}$ admits a cofibrantly generated model structure.*

Definition 2.6.11
- The *fibrant* $\mathfrak{C}$–colored operads are those which are locally fibrant; ie $\mathsf{P}\binom{d}{\underline{c}}$ is fibrant in $\mathcal{M}$ for all profiles $(\underline{c}; d)$.
- A $\mathfrak{C}$–colored operad is called Σ*–cofibrant* if P is cofibrant as an object in $\mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M}) = \mathcal{M}^{\Sigma_{\mathfrak{C}}^{\mathrm{op}} \times \mathfrak{C}}$.

Every cofibrant operad is, in particular, Σ–cofibrant [3, Proposition 4.3].

Example 2.6.12 The associative operad $\mathbb{A}$ is the prototypical Σ–cofibrant operad which is not cofibrant. In sSet, the commutative operad $\mathbb{C}$ is neither Σ–cofibrant nor cofibrant.

3 Relative left properness of operads with fixed colors

In this section, we show that the model category structure of Corollary 2.6.10 satisfies a property close to that of left properness, to which we will refer as *relative* left properness.

Definition 3.0.1 The model category $\mathsf{Operad}^{\mathfrak{C}}$ is called left proper *relative to the class of Σ–cofibrant operads* if pushouts by cofibrations preserve weak equivalences whose domain and codomain are Σ–cofibrant.

3.1 The pushout filtration

Relative left properness of $\mathsf{Operad}^{\mathfrak{C}}$ comes down to a study of pushouts of $\mathfrak{C}$–colored operads where one of the defining maps is a free morphism of free operads (Lemma 3.1.6). To perform this analysis, we make use of the language of colored, planar trees such as those in [2, Section 5.8], [12] or [3, Section 3]. The following definition comes from [30, Chapter 3].

Definition 3.1.1 A *rooted n–tree* is a nonempty, finite, connected, directed graph with no directed cycles in which

(1) there are n distinguished vertices, called inputs, each with exactly one outgoing edge and no incoming edges;
(2) there is a distinguished vertex that is not an input, called the root, with exactly one incoming edge and no outgoing edges;
(3) each vertex away from the set of inputs and the root has exactly one outgoing edge.

A *planar rooted tree* is a rooted tree in which the set of incoming edges at each vertex is equipped with a linear ordering.

Remark 3.1.2 For a planar rooted tree T, we write $\mathrm{in}(T)$ for the set of its input edges. Since T is planar, the input edges (or leaves) have a linear order, and we write $\lambda(T)$ for the set of all such orderings

$$\{1,\dots,n\} \longrightarrow \mathrm{in}(T),$$

where $n = |\mathrm{in}(T)|$. It is fairly easy to check that one can identify the set $\lambda(T)$ of all linear orderings of the input edges of T with the group of permutations Σ_n.

Definition 3.1.3 Let $A \in \mathsf{SymSeq}_{\mathfrak{C}}(\mathcal{M})$ (Definition 2.1.2), and suppose that $m \geq 1$ and $t, s_j \in \mathbb{N}(\mathfrak{C})$ for $1 \leq j \leq m$.

(1) Denote by $\mathsf{Tree}(t)$ the groupoid of directed, planar, rooted, $\mathfrak{C}$–colored trees in which the input-output profile is given by t. The morphisms in $\mathsf{Tree}(t)$ are nonplanar isomorphisms of $\mathfrak{C}$–colored trees.
(2) Denote by $\mathsf{Tree}(\{s_j\}_1^m; t)$ the groupoid of pairs (T, ds) such that
   - $T \in \mathsf{Tree}(t)$, and
   - $\mathsf{ds} \subseteq \mathsf{Vt}(T)$ such that the set of vertex profiles in ds is the set $\{s_j\}_1^m$.

   Vertices in ds are called *distinguished vertices*. Vertices in the complement

   $$\mathsf{n}(T) \equiv \mathsf{Vt}(T) \setminus \mathsf{ds}$$

   are called *normal vertices*. Isomorphisms of $\mathsf{Tree}(\{s_j\}; t)$ are isomorphisms of $\mathfrak{C}$–colored trees which preserve the distinguished vertices and colorings of edges.
(3) A pair $(T, \mathsf{ds}) \in \mathsf{Tree}(\{s_j\}; t)$ is said to be *well-marked* if every flag of a distinguished vertex is part of an internal edge whose other end vertex is normal.
(4) A pair $(T, \mathsf{ds}) \in \mathsf{Tree}(\{s_j\}; t)$ is said to be *reduced* if it is well-marked and there are no adjacent normal vertices, ie every vertex adjacent to a normal vertex is distinguished. The groupoid of such reduced trees is denoted by $\mathsf{rTree}(\{s_j\}; t)$.

(5) Given a vertex u in a tree T, write $A(u)$ for the component of the symmetric sequence A corresponding to the profiles of u. In other words, if the profiles of u are $(\underline{c}; d) \in \mathbb{N}(\mathfrak{C})$, then $A(u) = A\binom{d}{\underline{c}}$. We also say that $A(u)$ is a *decoration* of u by A and that u is *A–decorated*. A tree with each vertex decorated by A is said to be *A–decorated*.

Definition 3.1.4 Suppose that $f\colon H \to G$ is a homomorphism of groups. Then there is an adjoint pair

$$(-) \cdot_H G\colon \mathcal{M}^{H^{\mathrm{op}}} \rightleftarrows \mathcal{M}^{G^{\mathrm{op}}} : f^*,$$

which is actually a Quillen adjunction [3, Lemma 2.5.1]. If f is a subgroup inclusion and $X \in \mathcal{M}$ is an object with a right H action (ie $X \in \mathcal{M}^{H^{\mathrm{op}}}$), we have

$$X \cdot_H G \cong \coprod_{G/H} X,$$

where the coproduct is indexed over the cosets of H in G.

The following definition appears in [14, Definition 7.10].

Definition 3.1.5 (Q–construction) Suppose there is a map $i\colon X \to Y \in \mathcal{M}$. The object $Q^t_q \in \mathcal{M}^{\Sigma_t}$ is given as follows.

- $Q^t_0 = X^{\otimes t}$.
- $Q^t_t = Y^{\otimes t}$.
- For $0 < q < t$, there is a pushout in $\mathcal{M}^{\Sigma_t}$:

(3.1.5.1)
$$\begin{tikzcd}
{[X^{\otimes(t-q)} \otimes Q^q_{q-1}] \cdot_{\Sigma_{t-q} \times \Sigma_q} \Sigma_t} \arrow[r] \arrow[d, "{(\mathrm{id}, i_*)}"'] \arrow[dr, phantom, "\Searrow"] & Q^t_{q-1} \arrow[d] \\
{[X^{\otimes(t-q)} \otimes Y^{\otimes q}] \cdot_{\Sigma_{t-q} \times \Sigma_q} \Sigma_t} \arrow[r] & Q^t_q
\end{tikzcd}$$

Lemma 3.1.6 *For $A \in \mathsf{Operad}^{\mathfrak{C}}$ and a map $i\colon X \to Y$ in $\mathcal{M}$, regarded as a map in $\mathcal{M}^{\mathbb{N}(\mathfrak{C})}$ concentrated in the s–entry for some $s \in \mathbb{N}(\mathfrak{C})$, consider a pushout*

$$\begin{tikzcd}
\mathsf{Op}^{\mathfrak{C}} \circ X \arrow[r, "f"] \arrow[d, "i_*"'] \arrow[dr, phantom, "\Searrow"] & A \arrow[d, "h"] \\
\mathsf{Op}^{\mathfrak{C}} \circ Y \arrow[r] & A_\infty
\end{tikzcd}$$

in $\mathsf{Operad}^{\mathfrak{C}}$. *Then for a fixed orbit* $[r]$, *with* $r \in \mathbb{N}(\mathfrak{C})$, *the* $[r]$*–entry of the map* h *is a countable composition*

$$A([r]) = A_0([r]) \xrightarrow{h_1} A_1([r]) \xrightarrow{h_2} A_2([r]) \xrightarrow{h_3} \cdots \longrightarrow A_\infty([r]),$$

where, for $k \geq 1$, *the* h_k *are inductively defined as the following pushout in* $\mathcal{M}^{\Sigma_{[r]}}$:

$$\begin{tikzcd}
\coprod_{[T,\mathsf{ds}]}\{[\bigotimes_{u\in\mathsf{n}(T)} A(u)] \otimes Q^k_{k-1}\} \cdot_{\mathrm{Aut}(T,\mathsf{ds})} \Sigma_{[r]} \arrow[r, "f^{k-1}_*"] \arrow[d, "\amalg(\mathrm{id}\otimes i^{\square k})\otimes_{\mathrm{Aut}(T,\mathsf{ds})}\mathrm{id}"'] & A_{k-1}([r]) \arrow[d, "h_k"] \\
\coprod_{[T,\mathsf{ds}]}\{\underbrace{[\bigotimes_{u\in\mathsf{n}(T)} A(u)] \otimes Y^{\otimes k}}_{\text{normal/dist. vertex decorations}}\} \cdot_{\mathrm{Aut}(T,\mathsf{ds})} \underbrace{\Sigma_{[r]}}_{\text{input labeling}} \arrow[r, "\xi_k"'] \arrow[ur, phantom, "\Searrow"] & A_k([r])
\end{tikzcd} \tag{3.1.6.1}$$

In this pushout:

(1) *The top horizontal map* f^{k-1}_* *is induced by* f *and the operad structure map of* A.

(2) *Each coproduct on the left is indexed by the set of weak isomorphism classes of reduced trees* (T,ds) *such that*

- *the input profile of* T *is in the orbit* $[r]$, *and*
- ds *consists of* k *distinguished vertices, all with profiles in the orbit* $[s]$.

Proof This theorem is a special case of Proposition 4.3.16 in [29] by taking $\mathsf{O} = \mathsf{Op}^{\mathfrak{C}}$; we sketch the proof. For each $r \in \mathbb{N}(\mathfrak{C})$, define

$$B([r]) = \operatorname*{colim}_k A_k([r]).$$

Then B has a canonical $\mathfrak{C}$–colored operad structure given as follows.

- Its colored units are those of A; ie $\mathbb{I} \to A\binom{c}{c} \to B\binom{c}{c}$ for each $c \in \mathfrak{C}$.
- The operadic $\circ_i$ compositions are given by the grafting of reduced trees, where the colored operad structure of A is used to bring the grafted tree to a reduced one if necessary.
- Its equivariant structure is given by the factors $\Sigma_{|\mathrm{in}(T)|}$.

The operad map $A \to B$ is induced by $A_0 \to B$. The map $Y \to B$ is induced by ξ_1 (for the s–corolla whose only vertex is distinguished) and $A_1 \to B$. That B is the pushout A_∞ follows from its inductive definition. □

For any finite group G, the category of G–objects, $\mathcal{M}^G$, has a natural structure of a cofibrantly generated model category, where weak equivalences and fibrations are defined entrywise, as described in [15, Proposition 11.1.10]. In this model category, a generating (acyclic) cofibration is a G–equivariant (acyclic) cofibration in the category of $\mathcal{M}$–objects with G–action. Because it will be important to keep track of which group we are working with, we will denoted these sets of generating cofibrations by $\mathsf{I}[G]$ and generating acyclic cofibrations by $\mathsf{J}[G]$.

The following lemma, due to Berger and Moerdijk [2, Lemma 5.10] and Spitzweck [28, Lemma 4], gives an equivariant version of the pushout product axiom.

Lemma 3.1.7 *Let G and Γ be finite groups with Γ acting from the right on G. For any Γ–cofibration $i\colon X \to Y$ and any map of right $G \rtimes \Gamma$–objects $A \to B$ whose underlying map is a cofibration in a nice monoidal model category $\mathcal{M}$, the induced map*

$$(X \otimes B) \amalg_{X \otimes A} (Y \otimes A) \longrightarrow Y \otimes B$$

is a $G \rtimes \Gamma$–cofibration, where $G \rtimes \Gamma$ acts on $Y \otimes B$ by $(y \otimes b)^{(g,\gamma)} = y^{\gamma} \otimes b^{(g,\gamma)}$.

In practice, Γ will be the symmetric group acting on the inputs of a tree T in rTree.

Lemma 3.1.8 *In the context of Lemma 3.1.6, suppose that*

- *$\mathcal{M}$ is a nice monoidal model category,*
- *$i\colon X \to Y \in \mathcal{M}$ is a cofibration, and*
- *A is a Σ–cofibrant operad.*

Then each map

$$\begin{array}{c}\left[\bigotimes_{u \in \mathsf{n}(T)} A(u)\right] \otimes Q^k_{k-1} \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes i^{\square k}} \\ \left[\bigotimes_{u \in \mathsf{n}(T)} A(u)\right] \otimes Y^{\otimes k}\end{array}$$

is an $\mathrm{Aut}(T, \mathsf{ds})$–cofibration.

Proof As in [2, Lemma 5.9], each (T, ds) has a grafting decomposition as

$$(T, \mathsf{ds}) = t_n((T_1, \mathsf{ds}_1), \ldots, (T_n, \mathsf{ds}_n)),$$

where

- t_n is the n–corolla,
- $\mathsf{ds} = \mathsf{ds}_1 \amalg \cdots \amalg \mathsf{ds}_n$ if the top vertex is not distinguished, and
- $\mathsf{ds} = \mathsf{ds}_1 \amalg \cdots \amalg \mathsf{ds}_n \amalg t_n$ if the top vertex is distinguished.

Let

$$(T_{j_1}, \mathsf{ds}_{j_1}), \dots, (T_{j_k}, \mathsf{ds}_{j_k}) \in \{(T_1, \mathsf{ds}_1), \dots, (T_n, \mathsf{ds}_n)\}$$

be such that each $(T_\ell, \mathsf{ds}_\ell)$ is isomorphic to exactly one $(T_{j_i}, \mathsf{ds}_{j_i})$, and let

$$n_i = \left|\left\{(T_\ell, \mathsf{ds}_\ell) \mid (T_\ell, \mathsf{ds}_\ell) \cong (T_{j_i}, \mathsf{ds}_{j_i})\right\}\right|.$$

There is a decomposition of the automorphism group,

$$\mathrm{Aut}(T, \mathsf{ds}) \cong \underbrace{\left(\prod_{i=1}^{k} \mathrm{Aut}(T_{j_i}, \mathsf{ds}_{j_i})^{\times n_i}\right)}_{G} \rtimes \underbrace{\left(\prod_{i=1}^{k} \Sigma_{n_i}\right)}_{\Gamma},$$

where each $n_i \geq 1$ and $n_1 + \cdots + n_k = n$.

(1) The map $i^{\square k}$ is a cofibration in $\mathcal{M}$ by the pushout-product axiom. Furthermore, it has a right $\mathrm{Aut}(T, \mathsf{ds})$–action (ie a $G \rtimes \Gamma$–action) because isomorphisms preserve distinguished vertices.

(2) Since $A(r)$ is Γ–cofibrant (where r is the vertex at the root) and Γ acts on $\bigotimes_{n(T)\setminus r} A(u)$ by permuting tensor factors, we know that $\bigotimes_{n(T)} A(u)$ is Γ–cofibrant.

These two facts and Lemma 3.1.7 together imply that

$$\mathrm{id} \otimes i^{\square k} = \left[\varnothing \longrightarrow \bigotimes A(u)\right] \square\, i^{\square k}$$

is a $G \rtimes \Gamma$–cofibration. $\square$

Lemma 3.1.9 *Suppose that $\mathcal{M}$ is a nice monoidal model category, and that $i\colon X \to Y$ is a cofibration in $\mathcal{M}$, regarded as a map in $\mathcal{M}^{\mathbb{N}(\mathfrak{C})}$ concentrated at the s–entry for some $s \in \mathbb{N}(\mathfrak{C})$. Suppose we have a diagram*

(3.1.9.1)
$$\begin{tikzcd}
\mathsf{Op}^{\mathfrak{C}} \circ X \arrow[r] \arrow[d, "i_*"'] & A \arrow[r, "f", "\sim"'] \arrow[d, "h^A"] & B \arrow[d, "h^B"] \\
\mathsf{Op}^{\mathfrak{C}} \circ Y \arrow[r] \arrow[ur, phantom, "\Searrow"] & A_\infty \arrow[r, "f_\infty"] \arrow[ur, phantom, "\Searrow"] & B_\infty
\end{tikzcd}$$

in $\mathsf{Alg}(\mathsf{Op}^{\mathfrak{C}}) \cong \mathsf{Operad}^{\mathfrak{C}}$ in which both squares are pushouts and $f\colon A \to B$ is a weak equivalence between Σ–cofibrant operads. Then f_∞ is also a weak equivalence between Σ–cofibrant operads.

Proof Weak equivalences in $\mathsf{Alg}(\mathsf{Op}^{\mathfrak{C}})$ are created entrywise in $\mathcal{M}$. The outer rectangle in (3.1.9.1) is also a pushout. It follows that h_k^A and h_k^B are filtered in such a way that for each orbit $[r]$, the $[r]$–entry of the k^{th} map is a pushout as in (3.1.6.1). There is a commutative ladder diagram in $\mathcal{M}^{\Sigma_{[r]}}$:

$$\begin{tikzcd}
A([r]) \arrow[r, equal] \arrow[d, "f"'] & A_0([r]) \arrow[r, "h_1^A"] \arrow[d, "f_0"'] & A_1([r]) \arrow[r, "h_2^A"] \arrow[d, "f_1"'] & \cdots \arrow[r] & \mathsf{colim}\ A_k([r]) = A_\infty([r]) \arrow[d, "f_\infty"] \\
B([r]) \arrow[r, equal] & B_0([r]) \arrow[r, "h_1^B"] & B_1([r]) \arrow[r, "h_2^B"] & \cdots \arrow[r] & \mathsf{colim}\ B_k([r]) = B_\infty([r])
\end{tikzcd}$$

We now argue that all the horizontal maps h_k^A and h_k^B are cofibrations in $\mathcal{M}^{\Sigma_{[r]}}$, and all the objects in the ladder diagram are cofibrant in $\mathcal{M}^{\Sigma_{[r]}}$. Each coproduct summand map on the left of (3.1.6.1) is a $\Sigma_{[r]}$–cofibration since

$$(-) \cdot_{\mathrm{Aut}(T,\mathsf{ds})} \Sigma_{[r]} \colon \mathcal{M}^{\mathrm{Aut}(T,\mathsf{ds})} \longrightarrow \mathcal{M}^{\Sigma_{[r]}}$$

is a left Quillen functor and each $\mathrm{id} \otimes i^{\square k}$ is an $\mathrm{Aut}(T,\mathsf{ds})$–cofibration by Lemma 3.1.8. But cofibrations are closed under coproducts and pushouts, so each h_k^A and h_k^B is a cofibration in $\mathcal{M}^{\Sigma_{[r]}}$. The fact that all objects are cofibrant now follows from the Σ–cofibrancy of A and B.

By [15, Proposition 15.10.12(1)], in order to show that the map f_∞ is a weak equivalence between cofibrant objects in $\mathcal{M}^{\Sigma_{[r]}}$, it suffices to show that all the vertical maps f_k, with $0 \le k < \infty$, are weak equivalences by induction on k.

The map f_0 is a weak equivalence by assumption. Suppose $k \ge 1$. Consider the commutative cube in $\mathcal{M}^{\Sigma_{[r]}}$, where the coproducts are taken over the same sets of trees as in (3.1.6.1):

$$\begin{tikzcd}[column sep=small]
\coprod\{[\bigotimes A(u)]\otimes Q_{k-1}^k\}\cdot_{\mathrm{Aut}(T,\mathsf{ds})}\Sigma_{[r]} \arrow[rr] \arrow[dd, "\coprod(\mathrm{Id}\otimes i^{\square k})_*"'] \arrow[dr, bend right, "f_*"] & & A_{k-1}([r]) \arrow[dr, "f_{k-1}"] \arrow[dd] & \\
& \coprod\{[\bigotimes B(u)]\otimes Q_{k-1}^k\}\cdot_{\mathrm{Aut}(T,\mathsf{ds})}\Sigma_{[r]} \arrow[rr, crossing over] & & B_{k-1}([r]) \arrow[dd] \\
\coprod\{[\bigotimes A(u)]\otimes Y^{\otimes k}\}\cdot_{\mathrm{Aut}(T,\mathsf{ds})}\Sigma_{[r]} \arrow[rr] \arrow[dr, bend right, "f_*"] & & A_k([r]) \arrow[dr, "f_k"] & \\
& \coprod\{[\bigotimes B(u)]\otimes Y^{\otimes k}\}\cdot_{\mathrm{Aut}(T,\mathsf{ds})}\Sigma_{[r]} \arrow[rr] \arrow[from=uu, crossing over] & & B_k([r])
\end{tikzcd}$$

Both the back face (with As) and the front face (with Bs) are pushout squares, and the maps from the back square to the front square are all induced by f. Moreover, f_{k-1} is a weak equivalence by the induction hypothesis. By Lemma 3.1.8, all the objects in the diagram are cofibrant in $\mathcal{M}^{\Sigma_{[r]}}$, and the vertical and diagonal maps are $\Sigma_{[r]}$–cofibrations. To see that f_k in the above diagram is a weak equivalence, it is enough to show, by the cube lemma [16, Lemma 5.2.6], that both maps labeled as f_* are weak equivalences.

To see that the top f_* in the above diagram is a weak equivalence, note that a coproduct of weak equivalences between cofibrant objects is again a weak equivalence by Ken Brown's lemma [16, Lemma 1.1.12]. The left Quillen functor (Definition 3.1.4) $(-)\cdot_{\mathrm{Aut}(T,\mathsf{ds})}\Sigma_{[r]}$ takes $\mathrm{Aut}(T,\mathsf{ds})$–cofibrations between $\mathrm{Aut}(T,\mathsf{ds})$–cofibrant objects to $\Sigma_{[r]}$–cofibrations between $\Sigma_{[r]}$–cofibrant objects. Now Ken Brown's lemma again says that it is enough to show that within each coproduct summand, the map

$$\Big[\bigotimes A(u)\Big]\otimes Q^k_{k-1} \xrightarrow{f_*} \Big[\bigotimes B(u)\Big]\otimes Q^k_{k-1} \tag{3.1.9.2}$$

is a weak equivalence between $\mathrm{Aut}(t,\mathsf{ds})$–cofibrant objects. Recall that weak equivalences in any diagram category in $\mathcal{M}$ are defined entrywise. The map

$$\Big[\bigotimes A(u)\Big] \xrightarrow{f_*} \Big[\bigotimes B(u)\Big]$$

is a finite tensor product of entries of f, each of which is a weak equivalence in $\mathcal{M}$. So this f_* is a weak equivalence between cofibrant objects, and tensoring this map with Q^k_{k-1} yields a weak equivalence.

A similar argument with $Y^{\otimes k}$ in place of Q^k_{k-1} shows that the bottom f_* in the commutative diagram is also a weak equivalence. Therefore, as discussed above, f_k is a weak equivalence, finishing the induction. $\square$

Theorem 3.1.10 *If $\mathcal{M}$ is a nice monoidal model category, then the cofibrantly generated model structure on $\mathsf{Alg}(\mathsf{Op}^{\mathfrak{C}}) \cong \mathsf{Operad}^{\mathfrak{C}}$ in Corollary 2.6.10 is left proper relative to the class of Σ–cofibrant operads.*

Proof The set of generating cofibrations in $\mathsf{Alg}(\mathsf{Op}^{\mathfrak{C}}) \cong \mathsf{Operad}^{\mathfrak{C}}$ is $\mathsf{Op}^{\mathfrak{C}}\circ \mathsf{I}$, where I is the set of generating cofibrations in $\mathcal{M}^{\mathbb{N}(\mathfrak{C})}$, each of which is concentrated in one entry and is a generating cofibration of $\mathcal{M}$ there. A general cofibration in $\mathsf{Alg}(\mathsf{Op}^{\mathfrak{C}})$ is a retract of a relative $(\mathsf{Op}^{\mathfrak{C}}\circ \mathsf{I})$–cell complex. So a retract and transfinite induction argument reduces the proof to the situation in Lemma 3.1.9. $\square$

Remark 3.1.11 An anonymous referee has pointed out that an alternative proof of Lemma 3.1.9 and Theorem 3.1.10 can be obtained using the machinery developed in [1]. Specifically, a modification of the proof of [1, Theorem 8.1], together with [1, Theorem 2.11, Proposition 2.14] would reproduce these results. The filtration on the pushout (3.1.9.1) would be different from the one we have used here, instead being based on "classifiers."

4 Categories of operads are not left proper

In this section, we present an illuminating counterexample to the category of $\mathfrak{C}$–colored operads being left proper. The example is due to Bill Dwyer, and we thank him for allowing us to present it in this paper.

Let $\mathcal{M}$ be the category of simplicial sets with the standard (Kan) model category structure, and fix $\mathfrak{C} = \{*\}$. In other words, we are working in just regular simplicial operads. Let $\varnothing$ denote the initial operad, and let $\varnothing_+$ denote the operad constructed by attaching a singleton in arity 0. In other words,

$$\varnothing(n) = \begin{cases} \{\mathrm{id}\} & n = 1, \\ \varnothing & n \neq 1, \end{cases} \qquad \varnothing_+(n) = \begin{cases} * & n = 0, \\ \{\mathrm{id}\} & n = 1, \\ \varnothing & n > 1. \end{cases}$$

The inclusion $i\colon \varnothing \to \varnothing_+$ is a cofibration of operads.

Given an operad A, we can construct the pushout

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, "i"] & A \arrow[d] \\ \varnothing_+ \arrow[r] & A_+ \end{tikzcd}$$

where $A_+(0) = \coprod_j A(j)/\Sigma_j$, and the map $A \to A_+$ is a cofibration of simplicial operads. If $\mathsf{Operad}^{\mathfrak{C}}$ were left proper, then in the pushout diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, "i"] & A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \varnothing_+ \arrow[r] & A_+ \arrow[r, "f_+"] & B_+ \end{tikzcd}$$

we would have that if f is a weak equivalence, then f_+ is a weak equivalence. Taking A to be an E_∞–operad and B to be the commutative operad, we know that $f\colon A \to B$ is a weak equivalence. On the other hand, in arity 0, f_+ is the map

$$f_+(0)\colon \coprod_j A(j)/\Sigma_j = \coprod_j E\Sigma_j/\Sigma_j = \coprod_j B\Sigma_j \longrightarrow \coprod_j B(j)/\Sigma_j = \coprod_j *.$$

This is not a weak equivalence since $B\Sigma_j$ is not contractible for $j > 1$.

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Function spaces and classifying spaces of algebras over a prop

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The goal of this paper is to prove that the classifying spaces of categories of algebras governed by a prop can be determined by using function spaces on the category of props. We first consider a function space of props to define the moduli space of algebra structures over this prop on an object of the base category. Then we mainly prove that this moduli space is the homotopy fiber of a forgetful map of classifying spaces, generalizing to the prop setting a theorem of Rezk.

The crux of our proof lies in the construction of certain universal diagrams in categories of algebras over a prop. We introduce a general method to carry out such constructions in a functorial way.

18D10, 18D50, 18G55, 55U10

Introduction

Associative algebras, Lie algebras, Poisson algebras and their variants play a key role in algebra, topology, category theory, differential and algebraic geometry, mathematical physics. They all share the common feature of being defined by operations with several inputs and one single output (the associative product, the Lie bracket, the Poisson bracket). A powerful device to handle such kinds of algebraic structure is the notion of operad, which has proven to be a fundamental tool to study algebras such as the aforementioned examples, feeding back important outcomes in these various fields. However, algebraic structures equipped not only with products but also with coproducts play a crucial role in various places in mathematics. It is worth mentioning, for instance, the following important examples: Hopf algebras in representation theory and mathematical physics, Frobenius algebras encompassing the Poincaré duality phenomenon in algebraic topology (which corresponds to unitary and counitary Frobenius bialgebras, see Kock [16]), Lie bialgebras introduced by Drinfeld in quantum group theory (see Drinfel'd [6; 7]), involutive Lie bialgebras originally encoding operations on free loops on surfaces in the work of Turaev [24] and then generalized to higher dimensional manifolds by Chas and Sullivan [4] in string topology [3]. A convenient way to handle such kinds of structure is to use the formalism of props, a generalization of operads

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encoding algebraic structures based on operations with several inputs and several outputs. A dg prop is a collection of complexes $P = \{P(m,n)\}_{m,n\in\mathbb{N}}$, where each $P(m,n)$ represents formal operations with m inputs and n outputs. This collection P is equipped with composition products grafting and concatenating these operations in a compatible way.

This paper is a follow-up to Yalin [25], where we set up a homotopy theory for algebras over (possibly colored) differential graded (dg for short) props. The crux of our approach lies in the proof that the Dwyer–Kan classifying spaces attached to categories of algebras over dg props are homotopy invariants of the dg prop. Such spaces have been introduced by Dwyer and Kan in their seminal work on simplicial localization of categories; see [10; 8; 9]. Recall from these papers that the classifying space of a category with weak equivalences (for instance a model category) is the nerve of its subcategory of weak equivalences. It encodes information about homotopy types and internal symmetries of the objects, ie their homotopy automorphisms. The goal of the present paper is to give another description of these classifying spaces, in terms of function spaces of dg prop morphisms, in order to make their homotopy theory accessible to computation. These function spaces are moduli spaces of algebra structures, that is, simplicial sets $P\{X\}$ whose vertices are dg prop morphisms $P \to \mathrm{End}_X$ representing a P–algebra structure on an object X of the base category. For us, the base category is the category $\mathcal{C}h$ of $\mathbb{Z}$–graded chain complexes over a field $\mathbb{K}$. Let $\mathcal{C}h^P$ be the category of P–algebras and $w\mathcal{C}h^P$ be its subcategory obtained by restriction to morphisms which are quasi-isomorphisms in $\mathcal{C}h$. Let us denote by $\mathcal{N}(-)$ the nerve of a category. Our main theorem reads as follows.

Theorem 0.1 *Let P be a cofibrant dg prop defined in the category $\mathcal{C}h$ of chain complexes and let $X \in \mathcal{C}h$. The commutative square*

$$\begin{tikzcd} P\{X\} \arrow[r] \arrow[d] & \mathcal{N}w\mathcal{C}h^P \arrow[d] \\ \{X\} \arrow[r] & \mathcal{N}w\mathcal{C}h \end{tikzcd}$$

is a homotopy pullback of simplicial sets.

As a consequence, we get the following decomposition of function spaces in terms of homotopy automorphisms.

Corollary 0.2 *We have*

$$P\{X\} \sim \coprod_{[X]} L\, w\mathcal{C}h^P(X,X),$$

where $L(-)$ is the simplicial localization functor of Dwyer–Kan [8], and $[X]$ ranges over the weak equivalence classes of P–algebras having X as underlying object. In particular, the simplicial monoids of homotopy automorphisms $L\,w\mathcal{C}h^{P}(X,X)$ are homotopically small in the sense of Dwyer–Kan, that is, their homotopy groups are all small as sets.

Theorem 0.1 is a broad generalization of the first main result of Rezk's thesis [21, Theorem 1.1.5], which concerns the case of operads in simplicial sets and simplicial modules. However, the method of [21] relies on the existence of a model category structure on algebras over operads, which does not exist anymore for algebras over dg props. The crux of the proof of Theorem 0.1 lies in the construction of functorial diagram factorizations in categories of algebras over dg props. We use a new approach, relying on universal categories of algebras over dg props, to perform such constructions in our context. This method enables us to get around the lack of model structure.

We would like to emphasize some links with two other objects encoding algebraic structures and their deformations. Theorem 0.1 asserts that we can use a function space of dg props, the moduli space $P\{X\}$, to determine classifying spaces of categories of algebras over dg props $\mathcal{N}\,w\mathcal{C}h^{P}$. The homotopy groups of this moduli space, in turn, can be approached by means of a Bousfield–Kan-type spectral sequence. The E_2–page of this spectral sequence is identified with the cohomology of certain deformation complexes. These complexes have been studied in Frégier, Markl and Yau [11], Markl [18] and Merkulov and Vallette [19]. These papers prove the existence of an L_∞–structure on such complexes which generalizes the intrinsic Lie bracket of Schlessinger and Stasheff [23]. We aim to apply this spectral sequence technique and provide new results about the deformation theory of bialgebras in an ongoing work. To complete this outlook, we point out that Ciocan-Fontanine and Kapranov in [5] used a similar approach to that of Rezk in order to define a derived moduli space of algebras structures in the formalism of dg schemes. The author recently proved in [27], by different methods, that the simplicial moduli spaces considered in the present paper are also the global points of derived moduli stacks in the setting of Toën and Vezzosi's derived algebraic geometry, and that the deformation complexes of [19] really compute the tangent complexes of these stacks.

Organization In Section 1, we briefly recall several properties of dg props and their algebras, and we define the notion of moduli space of algebraic structures. In Section 2, we revisit the notion of a colored dg prop as a symmetric monoidal dg category generated by words of colors. Then we carry out the main technical construction of this section: a dg category associated to the data of a small category $\mathcal{J}$ and a colored dg prop $P_{\mathcal{I}}$ encoding $\mathcal{I}$–diagrams of P–algebras, where $\mathcal{I}$ is a subcategory of $\mathcal{J}$. This category of "formal variables" is used to explain how a functorial $\mathcal{I}$–diagram of P–algebras can be extended to a functorial $\mathcal{J}$–diagram of P–algebras under several technical assumptions. This construction applies in particular to the functorial factorizations of morphisms provided by the axioms of model categories. In Section 3, we prove that the classifying space of quasi-isomorphisms of P–algebras is weakly equivalent to the classifying space of acyclic surjections of P–algebras. For this, we need to examine in Section 3 how the internal and external tensor products of a diagram category behave with respect to its injective and projective model structures. The projective case is more subtle and does not appear in the literature. Then we combine the results of Section 2, those of Section 3 and Quillen's Theorem A to provide this weak equivalence (induced by an inclusion of categories). Finally, in Section 4 we rely on the previous results to generalize [21, Theorem 1.1.5] to the dg prop setting.

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1 Props, algebras and moduli spaces

Throughout this paper, we work in the category $\mathcal{C}h$ of $\mathbb{Z}$–graded chain complexes over a field $\mathbb{K}$. We write "dg" as an abbreviation for "differential graded". We briefly recall our conventions and the main definitions concerning dg props in this section. We refer to [12] for a more comprehensive account.

1.1 Background on props and their algebras

An $\mathbb{S}$–biobject in $\mathcal{C}h$ is a double sequence $\{M(m,n) \in \mathcal{C}h\}_{(m,n)\in\mathbb{N}^2}$, where each $M(m,n)$ is equipped with a right action of the symmetric group on m letters Σ_m, a left action of the symmetric group on n letters Σ_n, such that these actions commute with each other.

Definition 1.1 A dg prop is an $\mathbb{S}$–biobject in $\mathcal{C}h$ endowed with associative horizontal composition products

$$\circ_h\colon P(m_1,n_1)\otimes P(m_2,n_2)\to P(m_1+m_2,n_1+n_2),$$

vertical associative composition products

$$\circ_v\colon P(k,n)\otimes P(m,k)\to P(m,n)$$

and units $\eta\colon \mathbb{K}\to P(n,n)$. These products satisfy the exchange law

$$(f_1\circ_h f_2)\circ_v(g_1\circ_h g_2)=(-1)^{|g_1||f_2|}(f_1\circ_v g_1)\circ_h(f_2\circ_v g_2)$$

and are compatible with the actions of symmetric groups and with the differentials. Morphisms of dg props are equivariant morphisms of collections compatible with the composition products. We denote by Prop the category of dg props.

The following definition shows how a given dg prop encodes algebraic operations on the tensor powers of a chain complex.

Definition 1.2 (1) The endomorphism dg prop of a chain complex X is given by $\mathrm{End}_X(m,n) = \mathrm{Hom}_{\mathcal{C}h}(X^{\otimes m}, X^{\otimes n})$, where $\mathrm{Hom}_{\mathcal{C}h}(-,-)$ is the internal hom bifunctor of $\mathcal{C}h$. The horizontal composition is given by the tensor product of homomorphisms and the vertical composition is given by the composition of homomorphisms.

(2) Let P be a dg prop. A P–algebra is a chain complex X equipped with a dg prop morphism $P \to \mathrm{End}_X$.

Hence any "abstract" operation of P is sent to an operation on X, and the way abstract operations compose under the composition products of P tells us the relations satisfied by the corresponding algebraic operations on X.

One can carry out similar constructions in the category of colored $\mathbb{S}$–biobjects in order to define colored dg props and their algebras.

Definition 1.3 Let C be a non-empty set, called the set of colors.

(1) A C–colored $\mathbb{S}$–biobject M is a double sequence of chain complexes

$$\{M(m,n)\}_{(m,n)\in\mathbb{N}^2},$$

where each $M(m,n)$ admits commuting right Σ_m action and left Σ_n action as well as a decomposition

$$M(m,n) = \bigoplus_{c_i,d_i\in C} M(c_1,\dots,c_m;d_1,\dots,d_n)$$

compatible with these actions. The objects $M(c_1,\dots,c_m;d_1,\dots,d_n)$ should be thought of as spaces of operations with colors $c_1,\dots,c_m$ indexing the m inputs and colors $d_1,\dots,d_n$ indexing the n outputs.

(2) A C–colored dg prop P is a C–colored $\mathbb{S}$–biobject endowed with a horizontal composition

$$\begin{aligned}\circ_h\colon P(c_{11},\dots,c_{1m_1};d_{11},\dots,d_{1n_1})\otimes\cdots\otimes P(c_{k1},\dots,c_{km_k};d_{k1},\dots,d_{kn_1})\\ \to P(c_{11},\dots,c_{km_k};d_{k1},\dots,d_{kn_k})\subseteq P(m_1+\cdots+m_k,n_1+\cdots+n_k)\end{aligned}$$

and a vertical composition

$$\circ_v\colon P(c_1,\dots,c_k;d_1,\dots,d_n)\otimes P(a_1,\dots,a_m;b_1,\dots,b_k) \\ \to P(a_1,\dots,a_m;d_1,\dots,d_n)\subseteq P(m,n),$$

which is equal to zero unless $b_i=c_i$ for $1\le i\le k$. These two compositions satisfy associativity axioms (we refer the reader to [14] for details).

Definition 1.4 (1) Let $\{X_c\}_C$ be a collection of chain complexes. The C–colored endomorphism dg prop $\mathrm{End}_{\{X_c\}_C}$ is defined by

$$\mathrm{End}_{\{X_c\}_C}(c_1,\dots,c_m;d_1,\dots,d_n)=\mathrm{Hom}_{\mathcal{C}h}(X_{c_1}\otimes\cdots\otimes X_{c_m},X_{d_1}\otimes\cdots\otimes X_{d_n}).$$

(2) Let P be a C–colored dg prop. A P–algebra is the data of a collection of objects $\{X_c\}_C$ and a C–colored dg prop morphism $P\to\mathrm{End}_{\{X_c\}_C}$.

Remark 1.5 Let $\mathcal{I}$ be a small category and let P be a dg prop. We can build an $\mathrm{ob}(\mathcal{I})$–colored dg prop $P_{\mathcal{I}}$ such that the $P_{\mathcal{I}}$–algebras are the $\mathcal{I}$–diagrams of P–algebras in $\mathcal{C}h$ in the same way as in [17]. We refer the reader to Definition 2.3 where this construction is made explicit.

We provide $\mathcal{C}h$ with the model category structure such that the weak equivalences are quasi-isomorphisms and fibrations are degreewise surjections. The category of $\mathbb{S}$–biobjects is a diagram category over $\mathcal{C}h$, so it inherits a cofibrantly generated model category structure in which weak equivalences and fibrations are defined componentwise. The free dg prop functor [12, Appendix A] gives rise to an adjunction $\mathcal{C}h^{\mathbb{S}}\rightleftarrows\mathrm{Prop}$ between the category of $\mathbb{S}$–biobjects $\mathcal{C}h^{\mathbb{S}}$ and the category of dg props Prop, which transfers this model category structure to the category of dg props.

Theorem 1.6 (see [12, Theorem 4.9] and [14, Theorem 1.1]) (1) *Suppose that* $\mathrm{char}(\mathbb{K})>0$. *The category* Prop_0 *of dg props with non-empty inputs (or outputs) equipped with the classes of componentwise weak equivalences and componentwise fibrations forms a cofibrantly generated semi-model category.*

(2) *Suppose that* $\mathrm{char}(\mathbb{K})=0$. *Then the entire category of dg props inherits a cofibrantly generated model category structure with the weak equivalences and fibrations as above.*

(3) *Suppose that* $\mathrm{char}(\mathbb{K})=0$. *Let C be a non-empty set. Then the category* Prop_C *of C–colored dg props forms a cofibrantly generated model category with fibrations and weak equivalences defined componentwise.*

A semi-model category structure is a slightly weakened version of a model category structure where the lifting axioms only work for cofibrations with cofibrant domain, and where the factorization axioms only work for a map with a cofibrant domain (see the relevant section of [12]). The notion of a semi-model category is sufficient to apply the usual constructions of homotopical algebra. A dg prop P has non-empty inputs if it satisfies

$$P(0,n) = \begin{cases} \mathbb{K} & \text{if } n=0, \\ 0 & \text{otherwise.} \end{cases}$$

We define in a symmetric way a dg prop with non-empty outputs. Such dg props usually encode algebraic structures without unit or without counit, for instance Lie bialgebras.

We will use all the time the existence of a (semi)-model category structure on dg props. Our results hold over a field of any characteristic: we can work alternatively with every dg prop in characteristic zero or with dg props with non-empty inputs/outputs in positive characteristic.

Finally, we recall from [12] the construction of the endomorphism dg prop associated to a diagram $F\colon \mathcal{J} \to \mathcal{C}h$,

$$\mathrm{End}_F(m,n) := \int_{i\in\mathcal{J}} \mathrm{Hom}_{\mathcal{C}h}(X_i^{\otimes m}, X_i^{\otimes n}),$$

where $X_i = F(i)$. This end can equivalently be defined as a coreflexive equalizer

$$\begin{tikzcd} \mathrm{End}_F(m,n) \arrow[r] & \prod_{i\in\mathcal{J}} \mathrm{Hom}_{\mathcal{C}h}(X_i^{\otimes m}, X_i^{\otimes n}) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \prod_{u\colon i\to j} \mathrm{Hom}_{\mathcal{C}h}(X_i^{\otimes m}, X_j^{\otimes n}) \arrow[l, bend left=40, "s_0"] \end{tikzcd}$$

where d_0 is the product of the maps

$$u_* = (F(u)^{\otimes n} \circ -)\colon \mathrm{Hom}_{\mathcal{C}h}(X_i^{\otimes m}, X_i^{\otimes n}) \to \mathrm{Hom}_{\mathcal{C}h}(X_i^{\otimes m}, X_j^{\otimes n})$$

induced by the morphisms $u\colon i \to j$ of $\mathcal{J}$, and d_1 is the product of the maps

$$u^* = (- \circ F(u)^{\otimes m})\colon \mathrm{Hom}_{\mathcal{C}h}(X_j^{\otimes m}, X_j^{\otimes n}) \to \mathrm{Hom}_{\mathcal{C}h}(X_i^{\otimes m}, X_j^{\otimes n}).$$

The section s_0 is the projection on the factors associated to the identities $\mathrm{id}\colon i \to i$. This construction allows us to characterize a diagram of P–algebras $F\colon \mathcal{J} \to \mathcal{C}h^P$, where $\mathcal{C}h^P$ is the category of P–algebras in chain complexes, as a dg prop morphism

$$P \to \mathrm{End}_{U(F)},$$

where $U(F)$ is the diagram of chain complexes underlying F.

1.2 Moduli spaces of algebra structures

Throughout the text, we use the Kan–Quillen model category structure on simplicial sets. A moduli space of algebra structures over a dg prop P, on a given chain complex X, is a simplicial set whose points are the dg prop morphisms $P \to \mathrm{End}_X$ and connected components are homotopy classes of P–algebra structures on X. Such a moduli space can be more generally defined on diagrams of chain complexes. We then deal with endomorphism dg props of diagrams. To properly construct such a simplicial set and give its first fundamental properties, we have to recall some results about cosimplicial and simplicial resolutions in a model category. For the sake of brevity and clarity, we refer the reader to [13, Chapter 16] for a complete treatment of the notions of simplicial resolutions, cosimplicial resolutions and Reedy model categories.

Definition 1.7 Let $\mathcal{M}$ be a model category and let X be an object of $\mathcal{M}$.

(1) A *cosimplicial resolution* of X is a cofibrant approximation to the constant cosimplicial object cc_*X in the Reedy model category structure on cosimplicial objects $\mathcal{M}^{\Delta}$ of $\mathcal{M}$.

(2) A *simplicial resolution* of X is a fibrant approximation to the constant simplicial object cs_*X in the Reedy model category structure on simplicial objects $\mathcal{M}^{\Delta^{op}}$ of $\mathcal{M}$.

Definition 1.8 Let $\mathcal{M}$ be a model category and let X be an object of $\mathcal{M}$.

(1) A *cosimplicial frame* on X is a cosimplicial object $\widetilde{X}$ in $\mathcal{M}$, together with a weak equivalence $\widetilde{X} \to cc_*X$ in the Reedy model category structure of $\mathcal{M}^{\Delta}$. It has to satisfy the two following properties: the induced map $\widetilde{X}^0 \to X$ is an isomorphism, and if X is cofibrant in $\mathcal{M}$ then $\widetilde{X}$ is cofibrant in $\mathcal{M}^{\Delta}$.

(2) A *simplicial frame* on X is a simplicial object $\widetilde{X}$ in $\mathcal{M}$, together with a weak equivalence $cs_*X \to \widetilde{X}$ in the Reedy model category structure of $\mathcal{M}^{\Delta}$. It has to satisfy the following two properties: the induced map $X \to \widetilde{X}^0$ is an isomorphism, and if X is fibrant in $\mathcal{M}$ then $\widetilde{X}$ is fibrant in $\mathcal{M}^{\Delta^{op}}$.

Proposition 1.9 [13, Proposition 16.1.9] *Let $\mathcal{M}$ be a model category. There exist functorial simplicial resolutions and functorial cosimplicial resolutions in $\mathcal{M}$.*

Proposition 1.10 [13, Proposition 16.6.3] *Let X be an object of $\mathcal{M}$.*

(1) *If X is cofibrant then every cosimplicial frame of X is a cosimplicial resolution of X.*

(2) *If X is fibrant then every simplicial frame of X is a simplicial resolution of X.*

In a model category $\mathcal{M}$, one can define homotopy mapping spaces $\mathrm{Map}_{\mathcal{M}}(-,-)$, which are simplicial sets equipped with a composition law defined up to homotopy. There are two possible definitions. We can take either $\mathrm{Map}_{\mathcal{M}}(X,Y) = \mathrm{Mor}_{\mathcal{M}}(X \otimes \Delta^{\bullet}, Y)$, where $(-)\otimes\Delta^{\bullet}$ is a cosimplicial resolution, or $\mathrm{Map}_{\mathcal{M}}(X,Y) = \mathrm{Mor}_{\mathcal{M}}(X, Y^{\Delta^{\bullet}})$, where $(-)^{\Delta^{\bullet}}$ is a simplicial resolution. When X is cofibrant and Y is fibrant, these two definitions give the same homotopy type of mapping space and have also the homotopy type of Dwyer and Kan's hammock localization $L^H(\mathcal{M}, w\mathcal{M})(X,Y)$, where $w\mathcal{M}$ is the subcategory of weak equivalences of $\mathcal{M}$; see [9]. Moreover, the set of connected components $\pi_0 \mathrm{Map}_{\mathcal{M}}(X,Y)$ is the set of homotopy classes $[X,Y]_{\mathcal{M}}$ in $\mathrm{Ho}(\mathcal{M})$.

Proposition 1.11 [13, Corollaries 16.5.3 and 16.5.4] *Let $\mathcal{M}$ be a model category and C a cosimplicial resolution in $\mathcal{M}$.*

1. *If Y is a fibrant object of $\mathcal{M}$, then $\mathrm{Mor}_{\mathcal{M}}(C,Y)$ is a fibrant simplicial set.*
2. *If $p\colon X \twoheadrightarrow Y$ is a fibration in $\mathcal{M}$, then $p_*\colon \mathrm{Mor}_{\mathcal{M}}(C,X) \twoheadrightarrow \mathrm{Mor}_{\mathcal{M}}(C,Y)$ is a fibration of simplicial sets, acyclic if p is so.*
3. *If $p\colon X \xrightarrow{\sim} Y$ is a weak equivalence of fibrant objects in $\mathcal{M}$, then*
$$p_*\colon \mathrm{Mor}_{\mathcal{M}}(C,X) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{M}}(C,Y)$$
is a weak equivalence of fibrant simplicial sets.

Definition 1.12 Let P be a cofibrant dg prop in $\mathcal{C}h$. Let X be a chain complex. The *moduli space of P–algebra structures* on X is the simplicial set defined by
$$P\{X\} = \mathrm{Mor}_{\mathrm{Prop}}(P \otimes \Delta^{\bullet}, \mathrm{End}_X),$$
where $(-)\otimes\Delta^{\bullet}$ is a functorial cosimplicial frame on Prop. We get a functor
$$\mathrm{Prop} \to \mathrm{sSet}, \quad P \mapsto P\{X\},$$
where sSet is the category of simplicial sets.

We can already get two interesting properties of these moduli spaces.

Lemma 1.13 *Let P be a cofibrant dg prop. For any chain complex X, the moduli space $P\{X\}$ is a fibrant simplicial set.*

Proof Every chain complex is fibrant, and fibrations of dg props are defined componentwise, so End_X is a fibrant dg prop. Given that P is cofibrant, the mapping space $P\{X\}$ is fibrant. □

In this case, the connected components of this moduli space are exactly the homotopy classes of P–algebra structures on X.

To conclude, let us note that these moduli spaces are a well defined homotopy invariant of algebraic structures over a given object.

Lemma 1.14 *Let X be a chain complex. Every weak equivalence of cofibrant dg props $P \xrightarrow{\sim} Q$ gives rise to a weak equivalence of fibrant simplicial sets*

$$Q\{X\} \xrightarrow{\sim} P\{X\}.$$

Proof Let $\varphi\colon P \to Q$ be a weak equivalence of cofibrant dg props. According to [13, Proposition 16.1.24], the map φ induces a Reedy weak equivalence of cosimplicial resolutions $P \otimes \Delta^\bullet \xrightarrow{\sim} Q \otimes \Delta^\bullet$. The dg prop End_X is fibrant, so we conclude by [13, Corollary 16.5.5] that this weak equivalence of cosimplicial resolutions induces a weak equivalence between the corresponding moduli spaces. □

Remark 1.15 The reader may have noticed that, using the existence of functorial cosimplicial resolutions, Definition 1.12, Lemma 1.13 and Lemma 1.14 could have been stated without the cofibrancy assumption on P. In this case, let

$$P^\bullet \xrightarrow{\sim} cc^\bullet P$$

be such a cosimplicial resolution of a dg prop P, and

$$\widetilde{P\{X\}} = \mathrm{Mor}_{\mathrm{Prop}}(P^\bullet, \mathrm{End}_X)$$

be this alternative construction of the moduli space. Let

$$P_\infty \xrightarrow{\sim} P$$

be a functorial cofibrant resolution of P. Then a cosimplicial frame on P_∞ is a cosimplicial resolution of P_∞ by Proposition 1.10, hence a cosimplicial resolution of P as well. By [13, Proposition 16.1.13], any two cosimplicial resolutions of a given object are related by a zigzag whose middle object is a fibrant cosimplicial resolution, and by [13, Corollary 16.5.5] a Reedy weak equivalence of cosimplicial resolutions induces a weak equivalence of mapping spaces, hence

$$\widetilde{P\{X\}} = \mathrm{Mor}_{\mathrm{Prop}}(P^\bullet, \mathrm{End}_X) \simeq \mathrm{Mor}_{\mathrm{Prop}}(P_\infty \otimes \Delta^\bullet, \mathrm{End}_X) = P_\infty\{X\}.$$

Our alternative construction of a moduli space directly from a dg prop P thus has the homotopy type of the moduli space of homotopy P–algebra structures constructed in Definition 1.12 from a cofibrant resolution of P.

2 Dg categories associated to colored dg props

2.1 Colored dg props as symmetric monoidal dg categories

We revisit the definition of colored dg props by explaining how they can alternatively be defined as symmetric monoidal dg categories "monoidally" generated by the set of colors. We start with two simple examples before explaining the general construction.

Example 2.1 Any dg prop in $\mathcal{C}h$ can alternatively be defined as a dg monoidal category $\mathtt{cat}(P)$ such that $\mathrm{ob}(\mathtt{cat}(P)) = \{x^{\otimes n}, n \in \mathbb{N}\}$ (where x is a formal variable), the tensor product is given by $x^{\otimes m} \otimes x^{\otimes n} = x^{\otimes(m+n)}$ and the complexes of morphisms by

$$\mathrm{Hom}_{\mathtt{cat}(P)}(x^{\otimes m}, x^{\otimes n}) = P(m,n).$$

The category of P–algebras consists of enriched symmetric monoidal functors

$$\mathtt{cat}(P) \to \mathcal{C}h$$

with their natural transformations.

Example 2.2 Let P be a (1–colored) dg prop. There exists a 2–colored dg prop $P_{x\to y}$ such that the category of $P_{x\to y}$–algebras is the category of morphisms $f\colon X \to Y$ in the category of P–algebras $\mathcal{C}h^P$. It has two colors x, y and it is generated for the composition products by $P(x,\dots,x;x,\dots,x) = P(m,n)$, by $P(y,\dots,y;y,\dots,y) = P(m,n)$, and by an element $f \in P(x,y)$ which represents an abstract arrow $f\colon x \to y$. The associated dg monoidal category $\mathtt{cat}(P_{x\to y})$ is defined in the following way. Let $\mathrm{Free}_{\mathrm{mon}}(x,y)$ be the monoid freely generated by the two generators x and y, ie the set of words in two letters $w \in \mathrm{Free}_{\mathrm{mon}}(x,y)$. Then the objects of $\mathtt{cat}(P_{x\to y})$ are the "monoidal words"

$$\mathrm{ob}(\mathtt{cat}(P_{x\to y})) = \{w_\otimes(x,y), w \in \mathrm{Free}_{\mathrm{mon}}(x,y)\},$$

where $w_\otimes(x,y)$ is the formal tensor product corresponding to the word w. The complexes of morphisms are

$$\mathrm{Hom}_{\mathtt{cat}(P_{x\to y})}(w_\otimes(x,y), v_\otimes(x,y)) = P_{x\to y}(\underline{w}, \underline{v})$$

where $\underline{w}$ is the ordered sequence of letters, ie colors, appearing in the word w. Algebras over $P_{x\to y}$ are enriched symmetric monoidal functors $\mathtt{cat}(P_{x\to y}) \to \mathcal{C}h$. A $P_{x\to y}$–algebra is equivalent to a diagram of P–algebras $\{\bullet \to \bullet\} \to \mathcal{C}h^P$.

These constructions can be generalized to arbitrary diagrams as follows.

Definition 2.3 Let $\mathcal{I}$ be a small category. Then there exists an $\mathrm{ob}(\mathcal{I})$–colored dg prop $P_{\mathcal{I}}$ consisting of abstract objects x_i associated to $i \in \mathcal{I}$, and the morphisms of $P_{\mathcal{I}}$ are generated by operations $p \in P_{\mathcal{I}}(x_i^{\otimes m}, x_i^{\otimes n})$ associated to each $p \in P(m,n)$ and each variable x_i, as well as abstract arrows $f\colon x_i \to x_j$ associated to the morphisms of $\mathcal{I}$. The corresponding dg monoidal category $\mathtt{cat}(P_{\mathcal{I}})$ is defined as follows:

$$\mathrm{ob}(\mathtt{cat}(P_{\mathcal{I}})) = \{w_{\otimes}(x_i, i \in \mathrm{ob}(\mathcal{I})),\ w \in \mathrm{Free}_{\mathrm{mon}}(x_i, i \in \mathrm{ob}(\mathcal{I}))\}.$$

The tensor product is defined by

$$w_{\otimes}(x_i,\ i \in \mathrm{ob}(\mathcal{I})) \otimes v_{\otimes}(x_i,\ i \in \mathrm{ob}(\mathcal{I})) = (w * v)_{\otimes}(x_i,\ i \in \mathrm{ob}(\mathcal{I})).$$

The complexes of morphisms are

$$\mathrm{Hom}_{\mathtt{cat}(P_{\mathcal{I}})}(w_{\otimes}(x_i,\ i \in \mathrm{ob}(\mathcal{I})), v_{\otimes}(x_i,\ i \in \mathrm{ob}(\mathcal{I}))) = P_{\mathcal{I}}(\underline{w}, \underline{v}).$$

The composition on the dg hom is the vertical composition product of $P_{\mathcal{I}}$, and the tensor product of morphisms is the horizontal composition product of $P_{\mathcal{I}}$.

In other words, the category $\mathtt{cat}(P_{\mathcal{I}})$ is a differential graded monoidal category monoidally generated on objects by the set of colors of $P_{\mathcal{I}}$. This can be generalized in any symmetric monoidal category, giving an alternative definition of a colored dg prop.

Definition 2.4 (1) A C–colored dg prop is a small symmetric monoidal dg category monoidally generated by C.

(2) A $P_{\mathcal{I}}$–algebra is a symmetric monoidal dg functor $\mathtt{cat}(P_{\mathcal{I}}) \to \mathcal{C}h$.

Proposition 2.5 *A $P_{\mathcal{I}}$–algebra corresponds to an $\mathcal{I}$–diagram of P–algebras.*

This result follows from the construction of $P_{\mathcal{I}}$ in terms of generators and relations. For more details we refer the reader to [17, Section 2], where such a construction is carried out in the case of colored dg operads.

2.2 Categories of universal twisted sums and functorial diagrams of algebras

Let $P_{\mathcal{J}}$ be a colored prop on a small category $\mathcal{J}$. The category $\mathtt{cat}(P_{\mathcal{J}})$ reflects the universal structures of the symmetric monoidal category defined by a P–algebra in the category of chain complexes. But for some constructions of homotopy theory, we need operations of the ambient category of chain complexes which lie outside the image of this category $\mathtt{cat}(P_{\mathcal{J}})$. Namely, we need to perform direct sums $C \oplus D$, suspensions ΣC, and twisted complexes (C, d) which we form by adding a twisting homomorphism

$d \in \mathrm{Hom}(C, C)$ to the internal differential of a chain complex $\delta\colon C \to C$. These operations can clearly not be formed within the image of $\mathtt{cat}(P_{\mathcal{J}})$ in the category of chain complexes in general. Therefore, we define a universal enriched category $\mathtt{TwSum}(P_{\mathcal{J}})$ generated by the formal image of the tensor products $w(x_j, j \in \mathcal{J}) \in \mathrm{ob}(\mathtt{cat}(P_{\mathcal{J}}))$ under such direct sum, suspension and twisting operations. If we put all these operations together, then we get the notion of a twisted direct sum which we formalize in our definition. Let us simply mention that we use formal tensor products $\mathbb{K}e \otimes V$, where $\mathbb{K}e$ is the free $\mathbb{K}$–module spanned by a homogeneous element of degree $d = \deg(e)$, to create a d–fold suspension operation $\Sigma^d\colon C \mapsto \Sigma^d C$. In the sequel, our idea is to define universal models of the homotopical construction which we need to work out our problems in this enriched category $\mathtt{TwSum}(P_{\mathcal{J}})$.

2.2.1 Construction of the category of universal twisted sums Let $\mathcal{J}$ be a small category and $P_{\mathcal{J}}$ the associated $\mathrm{ob}(\mathcal{J})$–colored dg prop. Our goal is to build from $\mathtt{cat}(P_{\mathcal{J}})$ a certain dg monoidal category $\mathtt{TwSum}(P_{\mathcal{J}})$ called its category of universal twisted sums. The objects are pairs

$$\Big(\bigoplus_{\underline{\alpha}\in A}(\mathbb{K}\mathbb{e}_{\underline{\alpha}}) \otimes (x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}), \mathtt{tw}\Big),$$

where

- the first term $\bigoplus_{\underline{\alpha}\in A}$ is a formal sum over a finite set A of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ of formal tensor products $(\mathbb{K}\mathbb{e}_{\underline{\alpha}}) \otimes (x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n})$, where $x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}$ is an object of $\mathtt{cat}(P_{\mathcal{J}})$ and we consider the graded $\mathbb{K}$–module $\mathbb{K}\mathbb{e}_{\underline{\alpha}}$ generated by a homogeneous element $\mathbb{e}_{\underline{\alpha}}$ of a certain degree $d_{\underline{\alpha}} = \deg(\mathbb{e}_{\underline{\alpha}})$;
- the second term represents a collection of homomorphisms

  $$\mathtt{tw}_{\underline{\alpha}\,\underline{\beta}} \in \mathbb{e}_{\underline{\alpha}} \otimes \mathbb{e}_{\underline{\beta}}^{\vee} \otimes \mathrm{Hom}_{\mathtt{cat}(P_{\mathcal{J}})}(x_{\beta_1} \otimes \cdots \otimes x_{\beta_m}, x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}),$$

  indexed by the couples $(\underline{\alpha}, \underline{\beta}) \in A^2$, homogeneous of degree -1, that satisfy the relation of twisting cochains

  $$\delta(\mathtt{tw}_{\underline{\alpha}\underline{\beta}}) + \sum_{\underline{\gamma}\in A} \mathtt{tw}_{\underline{\alpha}\underline{\gamma}} \circ \mathtt{tw}_{\underline{\gamma}\underline{\beta}} = 0$$

  in the dg-module

  $$\mathbb{e}_{\underline{\alpha}} \otimes \mathbb{e}_{\underline{\beta}}^{\vee} \otimes \mathrm{Hom}_{\mathtt{cat}(P_{\mathcal{J}})}(x_{\beta_1} \otimes \cdots \otimes x_{\beta_m}, x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}),$$

  for every couple $(\underline{\alpha}, \underline{\beta}) \in A^2$ of sequences of colors. The notation $\mathbb{e}_{\underline{\beta}}^{\vee}$ represents an element which is dual to $\mathbb{e}_{\underline{\beta}}$, homogeneous of degree $\deg(\mathbb{e}_{\underline{\beta}}^{\vee}) = -\deg(\mathbb{e}_{\underline{\beta}})$, and we use the relation $\mathbb{e}_{\underline{\beta}}^{\vee}(\mathbb{e}_{\underline{\beta}}) = 1$ when we form the composites $\mathtt{tw}_{\underline{\alpha}\,\underline{\gamma}} \circ \mathtt{tw}_{\underline{\gamma}\,\underline{\beta}}$.

We define the dg-modules of homomorphisms of $\mathtt{TwSum}(P_{\mathcal{J}})$ as the twisted sums of dg-modules; that is, for

$$L = \Big(\bigoplus_{\underline{\beta} \in B} (\mathbb{K}\mathbb{e}_{\underline{\beta}}) \otimes (x_{\beta_1} \otimes \cdots \otimes x_{\beta_m}), \mathtt{tw}_L \Big),$$
$$K = \Big(\bigoplus_{\underline{\alpha} \in A} (\mathbb{K}\mathbb{e}_{\underline{\alpha}}) \otimes (x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}), \mathtt{tw}_K \Big),$$

we define

$$\mathrm{Hom}_{\mathtt{TwSum}(P_{\mathcal{J}})}(L,K) := \Big(\bigoplus_{(\underline{\beta}\,\underline{\alpha}) \in B \times A} \mathbb{K}\mathbb{e}_{\underline{\alpha}} \otimes \mathbb{K}\mathbb{e}_{\underline{\beta}}^{\vee} \otimes \mathrm{Hom}_{\mathtt{cat}(P_{\mathcal{J}})}(x_{\beta_1} \otimes \cdots \otimes x_{\beta_m}, x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}), \partial \Big),$$

with twisting homomorphism ∂: $(f_{\underline{\beta}\,\underline{\alpha}}) \mapsto (\partial(f)_{\underline{\beta}\,\underline{\alpha}})$ such that

$$\partial(f)_{\underline{\beta}\,\underline{\alpha}} = \sum_{\underline{\gamma} \in B} \mathtt{tw}_{\underline{\beta}\,\underline{\gamma}} \circ f_{\underline{\gamma}\,\underline{\alpha}} - \sum_{\underline{\gamma} \in A} \mathrm{sign}(f) f_{\underline{\beta}\,\underline{\gamma}} \circ \mathtt{tw}_{\underline{\gamma}\,\underline{\alpha}}$$

for every couple $(\underline{\alpha}, \underline{\beta})$ of sequences of colors, where $\mathrm{sign}(f)$ is a sign depending on f.

Claim This endows $\mathtt{TwSum}(P_{\mathcal{J}})$ with a dg category structure.

Proof We equip this dg hom $\mathrm{Hom}_{\mathtt{TwSum}(P_{\mathcal{J}})}(K,L)$ with the total differential $\delta + \partial$, where δ is the internal differential induced by the differential of P and ∂ is the twisting homomorphism. The fact that $(\delta+\partial)^2 = 0$ follows from the relation of twisting cochains satisfied by the $\mathtt{tw}$ with respect to δ. Indeed, for each $\underline{\beta} \in B, \underline{\alpha} \in A$, we have

$$(\delta+\partial)^2(f)_{\underline{\beta},\underline{\alpha}} = (\delta(\partial) + \partial^2)(f)_{\underline{\beta},\underline{\alpha}},$$

where $\delta(\partial)$ is the usual differential of a homomorphism defined by the commutator

$$\delta(\partial) = \delta \circ \partial - (-1)^{\deg(\partial)} \partial \circ \delta = \delta \circ \partial + \partial \circ \delta.$$

We have

$$\delta(\partial)(f)_{\underline{\beta},\underline{\alpha}} = \delta(\partial(f)_{\underline{\beta},\underline{\alpha}}) + \partial(\delta(f))_{\underline{\beta},\underline{\alpha}} = \delta(\mathtt{tw}_{\underline{\beta},\underline{\alpha}})(f),$$

and

$$\begin{aligned}\partial^2(f)_{\underline{\beta},\underline{\alpha}} = \partial(\partial(f))_{\underline{\beta},\underline{\alpha}} &= \sum_{\underline{\gamma}\in B} \mathtt{tw}_{\underline{\beta}\,\underline{\gamma}} \circ \partial(f)_{\underline{\gamma}\,\underline{\alpha}} - \sum_{\underline{\gamma}\in A} \operatorname{sign}(\partial(f))\partial(f)_{\underline{\beta}\,\underline{\gamma}} \circ \mathtt{tw}_{\underline{\gamma}\,\underline{\alpha}} \\ &= \sum_{\underline{\gamma}\in B} \mathtt{tw}_{\underline{\beta}\,\underline{\gamma}} \circ \partial(f)_{\underline{\gamma}\,\underline{\alpha}} + \sum_{\underline{\gamma}\in A} \operatorname{sign}(f)\partial(f)_{\underline{\beta}\,\underline{\gamma}} \circ \mathtt{tw}_{\underline{\gamma}\,\underline{\alpha}} \\ &= \Big(\sum_{\underline{\gamma}\in B} \mathtt{tw}_{\underline{\beta}\,\underline{\gamma}} \circ \mathtt{tw}_{\underline{\gamma}\,\underline{\alpha}}\Big)(f)\end{aligned}$$

because $\operatorname{sign}(\partial(f)) = \operatorname{sign}(f) - 1$ (the homomorphism ∂ is of degree -1), so

$$(\delta+\partial)^2(f)_{\underline{\beta},\underline{\alpha}} = \Big(\delta(\mathtt{tw}_{\underline{\beta},\underline{\alpha}}) + \sum_{\underline{\gamma}\in B} \mathtt{tw}_{\underline{\beta}\,\underline{\gamma}} \circ \mathtt{tw}_{\underline{\gamma}\,\underline{\alpha}}\Big)(f) = 0.$$

For each object

$$K = \Big(\bigoplus_{\underline{\alpha}} \mathbb{K}\mathbb{e}_{\underline{\alpha}} \otimes (x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}), \mathtt{tw}_K\Big)$$

of $\mathtt{TwSum}(P_{\mathcal{J}})$, the associated identity element in the dg hom $\operatorname{Hom}_{\mathtt{TwSum}(P_{\mathcal{J}})}(K,K)$ is the 0–cycle defined by

$$\bigoplus_{\underline{\alpha}} (\mathbb{K}\mathbb{e}_{\underline{\alpha}}) \otimes \mathbb{K}\mathbb{e}_{\underline{\alpha}}^{\vee} \otimes \mathrm{id}_{x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_n}},$$

where $\mathrm{id}_{x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_n}}$ is the identity on the object $x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}$ of $\mathtt{cat}(P_{\mathcal{J}})$. The composition law

$$\operatorname{Hom}_{\mathtt{TwSum}(P_{\mathcal{J}})}(K,L) \otimes \operatorname{Hom}_{\mathtt{TwSum}(P_{\mathcal{J}})}(L,M) \to \operatorname{Hom}_{\mathtt{TwSum}(P_{\mathcal{J}})}(K,M)$$

on such dg homs is then defined by the composition of dg homs in $\mathtt{cat}(P_{\mathcal{J}})$ and the relation $\mathbb{e}_{\underline{\alpha}}^{\vee}(\mathbb{e}_{\underline{\alpha}}) = 1$ on matching colors. The compatibility of this composition with the twisted differentials of the dg homs is automatic. □

2.2.2 The tensor structure on a category of universal twisted sums The category $\mathtt{TwSum}(P_{\mathcal{J}})$ is equipped with a dg enriched symmetric monoidal structure, defined by the natural distribution formula at the level of objects; that is, for

$$K = \Big(\bigoplus_{\underline{\alpha}} (\mathbb{K}\mathbb{e}_{\underline{\alpha}}) \otimes (x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_m}), \mathtt{tw}_K\Big),$$

$$L = \Big(\bigoplus_{\underline{\beta}} (\mathbb{K}\mathbb{e}_{\underline{\beta}}) \otimes (x_{\beta_1} \otimes \cdots \otimes x_{\beta_n}), \mathtt{tw}_L\Big),$$

we define

$$K\otimes L := \left(\bigoplus_{\underline{\alpha},\underline{\beta}}(\mathbb{K}\mathbb{e}_{\underline{\alpha}}\otimes\mathbb{e}_{\underline{\beta}})\otimes(x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_m}\otimes x_{\beta_1}\otimes\cdots\otimes x_{\beta_n}), \mathtt{tw}_K\otimes\mathrm{id}+\mathrm{id}\otimes\mathtt{tw}_L\right),$$

where we use the horizontal compositions

$$\begin{aligned}&\big(\mathbb{K}\mathbb{e}_{\underline{\gamma}}\otimes\mathbb{K}\mathbb{e}_{\underline{\alpha}}^{\vee}\otimes\mathrm{Hom}_{\mathtt{cat}(P_{\mathcal{J}})}(x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_m}, x_{\gamma_1}\otimes\cdots\otimes x_{\gamma_p})\big)\\ &\qquad\qquad\otimes\big(\mathbb{K}\mathbb{e}_{\underline{\delta}}\otimes\mathbb{K}\mathbb{e}_{\underline{\beta}}^{\vee}\otimes\mathrm{Hom}_{\mathtt{cat}(P_{\mathcal{J}})}(x_{\beta_1}\otimes\cdots\otimes x_{\beta_n}, x_{\delta_1}\otimes\cdots\otimes x_{\delta_q})\big)\\ &\xrightarrow{\otimes}(\mathbb{K}\mathbb{e}_{\underline{\gamma}}\otimes\mathbb{K}\mathbb{e}_{\underline{\delta}})\otimes(\mathbb{K}\mathbb{e}_{\underline{\alpha}}\otimes\mathbb{K}\mathbb{e}_{\underline{\beta}})^{\vee}\otimes\mathrm{Hom}_{\mathtt{cat}(P_{\mathcal{J}})}(x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_m}\otimes x_{\beta_1}\otimes\cdots\otimes x_{\beta_n},\\ &\qquad\qquad x_{\gamma_1}\otimes\cdots\otimes x_{\gamma_p}\otimes x_{\delta_1}\otimes\cdots\otimes x_{\delta_q})\end{aligned}$$

to define the formal twisted cochain $\mathtt{tw}_K\otimes\mathrm{id}+\mathrm{id}\otimes\mathtt{tw}_L$ of this object $K\otimes L$. An analogous construction holds at the level of homomorphisms.

Each object $x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_n}\in\mathtt{cat}(P_{\mathcal{J}})$ is naturally identified with the trivial twisted sum $K=(\mathbb{K}\mathbb{e}_0\otimes(x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_n}),0)$ in $\mathtt{TwSum}(P_{\mathcal{J}})$, where $\deg(\mathbb{e}_0)=0\Rightarrow\mathbb{K}\mathbb{e}_0=\mathbb{K}$. In particular, to each x_{α_i} corresponds a trivial twisted sum $K_{\alpha_i}=(\mathbb{K}\mathbb{e}_0\otimes x_{\alpha_i},0)$. This defines a functor

$$\mathtt{cat}(P_{\mathcal{J}})\to\mathtt{TwSum}(P_{\mathcal{J}}).$$

The category of universal twisted sums satisfies the following universal property with respect to this functor.

Lemma 2.6 *For every symmetric monoidal dg functor R: $\mathtt{cat}(P_{\mathcal{J}})\to\mathcal{C}h$ (that is, every $P_{\mathcal{J}}$–algebra), there exists a canonical factorization:*

$$\begin{tikzcd}\mathtt{cat}(P_{\mathcal{J}}) \arrow[r, "R"] \arrow[d] & \mathcal{C}h \\ \mathtt{TwSum}(P_{\mathcal{J}}) \arrow[ur, "\widetilde{R}"'] & \end{tikzcd}$$

Proof We construct $\widetilde{R}$ by first setting $\widetilde{R}(K_{\alpha_i})=R(x_{\alpha_i})$ so that the diagram commutes. Then, for any object

$$\left(\bigoplus_{\underline{\alpha}}(\mathbb{K}\mathbb{e}_{\underline{\alpha}})\otimes(x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_n}),\mathtt{tw}\right)$$

of $\mathtt{TwSum}(P_{\mathcal{J}})$, we define

$$\widetilde{R}\Big(\bigoplus_{\underline{\alpha}}(\mathbb{K}\mathbb{e}_{\underline{\alpha}})\otimes(x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_n}),\mathtt{tw}\Big) = \Big(\bigoplus_{\underline{\alpha}}(\mathbb{K}\mathbb{e}_{\underline{\alpha}})\otimes(R(x_{\alpha_1})\otimes\cdots\otimes R(x_{\alpha_n})),R(\mathtt{tw})\Big),$$

where the left-hand term is built with the direct sum and tensor product of $\mathcal{C}h$. The differential of $\widetilde{R}\big(\bigoplus_{\underline{\alpha}}(\mathbb{K}\mathbb{e}_{\underline{\alpha}})\otimes(x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_n}),\mathtt{tw}\big)$ is then defined on each component of this direct sum by the sum of the differential of $R(x_{\alpha_1})\otimes\cdots\otimes R(x_{\alpha_n})$ with a twisting cochain $R(\mathtt{tw})$ defined as follows. Since R is a symmetric monoidal dg functor, it induces a morphism of chain complexes

$$\begin{aligned}R_{x_{\beta_1}\otimes\cdots\otimes x_{\beta_m},x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_n}}\colon \mathrm{Hom}_{\mathtt{cat}(P_{\mathcal{J}})}(x_{\beta_1}\otimes\cdots\otimes x_{\beta_m},x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_n})&\\ \to \mathrm{Hom}_{\mathcal{C}h}(R(x_{\beta_1})\otimes\cdots\otimes R(x_{\beta_m}),R(x_{\alpha_1})\otimes\cdots\otimes R(x_{\alpha_n}))&,\end{aligned}$$

so that the collection $R(\mathtt{tw})=\{R(\mathtt{tw}_{\underline{\alpha}\,\underline{\beta}})\}_{\underline{\alpha}\,\underline{\beta}}$ is well defined by

$$\begin{aligned}R(\mathtt{tw}_{\underline{\alpha}\,\underline{\beta}}) &= R_{x_{\beta_1}\otimes\cdots\otimes x_{\beta_m},x_{\alpha_1}\otimes\cdots\otimes x_{\alpha_n}}(\mathtt{tw}_{\underline{\alpha}\,\underline{\beta}})\\ &\in \mathbb{e}_{\underline{\alpha}}\otimes\mathbb{e}_{\underline{\beta}}^{\vee}\otimes\mathrm{Hom}_{\mathcal{C}h}(R(x_{\beta_1})\otimes\cdots\otimes R(x_{\beta_m}),R(x_{\alpha_1})\otimes\cdots\otimes R(x_{\alpha_n})).\end{aligned}$$

This collection satisfies the relation of twisting cochains because R is a symmetric monoidal dg functor and the collection $\{\mathtt{tw}_{\underline{\alpha}\,\underline{\beta}}\}_{\underline{\alpha}\,\underline{\beta}}$ satisfies the relation of twisting cochains in $\mathtt{TwSum}(P_{\mathcal{J}})$. □

2.2.3 Functorial diagrams of algebras Our purpose is to use categories of universal twisted sums to construct diagrams of dg P–algebras "functorial in their variables" in a suitable sense.

Recall that the colored dg prop $P_{\mathcal{J}}$ parametrizing $\mathcal{J}$–diagrams of P–algebras is equivalent to the datum of a symmetric monoidal dg category $\mathtt{cat}(P_{\mathcal{J}})$. Algebras over $P_{\mathcal{J}}$ are then strict symmetric monoidal dg functors $\mathtt{cat}(P_{\mathcal{J}})\to\mathcal{C}h$, and morphisms of $P_{\mathcal{J}}$–algebras are natural transformations preserving the strict symmetric monoidal dg structures. Such a natural transformation corresponds to a natural transformation of $\mathcal{J}$–diagrams of P–algebras.

Now let $A,B\colon \mathtt{cat}(P_{\mathcal{J}})\to\mathcal{C}h$ be two such algebras, and $\phi\colon A\Rightarrow B$ be a strict symmetric monoidal dg natural transformation. Recall that, according to Lemma 2.6, such functors lift to strict symmetric monoidal dg functors $\widetilde{A},\widetilde{B}\colon \mathtt{TwSum}(P_{\mathcal{J}})\to\mathcal{C}h$. We want to prove that such a lift works similarly for symmetric monoidal dg natural transformations between such functors.

Lemma 2.7 *The natural transformation ϕ lifts to a strict symmetric monoidal dg natural transformation $\widetilde{\phi}\colon \widetilde{A} \Rightarrow \widetilde{B}$.*

Proof To see this, let us first recall from [15] the notion of enriched natural transformation in the case where the categories are enriched over $\mathcal{C}h$. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two dg functors and $\mathrm{Hom}_{\mathcal{C}}(-,-)$, $\mathrm{Hom}_{\mathcal{D}}(-,-)$ be respectively the dg homs of $\mathcal{C}$ and $\mathcal{D}$. A dg natural transformation $\tau\colon F \Rightarrow G$ is a collection of chain morphisms

$$\{\tau(x)\colon \mathbb{K} \to \mathrm{Hom}_{\mathcal{D}}(F(x), G(x))\}_{x\in \mathrm{ob}(\mathcal{C})},$$

that is, of 0–cycles in the complexes $\mathrm{Hom}_{\mathcal{D}}(F(x), G(x))$ indexed by the objects x of $\mathcal{C}$. For every $x, y \in \mathrm{ob}(\mathcal{C})$, this collection makes the following diagram commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{C}}(x,y)\otimes\mathbb{K} \arrow[d, "G_{x,y}\otimes\tau(x)"] \\
\mathbb{K}\otimes\mathrm{Hom}_{\mathcal{D}}(x,y) \arrow[d, "\tau(y)\otimes F_{x,y}"'] & \mathrm{Hom}_{\mathcal{D}}(G(x),G(y))\otimes\mathrm{Hom}_{\mathcal{D}}(F(x),G(x)) \arrow[d, "\circ_{\mathcal{D}}"] \\
\mathrm{Hom}_{\mathcal{D}}(F(y),G(y))\otimes\mathrm{Hom}_{\mathcal{D}}(F(x),F(y)) \arrow[r, "\circ_{\mathcal{D}}"] & \mathrm{Hom}_{\mathcal{D}}(F(x),G(y))
\end{tikzcd}$$

For any object

$$K = \left(\bigoplus_{\underline{\alpha}} (\mathbb{K}e_{\underline{\alpha}}) \otimes (x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}), \mathtt{tw}_K\right)$$

of $\mathtt{TwSum}(P_{\mathcal{J}})$, we define the associated 0–cycle $\widetilde{\phi}$ in

$$\mathrm{Hom}_{\mathcal{C}h}\left(\left(\bigoplus_{\underline{\alpha}} (\mathbb{K}e_{\underline{\alpha}}) \otimes (A(x_{\alpha_1}) \otimes \cdots \otimes A(x_{\alpha_n})), A(\mathtt{tw}_K)\right),\right.$$
$$\left.\left(\bigoplus_{\underline{\alpha}} (\mathbb{K}e_{\underline{\alpha}}) \otimes (B(x_{\alpha_1}) \otimes \cdots \otimes B(x_{\alpha_n})), B(\mathtt{tw}_K)\right)\right)$$

by

$$\widetilde{\phi}(K) = \bigoplus_{\underline{\alpha}} (\mathbb{K}e_{\underline{\alpha}}) \otimes (\phi(x_{\alpha_1}) \otimes \cdots \otimes \phi(x_{\alpha_n})).$$

We have to prove that this form a 0–cycle, thus that

$$(\delta + B(\mathtt{tw}_K)) \circ \widetilde{\phi}(K) = \widetilde{\phi}(K) \circ (\delta + A(\mathtt{tw}_K)).$$

The equality

$$\delta \circ \widetilde{\phi}(K) = \widetilde{\phi}(K) \circ \delta$$

follows from the fact that each $\phi(x_{\alpha_i})\colon A(x_{\alpha_i}) \to B(x_{\alpha_i})$ is a morphism of chain complexes and the differential δ is the differential of $B(x_{\alpha_1}) \otimes \cdots \otimes B(x_{\alpha_n})$ on the left-hand side of the equality and of $A(x_{\alpha_1}) \otimes \cdots \otimes A(x_{\alpha_n})$ on the right-hand side. The equality

$$B(\mathtt{tw}_K) \circ \widetilde{\phi}(K) = \widetilde{\phi}(K) \circ A(\mathtt{tw}_K)$$

follows from the definition of $A(\mathtt{tw}_K)$ as

$$A(\mathtt{tw}_K) = \{A_{x_{\beta_1} \otimes \cdots \otimes x_{\beta_m}, x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}}((\mathtt{tw}_K)_{\underline{\alpha}\,\underline{\beta}})\}_{\underline{\alpha}\,\underline{\beta}}$$

(and the same for $B(\mathtt{tw}_K)$), the fact that ϕ is a dg natural transformation between A and B, and the definition of $\widetilde{\phi}(K)$ in terms of the $\phi(x_{\alpha_i})$.

Concerning the monoidality of our collection $\{\widetilde{\phi}(K)\}_{K \in \mathrm{ob}(\mathtt{TwSum}(P_{\mathcal{J}}))}$ of 0–cycles, recall from Section 2.2.2 that the tensor product of two objects of $\mathtt{TwSum}(P_{\mathcal{J}})$ is defined by

$$\underbrace{\Big(\bigoplus_{\underline{\alpha}} (\mathbb{K}\mathbb{e}_{\underline{\alpha}}) \otimes (x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_m}), \mathtt{tw}_K\Big)}_{K} \otimes \underbrace{\Big(\bigoplus_{\underline{\beta}} (\mathbb{K}\mathbb{e}_{\underline{\beta}}) \otimes (x_{\beta_1} \otimes \cdots \otimes x_{\beta_n}), \mathtt{tw}_L\Big)}_{L}$$
$$:= \underbrace{\Big(\bigoplus_{\underline{\alpha},\underline{\beta}} (\mathbb{K}\mathbb{e}_{\underline{\alpha}} \otimes \mathbb{e}_{\underline{\beta}}) \otimes (x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_m} \otimes x_{\beta_1} \otimes \cdots \otimes x_{\beta_n}), \mathtt{tw}_K \otimes \mathrm{id} + \mathrm{id} \otimes \mathtt{tw}_L\Big)}_{= K \otimes L}$$

and that the functors $\widetilde{A}, \widetilde{B}\colon \mathtt{TwSum}(P_{\mathcal{J}}) \to \mathcal{C}h$ associated to $A, B\colon \mathtt{cat}(P_{\mathcal{J}}) \to \mathcal{C}h$ are defined by

$$\widetilde{A}\Big(\bigoplus_{\underline{\alpha}} (\mathbb{K}\mathbb{e}_{\underline{\alpha}}) \otimes (x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}), \mathtt{tw}\Big)$$
$$= \Big(\bigoplus_{\underline{\alpha}} (\mathbb{K}\mathbb{e}_{\underline{\alpha}}) \otimes (A(x_{\alpha_1}) \otimes \cdots \otimes A(x_{\alpha_n})), A(\mathtt{tw})\Big).$$

We have natural isomorphisms

$$a_{K \otimes L}\colon \widetilde{A}(K \otimes L) \xrightarrow{\cong} \widetilde{A}(K) \otimes \widetilde{A}(L), \quad b_{K \otimes L}\colon \widetilde{B}(K \otimes L) \xrightarrow{\cong} \widetilde{B}(K) \otimes \widetilde{B}(L),$$

induced by natural isomorphisms

$$A(\cdot \otimes \cdot) \xrightarrow{\cong} A(\cdot) \otimes A(\cdot), \quad B(\cdot \otimes \cdot) \xrightarrow{\cong} B(\cdot) \otimes B(\cdot),$$

since A and B are symmetric monoidal functors. We have to check the commutativity of the square below:

$$\begin{tikzcd}
\widetilde{A}(K \otimes L) \arrow[r, "\widetilde{\phi}(K\otimes L)"] \arrow[d, "a_{K\otimes L}"'] & \widetilde{B}(K \otimes L) \arrow[d, "b_{K\otimes L}"] \\
\widetilde{A}(K) \otimes \widetilde{A}(L) \arrow[r, "\widetilde{\phi}(K)\otimes\widetilde{\phi}(L)"] & \widetilde{B}(K) \otimes \widetilde{B}(L)
\end{tikzcd}$$

By construction of $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{\phi}$, which are defined by applying A, B and ϕ to each variable of the tensor powers defining the objects of $\mathtt{TwSum}(P_{\mathcal{J}})$, this boils down to the commutativity of such a monoidality square for A, B and ϕ, which holds because ϕ is a monoidal natural transformation.

The naturality of $\{\widetilde{\phi}(K)\}_{K\in \mathrm{ob}(\mathtt{TwSum}(P_{\mathcal{J}}))}$ follows directly from the naturality of ϕ. □

We consequently get two functors

$$\widetilde{A}_*, \widetilde{B}_*\colon \mathtt{TwSum}(P_{\mathcal{J}})^P \to \mathcal{C}h^P,$$

that carry any P–algebra in $\mathtt{TwSum}(P_{\mathcal{J}})$, represented by a symmetric monoidal functor $\widetilde{C}\colon \mathtt{cat}(P) \to \mathtt{TwSum}(P_{\mathcal{J}})$, to the P–algebra in $\mathcal{C}h$ represented by the composite functors $\widetilde{A}\widetilde{C}, \widetilde{B}\widetilde{C}\colon \mathtt{cat}(P) \to \mathcal{C}h$. We also have a natural transformation $\widetilde{\phi}_*\colon \widetilde{A}_* \Rightarrow \widetilde{B}_*$ between these functors on P–algebras.

For any small category $\mathcal{I}$, we get strict symmetric monoidal dg functors

$$\widetilde{A}_*, \widetilde{B}_*\colon (\mathtt{TwSum}(P_{\mathcal{J}})^P)^{\mathcal{I}} \to (\mathcal{C}h^P)^{\mathcal{I}}$$

and a strict symmetric monoidal dg natural transformation $\widetilde{\phi}_*\colon \widetilde{A}_* \Rightarrow \widetilde{B}_*$. This transformation consists in a collection of natural transformations of $\mathcal{I}$–diagrams of dg P–algebras

$$\widetilde{\phi}_*(Y)\colon \widetilde{A}_*(Y) \Rightarrow \widetilde{B}_*(Y)$$

for every $Y \in (\mathtt{TwSum}(P_{\mathcal{J}})^P)^{\mathcal{I}}$.

Thus, whenever we have an $\mathcal{I}$–diagram of P–algebras in $\mathtt{TwSum}(P_{\mathcal{J}})$, say Y, we can associate an $\mathcal{I}$–diagram of dg P–algebras $\widetilde{A}_*(Y)$ to any $\mathcal{J}$–diagram of dg P–algebras A, and a natural transformation of $\mathcal{I}$–diagrams of dg P–algebras $\widetilde{\phi}_*(Y)\colon \widetilde{A}_*(Y) \Rightarrow \widetilde{B}_*(Y)$ to any natural transformation of $\mathcal{J}$–diagrams of dg P–algebras $\phi\colon A \Rightarrow B$. This result is equivalent to the following statement.

Proposition 2.8 *Given an $\mathcal{I}$–diagram Y of P–algebras in $\mathtt{TwSum}(P_{\mathcal{J}})$, the above construction determines a functor*

$$(\mathcal{C}h^P)^{\mathcal{J}} \to (\mathcal{C}h^P)^{\mathcal{I}}.$$

The main example to which we want to apply this construction is the following. Let $f\colon X \to Y$ be a morphism of chain complexes. Then it admits a functorial factorization by an acyclic cofibration (ie acyclic injection) followed by a fibration (ie a surjection). This factorization is explicitly given by

$$\Xi(f\colon X \to Y): \quad \begin{tikzcd} & & X \\ X \arrow[r, tail, "\sim", "i"'] \arrow[urr, bend left, "\mathrm{id}_X"] \arrow[drr, bend right, "f"'] & Z \arrow[ur, two heads, "s"'] \arrow[dr, two heads, "p"] & \\ & & Y \end{tikzcd}$$

where

$$Z = (\mathbb{K}\mathbb{e}_0 \otimes X \oplus \mathbb{K}\mathbb{e}_{01} \otimes Y \oplus \mathbb{K}\mathbb{e}_1 \otimes Y, d_Z),$$

with $\deg(\mathbb{e}_0) = \deg(\mathbb{e}_1) = 0$ and $\deg(\mathbb{e}_{01}) = -1$. The differential d_Z can be expressed in this direct sum by the matrix

$$\begin{pmatrix} d_X & 0 & 0 \\ f & -d_Y & -\mathrm{id} \\ 0 & 0 & d_Y \end{pmatrix} = \begin{pmatrix} d_X & 0 & 0 \\ 0 & -d_Y & 0 \\ 0 & 0 & d_Y \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ f & 0 & -\mathrm{id} \\ 0 & 0 & 0 \end{pmatrix},$$

where the first matrix of the sum is the differential of the direct sum

$$\mathbb{K}\mathbb{e}_0 \otimes X \oplus \mathbb{K}\mathbb{e}_{01} \otimes Y \oplus \mathbb{K}\mathbb{e}_1 \otimes Y$$

and the second is a twisting $\mathtt{tw}_Z$, a map of degree -1 satisfying $\mathtt{tw}_Z^2 = 0$. The map i sends $x \in X$ to $x \oplus 0 \oplus f(x)$ and s and p are respectively projections on the first and the third factor; that is, we have

$$i = \begin{pmatrix} \mathrm{id} & 0 & f \end{pmatrix}, \quad s = \begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 0 \\ 0 \\ \mathrm{id} \end{pmatrix}.$$

There is a diagram of chain complexes

$$\Xi\colon \mathrm{Mor}(\mathcal{C}h) \to \mathrm{Fun}(\mathcal{Y}, \mathcal{C}h),$$

functorial in its variables, where $\mathrm{Mor}(\mathcal{C}h)$ is the category whose objects are morphisms of chain complexes and morphisms are commutative squares, and $\mathcal{Y}$ is the small category whose objects and arrows are given by

$$\mathcal{Y} := \left\{ \begin{tikzcd} & & \bullet \\ \bullet \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & \bullet \arrow[ur] \arrow[dr] & \\ & & \bullet \end{tikzcd} \right\}.$$

Our goal is to prove that for any cofibrant dg prop P, this functor induces a functor

$$\Xi\colon \mathrm{Mor}(\mathcal{C}h^{P}) \to \mathrm{Fun}(\mathcal{Y}, \mathcal{C}h^{P}),$$

that is, a functor

$$\Xi\colon (\mathcal{C}h^{P})^{\bullet\to\bullet} \to (\mathcal{C}h^{P})^{\mathcal{Y}}.$$

This means the following.

Theorem 2.9 *Let P be a cofibrant dg prop. The functorial factorization of morphisms of chain complexes described above lifts to a functorial factorization of P–algebra morphisms into an acyclic injection followed by a surjection.*

Proof The general strategy is to prove that the diagram in $\mathtt{TwSum}(P_{x\to y})$ associated to $\Xi(f\colon X \to Y)$ is actually a diagram in $\mathtt{TwSum}(P_{x\to y})^{P}$, and then apply Proposition 2.8.

Let $f\colon X \to Y$ be a morphism of chain complexes and $P_{x\to y}$ the 2–colored dg prop of P–algebra morphisms. In this proof, we will use the short notation

$$\mathtt{Tw} := \mathtt{TwSum}(P_{x\to y}).$$

We can associate to the diagram of chain complexes $\Xi(f\colon X \to Y)$ a diagram $\Xi(f\colon x \to y)$ in $\mathtt{Tw}$ so that Proposition 2.8 applies. For this, recall that the colors x and y are embedded into $\mathtt{Tw}$ as the objects $(\mathbb{K}\mathbb{e}_0 \otimes x, 0)$ and $(\mathbb{K}\mathbb{e}_1 \otimes y, 0)$. We will denote by f both the operation of $P_{x\to y}$ corresponding to f and the morphism $(\mathbb{K}\mathbb{e}_0 \otimes x, 0) \to (\mathbb{K}\mathbb{e}_1 \otimes y, 0)$ in $\mathtt{Tw}$. The object z of $\mathtt{Tw}$ corresponding to Z is defined to be

$$(\mathbb{K}\mathbb{e}_0 \otimes x \oplus \mathbb{K}\mathbb{e}_{01} \otimes y \oplus \mathbb{K}\mathbb{e}_1 \otimes y, \mathtt{tw}_z),$$

with

$$\mathtt{tw}_z = \begin{pmatrix} \mathtt{tw}_{0,0} & \mathtt{tw}_{01,0} & \mathtt{tw}_{1,0} \\ \mathtt{tw}_{0,01} & \mathtt{tw}_{01,01} & \mathtt{tw}_{1,01} \\ \mathtt{tw}_{0,1} & \mathtt{tw}_{01,1} & \mathtt{tw}_{1,1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \mathbb{e}_{01} \otimes \mathbb{e}_0^{\vee} \otimes f & 0 & \mathbb{e}_{01} \otimes \mathbb{e}_1^{\vee} \otimes -\mathrm{id} \\ 0 & 0 & 0 \end{pmatrix}$$

representing the twisting part $\mathtt{tw}_Z$ of Z. The maps i and p of $\Xi(f\colon x \to y)$ are then defined similarly to those of $\Xi(f\colon X \to Y)$.

The endomorphism dg prop $\mathrm{End}_{(\Xi(f\colon x\to y),\mathtt{Tw})}$ projects to the endomorphism dg prop $\mathrm{End}_{(f,\mathtt{Tw})}$ of the subdiagram $f\colon x \to y$, hence we have a fibration of dg props

$$\mathrm{End}_{(\Xi(f\colon x\to y),\mathtt{Tw})} \twoheadrightarrow \mathrm{End}_{(f,\mathtt{Tw})}\,.$$

We will denote these dg props by $\mathrm{End}_{\Xi(f\colon x\to y)}$ and End_f for short. We have to prove that this fibration is acyclic. For this, we consider the following commutative diagram of $\mathbb{S}$–biobjects:

$$\begin{tikzcd}
\mathrm{End}_z = \mathrm{Hom}_{zz} \arrow[dr, "{(i^*,p_*)}"] \arrow[drr, bend left, "p_*"] \arrow[ddr, bend right, "i^*"'] & & \\
& \mathrm{Hom}_{xz} \times_{\mathrm{Hom}_{xy}} \mathrm{Hom}_{zy} \arrow[r] \arrow[d] & \mathrm{Hom}_{zy} \arrow[d, "i^*"] \\
& \mathrm{Hom}_{xz} \arrow[r, "p_*"'] & \mathrm{Hom}_{xy}
\end{tikzcd}$$

where $\mathrm{Hom}_{zz}(m,n) = \mathrm{Hom}_{\mathtt{Tw}}(z^{\otimes m}, z^{\otimes n})$. Limits of $\mathbb{S}$–biobjects are created pointwise, so for every $(m,n) \in \mathbb{N}^2$ we have a commutative diagram as follows:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathtt{Tw}}(z^{\otimes m}, z^{\otimes n}) \arrow[dr, "{((i^{\otimes m})^*,(p^{\otimes n})_*)}"] \arrow[drr, bend left, "(p^{\otimes n})_*"] \arrow[ddr, bend right, "(i^{\otimes m})^*"'] & & \\
& \text{pullback} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathtt{Tw}}(z^{\otimes m}, y^{\otimes n}) \arrow[d, "(i^{\otimes m})^*"] \\
& \mathrm{Hom}_{\mathtt{Tw}}(x^{\otimes m}, z^{\otimes n}) \arrow[r, "(p^{\otimes n})_*"'] & \mathrm{Hom}_{\mathtt{Tw}}(x^{\otimes m}, y^{\otimes n})
\end{tikzcd}$$

We have to check that $((i^{\otimes m})^*, (p^{\otimes n})_*)$ is an acyclic fibration. Since acyclic fibrations of $\mathbb{S}$–biobjects are determined pointwise, we deduce that

$$(i^*, p_*)\colon \mathrm{End}_z \xrightarrow{\sim}\mathrel{\mkern-14mu}\rightarrow \mathrm{Hom}_{xz} \times_{\mathrm{Hom}_{xy}} \mathrm{Hom}_{yz}$$

is an acyclic fibration of Σ–objects. Let us consider now the base extensions

$$\mathrm{End}_x \times_{\mathrm{Hom}_{xz}} \mathrm{End}_z \times_{\mathrm{Hom}_{zy}} \mathrm{End}_y = \mathrm{End}_{\Xi(f\colon x\to y)},$$

$$\mathrm{End}_x \times_{\mathrm{Hom}_{xz}} (\mathrm{Hom}_{xz} \times_{\mathrm{Hom}_{xy}} \mathrm{Hom}_{zy}) \times_{\mathrm{Hom}_{zy}} \mathrm{End}_y = \mathrm{End}_f\,.$$

Acyclic fibrations are stable under base extensions, and acyclic fibrations of dg props are determined in the category of $\mathbb{S}$–biobjects under the forgetful functor, so we finally get the desired acyclic fibration of dg props

$$\mathrm{End}_x \times_{\mathrm{Hom}_{xz}} (i^*, p_*) \times_{\mathrm{Hom}_{zy}} \mathrm{End}_y \colon \mathrm{End}_{\Xi(f\colon x\to y)} \xrightarrow{\sim} \mathrm{End}_f .$$

Now let us denote $X_b = \mathbb{K}\mathbb{e}_0$, $Y_b = \mathbb{K}\mathbb{e}_1$ and $f_b\colon X_b \to Y_b$ the morphism sending $\mathbb{e}_0$ to $\mathbb{e}_1$. This morphism admits a factorization

$$X_b \underset{i_b}{\overset{\sim}{\rightarrowtail}} Z_b \underset{p_b}{\twoheadrightarrow} Y_b.$$

Our goal is to prove that for all natural integers m and n, we have isomorphisms of chain complexes

$$\begin{aligned}
\mathrm{Hom}_{\mathtt{Tw}}(z^{\otimes m}, z^{\otimes n}) &\cong \mathrm{Hom}_{Ch}(Z_b^{\otimes m}, Z_b^{\otimes n}) \otimes P(m,n),\\
\mathrm{Hom}_{\mathtt{Tw}}(z^{\otimes m}, y^{\otimes n}) &\cong \mathrm{Hom}_{Ch}(Z_b^{\otimes m}, Y_b^{\otimes n}) \otimes P(m,n),\\
\mathrm{Hom}_{\mathtt{Tw}}(x^{\otimes m}, z^{\otimes n}) &\cong \mathrm{Hom}_{Ch}(X_b^{\otimes m}, Z_b^{\otimes n}) \otimes P(m,n).
\end{aligned}$$

The method is exactly the same for the three cases, so we just write the argument for the third isomorphism. We need to determine the tensor powers of z. For every natural integer n, the object $z^{\otimes n}$ is given by the direct sum of shuffles

$$\bigoplus_{\substack{1\leq j\leq i\leq n\\ \sigma\in\mathrm{Sh}(i,m-i)\\ \tau\in\mathrm{Sh}(j,m-j)}} \sigma_*\big((\mathbb{K}\mathbb{e}_0\otimes x)^{\otimes n-i}, \tau_*((\mathbb{K}\mathbb{e}_{01}\otimes y)^{\otimes j}, (\mathbb{K}\mathbb{e}_1\otimes y)^{\otimes i-j})\big),$$

where the action $\sigma_*(A^{\otimes k}, B^{\otimes l})$ of a (k,l)–shuffle σ on a pair of tensor powers $(A^{\otimes k}, B^{\otimes l})$ permutes the variables of the tensor product $A^{\otimes k}\otimes B^{\otimes l}$. The twisting of $z^{\otimes n}$ is determined by

$$\mathtt{tw}_{0,01}^{\otimes n} = \mathbb{e}_{01}^{\otimes n}\otimes(\mathbb{e}_0^{\vee})^{\otimes n}\otimes f^{\circ_h n} \quad\text{and}\quad \mathtt{tw}_{1,01}^{\otimes n} = \mathbb{e}_{01}^{\otimes n}\otimes(\mathbb{e}_1^{\vee})^{\otimes n}\otimes(-\mathrm{id})^{\circ_h n},$$

where $\circ_h$ is the horizontal composition product of the dg prop $P_{x\to y}$. We get:

$$\begin{aligned}
&\mathrm{Hom}_{\mathtt{Tw}}(x^{\otimes m}, z^{\otimes n})\\
&= \bigoplus_{\substack{1\leq j\leq i\leq n\\ \sigma\in\mathrm{Sh}(i,m-i)\\ \tau\in\mathrm{Sh}(j,m-j)}} \mathrm{Hom}_{\mathtt{Tw}}\big(x^{\otimes m}, \sigma_*((\mathbb{K}\mathbb{e}_0\otimes x)^{\otimes n-i}, \tau_*((\mathbb{K}\mathbb{e}_{01}\otimes y)^{\otimes j}, (\mathbb{K}\mathbb{e}_1\otimes y)^{\otimes i-j}))\big)\\
&\cong \bigoplus_{\substack{1\leq j\leq i\leq n\\ \sigma\in\mathrm{Sh}(i,m-i)\\ \tau\in\mathrm{Sh}(j,m-j)}} \mathbb{K}\mathbb{e}_0^{\otimes n-i}\otimes\mathbb{K}\mathbb{e}_{01}^{\otimes j}\otimes\mathbb{K}\mathbb{e}_1^{\otimes i-j}\otimes\mathrm{Hom}_{\mathtt{Tw}}\big(x^{\otimes m}, \sigma_*(x^{\otimes n-i}, \tau_*(y^{\otimes j}, y^{\otimes i-j}))\big)
\end{aligned}$$

Moreover, we have

$$\mathrm{Hom}_{\mathtt{Tw}}\big(x^{\otimes m}, \sigma_*(x^{\otimes n-i}, \tau_*(y^{\otimes j}, y^{\otimes i-j}))\big) \\ = P_{x\to y}(x,\dots,x;\sigma_*(x,\dots,x,\tau_*(y,\dots,y))),$$

where $P_{x\to y}(x,\dots,x;\sigma_*(x,\dots,x,\tau_*(y,\dots,y)))$ has m copies of the color x as input, and as output $n-i$ copies of color x and i copies of color y permuted by the shuffles σ and τ. We want to build an isomorphism

$$\bigoplus_{\substack{1\le j\le i\le n\\ \sigma\in\mathrm{Sh}(i,m-i)\\ \tau\in\mathrm{Sh}(j,m-j)}} \mathbb{K}\mathbb{e}_0^{\otimes n-i}\otimes\mathbb{K}\mathbb{e}_{01}^{\otimes j}\otimes\mathbb{K}\mathbb{e}_1^{\otimes i-j}\otimes P_{x\to y}(x,\dots,x;\sigma_*(x,\dots,x,\tau_*(y,\dots,y))) \\ \cong \bigoplus_{\substack{1\le j\le i\le n\\ \sigma\in\mathrm{Sh}(i,m-i)\\ \tau\in\mathrm{Sh}(j,m-j)}} \mathrm{Hom}_{\mathcal{C}h}\big(X_b^{\otimes m}, \sigma_*(X_b^{\otimes n-i}, \tau_*(Y_b[-1]^{\otimes j}, Y_b^{\otimes i-j}))\big)\otimes P(m,n),$$

where $[-1]$ is the degree shift applied to the chain complex Y_b. For this, we define in each component (i,j,σ,τ) of the direct sum an isomorphism

$$\mathbb{K}\mathbb{e}_0^{\otimes n-i}\otimes\mathbb{K}\mathbb{e}_{01}^{\otimes j}\otimes\mathbb{K}\mathbb{e}_1^{\otimes i-j}\otimes P_{x\to y}(x,\dots,x;\sigma_*(x,\dots,x,\tau_*(y,\dots,y))) \\ \to \mathrm{Hom}_{\mathcal{C}h}\big(X_b^{\otimes m}, \sigma_*(X_b^{\otimes n-i}, \tau_*(Y_b[-1]^{\otimes j}, Y_b^{\otimes i-j}))\big)\otimes P(m,n),$$

which sends any

$$\xi\in P_{x\to y}(x,\dots,x;\sigma_*(x,\dots,x,\tau_*(y,\dots,y)))$$

to

$$\sigma_*\tau_*\otimes\sigma_*(f^{\circ_h n-i}, \mathrm{id}^{\circ_h i})\circ_v\xi,$$

where $\sigma_*\tau_*$ is the unique homomorphism sending $\mathbb{e}_0^{\otimes m}$ to $\sigma_*(\mathbb{e}_0^{\otimes n-i}, \tau_*(\mathbb{e}_{01}^{\otimes j}, \mathbb{e}_1^{\otimes i-j}))$ and $\sigma_*(f^{\circ_h n-i}, \mathrm{id}^{\circ_h i})$ is the permutation of the variables in the iterated horizontal product $f\circ_h\cdots\circ_h f\circ_h \mathrm{id}\circ_h\cdots\circ_h \mathrm{id}$ by σ.

Finally, since $((i^{\otimes m})^*, (p^{\otimes n})_*)$ is the tensor product of $((i_b^{\otimes m})^*, (p_b^{\otimes n})_*)$ by $P(m,n)$, it remains to apply the methods of [12, Lemma 8.3] in the category of chain complexes, for X_b and Y_b, to prove that $((i_b^{\otimes m})^*, (p_b^{\otimes n})_*)$ is an acyclic fibration. We write the arguments here for the sake of clarity. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}h}(Z_b^{\otimes m}, Z_b^{\otimes n}) \arrow[dr, "((i_b^{\otimes m})^*{,}(p_b^{\otimes n})_*)"] \arrow[drr, bend left, "(p_b^{\otimes n})_*"] \arrow[ddr, bend right, "(i_b^{\otimes m})^*"'] & & \\
& \text{pullback} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}h}(Z_b^{\otimes m}, Y_b^{\otimes n}) \arrow[d, "(i_b^{\otimes m})^*"] \\
& \mathrm{Hom}_{\mathcal{C}h}(X_b^{\otimes m}, Z_b^{\otimes n}) \arrow[r, "(p_b^{\otimes n})_*"'] & \mathrm{Hom}_{\mathcal{C}h}(X_b^{\otimes m}, Y_b^{\otimes n})
\end{tikzcd}$$

Recall that chain complexes over a field are all cofibrant and fibrant in the model structure of $\mathcal{C}h$. The map i_b is a cofibration and X_b is cofibrant, so by the pushout-product axiom, for every integer n the map $i_b^{\otimes n}\colon X^{\otimes n} \to Z^{\otimes n}$ is a cofibration. The category $\mathcal{C}h$ satisfies the limit monoid axioms [12, Section 6] and Y_b is fibrant, so for every integer n the map $p_b^{\otimes n}\colon Z_b^{\otimes n} \to Y_b^{\otimes n}$ is a fibration [12, Proposition 6.7]. Moreover, by the pushout-product axiom, the tensor product preserves acyclic cofibrations between cofibrant objects, so by Brown's lemma it preserves weak equivalences between cofibrant objects. Given that Z_b and Y_b are cofibrant, it implies that $p_b^{\otimes n}$ is an acyclic fibration. According to the dual pushout-product axiom, the fact that $i_b^{\otimes m}$ is a cofibration and $p_b^{\otimes n}$ is an acyclic fibration implies that $((i_b^{\otimes m})^*, (p_b^{\otimes n})_*)$ is an acyclic fibration. $\square$

3 The subcategory of acyclic fibrations

The goal of this section is to prove that the classifying space of weak equivalences of P–algebras is weakly equivalent to the classifying space of acyclic fibrations of P–algebras.

Theorem 3.1 *Let P be a cofibrant dg prop. The inclusion $i\colon fw\mathcal{C}h^P \hookrightarrow w\mathcal{C}h^P$ of categories gives rise to a weak equivalence of simplicial sets, $\mathcal{N} fw\mathcal{C}h^P \xrightarrow{\sim} \mathcal{N} w\mathcal{C}h^P$.*

Remark 3.2 Actually, the methods of [26] can be transposed in our setting to prove the following much stronger statement. We refer the reader to the seminal papers [10; 8; 9] for the notions of simplicial localization, hammock localization and Dwyer–Kan equivalences of simplicial categories. The inclusion of categories $i\colon fw\mathcal{C}h^P \hookrightarrow w\mathcal{C}h^P$ induces a Dwyer–Kan equivalence of hammock localizations

$$L^H(\mathcal{C}h^P, fw\mathcal{C}h^P) \xrightarrow{\sim} L^H(\mathcal{C}h^P, w\mathcal{C}h^P).$$

We refer the reader to [26] for more details about this proof, which relies on the properties of several models of $(\infty, 1)$–categories (simplicial categories [2], relative categories [1] and complete Segal spaces [22]).

To prove this theorem, we use Quillen's Theorem A [20]: we have to check that for every chain complex X, the nerve of the comma category $(X \downarrow i)$ is contractible. For this aim, we prove the following more general result.

Proposition 3.3 *Let $\mathcal{I}$ be a small category. Every simplicial map $\mathcal{N}\mathcal{I} \to \mathcal{N}(X \downarrow i)$ is null up to homotopy.*

As a consequence we get:

Proposition 3.4 *The simplicial set $\mathcal{N}(X \downarrow i)$ is contractible.*

To prove Proposition 3.4, we apply Proposition 3.3, for every $n \in \mathbb{N}$, to the subdivision category of a simplicial model of the n–sphere S^n. We take $\partial\Delta^{n+1}$ as simplicial model of S^n and denote by $\operatorname{sd}\partial\Delta^{n+1}$ its subdivision category. We then use general arguments of homotopical algebra.

Proposition 3.5 *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction. It induces natural isomorphisms*

$$\mathrm{Map}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Map}_{\mathcal{C}}(X, G(Y)),$$

where X is a cofibrant object of $\mathcal{C}$ and Y a fibrant object of $\mathcal{D}$.

Proof We will use the definition of mapping spaces via cosimplicial frames. The proof works as well with simplicial frames. The adjunction (F, G) induces an adjunction at the level of diagram categories

$$F\colon \mathcal{C}^{\Delta} \rightleftarrows \mathcal{D}^{\Delta} : G.$$

Now let $\phi\colon A^{\bullet} \rightarrowtail B^{\bullet}$ be a Reedy cofibration between Reedy cofibrant objects of $\mathcal{C}^{\Delta}$. This is equivalent, by definition, to saying that for every integer r, the maps

$$(\lambda, \phi)_r\colon L^r B \coprod_{L^r A} A^r \rightarrowtail B^r$$

induced by ϕ and the latching object construction $L^{\bullet} A$ are cofibrations in $\mathcal{C}$. Let us consider the following pushout:

$$\begin{tikzcd}
L^r A \arrow[r] \arrow[d, "L^r \phi"'] & A^r \arrow[d] \\
L^r B \arrow[r] & L^r B \coprod_{L^r A} A^r
\end{tikzcd}$$

The fact that ϕ is a Reedy cofibration implies that for every r, the map $L^r\phi$ is a cofibration. Since cofibrations are stable under pushouts, the map $A^r \to L^r B \coprod_{L^r A} A^r$ is also a cofibration. By assumption, the cosimplicial object $A^\bullet$ is Reedy cofibrant, so it is, in particular, pointwise cofibrant. We deduce that $L^r B \coprod_{L^r A} A^r$ is cofibrant. Similarly, each B^r is cofibrant since $B^\bullet$ is Reedy cofibrant. The map $(\lambda, \phi)_r$ is a cofibration between cofibrant objects and F is a left Quillen functor, so $F((\lambda, \phi)_r)$ is a cofibration of $\mathcal{D}$ between cofibrant objects. Recall that the r^{th} latching object construction is defined by a colimit. As a left adjoint, the functor F commutes with colimits so we get a cofibration

$$L^r F(B^\bullet) \coprod_{L^r F(A^\bullet)} F(A^r) \rightarrowtail F(B^r).$$

This means that $F(\phi)$ is a Reedy cofibration in $\mathcal{D}^\Delta$. Now, given that Reedy weak equivalences are the pointwise equivalences, if ϕ is a Reedy weak equivalence between Reedy cofibrant objects then it is, in particular, a pointwise weak equivalence between pointwise cofibrant objects, hence $F(\phi)$ is a Reedy weak equivalence in $\mathcal{D}^\Delta$. We conclude that F induces a left Quillen functor between cosimplicial objects for the Reedy model structures. In particular, it sends any cosimplicial frame of a cofibrant object X of $\mathcal{C}$ to a cosimplicial frame of $F(X)$. □

Remark 3.6 The isomorphism above holds if the cosimplicial frame for the left-hand mapping space is chosen to be the image under F of the cosimplicial frame of the right-hand mapping space. But recall that cosimplicial frames on a given object are all weakly equivalent, so that for any choice of cosimplicial frame we get at least weakly equivalent mapping spaces.

Now, recall that the geometric realization functor and the singular complex functor induce a Quillen equivalence

$$|-|\colon \mathrm{sSet} \rightleftarrows \mathrm{Top} : \mathrm{Sing}_\bullet(-)$$

between topological spaces and simplicial sets. We have

$$\begin{aligned}
\mathrm{Map}_{\mathrm{sSet}}(\mathcal{N}\,\mathrm{sd}\,\partial\Delta^{n+1}, \mathcal{N}(X\downarrow i)) &\simeq \mathrm{Map}_{\mathrm{sSet}}(\mathcal{N}\,\mathrm{sd}\,\partial\Delta^{n+1}, \mathrm{Sing}_\bullet(|\mathcal{N}(X\downarrow i)|)) \\
&\simeq \mathrm{Map}_{\mathrm{Top}}(|\mathcal{N}\,\mathrm{sd}\,\partial\Delta^{n+1}|, |\mathcal{N}(X\downarrow i)|) \\
&\simeq \mathrm{Map}_{\mathrm{Top}}(S^n, |\mathcal{N}(X\downarrow i)|),
\end{aligned}$$

hence

$$\left|\mathrm{Map}_{\mathrm{sSet}}(\mathcal{N}\,\mathrm{sd}\,\partial\Delta^{n+1}, \mathcal{N}(X\downarrow i))\right| \simeq \left|\mathrm{Map}_{\mathrm{Top}}(S^n, |\mathcal{N}(X\downarrow i)|)\right|;$$

in particular

$$\pi_0\left|\mathrm{Map}_{\mathrm{sSet}}(\mathcal{N}\,\mathrm{sd}\,\partial\Delta^{n+1}, \mathcal{N}(X\downarrow i))\right| \cong [S^n, |\mathcal{N}(X\downarrow i)|]_{\mathrm{Ho(Top)}}.$$

Proposition 3.3 means that for every integer n, the space

$$\left|\mathrm{Map}_{\mathrm{sSet}}(\mathcal{N}\,\mathrm{sd}\,\partial\Delta^{n+1}, \mathcal{N}(X\downarrow i))\right|$$

has only one connected component (the component of the zero map); that is, the homotopy groups of $|\mathcal{N}(X\downarrow i)|$ are trivial.

Proof of Proposition 3.3 The category $(X\downarrow i)$ has weak equivalences $X\xrightarrow{\sim} Y$ as objects and acyclic fibrations as morphisms. It contains the initial object $X\xrightarrow{=} X$ of $(X\downarrow \mathcal{C}h)$.

Every simplicial map $\mathcal{N}\mathcal{I}\to\mathcal{N}(X\downarrow i)$ comes from a functor $\mathcal{I}\to(X\downarrow i)$, ie an $\mathcal{I}$–diagram in $(X\downarrow i)$. Let F be such a functor. Let $\overline{X}$ be the initial $\mathcal{I}$–diagram, that is the constant diagram on $X\xrightarrow{=} X$. In order to simplify notation, we write Y for a morphism $X\to Y$ (an object of $(X\downarrow\mathcal{C}h)$) and $Y\to Y'$ for a commutative triangle relating $X\to Y$ to $X\to Y'$ (a morphism of $(X\downarrow\mathcal{C}h)$). The diagram $F\times\overline{X}\colon\mathcal{I}\to(X\downarrow\mathcal{C}h)$ is defined on objects by $F\times\overline{X}(k)=F(k)\times X$ and on arrows by $F\times\overline{X}(\phi)=F(\phi)\times\mathrm{id}_X$. Applying the functorial factorization of Theorem 2.9 to the unique initial morphism $\overline{X}\to F\times\overline{X}$, we get a decomposition in $(X\downarrow\mathcal{C}h)^P$ into a diagram $\mathcal{Y}$ given by

$$\begin{tikzcd} & & \overline{X} \\ \overline{X} \arrow[r, tail, "\sim", "i"'] \arrow[urr, bend left, "="] \arrow[drr, bend right] & G \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_2"] & \\ & & F \end{tikzcd}$$

where the functor G is defined pointwise by the functorial factorization of Theorem 2.9. The map $(p_1,p_2)\colon G\twoheadrightarrow F\times\overline{X}$ is a pointwise fibration and i is a pointwise acyclic cofibration of chain complexes. Since the map $(p_1,p_2)\colon G\twoheadrightarrow F\times\overline{X}$ is a pointwise fibration and F and $\overline{X}$ are pointwise fibrant, the maps p_1 and p_2 are pointwise acyclic fibrations: the product $F\times\overline{X}$ is given by the pullback

$$\begin{tikzcd} F\times\overline{X} \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d] \\ F \arrow[r] & \bullet \end{tikzcd}$$

and pointwise fibrations are stable under pullbacks so p_1 and p_2 are pointwise fibrations. Since $\mathrm{id}_{\overline{X}}=p_1\circ i$ and $\overline{X}\to F=p_2\circ i$ are weak equivalences, the maps p_1 and p_2 are acyclic by the two-out-of-three property.

The functors $\overline{X}$ and F take their values in $(X \downarrow i)$ by definition. This implies that the functor G sends morphisms of $\mathcal{I}$ to acyclic fibrations by definition of the functorial factorization in chain complexes. We consequently obtain a zigzag of natural transformations $\overline{X} \leftarrow G \rightarrow F$ of functors $\mathcal{I} \rightarrow (X \downarrow i)$. This zigzag implies that $\mathcal{N}F$ is homotopic to $\mathcal{N}\overline{X}$, which is itself null up to homotopy. This concludes the proof of Proposition 3.3. □

4 Moduli spaces of algebraic structures as homotopy fibers

4.1 Moduli spaces of algebra structures on fibrations

The results of this subsection holds for algebras in $\mathcal{E}$ over a prop in $\mathcal{C}$, where the category $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category and the category $\mathcal{E}$ is a cofibrantly generated symmetric monoidal model category over $\mathcal{C}$. However, for the sake of simplicity we explain only the case $\mathcal{E} = \mathcal{C} = \mathcal{C}h$.

We start by recalling [12, Lemma 7.2]. Let $f\colon A \rightarrow B$ be a morphism of $\mathcal{C}h$. Then we have a pullback

$$\begin{tikzcd}
\mathrm{End}_{\{A \xrightarrow{f} B\}} \arrow[r, "d_0"] \arrow[d, "d_1"'] & \mathrm{End}_B \arrow[d, "f^*"] \\
\mathrm{End}_A \arrow[r, "f_*"'] & \mathrm{Hom}_{AB}
\end{tikzcd}$$

where Hom_{AB} is defined by $\mathrm{Hom}_{AB}(m,n) = \mathrm{Hom}_{\mathcal{C}h}(A^{\otimes m}, B^{\otimes n})$.

Lemma 4.1 [12, Lemma 7.2] (1) *If f is a (acyclic) fibration then so is d_0.*

(2) *If f is a cofibration, then d_1 is a fibration. If f is also acyclic then d_1 is an acyclic fibration and d_0 a weak equivalence.*

Remark 4.2 Lemma 4.1 is a generalization in the prop context of [21, Propositions 4.1.7 and 4.1.8].

Lemma 4.3 *Let $X_n \twoheadrightarrow \cdots \twoheadrightarrow X_1 \twoheadrightarrow X_0$ be a chain of fibrations of chain complexes. For every $0 \leq k \leq n-1$, the map d_0 in the pullback*

$$\begin{tikzcd}
\mathrm{End}_{\{X_n \twoheadrightarrow \cdots \twoheadrightarrow X_0\}} \arrow[r, "d_0"] \arrow[d, "d_1"'] & \mathrm{End}_{\{X_k \twoheadrightarrow \cdots \twoheadrightarrow X_0\}} \arrow[d] \\
\mathrm{End}_{\{X_n \twoheadrightarrow \cdots \twoheadrightarrow X_{k+1}\}} \arrow[r] & \mathrm{Hom}_{X_{k+1}X_k}
\end{tikzcd}$$

is a fibration. Moreover, if the fibrations in the chain $X_n \twoheadrightarrow \cdots \twoheadrightarrow X_1 \twoheadrightarrow X_0$ are acyclic then so is d_0.

Proof We prove this lemma by induction. The case $n=1$ is Lemma 4.1. Now suppose that our lemma is true for a given integer $n \geq 1$. Let $X_{n+1} \twoheadrightarrow \cdots \twoheadrightarrow X_1 \twoheadrightarrow X_0$ be a chain of fibrations of complexes. We distinguish two cases:

Case $k = n$ We have the pullback

$$\begin{tikzcd}
\mathrm{End}_{\{X_{n+1} \twoheadrightarrow \cdots \twoheadrightarrow X_0\}} \arrow[r, "d_0"] \arrow[d, "d_1"'] & \mathrm{End}_{\{X_n \twoheadrightarrow \cdots \twoheadrightarrow X_0\}} \arrow[d] \\
\mathrm{End}_{X_{n+1}} \arrow[r, "f_*"'] & \mathrm{Hom}_{X_{n+1} X_n}
\end{tikzcd}$$

where $f\colon X_{n+1} \twoheadrightarrow X_n$. The fact that f is a fibration implies that f_* is a fibration, so d_0 is a fibration because of the stability of fibrations under pullback, and the acyclicity of f implies the acyclicity of d_0. The detailed proof of these statements is done in the proof of [12, Lemma 7.2].

Case $0 \leq k \leq n-1$ We have that

$$d_0 = \mathrm{End}_{\{X_{n+1} \twoheadrightarrow \cdots \twoheadrightarrow X_0\}} \to \mathrm{End}_{\{X_n \twoheadrightarrow \cdots \twoheadrightarrow X_0\}} \to \mathrm{End}_{\{X_k \twoheadrightarrow \cdots \twoheadrightarrow X_0\}}$$

is the composite of an map satisfying the induction hypothesis with the map of the case $k=n$, so the conclusion follows. □

Remark 4.4 This lemma is the generalization of [21, Proposition 4.1.9] in the prop context.

We deduce from Lemmata 4.1 and 4.3 the following properties of our moduli spaces.

Proposition 4.5 *Let $f\colon X \to Y$ be a chain complex morphism and P be a cofibrant dg prop. The pullback of Lemma 4.1 gives rise to the following diagram of simplicial sets:*

$$P\{X\} \xleftarrow{(d_1)_*} P\{f\} \xrightarrow{(d_0)_*} P\{Y\}$$

1. *If f is a cofibration then $(d_1)_*$ is a fibration. Moreover, if f is acyclic then $(d_0)_*$ and $(d_1)_*$ are weak equivalences.*
2. *If f is a fibration then $(d_0)_*$ is a fibration. Moreover, if f is acyclic then $(d_0)_*$ and $(d_1)_*$ are weak equivalences.*

Proof (1) If f is a cofibration then d_1 is a fibration. So $(d_1)_*$ is a fibration of simplicial sets according to Proposition 1.11. If f is acyclic, then d_0 and d_1 are weak equivalences. Every chain complex is fibrant and cofibrant, and fibrations of props are determined componentwise, so End_X and End_Y are fibrant props. This implies that $\mathrm{End}_{\{f\}}$ is also fibrant. We deduce from this and Proposition 1.11 that $(d_0)_*$ and $(d_1)_*$ are weak equivalences.

(2) The proof is the same as in the previous case. □

By induction we can also prove the following proposition.

Proposition 4.6 *Let $X_n \overset{\sim}{\twoheadrightarrow} \cdots \overset{\sim}{\twoheadrightarrow} X_1 \overset{\sim}{\twoheadrightarrow} X_0$ be a chain of acyclic fibrations and P be a cofibrant dg prop. For every $0 \leq k \leq n-1$, the map $(d_0)_*$ is an acyclic fibration and $(d_1)_*$ a weak equivalence in the diagram below:*

$$P\{X_n \overset{\sim}{\twoheadrightarrow} \cdots \overset{\sim}{\twoheadrightarrow} X_{k+1}\} \xleftarrow{(d_1)_*} P\{X_n \overset{\sim}{\twoheadrightarrow} \cdots \overset{\sim}{\twoheadrightarrow} X_0\} \xrightarrow{(d_0)_*} P\{X_k \overset{\sim}{\twoheadrightarrow} \cdots \overset{\sim}{\twoheadrightarrow} X_1\}$$

Remark 4.7 Propositions 4.5 and 4.6 are generalizations in the prop context of [21, Propositions 4.1.11, 4.1.12 and 4.1.13].

4.2 Proof of Theorem 0.1

We have now all the key results to generalize Rezk's theorem to algebras over dg props. The remaining arguments are the same as those of Rezk, so we will not repeat them with all details but essentially show how our Theorem 3.1, as well as the main theorem of [25], fit in the proof.

Let P be a cofibrant dg prop, and $\mathcal{N}wCh^{P\otimes\Delta^\bullet}$ the bisimplicial set defined by

$$\left(\mathcal{N}wCh^{P\otimes\Delta^\bullet}\right)_{m,n} = \left((\mathcal{N}wCh^{cf})^{P\otimes\Delta^n}\right)_m.$$

The dg prop P is cofibrant, thus so is $P\otimes\Delta^n$ for every $n \geq 0$. According to Theorem 3.1, we have a weak equivalence induced by an inclusion of categories

$$\mathcal{N}fwCh^{P\otimes\Delta^n} \xrightarrow{\sim} \mathcal{N}wCh^{P\otimes\Delta^n}.$$

Moreover, for every $n, n' \geq 0$, the map $\Delta^n \to \Delta^{n'}$ induces a weak equivalence of cofibrant dg props $P\otimes\Delta^n \to P\otimes\Delta^{n'}$ and thereby a weak equivalence of simplicial sets

$$\mathcal{N}wCh^{P\otimes\Delta^{n'}} \xrightarrow{\sim} \mathcal{N}wCh^{P\otimes\Delta^n}$$

according to [25, Theorem 0.1]. We obtain a zigzag of weak equivalences

$$\mathrm{diag}\,\mathcal{N}fwCh^{P\otimes\Delta^\bullet} \xrightarrow{\sim} \mathrm{diag}\,\mathcal{N}wCh^{P\otimes\Delta^\bullet} \xleftarrow{\sim} \mathcal{N}wCh^P.$$

We use an adaptation of a slightly modified version of Quillen's Theorem B (see [20]), namely [21, Lemma 4.2.2], in order to determine the homotopy fiber of the map $\operatorname{diag}\mathcal{N} fwCh^{P\otimes\Delta^\bullet} \to \mathcal{N} fwCh$. To prove that our map verifies the hypotheses of this lemma we use the propositions of Section 4.1 exactly in the same way as Rezk in the operadic case. Then we check that $\operatorname{diag}(U\downarrow X)\simeq P\{X\}$, where $U\colon fwCh^{P\otimes\Delta^\bullet}\to fwCh$ is the forgetful functor (by using again the propositions of Section 4.1) and finally we get the following diagram:

$$\begin{tikzcd}
P\{X\} \arrow[r] \arrow[d] & \operatorname{diag}\mathcal{N} fwCh^{P\otimes\Delta^\bullet} \arrow[r, "\sim"] \arrow[d] & \operatorname{diag}\mathcal{N} wCh^{P\otimes\Delta^\bullet} & \mathcal{N} wCh^{P} \arrow[l, "\sim"'] \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathcal{N} fwCh \arrow[rr, "\sim"] & & \mathcal{N} wCh
\end{tikzcd}$$

The proof of Theorem 0.1 is complete.

Remark 4.8 Note that we can recover the transfer theorem of bialgebra structures of [12, Theorem A] as a consequence of Theorem 0.1. Indeed, let P be a cofibrant dg prop in Ch. Let $X\xrightarrow{\sim} Y$ be a morphism of Ch such that Y is endowed with a P–algebra structure. We have a homotopy pullback of simplicial sets

$$\begin{tikzcd}
P\{X\} \arrow[r, "p"] \arrow[d] & \mathcal{N} wCh^{P} \arrow[d, "\mathcal{N}U"] \\
\{X\} \arrow[r] & \mathcal{N} wCh
\end{tikzcd}$$

which induces an exact sequence of pointed sets

$$\pi_0 P\{X\}\to \pi_0\mathcal{N} wCh^{P}\to \pi_0\mathcal{N} wCh\,.$$

The base point of the set $\pi_0\mathcal{N} wCh$ is the weak equivalence class of X, denoted by $[X]$. The weak equivalence $X\xrightarrow{\sim} Y$ in Ch implies that we have the equality $[Y]=[X]$ and thus $\pi_0\mathcal{N}U([Y]_P)=[X]$, where $[Y]_P$ is the weak equivalence class of Y in Ch^P. The exactness of the above sequence implies that $\pi_0 p(P\{X\})=(\pi_0\mathcal{N}U)^{-1}([X])$ so $[Y]_P\in\pi_0 p(P\{X\})$. This means that there exists a P–algebra structure on X such that we have a zigzag of P–algebra morphisms

$$X\xleftarrow{\sim}\cdots\xrightarrow{\sim} Y,$$

which are weak equivalences of Ch.

Remark 4.9 We do not address the case of simplicial sets. However, [14, Theorem 1.4] endows the algebras over a colored prop in simplicial sets with a model category structure. Moreover, the free algebra functor exists in this case. Therefore one can transpose the methods used in the operadic setting to obtain a simplicial version of [25, Theorem 0.1]. Theorem 0.1 in simplicial sets can be proved by following Rezk's original proof step by step. We also conjecture that our results have a version in simplicial modules which would follow from arguments similar to ours.

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Solvable Lie flows of codimension 3

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In Appendix E of *Riemannian foliations* [Progress in Mathematics 73, Birkhäuser, Boston (1988)], É Ghys proved that any Lie $\mathfrak{g}$–flow is homogeneous if $\mathfrak{g}$ is a nilpotent Lie algebra. In the case where $\mathfrak{g}$ is solvable, we expect any Lie $\mathfrak{g}$–flow to be homogeneous. In this paper, we study this problem in the case where $\mathfrak{g}$ is a 3–dimensional solvable Lie algebra.

57R30; 53C12, 22E25

1 Introduction

Throughout this paper, we suppose that all manifolds are connected, smooth and orientable and all foliations are smooth and transversely orientable. In this paper, flows mean orientable 1–dimensional foliations.

Lie foliations were first defined by E Fedida [4]. A classical example of a Lie foliation is a homogeneous one. Through the results of several authors, it is recognized that the class of homogeneous Lie foliations is a large class in the class of Lie foliations, though of course these classes do not coincide. Therefore deciding which Lie foliations belong to the class of homogeneous Lie foliations is an important problem in Lie foliation theory.

P Caron and Y Carrière [2] proved that any Lie $\mathbb{R}^q$–flow without closed orbits is diffeomorphic to a linear flow on the $(q+1)$–dimensional torus, which is homogeneous. Carrière [3] proved that any Lie $\mathfrak{a}(2)$–flow is homogeneous. S Matsumoto and N Tsuchiya [13] proved that any Lie $\mathfrak{a}(2)$–foliation of a 4–dimensional manifold or its double covering is homogeneous.

In the case where $\mathfrak{g}$ is semisimple, M Llabrés and A Reventós constructed an example of Lie $\mathfrak{sl}_2(\mathbb{R})$–flow which is not homogeneous [12, Example 5.3].

In the case where $\mathfrak{g}$ is nilpotent, É Ghys [7] proved that any Lie $\mathfrak{g}$–flow is homogeneous. In the case where $\mathfrak{g}$ is solvable, we conjecture that any Lie $\mathfrak{g}$–flow is homogeneous.

In this paper, we study this problem in the case where $\mathfrak{g}$ is a 3–dimensional solvable Lie algebra.

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If a Lie $\mathfrak{g}$–flow $\mathcal{F}$ on M has a closed orbit, then any orbit is closed and M is an oriented S^1–bundle. In this case, the base space is diffeomorphic to a homogeneous space $\Gamma\backslash G$ and hence $\mathfrak{g}$ is unimodular. The total space M is, in general, not diffeomorphic to a homogeneous space. However, in the case where $\mathfrak{g}$ is of type (R) or $\mathfrak{g}$ is 3–dimensional, we can prove that the total space is a homogeneous space. More precisely, we obtain the following theorem.

Theorem A *Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathcal{F}$ be a Lie $\mathfrak{g}$–flow on a closed manifold M. Suppose that $\mathcal{F}$ has a closed orbit.*

(i) *If $\mathfrak{g}$ is of type (R) and unimodular, then $\mathcal{F}$ is diffeomorphic to the flow in Example 3.1.*

(ii) *If the dimension of $\mathfrak{g}$ is three and $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^0$, then $\mathcal{F}$ is diffeomorphic to the flow in Example 3.1.*

In particular, if $\mathfrak{g}$ is a 3–dimensional solvable Lie algebra and $\mathcal{F}$ has a closed orbit, then $\mathcal{F}$ is diffeomorphic to the flow in Example 3.1.

In the case where $\mathcal{F}$ has no closed orbits, we obtain the following theorem.

Theorem B *Let $\mathfrak{g}$ be a 3–dimensional solvable Lie algebra and $\mathcal{F}$ be a Lie $\mathfrak{g}$–flow on a closed manifold. Suppose that $\mathcal{F}$ has no closed orbits.*

(i) *If $\mathfrak{g}$ is isomorphic to either $\mathbb{R}^3$ or $\mathfrak{n}(3)$, then $\mathcal{F}$ is diffeomorphic to the flow in Example 3.1.*

(ii) *If $\mathfrak{g}$ is isomorphic to $\mathfrak{a}(3)$, then $\mathcal{F}$ is isomorphic to the flow in Example 3.3.*

(iii) *If $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_2^k$, then $\mathcal{F}$ is isomorphic to the flow in Example 3.4.*

(iv) *If $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^h$ and $h \neq 0$, then $\mathcal{F}$ is isomorphic to the flow in Example 3.5.*

(v) *If $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^0$, then $\mathcal{F}$ is isomorphic to the flow in Example 3.6.*

Since (see Llabrés and Reventós [12]) there does not exist a Lie $\mathfrak{g}_1$–flow on a closed manifold, we have the following corollary.

Corollary 1.1 *For any 3–dimensional solvable Lie algebra $\mathfrak{g}$, any Lie $\mathfrak{g}$–flow on a closed manifold is homogeneous.*

The contents of this paper are the following: Section 2 is devoted to recalling some basic definitions and properties of Lie foliations and Lie algebras. In Section 2A, we recall some basic definitions of Lie algebras. In Section 2B, we recall the classification of 3–dimensional solvable Lie algebras. In Section 2C, we recall the definition and some properties of Lie foliations. In Section 3, we construct some important examples of Lie $\mathfrak{g}$–flows, which are models of codimension-3 solvable Lie $\mathfrak{g}$–flows. In Section 4, we prove Theorem A. In Section 5, we construct a diffeomorphism between Lie flows without closed orbits according to the construction of Ghys [7]. In Section 6, by using the diffeomorphism constructed in Section 5, we prove Theorem B.

2 Preliminaries

2A Solvable Lie groups and solvable Lie algebras

Let $\mathfrak{g}$ be a q–dimensional real Lie algebra. The descending central series of $\mathfrak{g}$ is defined inductively by

$$C^0\mathfrak{g} = \mathfrak{g} \quad \text{and} \quad C^k\mathfrak{g} = [\mathfrak{g}, C^{k-1}\mathfrak{g}].$$

Similarly the derived series of $\mathfrak{g}$ is defined inductively by

$$D^0\mathfrak{g} = \mathfrak{g} \quad \text{and} \quad D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}].$$

A Lie algebra $\mathfrak{g}$ is nilpotent if there exists an integer k such that $C^k\mathfrak{g} = \{0\}$, and a connected Lie group G is nilpotent if the Lie algebra of G is nilpotent. Similarly a Lie algebra $\mathfrak{g}$ is solvable if there exists an integer k such that $D^k\mathfrak{g} = \{0\}$, and a connected Lie group G is solvable if the Lie algebra of G is solvable.

Let H and G be Lie groups, and let $\Phi\colon H \to \mathrm{Aut}(G)$ be a homomorphism. Then we can construct a new Lie group $H \ltimes_\Phi G$, which is called the semidirect product of H and G with respect to Φ, as follows. The semidirect product $H \ltimes_\Phi G$ is the direct product of the sets H and G endowed with the group structure via

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \cdot h_2,\ g_1 \cdot \Phi(h_1)(g_2)).$$

The Lie group H is naturally a subgroup of $H \ltimes_\Phi G$, and G is naturally a normal subgroup of $H \ltimes_\Phi G$.

Let $\mathfrak{g}$ be a Lie algebra and $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the adjoint representation of $\mathfrak{g}$.

Definition 2.1 A solvable Lie algebra $\mathfrak{g}$ is said to be of type (R) if all the eigenvalues of $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ are real for any $X \in \mathfrak{g}$. A simply connected solvable Lie group is said to be of type (R) if the Lie algebra of G is of type (R).

It is well known that simply connected solvable Lie groups of type (R) have similar properties of simply connected nilpotent Lie groups; see [9].

2B Classification of 3–dimensional solvable Lie algebras

It is well known that 1–dimensional Lie algebras are isomorphic to $\mathbb{R}$ and that 2–dimensional Lie algebras are isomorphic to either $\mathbb{R}^2$ or $\mathfrak{a}(2)$, where

$$\mathfrak{a}(2) = \left\{ \begin{pmatrix} t & x \\ 0 & 0 \end{pmatrix} \,\middle|\, t, x \in \mathbb{R} \right\}$$

is the Lie algebra of $A(2)$, which is the affine transformation group of the real line.

Let $V = \langle T, X, Y \rangle_{\mathbb{R}}$ be a 3–dimensional vector space and consider the following Lie brackets on V:

- $\mathbb{R}^3$ (abelian): $[T, X] = [T, Y] = [X, Y] = 0$;
- $\mathfrak{n}(3)$ (Heisenberg): $[T, Y] = X$ and $[T, X] = [X, Y] = 0$;
- $\mathfrak{a}(3)$ (affine): $[T, X] = X$ and $[T, Y] = [X, Y] = 0$;
- $\mathfrak{g}_1$: $[T, X] = X + Y$, $[T, Y] = Y$ and $[X, Y] = 0$;
- $\mathfrak{g}_2^k$: $[T, X] = X$, $[T, Y] = kY$ and $[X, Y] = 0$ where $k \neq 0$;
- $\mathfrak{g}_3^h$: $[T, X] = Y$, $[T, Y] = -X + hY$ and $[X, Y] = 0$ where $h^2 < 4$.

Then any 3–dimensional solvable Lie algebra is isomorphic to one of the above Lie algebras.

It is well known that $\mathfrak{n}(3)$ is the Lie algebra of the 3–dimensional Heisenberg group

$$N(3) = \left\{ \begin{pmatrix} 1 & t & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

We will need an explicit description of simply connected Lie groups corresponding to the Lie algebras $\mathfrak{a}(3)$, $\mathfrak{g}_2^k$ and $\mathfrak{g}_3^h$. These Lie groups are given by

$$A(3) = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, t, x, y \in \mathbb{R} \right\},$$

$$G_2^k = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{kt} & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, t, x, y \in \mathbb{R} \right\},$$

$$G_3^{h=0} = \widetilde{\mathrm{SO}(2) \ltimes \mathbb{R}^2},$$

and

$$G_3^{h\neq 0}=\left\{\begin{pmatrix} c(t)\cos(\phi+t) & -c(t)\sin t & x\\ c(t)\sin t & c(t)\cos(\phi-t) & y\\ 0 & 0 & 1\end{pmatrix}\;\middle|\; t,x,y\in\mathbb{R}\right\},$$

where $\widetilde{\mathrm{SO}(2)\ltimes\mathbb{R}^2}$ is the universal covering of the group of rigid motions $\mathrm{SO}(2)\ltimes\mathbb{R}^2$, $c(t)=(2/\alpha)e^{\beta t}$, $\beta=\tan\phi=h/\alpha$, and $\alpha=\sqrt{4-h^2}$; see [5].

Note that G_3^0 is isomorphic to the semidirect product $\mathbb{R}\ltimes_\rho\mathbb{R}^2$, where $\rho\colon\mathbb{R}\to\mathrm{Aut}(\mathbb{R}^2)$ is given by

$$\rho(t)=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}.$$

Note also that the Lie group $G_3^{h\neq 0}$ has another description

$$G_3^{h\neq 0}=\left\{\begin{pmatrix} e^{\beta t}\cos t & -e^{\beta t}\sin t & x\\ e^{\beta t}\sin t & e^{\beta t}\cos t & y\\ 0 & 0 & 1\end{pmatrix}\;\middle|\; t,x,y\in\mathbb{R}\right\},$$

where $\beta=\tan\phi=h/\alpha$ and $\alpha=\sqrt{4-h^2}$. In this paper, we will use this description.

Lie algebras $\mathfrak{g}_2^k$ and $\mathfrak{g}_2^{k'}$ are isomorphic if and only if $k=k'$ or $k=1/k'$ and the Lie algebras $\mathfrak{g}_3^h$ and $\mathfrak{g}_3^{h'}$ are isomorphic if and only if $h=h'$ or $h=-h'$. The Lie algebra $\mathfrak{g}_2^k$ is unimodular if and only if $k=-1$. The Lie algebra $\mathfrak{g}_3^h$ is unimodular if and only if $h=-1$. The Lie algebra $\mathfrak{g}_3^h$ is not of type (R) for any h and the other 3–dimensional solvable Lie algebras are of type (R).

2C Lie foliations

Let $\mathcal{F}$ be a codimension-q foliation of a closed manifold M and $\mathfrak{g}$ be a q–dimensional real Lie algebra. A $\mathfrak{g}$–valued 1–form ω on M is said to be a Maurer–Cartan form if ω satisfies the equation $d\omega+\frac{1}{2}[\omega,\omega]=0$ and nonsingular if $\omega_x\colon T_xM\to\mathfrak{g}$ is surjective for each $x\in M$.

Definition 2.2 A codimension-q foliation $\mathcal{F}$ is a Lie $\mathfrak{g}$–foliation if there exists a nonsingular $\mathfrak{g}$–valued Maurer–Cartan form ω such that $\mathrm{Ker}(\omega)=T\mathcal{F}$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be foliations of M_1 and M_2, respectively. A smooth map $f\colon M_1\to M_2$ preserves foliations if $f(L)\in\mathcal{F}_2$ for every leaf $L\in\mathcal{F}_1$. We denote such a map by $f\colon(M_1,\mathcal{F}_1)\to(M_2,\mathcal{F}_2)$. We call two foliations $\mathcal{F}_1$ of M_1 and $\mathcal{F}_2$ of M_2 diffeomorphic if there exists a foliation preserving map $f\colon(M_1,\mathcal{F}_1)\to(M_2,\mathcal{F}_2)$ such that $f\colon M_1\to M_2$ is a diffeomorphism.

In this paper, we call two Lie $\mathfrak{g}$–foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ diffeomorphic if $\mathcal{F}_1$ is diffeomorphic to $\mathcal{F}_2$ as a foliation.

Fedida [4] proved that Lie $\mathfrak{g}$–foliations have a special property.

Theorem 2.3 [4] *Let $\mathcal{F}$ be a codimension-q Lie $\mathfrak{g}$–foliation of a closed manifold M and G be the simply connected Lie group of $\mathfrak{g}$. Let $p\colon \widetilde{M} \to M$ be the universal covering of M. Fix a Maurer–Cartan form $\omega \in A^1(M;\mathfrak{g})$ of $\mathcal{F}$. Then there exists a locally trivial fibration $D\colon \widetilde{M} \to G$ and a homomorphism $h\colon \pi_1(M) \to G$ such that*

(1) $D(\alpha \cdot \tilde{x}) = h(\alpha) \cdot D(\tilde{x})$ *for any $\alpha \in \pi_1(M)$ and any $\tilde{x} \in \widetilde{M}$, and*

(2) *the lifted foliation $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ coincides with the fibers of the fibration D.*

The fibration D is called the developing map, the homomorphism h is called the holonomy homomorphism and the image of h is called the holonomy group of the Lie $\mathfrak{g}$–foliation $\mathcal{F}$ with respect to the Maurer–Cartan form ω.

Conversely, if there exist D and h which satisfy condition (1) above, then the set of fibers of D defines a Lie $\mathfrak{g}$–foliation $\mathcal{F}$ of M such that the developing map is D and the holonomy homomorphism is h.

Example 2.4 Let G be a simply connected Lie group and $\widetilde{G}$ a simply connected Lie group with a uniform lattice Δ. Suppose that there exists a short exact sequence

$$0 \to K \to \widetilde{G} \xrightarrow{D_0} G \to 0.$$

Then the map D_0 defines a Lie $\mathfrak{g}$–foliation $\mathcal{F}_0$ of the homogeneous space $\Delta\backslash\widetilde{G}$.

We call Lie $\mathfrak{g}$–foliations constructed as in Example 2.4 homogeneous Lie $\mathfrak{g}$–foliations.

Definition 2.5 A Lie $\mathfrak{g}$–foliation $\mathcal{F}$ of a closed manifold M is homogeneous if $\mathcal{F}$ is diffeomorphic to a homogeneous Lie $\mathfrak{g}$–foliation.

Let $D\colon \widetilde{M} \to G$ be the developing map and $h\colon \pi_1(M) \to G$ be the holonomy homomorphism of a Lie $\mathfrak{g}$–foliation $\mathcal{F}$. Let $\Gamma = h(\pi_1(M))$ be the holonomy group of $\mathcal{F}$. Since the developing map D is h–equivariant, the map D induces a fibration

$$\overline{D}\colon M \to \overline{\Gamma}\backslash G,$$

where $\overline{\Gamma}$ is the closure of Γ. This fibration $\overline{D}$ is called the basic fibration, the homogeneous space $\overline{\Gamma}\backslash G$ the basic manifold, and the dimension of $\overline{\Gamma}\backslash G$ the basic dimension of $\mathcal{F}$.

Let $\overline{\mathcal{F}}$ be the foliation of M defined by the fibers of the fibration $\overline{D}$. By the definition of $\overline{D}$, we can see that any leaf F of $\overline{\mathcal{F}}$ is saturated by $\mathcal{F}$ and the foliation $\mathcal{F}|_F$ is a minimal foliation of F. Moreover the basic fibration $\overline{D}\colon M \to \overline{\Gamma}\backslash G$ induces a diffeomorphism from the leaf space $M/\overline{\mathcal{F}}$ to $\overline{\Gamma}\backslash G$.

3 Models of Lie flows

In this section, we construct some examples of homogeneous Lie flows which are important examples in this paper.

Example 3.1 Let

$$1 \to \mathbb{R} \to \widetilde{G} \xrightarrow{D_0} G \to 1$$

be a central exact sequence of Lie groups and Δ be a uniform lattice of $\widetilde{G}$. Then the surjective homomorphism $D_0\colon \widetilde{G} \to G$ defines a Lie $\mathfrak{g}$–flow $\mathcal{F}_0$ on $\Delta\backslash\widetilde{G}$.

The Lie $\mathfrak{g}$–flow construction in Example 3.1 is a special case of the construction of homogeneous Lie $\mathfrak{g}$–flows.

In the case in which the dimension of $\mathfrak{g}$ is three, by using the classification of 4–dimensional solvable Lie algebras (see [1]), we have more explicit descriptions of $\widetilde{G}$ and D_0.

Example 3.2 Let $\mathfrak{g}$ be a unimodular 3–dimensional solvable Lie algebra and G be the simply connected Lie group with the Lie algebra $\mathfrak{g}$. Then any central extension

$$1 \to \mathbb{R} \to \widetilde{G} \xrightarrow{D_0} G \to 1$$

of G by $\mathbb{R}$ is given as follows:

(1) If $\mathfrak{g}$ is abelian, then $\widetilde{G}$ is isomorphic to either $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ or $\mathbb{R} \times N(3)$. If $\widetilde{G}$ is isomorphic to $\mathbb{R}^4$, then $D_0\colon \mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ is given by the natural projection

$$D_0\colon (t, x, y, z) \mapsto (x, y, z).$$

If $\widetilde{G}$ is isomorphic to $\mathbb{R} \times N(3)$, then $D_0\colon \mathbb{R} \times N(3) \to \mathbb{R}^3$ is given by

$$D_0\colon \left(s, \begin{pmatrix} 1 & t & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) \mapsto A \begin{pmatrix} s \\ t \\ y \end{pmatrix},$$

where $A \in \mathrm{GL}(3; \mathbb{R})$.

(2) If $\mathfrak{g}$ is isomorphic to $\mathfrak{n}(3)$, then $\widetilde{G}$ is isomorphic to $\mathbb{R} \times N(3)$ or to

$$N(4) = \left\{ \begin{pmatrix} 1 & t & \frac{1}{2}t^2 & x \\ 0 & 1 & t & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \,\middle|\, t, x, y, z \in \mathbb{R} \right\}.$$

If $\widetilde{G}$ is isomorphic to $\mathbb{R}\times N(3)$, then $D_0\colon \mathbb{R}\times N(3)\to N(3)$ is given by the natural projection. If $\widetilde{G}$ is isomorphic to $N(4)$, then $D_0\colon N(4)\to N(3)$ is given by

$$\begin{pmatrix}1 & t & \frac{1}{2}t^2 & x\\ 0 & 1 & t & y\\ 0 & 0 & 1 & z\\ 0 & 0 & 0 & 1\end{pmatrix}\mapsto\begin{pmatrix}1 & at+cz & \frac{1}{2}(abt^2+cdz^2)+(ad-1)tz+y\\ 0 & 1 & bt+dz\\ 0 & 0 & 1\end{pmatrix},$$

where either

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\qquad\text{or}\qquad\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & 0\\ k & 1\end{pmatrix}\quad\text{for } k\in\mathbb{R}.$$

(3) If $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_2^{-1}$, then $\widetilde{G}$ is isomorphic to $\mathbb{R}\times G_2^{-1}$ or the semidirect product $\mathbb{R}\ltimes_\Phi N(3)$ with respect to the homomorphism $\Phi\colon\mathbb{R}\to\operatorname{Aut}(N(3))$ defined by

$$\Phi(s)\colon\begin{pmatrix}1 & t & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\mapsto\begin{pmatrix}1 & e^st & x\\ 0 & 1 & e^{-s}y\\ 0 & 0 & 1\end{pmatrix}.$$

If $\widetilde{G}$ is isomorphic to $\mathbb{R}\times G_2^{-1}$, then $D_0\colon\mathbb{R}\times G_2^{-1}\to G_2^{-1}$ is given by the natural projection. If $\widetilde{G}$ is isomorphic to $\mathbb{R}\ltimes_\Phi N(3)$, then the homomorphism

$$D_0\colon\mathbb{R}\ltimes_\phi N(3)\to G_2^{-1}$$

is given by

$$D_0\colon\left(s,\begin{pmatrix}1 & t & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\right)\mapsto\begin{pmatrix}e^s & 0 & t\\ 0 & e^{-s} & y\\ 0 & 0 & 1\end{pmatrix}.$$

(4) If $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^0$, then $\widetilde{G}$ is isomorphic to either $\mathbb{R}\times G_3^0$ or the semidirect product $\mathbb{R}\ltimes_\Psi N(3)$ with respect to the homomorphism $\Psi\colon\mathbb{R}\to\operatorname{Aut}(N(3))$ defined by

$$\begin{pmatrix}1 & t & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\mapsto\begin{pmatrix}1 & t\cos s-y\sin s & x-ty\sin^2 s+\frac{1}{4}(t^2-y^2)\sin 2s\\ 0 & 1 & t\sin s+y\cos s\\ 0 & 0 & 1\end{pmatrix}.$$

If $\widetilde{G}$ is isomorphic to $G_3^0\times\mathbb{R}$, then $D_0\colon\mathbb{R}\times G_3^0\to G_3^0$ is given by the natural projection. If $\widetilde{G}$ is isomorphic to $\mathbb{R}\ltimes_\Psi N(3)$, then the homomorphism

$$D_0\colon\mathbb{R}\ltimes_\Psi N(3)\to G_3^0=\mathbb{R}\ltimes_\rho\mathbb{R}^2$$

is given by

$$D_0\colon\left(s,\begin{pmatrix}1 & t & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\right)\mapsto\left(s,\begin{pmatrix}t\\ y\end{pmatrix}\right).$$

Define two classes of 4–dimensional solvable Lie groups $\widetilde{G}_2^k$ and $\widetilde{G}_3^h$ by

$$\widetilde{G}_2^k = \left\{ \begin{pmatrix} e^t & 0 & 0 & x \\ 0 & e^{kt} & 0 & y \\ 0 & 0 & e^{-(1+k)t} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, t, x, y, z \in \mathbb{R} \right\}$$

and

$$\widetilde{G}_3^h = \left\{ \begin{pmatrix} e^{\beta t}\cos t & -e^{\beta t}\sin t & 0 & x \\ e^{\beta t}\sin t & +e^{\beta t}\cos t & 0 & y \\ 0 & 0 & d^t & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, t, x, y, z \in \mathbb{R} \right\},$$

where $k \in \mathbb{R}$, $0 < h^2 < 4$, and $d = e^{-2\beta} \in \mathbb{R}$. We construct homogeneous Lie $\mathfrak{g}$–flows on the homogeneous spaces $\Delta\backslash\widetilde{G}_2^k$ and $\Delta\backslash\widetilde{G}_3^h$.

Example 3.3 Let Δ be a uniform lattice of $\widetilde{G}_2^0$. Define a homomorphism

$$D_0\colon \widetilde{G} \to A(3) \quad \text{by} \quad \begin{pmatrix} e^t & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & e^{-t} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e^t & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then D_0 defines a homogeneous Lie $\mathfrak{a}(2)$–flow on $\Delta\backslash\widetilde{G}$.

Example 3.4 Assume that the Lie group $\widetilde{G}_2^k$ has a uniform lattice Δ. Define a homomorphism $D_0\colon \widetilde{G}_2^k \to G_2^k$ by

$$D_0\colon \begin{pmatrix} e^t & 0 & 0 & x \\ 0 & e^{kt} & 0 & y \\ 0 & 0 & e^{-(1+k)t} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e^t & 0 & x \\ 0 & e^{kt} & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then D_0 defines a homogeneous Lie $\mathfrak{g}_2^k$–flow on $\Delta\backslash\widetilde{G}$.

Example 3.5 We assume that $\widetilde{G}_3^h$ has a uniform lattice Δ. Define a homomorphism $D_0\colon \widetilde{G}_3^h \to G_3^h$ by

$$D_0\colon \begin{pmatrix} e^{\beta t}\cos t & -e^{\beta t}\sin t & 0 & x \\ e^{\beta t}\sin t & e^{\beta t}\cos t & 0 & y \\ 0 & 0 & d^t & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e^{\beta t}\cos t & -e^{\beta t}\sin t & x \\ e^{\beta t}\sin t & e^{\beta t}\cos t & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then D_0 defines a homogeneous Lie $\mathfrak{g}_3^{h\neq 0}$–flow on $\Delta\backslash\widetilde{G}_3^h$.

Example 3.6 Let Δ be a uniform lattice of $G_3^0 \times \mathbb{R}$. Let $D_0\colon G_3^0 \times \mathbb{R} \to G_3^0$ be the natural homomorphism. Then D_0 defines a homogeneous Lie $\mathfrak{g}_3^0$–flow on $\Delta \backslash G_3^0$.

Remark We can extend the definition of $\widetilde{G}_3^h$ to the case when $h = 0$. Then $\widetilde{G}_3^0$ coincides with $\mathrm{SO}(2) \ltimes \mathbb{R}^2 \times \mathbb{R}$, which is not simply connected. The homomorphism $D_0\colon G_3^0 \times \mathbb{R} \to G_3^0$ defined in Example 3.6 coincides with the lifted homomorphism $D_0\colon \mathrm{SO}(2) \ltimes \mathbb{R}^2 \times \mathbb{R} \to \mathrm{SO}(2) \ltimes \mathbb{R}^2$ defined in Example 3.5.

4 Proof of Theorem A

Let $\mathcal{F}$ be a Lie $\mathfrak{g}$–flow on a closed manifold M. Assume that $\mathcal{F}$ has a closed orbit. Then any orbit of $\mathcal{F}$ is closed.

Let $D\colon \widetilde{M} \to G$ be the developing map and $h\colon \pi_1(M) \to G$ be the holonomy homomorphism. Since any orbit of $\mathcal{F}$ is closed, the holonomy group Γ is discrete in G and the basic fibration $\overline{D}\colon M \to \Gamma \backslash G$ is an oriented S^1–bundle over the homogeneous space $\Gamma \backslash G$.

Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, which is naturally identified with the set of left-invariant 1–forms on G. Consider the inclusion map

$$\iota\colon \textstyle\bigwedge^* \mathfrak{g} \to A^*(\Gamma \backslash G)$$

and the induced map

$$\iota\colon H^*(\mathfrak{g}) \to H^*_{\mathrm{dR}}(\Gamma \backslash G),$$

where $H^*(\mathfrak{g})$ is the cohomology of the Lie algebra $\mathfrak{g}$, $A^*(\Gamma \backslash G)$ is the de Rham complex of $\Gamma \backslash G$, and $H^*_{\mathrm{dR}}(\Gamma \backslash G)$ is the de Rham cohomology of $\Gamma \backslash G$. We call a k–form $\omega \in A^k(\Gamma \backslash G)$ algebraic if ω is in $\iota(A^k(\mathfrak{g}))$.

Let $e(\overline{D}) \in H^2_{\mathrm{dR}}(\Gamma \backslash G)$ be the real Euler class of the S^1–bundle $\overline{D}$. We use the following lemma, which is a special case of [12, Theorem 5.1].

Lemma 4.1 *If $e(\overline{D})$ is represented by an algebraic 2–form, then $\mathcal{F}$ is homogeneous.*

Suppose the Euler class $e(\overline{D})$ is represented by an algebraic 2–form $\iota(\beta) \in A^2(\Gamma \backslash G)$. Then there exists a homogeneous Lie $\mathfrak{g}$–flow $(\Delta \backslash \widetilde{G}, \mathcal{F}_0)$ which is diffeomorphic to $(M, \mathcal{F})$. By the proof of [12, Theorem 5.1], the Lie algebra $\tilde{\mathfrak{g}}$ of $\widetilde{G}$ coincides with the central extension

$$0 \to \mathbb{R} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of $\mathfrak{g}$ by $\mathbb{R}$ with the Euler class $-2[\beta] \in H^2(\mathfrak{g})$. Hence $\widetilde{G}$ is a central extension of G by $\mathbb{R}$. Therefore we have the following proposition.

Proposition 4.2 *If $e(\overline{D})$ is represented by an algebraic 2–form, then $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}$–flow in Example 3.1.*

Proof of Theorem A First, we assume that $\mathfrak{g}$ is a solvable Lie algebra of type (R). Let $e(\overline{D}) \in H^2_{\mathrm{dR}}(\Gamma\backslash G)$ be the real Euler class of the oriented S^1–bundle.

Since $\mathfrak{g}$ is of type (R), by Hattori [9, Theorem 4.1], the homomorphism

$$\iota\colon H^*(\mathfrak{g}) \to H^*_{\mathrm{dR}}(\Gamma\backslash G)$$

is an isomorphism. Therefore $e(\overline{D})$ is represented by an algebraic 2–form. Hence, by Proposition 4.2, $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}$–flow in Example 3.1.

Next, we assume that $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^0$. By the classification of uniform lattices of G_3^0, the homogeneous space $\Gamma\backslash G_3^0$ is isomorphic to the mapping torus

$$T_A^3 = T^2 \times \mathbb{R}/(\boldsymbol{x}, t+1) \sim (A\boldsymbol{x}, t),$$

where $A \in \mathrm{SL}(2;\mathbb{Z})$ such that $A^p = I$ for some $p \in \mathbb{Z}$. If $A = I$, then the mapping torus T_A^3 is the three-dimensional torus T^3.

Define left-invariant 1–forms θ_T, θ_X and θ_Y on $G_3^0 = \mathbb{R} \ltimes_\rho \mathbb{R}^2$ by

$$\theta_T = dt, \quad \theta_X = \cos t \cdot dx + \sin t \cdot dy \quad \text{and} \quad \theta_Y = -\sin t \cdot dx + \cos t \cdot dy.$$

Then the second cohomology $H^2(\mathfrak{g}_3^0)$ of the Lie algebra $\mathfrak{g}_3^0$ is generated by the cohomology class $[\theta_X \wedge \theta_Y] = [dx \wedge dy]$.

On the other hand, we have

$$H^2_{\mathrm{dR}}(T_A^3) = \begin{cases} \mathbb{R}[dt \wedge dx] \oplus \mathbb{R}[dt \wedge dy] \oplus \mathbb{R}[dx \wedge dy] & \text{if } A = I, \\ \mathbb{R}[dx \wedge dy] & \text{if } A \neq I. \end{cases}$$

Therefore if $A \neq I$, then $H^2_{\mathrm{dR}}(T_A^3)$ is isomorphic to $H^2(\mathfrak{g}_3^0)$ and the Euler class $e(\overline{D})$ is represented by an algebraic 2–form. Hence $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}_3^0$–flow in Example 3.1.

In the case where $A = I$, by the following lemma, there exists a diffeomorphism $f\colon T^3 \to T^3$ such that the pullback $f^* e(\overline{D})$ is represented by an algebraic 2–form. Then the S^1–bundle M is diffeomorphic to the S^1–bundle f^*M, which is diffeomorphic to the Lie $\mathfrak{g}$–flow in Example 3.1. □

Lemma 4.3 *For any $[\omega] = a[dt \wedge dx] + b[dt \wedge dy] + c[dx \wedge dy] \in H^2(T^3;\mathbb{Z})$, there exists an integer matrix $A \in \mathrm{SL}(3;\mathbb{Z}) \subset \mathrm{Diff}(T^3)$ such that $A^*[\omega] \in \mathbb{Z}[dx \wedge dy]$.*

Proof By the Smith normal form, we can show that there exist an integer d and an invertible 3×3 integer matrix B such that

$$B\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}.$$

Therefore there exists $A \in \mathrm{SL}(3;\mathbb{Z})$ and an integer n such that $A^*[\omega] = n[dx\wedge dy]$ for some $n\in\mathbb{Z}$. $\square$

5 A construction of a diffeomorphism of flows

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Lie $\mathfrak{g}$–flows on closed manifolds M_1 and M_2 and let Γ_1 and Γ_2 be the holonomy groups of $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ have no closed orbits and Γ_1 is conjugate to Γ_2 in G.

By replacing the developing map $D_1\colon \widetilde{M}_1 \to G$ and the holonomy homomorphism $h_1\colon \pi_1(M_1)\to G$ of $\mathcal{F}_1$ by

$$g\cdot D_1\colon \widetilde{M}_1 \to G \qquad \text{and} \qquad g^{-1}\cdot h_1\cdot g\colon \pi_1(M_1)\to G$$

for some $g\in G$, we may assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same holonomy group Γ. The aim of this section is to construct a diffeomorphism between $(M_1,\mathcal{F}_1)$ and $(M_2,\mathcal{F}_2)$ according to Ghys's method; see [7; 6; 14].

By results of Haefliger [8, Section 3], a Lie $\mathfrak{g}$–foliation of a closed manifold M is a classifying space for (G,Γ) if every leaf of $\mathcal{F}$ is contractible. Thus $(M_1,\mathcal{F}_1)$ and $(M_2,\mathcal{F}_2)$ are classifying spaces for (G,Γ). By the uniqueness of classifying spaces, there exists a homotopy equivalence $f\colon M_1\to M_2$, which we may assume is smooth, such that $f^*\mathcal{F}_2=\mathcal{F}_1$. In general, this map f is not a diffeomorphism. However, by using the averaging technique (see [7; 6]), we can modify f to a diffeomorphism from $(M_1,\mathcal{F}_1)$ to $(M_2,\mathcal{F}_2)$.

Parametrize $\mathcal{F}_1$ and $\mathcal{F}_2$ by ϕ_1^t and ϕ_2^t, respectively. Then we can define a smooth function

$$u\colon M_1\times\mathbb{R}\to\mathbb{R}$$

by the equation

$$f(\phi_1^t(x)) = \phi_2^{u(x,t)}(f(x)).$$

The function u satisfies the cocycle condition

$$u(x,s+t) = u(x,t) + u(\phi_1^t(x),s).$$

By this equation and by the compactness of M, we obtain the following lemma.

Lemma 5.1 *There exists a constant* $C > 0$ *such that*

$$\left| \frac{\partial}{\partial t} u(x,t) \right| < C$$

for any $x \in M$ *and* $t \in \mathbb{R}$.

Let $\pi_1 = \overline{D}_1 \colon M_1 \to Q_1 = \overline{\Gamma} \backslash G$ be the basic fibration of $\mathcal{F}_1$. Fix a fiber F_1 of $\pi_1 \colon M_1 \to Q_1$. Let $\{(U_i, g_i)\}_{i=1}^k$ be a local trivialization of π_1, where $\{U_i\}_{i=1}^k$ is an open covering of Q_1 and $g_i \colon \pi_1^{-1}(U_i) \to U_i \times F_1$ is a diffeomorphism as fiber bundles. Let $\{V_1, \dots, V_k\}$ be a refinement of $\{U_1, \dots, U_k\}$ such that the closure $\overline{V}_i$ is contained in U_i for each $i = 1, \dots, k$.

Since f is transverse to the flow $\mathcal{F}_2$, by replacing $V_1, \dots, V_k$ with smaller ones if necessary, for each i there exists a codimension-one open ball $D_i \subset F_1$ such that

(1) $N_i := g_i^{-1}(V_i \times D_i)$ is transverse to $\mathcal{F}_1$,

(2) $f(N_i)$ is transverse to $\mathcal{F}_2$, and

(3) the restriction $f|_{N_i} \colon N_i \to f(N_i)$ is a diffeomorphism.

Lemma 5.2 *There exists* $T_0 > 0$ *such that, for any* $i = 1, \dots, k$ *and any* $x \in \pi_1^{-1}(\overline{V}_i)$, *the orbit* $O_{\phi_1}(x; (0, T_0)) := \{\phi_1^t(x) \mid 0 < t < T_0\}$ *intersects* N_i.

Proof Define a function $r_i \colon \pi_1^{-1}(U_i) \to \mathbb{R}$ by

$$r_i(x) = \inf\{t > 0 \mid \phi_1^t(x) \in N_i\}.$$

Since $\mathcal{F}_1$ is minimal on each fiber, the function r_i is well-defined and upper semicontinuous. Since $\pi_1^{-1}(\overline{V}_i)$ is compact, for each $i \in \{1, \dots, k\}$, there exists an upper bound T_i. Then we should take $T_0 = \max\{T_1, \dots, T_k\} + 1$. □

For any $x \in N_i$, we define $s_i(x) \in \mathbb{Z}$ to be the number of times that the orbit $O_{\phi_2}\big(f(x); (-2CT_0 - \delta, 2CT_0 + \delta)\big)$ intersects $f(N_i)$, where δ is a small positive number and C and T_0 are the constants in Lemmas 5.1 and 5.2, respectively. By the choice of N_i, the function $s \colon N_i \to \mathbb{Z}$ is bounded.

For any $x \in N_i$, we consider the set

$$T_i(x) = \big\{t \in \mathbb{R} \bigm| \phi_1^t(x) \in N_i \text{ and } |u(x,t)| < 2CT + \delta\big\}.$$

If t and t' satisfy $u(x,t) = u(x,t')$ and $\phi_1^t(x)$ and $\phi_1^{t'}(x)$ are in N_i, then we have $f(\phi_1^t(x)) = f(\phi_1^{t'}(x))$. Since $f|_{N_i}$ is a diffeomorphism, we have $\phi_1^t(x) = \phi_1^{t'}(x)$. Since $\mathcal{F}_1$ has no closed orbits, this implies that $t = t'$. Hence, if t and t' are distinct points of $T_i(x)$, then $u(x,t) \neq u(x,t')$. Therefore the number of elements of $T_i(x)$ is less than $s_i(x)$ and hence bounded.

For an arbitrary point $x \in N_i$, we can take a sufficiently small connected neighborhood A_x of x in N_i and a sufficiently large number $t_x > 0$ such that t_x is an upper bound of $T_i(y)$ for any $y \in A_x$. Then we have

$$|u(y,t)| \geq 2CT_0 + \delta > 2CT_0$$

if $t > t_x$, $y \in A_x$ and $\phi_1^t(y) \in N_i$. Since the basic manifold $Q_1 = \overline{\Gamma}\backslash G$ is compact, we can choose connected open subsets $A_{i_1}, \dots, A_{i_{k_i}}$ of N_i and large numbers t_{i_j} such that

(1) $\{\pi_1(A_{i_j}) \mid i = 1, \dots, k,\ j = 1, \dots, k_i\}$ is a refinement of $\{V_1, \dots, V_k\}$, and

(2) t_{i_j} is an upper bound of $T_i(y)$ for any $y \in A_{i_j}$.

Let $t_0 > \max\{t_{i_j} \mid i = 1, \dots, k,\ j = 1, \dots, k_i\}$ be an arbitrary number. Then, for any $x \in A_{i_j}$, we have

$$|u(x,t)| > 2CT_0$$

if $t \geq t_0$ and $\phi_1^t(x) \in N_i$.

Lemma 5.3 *For any i_j, one of the following holds:*

(a) $u(x,t) > CT_0$ *for any* $x \in A_{i_j}$ *and any* $t \geq t_0$.

(b) $u(x,t) < -CT_0$ *for any* $x \in A_{i_j}$ *and any* $t \geq t_0$.

Proof Let $x \in A_{i_j}$ be an arbitrary point. Let $t_0 = s_0 < s_1 < s_2 < \cdots$ be the maximal sequence such that $\phi_1^{s_l}(x) \in N_i$ for $l \geq 1$. By Lemma 5.2, we obtain $s_{l+1} - s_l < T_0$ for any $l \geq 0$. On the other hand, we have $|u(x,s_l)| > 2CT_0$ for any $l \geq 1$.

Lemma 5.1 implies that

$$|u(x,s_{l+1}) - u(x,s_l)| < C(s_{l+1} - s_l) < CT_0.$$

Hence we have either $u(x,s_l) > 2CT_0$ for any $l \geq 1$ or $u(x,s_l) < -2CT_0$ for any $l \geq 1$.

For any $t \geq t_0$, there exists $l \geq 0$ such that $s_l \leq t \leq s_{l+1}$. By Lemma 5.1, we have

$$|u(x,s_{l+1}) - u(x,t)| \leq C(s_{l+1} - t) < C(s_{l+1} - s_l) < CT_0.$$

Therefore we have either $u(x,t) > CT_0$ for any $t \geq t_0$ or $u(x,t) < -CT_0$ for any $t \geq t_0$. By the continuity of u, we have either $u(x,t) > CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$ or $u(x,t) < -CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$. □

Let $W_{i_j} = \pi_1(A_{i_j})$ and $E_{i_j} = \pi_1^{-1}(W_{i_j})$.

Lemma 5.4 *There exists τ_0 such that, for each E_{i_j}, one of the following holds:*

(a) $u(x,t) > 0$ *for any* $x \in E_{i_j}$ *and any* $t \geq \tau_0$.

(b) $u(x,t) < 0$ *for any* $x \in E_{i_j}$ *and any* $t \geq \tau_0$.

Proof By Lemma 5.2, we have

$$S_{i_j} = \sup_{x \in E_{i_j}} \inf\{t > 0 \mid \phi_1^t(x) \in A_{i_j}\} < \infty$$

for any i_j. Let

$$\begin{aligned} S &= \max\{S_{i_j} \mid 1 \leq i \leq k,\ 1 \leq j \leq k_i\}, \\ \alpha_+ &= \max\{u(x,t) \mid x \in M,\ 0 \leq t \leq S\}, \\ \alpha_- &= \max\{-u(x,t) \mid x \in M,\ 0 \leq t \leq S\}, \end{aligned}$$

and

$$\alpha = \max\{\alpha_+, \alpha_-\}.$$

Take an integer n and a constant τ_0 satisfying

$$nCT_0 > \alpha \quad \text{and} \quad \tau_0 > n(t_0 + S).$$

Fix i_j. By Lemma 5.3, we have either $u(y,t) > CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$ or $u(x,t) < -CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$.

First, we suppose that $u(x,t) > CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$. Fix an arbitrary point $x \in E_{i_j}$ and any $t \geq \tau_0$. Define a sequence $0 \leq v_1 < v_2 < \cdots < v_n$ inductively as follows: Let v_1 be the first arrival time of x to A_{i_j}. Thus we have $\phi_1^{v_1}(x) \in A_{i_j}$ and $0 \leq v_1 \leq S$. For $l \geq 1$, let v_{l+1} be the first arrival time to A_{i_j} of x after the time $v_l + t_0$. Thus we have $\phi_1^{v_{l+1}} \in A_{i_j}$ and $v_l + t_0 \leq v_{l+1} \leq v_l + t_0 + S$.

Since $v_1 \leq S$ and $v_{l+1} - v_l \leq t_0 + S$, we have

$$v_n \leq S + (n-1)(t_0 + S) < \tau_0 - t_0 \leq t - t_0.$$

Since $v_{l+1} - v_l \geq t_0$ and $t - v_n > t_0$, we have

$$\begin{aligned} u(x,t) &= u\Big(x, v_1 + \sum_{l=1}^{n-1}(v_{l+1} - v_l) + t - v_n\Big) \\ &= u(x, v_1) + \sum_{l=1}^{n-1} u(\phi_1^{v_l}(x), v_{l+1} - v_l) + u(\phi_1^{v_n}(x), t - v_n) \\ &> -\alpha_- + (n-1)CT_0 + CT_0 \\ &> -\alpha + nCT_0 > 0. \end{aligned}$$

In the case where $u(x,t) < -CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$, by the same argument, we have

$$\begin{aligned}
u(x,t) &= u\left(x, v_1 + \sum_{l=1}^{n-1}(v_{l+1} - v_l) + t - v_n\right) \\
&= u(x, v_1) + \sum_{l=1}^{n-1} u(\phi_1^{v_l}(x), v_{l+1} - v_l) + u(\phi_1^{v_n}(x), t - v_n) \\
&< \alpha_+ - (n-1)CT_0 - CT_0 \\
&< \alpha - nCT_0 < 0.
\end{aligned}$$

□

Finally, we prove the following lemma.

Lemma 5.5 *One of the following holds:*

(a) $u(x,t) > 0$ *for any* $x \in M_1$ *and any* $t \geq \tau_0$.

(b) $u(x,t) < 0$ *for any* $x \in M_1$ *and any* $t \geq \tau_0$.

Proof By the continuity of u and the connectedness of M, if there exists i_j such that $u(x,t) > 0$ for any $x \in E_{i_j}$ and any $t \geq \tau_0$, then $u(x,t) > 0$ for any $x \in M$ and any $t \geq \tau_0$. Similarly, if there exists i_j such that $u(x,t) < 0$ for any $x \in E_{i_j}$ and any $t \geq \tau_0$, then $u(x,t) < 0$ for any $x \in M$ and any $t \geq \tau_0$. □

We construct a diffeomorphism from $(M_1, \mathcal{F}_1)$ to $(M_2, \mathcal{F}_2)$. Let $T \in \mathbb{R}$ be a positive constant such that $T \geq \tau_0$. Define $\nu_T\colon M_1 \to \mathbb{R}$ and $f_T\colon M_1 \to M_2$ by

$$\nu_T(x) = \frac{1}{T}\int_0^T u(x,\tau)\,d\tau \quad \text{and} \quad f_T(x) = \phi_2^{\nu(x)}(f(x)).$$

By the equation

$$\begin{aligned}
\nu_T(\phi_1^t(x)) &= \frac{1}{T}\int_0^T u(\phi_1^t(x),\tau)\,d\tau \\
&= \frac{1}{T}\int_0^T \{u(x,t+\tau) - u(x,t)\}\,d\tau,
\end{aligned}$$

we have

$$f_T(\phi_1^t(x)) = \phi_2^{\frac{1}{T}\int_0^T u(x,t+\tau)\,d\tau}(f(x)).$$

Therefore, for any $x \in M_1$, we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{T}\int_0^T u(x,t+\tau)\,d\tau\right) &= \frac{1}{T}(u(x,t+T) - u(x,t)) \\
&= \frac{1}{T}u(\phi_1^t(x),T).
\end{aligned}$$

By Lemma 5.5, $u(x,T) \neq 0$ for any $x \in M_1$. Therefore $f_T\colon M_1 \to M_2$ is a local diffeomorphism. Since M_1 is closed, f_T is a covering map. Since f_T is homotopic to f via

$$F\colon M_1 \times [0,1] \to M_2 \quad \text{defined by } F(x,t) = \phi_2^{tv(x)}(f(x))$$

and f is a homotopy equivalence, the map f_T is a diffeomorphism.

Therefore we obtain the following theorem.

Theorem 5.6 [7] *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Lie $\mathfrak{g}$–flows on closed manifolds M_1 and M_2, respectively. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ have no closed orbits and the holonomy group of $\mathcal{F}_1$ is conjugate to the holonomy group of $\mathcal{F}_2$ in G. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are diffeomorphic.*

6 Proof of Theorem B

Let $\mathfrak{g}$ be a 3–dimensional solvable Lie algebra and let $\mathcal{F}$ be a Lie $\mathfrak{g}$–flow on a closed manifold M which has no closed orbits. Let Γ be the holonomy group of $\mathcal{F}$. Since $\mathcal{F}$ has no closed orbits, the holonomy homomorphism $h\colon \pi_1(M) \to G$ is injective. Hence the fundamental group $\pi_1(M)$ is isomorphic to the holonomy group Γ.

If the Lie algebra $\mathfrak{g}$ is nilpotent, then the Lie algebra $\mathfrak{g}$ is isomorphic to either $\mathbb{R}^3$ or $\mathfrak{n}(3)$. By the theorem of Ghys [7, Section 2], $(M,\mathcal{F})$ is diffeomorphic to a homogeneous Lie $\mathfrak{g}$–flow $(\Delta\backslash\widetilde{G}, \mathcal{F}_0)$, where $\widetilde{G}$ is a simply connected nilpotent Lie group. Since any 1–dimensional ideal of a nilpotent Lie algebra is contained in its center, the kernel of the induced homomorphism

$$dD_0\colon \tilde{\mathfrak{g}} \to \mathfrak{g}$$

is contained in the center of $\tilde{\mathfrak{g}}$. Hence $\widetilde{G}$ is a central extension of G by $\mathbb{R}$ and $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}$–flow in Example 3.1.

We suppose that $\mathfrak{g}$ is not nilpotent. First, we consider the case where $\mathfrak{g}$ is isomorphic to $\mathfrak{a}(3)$.

6A $\mathfrak{a}(3)$ case

Let $\mathcal{F}$ be a Lie $\mathfrak{a}(3)$–flow on a closed manifold M without closed orbits, and fix a nonsingular $\mathfrak{a}(3)$–valued Maurer–Cartan form ω of $\mathcal{F}$. The Lie algebra $\mathfrak{a}(3)$ has the explicit description

$$\mathfrak{a}(3) = \left\{ \begin{pmatrix} t & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, t, x, y \in \mathbb{R} \right\}.$$

Then there exist nonsingular 1–forms ω_T, ω_X and ω_Y on M such that

$$\omega = \begin{pmatrix} \omega_T & 0 & \omega_X \\ 0 & 0 & \omega_Y \\ 0 & 0 & 0 \end{pmatrix}.$$

Since ω is a Maurer–Cartan form, we have the following equations:

$$d\omega_T = 0 \quad \text{and} \quad d\omega_X = \tfrac{1}{2}\omega_T \wedge \omega_X \quad \text{and} \quad d\omega_Y = 0.$$

Therefore ω_T and ω_Y are nonsingular closed 1–forms and

$$\omega' = \begin{pmatrix} \omega_T & \omega_X \\ 0 & 0 \end{pmatrix}$$

is a nonsingular $\mathfrak{a}(2)$–valued Maurer–Cartan form of M. The nonsingular closed 1–form ω_T and the nonsingular Maurer–Cartan form ω' define two foliations $\mathcal{G}$ and $\mathcal{F}'$ of M whose codimensions are one and two, respectively. Since ω' is a $\mathfrak{a}(2)$–valued Maurer–Cartan form, the foliation $\mathcal{F}'$ is a Lie $\mathfrak{a}(2)$–foliation. By an observation of Matsumoto and Tsuchiya [13, Section 7], we can see that the closed 1–form ω_T is a rational form. Therefore each leaf of $\mathcal{G}$ is compact and the leaf space $M/\mathcal{G}$ is diffeomorphic to S^1.

Let $\pi\colon M \to S^1 = M/\mathcal{G}$ be the natural projection and fix a fiber N of π. Since the tangent bundle $T\mathcal{F}$ coincides with $\mathrm{Ker}(\omega)$ and $\mathrm{Ker}(\omega_T)$ includes $\mathrm{Ker}(\omega)$, each orbit of the Lie $\mathfrak{a}(3)$–flow $\mathcal{F}$ is tangent to the fibers of π.

Let $\mathcal{F}|_N$ be the foliation defined by the restriction of $\mathcal{F}$ to the fiber N.

Lemma 6.1 *The fiber N is diffeomorphic to the 3–dimensional torus and the flow $\mathcal{F}|_N$ is diffeomorphic to a linear flow.*

Proof The tangent bundle TN coincides with $\mathrm{Ker}(\omega_T|_N)$. By the equation

$$d\omega_X = \tfrac{1}{2}\omega_T \wedge \omega_X,$$

the 1–form $\omega_X|_N$ on N is a nonsingular closed 1–form. Since $T\mathcal{F}$ coincides with

$$\mathrm{Ker}(\omega) = \mathrm{Ker}(\omega_T) \cap \mathrm{Ker}(\omega_X) \cap \mathrm{Ker}(\omega_Y),$$

$T\mathcal{F}|_N$ coincides with $\mathrm{Ker}(\omega_X|_N) \cap \mathrm{Ker}(\omega_Y|_N)$. Since the 1–forms $\omega_X|_N$ and $\omega_Y|_N$ are closed, the nonsingular $\mathbb{R}^2$–valued 1–form

$$\eta_N = \begin{pmatrix} \omega_X|_N \\ \omega_Y|_N \end{pmatrix}$$

is a Maurer–Cartan form. Hence the flow $\mathcal{F}|N$ is a Lie $\mathbb{R}^2$–flow on N. By the theorem of Caron and Carrière [2, Theorem 1], the manifold N is diffeomorphic to T^3 and the flow $\mathcal{F}|_N$ is diffeomorphic to a linear flow. $\square$

By Lemma 6.1, the manifold M is a T^3–bundle over S^1. Let $F \in \mathrm{Diff}_+(T^3)$ be the monodromy map of the T^3–bundle $\pi\colon M \to S^1$. Fix generators α_1, α_2 and α_3 of $\pi_1(T^3) \simeq \mathbb{Z}^3$ and an element β of $\pi_1(M)$ such that $\pi_*(\beta) \in \pi_1(S^1)$ is a generator. Then the induced map $F_*\colon \pi_1(T^3) \to \pi_1(T^3)$ defines an integer matrix $A \in \mathrm{SL}(3;\mathbb{Z})$ and the fundamental group $\pi_1(M)$ is isomorphic to $\mathbb{Z} \ltimes_A \mathbb{Z}^3$.

Set $A = (a_{ij})$. Then we have

$$\beta\alpha_j\beta^{-1} = \alpha_1^{a_{1j}}\alpha_2^{a_{2j}}\alpha_3^{a_{3j}} \quad \text{for } j = 1,2,3.$$

Since $\mathcal{F}$ has no closed orbits, the holonomy homomorphism $h\colon \pi_1(M) \to \Gamma$ is an isomorphism. Let Γ' be the abelian subgroup of Γ generated by $h(\alpha_1)$, $h(\alpha_2)$ and $h(\alpha_3)$. Since $\pi_1(N)$ is a normal subgroup of $\pi_1(M)$, Γ' is a normal subgroup of Γ.

Lemma 6.2 *Let H be an abelian subgroup of $A(3)$. Then H is contained in either*

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle|\, x, y \in \mathbb{R} \right\} \quad \text{or} \quad H_{(t_0,x_0)} = \left\{ \begin{pmatrix} e^t & 0 & \frac{1-e^t}{1-e^{t_0}}x_0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle|\, t, y \in \mathbb{R} \right\}$$

for some $x_0 \in \mathbb{R}$ and $t_0 \neq 0$.

Proof Suppose that H is not contained in $\mathbb{R}^2$. Then there exists

$$g_0 = \begin{pmatrix} e^{t_0} & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \in H$$

such that $t_0 \neq 0$. Let

$$g = \begin{pmatrix} e^t & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in H$$

be an arbitrary element. Since H is abelian, we have $g_0 g = g g_0$. Then we obtain the equation

$$x = \frac{1-e^t}{1-e^{t_0}}x_0. \qquad \square$$

Lemma 6.3 *The abelian subgroup Γ' of Γ is contained in $\mathbb{R}^2$.*

Proof Suppose that Γ' is not contained in $\mathbb{R}^2$. By Lemma 6.2, Γ' is contained in $H_{(t_0,x_0)}$ for some $x_0 \in \mathbb{R}$ and $t_0 \neq 0$. Since $\Gamma' \not\subset \mathbb{R}^2$, there exists

$$g = \begin{pmatrix} e^t & 0 & \frac{1-e^t}{1-e^{t_0}} x_0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma'$$

such that $t \neq 0$.

Set

$$h(\beta) = \begin{pmatrix} e^{t'} & 0 & x' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} \in A(3).$$

Since Γ' is normal in Γ, we have

$$h(\beta) g h(\beta)^{-1} \in \Gamma' \subset H_{(t_0,x_0)}.$$

Then we obtain the equation

$$(1-e^t)\{(1-e^{t_0})x' - (1-e^{t'})x_0\} = 0.$$

Since $t \neq 0$, this equation implies that $h(\beta) \in H_{(t_0,x_0)}$. Thus Γ is contained in $H_{(t_0,x_0)}$. However, this contradicts the fact that the holonomy group Γ is uniform in $A(3)$. Therefore Γ' is contained in $\mathbb{R}^2$. $\square$

Set

$$h(\alpha_i) = \begin{pmatrix} 1 & 0 & x_i \\ 0 & 1 & y_i \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } i = 1,2,3 \qquad \text{and} \qquad h(\beta) = \begin{pmatrix} e^{t_0} & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since Γ is uniform in $A(3)$, we have $t_0 \neq 0$. Moreover, by conjugating in $A(3)$, we may assume that $x_0 = 0$.

Lemma 6.4 *A is conjugate to the matrix*

$$\begin{pmatrix} e^{t_0} & 0 & 0 \\ 0 & e^{-t_0} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof By the equation

$$\beta \alpha_j \beta^{-1} = \alpha_1^{a_{1j}} \alpha_2^{a_{2j}} \alpha_3^{a_{3j}},$$

we have

$$\begin{pmatrix} 1 & 0 & e^{t_0} x_j \\ 0 & 1 & y_j \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a_{1j} x_1 + a_{2j} x_2 + a_{3j} x_3 \\ 0 & 1 & a_{1j} y_1 + a_{2j} y_2 + a_{3j} y_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus we have the equations

$${}^t A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = e^{t_0} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad {}^t A \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Since Γ is uniform in $A(3)$, we can show that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq \mathbf{0} \quad \text{and} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \neq \mathbf{0}.$$

Therefore e^{t_0}, 1 and e^{-t_0} are the eigenvalues of A. □

Define elements $\widehat{\alpha}_i$ and $\widehat{\beta}$ of $\widetilde{G}_2^{-1}$ by

$$\widehat{\alpha}_i = \begin{pmatrix} 1 & 0 & 0 & x_i \\ 0 & 1 & 0 & y_i \\ 0 & 0 & 1 & z_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \widehat{\beta} = \begin{pmatrix} e^{t_0} & 0 & 0 & 0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & e^{-t_0} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

is an eigenvector of A corresponding to the eigenvalue e^{t_0}. Let Δ be the subgroup of $\widetilde{G}_2^{-1}$ generated by $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$ and $\widehat{\beta}$. Since

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

are eigenvectors of $A \in \mathrm{SL}(3;\mathbb{Z})$ corresponding to the eigenvalues e^{t_0}, 1 and e^{-t_0}, respectively, the subgroup Δ is discrete in $\widetilde{G}_2^{-1}$. Therefore Δ is a uniform lattice of $\widetilde{G}_2^{-1}$.

Define an submersion homomorphism $D_0\colon \widetilde{G}_2^{-1} \to A(3)$ by

$$D_0\colon \begin{pmatrix} e^t & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & e^{-t} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e^t & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then D_0 defines a Lie $\mathfrak{a}(3)$–flow $\mathcal{F}_0$ on $\Delta\backslash\widetilde{G}_2^{-1}$ whose holonomy group coincides with Γ. Therefore, by Theorem 5.6, the Lie $\mathfrak{a}(3)$–flow $\mathcal{F}$ is diffeomorphic to $\mathcal{F}_0$. Hence $\mathcal{F}$ is diffeomorphic to the flow in Example 3.3.

6B $\mathfrak{g}_2^k$ case

We consider the case where $\mathfrak{g}$ is isomorphic $\mathfrak{g}_2^k$. In this case, the basic dimension of $\mathcal{F}$ is one; see [5; 11]. Hence the manifold M is diffeomorphic to a T^3–bundle over S^1.

Let α_1, α_2, α_3 and β be the same as defined in Section 6A. Then there exists an integer matrix $A=(a_{ij})\in \mathrm{SL}(3;\mathbb{Z})$ such that the fundamental group $\pi_1(M)$ is isomorphic to $\mathbb{Z}\ltimes_A \mathbb{Z}^3$.

Let Γ' be the normal abelian subgroup of Γ generated by $h(\alpha_1)$, $h(\alpha_2)$ and $h(\alpha_3)$.

Lemma 6.5 *Let H be an abelian subgroup of G_2^k. Then H is contained in either*

$$\mathbb{R}^2=\left\{\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right\} \quad \text{or} \quad H_{(t_0,x_0,y_0)}=\left\{\begin{pmatrix} e^t & 0 & \frac{1-e^t}{1-e^{t_0}}x_0 \\ 0 & e^{kt} & \frac{1-e^{kt}}{1-e^{kt_0}}y_0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; t\in\mathbb{R}\right\}$$

for some $x_0, y_0\in\mathbb{R}$ and $t_0\neq 0$.

Proof Suppose that H is not contained in $\mathbb{R}^2$. Then there exists

$$g_0=\begin{pmatrix} e^{t_0} & 0 & x_0 \\ 0 & e^{kt_0} & y_0 \\ 0 & 0 & 1 \end{pmatrix}\in H$$

such that $t_0\neq 0$. Let

$$g=\begin{pmatrix} e^{t} & 0 & x \\ 0 & e^{kt} & y \\ 0 & 0 & 1 \end{pmatrix}\in H$$

be an arbitrary element. Then $gg_0=g_0g$ implies the equations

$$(1-e^{t_0})x=(1-e^t)x_0 \quad \text{and} \quad (1-e^{t_0})x=(1-e^t)y_0.$$

Since $t_0\neq 0$, these equations imply that $g\in H_{(t_0,x_0,y_0)}$. □

Lemma 6.6 *The normal abelian subgroup* Γ' *of* Γ *is contained in* $\mathbb{R}^2$*.*

Proof Suppose that Γ' is not contained in $\mathbb{R}^2$. By Lemma 6.5, there exist $x_0, y_0 \in \mathbb{R}$ and $t_0 \neq 0$ such that Γ' is contained in $H_{(t_0,x_0,y_0)}$. Since $\Gamma' \not\subset \mathbb{R}^2$, there exists

$$g = \begin{pmatrix} e^t & 0 & x \\ 0 & e^{kt} & y \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma'$$

such that $t \neq 0$.

Set

$$h(\beta) = \begin{pmatrix} e^{t'} & 0 & x' \\ 0 & e^{kt'} & y' \\ 0 & 0 & 1 \end{pmatrix}.$$

Since Γ' is a normal subgroup of Γ, we have

$$h(\beta) g h(\beta)^{-1} \in \Gamma' \subset H_{(t_0,x_0,y_0)}.$$

Thus we obtain the equations

$$(1-e^t)\{(1-e_0^t)x_0-(1-e^{t'})x_0\}=0 \quad \text{and} \quad (1-e^{kt})\{(1-e^{kt})x'-(1-e^{kt'})x_0\}=0.$$

Since $t \neq 0$, these equations imply that $h(\beta) \in H_{(t_0,x_0,y_0)}$. This contradicts the fact that Γ is uniform in G_2^k. Therefore we have that Γ' is contained in $\mathbb{R}^2$. □

Set

$$h(\alpha_j) = \begin{pmatrix} 1 & 0 & x_j \\ 0 & 1 & y_j \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad h(\beta) = \begin{pmatrix} e^{t_0} & 0 & x_0 \\ 0 & e^{kt_0} & y_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since Γ is uniform in G_2^k, we have $t_0 \neq 0$. Moreover, by conjugating in G_2^k, we may assume that $x_0 = 0$ and $y_0 = 0$.

By the same argument as the proof of Lemma 6.4, we can prove the following lemma.

Lemma 6.7 $A \in \mathrm{SL}(3;\mathbb{Z})$ *is conjugate to the matrix*

$$\begin{pmatrix} e^{t_0} & 0 & 0 \\ 0 & e^{kt_0} & 0 \\ 0 & 0 & e^{(-1-k)t_0} \end{pmatrix}.$$

Define elements $\widehat{\alpha}_i$ and $\widehat{\beta}$ of $\widetilde{G}_2^k$ by

$$\widehat{\alpha}_i = \begin{pmatrix} 1 & 0 & 0 & x_i \\ 0 & 1 & 0 & y_i \\ 0 & 0 & 1 & z_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \widehat{\beta} = \begin{pmatrix} e^{t_0} & 0 & 0 & 0 \\ 0 & e^{kt_0} & 0 & 0 \\ 0 & 0 & e^{-(1+k)t_0} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

is an eigenvector of A corresponding to the eigenvalue e^{t_0}. Let Δ be the subgroup of $\widetilde{G}_2^k$ generated by $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$ and $\widehat{\beta}$. Then Δ is a uniform lattice of $\widetilde{G}_2^k$. Since Δ coincides with Γ via the homomorphism $D\colon \widetilde{G}_2^k \to G_2^k$ defined in Example 3.4, the Lie $\mathfrak{g}_2^k$–flow $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}_2^k$–flow in Example 3.4.

6C $\mathfrak{g}_3^{h\neq 0}$ case

In the case in which $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^{h\neq 0}$, the basic dimension of $\mathcal{F}$ is one; see [5; 11]. Hence the manifold M is diffeomorphic to a T^3–bundle over S^1 and $\pi_1(M) = \mathbb{Z} \ltimes_A \mathbb{Z}^3$ for some $A \in \mathrm{SL}(3;\mathbb{Z})$.

By the same argument as in Section 6B, we can prove the following lemma.

Lemma 6.8 *The normal subgroup Γ' of Γ is contained in $\mathbb{R}^2$.*

By Lemma 6.8, we have

$$h(\alpha_i) = \begin{pmatrix} 1 & 0 & x_i \\ 0 & 1 & y_i \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{\beta t_0}\cos t_0 & -e^{\beta t_0}\sin t_0 & x_0 \\ e^{\beta t_0}\sin t_0 & e^{\beta t_0}\cos t_0 & y_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since Γ is uniform in G_3^h, we have $t_0 \neq 0$. By conjugating in G_3^h, we may assume that $x_0 = 0$ and $y_0 = 0$.

By the equation

$$\beta\alpha_j\beta^{-1} = \alpha_1^{a_{1j}}\alpha_2^{a_{2j}}\alpha_3^{a_{3j}},$$

we have

$$A\boldsymbol{x} = e^{\beta t_0}(\cos t_0\boldsymbol{x} - \sin t_0\boldsymbol{y}) \quad \text{and} \quad A\boldsymbol{y} = e^{\beta t_0}(\sin t_0\boldsymbol{x} + \cos t_0\boldsymbol{y}),$$

where

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

By an easy calculation, we can show that $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{y} \neq \boldsymbol{0}$. Hence $\boldsymbol{x} + i\,\boldsymbol{y}$ and $\boldsymbol{x} - i\,\boldsymbol{y}$ are eigenvectors corresponding to the eigenvalues $e^{(\beta+i)t_0}$ and $e^{(\beta-i)t_0}$, respectively.

Let $d^{t_0} = e^{-2\beta t_0}$ be the other eigenvalue of A, and let

$$\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

be an eigenvector of A corresponding to the eigenvalue d^{t_0}. Let

$$\hat{\alpha}_i = \begin{pmatrix} 1 & 0 & 0 & x_i \\ 0 & 1 & 0 & y_i \\ 0 & 0 & 1 & z_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{\beta} = \begin{pmatrix} e^{\beta t_0} \cos t_0 & -e^{\beta t_0} \sin t_0 & 0 & 0 \\ e^{\beta t_0} \sin t_0 & e^{\beta t_0} \cos t_0 & 0 & 0 \\ 0 & 0 & d^{t_0} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

be elements of $\widetilde{G}_3^h$ and Δ be the subgroup of $\widetilde{G}_3^h$ generated by $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$ and $\hat{\beta}$. Then the subgroup Δ is a uniform lattice of $\widetilde{G}_3^h$

Since Δ coincides with Γ via the homomorphism $D\colon \widetilde{G}_3^h \to G_3^h$ defined in Example 3.5, the Lie $\mathfrak{g}_3^h$–flow $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}_3^h$–flow in Example 3.5.

6D $\mathfrak{g}_3^0$ case

Suppose that $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^0$. By [10, Corollaries 2.4 and 2.7] and the theorem of Caron and Carrière [2, Theorem 1], the manifold M is diffeomorphic to the 4–dimensional torus T^4.

Fix generators α_1, α_2, α_3 and α_4 of $\pi_1(M) \simeq \mathbb{Z}^4$ and set

$$h(\alpha_i) = \left(t_i, \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right) \in G_3^0 = \mathbb{R} \ltimes_\rho \mathbb{R}^2.$$

Since Γ is uniform in G_3^0, Γ is not contained in $\{0\} \times \mathbb{R}^2$. Hence we may assume that $t_1 \neq 0$.

Lemma 6.9 *$t_i \in 2\pi\mathbb{Z}$, for each i.*

Proof Suppose that there exists i such that $t_i \notin 2\pi\mathbb{Z}$. We may assume that $t_1 \notin 2\pi\mathbb{Z}$. Since Γ is abelian, we can show that Γ is contained in H, where

$$H = \left\{ \left(t, \begin{pmatrix} 1-\cos t & \sin t \\ -\sin t & 1-\cos t \end{pmatrix} \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} \right) \,\middle|\, t \in \mathbb{R} \right\}$$

is a simply connected 1–dimensional closed subgroup of G_3^0 and

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \begin{pmatrix} 1-\cos t_1 & \sin t_1 \\ -\sin t_1 & 1-\cos t_1 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

Since Γ is uniform in G_3^0, the homogeneous space $H\backslash G_3^0$ is compact.

On the other hand, $H\backslash G_3^0$ is homeomorphic to $\mathbb{R}^2$, since H is a simply connected 1–dimensional closed subgroup of G_3^0. This is a contradiction. □

By Lemma 6.9, we have $t_i = 2\pi n_i$. Define a diffeomorphism $F\colon G_3^0 \to \mathbb{R}^3$ by

$$F\colon \left(t, \begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto \begin{pmatrix} t \\ x \\ y \end{pmatrix}.$$

Then $F|_\Gamma\colon \Gamma \to \mathbb{R}^3$ is a homomorphism and F is $F|_\Gamma$–equivariant, that is,

$$F(\gamma \cdot g) = F|_\Gamma(\gamma) \cdot F(g)$$

for any $\gamma \in \Gamma$ and any $g \in G_3^0$. Therefore the rank of the matrix

$$\begin{pmatrix} 2\pi n_1 & 2\pi n_2 & 2\pi n_3 & 2\pi n_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}$$

is three.

We may assume that

$$\begin{pmatrix} 2\pi n_1 \\ x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} 2\pi n_2 \\ x_2 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} 2\pi n_3 \\ x_3 \\ y_3 \end{pmatrix}$$

are linearly independent. Consider the subgroup Δ of $G_3^0 \times \mathbb{R} = \mathbb{R} \ltimes_\rho \mathbb{R}^2 \times \mathbb{R}$ generated by $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$ and $\widehat{\alpha}_4$, where

$$\widehat{\alpha}_i = \left(2\pi n_i, \begin{pmatrix} x_i \\ y_i \end{pmatrix}, 0 \right) \quad \text{for } i = 1,2,3 \quad \text{and} \quad \widehat{\alpha}_4 = \left(2\pi n_4, \begin{pmatrix} x_4 \\ y_4 \end{pmatrix}, 1 \right).$$

Then Δ is a uniform lattice of $G_3^0 \times \mathbb{R}$ and Δ coincides with Γ via the homomorphism $D\colon G_3^0 \times \mathbb{R} \to G_3^0$ in Example 3.6. Therefore the Lie $\mathfrak{g}_3^0$–flow is diffeomorphic to the flow in Example 3.6.

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Homological stability for families of Coxeter groups

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We prove that certain families of Coxeter groups and inclusions $W_1 \hookrightarrow W_2 \hookrightarrow \cdots$ satisfy homological stability, meaning that in each degree the homology $H_*(BW_n)$ is eventually independent of n. This gives a uniform treatment of homological stability for the families of Coxeter groups of type A, B and D, recovering existing results in the first two cases, and giving a new result in the third. The key step in our proof is to show that a certain simplicial complex with W_n–action is highly connected. To do this we show that the barycentric subdivision is an instance of the "basic construction", and then use Davis's description of the basic construction as an increasing union of chambers to deduce the required connectivity.

20F55; 20J06

1 Introduction

1.1 Overview

A family of groups $G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots$ is said to satisfy *homological stability* if the induced maps $H_i(BG_{n-1}) \to H_i(BG_n)$ are isomorphisms when n is sufficiently large relative to i. Homological stability is known for many families of groups, including symmetric groups (see Nakaoka [23]), general linear groups (see Quillen [24]), mapping class groups of surfaces (see Harer [11]) and 3–manifolds (see Hatcher and Wahl [15]), diffeomorphism groups of highly connected manifolds (see Galatius and Randal-Williams [10]), and automorphism groups of free groups (see Hatcher [12] and Hatcher and Vogtmann [14]). *Coxeter groups* are abstract reflection groups, appearing in many areas of mathematics, such as root systems and Lie theory, geometric group theory, and combinatorics. See the books of Bourbaki [3], Davis [8] and Björner and Brenti [1] for introductions to Coxeter groups from each of these three viewpoints. In this paper we will show that homological stability holds for certain families of Coxeter groups.

Recall that a *Coxeter matrix* on a set S is an $S \times S$ symmetric matrix M, with values in $\mathbb{N} \cup \{\infty\}$, satisfying $m_{st} = 1$ if $s = t$ and $m_{st} \geqslant 2$ otherwise. The corresponding *Coxeter group* is the group generated by the elements of S, subject to the relations $(st)^{m_{st}} = e$ for $s, t \in S$. (When $m_{st} = \infty$ no relation is imposed.) It is common to

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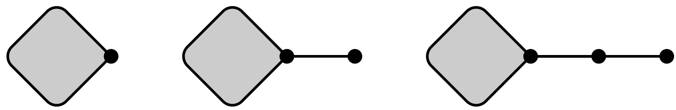
represent a Coxeter matrix by the equivalent *Coxeter diagram*. This is the graph with vertices S and edges $\{s,t\}$ for $m_{st} \geqslant 3$. The edge $\{s,t\}$ is labelled m_{st} if $m_{st} \geqslant 4$.

Now consider a sequence of finite Coxeter diagrams $(\Gamma_n)_{n\geqslant 1}$ of the form



Γ_1 Γ_2 Γ_3

where every diagram has a preferred vertex, and each diagram is obtained from its predecessor by attaching a new preferred vertex to the old one by an unlabelled edge. Writing W_n for the Coxeter group determined by Γ_n, the inclusion $\Gamma_{n-1} \hookrightarrow \Gamma_n$ induces an inclusion $W_{n-1} \hookrightarrow W_n$, and our main result states that the family

$$W_1 \hookrightarrow W_2 \hookrightarrow W_3 \hookrightarrow W_4 \hookrightarrow \cdots$$

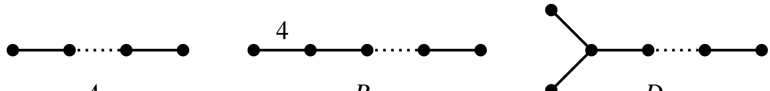
satisfies homological stability.

Main Theorem *The map $H_*(BW_{n-1}) \to H_*(BW_n)$ is an isomorphism in degrees satisfying $2* \leqslant n$. Here homology is taken with arbitrary constant coefficients.*

Observe that while the diagrams Γ_n are assumed to be finite, it is not necessary for the groups W_n to be finite.

1.2 Homological stability for Coxeter groups of type A, B and D

The Main Theorem gives a uniform treatment of homological stability for the families of Coxeter groups of type A_n, B_n and D_n. Recall that these are the Coxeter groups corresponding to the following diagrams, in which n always denotes the total number of vertices:



A_n B_n D_n

These families have an important place in the theory of Coxeter groups, since the classification of finite Coxeter groups states that a finite irreducible Coxeter group has type A_n, B_n or D_n, or is dihedral, or is one of six exceptional examples. (See Appendix C of [8].) The sequences $(A_n)_{n\geqslant 1}$, $(B_{n+1})_{n\geqslant 1}$ and $(D_{n+2})_{n\geqslant 1}$ all have the form $(\Gamma_n)_{n\geqslant 1}$ described above, with the rightmost vertex taken as the preferred vertex, and therefore we may apply the main theorem to each one. In what follows we will use concrete descriptions of the groups of type A_n, B_n and D_n that can be found in Section 6.7 of [8].

1.2.1 Coxeter groups of type A For the sequence of diagrams $(A_n)_{n\geqslant 1}$, the corresponding sequence of Coxeter groups is

$$\Sigma_2 \hookrightarrow \Sigma_3 \hookrightarrow \Sigma_4 \hookrightarrow \Sigma_5 \hookrightarrow \cdots$$

where Σ_n is the symmetric group on n letters and the inclusions are given by extending permutations by the identity. Applying the Main Theorem, we recover the following classical result.

Corollary (Nakaoka) *The map $H_*(B\Sigma_n) \to H_*(B\Sigma_{n+1})$ is an isomorphism in degrees $2* \leqslant n$. Here homology is taken with arbitrary constant coefficients.*

In fact, Nakaoka computed $H_*(B\Sigma_n;\mathbb{F}_p)$ for all primes p in Theorem 6.3 of [23]. From this he deduced stability with $\mathbb{F}_p$ coefficients in Corollary 6.7 of [23]. The case of arbitrary coefficients follows. Nakaoka's computations can be used to show that $H_k(B\Sigma_{2k-1};\mathbb{F}_2) \to H_k(B\Sigma_{2k};\mathbb{F}_2)$ is not surjective for $k \geqslant 1$, so that the bound $2* \leqslant n$ appearing in the corollary is sharp. Alternative proofs of Nakaoka stability, that do not rely on complete computations of $H_*(B\Sigma_n;\mathbb{F}_p)$, can be found in the Ph D thesis of Maazen [21] and the papers of Kerz [19] and Randal-Williams [25].

1.2.2 Coxeter groups of type B For the sequence of diagrams $(B_{n+1})_{n\geqslant 1}$, the corresponding sequence of Coxeter groups

$$C_2 \wr \Sigma_2 \hookrightarrow C_2 \wr \Sigma_3 \hookrightarrow C_2 \wr \Sigma_4 \hookrightarrow C_2 \wr \Sigma_5 \hookrightarrow \cdots$$

consists of the wreath products of the symmetric groups with the group C_2 of order 2, and the inclusions are again given by extending permutations by the identity. Applying the Main Theorem gives the following result.

Corollary *The map $H_*(B(C_2 \wr \Sigma_n)) \to H_*(B(C_2 \wr \Sigma_{n+1}))$ is an isomorphism in degrees $2* \leqslant n$. Here homology is taken with arbitrary constant coefficients.*

This result can be found in a number of places in the literature. In particular, May computed $H_*(B(C_2 \wr \Sigma_n);\mathbb{F}_p)$ for all $n \geqslant 1$ and all primes p. (See Cohen, Lada and May [7, Chapter I, Theorem 4.1] in the case $X = BC_2 \sqcup \{*\}$.) From this computation one obtains the corollary above in the case of $\mathbb{F}_p$ coefficients, and the case of arbitrary coefficients follows. The corollary also follows from existing stability results such as Theorem A of [25] and Proposition 1.6 of Hatcher and Wahl's paper [15]. Observe that the bound $2* \leqslant n$ is again sharp, since $C_2 \wr \Sigma_n$ is a split extension of Σ_n.

1.2.3 Coxeter groups of type D For the sequence of diagrams $(D_{n+2})_{n\geqslant 1}$, the corresponding sequence of Coxeter groups is

$$H_3 \hookrightarrow H_4 \hookrightarrow H_5 \hookrightarrow H_6 \hookrightarrow \cdots$$

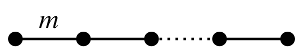
where H_n denotes the kernel of the homomorphism $C_2 \wr \Sigma_n \to C_2$ that takes the product of the C_2–components. (We regard C_2 as the set $\{\pm 1\}$ under multiplication.) The Main Theorem gives the following result.

Corollary *Let H_n denote the Coxeter group of type D_n. Then the inclusion map $H_{n+1} \hookrightarrow H_{n+2}$ induces an isomorphism $H_*(BH_{n+1}) \to H_*(BH_{n+2})$ in degrees where $2* \leqslant n$. Here homology is taken with arbitrary constant coefficients.*

We believe that the result is new in the stated generality. However, Swenson [27] gave a generating set for the ring $H^*(BH_n;\mathbb{F}_2)$, and deduced that the map $H^k(BH_{2k+1};\mathbb{F}_2)\to H^k(BH_{2k};\mathbb{F}_2)$ is not surjective. (See Theorem 6.4.1 and the paragraph that follows it in [27].) It follows that $H_k(BH_{2k};\mathbb{F}_2) \to H_k(BH_{2k+1};\mathbb{F}_2)$ is not injective, so that the bound $2* \leqslant n$ in the corollary is sharp.

1.3 The superideal simplex reflection groups

The Main Theorem applies to interesting families besides those of type A_n, B_n and D_n already considered. For example, if we fix an integer $m \geqslant 7$, then the main theorem shows that homological stability holds for the family of Coxeter groups associated to the sequence of diagrams $(\Gamma_n)_{n\geqslant 1}$



Γ_n

in which Γ_n has a total of $(n+1)$ vertices, the rightmost one preferred. These are the *superideal simplex reflection groups* that appear in recent work of Calegari [6]. The first group is finite, while the rest are all infinite hyperbolic.

It is not difficult to construct other sequences of hyperbolic groups to which our main theorem applies. For example, we can construct sequences $(W_n)_{n\geqslant 1}$ in which the W_n are all hyperbolic and have the same, arbitrary, virtual cohomological dimension (vcd). To do this we choose for W_1 an arbitrary hyperbolic right-angled Coxeter group with the desired vcd (see Januszkiewicz and Świątkowski [18]). Then by choosing a preferred element of the generating set of W_1 we extend to a sequence $(W_n)_{n\geqslant 1}$ of the kind appearing in the main theorem. By Moussong's condition [22, Theorem 17.1] (or see [8, Corollary 12.6.3]), the W_n are all hyperbolic. By construction, the nerves L_n of the W_n satisfy $L_{n+1} \cong CL_n$. (See [8, Section 7.1] for the definition of the nerve.) Then by Davis's computation of the vcd of Coxeter groups [8, Corollary 8.5.5] they all have the same vcd.

1.4 Homology of Coxeter groups in low degrees

The Main Theorem was to some extent inspired by existing results on the homology of Coxeter groups in degree 1 and 2, as we now explain.

In degree 1 our main theorem states that the map $H_1(BW_{n-1}) \to H_1(BW_n)$ is an isomorphism for $n \geqslant 2$. This result has a simple proof. Let W be a Coxeter group corresponding to Coxeter diagram Γ. Then one sees from the presentation of W that the abelianization W_{ab} is naturally isomorphic to the elementary abelian 2–group on the path-components of the graph obtained from Γ by deleting the edges with even or infinite label. In our situation Γ_n is obtained from Γ_{n-1} by attaching a single new vertex using an edge with label 3, so that $(W_{n-1})_{\mathrm{ab}} \to (W_n)_{\mathrm{ab}}$ is an isomorphism, and our stability result in degree 1 follows.

In degree 2 our main theorem states that the map $H_2(BW_{n-1}) \to H_2(BW_n)$ is an isomorphism for $n \geqslant 4$. The second homology groups $H_2(BW;\mathbb{Z})$ of the finite Coxeter groups were computed by Ihara and Yokonuma in [17]. They showed that the result is an elementary abelian 2–group, and computed its rank. In particular, they observed that for the groups of type A, B and C the rank of $H_2(BW;\mathbb{Z})$ stabilizes, and the stability range exactly corresponds to our result. Howlett [16] extended the work of Ihara and Yokonuma to arbitrary Coxeter groups. In our situation, his result shows that $H_2(BW_n;\mathbb{Z})$ is an elementary abelian 2–group whose rank is constant for $n \geqslant 3$, so that the isomorphism type of $H_2(BW_n)$ (now with arbitrary coefficients) is constant for $n \geqslant 3$. Thus Howlett's result almost implies our stability result in degree 2, since it shows that the domain and range of the map in question are isomorphic.

1.5 Outline of the proof of the main theorem

The proof of the Main Theorem is modelled closely on existing techniques for proving Nakaoka's stability result for symmetric groups, which is the statement that the map $H_*(B\Sigma_n) \to H_*(B\Sigma_{n+1})$ is an isomorphism for $2* \leqslant n$. So we begin by explaining an approach to Nakaoka stability.

The proof of Nakaoka stability is by induction on n, the initial case $n = 0$ being trivial. The inductive step uses the "complex of injective words", which we denote by X. This is the semisimplicial set whose k–simplices are ordered $(k+1)$–tuples in $\{1, \dots, n+1\}$, with each element appearing at most once. It admits an action of Σ_{n+1}, and this action is transitive on k–simplices with stabilizer Σ_{n-k-1}. Moreover, the realization $\|X\|$ is $(n-1)$–connected. We now consider the spectral sequence arising from the filtration of $EW_n \times_{W_n} \|X\|$ induced by the skeleta of $\|X\|$. Properties of the action of Σ_{n+1} on X allow us to identify the E^1–page of this spectral sequence in terms of the $H_*(B\Sigma_{n-k-1})$ and the stabilization maps between them. The inductive hypothesis then allows us to compute the remaining pages of the spectral sequence in a range of degrees. The connectivity of $\|X\|$ guarantees that the sequence converges to $H_*(B\Sigma_n)$ in a range of degrees. From that point the result follows easily.

The hardest step here is the proof that $\|X\|$ is $(n-1)$–connected. There are several proofs of this in the literature; see Remark 39. The approach relevant to us is the following. Observe that X is isomorphic to $(\Delta^n)^{\mathrm{ord}}$, the semisimplicial set of simplices of Δ^n equipped with an ordering of their vertices. Now Δ^n is weakly Cohen–Macaulay of dimension n, meaning that it and the links of simplices within it satisfy certain connectivity bounds. A result of Randal-Williams [28, Proposition 7.9] states that if a complex C is weakly Cohen–Macaulay of dimension n, then the realization $\|C^{\mathrm{ord}}\|$ is $(n-1)$–connected. Applying this to Δ^n, we obtain the connectivity of $\|X\|$.

Now here is a sketch of the proof of the main theorem. It follows the sketch proof of Nakaoka stability given above, and reduces to it in the case of Coxeter groups of type A.

(1) We construct a simplicial complex $\mathcal{C}^n$ with an action of W_n. For Coxeter groups of type A, the complex $\mathcal{C}^n$ is the n–simplex Δ^n. We prove that $\mathcal{C}^n$ is weakly Cohen–Macaulay of dimension n.

(2) We form a semisimplicial set $\mathcal{D}^n$ with an action of W_n. For Coxeter groups of type A, this is the complex of injective words X. We show that $\mathcal{D}^n$ is the semisimplicial set of ordered simplices in $\mathcal{C}^n$ and conclude that it is $(n-1)$–connected.

(3) Third, we use the spectral sequence associated to the filtration of $EW_n \times_{W_n} \|\mathcal{D}^n\|$ induced by the skeleta of $\|\mathcal{D}^n\|$ to prove the theorem.

For Coxeter groups of type A, the proof that $\mathcal{C}^n$ is weakly Cohen–Macaulay of dimension n is trivial. In general, we prove it as follows. We first prove that links of simplices in $\mathcal{C}^n$ are copies of $\mathcal{C}^m$ for appropriate $m < n$, so that the required connectivity bounds all follow if we can show that $\mathcal{C}^n$ is $(n-1)$–connected. To prove the latter, we make use of the "basic construction", a technique for constructing spaces with actions of Coxeter groups. (See [8, Chapters 5 and 8] and Section 2.5 below.) We identify the barycentric subdivision of $\mathcal{C}^n$ as an instance of the basic construction, and then use results of Davis on the topology of the basic construction (see Section 2.6) to conclude that it is $(n-1)$–connected.

Outline of the paper In Section 2 we recall background material on Coxeter groups, then in Section 3 we establish some notation and discuss the groups of type A, B and D in detail. In Section 4 we study the subgroups W_i of W_n, establishing important properties that will be used in the rest of the paper. Next, we move on to the simplicial complex $\mathcal{C}^n$: in Section 5 we define it, in Section 6 we study the links of its simplices, and in Section 7 we show that $|\mathrm{sd}\,\mathcal{C}^n|$ is $(n-1)$–connected. Then we define $\mathcal{D}^n$ in Section 8, we show that it is isomorphic to the semisimplicial set of ordered simplices in $\mathcal{C}^n$, and conclude that it is $(n-1)$–connected. The proof of the Main Theorem is completed in Section 9.

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2 Background on Coxeter groups

Here we will recall some of the basic facts about the theory of Coxeter groups, giving references to [8] where possible. Hopefully this covers all of the material we will use in the rest of the paper. Alternative introductions to Coxeter groups are [3; 1].

2.1 Coxeter systems

In Section 1.1 we defined *Coxeter matrices*, *Coxeter diagrams*, and the *Coxeter group* associated to a Coxeter matrix or diagram. A *Coxeter system* is a pair (W, S) consisting of a group W and a collection of involutions $S \subset W$ satisfying the following property: Let $\widetilde{W}$ denote the Coxeter group associated to the Coxeter matrix M on S defined by

$$m_{st} = \text{order of } st.$$

Then the homomorphism $\widetilde{W} \to W$ extending the identity $S \to S$ is an isomorphism. See Section 3.3 of [8].

2.2 Words

Let (W, S) be a Coxeter system. A *word* in S is an ordered tuple $(t_1, \dots, t_r)$ of elements of S. The word $(t_1, \dots, t_r)$ has *length* r and it *represents* the element $w = t_1 \cdots t_r$ of W. Every element $w \in W$ is represented by some word, and its *length* $\ell(w)$ is the minimum length of a word representing it. A word is *reduced* if it has minimum length for the element of W it represents.

2.3 The word problem

An M–*operation* on a word in S is a composite of the following *elementary M–operations*:

- Delete a subword (s,s).
- Replace an alternating subword $(s,t,\dots)$ of length $m(s,t)$ with the subword $(t,s,\dots)$ of the same length.

Observe that these operations do not alter the element of W represented by the word, since all elements of S are involutions, and since the relation $(st)^{m(s,t)}=e$ can be rewritten as $(st\cdots)=(ts\cdots)$, where each side is an alternating word of length $m(s,t)$. Observe also that these operations either preserve or reduce the length of a word. Tits' solution to the word problem in Coxeter groups states that a word is reduced if and only if it cannot be shortened by an M–operation, and that two reduced words represent the same element if and only if they are related by a sequence of elementary M–operations of the second kind. See Section 3.4 of [8].

2.4 Special subgroups

Let (W,S) be a Coxeter system. Given $T\subset S$, we denote by W_T the subgroup of W generated by T, and we refer to W_T as a *special subgroup* of W. Then (W_T,T) is again a Coxeter system. (This is why our sequence of homomorphisms $W_1\to W_2\to\cdots$ is in fact a sequence of inclusions.) See Section 4.1 of [8].

Given $T,U\subset S$ we say that $w\in W$ is (T,U) *reduced* if it cannot be represented by a reduced word starting with an element of T or ending with an element of U. If x is (T,U) reduced then a result of Kilmoyer, Solomon and Tits shows that $W_T\cap xW_Ux^{-1}=W_V$, where $V=T\cap xUx^{-1}$. See Lemma 2 of [26]. In particular, this shows that if $T,U\subset S$ then $W_T\cap W_U=W_{T\cap U}$. See Theorem 4.1.6 of [8] for a proof of this special case.

2.5 The basic construction

The "basic construction" is a method for building spaces with an action of a Coxeter group. It can be used, for example, to study the topology of the Coxeter complex and Davis complex of a Coxeter group. (Our discussion is tailored to the case of Coxeter groups. For an approach to the basic construction that applies to more general groups see Chapter II.12 of [5].)

Let (W,S) be a Coxeter system. A *mirrored space* over S is a space X together with subspaces $X_s\subset X$, called *mirrors*, one for each $s\in S$. We assume that X is a CW–complex and that the mirrors are subcomplexes. The *basic construction* is the space

$$\mathcal{U}(W, X) = (W \times X)/\sim,$$

where $(v, x) \sim (w, y)$ if and only if $x = y$ and $v^{-1}w$ belongs to the subgroup generated by the $s \in S$ for which $x \in X_s$. The basic construction is equipped with the action of W by left translation, and we identify X with the image of $\{e\} \times X$ in $\mathcal{U}(W, X)$. Observe that $\mathcal{U}(W, X)$ has the structure of a CW–complex in which each translate wX is a subcomplex. See Section 5.1 of [8].

2.6 The increasing union of chambers

For us the most important feature of the basic construction is that it can be described as an increasing union of chambers, meaning copies of X, as we now recall from Section 8.1 of [8]. Given $w \in W$, let

$$\mathrm{In}(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$$

denote the set of letters with which a reduced expression for w can end, and let

$$X^{\mathrm{In}(w)} = \bigcup_{s \in \mathrm{In}(w)} X_s$$

denote the corresponding union of mirrors. Order the elements of W as $w_0, w_1, w_2, \ldots,$ where $w_0 = e$ and $\ell(w_m) \leqslant \ell(w_{m+1})$ for $m \geqslant 0$. Define

$$P_m = \bigcup_{i=0}^{m} w_i X,$$

so that $\mathcal{U}(W, X)$ is the increasing union of the subcomplexes P_m. Then

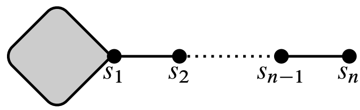
$$P_m = P_{m-1} \cup w_m X \quad \text{and} \quad P_{m-1} \cap w_m X = w_m X^{\mathrm{In}(w_m)}.$$

The latter equation is by Lemma 8.1.1 of [8]. It will be useful to us since it specifies exactly how each chamber is attached to its predecessor, so that we can study the topology of $\mathcal{U}(W, X)$ inductively by adding one chamber at a time.

3 Notation and examples

In this section we establish some notation that will be used throughout the rest of the paper. We also establish in more detail the Coxeter groups of type A, B and D, which will be used for illustration throughout the rest of the paper. Fix a sequence $(\Gamma_n)_{n \geqslant 1}$ of the kind described in the introduction.

Definition 1 (the elements $s_1, \dots, s_n$) For $n \geqslant 1$ we define s_n to be the preferred vertex of Γ_n, as in the following diagram:



Thus the special subgroup of W_n generated by $s_1, \dots, s_n$ is a copy of the Coxeter group of type A_n, and so is isomorphic to Σ_{n+1}. See Example 2 below.

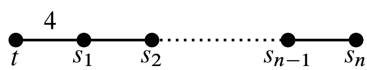
Example 2 (groups of type A) Consider the sequence of diagrams $(A_n)_{n\geq 1}$:

A_n

In this case the group W_n may be identified with Σ_{n+1}, the symmetric group on letters $1, \dots, (n+1)$, where s_i is the adjacent transposition $(i\ i+1)$. See Example 6.7.1 of [8].

Example 3 (groups of type B) Consider the sequence of diagrams $(B_{n+1})_{n\geq 1}$:

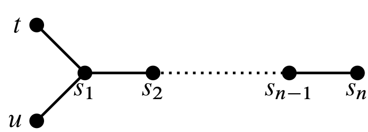
B_{n+1}



The group W_n may be identified with the wreath product $C_2 \wr \Sigma_{n+1}$, where t is identified with the generator of C_2 and s_i is again identified with the adjacent transposition $(i\ i+\ 1)$. For concreteness, we further identify $C_2 \wr \Sigma_{n+1}$ with the set of permutations σ of $\{\pm 1, \dots, \pm(n+1)\}$ that satisfy $\sigma(-i) = -\sigma(i)$ for all i. In this setting t is the permutation that sends ± 1 to ∓ 1 and fixes all other elements, while s_i is the permutation that sends $\pm i$ to $\pm(i+1)$ and vice versa and fixes all other elements. See Example 6.7.2 of [8].

Example 4 (groups of type D) Consider the sequence of diagrams $(D_{n+2})_{n\geq 1}$:

D_{n+2}



The group W_n may be identified as the kernel of the homomorphism $C_2 \wr \Sigma_{n+2} \to C_2$ that takes the product of the C_2–components. Regarding $C_2 \wr \Sigma_{n+2}$ as a group of permutations of $\{\pm 1, \dots, \pm(n+2)\}$, this kernel consists of the permutations that send an even number of the positive elements to negative ones. Under this identification, t corresponds to the element that negates ± 1 and ± 2 and fixes all other elements;

u transposes 1 and 2, -1 and -2, and fixes all other elements; and s_i sends $\pm(i+1)$ to $\pm(i+2)$ and vice versa and fixes all other elements. See Example 6.7.3 of [8].

Definition 5 (the additional diagrams Γ_0 and Γ_{-1}) We extend the sequence $(\Gamma_n)_{n\geqslant 1}$ to the left by two terms as follows. Define Γ_0 to be the diagram obtained from Γ_1 by deleting the preferred vertex, and define Γ_{-1} to be the diagram obtained from Γ_1 by deleting the preferred vertex and all vertices that shared an edge with it.

Example 6 (Γ_0 and Γ_{-1} for Coxeter groups of type A, B and D) For the sequence $(A_n)_{n\geqslant 1}$, the diagrams A_0 and A_{-1} are both empty. For $(B_{n+1})_{n\geqslant 1}$, the diagram B_{0+1} consists of the single vertex t and B_{-1+1} is empty. For $(D_{n+2})_{n\geqslant 1}$, the diagram D_{0+2} consists of the two vertices t and u with no edge, and D_{-1+2} is empty.

Definition 7 (the generating sets S_n) Let $(\Gamma_n)_{n\geqslant 1}$ be a sequence of the kind described in the introduction, and let $(\Gamma_n)_{n\geqslant -1}$ be the extension just described. Then for $n\geqslant -1$ we define S_n to be the set of vertices of Γ_n. Thus (W_n, S_n) is a Coxeter system for each $n\geqslant -1$.

4 The subgroups $W_{-1}\subset W_0\subset W_1\subset\cdots\subset W_n$

From this point onwards, unless stated otherwise, we fix a sequence $(\Gamma_n)_{n\geqslant 1}$ and an integer $n\geqslant 1$.

This section will study the sequence of subgroups $W_{-1}\subset W_0\subset\cdots\subset W_n$, and in particular the cosets of W_k in W_n for $k<n$. We do this now because the geometric objects that will appear later in the paper are constructed by considering these cosets. Throughout the section we will illustrate the results using the sequences $(A_n)_{n\geqslant 1}$, $(B_{n+1})_{n\geqslant 1}$ and $(D_{n+2})_{n\geqslant 1}$ that were explained in Examples 2, 3 and 4. The key idea to bear in mind is that W_n/W_{n-1} is "the natural set for W_n to act on". For example, for groups of type A, where $W_n=\Sigma_{n+1}$, we will see that W_n/W_{n-1} is isomorphic to $\{1,\dots,n+1\}$ with the permutation action.

Proposition 8 (left cosets of W_{n-1}) (1) *Let i lie in the range $1\leqslant i\leqslant n$. Then left-multiplication by the element s_i fixes the set*

$$\{s_1\cdots s_n W_{n-1},\ s_2\cdots s_n W_{n-1},\ \dots,\ s_n W_{n-1},\ W_{n-1}\}.$$

It acts on the set by transposing $s_i\cdots s_n W_{n-1}$ and $s_{i+1}\cdots s_n W_{n-1}$, and fixing the remaining elements. Here the product $(s_{i+1}\cdots s_n)$ is omitted when $i=n$.

(2) *For $c \in W_n$ the cosets*

$$c(s_1 \cdots s_n)W_{n-1}, \quad \dots, \quad cs_n W_{n-1}, \quad cW_{n-1}$$

are pairwise distinct.

Proof One can verify the identities

$$\begin{aligned} s_i(s_j \cdots s_n) &= (s_j \cdots s_n)s_i \quad \text{for } i < j-1, \\ s_i(s_{i+1} \cdots s_n) &= s_i \cdots s_n, \\ s_i(s_i \cdots s_n) &= s_{i+1} \cdots s_n, \\ s_i(s_j \cdots s_n) &= (s_j \cdots s_n)s_{i-1} \quad \text{for } i > j, \end{aligned}$$

and then the first part follows immediately. (The product $s_{i+1} \cdots s_n$ is omitted when $i = n$.) For the second part, if $cs_j \cdots s_n W_{n-1} = cs_k \cdots s_n W_{n-1}$ with $j < k$, then $(s_n \cdots s_j)(s_k \cdots s_n) \in W_{n-1}$. But

$$(s_n \cdots s_j)(s_k \cdots s_n) = (s_{k-1} \cdots s_n \cdots s_{k-1})(s_{k-2} \cdots s_j),$$

where the second factor on the right is omitted in the case $j = k-1$. This implies that $s_n \in W_{n-1}$, which is a contradiction. □

Example 9 (W_n/W_{n-1} for groups of type A, B and D) We illustrate Proposition 8 for the sequence $(A_n)_{n \geqslant 1}$ here. As explained in Example 2, we may regard W_n as the group Σ_{n+1} of permutations of the set $\{1, \dots, n+1\}$. This allows us to identify W_n/W_{n-1} via the isomorphism

$$W_n/W_{n-1} \xrightarrow{\cong} \{1, \dots, n+1\}, \quad \sigma W_{n-1} \longmapsto \sigma(n+1),$$

which respects the W_n action on each side, and which maps the coset $s_i \cdots s_n W_{n-1}$ to the letter i. (When $i = n+1$ the product $s_i \cdots s_n$ is omitted.) So for these groups, the first part of the proposition above amounts to the fact that s_i transposes the elements i and $i+1$. For $\sigma \in W_n$, the cosets

$$\sigma(s_1 \cdots s_n)W_{n-1}, \quad \dots, \quad \sigma s_n W_{n-1}, \quad \sigma W_{n-1}$$

correspond under the isomorphism above to the elements $\sigma(1), \dots, \sigma(n+1)$. So the second part of the proposition amounts to the fact that these elements are distinct since σ is a permutation.

A similar account can be given for the sequences $(B_{n+1})_{n \geqslant 1}$ and $(D_{n+2})_{n \geqslant 1}$, this time using Examples 3 and 4. For $(B_{n+1})_{n \geqslant 1}$ the account is identical after replacing $\{1, \dots, n+1\}$ with $\{\pm 1, \dots, \pm(n+1)\}$. For $(D_{n+2})_{n \geqslant 1}$ the set being acted on is now $\{\pm 1, \dots, \pm(n+2)\}$, and the isomorphism sends σW_n to $\sigma(n+2)$.

Proposition 10 $W_{i-1} \cap (s_i \cdots s_n W_{n-1} s_n \cdots s_i) = W_{i-2}$ for $1 \leqslant i \leqslant n$.

Proof The element $s_i \cdots s_n$ is (W_{i-1}, W_{n-1})–reduced, meaning that it does not have a reduced representative beginning with an element of W_{i-1} or ending with a representative of W_{n-1}. Thus, as we recalled in Section 2.4, the intersection

$$W_{i-1} \cap\ s_i \cdots s_n W_{n-1} s_n \cdots s_i$$

is the subgroup generated by $T = S_{i-1} \cap (s_i \cdots s_n S_{n-1} s_n \cdots s_i)$. So it will be enough to show that $T = S_{i-2}$. If $j \leqslant i-2$ then $s_n \cdots s_i s_j s_i \cdots s_n = s_j$, and consequently $S_{i-2} \subset T$. So suppose that $t \in T \setminus S_{i-2}$. Thus $t \in S_{i-1} \setminus S_{i-2}$ and $s_n \cdots s_i t s_i \cdots s_n \in S_{n-1}$. By the first condition we have $m_{s_i t} \geqslant 3$. By the second condition the word $(s_n, \ldots, s_i, t, s_i, \ldots, s_n)$ represents an element of S_{n-1}, so is not reduced. By the solution to the word problem recalled in Section 2.3, we must therefore be able to apply an M–operation to this word, but by inspection this is only possible if $m_{s_i t}$ is exactly 3. But in that case $(s_n, \ldots, s_i, t, s_i, \ldots, s_n)$ is already reduced, contradicting the second condition (see Sections 2.2 and 2.3). □

Example 11 In the case of the sequence $(A_n)_{n\geqslant 1}$, the previous proposition can be explained as follows. The group W_n is identified with the symmetric group Σ_{n+1} on the set $\{1, \ldots, n+1\}$, and W_{k-1} is the subgroup that fixes $(k+1), \ldots, (n+1)$. Thus:

- W_{i-1} is the subgroup that fixes $(i+1), \ldots, (n+1)$.
- $s_i \cdots s_n W_{n-1} s_n \cdots s_i$ is the subgroup that fixes i.
- W_{i-2} is the subgroup that fixes $i, \ldots, (n+1)$.

This makes the proposition's claim that $W_{i-1} \cap (s_i \cdots s_n W_{n-1} s_n \cdots s_i) = W_{i-1}$ immediate. For the sequences $(B_{n+1})_{n\geqslant 1}$ and $(D_{n+2})_{n\geqslant 1}$ one can give a similar account.

Proposition 12 *Let i lie in the range $1 \leqslant i \leqslant n$. If $\sigma, \tau \in W_n$ satisfy*

$$\sigma s_j \ldots s_n W_{n-1} = \tau s_j \ldots s_n W_{n-1} \qquad \text{for} \quad j = i, \ldots, n+1,$$

then $\sigma^{-1}\tau \in W_{i-2}$. Here the product $(s_j \cdots s_n)$ is omitted when $j = n+1$.

Proof The proposition is equivalent to the claim that

$$W_{n-1} \cap (s_n W_{n-1} s_n) \cap \cdots \cap (s_i \cdots s_n W_{n-1} s_n \cdots s_i) = W_{i-2},$$

which is proved by downward induction on i. The initial case $i = n+1$ is immediate, and the induction step follows from Proposition 10. □

5 The simplicial complex $\mathcal{C}^n$

Now we introduce the simplicial complex $\mathcal{C}^n$ that will be central to our proof of the Main Theorem. The definition of $\mathcal{C}^n$ is motivated by the case of Coxeter groups of type A, where W_n is the symmetric group Σ_{n+1}, and $\mathcal{C}^n$ is nothing other than the n–simplex. As explained in Section 1.5, this is relevant since the semisimplicial set of ordered simplices in the n–simplex is the "complex of injective words", which appears in several existing proofs of homological stability for the symmetric groups. In the general case, $\mathcal{C}^n$ is designed so that its semisimplicial set of ordered simplices can play the role of the complex of injective words in a proof of homological stability for the sequence $(W_n)_{n\geqslant 1}$.

The main result of this section is that $\mathcal{C}^n$ is weakly Cohen–Macaulay of dimension n. The proof relies on propositions that will be established in the following two sections.

Definition 13 (the simplicial complex $\mathcal{C}^n$) Given $n \geqslant 0$, we define $\mathcal{C}^n$ to be the n–dimensional simplicial complex with vertex set W_n/W_{n-1} and with k–simplices given by the subsets

$$C = \{c(s_{n-k+1}\cdots s_n)W_{n-1},\ \ldots,\ cs_nW_{n-1},\ cW_{n-1}\}$$

for $0 \leqslant k \leqslant n$ and $c \in W_n$. Proposition 8 shows that C does indeed have cardinality $(k+1)$. In this situation we call c a *lift* of the simplex C.

Remark 14 We chose the name "lift" to emphasize the formal similarity with the concept of the same name that appears in Definition 2.1 of Wahl [28].

A given simplex can have many lifts. Choosing a lift for a simplex induces an ordering of its vertices, and all orderings occur in this way. For if c lifts a k–simplex C then so does $cs_{n-k+i+1}$, and the induced orderings differ by transposition of the i^{th} and $(i+1)^{\text{st}}$ vertices (see Proposition 8). This makes it simple to verify that $\mathcal{C}^n$ is indeed a simplicial complex, for if C is a simplex of $\mathcal{C}^n$ and $D \subset C$ is a nonempty subset, then we may choose a lift c of C such that D is a terminal segment in the induced ordering. Then c is also a lift of D.

The natural action of W_n on W_n/W_{n-1} extends to an action on $\mathcal{C}^n$. For if C is a simplex of $\mathcal{C}^n$ with lift c, and if $w \in W_n$, then wC is a simplex of $\mathcal{C}^n$ with lift wc.

We now give a concrete description of $\mathcal{C}^n$ for the families of Coxeter groups of type A, B and D. See Examples 2, 3 and 4 for the description of these groups, and Example 9 for a description of W_n/W_{n-1} in each case.

Example 15 ($\mathcal{C}^n$ for groups of type A) For the sequence of diagrams $(A_n)_{n\geqslant 1}$ we saw in Example 2 that $W_n = \Sigma_{n+1}$ is the symmetric group on $(n+1)$ letters. Then $\mathcal{C}^n$ is the n–dimensional simplex Δ^n with the action of Σ_{n+1} that permutes the vertices. For as in Example 9 the vertex set W_n/W_{n-1} of $\mathcal{C}^n$ is isomorphic to $\{1,\dots,n+1\}$ via the map that sends σW_{n-1} to $\sigma(n+1)$. Under this isomorphism, an element $\sigma \in W_n$ is a lift of the k–simplex

$$C = \{\sigma(n-k+1),\dots,\sigma(n+1)\},$$

and every subset of $\{1,\dots,n+1\}$ arises in this way.

Example 16 ($\mathcal{C}^n$ for groups of type B) For the sequence of diagrams $(B_{n+1})_{n\geqslant 1}$ we saw in Example 3 that W_n is the group of permutations σ of the set $\{\pm 1,\dots,\pm(n+1)\}$ satisfying the rule $\sigma(-i) = -\sigma(i)$ for all i. In this case $\mathcal{C}^n$ is isomorphic to the *hyperoctahedron of dimension n*, which is the simplicial complex whose vertex set is $\{\pm 1,\dots,\pm(n+1)\}$ and whose simplices are the subsets containing at most one element from each pair $\{i,-i\}$. In particular, its realization is homeomorphic to the n–sphere. To obtain this description, we use the isomorphism $W_n/W_{n-1} \to \{\pm 1,\dots,\pm(n+1)\}$ sending σW_{n-1} to $\sigma(n+1)$, as in Example 9. Under this isomorphism an element σ lifts the k–simplex

$$C = \{\sigma(n-k+1),\dots,\sigma(n+1)\},$$

so that a subset of $\{\pm 1,\dots,\pm(n+1)\}$ spans a simplex of $\mathcal{C}^n$ if and only if it does not contain any element and its negative.

Example 17 ($\mathcal{C}^n$ for groups of type D) For the sequence of diagrams $(D_{n+2})_{n\geqslant 1}$ we saw in Example 4 that W_n is the group of permutations of $\{\pm 1,\dots,\pm(n+2)\}$ that satisfy the rule $\sigma(-i) = -\sigma(i)$ and that send an even number of the positive elements to negative ones. In this case $\mathcal{C}^n$ is the n–skeleton of the $(n+1)$–dimensional hyperoctahedron. In other words, it is the simplicial complex with vertex set $\{\pm 1,\dots,\pm(n+2)\}$, and whose simplices are the subsets of size at most $n+1$ containing at most one element from each pair $\{i,-i\}$. (Compare with Example 16.) In particular, the realization of $\mathcal{C}^n$ has the homotopy type of the wedge of (2^n-1) copies of the n–dimensional sphere. To obtain this description recall from Example 9 that the vertex set W_n/W_{n-1} is identified with $\{\pm 1,\dots,\pm(n+2)\}$ via the map sending σW_{n-1} to $\sigma(n+2)$. Under this identification the k–simplex with lift σ is

$$C = \{\sigma(n-k+2),\dots,\sigma(n+2)\},$$

so that a subset of $\{\pm 1,\dots,\pm(n+2)\}$ spans a simplex if and only if it does not contain any element and its negative and has size at most n.

Recall from Definition 3.4 of [15] that a simplicial complex is called *weakly Cohen–Macaulay of dimension* n if it is $(n-1)$–connected and the link of each p–simplex is $(n-p-2)$–connected. In each of the three examples above, $\mathcal{C}^n$ has the homotopy type of a wedge of n–dimensional spheres, and so is $(n-1)$–connected. In fact, it is not hard to see that in these examples $\mathcal{C}^n$ is weakly Cohen–Macaulay of dimension n. This is an instance of the following general fact.

Theorem 18 *$\mathcal{C}^n$ is weakly Cohen–Macaulay of dimension n.*

The proof of this theorem relies on Propositions 19, 26 and 27, which are proved over the course of the next two sections.

Proof By Proposition 19, if C is a p–simplex of $\mathcal{C}^n$ then $\mathrm{lk}_{\mathcal{C}^n}(C)\cong\mathcal{C}^{n-p-1}$. It therefore suffices to show that $\mathcal{C}^n$ is $(n-1)$–connected for all n, or equivalently that the barycentric subdivision $\mathrm{sd}\,\mathcal{C}^n$ is $(n-1)$–connected for all n. Now Proposition 26 shows that $|\mathrm{sd}\,\mathcal{C}^n|$ is homeomorphic to the basic construction $\mathcal{U}(W_n,|\Delta|)$, while Proposition 27 shows that $\mathcal{U}(W_n,|\Delta|)$ is $(n-1)$–connected. $\square$

6 Links of simplices of $\mathcal{C}^n$

The aim of this section is to prove the following proposition, which was used in the proof of Theorem 18 above.

Proposition 19 *Let C be a p–simplex of $\mathcal{C}^n$. Then $\mathrm{lk}_{\mathcal{C}^n}(C)\cong\mathcal{C}^{n-p-1}$.*

In the next section we prove that $\mathcal{C}^n$ is $(n-1)$–connected. This, combined with the proposition above, shows that the links of p–simplices in $\mathcal{C}^n$ are $(n-p-2)$–connected, and consequently that $\mathcal{C}^n$ is weakly Cohen–Macaulay of dimension n.

Example 20 (links in $\mathcal{C}^n$ for groups of type A, B and D) In Examples 15, 16 and 17 we gave concrete descriptions of $\mathcal{C}^n$ for each of the sequences $(A_n)_{n\geqslant 1}$, $(B_{n+1})_{n\geqslant 1}$ and $(D_{n+2})_{n\geqslant 1}$. These descriptions can be used to illustrate Proposition 19. For example, if we take the sequence $(B_{n+1})_{n\geqslant 1}$, then $\mathcal{C}^n$ is the hyperoctahedron of dimension n, ie, the simplicial complex with vertices $\{\pm 1,\dots,\pm(n+1)\}$ in which a subset of the vertices spans a simplex if and only if it does not contain any element and its negative. Thus $\mathcal{C}^2$, $\mathcal{C}^1$ and $\mathcal{C}^0$ are as shown in Figure 1 (in $\mathcal{C}^1$ and $\mathcal{C}^0$ the dashed parts are not included). We see that in $\mathcal{C}^2$ the link of the vertex $\{3\}$ is a copy of $\mathcal{C}^1$, while the link of the edge $\{2,3\}$ is a copy of $\mathcal{C}^0$.

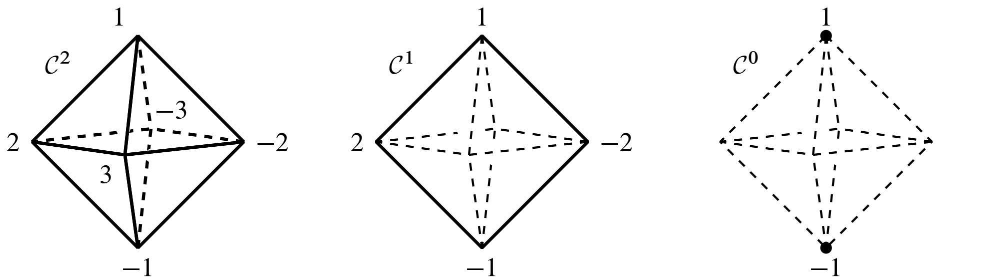


Figure 1: The hyperoctohedra $\mathcal{C}^2, \mathcal{C}^1, \mathcal{C}^0$

Proof of Proposition 19 Choose a lift c of C. Define

$$\phi\colon W_{n-p-1}/W_{n-p-2} \to W_n/W_{n-1}$$

by $\phi(dW_{n-p-2}) = cds_{n-p}\cdots s_n W_{n-1}$ for $d \in W_{n-p-1}$. This is well defined since every generator of W_{n-p-2} commutes with $s_{n-p}, \dots, s_n$. Observe that the domain and range of ϕ are the vertex sets of $\mathcal{C}^{n-p-1}$ and $\mathcal{C}^n$ respectively.

Claim 1 *The map ϕ is an injection.*

To prove this claim, let $d, d' \in W_{n-p-1}$ satisfy

$$cd(s_{n-p}\cdots s_n)W_{n-1} = cd's_{n-p}\cdots s_n W_{n-1}.$$

Then

$$d^{-1}d' \in W_{n-p-1} \cap (s_{n-p}\cdots s_n)W_{n-1}(s_n \cdots s_{n-p}) = W_{n-p-2},$$

the latter equation by Proposition 10. Thus $d'W_{n-p-2} = dW_{n-p-2}$.

Claim 2 *The map ϕ sends simplices of $\mathcal{C}^{n-p-1}$ to simplices of $\mathrm{lk}_{\mathcal{C}^n}(C)$.*

To prove this, suppose that D is an i–simplex of $\mathcal{C}^{n-p-1}$. Let $d \in W_{n-p-1}$ be a lift of D. Then

$$\phi D = \{cds_{n-p-i}\cdots s_n W_{n-1}, \dots, cds_{n-p}\cdots s_n W_{n-1}\}$$

while

$$\begin{aligned} C &= \{cs_{n-p+1}\cdots s_n W_{n-1}, \dots, cs_n W_{n-1}, cW_{n-1}\}, \\ &= \{cds_{n-p+1}\cdots s_n W_{n-1}, \dots, cds_n W_{n-1}, cdW_{n-1}\}. \end{aligned}$$

Thus $\phi D \cap C = \varnothing$ by Proposition 8, and $\phi D \cup C$ is a simplex of $\mathcal{C}^n$ with lift cd, so that ϕD is a simplex of $\mathrm{lk}_{\mathcal{C}^n}(C)$ as claimed.

Claim 3 *Every simplex of* $\mathrm{lk}_{\mathcal{C}^n}(C)$ *has the form* ϕD *for some simplex* D *of* $\mathcal{C}^{n-p-1}$.

To prove this, suppose that $\overline{D}$ is an i–simplex of $\mathrm{lk}_{\mathcal{C}^n}(C)$. Then $\overline{D}\cap C=\varnothing$ and $\overline{D}\cup C$ is a simplex of $\mathcal{C}^n$. Let c' be a lift of $\overline{D}\cup C$, and assume without loss that the ordering it induces on $\overline{D}\cup C$ contains $\overline{D}$ as an initial segment and C as a terminal segment with the ordering induced by c. Thus

$$\overline{D}=\{c'(s_{n-p-i}\cdots s_n)W_{n-1},\dots,c'(s_{n-p}\cdots s_n)W_{n-1}\}$$

and

$$c'(s_{n-p+k}\cdots s_n)W_{n-1}=c(s_{n-p+k}\cdots s_n)W_{n-1}$$

for $k=1,\dots,p+1$, where the product $(s_{n-p+k}\cdots s_n)$ is omitted for $k=p+1$. The latter gives $c^{-1}c'\in W_{n-p-1}$ by Proposition 12, so that $c'=cd$ for some $d\in W_{n-p-1}$. Then $\overline{D}=\phi D$, where D is the i–simplex of $\mathcal{C}^{n-p-1}$ with lift d.

We can now prove the proposition. Combining the first claim with the third in the case of 0–simplices, we see that ϕ is an isomorphism between the vertex sets of $\mathcal{C}^{n-p-1}$ and $\mathrm{lk}_{\mathcal{C}^n}(C)$. The second and third claims then show that ϕ induces an isomorphism of simplicial complexes from $\mathcal{C}^{n-p-1}$ to $\mathrm{lk}_{\mathcal{C}^n}(C)$. □

7 The barycentric subdivision of $\mathcal{C}^n$ and the basic construction

Our aim now is to complete the proof of Theorem 18 by proving Propositions 26 and 27 below. These results make use of the basic construction, whose definition we now recall from Section 2.5. Let (W,S) be a Coxeter system. A *mirrored space* over S is a space X equipped with a *mirror* $X_s\subset X$ for each $s\in S$. Given such a mirrored space, the *basic construction* $\mathcal{U}(W,X)$ is then the quotient $(W\times X)/\sim$, where $(w,x)\sim(v,y)$ if and only if $x=y$ and $w^{-1}v$ lies in the subgroup generated by those $s\in S$ for which $x\in X_s$.

We will show in Proposition 26 that $|\mathrm{sd}\,\mathcal{C}^n|$ is the basic construction $\mathcal{U}(W_n,X)$ for an appropriate choice of mirrored space X. Then in Proposition 27 we will show that $\mathcal{U}(W_n,X)$ is $(n-1)$–connected. Together these show that the barycentric subdivision $\mathrm{sd}\,\mathcal{C}^n$ is $(n-1)$–connected, completing the proof of Theorem 18.

We begin by defining the required mirrored space X over S_n. To do this we will identify a simplex Δ of $\mathrm{sd}\,\mathcal{C}^n$ and make its realization $|\Delta|$ into a mirrored space.

Definition 21 (the simplex Δ) For $i = 0, \dots, n$, let a_i denote the $(n-i)$–simplex of $\mathcal{C}^n$ defined by

$$a_i = \{(s_{i+1} \cdots s_n)W_{n-1}, \dots, s_n W_{n-1}, eW_{n-1}\}.$$

Each a_i has lift $e \in W_n$. Now let Δ denote the n–simplex of $\operatorname{sd}\mathcal{C}^n$ defined by

$$\Delta = \{a_0, \dots, a_n\}.$$

It is a simplex of $\operatorname{sd}\mathcal{C}^n$ since $a_0 \supset \cdots \supset a_n$.

Definition 22 (the subcomplexes Δ_s) For each $s \in S_n$, we define a subcomplex Δ_s of Δ as follows. If $s = s_i$ for $i = 1, \dots, n$, then Δ_{s_i} is the face

$$\Delta_{s_i} = \{a_0, \dots, \widehat{a_i}, \dots, a_n\}$$

of Δ. If $s \in S_0 \setminus S_{-1}$ then Δ_s is the face

$$\Delta_s = \{a_1, \dots, a_n\}$$

of Δ. And finally, if $s \in S_{-1}$ then

$$\Delta_s = \Delta.$$

Definition 23 (the mirrored space $|\Delta|$) We make $|\Delta|$ into a mirrored space over S_n by defining $|\Delta|_s = |\Delta_s| \subset |\Delta|$ for $s \in S_n$.

Example 24 (Δ and Δ_s for Coxeter groups of type A) For the sequence $(A_n)_{n\geqslant 1}$, we saw in Example 2 that W_n is the symmetric group Σ_{n+1}, and in Example 15 that $\mathcal{C}^n$ can be identified with the n–simplex Δ^n. Under this identification the vertex $s_{i+1} \cdots s_n W_{n-1}$ is identified with i. Thus

$$a_i = \{s_{i+1} \cdots s_n W_{n-1}, \dots, s_n W_{n-1}, W_{n-1}\} = \{i+1, \dots, n\}.$$

Consequently Δ is the n–simplex of $\operatorname{sd}\Delta^n$ with vertices

$$\{1, \dots, n+1\}, \quad \dots, \quad \{n, n+1\}, \quad \{n+1\}.$$

This is illustrated in Figure 2 in the case $n = 2$. Observe that in this case every 2–simplex of $\operatorname{sd}\Delta^2$ is a translate of Δ by an element of Σ_3, and that every simplex of $\operatorname{sd}\Delta^2$ is a face of such a translate.

The subcomplex Δ_{s_i} of Δ is simply the face opposite the vertex $a_i = \{i+1, \dots, n\}$. This is illustrated in Figure 3 in the case $n = 2$. Observe that s_1 fixes Δ_{s_1} vertexwise, and that s_2 fixes Δ_{s_2} vertexwise.

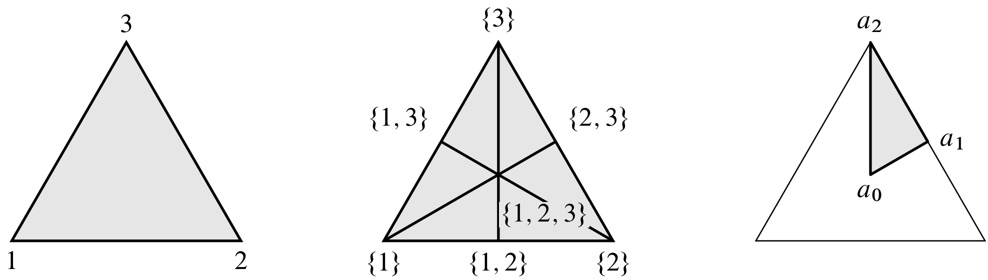


Figure 2: The simplex Δ^2 (left), its subdivision $\mathrm{sd}\,\Delta^2$ (middle) and the simplex Δ (right)

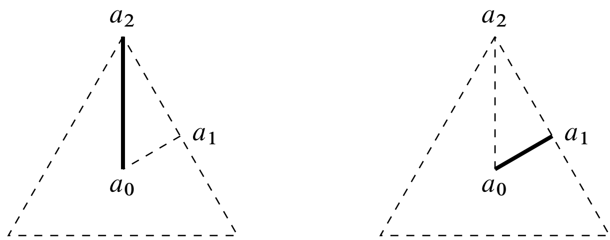


Figure 3: The faces Δ_{s_1} (left) and Δ_{s_2} (right)

The inclusion $|\Delta| \hookrightarrow |\mathrm{sd}\,\mathcal{C}^n|$ extends uniquely to a W_n–equivariant map $W_n \times |\Delta| \to |\mathrm{sd}\,\mathcal{C}^n|$. We want this to reduce to a map

$$\mathcal{U}(W_n, |\Delta|) \to |\mathrm{sd}\,\mathcal{C}^n|,$$

and so we must check that it respects the equivalence relation $\sim$ on $W_n \times |\Delta|$ that defines $\mathcal{U}(W_n, |\Delta|)$. This is an immediate consequence of the following lemma.

Lemma 25 *Under the action of W_n on $|\mathrm{sd}\,\mathcal{C}^n|$, the mirror $|\Delta|_s \subset |\Delta| \subset |\mathrm{sd}\,\mathcal{C}^n|$ is fixed pointwise by s.*

Proof Let $i \geqslant 0$. If $s \in S_i \setminus S_{i-1}$, then s fixes a_j for $j \neq i$. For $i \geqslant 1$ this follows from Proposition 8, and for $i = 0$ it follows because s commutes with s_k for $k \geqslant 2$. Similarly, if $s \in S_{-1}$ then s fixes a_j for all j. In all cases it follows that s fixes every vertex of Δ_s, so that s fixes $|\Delta|_s = |\Delta_s|$ pointwise. □

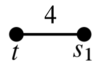
We can now state the main results of this section.

Proposition 26 *The map $\mathcal{U}(W_n, |\Delta|) \to |\mathrm{sd}\,\mathcal{C}^n|$ is a homeomorphism.*

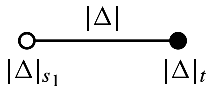
Proposition 27 *$\mathcal{U}(W_n, |\Delta|)$ is $(n-1)$–connected.*

Example 28 (the map $\mathcal{U}(W_1, |\Delta|) \to |\mathrm{sd}\, \mathcal{C}^1|$ for groups of type B) Let us illustrate Proposition 26 in the case of the sequence $(B_{n+1})_{n \geqslant 1}$ and $n = 1$. As in Example 3, the Coxeter diagram of W_1 is as follows:

B_2



In this case Δ is the simplex with vertex set $\{a_0, a_1\}$, while Δ_{s_1} and Δ_t are the faces with vertices a_0 and a_1 respectively. Thus $|\Delta|$ is an interval and $|\Delta|_{s_1}$ and $|\Delta_t|$ are its endpoints. We draw $|\Delta|$ as follows, with $|\Delta|_{s_1}$ represented by a hollow vertex and $|\Delta|_t$ represented by a solid vertex:



Then by definition $\mathcal{U}(W_1, |\Delta|)$ is the union of the translates of $|\Delta|$ by elements of W_1, where for each $w \in W_1$, the solid vertices of $w|\Delta|$ and $wt|\Delta|$ are identified, as are the hollow vertices of $w|\Delta|$ and $ws_1|\Delta|$. Thus $\mathcal{U}(W_1, |\Delta|)$ is as shown on the left of Figure 4.

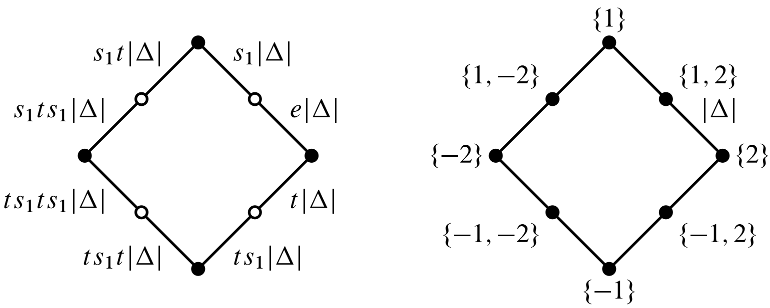


Figure 4: The spaces $\mathcal{U}(W_1, |\Delta|)$ and $|\mathrm{sd}\, \mathcal{C}^1|$

Recall from Example 16 that $\mathcal{C}^1$ is the square with vertices $\{\pm 1, \pm 2\}$, with each vertex being opposite to its negative, where t and s_1 act as the permutations $(1, -1)$ and $(1,2)(-1,-2)$ respectively. Thus $|\mathrm{sd}\, \mathcal{C}^1|$ is as shown on the right of Figure 4 with the subspace $|\Delta|$ labelled. Now the map $\mathcal{U}(W_1, |\Delta|) \to |\mathrm{sd}\, \mathcal{C}^1|$ is the one that is evident from the drawings. It is the identity on the copy of $|\Delta|$ within each space, it is equivariant with respect to the W_1–actions, and it is clearly a homeomorphism.

We now work towards the proof of Proposition 26. It relies on the following two lemmas. Roughly speaking, these correspond to surjectivity and injectivity of the map $\mathcal{U}(W_n, |\Delta|) \to |\mathrm{sd}\, \mathcal{C}^n|$ respectively.

Lemma 29 *Every n–simplex of $\operatorname{sd}\mathcal{C}^n$ is a translate of Δ, and every simplex of $\operatorname{sd}\mathcal{C}^n$ is a face of such a translate.*

Proof Let $\boldsymbol{C}=\{C_0,\dots,C_n\}$ be an n–simplex of $\operatorname{sd}\mathcal{C}^n$ with $C_0\subset\cdots\subset C_n$. Then $\boldsymbol{C}$ induces a natural ordering of the vertices of C_n by declaring that each C_i consists of an initial segment. Let c be a lift of C_n that induces this ordering. Then, by inspecting the definition of the induced order, one sees that $\boldsymbol{C}=c\cdot\Delta$ as required. This proves the first part. Now observe that every simplex of $\mathcal{C}^n$ is a face of an n–simplex, since a simplex with a given lift is a face of the n–simplex with that lift. The second part follows. □

Lemma 30 *Let F be a face of Δ. Then the stabilizer of F under the action of W_n is the subgroup generated by those $s\in S_n$ for which $F\subset\Delta_s$.*

Proof The stabilizer of a simplex of $\operatorname{sd}\mathcal{C}^n$ coincides with the intersection of the stabilizers of its vertices. To see this, let $w\in W_n$ and let $\boldsymbol{C}=\{C_0,\dots,C_k\}$ be a simplex of $\operatorname{sd}\mathcal{C}^n$ that is fixed by w. Then each C_i is a simplex of $\mathcal{C}^n$ itself, and without loss $C_0\subset\cdots\subset C_k$. The assumption $w\cdot\boldsymbol{C}=\boldsymbol{C}$ means that w permutes the C_i. But since each C_i has a different cardinality, this means that w must in fact fix each C_i. So the stabilizer of $\boldsymbol{C}$ is contained in the intersection of the stabilizers of its vertices. The converse is immediate. See [4, page 115].

For the purposes of this proof, given $i\geqslant 0$ we write $S_{=i}$ for the difference $S_i\setminus S_{i-1}$. So for $i\geqslant 1$ we have $S_{=i}=\{s_i\}$, while $S_{=0}$ is the set of elements of S_0 that do not commute with s_1.

Fix $i\geqslant 0$, and consider the vertex a_i of $\operatorname{sd}\mathcal{C}^n$. We will show that the stabilizer of a_i is the subgroup of W_n generated by

$$S_n\setminus S_{=i}=S_{i-1}\cup\{s_{i+1},\dots,s_n\}.$$

To see this, recall that the vertices of a_i (when a_i is regarded as a simplex of $\mathcal{C}^n$) are

$$s_{i+1}\cdots s_nW_{n-1},\dots,s_nW_{n-1},W_{n-1}.$$

Observe that if $s\in S_{i-1}$ then s commutes with $s_{i+1},\dots,s_n$, and so fixes the vertices of a_i, and so fixes a_i itself. And if $s\in\{s_{i+1},\dots,s_n\}$, then by Proposition 8, s permutes the vertices of a_i, and so fixes a_i itself. So the subgroup generated by $S_n\setminus S_{=i}$ fixes a_i. Conversely, suppose that $w\in W_n$ fixes a_i. Proposition 8 shows that any permutation of the vertices of a_i can be achieved using the subgroup generated by $s_{i+1},\dots,s_n$. So after left-multiplying w by an element of the subgroup generated by $s_{i+1},\dots,s_n$, we may assume that w fixes every vertex of a_i. Proposition 12 now shows that w lies in the subgroup generated by S_{i-1}, as required.

Let $F=\{a_{i_1},\dots,a_{i_r}\}$. Then according to the first paragraph, the stabilizer of F is the intersection of the stabilizers of the a_{i_j}. By the last paragraph this is the intersection of the subgroups generated by the sets $S_n\setminus S_{=i_j}$, and by the general results described in Section 2.4, this is the subgroup generated by $\bigcap(S_n\setminus S_{=i_j})=S_n\setminus\bigcup S_{=i_j}$.

It remains to show that $S_n\setminus\bigcup S_{=i_j}$ is the set of s such that $F\subset\Delta_s$. Now by the definition of Δ_s, we see that $F\subset\Delta_s$ for all $s\in S_{-1}$, and that $F\subset\Delta_s$ for $s\in S_{=i}$ if and only if $a_i\notin F$. Thus the set of s such that $F\subset\Delta_s$ is $S\setminus\bigcup S_{=i_j}$ as required. □

Example 31 Let us illustrate the proof of Lemma 30 for the sequence $(A_n)_{n\geqslant 1}$ and $n=2$. We described $\mathcal{C}^2$, $\operatorname{sd}\mathcal{C}^2$, Δ and the Δ_s for this case in Example 24. The following points correspond to the paragraphs of the proof.

- First observe that the action of $W_2=\Sigma_3$ on the vertices of $\operatorname{sd}\mathcal{C}^2$ has three orbits, namely the three vertices of the triangle (which are the 0–simplices of $\mathcal{C}^2$), the midpoints of the edges of the triangle (which are the 1–simplices of $\mathcal{C}^2$), and the barycentre of the triangle (which is the single 2–simplex of $\mathcal{C}^2$). Each simplex of $\operatorname{sd}\mathcal{C}^2$ contains at most one vertex from each orbit. So the stabilizer of a simplex is the intersection of the stabilizers of its orbits.
- We have $S_2=\{s_1,s_2\}$, $S_1=\{s_1\}$ and $S_0=\varnothing$. Thus $S_{=2}=\{s_2\}$, $S_{=1}=\{s_1\}$ and $S_{=0}=\varnothing$.
- Next, observe that the stabilizers of a_0, a_1 and a_2 are $\langle s_1,s_2\rangle$, $\langle s_2\rangle$ and $\langle s_1\rangle$ respectively, and these are indeed the subgroups generated by the sets $S_2\setminus S_{=0}$, $S_2\setminus S_{=1}$ and $S_2\setminus S_{=2}$ respectively. So s_i stabilizes a_j if and only if $i\neq j$.
- By the first point, the stabilizer of a face F of Δ is the intersection of the stabilizers of its vertices, and by the previous point this is the subgroup generated by the s_i for which a_i is not contained in F.
- On the other hand, $F\subset\Delta_{s_i}$ if and only if F does not contain a_i. This, combined with the previous point, shows that the stabilizer of F is generated by the s for which $F\subset\Delta_s$.

Proof of Proposition 26 The map is surjective because any point of $|\operatorname{sd}\mathcal{C}^n|$ is in a translate of $|\Delta|$. This follows from Lemma 29, which shows that every simplex of $\operatorname{sd}\mathcal{C}^n$ is a face of a translate of Δ.

To show that the map is injective, suppose that $[w,x],[v,y]\in\mathcal{U}(W_n,|\Delta|)$ have the same image in $|\operatorname{sd}\mathcal{C}^n|$, or in other words that $w\cdot x=v\cdot y$. We will show that $(w,x)\sim(v,y)$ so that $[w,x]=[v,y]$. First we show that $x=y$. There is a canonical map $\operatorname{sd}\mathcal{C}^n\to\Delta^n$ that sends a vertex C of $\operatorname{sd}\mathcal{C}^n$, or in other words a simplex C of $\mathcal{C}^n$, to the vertex $|C|$ of Δ^n. By construction this is W_n–invariant and restricts to an isomorphism $\Delta \to \Delta^n$. Taking realizations, we obtain a W_n–invariant map $|\mathrm{sd}\,\mathcal{C}^n| \to \Delta^n$ that restricts to a homeomorphism $|\Delta| \to \Delta^n$. Since $w \cdot x = v \cdot y$, we therefore have $x = y$. Next we show that $w^{-1}v$ lies in the subgroup generated by those $s \in S_n$ for which $x \in |\Delta|_s$. Write F for the unique face of Δ for which x lies in the interior of F. Then $x \in |\Delta|_s = |\Delta_s|$ if and only if $F \subset \Delta_s$. Moreover, since $w \cdot x = v \cdot y$ and $x = y$, we see that $w^{-1}v$ lies in the stabilizer of x, which is exactly the stabilizer of F. Then $w^{-1}v$ lies in the claimed subgroup by Lemma 30. Consequently $(w, x) \sim (v, y)$ as required.

The map is a homeomorphism because $|\mathrm{sd}\,\mathcal{C}^n|$ has the weak topology with respect to the realizations of its simplices. By Lemma 29 this coincides with the weak topology with respect to the realizations of its n–simplices. This is exactly the topology of $\mathcal{U}(W_n, |\Delta|)$. □

We now work towards the proof of Proposition 27. We will make use of the increasing union of chambers, which we described in Section 2.6. Recall in particular that if $w \in W_n$ then $\mathrm{In}(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$ is the set of letters with which a reduced expression for w can end. We begin with two lemmas.

Lemma 32 *For $w \in W_n$, $w \neq e$, the space $|\Delta|^{\mathrm{In}(w)}$ is $(n-2)$–connected.*

Proof The set $\mathrm{In}(w)$ is nonempty since $w \neq e$. Thus $|\Delta|^{\mathrm{In}(w)}$ is either $|\Delta|$, or it is a nonempty union of facets of $|\Delta|$. In the first case it is contractible, and in the second case it is either contractible (if not all facets are in the union) or it is $\partial|\Delta| \cong S^{n-1}$ (if all facets are in the union). In all cases it is $(n-2)$–connected. □

Lemma 33 *Let $n \geqslant 1$. Suppose that $(X; A, B)$ is a CW–triad in which A and B are $(n-1)$–connected and $C = A \cap B$ is $(n-2)$–connected. Then X is $(n-1)$–connected.*

Proof For $n = 1$ this is immediate since the union of two path-connected spaces with nonempty intersection is path-connected. So we assume that $n \geqslant 2$. The pairs (A, C) and (B, C) are $(n-1)$–connected, and C is path-connected, so Theorem 4.23 of [13] can be applied to show that $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < 2n-2$, and in particular for $i \leqslant (n-1)$. Thus (X, B) is $(n-1)$–connected, and the same then follows for X itself. □

Proof of Proposition 27 If $n = 0$ then the claim is that $\mathcal{U}(W_n, |\Delta|)$ is nonempty, which holds vacuously. So we may assume that $n \geqslant 1$.

As in Section 2.6, order the elements of W_n as $w_0, w_1, w_2, \dots$ starting with the identity and respecting the length. Then $\mathcal{U}(W_n, |\Delta|)$ is the union of subcomplexes $P_0 \subset P_1 \subset P_2 \subset \cdots$, where $P_0 = |\Delta|$ and

$$P_m = P_{m-1} \cup w_m|\Delta| \quad \text{with} \quad P_{m-1} \cap w_m|\Delta| = w_m|\Delta|^{\mathrm{In}(w_m)}.$$

It will suffice to show that each P_m is $(n-1)$–connected. We do this by induction on m.

In the initial case $m=0$ we have $P_0 = e|\Delta|$, which is contractible and so the claim holds. For the induction step we take $m \geqslant 1$ and assume that P_{m-1} is $(n-1)$–connected. Then $P_m = P_{m-1} \cup w_m|\Delta|$ is the union of the subcomplexes P_{m-1} and $w_m|\Delta|$, and their intersection $w_m|\Delta|^{\mathrm{In}(w_m)}$ is $(n-2)$–connected by Lemma 32. Thus $(P_m; P_{m-1}, w_m|\Delta|)$ is a CW–triad in which the subspaces P_{m-1} and $w_m|\Delta|$ are $(n-1)$–connected and their intersection is $(n-2)$–connected. It now follows from Lemma 33 that P_m is $(n-1)$–connected as required. $\square$

8 The semisimplicial set $\mathcal{D}^n$

In this section we introduce a semisimplicial set $\mathcal{D}^n$ with an action of W_n. It will be used in the next section to give the proof of the Main Theorem. As explained in Section 1.5, the definition of $\mathcal{D}^n$ is inspired by the "complex of injective words" (see Example 35) which is used in existing proofs of homological stability for symmetric groups, for example [21; 19; 25]. Indeed, we first obtained $\mathcal{D}^n$ by writing every aspect of the complex of injective words in terms of the symmetric groups and adjacent transpositions, and then abstracting this definition to our sequences of Coxeter groups $(W_n)_{n\geqslant 1}$. This may leave the definition of $\mathcal{D}^n$ a little unmotivated, but we hope that it will become apparent over this section and the next that $\mathcal{D}^n$ is precisely the object required to complete the proof of the main theorem.

The main result of the section is that the realization $\|\mathcal{D}^n\|$ is $(n-1)$–connected. This is proved by identifying $\mathcal{D}^n$ as the semisimplicial set of ordered simplices in $\mathcal{C}^n$, and then using the fact that $\mathcal{C}^n$ is weakly Cohen–Macaulay of dimension n to deduce that the geometric realization $\|\mathcal{D}^n\|$ is $(n-1)$–connected. We learned this approach from Wahl's paper[28], in particular Proposition 7.9, which is due to Randal-Williams.

In this section and the next we will use semisimplicial spaces and their realizations. The background material we require can be found in Section 2 of [25].

Definition 34 Let $\mathcal{D}^n$ denote the semisimplicial set with k–simplices

$$\mathcal{D}^n_k = \begin{cases} W_n/W_{n-k-1} & \text{for} \quad k \leqslant n, \\ \varnothing & \text{for} \quad k > n, \end{cases}$$

and with face maps

$$d_i\colon W_n/W_{n-k-1} \to W_n/W_{n-k},$$

defined by

$$d_i(cW_{n-k-1}) = c(s_{n-k+i}\cdots s_{n-k+1})W_{n-k}$$

for $i = 0,\dots,k$. Here the product $(s_{n-k+i}\cdots s_{n-k+1})$ is omitted when $i = 0$.

One can verify directly that the face maps d_i satisfy the relations $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$. Alternatively, it is a consequence of the proof of Proposition 37 below.

Example 35 ($\mathcal{D}^n$ for groups of type A, B and D) In order to illustrate the definition above, we recall the definition of the complex of injective words. Let L be a set. An *injective word* in L is a finite sequence of distinct elements of L. The *complex of injective words in* L is the semisimplicial set whose k–simplices are injective words in L of length $(k+1)$, and in which the face map d_i sends $(x_0,\dots,x_k)$ to $(x_0,\dots,\widehat{x_i},\dots,x_k)$.

For the family $(A_n)_{n\geqslant 1}$, the semisimplicial set $\mathcal{D}^n$ is the complex of injective words in $\{1,\dots,n+1\}$. Recall from Example 2 that W_n is the group of permutations of $\{1,\dots,n+1\}$. Thus W_n/W_{n-k-1} can be identified with the set of injective words of length $(k+1)$ in $\{1,\dots,n+1\}$ via the isomorphism

$$\sigma W_{n-k-1} \longmapsto (\sigma(n-k+1),\dots,\sigma(n+1)).$$

To see that the face map d_i corresponds to the map that deletes the i^{th} letter, we must show that

$$\begin{aligned}\big(\sigma s_{n-k+i}\cdots s_{n-k+1}(n-k+2),\dots,&\sigma s_{n-k+i}\cdots s_{n-k+1}(n+1)\big)\\ &= \big(\sigma(n-k+1),\dots,\widehat{\sigma(n-k+i)},\dots,\sigma(n+1)\big).\end{aligned}$$

This follows because $s_{n-k+i}\cdots s_{n-k+1}$ decreases each of $(n-k+2),\dots,(n-k+i)$ by one, sends $(n-k+i)$ to $(n-k+1)$, and fixes $(n-k+i+1),\dots,(n+1)$.

For the family $(B_{n+1})_{n\geqslant 1}$, $\mathcal{D}^n$ is the subset of the complex of injective words in $\{\pm 1,\dots,\pm(n+1)\}$ in which each word features at most one entry from each pair $\{i,-i\}$. For the family $(D_{n+2})_{n\geqslant 1}$, we have that $\mathcal{D}^n$ is the subset of the complex of injective words in $\{\pm 1,\dots,\pm(n+2)\}$ in which each word again features at most one entry from each pair $\{i,-i\}$. These two facts can be proved by the method of the previous example, this time making use of Examples 3 and 4.

Definition 36 Let X be a simplicial complex. By an *ordered simplex of* X, we mean a simplex of X equipped with an ordering of its vertices. The *semisimplicial set of ordered simplices in* X, denoted X^{ord}, has for its k–simplices the ordered k–simplices in X, with face maps d_i given by forgetting the i^{th} vertex of an ordered simplex.

Proposition 37 *$\mathcal{D}^n$ is isomorphic to $\mathcal{C}^{n,\mathrm{ord}}$.*

Proof We define $\phi_k\colon \mathcal{D}^n_k \to \mathcal{C}^{n,\mathrm{ord}}_k$ by

$$\phi_k(cW_{n-k-1}) = \{c(s_{n-k+1}\cdots s_n)W_{n-1},\ \dots,\ cs_nW_{n-1},\ cW_{n-1}\}$$

for $cW_{n-k-1} \in W_n/W_{n-k-1}$. In other words, $\phi_k(cW_{n-k-1})$ is the k–simplex with lift c, equipped with the ordering induced by c. The map ϕ_k is well defined because the generators of W_{n-k-1} all commute with $s_{n-k+1},\dots,s_n$. It is surjective because by definition every simplex admits a lift, and any ordering of a simplex is afforded by some lift (see the paragraph following Definition 13). It is injective because if $\phi_k(cW_{n-k-1}) = \phi_k(c'W_{n-k-1})$ then $cs_i\cdots s_nW_{n-1} = c's_i\cdots s_nW_{n-1}$ for $i = n-k+1,\dots,n+1$, so that $cW_{n-k-1} = c'W_{n-k-1}$ by Proposition 12.

To complete the proof we must show that the face maps in $\mathcal{C}^{n,\mathrm{ord}}$ and $\mathcal{D}^n$ are compatible under the ϕ_k. In other words, given $0 \leqslant i \leqslant k \leqslant n$, we must show that

$$\phi_{k-1}\circ d_i = d_i \circ \phi_k.$$

Observe from the definition of d_i in $\mathcal{D}^n$ that $d_i(cW_{n-k-1}) = d_{i-1}(cs_{n-k+i}W_{n-k+1})$ for $i \geqslant 1$. Proposition 8 shows that $\phi_k(cW_{n-k-1})$ and $\phi_k(cs_{n-k+i}W_{n-k-1})$ differ only by the transposition of their $(i-1)^{\mathrm{st}}$ and i^{th} vertices, so that $d_i(\phi_k(cW_{n-k-1})) = d_{i-1}(\phi_k(cs_{n-k+i}W_{n-k-1}))$. Thus the claim will follow by induction on i so long as we can show that

$$\phi_{k-1}\circ d_0 = d_0\circ\phi_k.$$

This follows by inspection. □

Corollary 38 *$\|\mathcal{D}^n\|$ is $(n-1)$–connected.*

Proof Theorem 18 shows that $\mathcal{C}^n$ is weakly Cohen–Macaulay of dimension n. It was shown in Proposition 7.9 of [28] that if a simplicial complex X is weakly Cohen–Macaulay of dimension n, then $\|X^{\mathrm{ord}}\|$ is $(n-1)$–connected. Consequently $\|\mathcal{C}^{n,\mathrm{ord}}\|$ is $(n-1)$–connected, and by Proposition 37 the same holds for $\|\mathcal{D}^n\|$. □

Remark 39 In the case of the sequence $(A_n)_{n\geqslant 1}$, when $\mathcal{D}^n$ is the complex of injective words in $\{1,\dots,n+1\}$, the connectivity of $\|\mathcal{D}^n\|$ is well-known: see [9; 2; 19; 25]. (Strictly speaking, the first and third references deal with the homology of $\|\mathcal{D}^n\|$, rather than its homotopy type.)

9 Completion of the proof

We now complete the proof of the Main Theorem. This section is modelled closely on Section 5 of [25], from which there is little essential difference. It is also similar to the proof of Theorem 2 of [19].

We regard $\mathcal{D}^n$ as a simplicial space by equipping its constituent sets with the discrete topology. Then we form a semisimplicial space

$$EW_n \times_{W_n} \mathcal{D}^n$$

by setting $(EW_n \times_{W_n} \mathcal{D}^n)_k = EW_n \times_{W_n} \mathcal{D}^n_k$ and using the face maps obtained from those of $\mathcal{D}^n$.

Lemma 40 *The projection $EW_n \times_{W_n} \mathcal{D}^n_0 \to BW_n$ makes $EW_n \times_{W_n} \mathcal{D}^n$ into an augmented simplicial space over BW_n, and the induced map $\|EW_n \times_{W_n} \mathcal{D}^n\| \to BW_n$ is $(n-1)$–connected.*

Proof The composites of the projection with d_0 and d_1 coincide, so that the projection is indeed an augmentation. Since the map $EW_n \to BW_n$ is a locally trivial principal W_n–bundle, it follows that $\|EW_n \times_{W_n} \mathcal{D}^n\| \to BW_n$ is a locally trivial bundle with fibre $\|W_n \times_{W_n} \mathcal{D}^n\| \cong \|\mathcal{D}^n\|$, which is $(n-1)$–connected by Corollary 38, so that the map itself is $(n-1)$–connected. □

Lemma 41 *There are homotopy equivalences $EW_n \times_{W_n} \mathcal{D}^n_k \simeq BW_{n-k-1}$ under which the face maps $d_i\colon EW_n \times_{W_n} \mathcal{D}^n_k \to EW_n \times_{W_n} \mathcal{D}^n_{k-1}$ are all homotopic to the stabilization map $BW_{n-k-1} \to BW_{n-k}$, and under which the composite*

$$EW_n \times_{W_n} \mathcal{D}^n_0 \to \|EW_n \times_{W_n} \mathcal{D}^n\| \to BW_n$$

is homotopic to the stabilization map $BW_{n-1} \to BW_n$.

Proof There is an isomorphism

$$EW_n \times_{W_n} \mathcal{D}^n_k = EW_n \times_{W_n} (W_n/W_{n-k-1}) \xrightarrow{\cong} EW_n/W_{n-k-1}$$

sending the orbit of (x, cW_{n-k-1}) to the orbit of $c^{-1}x$. This identifies d_i with the map

$$EW_n/W_{n-k-1} \to EW_n/W_{n-k}$$

sending the W_{n-k-1}–orbit of x to the W_{n-k}–orbit of $(s_{n-k+1}\cdots s_{n-k+i})x$. We claim that this map is homotopic to the one sending the W_{n-k-1}–orbit of x to the W_{n-k}–orbit of x. Indeed, EW_n is contractible, and W_{n-k-1} acts on it freely. Moreover, when we equip EW_n with its natural CW–structure as the realization of a simplicial set, then this action is cellular. It follows that any two W_{n-k-1}–equivariant maps from EW_n to itself are W_{n-k-1}–equivariantly homotopic. (This can be proved by induction on the cells. Alternatively, see [20, Definition 1.8 and Theorem 1.9] in the case where $G = W_{n-k-1}$ and $\mathcal{F}$ consists of the trivial subgroup.) Since $(s_{n-k+1}\cdots s_{n-k+i})$ commutes with every element of W_{n-k-1}, the map $EW_n \to EW_n$ given by left-multiplication by $(s_{n-k+1}\cdots s_{n-k+i})$ is W_{n-k-1}–equivariant, and is therefore W_{n-k-1}–equivariantly homotopic to the identity map. The claim now follows by taking W_{n-k-1}–orbits in the domain and W_{n-k}–orbits in the codomain.

Now the equivariant homotopy equivalences

$$EW_{n-k-1} \to EW_n \quad \text{and} \quad EW_{n-k} \to EW_n$$

induce homotopy equivalences

$$BW_{n-k-1} \to EW_n/W_{n-k-1} \quad \text{and} \quad BW_{n-k} \to EW_n/W_{n-k}$$

under which the map $EW_n/W_{n-k-1} \to EW_n/W_{n-k}$ just described becomes the stabilization map. □

The skeletal filtration of $\|EW_n \times_{W_n} \mathcal{D}^n\|$ leads to a first-quadrant spectral sequence

$$E^1_{k,l} = H_l(EW_n \times_{W_n} \mathcal{D}^n_k) \implies H_{k+l}(\|EW_n \times_{W_n} D^n\|),$$

in which the differential d^1 is given by the alternating sum $\sum_{i=0}^k (-1)^i (d_i)_*$ of the maps induced by the face maps. Lemma 41 allows us to identify the E^1–term of this spectral sequence: $E^1_{k,l} = H_l(BW_{n-k-1})$, and $d^1\colon E^1_{k,l} \to E^1_{k-1,l}$ is the map $H_l(BW_{n-k-1}) \to H_l(BW_{n-k})$ induced by stabilization if k is even, and is zero if k is odd.

Lemma 42 *For all $m<n$, assume that the stabilization map $H_l(BW_{m-1}) \to H_l(BW_m)$ is an isomorphism in degrees $2l \leqslant m$. Then the spectral sequence has the following properties:*

(1) $E^\infty_{0,l} = \cdots = E^2_{0,l} = E^1_{0,l}$ *for* $2l \leqslant n$.

(2) $E^\infty_{k,l} = 0$ *for* $k > 0$ *and* $2(k+l) \leqslant n$.

Proof The assumption allows us to deduce that $E^2_{k,l} = 0$ when $k \geqslant 1$ is odd and $2l + k + 1 \leqslant n$, and that $E^2_{k,l} = 0$ when $k \geqslant 2$ is even and $2l + k \leqslant n$. For in the first case

$$d^1\colon E^1_{k+1,l} \to E^1_{k,l}$$

is the stabilization map

$$H_l(BW_{n-k-2}) \to H_l(BW_{n-k-1}),$$

and in the second case

$$d^1\colon E^1_{k,l} \to E^1_{k-1,l}$$

is the stabilization map

$$H_l(BW_{n-k-1}) \to H_l(BW_{n-k}),$$

and our assumption means that both are isomorphisms in the given range. Figure 5 shows the E^1–page, where the left-hand shaded region consists of terms with total degree satisfying $2{*} \leqslant n$ and the right-hand shaded region consists of terms that vanish on the E^2-page.

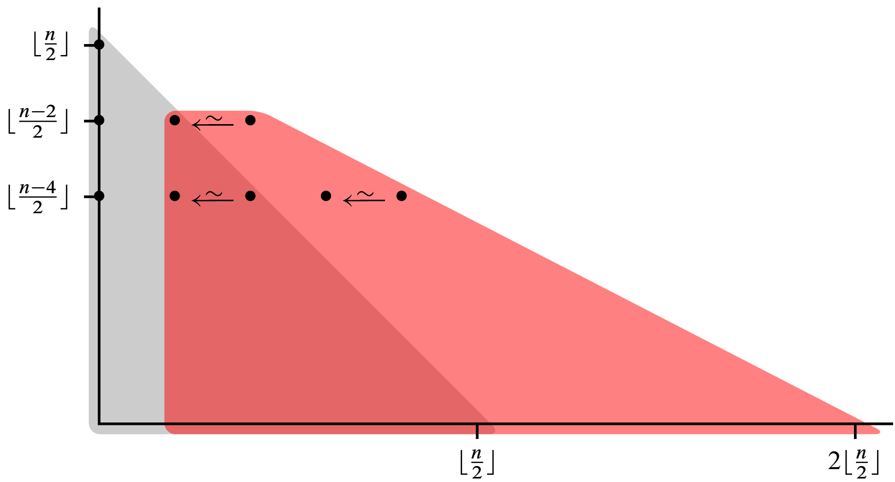


Figure 5: The spectral sequence $(E^r_{s,t})$

To prove the first claim, observe that, since $d^1\colon E^1_{1,l} \to E^1_{0,l}$ is zero, $E^2_{0,l} = E^1_{0,l}$. The remaining differentials with target in bidegree $(0,l)$ are

$$d^k\colon E^k_{k,l-k+1} \to E^k_{0,l}$$

with $k \geqslant 2$, and these have domain zero since

$$2(l-k+1)+k \leqslant 2(l-k+1)+k+1 = 2l-k+2 \leqslant 2l \leqslant n$$

so that $E^2_{k,l-k+1} = 0$. To prove the second claim, observe that if $2(k+l) \leqslant n$ and $k > 0$, then certainly

$$2l+k < 2l+k+1 \leqslant 2(l+k) \leqslant n,$$

so that $E^2_{k,l} = 0$. $\square$

We can now complete the proof of the main theorem, showing by induction on $n \geqslant 0$ that $H_l(BW_{n-1}) \to H_l(BW_n)$ is an isomorphism for $2l \leqslant n$. (In the main theorem this claim was made only for $n \geqslant 2$, but the proof by induction relies on the cases obtained by extending to the left.) For $n = 0$ the claim is that $H_0(BW_{-1}) \to H_0(BW_0)$ is an isomorphism, which is trivial since BW_{-1} and BW_0 are both path connected. Take $n \geqslant 1$ and suppose that the theorem holds for all smaller integers. Lemma 41 shows that the composite

$$H_l(BW_{n-1}) = E^1_{0,l} \to E^\infty_{0,l} \to H_l(\|EW_n \times_{W_n} \mathcal{D}^n\|) \to H_l(BW_n),$$

is the stabilization map, and we must show that this is an isomorphism for $2l \leqslant n$. Lemma 42 shows that the first two arrows are isomorphisms in this range, while Lemma 40 shows that the last map is an isomorphism for $l \leqslant n-1$, which holds since $2l \leqslant n$ and $n \geqslant 2$.

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The extended Goldman bracket determines intersection numbers for surfaces and orbifolds

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In the mid eighties Goldman proved that an embedded closed curve could be isotoped to not intersect a given closed geodesic if and only if their Lie bracket (as defined in that work) vanished. Goldman asked for a topological proof and about extensions of the conclusion to curves with self-intersection. Turaev, in the late eighties, asked about characterizing simple closed curves algebraically, in terms of the same Lie structure. We show how the Goldman bracket answers these questions for all finite type surfaces. In fact we count self-intersection numbers and mutual intersection numbers for all finite type orientable orbifolds in terms of a new Lie bracket operation, extending Goldman's. The arguments are purely topological, or based on elementary ideas from hyperbolic geometry.

These results are intended to be used to recognize hyperbolic and Seifert vertices and the gluing graph in the geometrization of three-manifolds. The recognition is based on the structure of the string topology bracket of three-manifolds.

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Dedicated with deep and grateful admiration to Bill Thurston (1946–2012)

1 Introduction

Goldman [10] discovered in the eighties an intriguing Lie algebra structure on the free abelian group generated by the set of free homotopy classes of closed directed curves on an oriented surface F. The definition of the Goldman bracket combines intersection structure with the usual based loop product in the following way: given two closed free homotopy classes a and b with representatives A and B respectively, intersecting only in transversal double points,

$$[a,b] = \sum_{P \in A\cap B} \operatorname{sign}(P)\,\widetilde{A \cdot_P B}, \tag{1}$$

where $\operatorname{sign}(p)$ is the sign of the intersection between the curves A and B at P, $A \cdot_p B$ is the loop product of A and B both viewed as based at P, and $\widetilde{C}$ is the free homotopy

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class of a curve C. This bracket is extended by linearity to the free module generated by free homotopy classes of curves. Goldman showed that this bracket is well-defined, skew-symmetric and satisfies the Jacobi identity.

Clearly, if a and b are free homotopy classes that have disjoint representatives, then $[a, b]$ is zero. Goldman [10] also showed (using Thurston's earthquakes) that this bracket has the remarkable property that if one of the classes, a or b, has a simple representative, then the bracket $[a, b]$ vanishes if and only if a and b can be represented by disjoint curves. Goldman asked for a topological proof and about extensions of the conclusion to curves with self-intersection. Turaev, in the late eighties, asked about characterizing simple closed curves algebraically in terms of this Lie structure.

Later on Chas [7] proved that if either a or b has a simple representative then the bracket of a and b counts the geometric intersection number between a and b (by geometric intersection number we mean the minimum number of points, counted with multiplicity, in which representatives of a and b intersect).

On the other hand, there are examples of classes a and b with no disjoint representatives and such that $[a, b] = 0$; see for instance [6, Example 9.1]. The bracket is a homotopy invariant like the set of conjugacy classes in the fundamental group which is, in some sense, simpler than the fundamental group itself. Since intersection and self-intersection numbers of closed curves on surfaces play such a critical role in several areas of low-dimensional topology, it is highly desirable to find characterizations of the intersection numbers. A result of this nature, obtained by Chas and Krongold [8], was that for the subset of compact orientable surfaces with non-empty boundary, the bracket $[a, a^3]$ determines the self-intersection number of a.

Finally, after twenty five years since Goldman's paper [10] we show here how the bracket answers the question about disjunction and simplicity of closed curves for all finite type surfaces. We also count self-intersection numbers and mutual intersection numbers for all finite type orientable orbifolds in terms of a new Lie bracket operation, extending Goldman's. Our results fill in most of the lacunae in partial results that have resisted extension over the intervening years. The arguments are purely topological, using group theory ideas of Freedman, Scott and Hass [17; 9], or they are based on elementary geometrical ideas from hyperbolic geometry.

By a *Fuchsian group* we mean a discrete group of orientation-preserving isometries of the hyperbolic plane. Below are the two main results of this paper.

Mutual Intersection Theorem *Let x and y be non-conjugate hyperbolic elements in a finitely generated Fuchsian group. Consider the generalized Goldman bracket $[\cdot, \cdot]$ of the p^{th} power of x with the q^{th} power of y, where p and q are such that the* *ratio p/q is different from the ratio of the translation length of x and the translation length of y. Then for all but finitely many values of p and q (which are explicit from the proof), the geometric intersection number of x and y is given by the number of terms in $[\langle x^p\rangle, \langle y^q\rangle]$, counted with multiplicity, divided by $p\cdot q$.*

Self-Intersection Theorem *For x a hyperbolic element in a finitely generated Fuchsian group, which is not a proper power of another element, the geometric self-intersection number of x is given by the number of terms (counted with multiplicity) divided by $p\cdot q$ of $[\langle x^p\rangle, \langle x^q\rangle]$, for all but finitely many pairs of distinct positive integers p and q. (Once more, the excluded pairs are determined explicitly by the proof.)*

Our proof is based on the word hyperbolicity of Fuchsian groups rather than small-cancellation theory as in [8]. By extending the result of [8] for surfaces with boundary to closed surfaces we complete the answer to Goldman's question [10, Subsection 5.17] as to whether his topological result (that if a and b are two free homotopy classes of curves on a surface such that a has a simple representative and $[a,b]=0$, then a and b have disjoint representatives) has a topological proof.

The main lemma of this work states that if at least one of p and q is sufficiently large and the lengths of x^p and y^q are different, then there is no cancellation of terms in the bracket $[\langle x^p\rangle, \langle y^q\rangle]$. In other words, if the representatives A and B intersect in the minimum number of points, then two intersection points P and Q with different sign do not give the same free homotopy class of curves, that is, $\widetilde{A\cdot_P B}\neq\widetilde{A\cdot_Q B}$.

We show this by constructing quasigeodesic representatives of a lift of a loop representing $A\cdot_P B$. These quasigeodesics are the concatenations of certain segments of translates of the axis of x and the axis of y. As quasigeodesics are not too far from geodesics, it follows that if two points of intersection give the same free homotopy class, then there is a pair of corresponding quasigeodesics that are close, which then implies that they are equal. We deduce that the two points correspond to terms with the same sign in the Goldman bracket.

For the final step (deducing that two points correspond to terms with the same sign), rather than using general δ–hyperbolicity arguments as sketched above, we use hyperbolic geometry and the fact that the quasigeodesic curves we construct are actually piecewise geodesic and are explicitly described. This gives a sharper result than one would get with general arguments: for our result, we only require that one of the exponents p and q is large, while coarser geometric arguments would require both to be large.

These results are intended to be applied to recognize hyperbolic and Seifert vertices and the gluing graph in the geometrization of three-manifolds. The recognition is based on the structure of the string topology bracket of three-manifolds.

For a typical irreducible three-manifold, the cyclic homology of the group ring of the fundamental group lives in two degrees: zero and one. Degree one is a Lie algebra and degree zero is a Lie module for degree one. The Lie algebra breaks into a direct sum corresponding to the pieces and the module structure tells how they are combined in the graph.

One can show that the Goldman bracket on the linear space with basis the set of free homotopy classes and the power operations on this basis determine the Fuchsian group of an orbifold. Thus, the Goldman bracket solves the "recognition problem" for two-dimensional orbifolds. More significantly, now that the proof of the Geometrization conjecture has enabled a classification of three-manifolds, there arises the need to calculate the geometrization in examples like knots, ie the "recognition problem for three-manifolds". Our work directly applies to that since the string topology bracket in three-manifolds will be used to describe the canonical graph of the geometrization picture as well as which vertices are hyperbolic and which are Seifert fibered spaces. This bracket is largely concentrated on the Seifert pieces. On these pieces it depends on the orbifold bracket defined here. The orbifold part of the story seemed sufficiently interesting to present independently with the details of the application to three-manifolds coming next.

We emphasize though that the above characterization is a new one for closed curves on closed surfaces, and should be of interest even in this case.

Others have considered string topology operations for orbifolds and manifold stacks in a more abstract setting, see for instance Ángel, Backelin and Uribe [1], Behrend, Ginot, Noohi and Xu [3], and Lupercio, Uribe and Xicotencatl [15]. It would be interesting to relate those results to the concrete results here.

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Outline In Section 2 we review the group theoretic definition of intersection number from [9] and [17] as well as the definition of the geometric intersection number of closed curves in a two-dimensional, orientable orbifold. Section 3 is devoted to the extension of the Goldman bracket to oriented orbifolds (a crucial part of this definition is the elementary geometric fact that if two hyperbolic transformations x and y have intersecting axes, then $x \cdot y$ is hyperbolic). In Section 4 we prove the Jacobi identity for the extension of the Goldman bracket (interestingly enough, this proof boils down to the proposition of geometry that if a line intersects a side of a triangle, then it intersects one of the other two sides). In Section 5 we give examples of the bracket in the modular surface (a beautiful and computable example of orbifolds). In Section 6 we show that geodesics are quantitatively separated for hyperbolic surfaces (and orbifolds): namely if two closed geodesics are sufficiently close and parallel after lifting to the universal cover, they must coincide. In Section 7 we prove the main non-cancellation lemma, stating that if the conjugacy classes of the two terms of the bracket coincide, then the two quasigeodesics associated to these two terms coincide. Finally in Section 8 we give the proofs of the intersection theorem and the self-intersection theorem.

2 The geometric intersection number and the group theoretic intersection number

Let G be a discrete subgroup of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}$. (The set of isometries of $\mathbb{H}$, $\mathrm{Isom}(\mathbb{H})$ has the compact-open topology.)

Each isometry g of the hyperbolic plane extends to the circle at infinity, where, if $g \neq 1$, it fixes at most 2 points. An isometry is called *elliptic, parabolic* or *hyperbolic* according as it fixes 0, 1 or 2 points respectively in the circle at infinity. A hyperbolic isometry g fixes the (hyperbolic) line joining its two fixed points at infinity. This line is called the *axis of* g. Further, the sets of fixed points at infinity of two isometries contained in a discrete subgroup G are either disjoint or coincide. If the sets of fixed points at infinity of a pair of elements of G coincide and are non-empty, then the isometries are both powers of the same element of G.

In this paper, an *orbifold* $\mathbb{H}/G$ is the quotient of the hyperbolic plane $\mathbb{H}$ by a discrete group of orientation-preserving isometries G, provided with the induced metric. The pertinent finer notion of free homotopy for orbifolds is described in Section 2.1. (Note that we are using the word "orbifold" in a narrower sense than the usual.)

In this section we review the definition of closed curves, homotopy and geometric intersection number for curves for an orbifold (Section 2.1), the group theoretic definition of intersection number in orbifolds (Section 2.2), and show these two definitions agree.

(The reader is referred to [18, Chapter 13], [4, Chapter 2] and [13, Section 6.2] for a more general definition of orbifolds and orbifold homotopy. See also [16, Section 13.3] for a formidable discussion of based orbifold homotopy in terms of charts.)

2.1 Orbifold homotopy and the geometric intersection number

A *cone point* P in $\mathbb{H}/G$ is the projection of a point in $\mathbb{H}$ which is fixed by some non-trivial element of G. The *order* of a cone point P is the cardinality of the maximal cyclic subgroup of G fixing P.

Consider the projection map, $\Pi\colon \mathbb{H} \to \mathbb{H}/G$. A *representative of a closed oriented curve in an orbifold* $\mathbb{H}/G$ is a continuous map $\alpha\colon \mathbb{S}^1 \to \mathbb{H}/G$ (with $\mathbb{H}/G$ thought of as a topological space), passing through at most finitely many cone points, together with the choice of a full lift $\hat{\alpha}\colon \mathbb{R} \to \mathbb{H}$, so that $\Pi \circ \hat{\alpha} = \alpha \circ \Theta$, where $\Theta\colon \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ is the usual projection. Two representatives of closed curves are equivalent if their full lifts are related by an element of the group G. A *closed curve on the orbifold* $\mathbb{H}/G$ is an equivalence class of representatives of closed curves.

Definition 2.1 Two closed oriented curves α and α' in $\mathbb{H}/G$ are $\mathbb{H}/G$*–homotopic* if they are related by a finite sequence of moves. Each of these moves is either a homotopy in the complement of the cone points or is one of the skein relations or moves depicted in Figures 1 and 2. There, the disk where the move happens contains exactly one cone point P, and n denotes the order of P. An arc with no self-intersection in the disk and passing through P is $\mathbb{H}/G$–homotopic relative to endpoints to an arc spiraling around P in either direction $(n-1)/2$ times if n is odd (Figure 2), or $n/2$ times if n is even (Figure 1). Also, if n is odd, the endpoints of the arc are antipodal and if n is even, the endpoints coincide.

Remark 2.2 The skein relations depicted in Figures 1 and 2 imply that a loop going n times in either direction around a point of order n can be "erased" from a closed curve (Figure 3). However, note that the skein relation in Figure 3 is less precise than Definition 2.1. Namely, this relation does not "tell" as Definition 2.1 does tell how to homotope a curve passing through a cone point. Since some geodesics do pass through cone points, we need the skein relation in Definition 2.1 that deals with those cases.

The proof of the next result is very similar to that of the (standard) proof of a bijection between free (usual) homotopy classes of closed curves on a path-connected space and conjugacy classes of the fundamental group of the space (see, for instance, [12, Chapter 1, Exercise 6]).

Theorem 2.3 *There is a natural bijection between the set of conjugacy classes of G and the set of $\mathbb{H}/G$–free homotopy classes of closed oriented curves in $\mathbb{H}/G$.*

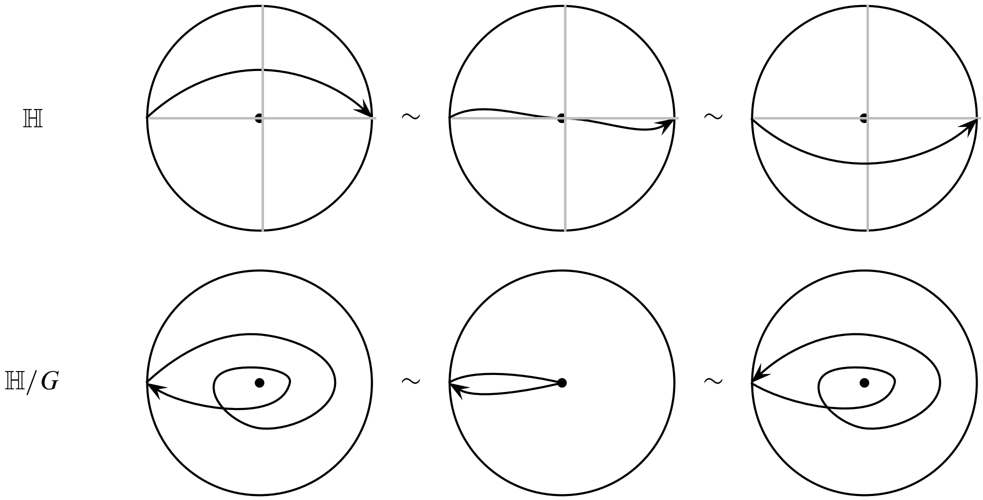


Figure 1: Skein relations for points of order $n = 4$ (bottom) and the corresponding lifts (top)

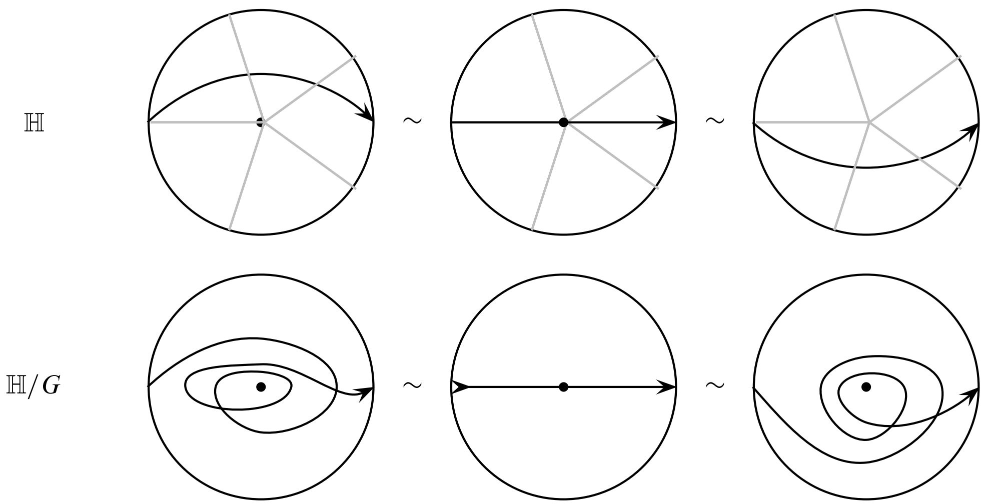
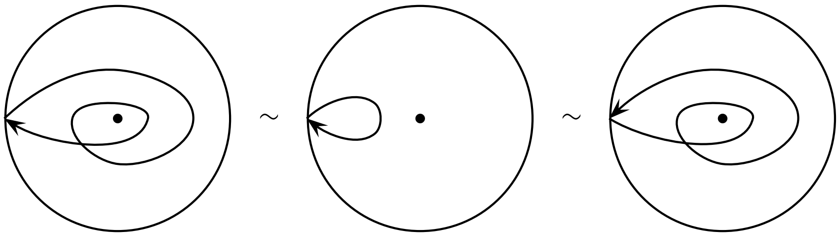


Figure 2: Skein relations for points of order $n = 5$ (bottom) and the corresponding lifts (top)

If a and b in are two elements of $\mathbb{H}/G$, *the intersection number of a and b* is the minimum number (counted with multiplicity) of transversal intersection points of pairs of loops representing a and b not passing through cone points.

Remark 2.4 If at least one of the elements, a or b, belongs to the conjugacy class of a non-hyperbolic element of G then the intersection number of a and b is zero.

Case $n = 2$



Case $n = 3$

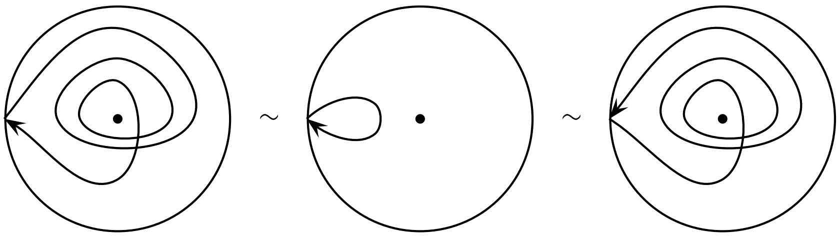


Figure 3: Consequence of skein relations for points of order $n = 2$ (top) and $n = 3$ (bottom)

(Conjugacy classes of elements of G are identified with free homotopy classes of curves on $\mathbb{H}/G$ via Theorem 2.3.)

2.2 Labeling intersection points: the group theoretic intersection number

A hyperbolic isometry x acts on its axis A_x by translation by a real number τ_x, the *translation length of* x. We orient the axis A_x so that for each point P in A_x, the direction from P to xP is positive.

Let $x, y \in G$. Denote by $X\backslash G/Y$ the space of double cosets XgY where $g \in G$, and X and Y denote the cyclic subgroups generated by x and y respectively. If x or y is not hyperbolic, set $I(x, y) = \varnothing$, otherwise, set

$$I(x, y) = \{XgY \in X\backslash G/Y \text{ such that } A_x \cap gA_y \neq \varnothing\}.$$

Scott [17] discusses intersection numbers of closed curves on compact surfaces. The next proposition can be proven by arguments completely analogous to those of Scott [17, Section 1]. The point is that $\mathbb{H}/G$–homotopy after lifting becomes exactly like usual homotopy in the universal cover. Thus our discussion and Scott's are the same, mutatis mutandis, as far as the proposition below is concerned. (In the next proposition, the identification of conjugacy classes in G and $\mathbb{H}/G$–free homotopy classes of closed curves in $\mathbb{H}/G$ given by Theorem 2.3 is used.)

Proposition 2.5 *Let x and y be elements of G. Then the intersection number of the conjugacy classes of x and y equals the cardinality of $I(x, y)$.*

3 The Goldman bracket for orbifolds

Recall that $\mathcal{C}$ denotes the set of conjugacy classes of elements in G. Consider $\mathbb{Z}[\mathcal{C}]$, the free module generated by $\mathcal{C}$. For $x \in G$, let $\langle x\rangle$ denote the conjugacy class of x. In particular, $\langle x\rangle \in \mathbb{Z}[\mathcal{C}]$.

In this section we will define a linear map $[\cdot,\cdot]\colon \mathbb{Z}[\mathcal{C}] \otimes \mathbb{Z}[\mathcal{C}] \to \mathbb{Z}[\mathcal{C}]$ and show in Section 4 that it is a Lie bracket. This bracket generalizes Goldman's to orientable two-dimensional orbifolds and will be defined (as Goldman's) on two elements of the basis of $\mathbb{Z}[\mathcal{C}]$ by considering the intersection points of a certain pair of representatives (see Section 2.2), assigning a signed free homotopy class to each of these points (the signed product at the intersection point) and adding up all those terms.

For elements a and x in G, let x^a denote axa^{-1}. If x is hyperbolic, the isometry x^a is also hyperbolic, has the same translation length as x, ie $\tau_{x^a} = \tau_x$, and the axis of x^a is given by $a \cdot A_x$. From now on, fix an orientation of $\mathbb{H}$. Also, for x and y in G set $\iota(x,y)$ to be zero if x or y are elliptic or parabolic or if the axes of x and y do not cross, and to be the sign of the crossing, otherwise. Finally, set

$$[\langle x\rangle, \langle y\rangle] = \sum_{XbY \in I(x,y)} \iota(x, y^b)\langle xy^b\rangle. \tag{2}$$

Notation 3.1 Let P be a point in the axis A_x of a hyperbolic transformation x. If r is a positive real number, $S(x,P,r)$ denotes the segment of A_x of length r starting (and including) P, but not the other endpoint, in the positive direction of A_x. If r is a negative number, $S(x,P,r)$ denotes the segment of A_x starting at a point Q at distance r from P in the negative direction, containing Q but not P.

Remark 3.2 Fix a point P in A_x and let r be the translation length of x. Let

$$J(x,y,P) = \{gY \in G/Y : S(x,P,r) \cap gA_y \neq \varnothing\}.$$

Then there is a bijection between $I(x,y)$ and $J(x,y,P)$. Since G is a discrete group, both sets have finite cardinality. Moreover,

$$[\langle x\rangle, \langle y\rangle] = \sum_{gY \in J(x,y,P)} \iota(x, y^g)\langle xy^g\rangle. \tag{3}$$

Remark 3.3 The conjugacy classes of elliptic and parabolic elements of G are in the center of the Lie algebra; that is, the bracket between these classes and all other classes is zero.

Remark 3.4 By [2, Theorem 7.38.6], if x and y are hyperbolic isometries whose axes intersect then xy is also hyperbolic. Moreover, the axis of xy and its translation length can be determined as follows (see [2] for details). Denote by P the intersection point of A_x and A_y. Denote by Q the point on A_x at distance $\tau_x/2$ from P in the positive direction of A_x and by R the point on A_y at distance $\tau_y/2$ from P in the negative direction of A_y. The axis of A_{xy} is the oriented line from R to Q and the translation length of xy equals twice the distance between R and Q. (See Figure 4; this is one of the "triangles" mentioned in the introduction which are used to unravel the Jacobi relation.)

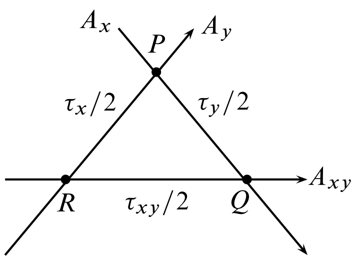


Figure 4: The axis of xy

Remark 3.5 Consider the set of pairs of cosets $G/X \times G/Y$. The group G acts on the set $G/X \times G/Y$ by $(gX, hY) \mapsto (agX, ahY)$, for each $a \in G$. Denote by $D(x, y)$ the quotient under this action. Set $f\colon D(x, y) \to X\backslash G/Y$ by mapping the equivalence class of (gX, hY) to $Xg^{-1}hY$. A straightforward computation shows that f is well defined and it is a bijection. Also, the preimage under f of an element XkY of $I(x, y)$ is the set of equivalence classes of pairs of cosets (gX, hY) such that $gA_x \cap hA_y \neq \varnothing$ and $g^{-1}h = k$. Moreover,

$$[\langle x\rangle, \langle y\rangle] = \sum_{(aX,bY)\in D(x,y)} \iota(x^a, y^b)\langle x^a y^b\rangle. \tag{4}$$

4 Triple brackets and the Jacobi identity

The Jacobi identity for the extended bracket can probably be proved by arguments analogous to those used by Goldman in his proof that the bracket of curves on surfaces satisfies it.

In this section we present a geometric proof of the Jacobi identity, that does not use transversality.

Let x be a hyperbolic isometry. The next result is stated using Notation 3.1.

Lemma 4.1 *The following equation holds (see Figure 5):*

$$[[\langle x\rangle,\langle y\rangle],\langle z\rangle] = \sum_{(XgY,XhZ)\in T} \iota(x,y^g)\iota(x,z^h)\langle xy^g z^h\rangle + \sum_{(XgY,YhZ)\in U} \iota(x,y^g)\iota(y^g,z^h)\langle xy^g z^h\rangle,$$

where

$$T=\big\{(XgY,XhZ) : \text{for some } P\in A_x,\ A_x\cap gA_y=\{P\}, S(x,P,\tau_x)\cap hA_z\neq\varnothing,\ hA_z\cap(S(y^g,P,-\tau_y/2)\cup S(y^{xg},xP,\tau_y/2)=\varnothing\big\},$$

$$U=\big\{(XgY,YhZ) : \text{for some } P\in A_x,\ A_x\cap gA_y=\{P\},\ (S(y^g,P,-\tau_y/2)\cup S(y^{xg},xP,\tau_y/2))\cap hA_z\neq\varnothing,\ S(x,P,\tau_x)\cap hA_z=\varnothing\big\}.$$

Proof Let $g\in G$ such that $A_x\cap gA_y\neq\varnothing$. We can retrace the steps of the construction described in Remark 3.4 to find A_{xy^g} (Figure 5). Next, we compute $[\langle xy^g\rangle,\langle z\rangle]$. Denote by P the intersection point between A_x and gA_y, by S the intersection point of A_x with A_{xy^g} and by R the intersection point of gA_y and A_{xy^g}. Finally, denote by Z the cyclic group generated by z. By Remark 3.2,

$$[\langle xy^g\rangle,\langle z\rangle] = \sum_{\substack{hZ\in G/Z,\\ S(xy^g,R,\tau_{xy})\cap hA_z\neq\varnothing}} \iota(xy^g,z^h)\langle xy^g z^h\rangle.$$

Let $hZ\in G/Z$. Observe that the inequality $S(xy^g,R,\tau_{xy})\cap hA_z\neq\varnothing$ holds if and only if hA_z crosses either the triangle with vertices R,P,S or the triangle with vertices S,xP,xy^gR (Figure 5). Thus, hA_z intersects $S(xy^g,R,\tau_{xy})$ if and only if exactly one of the following holds:

(1) $S(x,P,\tau_x)\cap hA_z\neq\varnothing$ and $(S(y^g,P,-\tau_y/2)\cup S(y^{xg},xP,\tau_y/2))\cap hA_z=\varnothing$, or

(2) $S(x,P,\tau_x)\cap hA_z=\varnothing$ and $(S(y^g,P,-\tau_y/2)\cup S(y^{xg},xP,\tau_y/2)\cap hA_z\neq\varnothing$.

The first pair of conditions corresponds to a term in the first sum, and the second pair of conditions corresponds to terms in the second sum.

This concludes the proof. □

A corollary is the Jacobi identity.

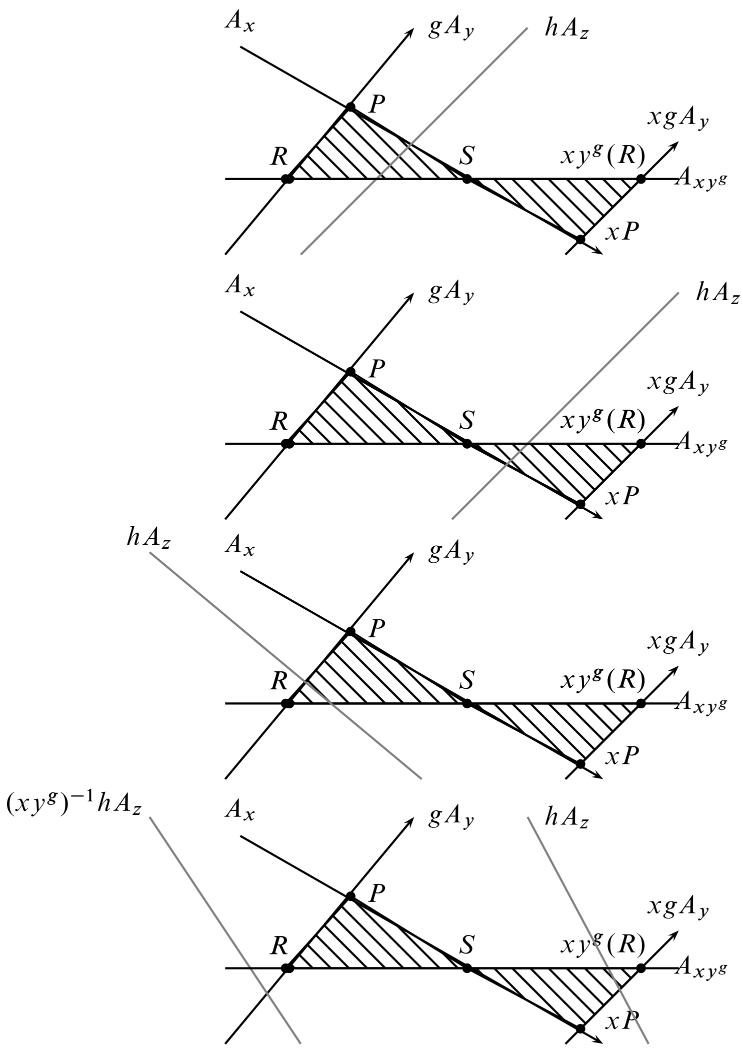


Figure 5: Jacobi identity

Theorem 4.2 *For $x, y, z \in G$,*

$$\big[[\langle x\rangle, \langle y\rangle], \langle z\rangle\big] + \big[[\langle y\rangle, \langle z\rangle], \langle x\rangle\big] + \big[[\langle z\rangle, \langle x\rangle], \langle y\rangle\big] = 0.$$

Therefore, $[\cdot\,,\cdot]\colon \mathbb{Z}[\mathcal{C}] \otimes \mathbb{Z}[\mathcal{C}] \to \mathbb{Z}[\mathcal{C}]$ is a Lie bracket.

Proof The three terms of the Jacobi relation after applying Lemma 4.1 decompose into six groups of terms. Among these, the pairs corresponding to the triangles of Figure 5 cancel. □

5 Examples

Consider the modular group $\mathrm{PSL}(2,\mathbb{Z})$, that is, the group consisting of all transformations $z \to (az+b)/(cz+d)$, where $a,b,c,d \in \mathbb{Z}$ and $ad-bc=1$. This group is generated by $T(z)=z+1$ and $S(z)=-1/z$, with relations $S^2=1$ and $(ST)^3=1$. The modular group is a finitely generated, discrete subgroup of orientation-preserving isometries of the hyperbolic plane. Therefore, the bracket can be defined on the free module generated by conjugacy classes.

Orient the hyperbolic plane clockwise.

By computing the traces, one can see that the elements $x=TSTT$ and $y=TTTSTTT$ of $\mathrm{PSL}(2,\mathbb{Z})$ are hyperbolic and not conjugate.

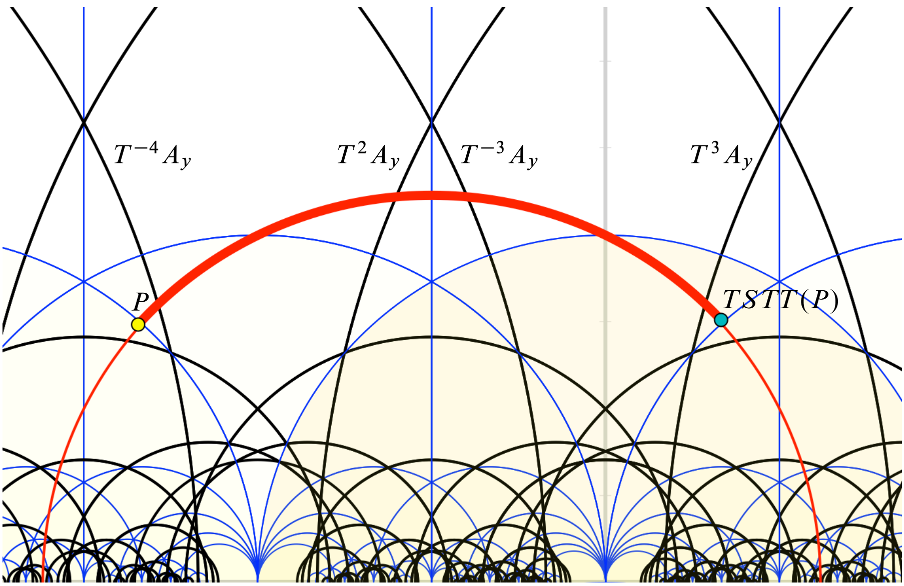


Figure 6: Translates of A_y (in black), and a fundamental domain of A_x (in thick red) where $x=TSTT$ and $y=TTTSTTT$

As shown in Figure 6, there are exactly four translates of y by $\mathrm{PSL}(2,\mathbb{Z})$ that intersect the segment of A_x from the point P to $TTST(P)$.

In this example, $I(x,y)=\{XT^{-4}Y, XT^{-3}Y, XT^2Y, XT^3Y\}$. The term corresponding to the double coset $XT^{-4}Y$ has positive sign and is the conjugacy class of ST^6 because $xT^{-4}yT^4 = TSTST^7 = ST^6$. The term corresponding to XT^3Y has negative sign and is the conjugacy class of $xT^3yT^{-3} = TST^8S$. This element is conjugate to $STST^8 = T^{-1}ST^{-1}T^8$. Thus the term corresponding to the double coset XT^3Y is $-\langle ST^6\rangle$ Also, the terms associated to $XT^{-3}Y$ and XT^2Y are $+\langle STTST^7\rangle$ and $-\langle STTST^7\rangle$. Thus $[\langle x\rangle,\langle y\rangle]=0$.

In order to study the brackets of $\langle x^p\rangle$ and $\langle y^q\rangle$ when p and q are larger than one, one can use the criteria given in [14] for conjugacy in $\mathrm{SL}(2,\mathbb{Z})$ (and therefore in $\mathrm{PSL}(2,\mathbb{Z})$). Doing so, one can check that $[\langle x\rangle,\langle y^3\rangle]\neq 0$. Moreover, the number of terms of the bracket $[\langle x\rangle,\langle y^3\rangle]$ (counted with multiplicity) equals twelve, which is three times the intersection number of $\langle x\rangle$ and $\langle y\rangle$.

In the same way one can see that $[\langle x\rangle,\langle x^2\rangle]=0$ and $[\langle x\rangle,\langle x^3\rangle]$ has 24 terms, which is six times the self-intersection number of $\langle x\rangle$.

The above calculations are computer-assisted: one looks at Figure 6 (done with Cinderella) to identify the terms, then uses Mathematica to calculate the terms, and study cancellation.

6 Quantitative separation of geodesics

From now on, we assume that the discrete subgroup G of $\mathrm{Isom}(\mathbb{H})$ is finitely generated.

Definition 6.1 Fix $\delta>0$, two geodesics Γ and Γ' and two (not necessarily distinct) points P and Q in Γ and Γ' respectively. We say that Γ and Γ' are *δ–close at P and Q* if $d(P,Q)<\delta$ and, if Υ denotes a geodesic passing through P and Q, then the absolute value of the difference between the corresponding angles between Υ and A_x and between Υ and A_y (in the positive direction of both axes) is less than δ. If there exist points P and Q such that two geodesics Γ and Γ' are δ–close at P and Q, then we say that *Γ and Γ' are δ–close*.

The next lemma is well known to experts but we include a proof here because we were unable to find one in the literature.

Lemma 6.2 *For each $L>0$ there exists a $\delta>0$ such that if x and y are two hyperbolic transformations in G such that $\tau_x\le L$ and $\tau_y\le L$ and A_x and A_y are δ–close, then $A_x=A_y$.*

Proof Denote by λ the hyperbolic convex hull of the limit set of G. (Recall that the limit set of G is the set of accumulation points of any G–orbit in $\mathbb{H}$.) Since G is finitely generated, by [11, Lemma 1.3.1 and Theorem 1.3.2], there exists a subset λ^* of λ, invariant under G, such that the quotient of λ^* by G is compact and the axis of every hyperbolic transformation in G intersects λ^*. Thus, there exists a compact, convex subset C of $\mathbb{H}$ such that $\lambda^*\subset G\cdot C$.

Fix a positive number L and denote by C' the closure of the $(L+1)$–neighborhood of C.

Claim 1 Given $\varepsilon > 0$ there exists a $\delta > 0$ such that if x and y are hyperbolic transformations whose axes are δ–close and whose transformation lengths are bounded above by L, then $d(R, [x, y]R) < \varepsilon$ for all $R \in C'$.

We argue by contradiction. Suppose that there exist $\varepsilon > 0$ and two sequences $\{x_n\}$ and $\{y_n\}$ of hyperbolic transformations with translation length bounded above by L and such that for each n, x_n and y_n are $1/n$–close, $A_{x_n} \neq A_{y_n}$ and there exists a point $R_n \in C'$ that satisfies $d(R_n, [x_n, y_n]R_n) > \varepsilon$.

Claim 2 For each n, we can assume that the points P_n and Q_n in A_{x_n} and A_{y_n} realizing Definition 6.1 are in C'.

Indeed, denote by P'_n and Q'_n the points in A_{x_n} and A_{y_n} realizing Definition 6.1.

The axis A_{x_n} projects to a closed geodesic a_n in $\mathbb{H}/G$. Since the translation length of x_n is bounded above by L, so is the length of a_n. On the other hand, A_{x_n} intersects $G \cdot C$. Hence, the projection of P'_n to $\mathbb{H}/G$ is at distance at most L from the projection of $G \cdot C$. Thus there is an element $g \in G$ such that gP'_n is at distance at most L from C. Since Q'_n is close to P'_n, we have that Q'_n is also in C'. The proof of Claim 2 is completed by replacing the sequences $\{x_n\}$ and $\{y_n\}$ by the sequences $\{gx_ng^{-1}\}$ and $\{gy_ng^{-1}\}$.

Claim 3 The sequences $\{x_n\}$ and $\{y_n\}$ have subsequences converging to hyperbolic transformations x and y respectively.

Consider the sequences $\{T_n\}$ and $\{S_n\}$ of endpoints of $\{A_{x_n}\}$ in the circle at infinity in the negative and positive directions respectively. Since the circle is compact, by taking subsequences, we can assume that $\{T_n\}$ and $\{S_n\}$ converge to T and S respectively. Since each A_{x_n} intersects the compact set C', we get $T \neq S$. Analogously, the sequence $\{\tau_{x_n}\}$ of translation lengths is bounded above by L. Therefore, it has a convergent subsequence. Thus, Claim 3 follows.

Since A_{x_n} and A_{y_n} are $1/n$–close, we get $A_x = A_y$. Hence, $[x, y]P = P$ for all $P \in \mathbb{H}$. On the other hand, by taking a convergent subsequence of $\{R_n\}$, we see that $d(R, [x, y]R) \geq \varepsilon$ for some $R \in C'$. This contradiction completes the proof of Claim 1.

To finish the proof of the lemma, observe that since G is discrete, there exists an open subset U of isometries of $\mathbb{H}$ such that the identity is the only element of G in U. Let

$$V_\eta = \{g \in \mathrm{PSL}(2, \mathbb{R}) \mid d(R, gR) < \eta \text{ for all } R \text{ in } C'\}.$$

There exists an $\varepsilon > 0$ such that $V_\varepsilon \subset U$. On the other hand, by Claim 1, there exists a $\delta > 0$ such that if the axes of x and y are δ–close, then $[x, y] \in V_\varepsilon$. Thus, the bracket $[x, y]$ equals the identity, which implies $A_x = A_y$. □

Corollary 6.3 *For each $L > 0$ and each $C > 0$ there exists a constant $M > 0$ such that for every pair of hyperbolic elements x and y in G with different axes and such that $\tau_x < L$ and $\tau_y < L$, the set $A_x \cap N_C(A_y)$ is a (possibly empty) geodesic segment of length at most M.*

Proof Let δ be as in Lemma 6.2 for L and G and let N be the length of the (possibly empty) segment $A_x \cap N_C(A_y)$.

If A_x and A_y intersect at an angle θ, then by Lemma 6.2, $\sin(\theta) \geq \sin(\delta)$. By the rule of sines, $\sinh(N/2) \leq \sinh(C)/\sin(\delta)$ (see Figure 7, left). Then N is bounded above by a constant depending on C and δ.

If A_x and A_y are parallel, by Lemma 6.2 they are at distance at least δ. Since the distance between A_x and A_y is realized, there is a quadrilateral as in Figure 7, right, with all angles except θ being right angles, $A \geq \delta$ and $B \leq C$.

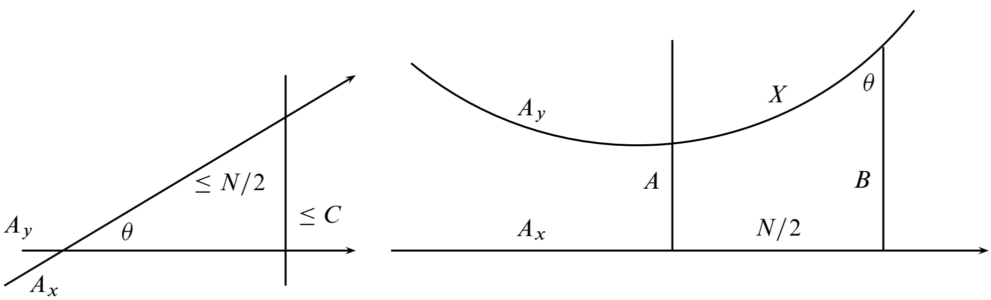


Figure 7: Proof of Corollary 6.3

By [2, Theorem 7.17.1(i)], $\sinh(N/2) = \cos(\theta)/\sinh(A) \leq 1/\sinh(\delta)$ (see Figure 7, right). This implies that $\cosh(N/2)$ is bounded above by a bound depending on δ. An elementary computation gives the desired result. □

7 The non-cancellation lemma

Let K be a real positive number. A piecewise-smooth embedding γ of $\mathbb{R}$ in the hyperbolic plane is a *K–quasigeodesic* if for any pair of points P and Q in γ, the length of the path in γ joining P and Q is at most $K \cdot d(P, Q)$.

Fix a pair of hyperbolic elements x and y in G whose axes intersect at a point P. We will describe the construction of a piecewise-smooth embedding γ of $\mathbb{R}$ (depending on x and y) and show it is a quasigeodesic.

Let $\alpha\colon [0,1] \to \mathbb{H}$ be the curve from $\alpha(0) = y^{-1}P$ to $\alpha(1) = xP$, whose image is given by the concatenation of the geodesic segment of A_y from $y^{-1}P$ to P with the geodesic segment of A_x from P to xP. Since $xy(\alpha(0)) = \alpha(1)$, α can be extended by periodicity to a map $\gamma(x,y)\colon \mathbb{R} \to \mathbb{H}$ such that $\gamma(x,y)(t) = \alpha(t)$ for $t \in [0,1]$ and $\gamma(x,y)(t+1) = xy\gamma(x,y)(t)$ for all t.

The map $\gamma(x,y)$ is a piecewise geodesic curve consisting of segments of length τ_x (included in the axes of conjugates of A_x by some power of xy) alternating with segments of length τ_y (included in the axes of conjugates of A_y by some power of xy).

We remark that we will be using more than just that $\gamma(x,y)$ is a quasigeodesic, but also its geometric nature. Indeed purely abstract results about quasigeodesics suffice to prove a weaker version of our result, where we need to assume that both p and q are large.

Lemma 7.1 *For each $L > 0$ there exists a constant $K > 0$ depending on G such that if x and y are hyperbolic transformations in G whose axes are distinct and intersect, and whose translation lengths are bounded above by L, then for each pair of positive integers p and q, the curve $\gamma(x^p, y^q)$ is a K–quasigeodesic. Moreover, the oriented angles between any pair of consecutive maximal segments of $\gamma(x^p, y^q)$ are congruent.*

Proof Fix p and q and repeat the construction of Remark 3.4 for the hyperbolic isometries x^p and y^q. The transformation x^p maps the angle determined by $y^{-q}P$, P, $x^p(P)$ to the angle $x^p y^{-q}P$, x^pP, $x^{2p}(P)$ (Figure 8). Thus, these two angles are congruent. The angle $x^p y^{-q}P$, x^pP, $x^{2p}(P)$ is congruent to the angle P, x^pP, $x^p y^q(P)$ because they are opposite at the intersection of A_x and $x^p y^q(A_y) = A_{x^p y x^{-p}}$. This implies that the angles determined by $y^{-q}P$, P, $x^p(P)$ and by P, $x^p(P)$, $y^q x^p(P)$ are congruent. Therefore the angles formed by the consecutive maximal segments of $\gamma(x^p, y^q)$ (labeled with θ_1 in Figure 8) are all congruent.

Denote by T the triangle with vertices $y^{-q}P$, P, $x^p(P)$ and by T' the triangle with vertices P, $x^p(P)$ and $x^p y^q(P)$, see Figure 8. Since T and T' have an angle and the two adjacent sides to the angle congruent, they are congruent.

Set $g = x^p y^q$. Then A_g is invariant under g, so A_g crosses the middle of the band $\bigcup_{k\in\mathbb{Z}} g^k(T \cup T')$.

To prove that $\gamma(x^p, y^q)$ is a quasigeodesic, observe that triangles

$$g^s(T), g^s(T'), g^{s+1}(T), g^{s+1}(T'), \dots, g(T), g(T'), \dots, g^r(T), g^r(T')$$

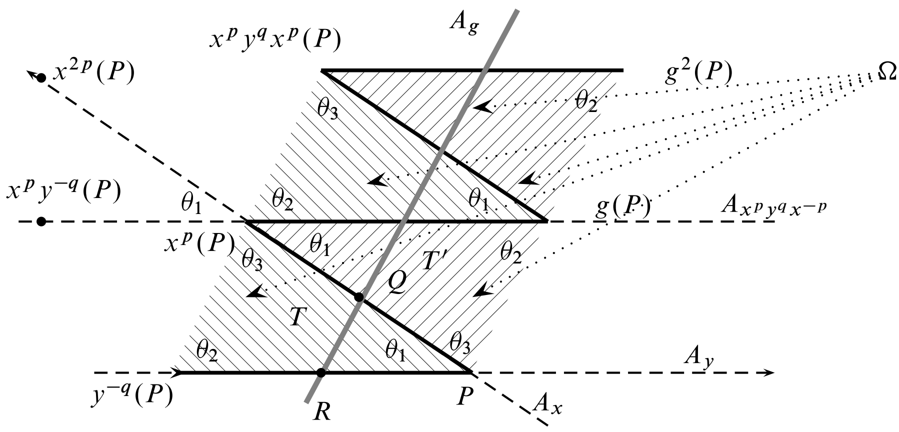


Figure 8: Quasigeodesic associated to x, y, p and q (here $g = x^p y^q$)

form a polygon Ω. On the other hand, since the angles θ_1, θ_2 and θ_3 (see Figure 8) are the interior angles of a triangle, they add up to at most π. This implies that the polygon Ω is convex. Therefore, the geodesic between two points in the curve γ is in the interior of Ω. By elementary hyperbolic geometry, there exists a positive constant K such that γ is a K–quasigeodesic. (Note that K can be taken so that it depends only on the lower bound of the angle between intersecting elements of axes of hyperbolic elements in G given by Lemma 6.2.) □

We can (and will) assume without loss of generality that $K \geq 1$.

Lemma 7.2 *Let $L > 0$ and let $K > 0$ be the constant of Lemma 7.1. Then there exists a constant $C > 0$ depending on G such that if x and y are hyperbolic transformations in G whose axes are distinct and intersect, and whose translation lengths are bounded above by L, then for each pair of positive integers p and q, the K–quasigeodesic $\gamma(x^p, y^q)$ satisfies $\gamma(x^p, y^q) \subset N_{C/2}(A_g)$ and $A_g \subset N_{C/2}(\gamma(x^p, y^q))$, where $g = x^p y^q$.*

Proof Denote by $d[p,q]$ the distance between P (the point in $A_x \cap A_y$) and A_g. Consider the region λ bounded by the axes A_x and A_y and the arc of the circle of center P and radius $d[p,q]$. The area of λ equals $2\theta_1 \sinh^2(d[p,q]/2)$. Also, λ is included in the triangle T, of area bounded above by $\pi - \theta_1$ (see Figure 9). Hence,

$$2\sinh^2(d[p,q]/2) \leq (\pi - \theta_1)/\theta_1 \leq \pi/\delta.$$

Therefore, there exists a constant $C_1 > 0$ such that $d[p,q] \leq C_1$ for all positive integers p and q. Observe (Figure 8) the distance between any point in $\gamma(x^p, y^q)$ and A_g is smaller than $d[p,q]$. This implies $\gamma(x^p, y^q) \subset N_{C_1}(A_g)$.

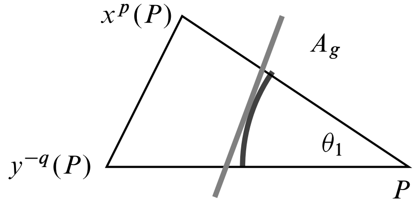


Figure 9: The region λ

Denote by R the intersection point of A_g with A_x and by Q the intersection point of A_g with A_y (see Figure 8).

Consider the triangle with vertices P, Q and R. Triangles in the hyperbolic plane $\mathbb{H}^2$ are $\ln(1+\sqrt{2})$–thin [5, Fact 4, page 90]. In particular, the side of the triangle included in A_g is at distance at most $\ln(1+\sqrt{2})$ from the union of the other two sides.

By taking $C = 2\max\{\ln(1+\sqrt{2}), C_1\}$ the desired result follows. □

Let x and y be two hyperbolic transformations in G whose axes intersect at a point P and whose length is less than L. Let p and q be positive integers. Denote by I the segment of A_x from P to $x^p(P)$.

For a subsegment J of I with endpoints S and R, we consider a *rectangular* neighborhood $U = U(J, C)$ defined as follows. Let s (resp. r) be the open half-plane bounded by the line perpendicular to A_x through S (resp. R), containing the point $x^p P$ (resp. P). Set $U = s \cap r \cap N_C(I)$.

Note that the boundary of U consists of *vertical* segments contained in the boundaries of s and r and *horizontal* segments contained in the boundary of $N_C(I)$. By elementary hyperbolic geometry, the distance between the vertical segments is the length of the geodesic J.

Lemma 7.3 *Let $L > 0$ and let x and y be two hyperbolic transformations in G whose axes intersect at a point P and whose length is less than L. Let p and q be positive integers such that $p \cdot \tau_x \geq 6KC$, where K and C are as in Lemmas 7.1 and 7.2. Denote by I the segment of A_x from P to $x^p(P)$.*

Let S and R be the points in A_x at distance $3KC$ from P and $x^p P$, and let J be the segment from S to R. Let $U = U(J, C)$ be the associated rectangular neighborhood.

Then $\operatorname{closure}(U) \cap N_C(\lambda) = \varnothing$ for all maximal geodesic segments of λ of $\gamma(x^p, y^q)$ distinct from I.

Proof Let $Q \in \lambda$, where λ is a maximal segment of $\gamma(x^p, y^q)$ different from I, and let $T \in I$ be a point. Then by construction the length of a path in $\gamma(x^p, y^q)$ from Q to T is at least $3KC$. As $\gamma(x^p, y^q)$ is a K–quasigeodesic, it follows that $d(Q, T) > 3C$. As Q was an arbitrary point of λ and $U \subset N_C(\lambda)$ it follows that closure$(U) \cap N_C(L) = \varnothing$. □

Observe that U contains the open subsegment J of length at least $p \cdot \tau_x - 6KC$.

The following lemma is key to the paper.

Lemma 7.4 *For each $L > 0$ there exists a positive integer p_0 such that for each pair of integers p and q satisfying $p \geq p_0$, and for each pair of hyperbolic transformations x, y and x_1, y_1 whose axes are distinct and intersect, and whose translation length is bounded above by L, if $x^p y^q = x_1^p y_1^q$, x_1 is conjugate to x, and y_1 is conjugate to y, then $\gamma(x^p, y^q) = \gamma(x_1^p, y_1^q)$.*

Proof We start by informally describing the two parts of the proof. First, in the situation above, the two corresponding quasigeodesics are such that one is in a C–neighborhood of the other. In particular, segments of one quasigeodesic are in C–neighborhoods of segments of the other quasigeodesic. By making the integer p large enough, we obtain a "long" geodesic segment in a C–neighborhood of other geodesic segment. This implies that these two segments intersect in an interval.

Second, we use the fact that the quasigeodesics are constructed by translating two consecutive maximal segments by powers of g, to show if the two intersecting segments are distinct, an impossible figure is obtained.

Here are the details of the proof. For each finitely generated, discrete subgroup G of Isom$(\mathbb{H})$, there exists a positive constant τ_0 such that for each hyperbolic transformation $x \in G$, one has $\tau_x \geq \tau_0$ (see, for instance, [11, Theorem 1.4.2])

Let C and K be as in Lemmas 7.1 and 7.2. Let M be the constant of Corollary 6.3. We will show that $p_0 = K(3M + 6C)/\tau_0$ gives the desired conclusion.

Since $x^p y^q = x_1^p y_1^q$, we have $A_{x^p y^q} = A_{x_1^p y_1^q}$. By Lemma 7.2,

$$\gamma(x_1^p, y_1^q) \subset N_{C/2}(A_g) \subset N_C(\gamma(x^p, y^q)).$$

Let U and J respectively be the neighborhood and the segment given by Lemma 7.3, so $J \subset U$, $J \subset I \subset \gamma(x^p, y^q)$ and the length of J is at least $p\tau_x - 6KC$.

Observe that $\gamma(x_1^p, y_1^q)$ must intersect U, for otherwise $\gamma(x_1^p, y_1^q)$ is included in $N_C(\gamma(x^p, y^q) \setminus J)$, which has two components. Furthermore, $\gamma(x_1^p, y_1^q)$ must intersect both components, contradicting the fact that $\gamma(x_1^p, y_1^q)$ is connected. By

Lemma 7.3, $N_C(L) \cap \mathrm{closure}(U) = \varnothing$ for all maximal segments λ of $\gamma(x^p, y^q)$ distinct from I. Hence, $\gamma(x_1^p, y_1^q)$ does not intersect the horizontal boundary components of U, as otherwise we obtain points in $\gamma(x_1^p, y_1^q)$ whose distance from $\gamma(x^p, y^q)$ is greater than C.

By hypothesis, the length of J is at least $p\tau_x - 6KC$ so it is at least $3KM$.

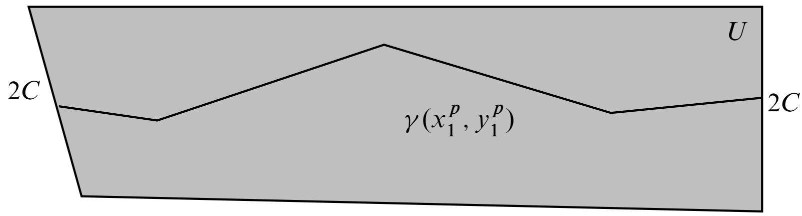


Figure 10: The intersection of neighborhood U of J with $\gamma(x_1^p, y_1^q)$

Thus, the components of the set $U \cap \gamma(x_1^p, y_1^q)$ are piecewise linear curves starting and ending at the vertical sides of U (see Figure 10). Let β be one of these components. We claim that β contains a segment l of length greater than M. Indeed, if β contains three or more vertices of $\gamma(x_1^p, y_1^q)$ then one segment of β is a maximal segment of $\gamma(x_1^p, y_1^q)$ included in a translate of x_1^p. Therefore, it must have length at least $p_0\tau_x$. Otherwise, β consists of at most three segments. Denote by m the length of the longest of these segments. As the distance between the vertical boundary components of U is the length of J,

$$3KM \le p\tau_x - 6KC \le 3m.$$

Since $K > 1$, $m > M$. Thus the claim is proved.

The segment l of β of length at least M is included in some segment I' of $\gamma(x_1^p, y_1^q)$. Thus $I' \cap N_C(J)$ contains a segment longer than M. By Corollary 6.3, I' intersects I in a subsegment. This concludes the first part of the proof. We will show that the assumption $I \neq I'$ leads to a contradiction.

If $I \neq I'$, by interchanging the roles of I and I' if necessary, we can assume that there is a vertex v of I which is not in I'. Let v' be the vertex of I' closest to v. Denote by λ (resp. λ') the maximal segment of $\gamma(x^p, y^q)$ (resp. $\gamma(x_1^p, y_1^q)$) such that I and λ (resp. I' and λ') are adjacent and intersect in v (resp. v').

Recall that $\gamma(x^p, y^q)$ (resp. $\gamma(x_1^p, y_1^q)$) is constructed by taking two consecutive maximal segments and translating them by powers of g. To simplify the notation, we write $g = x^p y^q$. The segment adjacent to λ (resp. λ') different from I (resp. I')

is $g(I)$ (resp. $g(I')$). Denote by u (resp. u') the other vertex of I (resp. I'). Note that v and $g(u)$ (resp. v' and $g(u')$) are the vertices of λ (resp. λ').

Suppose first that u is in I'. By Lemma 7.1, the angles u, v, $g(u)$ and v, $g(u)$, $g(v)$ are congruent. Hence there is a convex quadrilateral with vertices v, v', $g(u)$, $g(u')$, see Figure 11. By Lemma 7.1, the sum of the interior angles of this quadrilateral is 2π, a contradiction in hyperbolic geometry. This implies that u is not in I'.

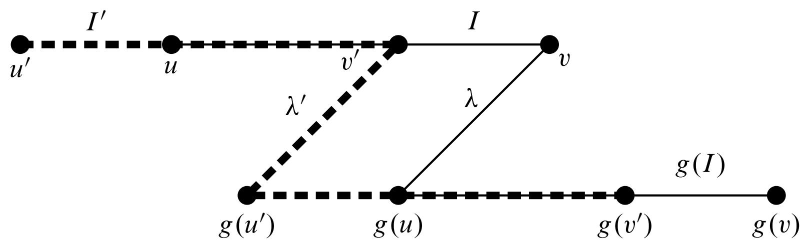


Figure 11: Length of I equals the length of I'

Denote by l the geodesic through v and $g(u)$. By Lemma 7.1, the angles u, v, $g(u)$ and v, $g(u)$, $g(v)$ are congruent. This implies that u and $g(v)$ are in different sides of l. On the other hand, u and v' (resp. $g(v)$ and $g(u')$) are on the same side of l. Then v' and $g(u')$ are on different sides of l. Hence λ intersects λ' and the quasigeodesics are arranged as in Figure 12.

In particular, the segments λ and λ' intersect at a point z. The triangles with vertices z, v', v and z, $g(u)$, $g(u')$ have congruent corresponding angles. Hence, these two triangles are congruent. Thus, z is the middle point of λ, and also of λ'. Since the segments with vertices u, u' and $g(u)$, $g(u')$ are congruent, the segments with vertices u, u' and v', v are congruent.

Denote by w the middle point of I. Observe that w is also the middle point of I' (as segments with vertices u, u' and v, v' are congruent). As x_1 is conjugate to x and y_1 is conjugate to y, the length of the arc of $\gamma(x_1^p, y_1^q)$ from w to z equals $(p\tau_x + q\tau_y)/2$. Also, the length of the arc of $\gamma(x^p, y^q)$ from w to z equals $(p\tau_x + q\tau_y)/2$. By the triangle inequality, this is impossible. Thus we conclude that $v = v'$, and hence also $u = u'$.

Thus, we see that $I = I'$ and $\lambda = \lambda'$. It follows that the quasigeodesics $\gamma(x^p, y^q)$ and $\gamma(x_1^p, y_1^q)$ coincide as they are the unions of translates under g of $I \cup \lambda$ and $I' \cup \lambda'$, respectively. $\square$

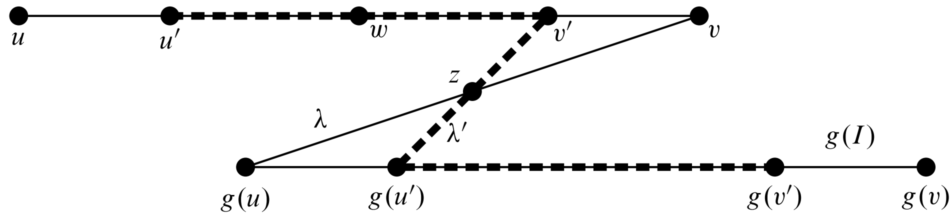


Figure 12: Length of I is larger than length of I'

Theorem 7.5 *For each $L > 0$ there exists a positive integer p_0 such that for each pair $p \geq p_0$ and q of positive integers, if x and y (resp. x_1 and y_1) are hyperbolic transformations whose axes are distinct and intersect, x is conjugate to x_1, y is conjugate to y_1, the translation lengths of x, x_1, y, y_1 are bounded above by L, $p\tau_x \neq q\tau_y$, and $x^p y^q = x_1^p y_1^q$, then there exists an $h \in G$ such that $x_1 = x^h$ and $y_1 = y^h$.*

Proof Since $x^p y^q = x_1^p y_1^q$, we have $A_{x^p y^q} = A_{x_1^p y_1^q}$. Moreover, both axes are oriented in the same direction. If p_0 is the positive integer given by Lemma 7.4, then $\gamma(x^p, y^q) = \gamma(x_1^p, y_1^q)$. Hence, by the definition of $\gamma(x^p, y^q)$, if $g = x^p y^q$ there exists an $n \in \mathbb{Z}$ such that one of the following holds:

(1) $x_1^p = (x^p)^{g^n}$ and $y_1^q = (y^q)^{g^n}$.

(2) $x_1^p = (y^q)^{g^{n+1}}$ and $y_1^q = (x^p)^{g^n}$.

Since $p\tau_x \neq q\tau_y$, (2) is impossible. Thus the result follows by taking $h = g^n$. □

8 Proof of the main theorem

An element z in $\mathbb{Z}[\mathcal{C}]$ can be uniquely represented as a sum $\sum_{i=1}^k n_i \langle x_i \rangle$ so that the conjugacy classes $\langle x_i \rangle$ are all distinct and the integers n_i are non-zero. We define the *Manhattan norm of z* by

$$M\left(\sum_{i=1}^k n_i \langle x_i \rangle\right) = \sum_{i=1}^k |n_i|.$$

We are now in a position to prove our main theorem. Denote by X_p and Y_q the cyclic groups generated by x^p and y^q respectively. Note that by definition

$$[\langle x^p \rangle, \langle y^q \rangle] = \sum_{X_p b Y_q \in I(x^p, y^q)} \iota(x^p, (y^q)^b) \langle x^p (y^q)^b \rangle.$$

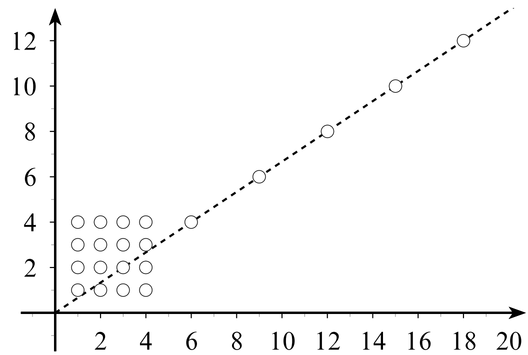


Figure 13: Values of p and q in the main theorem, $p_0 = 5$, $p/q \neq \frac{3}{2}$, p in x–axis

Our first step is to collate terms in this expression. There is a natural quotient map from $X_p \backslash G / Y_q$ to $X \backslash G / Y$, mapping $X_p \backslash g / Y_q$ to $X \backslash g / Y$. Observe that $\iota(x^p, (y^q)^b) = \iota(x, y^b)$. Further observe that if $X_p \backslash g / Y_q$ and $X_p \backslash g' / Y_q$ map to the same element in XgY, then $\langle x^p (y^q)^g \rangle = \langle x^p (y^q)^{g'} \rangle = \langle x^p (y^{g'})^q \rangle$. The lemma below follows by grouping terms corresponding to their images in $I(x, y)$.

Lemma 8.1 *We have*

$$[\langle x^p \rangle, \langle y^q \rangle] = pq \Bigg(\sum_{XbY \in I(x,y)} \iota(x, y^b) \langle x^p (y^b)^q \rangle \Bigg).$$

We are now ready to prove our main result.

Main Theorem *Let G be a finitely generated, discrete group of* $\mathrm{Isom}(\mathbb{H})$ *and let $L > 0$. There exists a p_0 such that if p and q are integers at least one of which is larger than p_0, then the following holds:*

(1) *If x and y are hyperbolic transformations in G such that neither is conjugate to a power of the other, with translation lengths bounded above by L and such that $p\tau(x) \neq q\tau(y)$, then $M[x^p, y^q]/(p \cdot q)$ equals the geometric intersection number of x and y.*

(2) *If $p \neq q$, and x is a hyperbolic transformation in G, not a proper power, and has translation length bounded above by L, then $M[x^p, x^q]/(2 \cdot p \cdot q)$ equals the geometric self-intersection number of x.*

Proof Interchanging x and y if necessary, we can assume that $p \geq p_0$.

Suppose that $\langle x^p (y^b)^q \rangle = \langle x^p (y^{b'})^q \rangle$. Then for some $h \in G$,

$$x^p (y^b)^q = (x^p (y^{b'})^q)^h = (x^p)^h (y^q)^{hb'} = (x^h)^p (y^{hb'})^q.$$

By Theorem 7.5, there is an element g that conjugates x to x^g and y^b to $y^{gb'}$. In particular, the signs $\iota(x, y^b)$ and $\iota(x^g, y^{gb'})$ coincide, so there is no cancellation. This concludes the proof. □

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Center of the Goldman Lie algebra

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We show that the center of the Goldman Lie algebra associated to a closed orientable surface is generated by the class of the trivial loop. For an orientable nonclosed surface of finite type, the center is generated by closed curves which are either homotopically trivial or homotopic to boundary components or punctures.

57M50; 57M07, 57M05

1 Introduction

Let F be an oriented surface. Given two free homotopy classes of oriented closed curves α and β, consider two oriented closed curves x and y representing α and β, respectively. Performing a small homotopy if necessary, we can assume that x and y intersect transversally in double points. Goldman [10] defined the bracket of α and β as the sum,

$$[\alpha, \beta] = \sum_{p \in x \cap y} \iota(p) \langle x *_p y \rangle,$$

where $x \cap y$ denotes the set of all intersection points between x and y, $\iota(p)$ denotes the sign of the intersection between x and y at p, $(x *_p y)$ denotes the loop product of x and y at p, and $\langle z \rangle$ denotes the free homotopy class of a curve z.

Let $\mathcal{C}$ be the set of all free homotopy classes of oriented closed curves in F. This bracket is extended by linearity to $\mathbb{Z}(\mathcal{C})$, the free module generated by $\mathcal{C}$. Goldman [10] showed that this bracket is well defined, skew-symmetric and satisfies the Jacobi identity. Therefore, this is a Lie bracket, and it gives a Lie algebra structure on $\mathbb{Z}(\mathcal{C})$, which we denote by $\mathcal{L}(F)$. Recall that the *center of a Lie algebra* $\mathcal{L}$ is the set of all elements x in $\mathcal{L}$ such that $[x, y] = 0$ for all y in $\mathcal{L}$. The main object of this paper is to study the center of $\mathcal{L}(F)$.

The structure of the Goldman Lie algebra for surfaces of nonnegative Euler characteristic is either trivial or well understood; see Chas [3, Lemma 7.6] for the torus case.

Chas and Sullivan conjectured that, for a closed surface F, the center of the Goldman algebra is generated by the trivial loop. It is natural to conjecture (see Chas [4,

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Open Problem 1], [3, Problem 13.1] and Kawazumi and Kuno [12, Section 8.3]) that, for a surface F with nonempty boundary, the center of $\mathcal{L}(F)$ is generated by the free homotopy classes of oriented closed curves which are either homotopic to a point, homotopic to a boundary, or homotopic to a puncture. In this paper, we prove these conjectures.

Main Theorem *The center of the Goldman Lie algebra of any closed orientable surface F is one-dimensional, and is generated by the class of the trivial loop. If F is an orientable surface of finite type with boundary, then the center of $\mathcal{L}(F)$ is generated by the set of all free homotopy classes of oriented closed curves which are homotopic to either a point, a boundary component, or a puncture.*

Remark 1 The closed case was done by Etingof in [8] using representation theory, but that proof did not address the case of surfaces with boundary. Our proof of both cases uses different ideas from hyperbolic geometry.

Goldman discovered this bracket while studying the Weil–Petersson symplectic form on Teichmüller spaces. Using Wolpert's [15] result on length and twist flow, he showed that if the Goldman bracket between two closed curves is zero and one of them has a simple representative, then their geometric intersection number is zero. The combinatorial structure of $\mathcal{L}(F)$ has also been studied. Using combinatorial topology and group theory, Chas [3] proved a stronger version of Goldman's result, namely if one of the curves has a simple representative, then the number of terms in the Goldman bracket is the same as their geometric intersection number. Chas and Krongold [6] proved that, for a compact surface with nonempty boundary, $[x, x^3]$ determines the self-intersection number of x. Using hyperbolic geometry, Chas and Gadgil [5] proved that there exists a positive integer m_0 such that, for all $m \geq m_0$, the geometric intersection number between x and y is the number of terms in $[x^m, y]$ divided by m. There is a Lie cobracket defined by Turaev [14] on $\mathbb{Z}(\mathcal{C})$ which is the dual object of the Goldman bracket. This structure has been studied by Chas and Krongold [7; 2].

Idea of the proof Our proof is based on hyperbolic geometry. Given an oriented surface of negative Euler characteristic, we fix a hyperbolic metric on it with geodesic boundary. There are two key ideas behind our proof.

The first idea is from [5]. Given two closed oriented curves x and y intersecting transversally, we construct lifts of $(x *_p y)$ in the hyperbolic plane $\mathbb{H}$ for each intersection point p. By [5, Lemma 7.1], the lifts are quasigeodesics. Hence they are homotopic to unique geodesics. Therefore, if two terms $(x *_p y)$ and $(x *_q y)$ cancel each other, then the corresponding geodesics will be the same. By [5, Main Theorem], there exists m_0 such that if we take a power $m \geq m_0$ of x, then we can ensure that if the geodesics are the same, then the quasigeodesics are also the same, and hence the terms have the same sign. Therefore, there is no cancellation between the terms of $[x^m, y]$.

The second key idea is that all lifts of a simple closed geodesic are disjoint. Now, if an element $y = \sum_{i=1}^{k} y_i$ of $\mathcal{L}(F)$ belongs to the center, then we consider a simple closed curve x which intersects at least one of the curves y_i nontrivially. Taking sufficiently large powers of x we can ensure that the same terms of $[x^m, y_i]$ have the same sign. Then, using that the lifts of x are disjoint, we show that if one term of $[x^m, y_i]$ and another term of $[x^m, y_j]$ are the same, then y_i and y_j are conjugate.

Therefore, if $[x, y]$ is zero for all closed curves x, then each y_i is disjoint from every simple closed curve, and hence each y_i is either homotopic to a point or to a boundary component or to a puncture.

Organization of the paper Throughout the paper we follow the notation and definitions from [5].

In Section 2, we recall some basic facts about hyperbolic surfaces and closed curves on hyperbolic surfaces. We also mention a well-known result about hyperbolic elements of the fundamental group of a hyperbolic surface.

In Section 3, we recall from [5] the algebraic definition of the Goldman bracket between conjugacy classes of elements. Throughout the paper, we use this as the definition of Goldman bracket.

In Section 4, we recall from [5] the description of the lifts of the terms of Goldman bracket. We also state the lemma that these lifts are quasigeodesic and, therefore, in a neighborhood of a geodesic, following [5].

In Section 5, we show that if we take a sufficiently high power of a simple closed curve, then there is no cancellation between the terms of the Goldman bracket with any other closed curve.

In Section 6, we mention a classical theorem and prove the main theorems.

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2 Closed curves on a hyperbolic surface

In this section, we recall some basic facts about hyperbolic surfaces and closed curves on hyperbolic surfaces. We use the same notation as [5]. References for the results mentioned in this section are [1; 5; 11; 13].

Let F be an orientable surface of finite type with negative Euler characteristic; ie F is a surface of genus g with b boundary components and n punctures such that $2-2g-b-n<0$. By [9, Theorem 1.2], we can endow F with a hyperbolic metric. By a hyperbolic surface we mean an orientable surface with negative Euler characteristic and with a given hyperbolic metric. Given a hyperbolic surface F, we identify the fundamental group $\pi_1(F)$ of F with a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, the group of orientation preserving isometries of the upper half plane $\mathbb{H}$. The action of $\pi_1(F)$ on $\mathbb{H}$ is properly discontinuous without fixed points, and the quotient space is isometric to F. Henceforth, by an isometry of $\mathbb{H}$ we mean an orientation preserving isometry, and by a closed curve we mean an oriented closed curve.

A homotopically nontrivial closed curve in F is called *essential* if it is not homotopic to a puncture. A closed curve is called *peripheral* if it is homotopic to a power of a simple closed curve bounding a once-punctured disc. By a *lift* of a closed curve γ to $\mathbb{H}$, we mean the image of a lift $\mathbb{R}\to\mathbb{H}$ of the map $\gamma\circ\pi$, where $\pi\colon\mathbb{R}\to S^1$ is the usual covering map.

There is a bijective correspondence between the set of all free homotopy classes of oriented closed curves in F and the set of all conjugacy classes in $\pi_1(F)$ [5, Theorem 2.3]. Given an oriented closed curve γ in F, we denote both its free homotopy class and the corresponding conjugacy class in $\pi_1(F)$ by $\langle\gamma\rangle$. Abusing notation, we sometimes denote the conjugacy class of γ by γ itself. Given $a, g\in\pi_1(F)$, we denote gag^{-1} by a^g and the translation length of a by τ_a. If a is hyperbolic, then a^g is also hyperbolic with $\tau_{a^g}=\tau_a$ and $A_{a^g}=gA_a$ for all $g\in\pi_1(F)$, where A_a denotes the axis of a.

The *geometric intersection number* between two free homotopy classes of closed curves x and y, denoted by $i(x,y)$, is defined to be the minimal number of intersection points between a representative curve in the class $\langle x\rangle$ and a representative curve in the class $\langle y\rangle$ which intersect transversally in double points.

Every free homotopy class of an essential closed curve contains a unique closed geodesic whose length is the same as the translation length of any element of the corresponding conjugacy class. By a slight abuse of notation, we denote the free homotopy classes of essential closed curves by their geodesic representatives.

The following lemma is a well-known result. See [5, Corollary 6.3] for a proof.

Lemma 2.1 *Let G be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Given two nonzero positive numbers L and C, there exists a constant $M > 0$ such that for every pair of hyperbolic elements x and y in G with $\tau_x \le L$ and $\tau_y \le L$, the set*

$$\{x \in A_x : d(x, A_y) < C\}$$

is either empty or a geodesic segment of length at most M.

3 Goldman bracket

In this section, we recall from [5] the algebraic definition of the Goldman bracket between two curves intersecting transversally (not necessarily in double points). For the equivalence of this definition with the previous one, see [5, Section 3].

Given two hyperbolic transformations x and y whose axes A_x and A_y, respectively, intersect in a point P, let $I(x, y)$ denote the segment of A_x of length τ_x starting from P in the positive direction of A_x, containing P but not containing xP.

Definition 3.1 Let $\langle x\rangle$ and $\langle y\rangle$ be two nontrivial conjugacy classes in $\pi_1(F)$. Define

$$[\langle x\rangle, \langle y\rangle] = \begin{cases} \displaystyle\sum_{gY \in J(x,y)} \iota(x, y^g)\langle x y^g\rangle & \text{if both } x \text{ and } y \text{ are hyperbolic,} \\ 0 & \text{if either } x \text{ or } y \text{ is parabolic,} \end{cases}$$

where Y is the cyclic subgroup generated by y,

$$J(x, y) = \{gY \in \pi_1(F)/Y : I(x, y) \cap gA_y \neq \varnothing\},$$

and $\iota(x, y)$ denotes the sign of intersection between the axes of x and y if they intersect and 0 otherwise.

Remark 2 This definition is independent of the type of the intersection points of the representative curves. Therefore, we can use the geodesic representatives of the corresponding conjugacy classes (which intersect transversally but not necessarily in double points). Henceforth, we use this as the definition of the Goldman bracket.

4 Terms of the Goldman bracket

In this section we recall the description of the lifts of the terms of the Goldman bracket from [5, Section 7].

Let x and y be two hyperbolic elements in $\pi_1(F)$ whose axes intersect at the point P. Denote the projections of A_x and A_y in F by x_1 and y_1 respectively. Let p be the

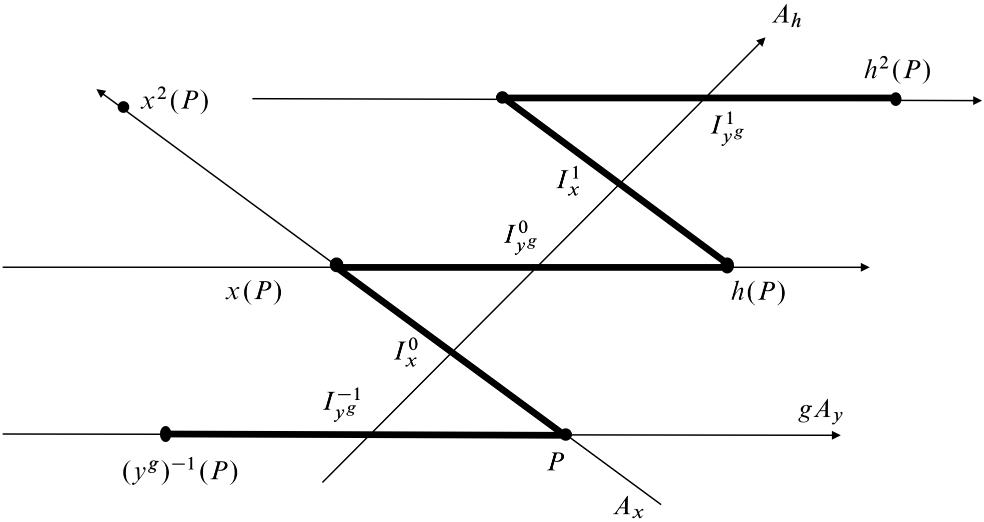

Figure 1: Lift of a term in the Goldman bracket

projection of P in F. By [5, Remark 3.2], there exists a unique $g \in \pi_1(F)$ such that $gY \in J(x,y)$ and gY corresponds to p. A lift of $(x_1 *_p y_1)$ is a bi-infinite piecewise geodesic passing through P, which we denote by $\gamma(x, y^g)$; see Figure 1.

Let $h = xy^g$. If we denote the geodesic arc from P to $x(P)$ by I^0_x and the geodesic segment from $x(P)$ to $h(P)$ by $I^0_{y^g}$, then $\gamma(x, y^g)$ consists of geodesic segments of the form $h^k(I^0_x)$ and $h^k(I^0_{y^g})$ occurring alternately.

Remark 3 Denote $h^k(I^0_x)$ by I^k_x and $h^k(I^0_{y^g})$ by $I^k_{y^g}$. From the definition, the length of I^k_x is τ_x and the length of $I^k_{y^g}$ is τ_y for all $k \in \mathbb{Z}$. Hence, by the description of the axis of the product of two isometries given in [5, Remark 3.4], A_h intersects I^k_x and $I^k_{y^g}$ in their midpoint for all $k \in \mathbb{Z}$.

For the definition of quasigeodesic and the proof of the following lemma, see [5, Section 7, Lemmas 7.1 and 7.2].

Lemma 4.1 *Given $L > 0$, there exist $K \geq 1$ and $C > 0$, depending on $\pi_1(F)$, such that if x and y are two hyperbolic elements in $\pi_1(F)$ whose axes are distinct with $\tau_x \leq L$ and $\tau_y \leq L$, then for any $g \in \pi_1(F)$ and $m \in \mathbb{N}$, if A_x and A_{y^g} intersect, then:*

(1) *$\gamma(x^m, y^g)$ is a K–quasigeodesic, and $\gamma(x^m, y^g)$ is homotopic to A_h, where $h = x^m y^g$;*

(2) *$\gamma(x^m, y^g) \subset N_{C/2}(A_h)$ and $A_h \subset N_{C/2}(\gamma(x^m, y^g))$, where $N_C(A_h)$ denotes the C neighborhood of A_h.*

5 Noncancellation lemma

Denote the length of a curve x by $l(x)$. For the proof of the following lemma, see [5, Lemma 7.3].

Lemma 5.1 *Let L, K and C be as in Lemma 4.1. For hyperbolic elements $x, y \in \pi_1(F)$ with $\tau_x \leq L$ and $\tau_y \leq L$, let m be a positive integer such that $m\tau_x > 6KC$.*

Let S and R be the points in $I^0_{x^m}$ at distance $3KC$ from P and $x^m P$ (see Figure 2). Let s (respectively r) be the open half-plane bounded by the line perpendicular to A_x through S (respectively R), containing the point $x^m P$ (respectively P).

Set $U = s \cap r \cap N_C(I^0_{x^m})$. Then U contains an open segment J of $I^0_{x^m}$ such that $N_C(I^0_{x^m}) \backslash U$ is disconnected, $l(J) \geq m\tau_x - 6KC$, $U \subset N_C(I^0_{x^m})$, $\bar{U} \cap N_C(I^k_{x^m}) = \varnothing$ for all $k \neq 0$ and $\bar{U} \cap N_C(I^k_y) = \varnothing$ for all $k \in Z$.

The following lemma is the main lemma of this paper. The proof is based on the proof of [5, Lemma 7.4, Claims 1 and 2] and the idea that lifts of simple closed geodesics are disjoint.

Lemma 5.2 *Let x be a hyperbolic element in $\pi_1(F)$ such that the geodesic representative in the free homotopy class of x is simple. Let $x_1 = x^h$ for some $h \in \pi_1(F)$. Suppose y and y_1 are two distinct hyperbolic elements in $\pi_1(F)$ whose axes are distinct and intersect the axes of x and x_1, respectively. Let L be a positive number such that the translation lengths of x, y and y_1 are bounded above by L. Then there exists m_0 such that for any $m > m_0$, we have $\gamma(x^m, y) = \gamma(x_1^m, y_1)$ whenever $x^m y = x_1^m y_1$. Moreover, there exists $u \in \pi_1(F)$ such that $x_1 = x^u$ and $y_1 = y^u$.*

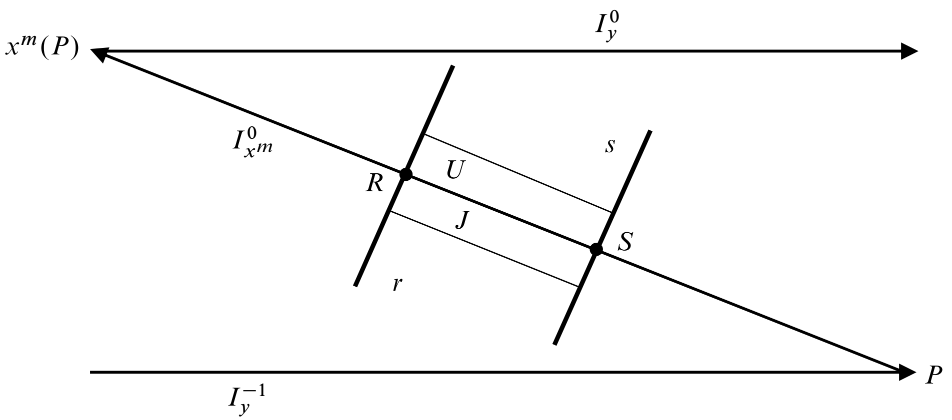

Figure 2: The open segment J as in Lemma 5.2

Proof Let τ_0 be the systole, ie the length of a shortest length closed geodesic of F.

Let C, K and M be the constants defined in Lemmas 4.1 and 2.1 and let $m > m_0$. Define $m_0 = K(3M + 10C)/\tau_0$.

Since $x^m y = x_1^m y_1$, we have $A_{x^m y} = A_{x_1^m y_1}$. Let $g = x^m y = x_1^m y_1$. By Lemma 5.1,

$$\gamma(x_1^m, y_1) \subset N_{C/2}(A_g) \subset N_C(\gamma(x^m, y)). \tag{1}$$

Let J and U be as in Lemma 5.1 corresponding to $\gamma(x^m, y)$. Therefore, $J \subset U$, $J \subset I^0_{x^m} \subset \gamma(x^m, y)$, and $\operatorname{length}(J) \geq m\tau_x - 6KC$.

Claim 1 *The curve $\gamma(x_1^m, y_1)$ intersects U and does not intersect the part of the boundary of U contained in the boundary of $N_C(\gamma(x^m, y))$.*

Proof of claim If $\gamma(x_1^m, y_1)$ does not intersect U, by (1), $\gamma(x_1^m, y_1)$ is contained in $N_C(\gamma(x^m, y)) \setminus U = N_C(\gamma(x^m, y) \setminus J)$, which is disconnected. Hence by (1), $\gamma(x_1^m, y_1)$ should intersect both components, which contradicts that $\gamma(x_1^m, y_1)$ is connected. By Lemma 5.1, $\bar{U} \cap N_C(I^k_{x^m}) = \varnothing$ for all $k \neq 0$ and $\bar{U} \cap N_C(I^k_y) = \varnothing$ for all $k \in Z$. Therefore, $\gamma(x_1^m, y_1)$ does not intersect the part of the boundary of U contained in the boundary of $N_C(\gamma(x^m, y))$. □

Therefore, any component of $U \cap \gamma(x_1^m, y_1)$ consists of piecewise geodesic arcs starting and ending at the sides of U of length $2C$.

Claim 2 *Let β be a component of $\gamma(x_1^m, y_1) \cap U$. Then β contains a geodesic segment l of length greater than M.*

Proof of claim **Case 1** Suppose β contains more than three vertices. Then β contains $I^k_{x_1^m}$ for some $k \in \mathbb{Z}$ and $\operatorname{length}(I^k_{x_1^m}) = m\tau_{x_1} > m_0\tau_0 = K(3M + 10C) > M$.

Case 2 Suppose β contains at most three vertices. Then β consists of at most three segments. Let ν be the longest segment of β and let $r = l(\nu)$. By hypothesis, $l(J) \geq m\tau_x - 6KC > 3KM + 4KC$. Using the triangle inequality and the properties of m and K, we have

$$(3M + 4C) \leq K(3M + 4C) < m\tau_x - 6KC \leq l(J) \leq 2C + 3r + 2C = 3r + 4C.$$

Hence $r > M$ which proves Claim 2. □

The geodesic segment ν is contained in $\gamma(x_1^m, y_1)$ and $\nu \subset N_C(I^0_{x^m})$. Therefore, by Lemma 2.1, ν intersects $I^0_{x^m}$ in a geodesic segment. Hence $I^0_{x^m}$ and $\gamma(x_1^m, y_1)$ intersect in a geodesic segment.

Claim 3 *If $\gamma(x^m, y)$ and $\gamma(x_1^m, y_1)$ intersect in a geodesic segment contained in $I^0_{x^m}$, then they are equal, and there exists $u \in \pi_1(F)$ such that $x_1 = x^u$ and $y_1 = y^u$.*

Proof of claim As $I^0_{x^m}$ intersects $\gamma(x_1^m, y_1)$ in a geodesic segment, $I^0_{x^m}$ intersects either $I^k_{x_1^m}$ or $I^k_{y_1}$ in a geodesic segment.

Now $l(I^k_{x_1^m}) = m\tau_{x_1} = m\tau_x = l(I^0_{x^m})$ for all $k \in \mathbb{Z}$, and A_g intersects $I^0_{x^m}$ and $I^k_{x_1^m}$ in their midpoints. So if $I^0_{x^m}$ intersects $I^k_{x_1^m}$ in a geodesic segment, then they are equal.

If $I^0_{x^m}$ intersects $I^k_{y_1}$ in a geodesic segment for some $k \in \mathbb{Z}$, then by the construction of $\gamma(x^m, y)$, we see that A_x intersects $I^{k+1}_{x_1^m}$, which lies in a translate of the geodesic A_{x_1} and hence in a translate of A_x (as x and x_1 are conjugates). As the geodesic representative in the free homotopy class of x is simple, all translates of A_x are disjoint. Hence $I^0_{x^m}$ cannot intersect $I^k_{y_1}$ for any $k \in \mathbb{Z}$.

Since $I^0_{x^m}$ can not intersect $I^k_{y_1}$ in a geodesic segment, $I^0_{x^m}$ intersects $I^k_{x_1^m}$ for some $k \in \mathbb{Z}$. Thus $I^0_{x^m} = I^k_{x_1^m}$. Since I^0_y and $I^k_{y_1}$ are the unique geodesic segments joining the end point of $I^0_{x^m} = I^k_{x_1^m}$ with the image of the starting point of $I^0_{x^m} = I^k_{x_1^m}$ under g, we see that $I^0_y = I^k_{y_1}$. By the periodic property of the definition of $\gamma(x^m, y)$ and $\gamma(x_1^m, y_1)$, they are equal. Since $g^n I^0_{x^m} = I^0_{x_1^m}$ and $g^n I^0_y = I^0_{y_1}$ for some n, taking $u = g^n$, we have $x_1 = x^u$ and $y_1 = y^u$. This proves the claim and thus the lemma. □

6 Center of the Goldman Lie algebra

Lemma 6.1 *Let F be a hyperbolic surface. Suppose x is an essential simple closed curve and y is an essential closed curve. If $i(x, y) \neq 0$, then there exists m_0 such that $[x^m, y] \neq 0$ for all $m > m_0$.*

Proof Let $L = \max\{\tau_x, \tau_y\}$ and m_0 be as in Lemma 5.2. If $m > m_0$, then

$$[x^m, y] = m\left(\sum_{kB \in J(x^m, y)} \iota(x^m, y^k)\langle x^m y^k\rangle\right).$$

Suppose $\langle x^m y^k\rangle = \langle x^m y^{k_1}\rangle$. Then for some $g \in \pi_1(F)$,

$$x^m y^k = (x^m y^{k_1})^g = (x^m)^g (y)^{k_1 g}.$$

By Lemma 5.2, there exists $u \in \pi_1(F)$ such that x is conjugate to x^g and y^k is conjugate to $y^{k_1 g}$ by the element u. Therefore,

$$\iota(x^m, y^k) = \iota((x^m)^u, (y^k)^u) = \iota(x^{mg}, y^{k_1 g}) = \iota(x^m, y^{k_1}).$$

Hence $[x^m, y] \neq 0$. □

The following lemma is a classical result.

Lemma 6.2 *Let F be a hyperbolic surface of finite type with geodesic boundary. Let γ be a closed curve whose geometric intersection number with any other nontrivial simple closed geodesic is zero. Then γ is either homotopically trivial or homotopic to a boundary curve or peripheral.*

Theorem 6.3 *Let F be a hyperbolic surface of finite type with geodesic boundary. Let $y = \sum_{i=1}^{n} c_i y_i \in \mathcal{L}(F)$, where each y_i is a geodesic and $y_i \neq y_j$ for $i \neq j$. If y belongs to the center of $\mathcal{L}(F)$, then $i(x, y_i) = \varnothing$ for every simple closed geodesic x and for all $i \in \{1, 2, \dots, n\}$.*

Proof We show that, given any simple closed geodesic x, if $i(x, y_i) \neq \varnothing$ for some $i \in \{1, 2, \dots, n\}$, then there exists $m \in \mathbb{N}$ such that $[x^m, y] \neq 0$.

Let x be a simple closed geodesic which intersects at least one y_i. If some y_k is disjoint from x, then the Goldman bracket between x and y_k is zero; therefore, without loss of generality, assume that x intersects y_j for all $j \in \{1, 2, \dots, n\}$. Let $L = \max\{\tau_x, \tau_{y_1}, \tau_{y_2}, \dots, \tau_{y_n}\}$. Hence by Lemma 5.2, there exists m_i for every $i \in \{1, 2, \dots, n\}$ such that, if $m > \max\{m_i\}$ with $x_1^m = (x^m)^h$ for some $h \in \pi_1(F)$, then $\gamma(x^m, y_i) = \gamma(x_1^m, y_j)$ whenever $x^m y_i = x_1^m y_j$. Also there exists $g \in \pi_1(F)$ such that $x_1 = x^g$ and $y_j = {y_i}^g$. Since $[x^m, \sum_{i=1}^{n} c_i y_i] = \sum_{i=1}^{n} c_i [x^m, y_i]$, by Lemma 6.1, it is enough to show that the terms of $[x^m, y_i]$ are distinct from the terms of $[x^m, y_j]$ for $i \neq j$.

Suppose $\langle x^m y_i^{k_i} \rangle = \langle x^m y_j^{k_j} \rangle$. Hence there exists $h \in \pi_1(F)$ such that

$$x^m y_i^{k_i} = (x^m y_j^{k_j})^h = (x^m)^h (y_j)^{k_j h}.$$

By Lemma 5.2, $y_i^{k_i}$ and $y_j^{k_j}$ are conjugates of each other in $\pi_1(F)$. Therefore, y_i and y_j are freely homotopic to each other. Hence the geodesic representative corresponding to y_i and y_j are the same, which contradicts the assumption. □

The Main Theorem follows at once from Lemma 6.2 and Theorem 6.3.

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The beta family at the prime two and modular forms of level three

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We use the orientation underlying the Hirzebruch genus of level three to map the beta family at the prime $p=2$ into the ring of divided congruences. This procedure, which may be thought of as the elliptic Greek letter beta construction, yields the f–invariants of this family.

55Q45; 11F11, 55Q51, 58J26

1 Introduction and statement of the results

The Adams–Novikov spectral sequence (ANSS) serves as a powerful tool to understand the structure of the stable homotopy groups of the sphere $\pi_* S^0$: working locally at a fixed prime p, we have

$$\mathrm{E}_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathrm{BP}_*\,\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \Rightarrow (\pi_{t-s} S^0)_{(p)},$$

and much insight can be gained by resolving its E_2–term into v_n–periodic components (see eg Ravenel's book [11]). In their seminal paper propagating this chromatic approach, Miller, Ravenel, and Wilson [10] introduced the so-called Greek letter map, and computed the 1–line (for all primes) and the 2–line (for odd primes), generated by the alpha and beta families, respectively. The computation of the 2–line for $p=2$, which we outline here, is due to Shimomura [12]. Let us concentrate on the beta elements at $p=2$ (where there are also products of alpha elements). Starting from certain elements $x_i \in v_2^{-1}\,\mathrm{BP}_*$ and $y_i \in v_1^{-1}\,\mathrm{BP}_*$, put

$$a_0 = 1, \quad a_1 = 2, \quad a_k = 3\cdot 2^{k-1} \text{ for } k \geq 2;$$

then, for $n \geq 0$, odd $s \geq 1$, $j \geq 1$, $i \geq 0$, subject to the conditions

$$n \geq i, \quad 2^i | j, \quad j \leq a_{n-i}, \quad \text{and} \quad j \leq 2^n \text{ if } s=1 \text{ and } i=0,$$

the simplest beta elements are given by [12, (1.3.1)]

$$\beta_{s\cdot 2^n/j,i+1} = \eta(x_n^s / 2^{i+1} v_1^j), \tag{1}$$

where η is the universal Greek letter map (see [10, (3.6)]). In fact, it is sometimes possible to improve divisibility: namely, for n, s, j, and i as above with the additional

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conditions that

$$n \geq i+1 \geq 2, \qquad j=2 \quad \text{and} \quad s \geq 3 \quad \text{if } n=2, \qquad j \leq a_{n-i-1} \quad \text{if } n \geq 3,$$

Shimomura defines [12, (1.3.2)]

$$\beta_{s\cdot 2^n/j,i+2} = \eta(x_n^s/2^{i+2}y_i^m), \qquad \text{where } m = j/2^i, \tag{2}$$

and shows the following relations between the beta elements given by (1) and (2) [12, Lemma 3.10]:

(i) $\beta_{s\cdot 2^n/j,i+2} = \beta_{s\cdot 2^n/j,(i+1)+1}$ if $2^{i+1}|j$,

(ii) $2\beta_{s\cdot 2^n/j,i+2} = \beta_{s\cdot 2^n/j,i+1}$.

There are striking number-theoretical patterns lurking in the stable stems which become visible from the chromatic point of view, eg the (nowadays) well-known relation between the 1–line and the (denominators of the) Bernoulli numbers. Concerning the 2–line, Behrens [1] established a precise relation between the beta family for primes $p \geq 5$ and the existence of modular forms satisfying appropriate congruences. On the other hand, using an injection of the 2–line into the ring of divided congruences (tensored with $\mathbb{Q}/\mathbb{Z}$), Laures [9] introduced the f–invariant as a higher analog of the e–invariant. Subsequent work (see Behrens and Laures [2]) has shown how these approaches can be merged and used to derive the f–invariant of the beta family, albeit still only for $p \geq 5$. A different route has been taken by Hornbostel and Naumann in [8], where the f–invariant is represented using Artin–Schreier theory; however, although no longer limited to primes $p \geq 5$, the calculations actually carried out in that reference only take care of two subfamilies (namely β_t for $p \nmid t$ and $\beta_{s2^n/2^n}$ for $p=2$).

Since there has been some progress on our geometrical understanding of the f–invariant through analytical techniques (see eg work of the author [3] and Bunke and Naumann [5]), it is desirable to have some sort of "comparison table"; to this end, we compute the f–invariant of the beta family[1] at the prime $p=2$. More precisely, we take a look at the following diagram for $p=2$ and $N=3$:

$$\begin{tikzcd}
\mathrm{Ext}^0(\mathrm{BP}_*, v_2^{-1}\,\mathrm{BP}_*/(p^\infty, v_1^\infty)) \arrow[r] \arrow[d] & \mathrm{Ext}^{2,*}(\mathrm{BP}_*, \mathrm{BP}_*) \arrow[d] \\
\mathrm{Ext}^0(E_*^{\Gamma_1(N)}, E_*^{\Gamma_1(N)}/(p^\infty, v_1^\infty)) \arrow[r] \arrow[dr, dotted] & \mathrm{Ext}^{2,*}(E_*^{\Gamma_1(N)}, E_*^{\Gamma_1(N)}) \arrow[d] \\
 & \underline{\underline{D}}_*^{\Gamma_1(N)} \otimes \mathbb{Q}/\mathbb{Z}_{(p)}
\end{tikzcd} \tag{3}$$

[1] The situation of products of permanent alpha elements has been studied in [4].

The upper horizontal arrow in this diagram produces the beta family; as a brief reminder of this construction (referring to [10; 12; 11] for details), note that an element of this family is annihilated by some power of 2, say 2^{i_0}. Thus, it is the image of an element in $\mathrm{Ext}^1(\mathrm{BP}_*, \mathrm{BP}_*/(2^{i_0}))$ under the connecting homomorphism associated to the short exact sequence $\mathrm{BP}_* \to \mathrm{BP}_* \to \mathrm{BP}_*/(2^{i_0})$. As this Ext^1–group admits multiplication by (suitable powers of) v_1, the same argument shows that its $v_1^{i_1}$–torsion elements can be obtained from $\mathrm{Ext}^0(\mathrm{BP}_*, \mathrm{BP}_*/(2^{i_0}, v_1^{i_1}))$. In particular, it turns out that, under the conditions quoted before, x_n^s is invariant mod $(2^{i+1}, v_1^j)$, thus giving rise to an element in $\mathrm{Ext}^0(\mathrm{BP}_*, \mathrm{BP}_*/(2^{i+1}, v_1^j))$ (despite the v_2^{-1} appearing in its definition) which in turn leads to the element $\beta_{s\cdot 2^n/j,i+1}$.

For the second row of the diagram, recall from Hirzebruch's book [7] that, for each level $N > 1$, there is a complex genus taking values in the ring of modular forms for the congruence subgroup $\Gamma_1(N) \subset \mathrm{SL}(2;\mathbb{Z})$; furthermore, as explained by the work of Franke [6], these genera give rise to complex oriented elliptic (co-)homology theories. Thus, working locally at the prime p, the orientation yields a map of coefficient rings $\mathrm{BP}_* \to E_*^{\Gamma_1(N)}$ and induces the upper vertical arrows.

The composition of the vertical arrows on the right-hand side can now be chosen to account for the algebraic portion of Laures' f–invariant [9], ie chosen to capture the p–local information of the second map in the factorization

$$f\colon \pi_{2k}S^0 \to \mathrm{Ext}^{2,2k+2}(\mathrm{MU}_*, \mathrm{MU}_*) \to \underline{\underline{D}}_{k+1}^{\Gamma_1(N)} \otimes \mathbb{Q}/\mathbb{Z}. \tag{4}$$

So, in order to compute the f–invariant of a member of the beta family, we chase its preimage through the composition of the vertical arrow on the left-hand side with the dotted arrow; put differently, we carry out (a sufficiently large portion of) an elliptic analog of the Greek letter construction explicitly. The result can be summarized as follows (where, as usual, we abbreviate $\beta_{k/j} = \beta_{k/j,1}$ and $\beta_k = \beta_{k/1}$):

Theorem 1 *The f–invariants of the beta elements of order two are given as follows:*

(i) *For odd $s \geq 3$,* $\quad f(\beta_s) \equiv \dfrac{1}{2}\left(\dfrac{E_1^2 - 1}{4}\right)^s \mod \underline{\underline{D}}_{3s-1}^{\Gamma_1(3)}$.

(ii) *For odd $s \geq 1$,* $\quad f(\beta_{2s/j}) \equiv \dfrac{1}{2}\left(\dfrac{E_1^2 - 1}{4}\right)^{2s} \mod \underline{\underline{D}}_{6s-j}^{\Gamma_1(3)}$.

(iii) *For $l \geq 0$ and odd $s \geq 1$,*

$$\begin{aligned} f(\beta_{4s\cdot 2^l/j}) &\equiv \frac{1}{2}\left(\frac{E_1^2 - 1}{4}\right)^{4s\cdot 2^l} + \frac{1}{2}\left(\frac{E_1^2 - 1}{4}\right)^{(4s-1)2^l} \mod \underline{\underline{D}}_{12s\cdot 2^l - j}^{\Gamma_1(3)} \\ &\equiv \frac{1}{2}\left(\frac{E_1^2 - 1}{4}\right)^{4s\cdot 2^l} \qquad \textit{if } j \leq 3\cdot 2^l. \end{aligned}$$

Theorem 2 *The f–invariants of the beta elements of higher order are given as follows:*

(i) *For odd $s \geq 1$,* $\quad f(\beta_{4s/2,2}) \equiv \frac{1}{4}\left(\frac{E_1^2-1}{4}\right)^{4s} \mod \underline{\underline{D}}^{\Gamma_1(3)}_{12s-2}.$

(ii) *For $l \geq 0$, $i \geq 1$, $j = m \cdot 2^i \leq a_{l+2}$, odd $s \geq 1$, and modulo $\underline{\underline{D}}^{\Gamma_1(3)}_{3s\cdot 2^{l+i+2}-j}$,*

$$\begin{aligned} f(\beta_{s\cdot 2^{l+i+2}/j,i+1}) &\equiv \frac{1}{2^{i+1}}\left(\frac{E_1^2-1}{4}\right)^{s\cdot 2^{l+i+2}} + \frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^{(s\cdot 2^{i+2}-1)2^l} \\ &\equiv \frac{1}{2^{i+1}}\left(\frac{E_1^2-1}{4}\right)^{s\cdot 2^{l+i+2}} \qquad \text{if } j \leq 3\cdot 2^l. \end{aligned}$$

(iii) *For $k \geq 2$,* $\quad f(\beta_{4k/2,3}) \equiv \frac{1+4k}{8}\left(\frac{E_1^2-1}{4}\right)^{4k} \mod \underline{\underline{D}}^{\Gamma_1(3)}_{12k-2}.$

(iv) *For $l \geq 0$, $i \geq 1$, $j = m \cdot 2^i \leq a_{l+2}$, odd $s \geq 1$, and modulo $\underline{\underline{D}}^{\Gamma_1(3)}_{3s\cdot 2^{l+i+3}-j}$,*

$$\begin{aligned} f(\beta_{s\cdot 2^{l+i+3}/j,i+2}) &\equiv \frac{1}{2^{i+2}}\left(\frac{E_1^2-1}{4}\right)^{s\cdot 2^{l+i+3}} + \frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^{(s\cdot 2^{i+3}-1)2^l} \\ &\equiv \frac{1}{2^{i+2}}\left(\frac{E_1^2-1}{4}\right)^{s\cdot 2^{l+i+3}} \qquad \text{if } j \leq 3\cdot 2^l. \end{aligned}$$

The proof presented in the following section turns out to be a pretty straightforward calculation: After a brief recollection of the relevant definitions, we study the image (under the orientation underlying the Hirzebruch genus) of the elements x_i and y_i occurring in the definition of the beta elements. Then, we sketch our approach to the argument given in [2, Section 4], ie we explain how to carry out the Greek letter construction on the level of (holomorphic) modular forms. The final step consists of performing this computation explicitly.

2 Proof of the theorems

2.1 Preliminaries

Working with the congruence subgroup $\Gamma_1(N) \subset \mathrm{SL}(2;\mathbb{Z})$ for a fixed level $N > 1$, modular forms will be thought of in terms of their q–expansions at the cusp $i\infty$, where $q = e^{2\pi i\tau}$. Setting $\mathbb{Z}^{\Gamma_1(N)} = \mathbb{Z}[\zeta_N, 1/N]$, where $\zeta_N = e^{2\pi i/N}$, we then denote by $M_*^{\Gamma_1(N)}$ the graded ring of modular forms with respect to $\Gamma_1(N)$ which expand integrally, ie which lie in $\mathbb{Z}^{\Gamma_1(N)}[\![q]\!]$. Now recall from [7, Section 7] that the power series associated to the Hirzebruch elliptic genus of level N may be expressed as

$$Q^{\Gamma_1(N)}(x) = x\frac{\Phi(\tau, x-2\pi i/N)}{\Phi(\tau,x)\Phi(\tau,-2\pi i/N)}, \tag{5}$$

where the Φ–function is given by

$$\Phi(\tau,z) = 2\sinh(z/2)\prod_{n\geq 1}\frac{(1-e^z q^n)(1-e^{-z}q^n)}{(1-q^n)^2}.$$

By the splitting principle, the power series (5) determines a homomorphism

$$\phi^{\Gamma_1(N)}\colon \mathrm{MU}_* \to M_*^{\Gamma_1(N)},$$

where MU_* is the coefficient ring of the complex cobordism spectrum, and integrality of the image follows by noting that each term in the q–expansion corresponds to a twisted Todd genus. Furthermore, as mentioned in the introduction, the Hirzebruch elliptic genus can be used to construct periodic complex oriented (co-)homology theories [6]: upon inverting, for example, the discriminant form, the Landweber exact functor theorem applies.

Finally, let us explain the map (4) in more detail. To this end, recall from [9] that the ring of *divided congruences* $D^{\Gamma_1(N)}$ consists of those rational combinations of modular forms which expand integrally, and that this ring can be filtered by setting

$$D_{k+1}^{\Gamma_1(N)} = \{f = \textstyle\sum_{i=0}^{k+1} f_i \mid f_i \in M_i^{\Gamma_1(N)}\otimes\mathbb{Q},\ f\in \mathbb{Z}^{\Gamma_1(N)}[\![q]\!]\};$$

furthermore, we put

$$\underline{\underline{D}}_{k+1}^{\Gamma_1(N)} = D_{k+1}^{\Gamma_1(N)} + M_0^{\Gamma_1(N)}\otimes\mathbb{Q} + M_{k+1}^{\Gamma_1(N)}\otimes\mathbb{Q}.$$

Temporarily switching to the ANSS based on MU, an element in the stable stems of positive even dimension is in second filtration; thus, it can be projected to the 2–line of the E_∞–page which in turn injects into the 2–line of the E_2–page (as there are no differentials hitting it). This explains the first part of the map (4). For the second part, we regard $\mathrm{Ext}^{2,*}(\mathrm{MU}_*,\mathrm{MU}_*)$ as a subquotient of $(\mathrm{MU}_*\otimes\mathbb{Q})^{\otimes 2}$ and consider a representative of an element; under a similar identification, its image under the orientation (determined by the Hirzebruch genus) is represented by a sum of tensor products of modular forms. This sum becomes a rational combination of modular forms (hence an element in the rationalized ring of divided congruences) by replacing, say, each second factor in the sum by the constant term in its q–expansion. Working locally at a prime $p\nmid N$, the induced composite map is injective by the results of [9], completing the definition of the f–invariant; on the other hand, switching back to BP, we arrive at the right-hand side of our diagram (3).

2.2 The image under the orientation

Henceforth, we fix $p = 2$ and $N = 3$, abbreviating $\Gamma = \Gamma_1(3)$. Then the ring of modular forms is given by (see eg [8, Section 3.2])

$$M_*^\Gamma = \mathbb{Z}^\Gamma[E_1, E_3],$$

where

$$E_1 = 1 + 6 \sum_{n=1}^{\infty} \sum_{d|n} \left(\frac{d}{3}\right) q^n, \quad E_3 = 1 - 9 \sum_{n=1}^{\infty} \sum_{d|n} \left(\frac{d}{3}\right) d^2\, q^n$$

are the odd Eisenstein series of the indicated weight at the level $N = 3$ (and $(\frac{\cdot}{\cdot})$ denotes the Legendre symbol); in passing, we note that $(E_1^3 - E_3)/27 \in \mathbb{Z}[\![q]\!]$. Furthermore, the following basic congruence can be read off from the q–expansions:

$$E_3 - 1 \equiv \frac{E_1^2 - 1}{4} \mod 2D_3^\Gamma. \tag{6}$$

Returning to the Hirzebruch elliptic genus, one may use [7, Appendix I, Theorem 6.2] (see also [3, Appendix C]) to verify that the first few terms of the power series (5), when expressed in terms of the generators E_1 and E_3 of M_*^Γ, read:

$$\begin{aligned} Q^\Gamma(x) = 1 &+ \frac{iE_1}{2\sqrt{3}}x + \frac{E_1^2}{12}x^2 + \frac{iE_1^3 - iE_3}{18\sqrt{3}}x^3 + \frac{13E_1^4 - 16E_1E_3}{2160}x^4 \\ &+ \frac{iE_1^2(E_1^3 - E_3)}{216\sqrt{3}}x^5 + \frac{121E_1^6 - 152E_1^3E_3 + 40E_3^2}{272160}x^6 \\ &+ \frac{iE_1}{\sqrt{3}} \frac{7E_1^6 - 11E_1^3E_3 + 4E_3^2}{19440}x^7 + O(x^8). \end{aligned}$$

Thus, the genus of the following complex projective spaces is readily evaluated:

$$\begin{aligned} w_1 &= \phi^\Gamma(\mathbb{CP}^1) = \frac{i}{\sqrt{3}}E_1, \\ w_3 &= \phi^\Gamma(\mathbb{CP}^3) = \frac{i}{\sqrt{3}} \frac{5E_1^3 - 2E_3}{9}, \\ w_7 &= \phi^\Gamma(\mathbb{CP}^7) = \frac{i}{\sqrt{3}} \frac{70E_1^4E_3 - 14E_1E_3^2 - 65E_1^7}{243}. \end{aligned}$$

We remind the reader that Hazewinkel's generators of BP_* are recursively defined, and that they are in fact integral, ie they live in MU_* (see eg [11, Appendix A2]). The same recursive procedure can be used to determine their respective images under the Hirzebruch genus, which, by abuse of notation, we still denote by v_i, leading to

$$\begin{aligned}
v_1 &= w_1 = \frac{i}{\sqrt{3}} E_1, \\
v_2 &= \frac{w_3 - w_1^3}{2} = \frac{i}{\sqrt{3}} \frac{4E_1^3 - E_3}{9}, \\
v_3 &= \frac{w_7}{4} - \frac{w_1^7 + w_1 w_3^2}{8} = \frac{iE_1}{\sqrt{3}} \frac{5E_1^3 E_3 - E_3^2 - 4E_1^6}{81};
\end{aligned}$$

in particular, we see that v_3 becomes decomposable in M_*^Γ:

$$\begin{aligned}
v_3 &= \frac{iE_1}{\sqrt{3}} \left(\frac{4E_1^3 E_3 - E_3^2}{81} - \frac{4E_1^6 - E_1^3 E_3}{81} \right) \\
&= \frac{iE_1}{\sqrt{3}} \left(\frac{i}{\sqrt{3}} \frac{4E_1^3 - E_3}{9} \right) \left(-\frac{i}{3\sqrt{3}} (E_3 - E_1^3) \right) \\
&= 3v_1 v_2 (v_2 + v_1^3).
\end{aligned} \tag{7}$$

Continuing with our abuse of notation, we now consider the x_i as elements in $v_2^{-1} M_*^\Gamma$, where, due to (7), their original definition [12, (1.1)] simplifies to

$$\begin{aligned}
x_0 &= v_2, \\
x_1 &= v_2^2 - v_1^2 v_2^{-1} v_3 = v_2^2 - 3v_1^3 (v_2 + v_1^3), \\
x_2 &= x_1^2 - v_1^3 v_2^3 - v_1^5 v_3 = v_2^4 - 7v_1^3 v_2^3 + 15 v_1^9 v_2 + 9 v_1^{12}, \\
x_i &= x_{i-1}^2 \quad i \geq 3,
\end{aligned} \tag{8}$$

showing that the (images of the) x_i are actually holomorphic. On the other hand, unless $i = 0$, this is not true for the $y_i \in v_1^{-1} M_*^\Gamma$, which read:

$$\begin{aligned}
y_0 &= v_1, \\
y_1 &= v_1^2 - 4v_1^{-1} v_2, \\
y_i &= y_{i-1}^2, \quad i \geq 2.
\end{aligned}$$

However, for $i \geq 1$ and $m \geq 1$, we may introduce

$$z_{i,m} = v_1^{m \cdot 2^i} - m \cdot 2^{i+1} v_1^{m \cdot 2^i - 3} v_2, \tag{9}$$

which are holomorphic for $m\cdot 2^i\geq 4$ and satisfy

$$\begin{aligned} z_{i,m} &\equiv y_i^m \mod 2^{i+2}v_1^{-1}M_*^\Gamma \\ &\equiv 1 \mod 2^{i+2}\mathbb{Z}^\Gamma[\![q]\!], \end{aligned}$$

the second line being an immediate consequence of (6).

2.3 Determining "elliptic" beta elements

Requiring $p>3$ and working with the full modular group, Behrens and Laures have shown in [2, Section 4] how an element in $\mathrm{Ext}^0(M_*, M_*/(p^\infty, E_{p-1}^\infty))$ gives rise to an element in $D\otimes\mathbb{Q}/D\big[\frac{1}{6}\big]+M_k\otimes\mathbb{Q}+\mathbb{Q}$; clearly, the other primes can be treated analogously by working with a smaller congruence subgroup. Let us rephrase their argument in a language closer to the original formulation of the Greek letter construction:

Still working at the prime $p=2$ and the level $N=3$, we choose a (holomorphic) modular form $\mu\in M_{|\mu|}^\Gamma$ and a pair of positive integers (i_0,i_1) such that

$$\mu^{i_1}\equiv 1 \mod 2^{i_0}D_{i_1|\mu|}^\Gamma;$$

in particular, this ensures that $(2^{i_0},\mu^{i_1})$ is regular on M_*^Γ.

Now, given a modular form $\widetilde{\varphi}_t\in M_t^\Gamma$, we can use the natural inclusion

$$M_t^\Gamma\hookrightarrow D_t^\Gamma$$

and ask whether $\widetilde{\varphi}_t$ satisfies

$$\widetilde{\varphi}_t\equiv \mu^{i_1}\varphi_{t/i_1|\mu|,i_0} \mod 2^{i_0}D_t^\Gamma \tag{10}$$

for some

$$\varphi_{t/i_1|\mu|,i_0}\in D_{t-i_1|\mu|}^\Gamma/2^{i_0}D_{t-i_1|\mu|}^\Gamma.$$

Let us call a modular form satisfying (10) *invariant mod* $(2^{i_0},\mu^{i_1})$. Moreover, we have the obvious composition:

$$\underline{\underline{(\cdot)}}\colon D_k^\Gamma/2^{i_0}D_k^\Gamma\cong D_k^\Gamma\otimes\mathbb{Z}/2^{i_0}\to D_k^\Gamma\otimes\mathbb{Q}/\mathbb{Z}\to \underline{\underline{D}}_k^\Gamma\otimes\mathbb{Q}/\mathbb{Z},\quad \varphi_k\mapsto\underline{\underline{\varphi}}_k.$$

Then it is easy to see that, for a modular form $\widetilde{\varphi}_t$ satisfying (10), the assignment

$$\widetilde{\varphi}_t\mapsto\underline{\underline{\varphi}}_{t/i_1|\mu|,i_0}$$

depends only on the reduction of $\widetilde{\varphi}_t \mod (2^{i_0}, \mu^{i_1})$, hence descends to a well-defined map

$$\ker(M_t^\Gamma/(2^{i_0}, \mu^{i_1}) \to D_t^\Gamma/(2^{i_0}, \mu^{i_1})) \to \underline{\underline{D}}_{t-i_1|\mu|}^\Gamma \otimes \mathbb{Q}/\mathbb{Z} \tag{11}$$

which we may think of as the *"elliptic" Greek letter beta map* and which corresponds to the dotted arrow in our diagram (3). More precisely, by removing the constant term of the q–expansion, we obtain another map

$$d\colon M_t^\Gamma \to D_t^\Gamma, \quad d(\widetilde{\varphi}_t) = \widetilde{\varphi}_t - q^0(\widetilde{\varphi}_t),$$

which might look like a more natural choice with respect to which invariance should be defined, see [2, Section 4]. However, we have $q^0(\widetilde{\varphi}_t) \equiv \mu^{i_1} q^0(\widetilde{\varphi}_t) \mod 2^{i_0} D_t^\Gamma$, hence both choices agree up to a shift of $\varphi_{t/i_1|\mu|, i_0}$ by the constant $q^0(\widetilde{\varphi}_t)$; as the latter maps to zero in $\underline{\underline{D}}_{t-i_1|\mu|}^\Gamma \otimes \mathbb{Q}/\mathbb{Z}$, our construction is visibly equivalent to the one leading to [2, Theorem 4.2].

2.4 Explicit computations

Computing the effect of the elliptic Greek letter map (11) on the preimage of Shimomura's beta elements now amounts to exhibiting appropriate congruences; the elements defined by (1) are dealt with easily, since $(2^{i+1}, v_1^j)$ is regular on M_*^Γ provided that $j = m \cdot 2^i$; moreover, for $k \geq 0$ this implies:

$$\left(\frac{E_1^2 - 1}{4}\right)^k \equiv v_1^j \left(\frac{E_1^2 - 1}{4}\right)^k \mod 2^{i+1} D_{2k+j}^\Gamma. \tag{12}$$

Furthermore, the following two results are useful:

Lemma 3 *For $i \geq 0$, $l \geq 0$, $m \cdot 2^i = j \leq 6 \cdot 2^l$, we have*

$$E_3^{s \cdot 2^{l+i+2}} \equiv \left(\frac{E_1^2 - 1}{4}\right)^{s \cdot 2^{l+i+2}} \mod 2^{i+1} D_{12s \cdot 2^{l+i}}^\Gamma + v_1^j \cdot M_{12s \cdot 2^{l+i} - j}^\Gamma.$$

Proof It is easy to see that for $l \geq 0$ and $i \geq 0$, we have

$$E_3^{2^{l+i+2}} \equiv (E_3 - v_1^3)^{2^{l+i+2}} + 2^{i+1}(v_1^6 E_3^2)^{2^l} E_3^{2^{l+2}(2^i - 1)} \mod (2^{i+2}, v_1^{12 \cdot 2^l}),$$

and the basic congruence (6) implies

$$(E_3 - v_1^3)^{2^k} \equiv \left(\frac{E_1^2 - 1}{4}\right)^{2^k} \mod 2^{k+1} D_{3 \cdot 2^k}^\Gamma.$$

This concludes the proof. □

Lemma 4 *For $i \geq 0$, $l \geq 0$, $1 \leq j \leq 6 \cdot 2^l$, we have*

$$\begin{aligned} E_3^{(s \cdot 2^{i+2}-1)2^l} &\equiv \left(\frac{E_1^2-1}{4}\right)^{(s \cdot 2^{i+2}-1)2^l} && \bmod 2D^{\Gamma}_{12s \cdot 2^{l+i}} + v_1^j \cdot M^{\Gamma}_{12s \cdot 2^{l+i}-j} \\ &\equiv 0 && \textit{if } j \leq 3 \cdot 2^l. \end{aligned}$$

Proof Noting that

$$\begin{aligned} E_3^{(s \cdot 2^{i+2}-1)2^l} &\equiv v_1^{3 \cdot 2^l}(E_3 - v_1^3)^{(s \cdot 2^{i+2}-1)2^l} && \bmod 2D^{\Gamma}_{12s \cdot 2^{l+i}} + v_1^j \cdot M^{\Gamma}_{12s \cdot 2^{l+i}-j} \\ &\equiv (E_3 - 1)^{(s \cdot 2^{i+2}-1)2^l} && \bmod 2D^{\Gamma}_{12s \cdot 2^{l+i}} + v_1^j \cdot M^{\Gamma}_{12s \cdot 2^{l+i}-j}, \end{aligned}$$

the claim follows from (6). □

Proof of Theorem 1 For part (i), we observe that

$$\begin{aligned} x_0^s &= v_2^s \\ &\equiv E_3^s && \bmod 2D^{\Gamma}_{3s} \\ &\equiv (E_3 - v_1^3)^s && \bmod 2D^{\Gamma}_{3s} + v_1 \cdot M^{\Gamma}_{3s-1} \\ &\equiv \left(\frac{E_1^2-1}{4}\right)^s && \bmod 2D^{\Gamma}_{3s} + v_1 \cdot M^{\Gamma}_{3s-1}. \end{aligned}$$

Similarly, for part (ii) we have:

$$\begin{aligned} x_1^s &\equiv v_2^{2s} && \bmod v_1^j \\ &\equiv E_3^{2s} && \bmod 2D^{\Gamma}_{6s} + v_1^j \cdot M^{\Gamma}_{6s-j} \\ &\equiv (E_3 - v_1^3)^{2s} && \bmod 2D^{\Gamma}_{6s} + v_1^j \cdot M^{\Gamma}_{6s-j} \\ &\equiv \left(\frac{E_1^2-1}{4}\right)^{2s} && \bmod 2D^{\Gamma}_{6s} + v_1^j \cdot M^{\Gamma}_{6s-j}, \end{aligned}$$

and since $j \leq a_{l+2} = 6 \cdot 2^l$ (and $j \leq 2^{l+2}$ if $s = 1$), for part (iii) we conclude

$$\begin{aligned} x_{2+l}^s &\equiv v_2^{4s \cdot 2^l} + v_1^{3 \cdot 2^l} v_2^{(4s-1)2^l} && \bmod (2, v_1^{a_{l+2}}) \\ &\equiv E_3^{4s \cdot 2^l} + E_3^{(4s-1)2^l} && \bmod 2D^{\Gamma}_{12s \cdot 2^l} + v_1^j \cdot M^{\Gamma}_{12s \cdot 2^l - j} \\ &\equiv \left(\frac{E_1^2-1}{4}\right)^{4s \cdot 2^l} + \left(\frac{E_1^2-1}{4}\right)^{(4s-1)2^l} && \bmod 2D^{\Gamma}_{12s \cdot 2^l} + v_1^j \cdot M^{\Gamma}_{12s \cdot 2^l - j}. \end{aligned}$$

In view of (12), this completes the proof. □

Remark 5 Since $x_0 = v_2$ is sent to zero under the map (11) with respect to $(2, v_1)$, we see that in order to obtain something interesting, we have to impose $s \geq 3$ in part (i). In a similar vein, the condition $j \leq 2^{l+2}$ if $s = 1$ in part (iii) is needed to ensure that $D^{\Gamma}_{8s\cdot 2^l+j} \subset D^{\Gamma}_{12s\cdot 2^l}$ when using (12).

Now we turn our attention to the elements $\beta_{4s\cdot 2^l/j,i+1}$ for $i \geq 1$.

Proof of Theorem 2(i) The choice $n = 2$ and $i = 1$ in (1) dictates $j = 2$, hence we compute

$$\begin{aligned} x_2^s &\equiv v_2^{4s} && \bmod (4, v_1^2) \\ &\equiv E_3^{4s} && \bmod 4D^{\Gamma}_{12s} + v_1^2 \cdot M^{\Gamma}_{12s-2} \\ &\equiv \left(\frac{E_1^2 - 1}{4}\right)^{4s} && \bmod 4D^{\Gamma}_{12s} + v_1^2 \cdot M^{\Gamma}_{12s-2}. \end{aligned}$$

Combined with (12), this yields the claim. □

Lemma 6 *For $l \geq 0$ and $i \geq 0$, we have*

$$x_{l+i+3} \equiv v_2^{2^{l+i+3}} + 2^{i+1} v_1^{3\cdot 2^l} v_2^{(2^{i+3}-1)2^l} \quad \bmod (2^{i+2}, v_1^{a_{l+2}}).$$

Proof Since $(a+b)^{2^{l+1}} \equiv a^{2^{l+1}} + b^{2^{l+1}} + 2(ab)^{2^l} \bmod 4$ for $l \geq 0$, we compute

$$x_{l+3} = x_2^{2^{l+1}} \equiv v_2^{8\cdot 2^l} + 2(v_1^3 v_2)^{2^l} v_2^{6\cdot 2^l} \quad \bmod (4, v_1^{a_{l+2}})$$

and use the binomial theorem. □

Proof of Theorem 2(ii) In order to treat the remaining cases of our computation of $x_n^s \bmod (2^{i+1}, v_1^j)$, we notice that since (1) requires $j = m \cdot 2^i \leq a_{n-i}$, and since all cases with $i = 0$ and the case $i = 1$ for $n = 2$ have already been taken care of, it suffices to consider $n = l + i + 2$ where $l \geq 0$ and $i \geq 1$; now, for odd $s \geq 1$ we have (by Lemma 6 in a reindexed form)

$$\begin{aligned} x^s_{l+i+2} &\equiv v_2^{s\cdot 2^{l+i+2}} + 2^i v_1^{3\cdot 2^l} v_2^{s\cdot 2^{l+i+2}-2^l} && \bmod (2^{i+1}, v_1^{a_{l+2}}) \\ &\equiv E_3^{s\cdot 2^{l+i+2}} + 2^i E_3^{s\cdot 2^{l+i+2}-2^l} && \bmod 2^{i+1} D^{\Gamma}_{12s\cdot 2^{l+i}} + v_1^j \cdot M^{\Gamma}_{12s\cdot 2^{l+i}-j}, \end{aligned}$$

from which the desired result follows. □

Finally, we treat the beta elements defined by (2):

Proof of Theorem 2(iii) In order to compute the f–invariant of $\beta_{4k/2,3}$, we are going to show that, although $z_{1,1} = y_1 = v_1^2 - 4v_1^{-1}v_2$ is not holomorphic, we can still make sense out of the map (11) with respect to $(8, z_{1,1})$ if $t = 12k \geq 24$. To this end, we observe

$$v_1^6 = z_{1,1}v_1^4 + 4v_1^3v_2 = z_{1,1}(v_1^4 + 4v_1v_2) + 16v_2^2,$$

hence we compute

$$\begin{aligned} x_2^k &\equiv v_2^{4k} + kv_1^3v_2^{4k-1} && \bmod (8, v_1^6) \\ &\equiv (1+4k)v_2^{4k} && \bmod (8, z_{1,1}) \\ &\equiv (1+4k)E_3^{4k} && \bmod 8D_{12k}^\Gamma + z_{1,1}\cdot M_{12k-2}^\Gamma, \end{aligned}$$

where $z_{1,1}\cdot M_{12k-2}^\Gamma \subset M_{12k}^\Gamma$ for dimensional reasons. Finally, we note that

$$\begin{aligned} E_3^{4k} &\equiv \left(\frac{E_1^2-1}{4}\right)^{4k} && \bmod 8D_{12k}^\Gamma + z_{1,1}\cdot M_{12k-2}^\Gamma \\ &\equiv \left(\frac{E_1^2-1}{4}\right)^{4k} v_1^4 z_{1,1} && \bmod 8D_{12k}^\Gamma + z_{1,1}\cdot M_{12k-2}^\Gamma \quad \text{if } k\geq 2; \end{aligned}$$

as $v_1^4 \equiv 1 \bmod 8$, the claim follows. □

Proof of Theorem 2(iv) Recall that in the definition (2) we have to impose $j = m\cdot 2^i \leq a_{n-i-l}$ for $n \geq 3$; since the situation $m = i = 1$ has already been dealt with in the previous part (iii), it is sufficient to consider the case $n = l+i+3$, $4 \leq m\cdot 2^i = j \leq a_{l+2}$, where $l \geq 0$, $i \geq 1$. In order to compute the f–invariants, we calculate the effect of the map (11) with respect to $(2^{i+2}, z_{i,m})$. Since

$$\begin{aligned} v_1^{6\cdot 2^l} &= z_{i,m}v_1^{6\cdot 2^l - j} + 2jv_1^{6\cdot 2^l-3}v_2, \\ v_1^{9\cdot 2^l} &= z_{i,m}(v_1^{9\cdot 2^l-j} + 2jv_1^{9\cdot 2^l-j-3}v_2) + 4j^2v_1^{9\cdot 2^l-6}v_2^2, \end{aligned} \tag{13}$$

we calculate, for $l \geq 0$, $i \geq 1$, and odd $s \geq 1$,

$$\begin{aligned} &x_{l+i+3}^s \\ &\equiv v_2^{s\cdot 2^{l+i+3}} + 2^{i+1}v_1^{3\cdot 2^l}v_2^{(s2^{i+3}-1)2^l} + 3s\cdot 2^i v_1^{6\cdot 2^l}v_2^{(s2^{i+3}-2)2^l} && \bmod (2^{i+2}, v_1^{9\cdot 2^l}) \\ &\equiv v_2^{s\cdot 2^{l+i+3}} + 2^{i+1}v_1^{3\cdot 2^l}v_2^{(s2^{i+3}-1)2^l} && \bmod (2^{i+2}, z_{i,m}), \end{aligned}$$

hence

$$x_{l+i+3}^s \equiv E_3^{s\cdot 2^{l+i+3}} + 2^{i+1}E_3^{(s\cdot 2^{i+3}-1)2^l} \quad \bmod 2^{i+2}D_{24s\cdot 2^{l+i}}^\Gamma + z_{i,m}\cdot M_{24s\cdot 2^{l+i}-j}^\Gamma.$$

Furthermore, the proof of Lemma 3 shows

$$E_3^{2^{l+(i+1)+2}} \equiv (E_3 - v_1^3)^{2^{l+(i+1)+2}} + 2^{(i+1)+1}(v_1^6 E_3^2)^{2^l} E_3^{2^{l+2}}(2^{i+1}-1) \mod (2^{(i+1)+2}, v_1^{12\cdot 2^l}).$$

Thus, it follows that

$$E_3^{2^{l+(i+1)+2}} \equiv (E_3 - v_1^3)^{2^{l+(i+1)+2}} \quad \mod (2^{i+2}, v_1^{12\cdot 2^l}),$$

and due to (13), application of Lemma 3 and Lemma 4 yields the claim. □

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On toric generators in the unitary and special unitary bordism rings

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We construct a new family of toric manifolds generating the unitary bordism ring. Each manifold in the family is the complex projectivisation of the sum of a line bundle and a trivial bundle over a complex projective space. We also construct a family of special unitary quasitoric manifolds which contains polynomial generators of the special unitary bordism ring with 2 inverted in dimensions > 8. Each manifold in the latter family is obtained from an iterated complex projectivisation of a sum of line bundles by amending the complex structure to make the first Chern class vanish.

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1 Introduction

Finding geometric representatives of bordism classes is a classical problem on the borders of geometry and topology. The theory of bordism and cobordism is one of the deepest and most influential parts of algebraic topology, which experienced a spectacular development in the 1960s. Although the original definition of bordism, going back to Pontryagin and Thom, was very geometric, it soon became clear that elaborate homotopy-theoretic, algebraic and number-theoretic techniques were required to obtain structural results on bordism groups and (co)bordism rings.

Most calculations of bordism rings of a point for the classical series of Lie groups were settled by coordinated efforts of many topologists by the end of the 1960s (with the notable exception of symplectic bordism, whose structure is still not described completely). These results were summarised in the monograph by Stong [16]. Nevertheless, it has remained a challenging task to describe particular geometric representatives for generators of bordism rings (which tend to be rings of polynomials when 2 is inverted) and other "special" bordism classes. The importance of this problem was much emphasised in the original works such as Conner and Floyd [7].

Over the rationals, the bordism rings are generated by projective spaces, but the integral generators are more subtle as they involve divisibility conditions on characteristic

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numbers. One of the few general results on geometric representatives for bordism classes known from the early 1960s is that the complex bordism ring Ω^{U}, which is an integral polynomial ring, can be generated by the so-called *Milnor hypersurfaces* $H(n_1,n_2)$. These are hyperplane sections of the Segre embeddings of products $\mathbb{C}P^{n_1}\times\mathbb{C}P^{n_2}$ of complex projective spaces. Similar generators exist for unoriented and oriented bordism rings.

The early progress was impeded by the lack of examples of higher-dimensional (stably) complex manifolds for which the characteristic numbers can be calculated explicitly. With the appearance of *toric varieties* in the late 1970s and subsequent development of toric topology (see Buchstaber and Panov [2]), a host of concrete examples of complex manifolds with large symmetry groups has been produced for which characteristic numbers can be calculated effectively using combinatorial-geometric techniques.

In [5], Buchstaber and Ray constructed a set of generators for Ω^{U} consisting entirely of complex projective toric manifolds $B(n_1,n_2)$, which are projectivisations of sums of line bundles over bounded flag manifolds. Later it was shown in Buchstaber, Panov and Ray [3] that one can get a geometric representative in *every* complex bordism class if toric manifolds are relaxed to *quasitoric* ones, the latter still have a “large torus” action, but are only stably complex instead of being complex. Characteristic numbers of toric manifolds satisfy quite restrictive conditions (eg their Todd genus is always 1) which prevent the existence of a toric representative in every bordism class; quasitoric manifolds enjoy more flexibility. We note that representing *polynomial* generators of Ω^{U} by toric manifolds remains open; some progress has been made by Wilfong[17].

Here we consider a family of projective toric manifolds obtained by iterated projectivisation of sums of line bundles, starting from a complex projective space. Such iterated projectivisations are also known as *generalised Bott manifolds* (see Masuda and Suh [12], and Buchstaber and Panov [2, Section 7.8]). Our first result (Theorem 3.8) shows that the complex bordism ring Ω^{U} can be generated by the most simple nontrivial two-stage projectivisations: manifolds $L(n_1,n_2)=\mathbb{C}P(\xi)$, where ξ is the sum of a tautological line bundle and an n_2–dimensional trivial bundle over $\mathbb{C}P^{n_1}$. This new toric generator set is somewhat simpler than either of the set of Milnor hypersurfaces $\{H(n_1,n_2)\}$ or Buchstaber and Ray's toric set $\{B(n_1,n_2)\}$.

We proceed by providing explicit families of quasitoric SU–manifolds which contain polynomial generators of the SU–bordism ring $\Omega^{\mathrm{SU}}\otimes\mathbb{Z}\left[\frac{1}{2}\right]$ (Theorem 4.19). In fact, our quasitoric SU–manifolds are genuinely indecomposable and indivisible elements in Ω^{SU} (integrally, without inverting any prime), however Ω^{SU} is not a polynomial ring.

We recall that a stably complex (or unitary) manifold M is *special unitary* (an SU–*manifold* for short) if $c_1(M)=0$. A renewed interest to this class of manifolds has been stimulated by the development of geometry motivated by physics; the notion of a *Calabi–Yau manifold* plays a central role here. By a Calabi–Yau manifold one usually understands a Kähler SU–manifold; it has a Ricci flat metric by the theorem of Yau. We note however that our SU–manifolds are rarely Kähler.

As was observed by Lü and Wang in [11], quasitoric SU–manifolds can be constructed by taking iterated complex projectivisations (which are projective toric manifolds) and then amending the stably complex structure so that the first Chern class becomes zero. The underlying smooth manifold of the result is still toric, but the stably complex structure is not the standard one. Examples of this sort were known to Conner and Floyd and used in their constructions [7], however the existence of a torus action was not emphasised and their amended stably complex structures were actually not SU.

The characteristic numbers of SU–manifolds satisfy intricate divisibility conditions. Ochanine's theorem [14] asserting that the signature of an $(8k+4)$–dimensional SU–manifold is divisible by 16 is one of the most famous examples. We therefore find it quite miraculous that polynomial generators for the SU–bordism ring Ω^{SU} occur within the most basic families of examples that one can produce using toric methods: two-stage complex projectivisations, and three-stage projectivisations with the first stage being just $\mathbb{C}P^1$. The proof of Theorem 4.19 involves calculating the characteristic numbers and checking various divisibility conditions. We use both classical and more recent results on binomial coefficients modulo a prime.

We note also that the existence of large torus actions indicates possible applications of our examples in the equivariant setting. Applicability of toric methods in equivariant bordism is currently being explored (see Buchstaber, Panov and Ray [4], Buchstaber and Panov [2, Chapter 9], and Lü [10]).

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2 Toric and quasitoric manifolds, cohomology and Chern classes

Here we collect the necessary information about toric varieties and quasitoric manifolds; the details can be found in [2].

A *toric variety* is a normal complex algebraic variety V containing an algebraic torus $(\mathbb{C}^\times)^n$ as a Zariski open subset in such a way that the natural action of $(\mathbb{C}^\times)^n$ on itself extends to an action on V. We only consider nonsingular complete (compact in the usual topology) toric varieties, also known as *toric manifolds*.

There is a bijective correspondence between the isomorphism classes of complex n–dimensional toric manifolds and complete regular fans in $\mathbb{R}^n$. A *fan* is a finite collection $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ of strongly convex cones σ_i in $\mathbb{R}^n$ such that every face of a cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each. A fan Σ is *regular* if each of its cones σ_j is generated by part of a basis of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ (we choose the standard lattice for simplicity). In particular, each 1–dimensional cone of Σ is generated by a primitive vector $\boldsymbol{a}_i \in \mathbb{Z}^n$. A fan Σ is *complete* if the union of its cones is the whole $\mathbb{R}^n$.

Projective toric varieties are particularly important. A projective toric manifold V is defined by a *lattice Delzant polytope* P. Given a simple n–dimensional polytope P with vertices in the lattice $\mathbb{Z}^n$, one defines the *normal fan* Σ_P as the fan whose n–dimensional cones σ_v correspond to the vertices v of P, and σ_v is generated by the primitive inside-pointing normals to the facets of P meeting at v. The polytope P is *Delzant* precisely when its normal fan Σ_P is regular. The fan Σ_P defines a projective toric manifold V_P. Different lattice Delzant polytopes with the same normal fan produce different projective embeddings of the same toric manifold.

Irreducible torus-invariant divisors on V are the toric subvarieties of complex codimension 1 corresponding to the 1–dimensional cones of Σ. When V is projective, they also correspond to the facets of P. We assume that there are m 1–dimensional cones (or facets), denote the corresponding primitive vectors by $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$, and denote the corresponding codimension-1 subvarieties by $V_1, \dots, V_m$.

Theorem 2.1 *Let V be a toric manifold of complex dimension n, with the corresponding complete regular fan Σ. The cohomology ring $H^*(V;\mathbb{Z})$ is generated by the degree-two classes v_i dual to the invariant submanifolds V_i, and is given by*

$$H^*(V;\mathbb{Z}) \cong \mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}, \quad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of the following two types:

(a) $v_{i_1} \cdots v_{i_k}$ *such that $\boldsymbol{a}_{i_1}, \dots, \boldsymbol{a}_{i_k}$ do not span a cone of Σ;*

(b) $\sum_{i=1}^m \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle v_i$, *for any vector $\boldsymbol{x} \in \mathbb{Z}^n$.*

It is convenient to consider the integer $n \times m$–matrix

$$\Lambda = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \tag{2-1}$$

whose columns are the vectors $\boldsymbol{a}_i$ written in the standard basis of $\mathbb{Z}^n$. Then the ideal (b) of Theorem 2.1 is generated by the n linear forms $a_{j1}v_1+\cdots+a_{jm}v_m$ corresponding to the rows of Λ.

Theorem 2.2 *There is the following isomorphism of complex vector bundles:*

$$\mathcal{T}V \oplus \underline{\mathbb{C}}^{m-n} \cong \rho_1 \oplus \cdots \oplus \rho_m,$$

where $\mathcal{T}V$ is the tangent bundle, $\underline{\mathbb{C}}^{m-n}$ is the trivial $(m-n)$–plane bundle, and ρ_i is the line bundle corresponding to V_i, with $c_1(\rho_i)=v_i$. In particular, the total Chern class of V is given by

$$c(V)=(1+v_1)\cdots(1+v_m).$$

Example 2.3 A basic example of a toric manifold is the complex projective space $\mathbb{C}P^n$. The cones of the corresponding fan are generated by proper subsets of the set of $m=n+1$ vectors $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n,-\boldsymbol{e}_1-\cdots-\boldsymbol{e}_n$, where $\boldsymbol{e}_i\in\mathbb{Z}^n$ is the i^{th} standard basis vector. It is the normal fan of the lattice simplex Δ^n with the vertices at $\mathbf{0}$ and $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$. The matrix (2-1) is given by

$$\begin{pmatrix}1&0&0&-1\\0&\ddots&0&\vdots\\0&0&1&-1\end{pmatrix}.$$

Theorem 2.1 gives the cohomology of $\mathbb{C}P^n$ as

$$H^*(\mathbb{C}P^n)\cong\mathbb{Z}[v_1,\dots,v_{n+1}]/(v_1\cdots v_{n+1},v_1-v_{n+1},\dots,v_n-v_{n+1})\cong\mathbb{Z}[v]/(v^{n+1}),$$

where v is any of the v_i. Theorem 2.2 gives the standard decomposition

$$\mathcal{T}\mathbb{C}P^n\oplus\underline{\mathbb{C}}\cong\overline{\eta}\oplus\cdots\oplus\overline{\eta}\quad(n+1\text{ summands}),$$

where $\eta=\mathcal{O}(-1)$ is the *tautological* (Hopf) line bundle over $\mathbb{C}P^n$, and $\overline{\eta}=\mathcal{O}(1)$ is its conjugate, or the line bundle corresponding to a hyperplane $\mathbb{C}P^{n-1}\subset\mathbb{C}P^n$.

Example 2.4 An example which will be important for our constructions is the complex projectivisation of a sum of line bundles over projective space.

Given two positive integers n_1, n_2 and a sequence of integers $(i_1,\dots,i_{n_2})$, consider the projectivisation $V=\mathbb{C}P(\eta^{\otimes i_1}\oplus\cdots\oplus\eta^{\otimes i_{n_2}}\oplus\underline{\mathbb{C}})$, where $\eta^{\otimes i}$ denotes the i^{th} tensor power of η over $\mathbb{C}P^{n_1}$ when $i\geqslant 0$ and the i^{th} tensor power of $\overline{\eta}$ otherwise. The manifold V is the total space of a bundle over $\mathbb{C}P^{n_1}$ with fibre $\mathbb{C}P^{n_2}$. It is also a projective toric manifold with the corresponding matrix (2-1) given by

$$\left(\begin{array}{cccccccc}
\multicolumn{3}{c}{\overbrace{\hphantom{1\quad 0\quad 0}}^{n_1}} & & & & & \\
1 & 0 & 0 & -1 & & & & \\
0 & \ddots & 0 & \vdots & & & 0 & \\
0 & 0 & 1 & -1 & & & & \\
 & & & i_1 & 1 & 0 & 0 & -1 \\
 & 0 & & \vdots & 0 & \ddots & 0 & \vdots \\
 & & & i_{n_2} & 0 & 0 & 1 & -1 \\
 & & & & \multicolumn{3}{c}{\underbrace{\hphantom{0\quad 0\quad 1}}_{n_2}} &
\end{array}\right).$$

The polytope P here is combinatorially equivalent to a product $\Delta^{n_1} \times \Delta^{n_2}$ of two simplices. Theorem 2.1 gives the cohomology of V as

$$H^*(V) \cong \mathbb{Z}[v_1, \dots, v_{n_1+1}, v_{n_1+2}, \dots, v_{n_1+n_2+2}]/\mathcal{I},$$

where $\mathcal{I}$ is generated by the elements

$$\begin{gathered} v_1 \cdots v_{n_1+1}, \quad v_{n_1+2} \cdots v_{n_1+n_2+2}, \quad v_1 - v_{n_1+1}, \quad \dots, \quad v_{n_1} - v_{n_1+1}, \\ i_1 v_{n_1+1} + v_{n_1+2} - v_{n_1+n_2+2}, \quad \dots, \quad i_{n_2} v_{n_1+1} + v_{n_1+n_2+1} - v_{n_1+n_2+2}. \end{gathered}$$

In other words,

$$H^*(V) \cong \mathbb{Z}[u, v]\big/\big(u^{n_1+1}, v(v - i_1 u) \cdots (v - i_{n_2} u)\big), \tag{2-2}$$

where $u = v_1 = \dots = v_{n_1+1}$ and $v = v_{n_1+n_2+2}$. Theorem 2.2 gives

$$c(V) = (1+u)^{n_1+1}(1 + v - i_1 u) \cdots (1 + v - i_{n_2} u)(1+v). \tag{2-3}$$

If $i_1 = \dots = i_{n_2} = 0$, we obtain $V = \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}$.

The same information can be retrieved from the following well-known description of the tangent bundle and the cohomology ring of a complex projectivisation:

Theorem 2.5 (Borel and Hirzebruch [1, Section 15]) *Let $p\colon \mathbb{C}P(\xi) \to X$ be the projectivisation of a complex n–plane bundle ξ over a complex manifold X, and let γ be the tautological line bundle over $\mathbb{C}P(\xi)$. Then there is an isomorphism of vector bundles*

$$\mathcal{T}\mathbb{C}P(\xi) \oplus \underline{\mathbb{C}} \cong p^*\mathcal{T}X \oplus (\bar{\gamma} \otimes p^*\xi),$$

where $\underline{\mathbb{C}}$ denotes a trivial line bundle over $\mathbb{C}P(\xi)$. Furthermore, the integral cohomology ring of $\mathbb{C}P(\xi)$ is the quotient of the polynomial ring $H^*(X)[v]$ on one generator $v = c_1(\bar{\gamma})$ with coefficients in $H^*(X)$ by the single relation

$$v^n + c_1(\xi)v^{n-1} + \dots + c_n(\xi) = 0. \tag{2-4}$$

The relation above is just $c_n(\overline{\gamma} \otimes p^*\xi) = 0$.

In the case considered above, $\xi = \eta^{\otimes i_1} \oplus \cdots \oplus \eta^{\otimes i_{n_2}} \oplus \underline{\mathbb{C}}$ over $X = \mathbb{C}P^{n_1}$. We then have $H^*(X) = \mathbb{Z}[u]/(u^{n_1+1})$ where $u = c_1(\overline{\eta})$, so that equation (2-4) becomes $v(v - i_1u)\cdots(v - i_{n_2}u) = 0$ and the ring $H^*(\mathbb{C}P(\xi))$ given by Theorem 2.5 is precisely (2-2). Further, the total Chern class of $p^*\mathcal{T}X \oplus (\overline{\gamma} \otimes p^*\xi)$ is given by (2-3).

The quotient of the projective toric manifold V_P by the action of the compact torus $T^n \subset (\mathbb{C}^\times)^n$ is the polytope P.

A *quasitoric manifold* over a combinatorial simple n–dimensional polytope P is a manifold M of dimension $2n$ with a locally standard action of T^n such that the quotient M/T^n is homeomorphic, as a manifold with corners, to P. (An action of T^n on M^{2n} is *locally standard* if every point $x \in M^{2n}$ is contained in a T^n–invariant neighbourhood equivariantly homeomorphic to an open subset in $\mathbb{C}^n$ with the standard coordinatewise action of T^n twisted by an automorphism of the torus; the orbit space of a locally standard action is a manifold with corners.) We therefore have a projection $\pi\colon M \to P$ whose fibres are orbits of the T^n–action.

Not every simple polytope can be the quotient of a quasitoric manifold. Nevertheless, quasitoric manifolds constitute a much larger family than projective toric manifolds, and enjoy more flexibility for topological applications.

If $F_1, \dots, F_m$ are facets of P, then each $M_i = \pi^{-1}(F_i)$ is a quasitoric submanifold of M of codimension 2, called a *characteristic submanifold*. The characteristic submanifolds $M_i \subset M$ are analogues of the invariant divisors V_i on a toric manifold V. Each M_i is fixed pointwise by a closed 1–dimensional subgroup (a subcircle) $T_i \subset T^n$ and therefore corresponds to a primitive vector $\lambda_i \in \mathbb{Z}^n$ defined up to a sign. Choosing a direction of λ_i is equivalent to choosing an orientation for the normal bundle $\nu(M_i \subset M)$ or, equivalently, choosing an orientation for M_i, provided that M itself is oriented. An *omniorientation* of a quasitoric manifold M consists of a choice of orientation for M and each characteristic submanifold M_i, $1 \leqslant i \leqslant m$.

The vectors λ_i are analogues of the generators $\boldsymbol{a}_i$ of the 1–dimensional cones of the fan corresponding to a toric manifold V (or analogues of the normal vectors to the facets of P when V is projective). However, the λ_i need not be the normal vectors to the facets of P in general.

There is an analogue of Theorem 2.1 for quasitoric manifolds:

Theorem 2.6 *Let M be an omnioriented quasitoric manifold of dimension $2n$ over a polytope P. The cohomology ring $H^*(M;\mathbb{Z})$ is generated by the degree-two* *classes v_i dual to the oriented characteristic submanifolds M_i, and is given by*

$$H^*(M;\mathbb{Z}) \cong \mathbb{Z}[v_1,\dots,v_m]/\mathcal{I}, \quad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of the following two types:

(a) $v_{i_1}\cdots v_{i_k}$ *such that* $F_{i_1}\cap\cdots\cap F_{i_k}=\varnothing$ *in* P;

(b) $\sum_{i=1}^m \langle\lambda_i,\boldsymbol{x}\rangle v_i$, *for any vector* $\boldsymbol{x}\in\mathbb{Z}^n$.

By analogy with (2-1), we consider the integer $n\times m$–matrix

(2-5) $$\Lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1m} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nm} \end{pmatrix}$$

whose columns are the vectors λ_i written in the standard basis of $\mathbb{Z}^n$. Changing a basis in the lattice results in multiplying Λ from the left by a matrix from $\mathrm{GL}(n,\mathbb{Z})$. The ideal (b) of Theorem 2.6 is generated by the n linear forms $\lambda_{j1}v_1+\cdots+\lambda_{jm}v_m$ corresponding to the rows of Λ. Also, Λ has the property that $\det(\lambda_{i_1},\dots,\lambda_{i_n})=\pm1$ whenever the facets $F_{i_1},\dots,F_{i_n}$ intersect at a vertex of P.

There is also an analogue of Theorem 2.2:

Theorem 2.7 *For a quasitoric manifold M of dimension $2n$, there is an isomorphism of real vector bundles:*

(2-6) $$\mathcal{T}M\oplus\underline{\mathbb{R}}^{2(m-n)}\cong\rho_1\oplus\cdots\oplus\rho_m,$$

where ρ_i is the real 2–plane bundle corresponding to the orientable characteristic submanifold $M_i\subset M$, so that $\rho_i|_{M_i}=\nu(M_i\subset M)$.

3 Unitary bordism

Here we provide a new set of toric generators for the unitary bordism ring. The general information about unitary (or complex) bordism can be found in [16].

Elements of the unitary bordism ring Ω^{U} are the complex bordism classes of stably complex manifolds. A *stably complex manifold* is a pair $(M,c_{\mathcal{T}})$ consisting of a smooth manifold M and a *stably complex structure* $c_{\mathcal{T}}$, where the latter is determined by a choice of an isomorphism

(3-1) $$c_{\mathcal{T}}\colon \mathcal{T}M\oplus\underline{\mathbb{R}}^N\xrightarrow{\cong}\xi$$

between the stable tangent bundle of M and a complex vector bundle ξ. We omit $c_{\mathcal{T}}$ in the notation when it is clear from the context. We denote by $[M] \in \Omega^{\mathrm{U}}$ the bordism class of a stably complex manifold M. The sum in Ω^{U} is the disjoint union, and the product is induced by the Cartesian product of manifolds. The ring Ω^{U} is graded by the dimension of manifolds.

A complex manifold M (in particular, a toric manifold) has a canonical stably complex structure arising from the complex structure on $\mathcal{T}M$. An omniorientation of a quasitoric manifold M gives it a stably complex structure by means of the isomorphism of Theorem 2.7, because a choice of orientation for each real 2–plane bundle ρ_i is equivalent to endowing it with a complex structure.

Example 3.1 The canonical stably complex structure on $\mathbb{C}P^n$ (as a complex manifold) is given by the isomorphism

$$\mathcal{T}\mathbb{C}P^n \oplus \underline{\mathbb{R}}^2 \cong \overline{\eta} \oplus \cdots \oplus \overline{\eta} \quad (n+1 \text{ summands}).$$

On the other hand, $\mathbb{C}P^n$, viewed as a quasitoric manifold over Δ^n, has $n+1$ characteristic submanifolds, and therefore 2^{n+2} different omniorientations. Each of these omniorientations gives rise to a stably complex structure, obtained by replacing some of the line bundles $\overline{\eta}$ above with η, or by reversing the global orientation. Some of these stably complex structures are equivalent, of course.

We have $H^*(BU(n)) \cong \mathbb{Z}[c_1, \dots, c_n]$, $\deg c_i = 2i$, where the c_i are the universal Chern characteristic classes. For any sequence $\omega = (i_1, \dots, i_n)$ of nonnegative integers, there is the monomial $c_\omega = c_1^{i_1} \cdots c_n^{i_n}$ of degree $2\|\omega\| = 2\sum_{k=1}^n k i_k$ and the corresponding characteristic class $c_\omega(\xi)$ of a complex n–plane bundle ξ. The corresponding tangential Chern *characteristic number* of a stably complex manifold M is defined by $c_\omega[M] = c_\omega(\mathcal{T}M)\langle M\rangle$. Here $\langle M\rangle$ is the fundamental homology class of M, and $\mathcal{T}M$ is regarded as a complex bundle via the isomorphism (3-1). The number $c_\omega[M]$ is assumed to be zero when $2\|\omega\| \neq \dim M$.

Theorem 3.2 *Two stably complex manifold M and N represent the same bordism classes in Ω^{U} if and only if their sets of Chern characteristic numbers coincide.*

Another important characteristic class is s_n. It is defined as the polynomial in $c_1, \dots, c_n$ obtained by expressing the symmetric polynomial $x_1^n + \cdots + x_n^n$ via the elementary symmetric functions $\sigma_i(x_1, \dots, x_n)$ and then replacing each σ_i by c_i. Define the corresponding characteristic number as $s_n[M] = s_n(\mathcal{T}M)\langle M\rangle$.

The ring Ω^{U} was described by Milnor and Novikov (see [13], and Stong [16]):

Theorem 3.3 *The ring Ω^{U} is a polynomial ring on generators in every even degree:*

$$\Omega^{\mathrm{U}} \cong \mathbb{Z}[a_i, i > 0], \quad \deg a_i = 2i.$$

Then the bordism class of a stably complex manifold M^{2i} may be taken to be the $2i$–dimensional generator a_i if and only if

$$s_i[M^{2i}] = \begin{cases} \pm 1 & \text{if } i+1 \neq p^s \text{ for any prime } p, \\ \pm p & \text{if } i+1 = p^s \text{ for some prime } p \text{ and integer } s > 0. \end{cases}$$

There is no universal description of connected manifolds representing the polynomial generators $a_n \in \Omega^{\mathrm{U}}$. However, there are known explicit families of manifolds whose bordism classes generate the whole ring Ω^{U}.

The classical family of generators for the ring Ω^{U} consists of the *Milnor hypersurfaces* $H(n_1, n_2)$. Each $H(n_1, n_2)$ is a hyperplane section of the Segre embedding $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \to \mathbb{C}P^{(n_1+1)(n_2+1)-1}$ and may be given explicitly by the equation

$$z_0 w_0 + \cdots + z_{n_1} w_{n_1} = 0$$

in the homogeneous coordinates $[z_0 : \cdots : z_{n_1}] \in \mathbb{C}P^{n_1}$ and $[w_0 : \cdots : w_{n_2}] \in \mathbb{C}P^{n_2}$, assuming that $n_1 \leqslant n_2$. Also, $H(n_1, n_2)$ can be identified with the projectivisation $\mathbb{C}P(\zeta)$ of a certain n_2–plane bundle over $\mathbb{C}P^{n_1}$. The bundle ζ is not a sum of line bundles when $n_1 > 1$, so $H(n_1, n_2)$ is *not* a toric manifold in this case (see [2, Section 9.1]).

Buchstaber and Ray [5] introduced a family $B(n_1, n_2)$ of *toric* generators of Ω^{U}. Each $B(n_1, n_2)$ is the projectivisation of a sum of n_2 line bundles over the *bounded flag manifold* BF_{n_1}. Then $B(n_1, n_2)$ is a toric manifold, because BF_{n_1} is toric and the projectivisation of a sum of line bundles over a toric manifold is toric.

We have $H(0, n_2) = B(0, n_2) = \mathbb{C}P^{n_2 - 1}$, so

$$s_{n_2-1}[H(0, n_2)] = s_{n_2-1}[B(0, n_2)] = n_2.$$

Furthermore,

$$s_{n_1+n_2-1}[H(n_1, n_2)] = s_{n_1+n_2-1}[B(n_1, n_2)] = -\binom{n_1+n_2}{n_1} \quad \text{for } n_1 > 1; \tag{3-2}$$

see [2, Section 9.1] for the details.

We shall need the following two facts from number theory:

Theorem 3.4 (Lucas) *Let p be a prime, and let*

$$n = n_0 + n_1 p + \cdots + n_{k-1} p^{k-1} + n_k p^k,$$
$$m = m_0 + m_1 p + \cdots + m_{k-1} p^{k-1} + m_k p^k$$

be the base p expansions of positive integers m and n. Then

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_k}{m_k} \mod p.$$

Here the standard convention $\binom{m}{n} = 0$ if $m < n$ is used.

For the proof, see eg [15, Lemma 2.6].

Proposition 3.5 *For any integer $n > 0$, we have*

$$\gcd\left\{\binom{n}{i},\ 0 < i < n\right\} = \begin{cases} 1 & \text{if} \quad n \neq p^s \quad \text{for any prime } p, \\ p & \text{if} \quad n = p^s \quad \text{for some prime } p \text{ and integer } s > 0. \end{cases}$$

Proof Assume $n = p^s$. Then each $\binom{n}{i}$ with $0 < i < n$ is divisible by p. On the other hand, $\binom{p^s}{p^{s-1}}$ is not divisible by p^2, eg by Kummer's theorem.

Now assume $n \neq p^s$. Write the base p expansion

$$n = n_0 + n_1 p + \cdots + n_{k-1} p^{k-1} + n_k p^k,$$

where we may assume $n_k > 0$. Take

$$i = n_0 + n_1 p + \cdots + n_{k-1} p^{k-1} + (n_k - 1) p^k.$$

Then $i \neq 0$ as otherwise $n = p^k$. By Theorem 3.4, $\binom{n}{i} \equiv n_k \not\equiv 0 \mod p$. □

The fact that each of the families $\{[H(n_1, n_2)]\}$ and $\{[B(n_1, n_2)]\}$ generates the unitary bordism ring Ω^{U} follows from (3-2), Proposition 3.5 and Theorem 3.3.

We proceed to describe another family of toric generators for Ω^{U}.

Construction 3.6 Given two positive integers n_1 and n_2, we define the manifold $L(n_1, n_2)$ as the projectivisation $\mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2})$, where η is the tautological line bundle over $\mathbb{C}P^{n_1}$. This $L(n_1, n_2)$ is a particular case of manifolds from Example 2.4, so it is a projective toric manifold with the corresponding matrix (2-1) given by

$$
\left(\begin{array}{ccccccccc}
\multicolumn{3}{c}{\overbrace{\hphantom{1\quad 0\quad 0}}^{n_1}} & & & & & & \\
1 & 0 & 0 & -1 & & & & & \\
0 & \ddots & 0 & \vdots & & & 0 & & \\
0 & 0 & 1 & -1 & & & & & \\
 & & & 1 & 1 & 0 & 0 & -1 \\
 & 0 & & 0 & 0 & \ddots & 0 & \vdots \\
 & & & 0 & 0 & 0 & 1 & -1 \\
 & & & & \multicolumn{3}{c}{\underbrace{\hphantom{0\quad 0\quad 1}}_{n_2}} &
\end{array}\right). \tag{3-3}
$$

The cohomology ring is given by

$$H^*(L(n_1,n_2)) \cong \mathbb{Z}[u,v]\big/(u^{n_1+1}, v^{n_2+1} - uv^{n_2}) \tag{3-4}$$

with $u^{n_1}v^{n_2}\langle L(n_1,n_2)\rangle = 1$. There is an isomorphism of complex bundles

$$\mathcal{T}L(n_1,n_2) \oplus \underline{\mathbb{C}}^2 \cong \underbrace{p^*\overline{\eta} \oplus \cdots \oplus p^*\overline{\eta}}_{n_1+1} \oplus (\overline{\gamma} \otimes p^*\eta) \oplus \underbrace{\overline{\gamma} \oplus \cdots \oplus \overline{\gamma}}_{n_2}, \tag{3-5}$$

where γ is the tautological line bundle over $L(n_1,n_2) = \mathbb{C}P(\eta \oplus \underline{\mathbb{C}}^{n_2})$. The total Chern class is

$$c(L(n_1,n_2)) = (1+u)^{n_1+1}(1+v-u)(1+v)^{n_2} \tag{3-6}$$

with $u = c_1(p^*\overline{\eta})$ and $v = c_1(\overline{\gamma})$. We also set $L(n_1,0) = \mathbb{C}P^{n_1}$ and $L(0,n_2) = \mathbb{C}P^{n_2}$, then the identities (3-4)–(3-6) still hold.

Lemma 3.7 *For $n_2 > 0$, we have*

$$s_{n_1+n_2}[L(n_1,n_2)] = \binom{n_1+n_2}{0} - \binom{n_1+n_2}{1} + \cdots + (-1)^{n_1}\binom{n_1+n_2}{n_1} + n_2.$$

Proof Using (3-6) and (3-4) we calculate

$$
\begin{aligned}
s_{n_1+n_2}(L(n_1,n_2)) &= (v-u)^{n_1+n_2} + n_2 v^{n_1+n_2} \\
&= \binom{n_1+n_2}{0} v^{n_1+n_2} - \binom{n_1+n_2}{1} uv^{n_1+n_2-1} + \cdots \\
&\qquad + (-1)^{n_1}\binom{n_1+n_2}{n_1} u^{n_1}v^{n_2} + n_2 v^{n_1+n_2} \\
&= \left(\binom{n_1+n_2}{0} - \binom{n_1+n_2}{1} + \cdots + (-1)^{n_1}\binom{n_1+n_2}{n_1} + n_2\right) u^{n_1}v^{n_2},
\end{aligned}
$$

and the result follows by evaluating at $\langle L(n_1,n_2)\rangle$. □

Theorem 3.8 *The bordism classes $[L(n_1,n_2)] \in \Omega^{\mathrm{U}}_{2(n_1+n_2)}$ generate the ring Ω^{U}.*

Proof Assuming $[L(n_1,n_2)]=0$ when $n_1<0$, we calculate using Lemma 3.7:

$$\begin{aligned} s_{n_1+n_2}\big[L(n_1,n_2)-2L(n_1-1,n_2+1)+L(n_1-2,n_2+2)\big]& \\ =(-1)^{n_1-1}\binom{n_1+n_2}{n_1-1}+(-1)^{n_1}\binom{n_1+n_2}{n_1}-2(-1)^{n_1-1}\binom{n_1+n_2}{n_1-1}& \\ =(-1)^{n_1}\binom{n_1+n_2+1}{n_1}.& \end{aligned}$$

The result follows from Proposition 3.5 and Theorem 3.3. □

Theorem 3.8 implies that any unitary bordism class can be represented by a disjoint union of products of projective toric manifolds. Products of toric manifolds are toric, but disjoint unions are not, as toric manifolds are connected. In bordism theory, a disjoint union may be replaced by a connected sum, representing the same bordism class. However, connected sum is not an algebraic operation, and a connected sum of two algebraic varieties is rarely algebraic. This can be remedied by appealing to quasitoric manifolds, as explained next. Recall that an omnioriented quasitoric manifold has an intrinsic stably complex structure, arising from the isomorphism of Theorem 2.7. One can form the equivariant connected sum of quasitoric manifolds, as explained in Davis and Januszkiewicz [8], but the resulting invariant stably complex structure does not represent the cobordism sum of the two original manifolds. A more intricate connected sum construction is needed, as outlined below. The details can be found in [3] or [2, Section 9.1].

Construction 3.9 The construction applies to two omnioriented $2n$–dimensional quasitoric manifolds M and M' over n–polytopes P and P' respectively. The connected sum will be taken at the fixed points of M and M' corresponding to vertices $v\in P$ and $v'\in P'$. We need to assume that v is the intersection of the first n facets of P, ie $v=F_1\cap\dots\cap F_n$, and the corresponding characteristic matrix (2-5) of M is in the *refined form*, ie

$$\Lambda=(I\mid\Lambda_\star)\ =\begin{pmatrix}1&0&0&\lambda_{1,n+1}&\cdots&\lambda_{1,m}\\ 0&\ddots&0&\vdots&\ddots&\vdots\\ 0&0&1&\lambda_{n,n+1}&\cdots&\lambda_{n,m}\end{pmatrix},$$

where I is the unit matrix and $\Lambda_\star$ is an $n\times(m-n)$–matrix. The same assumptions are made for M', P', v' and Λ'.

The next step depends on the *signs* of the fixed points, $\sigma(v)$ and $\sigma(v')$. The sign of v is determined by the omniorientation data; it is $+1$ when the orientation of $\mathcal{T}_vM$ induced from the global orientation of M coincides with the orientation arising from $\rho_1\oplus\dots\oplus\rho_n|_v$, and is -1 otherwise.

If $\sigma(v) = -\sigma(v')$, then we take the connected sum $M \# M'$ at v and v'. It is a quasitoric manifold over $P \# P'$ with the characteristic matrix $(\Lambda_\star \mid I \mid \Lambda'_\star)$.

If $\sigma(v) = \sigma(v')$, then we need an additional connected summand. Consider the quasitoric manifold $S = S^2 \times \cdots \times S^2$ over the n–cube I^n, where each S^2 is the quasitoric manifold over the segment I with the characteristic matrix $(1\,1)$. It represents zero in Ω^{U}, and may be thought of as $\mathbb{C}P^1$ with the stably complex structure given by the isomorphism $\mathcal{T}\mathbb{C}P^1 \oplus \underline{\mathbb{R}}^2 \cong \overline{\eta} \oplus \eta$. The characteristic matrix of S is therefore $(I \mid I)$. Now consider the connected sum $M \# S \# M'$. It is a quasitoric manifold over $P \# I^n \# P'$ with the characteristic matrix $(\Lambda_\star \mid I \mid I \mid \Lambda'_\star)$.

In either case, the resulting omnioriented quasitoric manifold $M \# M'$ or $M \# S \# M'$ with the canonical stably complex structure represents the sum of bordism classes $[M] + [M'] \in \Omega^{\mathrm{U}}_{2n}$.

The conclusion, which can be derived from the above construction and any of the toric generating sets $\{B(n_1, n_2)\}$ or $\{L(n_1, n_2)\}$ for Ω^{U}, is as follows:

Theorem 3.10 [3] *In dimensions > 2, every unitary bordism class contains a quasitoric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the torus action.*

4 Special unitary bordism

Basics

A *special unitary structure* (an SU–*structure* for short) on a manifold M is a stably complex structure $c_{\mathcal{T}}$ with a choice of an SU–structure on the complex bundle ξ; see (3-1). A stably complex manifold $(M, c_{\mathcal{T}})$ admits an SU–structure if and only if $c_1(\xi) = 0$. Bordism classes of SU-manifolds form the *special unitary bordism ring* Ω^{SU}.

The ring structure of Ω^{SU} is more subtle than that of Ω^{U}. Novikov [13] described $\Omega^{\mathrm{SU}} \otimes \mathbb{Z}[\frac{1}{2}]$ (it is a polynomial ring). The 2–torsion was described by Conner and Floyd [7]. For the description of the ring structure in Ω^{SU} (which is not a polynomial ring, even modulo torsion), see [16]. We shall need the following facts:

Theorem 4.1 (a) *The kernel of the forgetful map $\Omega^{\mathrm{SU}} \to \Omega^{\mathrm{U}}$ consists of torsion elements.*

(b) *Every torsion element in Ω^{SU} has order 2.*

(c) *$\Omega^{\mathrm{SU}} \otimes \mathbb{Z}[\frac{1}{2}]$ is a polynomial algebra on generators in every even degree > 2:*

$$\Omega^{\mathrm{SU}} \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[\tfrac{1}{2}][y_i : i > 1], \quad \deg y_i = 2i.$$

For the further analysis of the ring Ω^{SU} we need to consider an auxiliary ring $\mathcal{W}$, apparently named after C T C Wall. We describe it following [7] and [16].

Let $\partial\colon \Omega^{\mathrm{U}}_{2n} \to \Omega^{\mathrm{U}}_{2n-2}$ be the homomorphism sending a bordism class $[M^{2n}]$ to the bordism class $[V^{2n-2}]$ of a submanifold $V^{2n-2} \subset M$ dual to $c_1(M)$. There is a line bundle γ over M corresponding to $c_1(M)$, and the restriction of γ to V is the normal bundle $\nu(V \subset M)$. The stably complex structure on V is defined via the isomorphism $\mathcal{T}M|_V \cong \mathcal{T}V \oplus \nu(V \subset M)$. Then V is an SU–manifold, so $\partial^2 = 0$. The homomorphism ∂ is not a derivation of Ω^{U} though; it satisfies the identity

$$\partial(a \cdot b) = a \cdot \partial b + \partial a \cdot b - [\mathbb{C}P^1] \cdot \partial a \cdot \partial b.$$

Let $\mathcal{W}_{2n}$ be the subgroup of Ω^{U}_{2n} consisting of bordism classes $[M^{2n}]$ such that every Chern number of M^{2n} of which c_1^2 is a factor vanishes. The forgetful homomorphism decomposes as $\Omega^{\mathrm{SU}}_{2n} \to \mathcal{W}_{2n} \to \Omega^{\mathrm{U}}_{2n}$, and the restriction of the boundary homomorphism $\partial\colon \mathcal{W}_{2n} \to \mathcal{W}_{2n-2}$ is defined.

The direct sum $\mathcal{W} = \bigoplus_{i \geqslant 0} \mathcal{W}_{2i}$ is *not* a subring of Ω^{U}: one has $[\mathbb{C}P^1] \in \mathcal{W}_2$, but $c_1^2[\mathbb{C}P^1 \times \mathbb{C}P^1] = 8 \neq 0$, so $[\mathbb{C}P^1] \times [\mathbb{C}P^1] \notin \mathcal{W}_4$. However, $\mathcal{W}$ becomes a commutative ring with unit with respect to the *twisted product*

$$a * b = a \cdot b + 2[V^4] \cdot \partial a \cdot \partial b, \tag{4-1}$$

where $\cdot$ denotes the product in Ω^{U} and where V^4 is a stably complex manifold with $c_1^2[V^4] = -1$. One may take $V^4 = \mathbb{C}P^1 \times \mathbb{C}P^1 - \mathbb{C}P^2$ with the standard complex structure, or $V^4 = \overline{\mathbb{C}P^2}$ with the stably complex structure defined by the isomorphism $\mathcal{T}\mathbb{C}P^2 \oplus \underline{\mathbb{R}}^2 \cong \bar{\eta} \oplus \bar{\eta} \oplus \eta$.

We shall use the notation

$$m_i = \begin{cases} 1 & \text{if } i+1 \neq p^s \text{ for any prime } p, \\ p & \text{if } i+1 = p^s \text{ for some prime } p \text{ and integer } s > 0, \end{cases}$$

so that $[M^{2i}] \in \Omega^{\mathrm{U}}_{2i}$ represents a polynomial generator whenever $s_i[M^{2i}] = \pm m_i$.

Theorem 4.2 *$\mathcal{W}$ is a polynomial ring on generators in every even degree except 4:*

$$\mathcal{W} \cong \mathbb{Z}[x_1, x_i : i > 2], \quad x_1 = [\mathbb{C}P^1], \quad \deg x_i = 2i,$$

with $s_i[x_i] = m_i m_{i-1}$, and the boundary operator $\partial\colon \mathcal{W} \to \mathcal{W}$, $\partial^2 = 0$, given by

$$\partial x_1 = 2, \quad \partial x_{2i} = x_{2i-1},$$

satisfies the identity

$$\partial(a * b) = a * \partial b + \partial a * b - x_1 * \partial a * \partial b.$$

The forgetful map $\alpha\colon \Omega^{\mathrm{SU}} \to \mathcal{W}$ is a ring homomorphism; this follows from (4-1) because $\partial\alpha(x) = 0$ for any $x \in \Omega^{\mathrm{SU}}$.

The fundamental result relating Ω^{SU} and $\mathcal{W}$ is as follows:

Theorem 4.3 *There is an exact sequence of groups*

$$0 \longrightarrow \Omega^{\mathrm{SU}}_{2n-1} \xrightarrow{\theta} \Omega^{\mathrm{SU}}_{2n} \xrightarrow{\alpha} \mathcal{W}_{2n} \xrightarrow{\beta} \Omega^{\mathrm{SU}}_{2n-2} \xrightarrow{\theta} \Omega^{\mathrm{SU}}_{2n-1} \longrightarrow 0,$$

where θ is the multiplication by the generator $\theta \in \Omega^{\mathrm{SU}}_1 \cong \mathbb{Z}_2$, α is the forgetful homomorphism, and $\alpha\beta = -\partial$.

Analysing the exact sequence above, one obtains the following information about the free and torsion parts of Ω^{SU}:

Theorem 4.4 (a) $\mathrm{Torsion}(\Omega^{\mathrm{SU}}_n) = 0$ *unless $n = 8k+1$ or $8k+2$, in which case* $\mathrm{Torsion}(\Omega^{\mathrm{SU}}_n)$ *is a $\mathbb{Z}_2$–vector space of rank equal to the number of partitions of k.*

(b) $\Omega^{\mathrm{SU}}_{2i}/\,\mathrm{Torsion}$ *is isomorphic to* $\mathrm{Ker}(\partial\colon \mathcal{W} \to \mathcal{W})$ *if $2i \not\equiv 4 \bmod 8$ and is isomorphic to* $\mathrm{Im}(\partial\colon \mathcal{W} \to \mathcal{W})$ *if $2i \equiv 4 \bmod 8$.*

(c) *There exist* SU*–bordism classes $w_{4k} \in \Omega^{\mathrm{SU}}_{8k}$, $k \geqslant 1$, such that* $\mathrm{Im}\,\alpha/\,\mathrm{Im}\,\partial \cong \mathbb{Z}_2[w_{4k}]$. *Every torsion element of Ω^{SU} is uniquely expressible in the form $P\cdot\theta$ or $P\cdot\theta^2$ where P is a polynomial in w_{4k} with coefficients 0 or 1.*

Note that we have

$$\mathcal{W} \otimes \mathbb{Z}\bigl[\tfrac{1}{2}\bigr] \cong \mathbb{Z}\bigl[\tfrac{1}{2}\bigr][x_1, x_{2k-1}, 2x_{2k} - x_1x_{2k-1} : k > 1], \tag{4-2}$$

where $x_1^2 = x_1 * x_1$ is a ∂–cycle, and each x_{2k-1}, $2x_{2k} - x_1x_{2k-1}$ is a ∂–cycle.

Theorem 4.5 *There exist elements $y_i \in \Omega^{\mathrm{SU}}_{2i}$, $i > 1$, such that $s_i(y_i) = m_i m_{i-1}$ if i is odd, $s_2(y_2) = -48$, and $s_i(y_i) = 2m_i m_{i-1}$ if i is even and $i > 2$. These elements are mapped as follows under the forgetful homomorphism $\alpha\colon \Omega^{\mathrm{SU}} \to \mathcal{W}$:*

$$y_2 \mapsto 2x_1^2, \quad y_{2k-1} \mapsto x_{2k-1}, \quad y_{2k} \mapsto 2x_{2k} - x_1x_{2k-1}, \quad k > 1,$$

where the x_i are polynomial generators of $\mathcal{W}$. In particular, $\Omega^{\mathrm{SU}} \otimes \mathbb{Z}\bigl[\tfrac{1}{2}\bigr]$ embeds into (4-2) as the polynomial subring generated by x_1^2, x_{2k-1} and $2x_{2k} - x_1x_{2k-1}$.

Quasitoric SU–manifolds

Omnioriented quasitoric manifolds whose stably complex structures are SU can be detected using the following simple criterion:

Proposition 4.6 [4] *An omnioriented quasitoric manifold M has $c_1(M)=0$ if and only if there exists a linear function $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}$ such that $\varphi(\lambda_i)=1$ for $i=1,\dots,m$. Here the λ_i are the columns of matrix (2-5).*

In particular, if some n vectors of $\lambda_1,\dots,\lambda_m$ form the standard basis $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$, then M is SU *if and only if the column sums of Λ are all equal to 1.*

Proof By Theorem 2.7, $c_1(M)=v_1+\dots+v_m$. By Theorem 2.6, $v_1+\dots+v_m$ is zero in $H^2(M)$ if and only if

$$v_1+\dots+v_m=\sum_i \varphi(\lambda_i)v_i$$

for some linear function $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}$, whence the result follows. □

Corollary 4.7 *A toric manifold V cannot be* SU*.*

Proof If $\varphi(\lambda_i)=1$ for all i, then the vectors λ_i lie in the positive halfspace of φ, so they cannot span a complete fan. □

A more subtle result also rules out low-dimensional quasitoric manifolds:

Theorem 4.8 [4, Theorem 6.13] *A quasitoric* SU*–manifold M^{2n} represents 0 in $\varOmega^{\mathrm{U}}_{2n}$ whenever $n<5$.*

The reason for this is that the Krichever genus $\varphi_{\mathrm{K}}\colon \varOmega^{\mathrm{U}} \to R_{\mathrm{K}}$ vanishes on quasitoric SU–manifolds, but φ_{K} is an isomorphism in dimensions <10.

Examples of quasitoric SU–manifolds representing nonzero bordism classes in $\varOmega^{\mathrm{U}}_{2n}$ for all $n\geqslant 5$, except $n=6$, were constructed in [11]. We modify this construction to present two particular families of quasitoric SU–manifolds representing nonzero bordism classes in $\varOmega^{\mathrm{U}}_{2n}$ for all $n\geqslant 5$, including $n=6$.

Construction 4.9 Assume that $n_1=2k_1$ is positive even and $n_2=2k_2+1$ is positive odd, and consider the manifold $L(n_1,n_2)$ from Construction 3.6. We change the stably complex structure (3-5) to the following:

$$\begin{aligned}&\mathcal{T}L(n_1,n_2)\oplus\mathbb{R}^4\\ &\quad\cong \underbrace{p^*\overline{\eta}\oplus p^*\eta\oplus\dots\oplus p^*\overline{\eta}\oplus p^*\eta}_{2k_1}\oplus p^*\overline{\eta}\oplus(\overline{\gamma}\otimes p^*\eta)\oplus\underbrace{\overline{\gamma}\oplus\gamma\oplus\dots\oplus\overline{\gamma}\oplus\gamma}_{2k_2}\oplus\gamma\end{aligned}$$

and denote the resulting stably complex manifold by $\widetilde{L}(n_1,n_2)$. Its cohomology ring is given by the same formula (3-4), but

$$c(\widetilde{L}(n_1,n_2)) = (1-u^2)^{k_1}(1+u)(1+v-u)(1-v^2)^{k_2}(1-v), \tag{4-3}$$

so $\widetilde{L}(n_1,n_2)$ is an SU–manifold of dimension

$$2(n_1+n_2) = 4(k_1+k_2)+2.$$

Viewing $L(n_1,n_2)$ as a quasitoric manifold with the omniorientation coming from the complex structure, we see that changing a line bundle ρ_i in (2-6) to its conjugate results in changing λ_i to $-\lambda_i$ in (2-5). By applying this operation to the corresponding columns of (3-3) and then multiplying from the left by an appropriate matrix from $\mathrm{GL}(n,\mathbb{Z})$, we obtain that $\widetilde{L}(n_1,n_2)$ is the omnioriented quasitoric manifold over $\Delta^{n_1}\times\Delta^{n_2}$ corresponding to the matrix

$$\begin{pmatrix}
\begin{matrix}\overbrace{\begin{matrix}1&0&0&\cdots&0\\0&1&0&\cdots&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&0&0&1&0\\0&0&0&0&1\end{matrix}}^{n_1=2k_1}&\begin{matrix}1\\-1\\\vdots\\1\\-1\end{matrix}\end{matrix} & \mathbf{0}\\
\mathbf{0} & \begin{matrix}\begin{matrix}1\\ \\ \\ \\ \end{matrix}&\underbrace{\begin{matrix}1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\ddots&\ddots&0\\0&0&0&1\end{matrix}}_{n_2=2k_2+1}&\begin{matrix}1\\-1\\\vdots\\1\end{matrix}\end{matrix}
\end{pmatrix}.$$

The column sums of this matrix are 1 by inspection.

Construction 4.10 The previous construction can be iterated by considering projectivisations of sums of line bundles over $L(n_1,n_2)$. We shall need just one particular family of this sort.

Given positive even $n_1=2k_1$ and odd $n_2=2k_2+1$, consider the omnioriented quasitoric manifold $\widetilde{N}(n_1,n_2)$ over $\Delta^1\times\Delta^{n_1}\times\Delta^{n_2}$ with the characteristic matrix shown in Figure 1. The column sums are 1 by inspection, so $\widetilde{N}(n_1,n_2)$ is a quasitoric SU–manifold of dimension $2(1+n_1+n_2)=4(k_1+k_2)+4$.

It can be seen that $\widetilde{N}(n_1,n_2)$ is a projectivisation of a sum of n_2+1 line bundles over $\mathbb{C}P^1\times\mathbb{C}P^{n_1}$ with an amended stably complex structure.

$$\left(\begin{array}{cccccccccccccc}
1 & 1 & & & & & & & & & & & & \\
 & & 1 & 0 & 0 & \cdots & 0 & 1 & & & & & & \\
 & & 0 & 1 & 0 & \cdots & 0 & -1 & & & & & & \\
\mathbf{0} & & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & \mathbf{0} & & & \\
 & & 0 & 0 & 0 & 1 & 0 & 1 & & & & & & \\
 & & 0 & 0 & 0 & 0 & 1 & -1 & & & & & & \\
 & -1 & \multicolumn{5}{c}{n_1 = 2k_1} & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\
 & 1 & & & & & & 0 & 0 & 1 & 0 & \cdots & 0 & -1 \\
 & 0 & & & & & & 1 & 0 & 0 & 1 & \cdots & 0 & 1 \\
 & & & & \mathbf{0} & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & & & & & & 0 & 0 & 0 & 0 & 1 & 1 \\
 & & & & & & & & \multicolumn{5}{c}{n_2 = 2k_2 + 1} &
\end{array}\right).$$

Figure 1: Characteristic matrix for the omnioriented quasitoric manifold $\widetilde{N}(n_1, n_2)$ over $\Delta^1 \times \Delta^{n_1} \times \Delta^{n_2}$

The cohomology ring given by Theorem 2.6 is

$$H^*(\widetilde{N}(n_1, n_2)) \cong \mathbb{Z}[u, v, w]\big/\big(u^2, v^{n_1+1}, (w-u)^2(v+w)w^{n_2-2}\big) \tag{4-4}$$

with $uv^{n_1}w^{n_2}\langle\widetilde{N}(n_1, n_2)\rangle = 1$. The total Chern class is

$$c(\widetilde{N}(n_1, n_2)) = (1-v^2)^{k_1}(1+v)(1-(w-u)^2)(1-v-w)(1-w^2)^{k_2-1}(1+w). \tag{4-5}$$

Quasitoric representatives for polynomial generators of $\Omega^{SU} \otimes \mathbb{Z}\left[\frac{1}{2}\right]$

Our goal is to show that elements $y_i \in \Omega^{SU}_{2i}$ described in Theorem 4.5 can be represented by quasitoric SU–manifolds when $i \geqslant 5$. This will be done by calculating the characteristic numbers of $\widetilde{L}(n_1, n_2)$ and $\widetilde{N}(n_1, n_2)$ and then checking several divisibility conditions for binomial coefficients. We shall need the following generalisation of Lucas' theorem:

Theorem 4.11 [9, Theorem 1] *Suppose that prime power p^q and positive integers $m = n + r$ are given. Write*

$$n = n_0 + n_1 p + \cdots + n_{k-1} p^{k-1} + n_k p^k$$

in base p, and let

$$N_j = n_j + n_{j+1} p + \cdots + n_{j+q-1} p^{q-1}, \quad j \geqslant 0.$$

Also make the corresponding definitions for m_j, M_j, r_j, R_j. Let e_j be the number of indices $i \geqslant j$ for which $n_i < m_i$. Then

$$\frac{1}{p^{e_0}}\binom{n}{m} \equiv (\pm 1)^{e_{q-1}} \frac{N_0!_p}{M_0!_p\, R_0!_p} \cdot \frac{N_1!_p}{M_1!_p\, R_1!_p} \cdots \frac{N_k!_p}{M_k!_p\, R_k!_p} \mod p^q,$$

where ± 1 is -1 except if $p = 2$ and $q \geqslant 3$, and $n!_p$ denotes the product of those positive integers $\leqslant n$ that are not divisible by p.

Lemma 4.12 *For $n_1 = 2k_1 > 0$ and $n_2 = 2k_2 + 1 > 0$, we have*

$$s_{n_1+n_2}[\widetilde{L}(n_1,n_2)] = -\binom{n_1+n_2}{1} + \binom{n_1+n_2}{2} - \cdots - \binom{n_1+n_2}{n_1-1} + \binom{n_1+n_2}{n_1}.$$

Proof Using (4-3) and (3-4) we calculate

$$\begin{aligned}
&s_{n_1+n_2}(\widetilde{L}(n_1,n_2)) \\
&\qquad = (v-u)^{n_1+n_2} + (k_2+1)(-1)^{n_1+n_2} v^{n_1+n_2} + k_2 v^{n_1+n_2} \\
&\qquad = (v-u)^{n_1+n_2} - v^{n_1+n_2} \\
&\qquad = \Bigl(-\binom{n_1+n_2}{1} + \binom{n_1+n_2}{2} - \cdots - \binom{n_1+n_2}{n_1-1} + \binom{n_1+n_2}{n_1}\Bigr) u^{n_1} v^{n_2},
\end{aligned}$$

and the result follows by evaluating at $\langle \widetilde{L}(n_1,n_2) \rangle$. □

Note that $s_3(\widetilde{L}(2,1)) = 0$ in accordance with Theorem 4.8. On the other hand, $s_{2+n_2}(\widetilde{L}(2,n_2)) \neq 0$ for $n_2 > 1$, providing an example of a non-bounding quasitoric SU–manifold in each dimension $4k+2$ with $k > 1$.

Lemma 4.13 *For $k > 1$, there is a linear combination y_{2k+1} of SU–bordism classes $[\widetilde{L}(n_1,n_2)]$ with $n_1 + n_2 = 2k+1$ such that $s_{2k+1}(y_{2k+1}) = m_{2k+1}m_{2k}$.*

Proof By the previous lemma,

$$s_{n_1+n_2}[\widetilde{L}(n_1,n_2) - \widetilde{L}(n_1-2,n_2+2)] = \binom{n_1+n_2}{n_1} - \binom{n_1+n_2}{n_1-1}.$$

The result follows from the next lemma. □

Lemma 4.14 *For any integer $k > 1$, we have*

$$\gcd\Bigl\{\binom{2k+1}{2i} - \binom{2k+1}{2i-1},\ 0 < i \leqslant k\Bigr\} = m_{2k+1}m_{2k}.$$

Remark This result has been obtained independently in a recent work of Buchstaber and Ustinov on the coefficient rings of universal formal group laws [6, Section 9].

Proof of Lemma 4.14 We need to establish the following two facts:

(a) The largest power of 2 which divides each number $\binom{2k+1}{2i} - \binom{2k+1}{2i-1}$ with $0 < i \leqslant k$ is 2 if $2k+2 = 2^s$ and is 1 otherwise.

(b) The largest power of odd prime p which divides each number $\binom{2k+1}{2i} - \binom{2k+1}{2i-1}$ with $0 < i \leqslant k$ is p if $2k+1 = p^s$ and is 1 otherwise.

We prove (a) first.

Case 1 $(2k+2 = 2^s)$ Then $s > 2$, as $k > 1$. For $0 < i \leqslant k$, we have

$$\binom{2k+1}{2i} - \binom{2k+1}{2i-1} \equiv \binom{2^s-1}{2i} + \binom{2^s-1}{2i-1} = \binom{2^s}{2i} \equiv 0 \mod 2$$

by Proposition 3.5. On the other hand,

$$\binom{2^s-1}{2} - \binom{2^s-1}{1} = (2^s-1)(2^{s-1}-1-1) = 2(2^s-1)(2^{s-2}-1) \not\equiv 0 \mod 4.$$

Case 2 $(2k+2 \neq 2^s)$ Write the base 2 expansion

$$2k+2 = n_1 2 + \cdots + n_{l-1} 2^{l-1} + 2^l$$

with $n_i = 1$ or 0. Set $2i = n_1 2 + \cdots + n_{l-1} 2^{l-1}$. Then we have $2i \neq 0$, as otherwise $2k+2 = 2^l$. Then $\binom{2k+1}{2i} - \binom{2k+1}{2i-1} \equiv \binom{2k+2}{2i} \equiv 1 \bmod 2$ by Theorem 3.4.

Now we prove (b).

Case 1 $(2k+1 = p^s)$ Then $\binom{2k+1}{2i} - \binom{2k+1}{2i-1} \equiv 0 \bmod p$ for $0 < i \leqslant k$ by Proposition 3.5. On the other hand, setting $2i = p^{s-1}+1$, we get

$$\begin{aligned}\binom{2k+1}{2i} - \binom{2k+1}{2i-1} &= \frac{2k-4i+2}{2i}\binom{2k+1}{2i-1} \\ &= \frac{p^s - 2p^{s-1} - 1}{p^{s-1}+1}\binom{p^s}{p^{s-1}} \not\equiv 0 \mod p^2.\end{aligned}$$

This follows from the fact that $p^s - 2p^{s-1} - 1 > 0$ as $k > 1$, and $\binom{p^s}{p^{s-1}}$ is not divisible by p^2 by Kummer's theorem.

Case 2 $(2k+1 \neq p^s)$ Write the base p expansion

$$2k+1 = n_0 + n_1 p + \cdots + n_{l-1} p^{l-1} + n_l p^l$$

with $0 \leqslant n_i \leqslant p-1$ (for $i = 0, \dots, l$) and $n_l > 0$.

Assume that $n_0 > 1$. Then we set

$$2i = n_0 + n_1 p + \cdots + n_{l-1} p^{l-1} + (n_l - 1) p^l.$$

We have $\binom{2k+1}{2i} \equiv n_l \bmod p$ by Theorem 3.4. Also,

$$2i-1 = (n_0-1) + n_1 p + \cdots + n_{l-1} p^{l-1} + (n_l-1) p^l > 0$$

and $\binom{2k+1}{2i-1} \equiv n_l n_0 \bmod p$. Therefore, $\binom{2k+1}{2i} - \binom{2k+1}{2i-1} \equiv n_l(1-n_0) \not\equiv 0 \bmod p$.

Assume that $n_0 = 1$. Then we set $2i = 2k$. We have

$$\binom{2k+1}{2k} = 2k+1 \equiv 1 \mod p \qquad \text{and} \qquad \binom{2k+1}{2k-1} = k(2k+1) \equiv 0 \mod p,$$

so that

$$\binom{2k+1}{2k} - \binom{2k+1}{2k-1} \not\equiv 0 \mod p.$$

Finally, assume that $n_0 = 0$. Then we set

$$\begin{aligned} 2i &= n_0 + n_1 p + \cdots + n_{l-1} p^{l-1} + (n_l - 1) p^l \\ &= n_q p^q + \cdots + n_{l-1} p^{l-1} + (n_l-1) p^l, \end{aligned}$$

where $q > 0$ and $n_q > 0$. We have $2i > 0$, as otherwise $2k+1 = p^l$. Then $\binom{2k+1}{2i} \equiv n_l \bmod p$. Also,

$$2i-1 = (p-1) + (p-1)p + \cdots + (p-1)p^{q-1} + (n_q-1)p^q + \cdots + n_{l-1}p^{l-1} + (n_l-1)p^l,$$

and $\binom{2k+1}{2i-1} \equiv 0 \bmod p$ by Theorem 3.4. Therefore, $\binom{2k+1}{2i} - \binom{2k+1}{2i-1} \not\equiv 0 \bmod p$. □

Now we turn our attention to the manifolds $\widetilde{N}(n_1, n_2)$ from Construction 4.10.

Lemma 4.15 *For $n_1 = 2k_1 > 0$ and $n_2 = 2k_2 + 1 > 0$, set $n = n_1 + n_2 + 1$, so that $\dim \widetilde{N}(n_1, n_2) = 2n = 4(k_1 + k_2 + 1)$. Then*

$$s_n\big[\widetilde{N}(n_1,n_2)\big] = 2\Big(-\binom{n}{1} + \binom{n}{2} - \cdots - \binom{n}{n_1-1} + \binom{n}{n_1} - n_1\Big).$$

Proof Using (4-5) and (4-4) we calculate

$$\begin{aligned} s_n\big(\widetilde{N}(n_1,n_2)\big) &= 2(w-u)^n + (v+w)^n + (2k_2-1)w^n \\ &= 2w^n - 2nuw^{n-1} + w^n + \binom{n}{1} v w^{n-1} + \cdots \\ &\qquad + \binom{n}{2k_1} v^{2k_1} w^{2k_2+2} + (2k_2-1) w^n \\ &= -2nuw^{n-1} + (n-n_1)w^n + \binom{n}{1} v w^{n-1} + \cdots + \binom{n}{n_1} v^{n_1} w^{n-n_1}. \end{aligned} \tag{4-6}$$

Now we have to express each monomial above via $uv^{n_1}w^{n_2}$ using the identities in (4-4), namely

$$u^2 = 0, \quad v^{n_1+1} = 0, \quad w^{n_2+1} = 2uw^{n_2} - vw^{n_2} + 2uvw^{n_2-1}. \tag{4-7}$$

We have

$$\begin{aligned} uw^{n-1} &= uw^{n_1-1}w^{n_2+1} = uw^{n_1-1}(2uw^{n_2} - vw^{n_2} + 2uvw^{n_2-1}) \\ &= -uvw^{n-2} = \cdots = (-1)^j uv^j w^{n-j-1} = \cdots = uv^{n_1}w^{n_2}. \end{aligned} \tag{4-8}$$

Also, we show that

$$v^j w^{n-j} = (-1)^j 2uv^{n_1}w^{n_2}, \quad 0 \leqslant j \leqslant n_1, \tag{4-9}$$

by verifying the identity successively for $j = n_1, n_1 - 1, \dots, 0$. Indeed,

$$v^{n_1}w^{n-n_1} = v^{n_1}w^{n_2+1} = 2uv^{n_1}w^{n_2}$$

by (4-7). Now, we have

$$\begin{aligned} v^{j-1}w^{n-j+1} &= v^{j-1}w^{n_1+1-j}w^{n_2+1} = v^{j-1}w^{n_1+1-j}(2uw^{n_2} - vw^{n_2} + 2uvw^{n_2-1}) \\ &= 2uv^{j-1}w^{n-j} - v^j w^{n-j} + 2uv^j w^{n-1-j} = -v^j w^{n-j}, \end{aligned}$$

where the last identity holds because of (4-8). The identity (4-9) is therefore verified completely. Plugging (4-8) and (4-9) into (4-6) we obtain

$$\begin{aligned} &s_n(\widetilde{N}(n_1,n_2)) \\ &\quad = \left(-2n + 2(n-n_1) - 2\binom{n}{1} + 2\binom{n}{2} - \cdots - 2\binom{n}{n_1-1} + 2\binom{n}{n_1}\right)uv^{n_1}w^{n_2}. \end{aligned}$$

The result follows by evaluating at $\langle \widetilde{N}(n_1,n_2)\rangle$. □

Note that $s_4(\widetilde{N}(2,1)) = 0$ in accordance with Theorem 4.8. On the other hand, $s_n(\widetilde{N}(2,n_2)) = n^2 - 3n - 4 > 0$ for $n > 4$, providing an example of a non-bounding quasitoric SU–manifold in each dimension $4k$ with $k > 2$. This includes a 12–dimensional quasitoric SU–manifold $\widetilde{N}(2,3)$, which was missing in [11].

Lemma 4.16 *For $k > 2$, there is a linear combination y_{2k} of* SU*–bordism classes $[\widetilde{N}(n_1,n_2)]$ with $n_1 + n_2 + 1 = 2k$ such that $s_{2k}(y_{2k}) = 2m_{2k}m_{2k-1}$.*

Proof The result follows from Lemma 4.15 and Lemmata 4.17 and 4.18 below. □

Lemma 4.17 *For $k > 2$, the largest power of 2 which divides each number*

$$a_i = -\binom{2k}{1} + \binom{2k}{2} - \cdots - \binom{2k}{2i-1} + \binom{2k}{2i} - 2i, \quad 0 < i < k,$$

is 2 if $2k = 2^s$ and is 1 otherwise.

Proof First assume that $2k = 2^s$. Then $a_i \equiv 0 \bmod 2$ by Proposition 3.5. On the other hand, we have

$$a_1 = -2^s + 2^{s-1}(2^s - 1) - 2 \not\equiv 0 \mod 4,$$

because $s > 2$.

Now assume that $2k \neq 2^s$. We have $a_i - a_{i-1} \equiv \binom{2k}{2i} \bmod 2$, so it is enough to find i such that $\binom{2k}{2i} \neq 0 \bmod 2$. This was done in the proof of Lemma 4.14. □

Lemma 4.18 *For $k > 2$, the largest power of odd prime p which divides each*

$$a_i = -\binom{2k}{1} + \binom{2k}{2} + \cdots - \binom{2k}{2i-1} + \binom{2k}{2i} - 2i, \quad 0 < i < k,$$

is p if $2k+1 = p^s$ and is 1 otherwise.

Proof Using the identity

$$2 + \sum_{j=1}^{2k-1} (-1)^j \binom{2k}{j} = 0$$

we obtain $a_{k-1} = 0$ and

$$a_i + a_{k-i-1} = \binom{2k}{2i+1} - 2k, \quad 0 < i < k-1. \tag{4-10}$$

Case 1 $(2k+1 = p^s)$ We have

$$\binom{2k}{2i+1} = \binom{p^s-1}{2i+1} = \binom{p^s-1}{2i-1}\frac{(p^s-2i)(p^s-2i-1)}{2i(2i+1)} \equiv -1 \mod p$$

by induction starting from $i = 0$. Therefore,

$$a_i = a_{i-1} - \binom{2k}{2i-1} + \binom{2k}{2i} - 2 = a_{i-1} + \frac{p^s - 4i}{2i}\binom{p^s-1}{2i-1} - 2 \equiv 0 \mod p$$

by induction starting from $a_0 = 0$. In view of (4-10), it suffices to find i, where $0 < i < k-1$, such that

$$\binom{2k}{2i+1} - 2k \not\equiv 0 \bmod p^2.$$

If $s = 1$, then $p > 5$ as $k > 2$. We set $2i+1 = 3$, so that

$$\binom{2k}{2i+1} - 2k = \binom{p-1}{3} - (p-1) = \frac{p(p-1)(p-5)}{6} \not\equiv 0 \mod p^2.$$

Now assume that $s>1$. We set $2i+1=p^{s-1}$ and use Theorem 4.11 to calculate $\binom{2k}{2i+1} \bmod p^2$. In the notation of Theorem 4.11, we have $q=2$,

$$\begin{aligned} n=p^s-1&=n_0+n_1p+\cdots+n_{s-2}p^{s-2}+n_{s-1}p^{s-1}\\ &=(p-1)+(p-1)p+\cdots+(p-1)p^{s-2}+(p-1)p^{s-1},\\ N_0&=\cdots=N_{s-2}=p^2-1, \qquad N_{s-1}=p-1,\end{aligned}$$

$$\begin{aligned} m=p^{s-1}&=m_0+m_1p+\cdots+m_{s-2}p^{s-2}+m_{s-1}p^{s-1},\\ M_0&=\cdots=M_{s-3}=0, \qquad M_{s-2}=p, \qquad M_{s-1}=1,\end{aligned}$$

$$\begin{aligned} r=p^s-p^{s-1}-1&=r_0+r_1p+\cdots+r_{s-2}p^{s-2}+r_{s-1}p^{s-1}\\ &=(p-1)+(p-1)p+\cdots+(p-1)p^{s-2}+(p-2)p^{s-1},\\ R_0&=\cdots=R_{s-3}=p^2-1, \qquad R_{s-2}=p^2-p-1, \qquad R_{s-1}=p-2,\end{aligned}$$

and $e_0=e_1=0$. Therefore, Theorem 4.11 gives

$$\begin{aligned}\binom{p^s-1}{p^{s-1}} &\equiv \frac{(p^2-1)!_p}{p!_p\,(p^2-p-1)!_p}\cdot\frac{(p-1)!_p}{1!_p\,(p-2)!_p}=\frac{(p^2-1)\cdots(p^2-p+1)}{(p-1)!}\cdot(p-1)\\ &\equiv p-1 \mod p^2,\end{aligned}$$

and we obtain

$$\binom{2k}{2i+1}-2k=\binom{p^s-1}{p^{s-1}}-(p^s-1)\equiv p \mod p^2.$$

Case 2 $(2k+1\neq p^s)$ In view of (4-10), it suffices to find i, where $0<i<k-1$, such that $\binom{2k}{2i+1}-2k\not\equiv 0 \bmod p$. Write the base p expansion

$$2k=n_0+n_1p+\cdots+n_{l-1}p^{l-1}+n_lp^l$$

with $0\leqslant n_i\leqslant p-1$ (for $i=0,\dots,l$) and $n_l>0$. We have $2k\equiv n_0 \bmod p$.

Assume that $n_0=0$. Then we set

$$2i+1=n_0+n_1p+\cdots+n_{l-1}p^{l-1}+(n_l-1)p^l.$$

We have $\binom{2k}{2i+1}-2k\equiv n_l\not\equiv 0 \bmod p$.

Assume that $0<n_0<p-2$. If $2k<p$, then $n_0=2k>5$. We set $2i+1=3$, so that

$$\binom{2k}{2i+1}-2k\equiv\binom{n_0}{3}-n_0=\frac{n_0(n_0-4)(n_0+1)}{6}\not\equiv 0 \mod p. \tag{4-11}$$

If $2k > p$, then we set

$$2i+1=\begin{cases} n_0+1 & \text{if } n_0 \text{ is even,}\\ n_0+2 & \text{if } n_0 \text{ is odd.}\end{cases}$$

We have $2i+1<2k$ and $\binom{2k}{2i+1}-2k \equiv -n_0 \not\equiv 0 \bmod p$.

Assume that $n_0 = p-2$. If $p=3$, then $n_0=1$. We set $2i+1=5<2k$, so that

$$\binom{2k}{2i+1}-2k \equiv \binom{n_0}{2}\binom{n_1}{1}-1=-1\not\equiv 0 \bmod p.$$

If $p>3$, then we set $2i+1=3$, so that $\binom{2k}{2i+1}-2k \not\equiv 0 \bmod p$ by (4-11).

Assume that $n_0 = p-1$ and $n_l < p-1$. Then we set

$$2i+1 = n_0+n_1p+\cdots+n_{l-1}p^{l-1}+(n_l-1)p^l.$$

We have $\binom{2k}{2i+1}-2k \equiv n_l-n_0 \not\equiv 0 \bmod p$.

Finally, assume that $n_0=p-1$ and $n_l=p-1$. As $2k \neq p^s-1$, there exists j, where $0<j<l$, such that $n_j<p-1$. Then we set

$$2i+1 = n_0+n_1p+\cdots+n_{j-1}p^{j-1}+(n_j+1)p^j.$$

We have $2i+1<2k$ and $\binom{2k}{2i+1}-2k \equiv -n_0 \not\equiv 0 \bmod p$. □

We now can prove our main result:

Theorem 4.19 *There exist quasitoric* SU*–manifolds M^{2i}, $i \geqslant 5$, with $s_i(M^{2i}) = m_im_{i-1}$ if i is odd and $s_i(M^{2i}) = 2m_im_{i-1}$ if i is even. These quasitoric manifolds represent polynomial generators of $\Omega^{\mathrm{SU}} \otimes \mathbb{Z}[\frac{1}{2}]$.*

Proof It follows from Lemmata 4.13 and 4.16 that there exist linear combinations of SU–bordism classes represented by quasitoric SU–manifolds with the required properties. We observe that application of Construction 3.9 to two quasitoric SU–manifolds M and M' produces a quasitoric SU–manifold representing their bordism sum. Also, the SU–bordism class $-[M]$ can be represented by the omnioriented quasi-toric SU–manifold obtained by reversing the global orientation of M. Therefore, we can replace the linear combinations obtained using Lemmata 4.13 and 4.16 by appropriate connected sums, which are quasitoric SU–manifolds. □

Concluding remarks

By analogy with Theorem 3.10, we may ask the following:

Question 4.20 Which SU–bordism classes of dimension > 8 can be represented by quasitoric SU–manifolds?

Theorem 4.19 provides quasitoric representatives for the elements $y_i \in \Omega^{SU}_{2i}$ described in Theorem 4.5 for $i \geqslant 5$. The elements y_2, y_3, y_4 cannot be represented by quasitoric manifolds because of Theorem 4.8. Any polynomial in these elements cannot be represented by a quasitoric manifold for the same reason: the Krichever genus $\varphi_K\colon \Omega^U \to R_K$ vanishes on quasitoric SU–manifolds, but φ_K is nonzero on any polynomial in y_2, y_3, and y_4. We thank Michael Wiemeler for this observation.

The element $x_1^2 \in \mathcal{W}_4$ (see Theorem 4.2) is represented by $9\mathbb{C}P^1 \times \mathbb{C}P^1 - 8\mathbb{C}P^2$, which is also the bordism class of a toric manifold over a 12–gon, with characteristic numbers $c_1^2 = 0$ and $c_2 = 12$ (so $s_2 = -24$). The element $y_2 = 2x_1^2 \in \Omega^{SU}_4$ is represented by a *K3 surface*, but not by a toric manifold.

The 6–sphere S^6 has a T^2–invariant almost complex structure as the homogeneous space $G_2/SU(3)$ of the exceptional Lie group G_2 (see [1]), and therefore represents an SU–bordism class in Ω^{SU}_6. Its characteristic numbers are $c_1^3 = c_1c_2 = 0$ and $c_3 = 2$. Therefore, $s_3[S^6] = 6 = m_3m_2$, so S^6 represents $y_3 \in \Omega^{SU}_6$.

It would be interesting to find good geometric representatives for $y_4 \in \Omega^{SU}_8$, and also for the elements $w_{4k} \in \Omega^{SU}_{8k}$ that control the 2–torsion in Theorem 4.4(c). The image of w_{4k} under the forgetful homomorphism $\alpha\colon \Omega^{SU}_{8k} \to \mathcal{W}_{8k}$ is x_1^{4k}, so it is decomposable in Ω^U and has $s_{4k}[w_{4k}] = 0$. The conditions on the characteristic numbers specifying w_{4k} are given in [7, (19.3)].

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Higher rank lattices are not coarse median

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We show that symmetric spaces and thick affine buildings which are not of spherical type A_1^r have no coarse median in the sense of Bowditch. As a consequence, they are not quasi-isometric to a CAT(0) cube complex, answering a question of Haglund. Another consequence is that any lattice in a simple higher rank group over a local field is not coarse median.

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Introduction

A metric space (X,d) is called metric median if for each $(x,y,z) \in X^3$, the three intervals $I(x,y)$, $I(y,z)$ and $I(x,z)$ intersect in a single point, where the interval $I(x,y)$ is given by $\{p \in X \mid d(x,p)+d(p,y)=d(x,y)\}$. This point is called the median of x, y and z. The rank of (X,d) is then defined as the maximal dimension r of an embedded cube $\{0,1\}^r$. The relationship between groups and median metric spaces is rich and has been studied through many points of view, such as the Haagerup property, property (T), actions on a CAT(0) cube complex, and actions on a space with (measured) walls. (See Chepoi [12], Chatterji, Druțu and Haglund [8], Chatterji, Fernós and Iozzi [9], Chatterji and Niblo [10], and Bowditch [5; 6], for example.)

Bowditch recently introduced the notion of a coarse median on a metric space (see [3]), in order to gather in the same setting CAT(0) cube complexes and Gromov hyperbolic spaces. A metric space is Gromov-hyperbolic if and only if every finite subset admits a good metric comparison with a tree (see for instance Ghys and de la Harpe [14, Théorème 12, page 33]). Bowditch's definition of a coarse median is having a good metric comparison of every finite subset with a metric median space, or equivalently with a CAT(0) cube complex according to Chepoi (see [12]).

Definition (Bowditch) Let (X,d) be a metric space. A map $\mu \to X^3 \to X$ is called a *coarse median* if there exist $k \in [0,+\infty)$ and $h \to \mathbb{N} \to [0,+\infty)$ such that:

- For all $a,b,c,a',b',c' \in X$, we have
$$d\big(\mu(a,b,c),\mu(a',b',c')\big) \leqslant k\big(d(a,a')+d(b,b')+d(c,c')\big)+h(0).$$

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- For each finite nonempty set $A \subset X$, with $|A| \leqslant p$, there exists a finite median algebra (Π, μ_Π) and maps $\pi\colon A \to \Pi$ and $\lambda\colon \Pi \to X$ such that for every $x, y, z \in X$, we have
$$d(\lambda\mu_\Pi(x, y, z), \mu(\lambda x, \lambda y, \lambda z)) \leqslant h(p),$$
and for every $a \in A$, we have $d(a, \lambda(\pi(a))) \leqslant h(p)$.

If Π can be chosen (independently of p) to have rank at most r, we say that μ has rank at most r.

A finitely generated group is said to be coarse median if some Cayley graph has a coarse median (not necessarily equivariant under the group action). Bowditch showed that a coarse median group is finitely presented, and has at most quadratic Dehn function (see [3, Corollary 8.3]). Furthermore, Chatterji and Ruane's criterion (see [11]) applies to show that a coarse median group of finite rank has the *rapid decay* (RD) property (see [5, Theorem 9.1]). Moreover, if a group has a coarse median of rank at most r, there is no quasi-isometric embedding of $\mathbb{R}^{r+1}$ into that group. Bowditch also showed that the existence of a coarse median is a quasi-isometry invariant, that a group is Gromov hyperbolic if and only if it is coarse median of rank 1, and that a group hyperbolic relative to coarse median groups is itself coarse median (see [4]). Furthermore, Bowditch showed that the mapping class group of a surface of genus g with p punctures is coarse median of rank $3g-3+p$, hereby recovering Behrstock and Minsky's result that the mapping class group has property (RD) (see [2]), and the rank theorem (see Hammenstädt [16] and Behrstock and Minsky [1]).

Since most known examples of coarse median groups have some nonpositive curvature features, Bowditch asked in [3] whether higher rank symmetric spaces, or even CAT(0) spaces, admit coarse medians. In this article, we give a negative answer to this question.

Theorem A *Let X be a thick affine building of spherical type different from A_1^r. There is no locally convex Lipschitz median on X.*

By considering asymptotic cones and using work of Kleiner and Leeb, and of Bowditch, we deduce the following:

Theorem B *Let X be a symmetric space of noncompact type, or a thick affine building, of spherical type different from A_1^r. Then X has no coarse median.*

A consequence of this result is the classification of symmetric spaces of noncompact type and affine buildings which are coarse median.

Theorem C *Let X be a symmetric space of noncompact type, or a thick affine building. There exists a coarse median on X if and only if the spherical type of X is A_1^r.*

Note that the coarse median is not assumed to be equivariant by any group.

Haglund asked if a higher rank symmetric space or affine building is quasi-isometric to a $\mathrm{CAT}(0)$ cube complex, and we give a negative answer:

Theorem D *Let X be a symmetric space of noncompact type, or a discrete, thick affine building. Then X is quasi-isometric to a CAT(0) cube complex if and only if the spherical type of X is A_1^r*

Note that the $\mathrm{CAT}(0)$ cube complex is not assumed to be of finite dimension, and it could also be endowed with the L^p distance for any $p \in [1, \infty]$.

Also note that Theorem D still holds if we consider nondiscrete thick affine buildings and nondiscrete $\mathrm{CAT}(0)$ cube complexes.

Furthermore, for uniform lattices in semisimple Lie groups, property (RD) implies the Baum–Connes conjecture without coefficient (see Lafforgue [20]). Property (RD) has been proved notably for uniform lattices in $\mathrm{SL}(3, \mathbb{K})$, where $\mathbb{K}$ is a local field (see Ramagge, Robertson and Steger [23], Lafforgue [21] and Chatterji [7]). Valette conjectured that uniform lattices in semisimple Lie groups satisfy property (RD). Since being coarse median implies property (RD), one could ask if this could be a way to prove property (RD) for higher rank uniform lattices. Even though it might follow from [23] that looking only at coarse medians is not enough for $\mathrm{SL}(3, \mathbb{K})$, the following makes it clear.

Theorem E *Let $\mathbb{K}$ be a local field, let G be the group of $\mathbb{K}$–points of a simple algebraic group without compact factors and let Γ be a lattice in G. If Γ is coarse median, then G has $\mathbb{K}$–rank 1.*

Note that, due to property (T), higher rank lattices do not admit unbounded actions on median metric spaces (see [8]). But in the coarse median setting this is not a consequence of property (T), since, for instance, every hyperbolic group with property (T) is coarse median.

In the $\mathbb{K}$–rank 1 case, finding which nonuniform lattices are coarse median is harder. Here we summarize what is known.

Proposition F *Let $\mathbb{K}$ be a local field, let G be the group of $\mathbb{K}$–points of a simple algebraic group without compact factors of $\mathbb{K}$–rank 1, and let Γ be a lattice in G.*

- *If Γ is uniform in G, then Γ is coarse median.*
- *If G is locally isomorphic to $\mathrm{SO}_0(n,1)$ for some $n \geqslant 2$, then Γ is coarse median.*
- *If G is locally isomorphic to $\mathrm{SU}(2,1)$, then Γ is not coarse median.*

In the proof of Theorem A, we establish the following result, which is of independent interest:

Proposition G *Let X be a connected metric space, with a Lipschitz locally convex median of rank r. There exists a median, bi-Lipschitz embedding of the r–cube $[0,1]^r$ into X with convex image.*

Organization of the paper In Section 1, we recall the general definitions of median algebras. In Section 2, we recall work of Kleiner and Leeb and of Bowditch on asymptotic cones, and we prove that Theorem A implies Theorems B and C.

Sections 3 and 4 are devoted to the proof of Theorem A. We consider a thick affine building X which has a locally convex Lipschitz median. In Section 3 we prove Proposition G, which provides us with a convex cube in X. In Section 4, by considering a tangent cone of X in the cube we can assume that some apartment F of X is isomorphic to a vector space with the standard L^1 median. Considering singular geodesics in F, we prove that X has spherical type A_1^r.

Finally in Section 5, we prove the main consequences of Theorem C, which are Theorem D, Theorem E and Proposition F.

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1 Median algebras

Definition 1.1 Let X be a set. A map $\mu\colon X^3 \to X$ is called a *median* on X if for all a, b, c, d, e in X, it satisfies

$$\mu(a,b,c) = \mu(b,a,c) = \mu(b,c,a), \text{ ie } \mu \text{ is symmetric,}$$

$$\mu(a,a,b) = a,$$

$$\mu(a,b,\mu(c,d,e)) = \mu(\mu(a,b,c),\mu(a,b,d),e). \tag{1}$$

The pair (X, μ) is called a *median algebra*.

Remark
- There is a unique median on the set $\{0, 1\}$.
- We can consider the product median on the n–cube $\{0, 1\}^n$.

Definition 1.2 Let (X, μ) and (X', μ') be median algebras. A map $f\colon X \to X'$ is called *median* if for every $x, y, z \in X$, we have $\mu'(f(x), f(y), f(z)) = f(\mu(x, y, z))$. If furthermore f is injective, it is called a *median embedding*.

Definition 1.3 Let (X, μ) be a median algebra. If every median embedding of an n–cube $\{0, 1\}^n \to X$ satisfies $n \leqslant r$, we say that X has *rank* at most r.

Definition 1.4 Let (X, μ) be a median algebra. If $a, b \in X$, the *interval* between a and b is $I(a, b) = \{c \in X \mid \mu(a, b, c) = c\}$. A subset $C \subset X$ is called *convex* if for every $a, b \in C$, we have $I(a, b) \subset C$.

If (X, d) is a metric space, a weakening of the notion of metric median is the following:

Definition 1.5 Let (X, d) be a metric space. An abstract median μ on X is called

- *continuous* if $\mu\colon X^3 \to X$ is a continuous map,
- *Lipschitz* if there exists a constant $k \in [0, +\infty)$ such that μ is k–Lipschitz with respect to each variable, ie for every $a, b, c, a', b', c' \in X$, we have
$$d\big(\mu(a, b, c), \mu(a', b', c')\big) \leqslant k\big(d(a, a') + d(b, b') + d(c, c')\big),$$
- *locally convex* if each point of X has a basis of neighborhoods consisting of convex subsets of X.

Here is an example of a continuous median on $\mathbb{R}^2$ which is not Lipschitz: consider the image μ of the standard L^1 median by some non-Lipschitz diffeomorphism of $\mathbb{R}^2$. If we consider $\mathbb{R}^2$ with the standard L^1 distance and the new median μ, then μ is a continuous (even differentiable) median, but it is not Lipschitz.

Definition 1.6 Let (X, d) be a metric space, let μ be a continuous median on X, and let $C \subset X$ be a nonempty closed, locally compact convex subset of X. Then for each $x \in X$, there exists a unique $\pi_C(x) \in C$, called the *gate projection* of x onto C, such that for every $y \in C$, we have $\pi_C(x) \in I(x, y)$. The map $\pi_C\colon X \to C$ is called the gate projection, it is a continuous map. If μ is k–Lipschitz, then π_C is a k–Lipschitz map.

Now we recall the definition of walls in a median algebra.

Definition 1.7 Let (X,μ) be a median algebra. Then a *wall* in X is defined to be a pair $W=\{H^+(W),H^-(W)\}$, where $H^+(W)$ and $H^-(W)$ are nonempty convex disjoint subsets of X whose union is equal to X.

Lemma 1.8 [3, Lemma 6.1] *Let (X,μ) be a median algebra, and let A,B be disjoint convex subsets of M. There exists a wall $W=\{H^+(W),H^-(W)\}$ in X separating A from B, ie such that $A\subset H^{\pm}(W)$ and $B\subset H^{\mp}(W)$.*

Lemma 1.9 [3, Lemma 7.3] *Let (X,d) be a metric space, and let μ be a continuous locally convex median on X. Let a,b be distinct points of X. There exists a wall $W=\{H^+(W),H^-(W)\}$ in X strongly separating a from b, ie such that $a\in X\backslash\overline{H^-(W)}$ and $b\in X\backslash\overline{H^+(W)}$.*

Lemma 1.10 [3, Lemma 7.5] *Let X be a metric space, and let μ be a continuous locally convex median on X. For each wall $W=\{H^+(W),H^-(W)\}$ in X, the subset $L(W)=\overline{H^+(W)}\cap\overline{H^-(W)}$ is a convex median subalgebra of X, of rank at most $r-1$ if the rank of μ is r.*

2 Ultralimits of spaces and coarse medians

In [18], Kleiner and Leeb developed a geometric definition of spherical and affine buildings, and in particular they studied their asymptotic cones.

Theorem 2.1 [18, Theorem 1.2.1] *Let X be a symmetric space of noncompact type or a thick affine building. Then any asymptotic cone of X is a thick affine building of the same spherical type as X.*

They also proved that any tangent cone of an affine building is an affine building:

Theorem 2.2 [18, Theorem 5.1.1] *Let (X,d) be an affine building, let ω be a nonprincipal ultrafilter on $\mathbb{N}$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $(0,+\infty)$ such that $\lim_\omega \lambda_n=+\infty$. Letting $(X_\infty,d_\infty,x_\infty)$ be the ω–ultralimit of $(X_n,\lambda_n d,x_n)$, then (X_∞,d_∞) is an affine building. Furthermore, if X is thick, then X_∞ is thick. The affine Weyl group of X_∞ acts transitively on each apartment of X_∞.*

One motivation for Bowditch's definition of coarse median is that it behaves well when one considers asymptotic cones.

Theorem 2.3 [3, Theorem 2.3] *Let (X,d) be a metric space, and let μ be a (k,h)–coarse median on X. Then on any asymptotic cone (X_∞, d_∞) of (X,d), μ defines a k–Lipschitz, locally convex median μ_∞.*

We can now prove that Theorem A implies Theorem B.

Proof Let X be a symmetric space of noncompact type, or a thick affine building, of spherical type different from A_1^r. By Theorem 2.1 any asymptotic cone X_∞ is a thick affine building. If there existed a coarse median μ on X, it would give rise by Theorem 2.3 to a locally convex Lipschitz median on X_∞, which contradicts Theorem A since the spherical type of X_∞ is not A_1^r. Hence there is no coarse median on X. □

We can also prove that Theorem B implies Theorem C.

Proof Let X be a symmetric space of noncompact type or an affine building of spherical type A_1^r. If X is a symmetric space, it is a product of rank 1 symmetric spaces, which are Gromov hyperbolic, so X has a coarse median. If X is an affine building, if we endow it with the L^1 metric it becomes a metric median space. In particular, this median is a coarse median with respect to any usual metric on X, which is equivalent to the L^1 metric. □

3 Existence of a convex cube

In this Section, we will prove Proposition G, which we recall here.

Proposition G *Let X be a connected metric space, with a Lipschitz locally convex median of rank r. There exists a median, bi-Lipschitz embedding of the r–cube $[0,1]^r$ into X with convex image.*

Fix X a geodesic metric space, and $\mu\colon X^3 \to X$ a Lipschitz median.

Definition 3.1 A continuous path $p\colon I \to X$, where $I \subset \mathbb{R}$ is an interval, is said to be *monotone* if for each $t_1 < t_2 < t_3$ in I, we have $\mu(p(t_1), p(t_2), p(t_3)) = p(t_2)$.

To prove Proposition G, we will need the following two lemmas:

Lemma 3.2 *Let X be a connected metric space, with a continuous locally convex median μ, and let $f\colon \{0,1\}^r \to X$ be a median embedding of the r–cube, and let W be a wall in X strongly separating $f(0,\dots,0)$ and $f(1,0,\dots,0)$. There exists a median embedding $g\colon \{0,1\}^r \to X$ such that for every $t \in \{0\}\times\{0,1\}^{r-1}$, we have $g(t) = f(t)$ and for every $t \in \{1\}\times\{0,1\}^{r-1}$, we have $g(t) \in L(W)$.*

Proof Note that if we knew that $L(W)$ was locally compact, projecting the half-cube $\{1\}\times\{0,1\}^{r-1}$ using the gate projection onto $L(W)$ would immediately give the result.

Intervals are connected, so we can consider $a \in I(f(0,\dots,0), f(1,0,\dots,0)) \cap L(W)$. Define

$$\begin{aligned} g\colon \{0,1\}^r &\to X, \\ t \in \{0\}\times\{0,1\}^{r-1} &\mapsto f(t), \\ t \in \{1\}\times\{0,1\}^{r-1} &\mapsto \mu(f(0,t_2,\dots,t_r), a, f(t)). \end{aligned}$$

Since $L(W)$ is convex, we deduce that for every $t \in \{1\}\times\{0,1\}^{r-1}$, we have that $g(t) \in L(W)$.

Using repeatedly property (1), we prove that g is a median map. As a consequence, if for some $t, t' \in \{0,1\}^r$ we have $g(t) = g(t')$, then

$$\begin{aligned} f(0,t_2,\dots,t_r) &= \mu(g(0,t_2,\dots,t_r), g(t), g(0,t'_2,\dots,t'_r)) \\ &= \mu(g(0,t_2,\dots,t_r), g(t'), g(0,t'_2,\dots,t'_r)) \\ &= f(0,t'_2,\dots,t'_r), \end{aligned}$$

hence $(0,t_2,\dots,t_r) = (0,t'_2,\dots,t'_r)$, so $t=t'$ and hence g is injective. □

Lemma 3.3 *Let (X,μ) be a median algebra. Assume there exists a median embedding of the r–cube $f\colon [0,1]^r \to X$, such that the image by f of any edge of $[0,1]^r$ is convex. Then the image of f is convex in X.*

Proof For each $k \in [\![1,r]\!]$, let $e_k = (0,\dots,0,1,0,\dots,0)$, where the 1 is in the k^{th} position. Let $x \in I(f(0), f(e_1+\dots+e_r))$. For each $k \in [\![1,r]\!]$, since the image by f of the edge $[0,e_k]$ is convex, we deduce that $I(f(0), f(e_k)) = f([0,e_k])$, so there exists $t_k \in [0,1]$ such that $\mu(f(0,\dots,0), x, f(e_k)) = f(t_k e_k)$. We will show by induction on $k \in [\![0,r]\!]$ that $f(t_1e_1+\dots+t_ke_k) = \mu(f(0), x, f(e_1+\dots+e_k))$.

For $k=0$ this is immediate using property (1), so assume that for some $k<r$ we have $f(t_1e_1+\dots+t_ke_k) = \mu(f(0), x, f(e_1+\dots+e_k))$. Then

$$\begin{aligned} &f(t_1e_1+\dots+t_{k+1}e_{k+1}) \\ &\qquad = \mu\big(f(t_1e_1+\dots+t_ke_k), f(t_{k+1}e_{k+1}), f(e_1+\dots+e_{k+1})\big) \\ &\qquad = \mu\big(\mu(f(0),x,f(e_1+\dots+e_k)), \mu(f(0),x,f(e_{k+1})), f(e_1+\dots+e_{k+1})\big) \\ &\qquad = \mu\big(f(0), x, \mu(f(e_1+\dots+e_k), f(e_{k+1}), f(e_1+\dots+e_{k+1}))\big) \\ &\qquad = \mu\big(f(0), x, f(e_1+\dots+e_{k+1})\big). \end{aligned}$$

As a consequence, for $k=r$ we deduce that

$$f(t_1e_1+\cdots+t_re_r)=\mu(f(0),x,f(e_1+\cdots+e_r))=x,$$

as $x\in I(f(0),f(e_1+\cdots+e_r))$. So we have proved that the image of f is equal to the interval $I(f(0),f(e_1+\cdots+e_r))$, which is convex. □

We can now prove Proposition G.

Proof of Proposition G Since the rank of the median μ is r, consider a median embedding $f\colon\{0,1\}^r\to X$. Applying Lemma 3.2 $2r$ times, up to replacing f by another median embedding of $\{0,1\}^r$ into X, we can assume that for each $i\in[\![1,r]\!]$ and $\varepsilon\in\{0,1\}$, the image under f of the codimension-1 face

$$\{0,1\}^{i-1}\times\{\varepsilon\}\times\{0,1\}^{r-1-i}$$

is included in a closed convex subspace $L(W_{i,\varepsilon})$ of X, where $W_{i,\varepsilon}$ is a wall of X. According to Lemma 1.10, each $L(W_{i,\varepsilon})$ has rank at most $r-1$, and since it contains the image by f of the $(r-1)$–cube $\{0,1\}^{i-1}\times\{\varepsilon\}\times\{0,1\}^{r-1-i}$, we deduce that each $L(W_{i,\varepsilon})$ has rank $r-1$.

For $i,j\in[\![1,r]\!]$ distinct and $\varepsilon,\varepsilon'\in\{0,1\}$, since

$$L(W_{i,\varepsilon})\cap L(W_{j,\varepsilon'})=L(L(W_{i,\varepsilon})\cap W_{j,\varepsilon'}),$$

where $L(W_{i,\varepsilon})\cap W_{j,\varepsilon'}$ is a wall in the rank $r-1$ median algebra $L(W_{i,\varepsilon})$, we deduce by Lemma 1.10 that $L(W_{i,\varepsilon})\cap L(W_{j,\varepsilon'})$ has rank $r-2$.

By induction, we prove that for each $p\in[\![1,r]\!]$, for each distinct $i_1,\dots,i_p\in[\![1,r]\!]$ and each $\varepsilon_1,\dots,\varepsilon_p\in\{0,1\}$, the intersection $\bigcap_{1\leqslant k\leqslant p}L(W_{i_k,\varepsilon_k})$ has rank $r-p$.

For each $k\in[\![1,r]\!]$, let $e_k=(0,\dots,0,1,0,\dots,0)$, where the 1 is in the k^{th} position. Hence for each $k\in[\![1,r]\!]$, the points $f(0)$ and $f(e_k)$ are contained in a convex rank 1 closed subspace. In particular, there exists an injective, monotone, bi-Lipschitz path p_k from $f(0)$ to $f(e_k)$, with convex image.

We will show by induction on $k\in[\![0,r]\!]$ that we can extend f to a bi-Lipschitz median embedding from $[0,1]^k\times\{0,1\}^{r-k}$ into X. The case $k=0$ is already true. Assume we have extended f to a bi-Lipschitz median embedding $f\colon[0,1]^k\times\{0,1\}^{r-k}\to X$ for some $k<r$. Define

$$\begin{aligned} f\colon [0,1]^k\times[0,1]\times\{0,1\}^{r-k-1}&\to X\\ (t,u,v)\in[0,1]^k\times[0,1]\times\{0,1\}^{r-k-1}&\mapsto \mu(f(t,0,v),p_{k+1}(u),f(1,\dots,1)).\end{aligned}$$

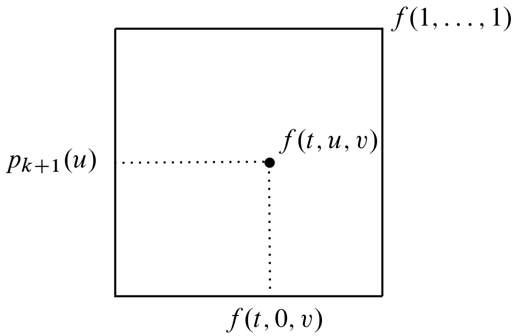

Figure 1: Extending f

See Figure 1. Since p_{k+1} and μ are bi-Lipschitz, we deduce that f is bi-Lipschitz on $[0,1]^k \times [0,1] \times \{0,1\}^{r-k-1}$.

So f is extended to a bi-Lipschitz map $f\colon [0,1]^r \to X$. If $t \in [0,1]^r$ and $k \in [\![1,r]\!]$, notice that in $[0,1]^r$ the median of $(t,0,e_k)$ is equal to $t_k e_k$. Therefore,

$$\mu(f(t), f(0), f(e_k)) = f(t_k e_k) = p_k(t_k).$$

Since each path $p_1,\dots,p_r$ is injective, we deduce that f itself is injective.

By using property (1) several times, we prove that f preserves the medians. Hence f is a median embedding, and by Lemma 3.3 the image of f is convex. □

Let us recall Bowditch's definition of the separation dimension of a space, which is a good notion of dimension when working with medians on a metric space.

Definition 3.4 (Bowditch) If X is a Hausdorff topological space, define the *separation dimension* of X inductively as follows:

- If $X = \varnothing$, then the separation dimension of X is -1.
- X has separation dimension at most $n \in \mathbb{N}$ if for any distinct $x, y \in X$, there exist closed subsets A, B of X such that $x \notin B$, $y \notin A$, $X = A \cup B$ and $A \cap B$ has dimension at most $n-1$.

Remark If X is a Hausdorff metric space, then the inductive dimension of X is at most equal to the separation dimension. Conversely, we have the following:

Lemma 3.5 [17, Section III.6] *If X is a locally compact Hausdorff metric space, then the inductive dimension of X equals the separation dimension.*

The following lemma is immediate:

Lemma 3.6 *Let X, Y be Hausdorff topological spaces, and let $f\colon X \to Y$ be a continuous injective map. Then the separation dimension of X is at most equal to the separation dimension of Y.*

We deduce the following:

Corollary 3.7 *Let X be a connected metric space, with a Lipschitz locally convex median of rank r. Then the separation dimension of X equals r.*

Proof According to [3, Theorem 2.2], the separation dimension of X is bounded above by r. According to Proposition G, there exists an embedding of $[0,1]^r$ into X, so according to Lemma 3.6 the separation dimension of X is precisely equal to r. □

Finally, for affine buildings, we have the following:

Corollary 3.8 *Let X be an affine building of rank r. Then any locally convex Lipschitz median on X has rank r.*

Proof According to [19, Theorem B], X has separation dimension equal to r. According to Corollary 3.7, any locally convex Lipschitz median on X has rank r. □

4 Proof of Theorem A

Consider a thick affine building X. Assume that there exists a k–Lipschitz, locally convex median μ on X. We will show that X has spherical type A_1^r.

Proposition 4.1 *There exists $x \in X$ such that in a tangent cone $(X_\infty, d_\infty, x_\infty, \mu_\infty)$ of (X, d, x, μ) at x, the ultralimit F_∞ of some apartment F containing x is convex and median-isomorphic to $(\mathbb{R}^r, L^1)$ by an affine isomorphism.*

Proof According to Corollary 3.8, the median μ has rank r, and according to Proposition G, there exists a bi-Lipschitz, median embedding f of $[0,1]^r$ into X with convex image. According to [18, Corollary 6.2.3], the image of f intersects finitely many apartments of X. Consider a nonempty open subset U of $[0,1]^r$ such that $f(U)$ lies in one apartment F of X. The map $f|_U\colon U \to F$ is bi-Lipschitz, hence it is differentiable almost everywhere: Pick a point $t \in U$ where f is differentiable. Since f is bi-Lipschitz, the differential of f at t is invertible. Then in any tangent cone of (X, d, x, μ) at $x = f(t)$, the ultralimit of F is convex and median-isomorphic to $(\mathbb{R}^r, L^1)$, by an affine isomorphism. □

According to Proposition 4.1, up to considering a tangent cone of X and using Theorem 2.2, we can assume that there exists a convex apartment F of X with a median, affine isomorphism with $(\mathbb{R}^r, L^1)$. Since F is convex, closed and locally compact, we can consider $\pi_F\colon X \to F$ the gate projection onto F.

Lemma 4.2 *For each $x \in X \setminus F$, and for each apartment F' of X containing x such that $F \cap F'$ is a half-apartment, we have $\pi_F(x) \in F \cap F'$.*

Proof By contradiction, assume that there exists such an $x \in X \setminus F$ and an apartment F' containing x such that $F \cap F'$ is a half-apartment, and such that $\pi_F(x) \notin F \cap F'$. Fix a Lipschitz embedding ι of the $(r-1)$–ball $\mathbb{B}^{r-1}_{\circ}$ into $F \cap F'$. Extend ι to a Lipschitz embedding of the half r–ball $\mathbb{B}^{r,+}$ into $F' \setminus F$, where $\mathbb{B}^{r-1}$ is the equatorial sphere of $\mathbb{B}^r$. Extend ι to a Lipschitz map $\iota\colon \mathbb{B}^r \to F \cup F'$ by setting $\iota(z) = \pi_F(\iota(-z))$, for $z \in \mathbb{B}^{r,-}$. Since $\pi_F(x) \in F \setminus F'$ and ι is Lipschitz, we deduce $(\iota(\mathbb{B}^r) \setminus \iota(\partial \mathbb{B}^r)) \cap F$ has nonempty interior.

For each $z \in (\partial \mathbb{B}^r)^+ = \mathbb{S}^{r-1,+}$, we have $\iota(-z) = \pi_F(\iota(z))$. Consider the following map:

$$\begin{aligned} \widetilde{\iota'}\colon \mathbb{S}^{r-1,+} \times [0,1] &\to X \\ (z,t) &\mapsto \mu\big(\iota(z), \pi_F(\iota(z)), [(1-t)\iota(z) + t\pi_F(\iota(z))]\big), \end{aligned}$$

where $[(1-t)\iota(z) + t\pi_F(\iota(z))]$ is the unique point on the CAT(0) geodesic segment between $\iota(z)$ and $\pi_F(\iota(z))$ at distance $t d(\iota(z), \pi_F(\iota(z)))$ from $\iota(z)$ (see Figure 2).

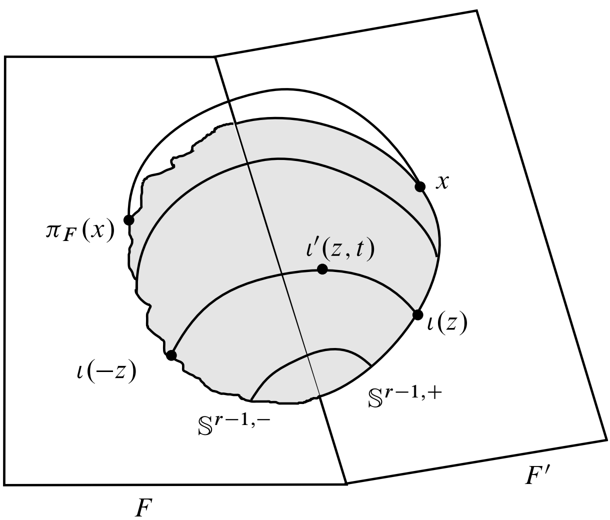

Figure 2: The sphere $\mathbb{S}^r$ in X

The map $\widetilde{\iota'}$ is Lipschitz and satisfies $\widetilde{\iota'}(z,t)) = \widetilde{\iota'}(z,t')$ for every $z \in \partial\mathbb{S}^{r-1,+} = \partial\mathbb{B}^{r-1}$ and every $t,t' \in [0,1]$, since $\iota(z) \in F \cap F'$ and hence $\pi_F(\iota(z)) = \iota(z)$. Consider the quotient of $\mathbb{S}^{r-1,+} \times [0,1]$ by the equivalence relation defined for every $z \in \partial\mathbb{S}^{r-1,+}$ and $t,t' \in [0,1]$ by $(z,t) \sim (z,t')$: it is a topological ball $\mathbb{B}^r$. So $\widetilde{\iota'}$ induces a Lipschitz map $\iota'\colon \mathbb{B}^r \to X$ such that $\iota|_{\mathbb{S}^{r-1}} = \iota'|_{\mathbb{S}^{r-1}}$. This defines a Lipschitz map $\alpha\colon \mathbb{S}^r \to X$.

In $\alpha(\mathbb{S}^r)$, if we collapse the complement of a small open ball in $F \backslash F'$, we obtain a topological sphere $\mathbb{S}^r$. As a consequence, $H_r(\alpha(\mathbb{S}^r)) \neq 0$. According to [19, Theorem B], X has topological dimension r, and since $\alpha(\mathbb{S}^r)$ is a compact subspace of X, we deduce that $H_r(\alpha(\mathbb{S}^r)) \to H_r(X)$ is an injection (see for instance [17, Theorem VIII.3']). Since X is contractible, this is a contradiction. □

We can now conclude the proof of Theorem A. By contradiction, assume that X has not spherical type A_1^r. Since X is thick, there exists a Weyl wall W in F, and two singular geodesics γ, γ' in X, each intersecting W, such that γ and γ' intersect in $X \backslash F$. Let $x = \gamma \cap \gamma' \in X \backslash F$. Since γ is singular, the intersection of all apartments F' containing γ such that $F' \cap F$ is a half-apartment is equal to γ. According to Lemma 4.2, we deduce that $\pi_F(x) \in \gamma \cap F$. Similarly, $\pi_F(x) \in \gamma' \cap F$. This contradicts the assumption that γ and γ' intersect in $X \backslash F$.

As a consequence, X has spherical type A_1^r. This concludes the proof of Theorem A, as well as Theorem B and Theorem C.

5 Proof of the main consequences

We will now prove the main consequences of Theorem C, namely Theorem D and Theorem E, and also give the proof of Proposition F.

Proof of Theorem D In one direction, assume that X is a symmetric space or affine building of spherical type A_1^r. If X is a discrete affine building of spherical type A_1^r, if we endow it with the L^1 metric, X becomes an actual CAT(0) cube complex. If X is a symmetric space, it is isometric to a product of rank 1 symmetric spaces. According to [15, Theorem 1.8], every word-hyperbolic group is quasi-isometric to a CAT(0) cube complex. So each rank 1 symmetric space is quasi-isometric to a CAT(0) cube complex, hence X itself is quasi-isometric to a CAT(0) cube complex.

Conversely, assume that the symmetric space or affine building X is quasi-isometric to a CAT(0) cube complex (Y, d_p), possibly of infinite dimension, endowed with the L^p distance for some $p \in [1, \infty]$. Since (Y, d_p) is quasi-isometric to the metric space X which has finite dimension, we deduce that (Y, d_p) is quasi-isometric to (Y, d_1). Since

(Y, d_1) is a metric median space, we deduce that there exists a coarse median on X. According to Theorem C, we deduce that the spherical type of X is A_1^r. □

Proof of Theorem E Assume that Γ is coarse median. Since nonuniform lattices do not have property (RD), Γ is cocompact in G. So Γ, endowed with a word metric, is quasi-isometric to G, endowed with a left G–invariant metric. Let X be the symmetric space of noncompact type of G (if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) or the Bruhat–Tits Euclidean building of G (if $\mathbb{K}$ is nonarchimedean). Then G is quasi-isometric to X, and so X has a coarse median. According to Theorem C, X has spherical type A_1^r, so G has relative type A_1^r. Since G is simple, $r = 1$, and G has $\mathbb{K}$–rank 1. □

We will now consider the rank 1 case, and give the proof of Proposition F.

Proof of Proposition F If Γ is a uniform lattice in G, then Γ is hyperbolic and hence coarse median.

If G is locally isomorphic to $SO_0(n, 1)$ for some $n \geqslant 2$, Γ is hyperbolic relative to a family $P_1, \dots, P_m$ of parabolic subgroups. Each parabolic subgroup P_i is virtually isomorphic to $\mathbb{Z}^{n-1}$. In particular, each P_i is coarse median, so by [4] Γ itself is coarse median.

If G is locally isomorphic to $SU(2, 1)$, Γ is hyperbolic relative to a family $P_1, \dots, P_m$ of parabolic subgroups. Each parabolic subgroup P_i is virtually isomorphic to the 3–dimensional Heisenberg group H_3, which has cubic Dehn function (see [13]). This implies that Γ has cubic Dehn function (see [22]), so by [5] Γ is not coarse median. □

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On a spectral sequence for the cohomology of infinite loop spaces

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We study the mod-2 cohomology spectral sequence arising from delooping the Bousfield–Kan cosimplicial space giving the 2–nilpotent completion of a connective spectrum X. Under good conditions its E_2–term is computable as certain nonabelian derived functors evaluated at $\mathrm{H}^*(X)$ as a module over the Steenrod algebra, and it converges to the cohomology of $\Omega^\infty X$. We provide general methods for computing the E_2–term, including the construction of a multiplicative spectral sequence of Serre type for cofibration sequences of simplicial commutative algebras. Some simple examples are also considered; in particular, we show that the spectral sequence collapses at E_2 when X is a suspension spectrum.

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1 Introduction

This paper explores the relationship between the $\mathbb{F}_2$–cohomology $\mathrm{H}^*E = \mathrm{H}^*(E;\mathbb{F}_2)$ of a connective spectrum E and that of its associated infinite loop space $\Omega^\infty E$.

The starting point is the stabilization map $\mathrm{H}^*(E) \to \mathrm{H}^*(\Omega^\infty E)$, induced by the adjunction counit $\Sigma^\infty\Omega^\infty E \to E$. This factors through the maximal unstable quotient $D\mathrm{H}^*(E)$ of the A–module $\mathrm{H}^*(E)$ (where A is the Steenrod algebra), and this map then extends over the free unstable algebra $UD\mathrm{H}^*(E)$. This construction provides the best approximation to $\mathrm{H}^*(\Omega^\infty E)$ functorial in the A–module $\mathrm{H}^*(E)$.

We study a spectral sequence that converges (for E connected and of finite type) to $\mathrm{H}^*(\Omega^\infty E)$ and has E_2–term given by the nonabelian derived functors of UD applied to $\mathrm{H}^*(X)$. This is the cohomology spectral sequence associated to the cosimplicial space obtained by applying Ω^∞ to a cosimplicial Adams (or Bousfield–Kan) resolution of the spectrum E.

This construction is analogous and in a sense dual to that of [21], where Miller constructed a spectral sequence that converges to $\mathrm{H}_*(E)$ by forming a simplicial resolution of E by suspension spectra and applying the zero-space functor Ω^∞. The

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best approximation to the homology of E functorial in the homology of the infinite loop space $\Omega^\infty E$ is given by the indecomposables of $\mathrm{H}_*(\Omega^\infty E)$ with respect to the Dyer–Lashof operations and products, which are annihilated by the natural map $\mathrm{H}_*(\Omega^\infty E) \to \mathrm{H}_*(E)$, and the E^2–term of the spectral sequence is given by the nonabelian left derived functors of these indecomposables applied to $\mathrm{H}_*(\Omega^\infty E)$.

The spectral sequence we study here is hardly new, and has been previously considered (in unpublished work) by Bill Dwyer, Paul Goerss, and no doubt others. Our main contribution here is related to the computation of the E_2–term, which is of the form $\pi_*(UV_\bullet)$, where $V_\bullet$ is a simplicial unstable A–module. We show that this is determined by a natural short exact sequence of graded unstable modules over the Steenrod algebra, in which the end terms are explicitly given in terms of the graded A–module $\pi_*(V_\bullet)$. This yields an explicit but mildly nonfunctorial description of the E_2–term.

This reduces the analysis of the E_2–term of the spectral sequence to the computation of the derived functors $\mathbb{L}_* D$. These derived functors of destabilization have been studied by many authors, including Singer [30; 31], Lannes and Zarati [19], Goerss [11], Kuhn and McCarty [18], and Powell [24].

As an outcome of our computation, we find that the spectral sequence must collapse when X is a connected suspension spectrum, $X = \Sigma^\infty B$ for B a connected space. While the spectral sequence collapses by construction when X is a mod-2 Eilenberg–Mac Lane spectrum, its collapse for suspension spectra is a bit of a surprise. This does not yet constitute an independent calculation of the cohomology of $\Omega^\infty\Sigma^\infty B$, however, since to prove that the spectral sequence collapses we simply compare the size of the E_2–term with that of the known homology of $\Omega^\infty\Sigma^\infty B$. It is possible that the collapse follows from Dwyer's description [9] of the behavior of differentials in a spectral sequence of this type.

It would be interesting to compare the spectral sequence we study to that arising from the Goodwillie–Taylor tower of the functor $\Sigma^\infty\Omega^\infty$, as studied by Kuhn and McCarty [18]. Those authors also relate their spectral sequence to derived functors of destabilization, though in a less direct way than they occur in our spectral sequence; we would like to better understand the relationship between these constructions, which seems analogous to the relationship between the Bousfield–Kan unstable Adams spectral sequence and the spectral sequence arising from the lower central series.

1.1 Overview

We construct the spectral sequence in Section 2, and review some background material on simplicial commutative $\mathbb{F}_2$–algebras in Section 3. Then in Section 4 we compute $\pi_* U(M)$ in terms of $\pi_* M$, where M is any simplicial A–module. We end by discussing some simple examples of the spectral sequence in Section 5.

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2 Definition and convergence of the spectral sequence

In this section we define the spectral sequence we are interested in, observe that its E_2–term is described by certain derived functors, and show that it converges under suitable finiteness and connectivity assumptions. More precisely, our goal is to prove the following:

Theorem 2.1 *Suppose X is a connected spectrum of finite type, ie $\pi_* X$ is 0 for $* \leq 0$ and is a finitely generated abelian group for $* > 0$. Then there is a convergent spectral sequence*

$$E_2^{s,t} = \mathbb{L}_{-s}(UD)(\mathrm{H}^* X)^t \Rightarrow \mathrm{H}^{t+s}(\Omega^\infty X).$$

Here $\mathbb{L}_*(UD)$ denotes the nonabelian derived functors of UD, which can be defined as $\pi_* UD(M_\bullet)$ where $M_\bullet$ is the simplicial free resolution of the A–modules $\mathrm{H}^* X$.

To define the spectral sequence, recall that for any spectrum X the Eilenberg–Mac Lane ring spectrum $\mathrm{H}\mathbb{F}_2$ gives a cosimplicial spectrum

$$P^\bullet := X \wedge \mathrm{H}\mathbb{F}_2^{\wedge(\bullet+1)}.$$

The homotopy limit of $P^\bullet$ is the 2*–nilpotent completion* $X_2^\wedge$ of X. Since the functor Ω^∞ preserves homotopy limits, the cosimplicial space $\Omega^\infty P^\bullet$ has homotopy limit $\Omega^\infty(X_2^\wedge)$. This gives a spectral sequence in cohomology,

$$E_2^{s,t} = \pi_{-s}\mathrm{H}^t(\Omega^\infty P^\bullet) \Rightarrow \mathrm{H}^{t+s}(\Omega^\infty(X_2^\wedge)).$$

Proposition 2.2 (Bousfield) *If X is a connected spectrum of finite type (ie all its homotopy groups are finitely generated), then this spectral sequence converges.*

Proof This follows from (the dual of) the convergence result of [3, Section 4.5]. □

Lemma 2.3 *Suppose X is a connected spectrum of finite type. Then the map $\Omega^\infty X \to \Omega^\infty(X_2^\wedge)$ exhibits $\Omega^\infty(X_2^\wedge)$ as the $\mathrm{H}\mathbb{F}_2$–localization of $\Omega^\infty X$. In particular, it induces an equivalence in $\mathrm{H}\mathbb{F}_2$–cohomology.*

Proof If X is connected, then by [2, Theorem 6.6] the 2–nilpotent completion $X_2^\wedge$ is equivalent to the $\mathrm{H}\mathbb{F}_2$–localization of X; in particular the natural map $X \to X_2^\wedge$ induces an equivalence in $\mathrm{H}\mathbb{F}_2$–cohomology.

Moreover, under the stated assumptions on X the map $X \to X_2^\wedge$ induces an isomorphism $(\pi_* X) \otimes \mathbb{Z}_2^\wedge \xrightarrow{\sim} \pi_* X_2^\wedge$, by [2, Proposition 2.5]. Since X is connected, the space $\Omega^\infty X$ is nilpotent, and so by [4, Example VI.5.2] the map $\Omega^\infty X \to (\Omega^\infty X)_2^\wedge$ also induces an isomorphism $(\pi_* \Omega^\infty X) \otimes \mathbb{Z}_2^\wedge \xrightarrow{\sim} \pi_*(\Omega^\infty X)_2^\wedge$. Since $\Omega^\infty(X_2^\wedge)$ is 2–complete, the map $\Omega^\infty X \to \Omega^\infty(X_2^\wedge)$ factors through $(\Omega^\infty X)_2^\wedge$; we know that two of the maps in the resulting commutative diagram

$$\begin{tikzcd} & \pi_*(\Omega^\infty X) \otimes \mathbb{Z}_2^\wedge \arrow[dl] \arrow[dr] & \\ \pi_*(\Omega^\infty X)_2^\wedge \arrow[rr] & & \pi_*(\Omega^\infty(X_2^\wedge)) \end{tikzcd}$$

are isomorphisms, hence the map $(\Omega^\infty X)_2^\wedge \to \Omega^\infty(X_2^\wedge)$ is a weak equivalence. The result follows since under our assumptions the map $\Omega^\infty X \to (\Omega^\infty X)_2^\wedge$ exhibits $(\Omega^\infty X)_2^\wedge$ as the $\mathrm{H}\mathbb{F}_2$–localization of $\Omega^\infty X$ by [4, Proposition VI.5.3]. □

Under these finiteness assumptions the spectral sequence thus converges to the mod-2 cohomology of $\Omega^\infty X$. To describe the E_2–term more algebraically, we appeal to Serre's computation of the cohomology of Eilenberg–Mac Lane spaces. To state this we must first recall some definitions:

Definition 2.4 Let Mod_A be the category of (graded) A–modules, and let $\mathcal{U}$ be the full subcategory of *unstable* modules, ie A–modules M such that if $x \in M_n$ then $\mathrm{Sq}^i x = 0$ for $i > n$. We define $D\colon \mathrm{Mod}_A \to \mathcal{U}$ to be the *destabilization* functor, which sends an A–module M to its quotient by the submodule generated by $\mathrm{Sq}^i x$ where $x \in M_n$ and $i > n$; the functor D is left adjoint to the inclusion $\mathcal{U} \hookrightarrow \mathrm{Mod}_A$.

Definition 2.5 Let $\mathcal{K}$ be the category of unstable algebras over the Steenrod algebra A, ie augmented commutative A–algebras R that are unstable as A–modules, with $x^2 = \mathrm{Sq}^n x$ for all $x \in R_n$. We define $U\colon \mathcal{U} \to \mathcal{K}$ to be the *free unstable algebra* functor, which sends $M \in \mathcal{U}$ to

$$S(M)/(x^2 - \mathrm{Sq}^{|x|} x),$$

where S is the free graded symmetric algebra functor; this functor is left adjoint to the forgetful functor $\mathcal{K} \to \mathcal{U}$.

Theorem 2.6 (Serre [29]) *If M is an Eilenberg–Mac Lane spectrum of finite type, then the natural map $\mathrm{H}^*(M) \to \mathrm{H}^*(\Omega^\infty M)$ induces an isomorphism*

$$UD(\mathrm{H}^* M) \xrightarrow{\sim} \mathrm{H}^*(\Omega^\infty M).$$

For any $n > 0$ the spectrum $X \wedge \mathrm{H}\mathbb{F}_2^{\wedge n}$ is a wedge of suspensions of Eilenberg–Mac Lane spectra, so this theorem allows us to rewrite the E_2–term of our spectral sequence as

$$E_2^{s,t} = \pi_{-s} UD(\mathrm{H}^* P^\bullet)^t.$$

But by the Künneth theorem $\mathrm{H}^*(P^n)$ is isomorphic to $\mathrm{H}^*(X) \otimes A^{\otimes n+1}$, and in fact the simplicial A–module $\mathrm{H}^*(P^\bullet)$ is the standard cotriple resolution of $\mathrm{H}^* X$. The A–modules $\pi_* UD(\mathrm{H}^* P^\bullet)$ can therefore be interpreted as the (nonabelian) derived functors $\mathbb{L}_*(UD)$ of UD evaluated at $\mathrm{H}^* X$. This completes the proof of Theorem 2.1.

Remark 2.7 Our spectral sequence is of the type considered by Dwyer in [9], so by [9, Proposition 2.3] it is a spectral sequence of A–algebras. By (the dual of) results of Hackney [15] it is actually a spectral sequence of Hopf algebras.

3 Simplicial commutative $\mathbb{F}_2$–algebras

In this section we first review some background material on simplicial commutative (graded) algebras: we recall the model category structure on simplicial commutative algebras in Section 3.1 and in Section 3.2 we review the higher divided square operations in the homotopy groups of simplicial commutative algebras. Then in Section 3.3 we discuss filtered algebras and modules from an abstract point of view, and finally in Section 3.4 we use this material to construct a "Serre spectral sequence" for cofiber sequences of simplicial commutative algebras.

3.1 Model category structure

We will make use of a model category structure on simplicial augmented commutative graded $\mathbb{F}_2$–algebras. This is an instance of a general class of model categories constructed by Quillen [25], and is also described by Miller [22]:

Theorem 3.1 *There is a simplicial model category structure on the category of simplicial augmented graded commutative $\mathbb{F}_2$–algebras where a morphism is a weak equivalence or fibration if the underlying map of simplicial sets is a weak equivalence or Kan fibration.*

Remark 3.2 For us *graded* will mean $\mathbb{N}$–graded rather than $\mathbb{Z}$–graded. To avoid confusion, let us also mention that we do not require that a graded $\mathbb{F}_2$–algebra A has $A_0 = \mathbb{F}_2$, as is sometimes assumed in the literature.

Remark 3.3 Since a simplicial graded commutative $\mathbb{F}_2$–algebra is a simplicial group, a morphism $f\colon A \to B$ is a fibration if and only if the induced map $A \to B \times_{\pi_0 B} \pi_0 A$ is surjective. In particular, every object is fibrant.

Theorem 3.4 (Rezk) *This model structure on simplicial augmented graded commutative $\mathbb{F}_2$–algebras is proper.*

Proof This follows from the properness criterion of [26, Theorem 9.1], since polynomial algebras are flat and thus tensoring with them preserves weak equivalences. □

We now recall Miller's description of the cofibrations in this model category:

Definition 3.5 A morphism $f\colon A \to B$ of simplicial augmented commutative $\mathbb{F}_2$–algebras is *almost-free* if for every $n \geq 0$ there is a subspace V_n of the augmentation ideal IB_n and maps

$$\delta_i\colon V_n \to V_{n-1} \quad \text{for } 1 \leq i \leq n,$$
$$\sigma_i\colon V_n \to V_{n+1} \quad \text{for } 0 \leq i \leq n,$$

so that the induced map $A_n \otimes S(V_n) \to B_n$ is an isomorphism for all n and the following diagrams commute:

$$\begin{tikzcd} A_n \otimes S(V_n) \arrow[r] \arrow[d, "d_i \otimes S(\delta_i)"'] & B_n \arrow[d, "d_i"] \\ A_{n-1} \otimes S(V_{n-1}) \arrow[r] & B_{n-1}, \end{tikzcd} \qquad \begin{tikzcd} A_n \otimes S(V_n) \arrow[r] \arrow[d, "s_i \otimes S(\sigma_i)"'] & B_n \arrow[d, "s_i"] \\ A_{n+1} \otimes S(V_{n+1}) \arrow[r] & B_{n+1}. \end{tikzcd}$$

In other words, all the face and degeneracy maps *except* d_0 are induced from maps between the V_n.

Remark 3.6 The definition of almost free morphisms in [22] is wrong and was corrected in [23].

Theorem 3.7 (Miller, [22, Corollary 3.5]) *A morphism of simplicial augmented commutative $\mathbb{F}_2$–algebras is a cofibration if and only if it is a retract of an almost-free morphism.*

3.2 Higher divided square operations

In this subsection we review the *higher divided square* operations on the homotopy groups of simplicial commutative $\mathbb{F}_2$–algebras. These operations were initially introduced by Cartan [5], and have subsequently also been studied by Bousfield [1] and Dwyer [10].

Definition 3.8 If V is a simplicial $\mathbb{F}_2$–vector space, we write $C(V)$ for the unnormalized chain complex of V (obtained by taking the alternating sum of the face maps as the differential) and $N(V)$ for the *normalized* chain complex, given by

$$N_k X = \bigcap_{i \neq 0} \ker(d_i\colon X_k \to X_{k-1})$$

with differential d_0. These chain complexes are quasi-isomorphic, and their homology groups are the same as the homotopy groups of V, regarded as a simplicial set.

Theorem 3.9 (Dwyer [10]) *Let A be a simplicial commutative $\mathbb{F}_2$–algebra.*

(i) *There are maps $\delta_i\colon C(A)_n \to N(A)_{n+i}$ for $i \geq 1$ that satisfy*

$$d\delta_i(a) = \begin{cases} \delta_i(da) & \text{if } n > i > 1, \\ \delta_1(da) + \phi(a) & \text{if } i = 1 \text{ and } n > 1, \\ a\,da & \text{if } i = n > 1, \\ a\,da + \phi(a) & \text{if } n = i = 1. \end{cases}$$

Here $\phi(a)$ denotes the image in $N(A)$ of the square a^2 of a in the multiplication on A_n.

(ii) *In particular, there are* higher divided square *operations $\delta_i\colon \pi_n(A) \to \pi_{n+i}(A)$ for $2 \leq i \leq n$. If $a^2 = 0$ for all $a \in A$, then there is also an operation $\delta_1\colon \pi_n(A) \to \pi_{n+1}(A)$ for $n \geq 1$.*

(iii) *These operations have the following properties:*

(1) *$\delta_i\colon \pi_n A \to \pi_{n+i} A$ is an additive homomorphism for $2 \le i < n$, and δ_n satisfies*

$$\delta_n(x+y) = \delta_n(x) + \delta_n(y) + xy.$$

(2) *δ_i acts on products as follows:*

$$\delta_i(xy) = \begin{cases} x^2\delta_i(y) & \text{if } x \in \pi_0 A, \\ y^2\delta_i(x) & \text{if } y \in \pi_0 A, \\ 0 & \text{otherwise.} \end{cases}$$

(3) *("Adém relations") If $i < 2j$ then*

$$\delta_i\delta_j(x) = \sum_{(i+1)/2 \le s \le (i+j)/3} \binom{j-i+s-1}{j-s} \delta_{i+j-s}\delta_s(x).$$

Remark 3.10 Part (i) is not quite true using Dwyer's definition of the chain-level operations in [10], but is correct for the variant due to Goerss [12].

Remark 3.11 The upper bound in the "Adém relation" above differs from that in [10], which does not give a sum of admissible operations; this form of the relation was proved by Goerss and Lada [13] and implies, by the same proof as for Steenrod operations, that composites of δ–operations are spanned by admissible composites:

Definition 3.12 A sequence $I = (i_1, \dots, i_k)$ is *admissible* if $i_t \ge 2i_{t+1}$ for all t. A composite $\delta_I := \delta_{i_1}\delta_{i_2}\cdots\delta_{i_k}$ is *admissible* if I is.

Corollary 3.13 *Any composite of δ–operations can be written as a sum of admissible ones.*

Remark 3.14 For any x in $\pi_* A$ in positive degree we have $x^2 = 0$. Therefore Theorem 3.9(iii)(1)–(2) imply that the top operation δ_n on π_n is a divided square, whence the name "higher divided squares" for the δ_i–operations.

Dwyer proves Theorem 3.9 by computing the homotopy groups in the universal case, namely the symmetric algebra $s(V)$ on a simplicial vector space V. We will now recall the result of this computation, as well as the analogous result for exterior algebras (both of which are originally due to Bousfield [1]). To state this we make use of the following theorem of Dold:

Theorem 3.15 (Dold [7, 5.17]) *Let* Vect *be the category of $\mathbb{F}_2$–vector spaces, and* grVect *that of graded $\mathbb{F}_2$–vector spaces. For any functor $F\colon \mathrm{Vect}^{\times n} \to \mathrm{Vect}$ there* *exists a functor* $\mathfrak{F}\colon \mathrm{grVect}^{\times n} \to \mathrm{grVect}$ *such that for* $V_1, \dots, V_n$ *simplicial* $\mathbb{F}_2$*–vector spaces there is a natural isomorphism*

$$\pi_* F(V_1, \dots, V_n) \cong \mathfrak{F}(\pi_* V_1, \dots, \pi_* V_n),$$

where on the left-hand side we take the homotopy of the diagonal of the multisimplicial $\mathbb{F}_2$*–vector space* $F(V_1, \dots, V_n)$.

Example 3.16 The Eilenberg–Zilber theorem implies that if F is the tensor product functor, then $\mathfrak{F}$ is the graded tensor product of graded vector spaces.

In the symmetric algebra case, the functor $\mathfrak{s}$ such that $\pi_* s(V) = \mathfrak{s}(\pi_* V)$ has the following description:

Theorem 3.17 (Bousfield [1], Dwyer [10]) *The functor* $\mathfrak{s}$ *sends a graded vector space* V *to that freely generated on* V *by a commutative product and operations* δ_i *satisfying the relations stated in Theorem 3.9 above as well as the relation* $x^2 = 0$ *for all* x *of positive degree.*

If B is a graded basis for V, then $\mathfrak{s}(\pi_*(V))$ is the free commutative algebra (modulo the relation $x^2 = 0$ for $|x| > 0$) generated by elements $\delta_I v$ in degree $|v| + i_1 + \cdots + i_k$ for admissible sequences $I = (i_1, \dots, i_k)$ with $i_k \geq 2$ of *excess* $e(I) := i_1 - i_2 - \cdots - i_k$ at most $|v|$, as v runs over B.

Let $s_k(V)$ be the subspace of the symmetric algebra $s(V)$ spanned by products of length k; it also is a functor $\mathrm{Vect} \to \mathrm{Vect}$. Implicit in Theorem 3.17 is the following description of the functor $\mathfrak{s}_k$ such that $\pi_* s_k(V) = \mathfrak{s}_k(\pi_* V)$.

Theorem 3.18 *Suppose* V *is a graded vector space. Define inductively a weight function on products* $\delta_{I_1}(v_1) \cdots \delta_{I_n}(v_n)$ *where* $v_i \in V$ *and the* I_i *are admissible sequences by*

$$\mathrm{wt}(v) = 1 \quad \text{for } v \text{ in } V,$$
$$\mathrm{wt}(xy) = \mathrm{wt}(x) + \mathrm{wt}(y),$$
$$\mathrm{wt}(\delta_i(x)) = 2\,\mathrm{wt}(x).$$

Then $\mathfrak{s}_k(V)$ *is the subspace of* $\mathfrak{s}(V)$ *spanned by elements of weight* k.

Let $e(V)$ denote the exterior algebra on V and $e_k(V)$ its subspace of products of length k. Then there are functors $\mathfrak{e}$ and $\mathfrak{e}_k$ such that $\pi_* e(V) = \mathfrak{e}(\pi_* V)$ and $\pi_* e_k(V) = \mathfrak{e}_k(\pi_* V)$ for a simplicial vector space V. These were also computed by Bousfield:

Theorem 3.19 (Bousfield [1]) *The functor $\mathfrak{e}$ sends a graded vector space V to that freely generated on V by a commutative product and operations δ_i (now with $i = 1$ allowed) satisfying the same relations as in the symmetric case, and with $x^2 = 0$ for all x. Thus $\mathfrak{e}(V)$ is generated by $v \in V$ and symbols $\delta_I v$ for admissible sequences $I = (i_1, \dots, i_k)$ (now with $i_k \geq 1$) of excess $\leq |v|$; the element $\delta_I v$ is again in degree $|v| + i_1 + \cdots + i_k$. Defining the weight of such a generator as before, the graded vector space $\mathfrak{e}_k V$ is the subspace of $\mathfrak{e}V$ spanned by elements of weight k.*

Remark 3.20 The same results hold in the graded case. We will use capital letters for the graded versions of the functors considered above: so $S(V)$ denotes the free graded symmetric algebra on the graded vector space V, etc. The higher divided power operations δ_i double the internal degree.

3.3 An abstract approach to filtered algebras and modules

Given filtered algebras A, B and C, and maps $A \to B$ and $A \to C$, we would like to construct a filtration on the relative tensor product $B \otimes_A C$ whose associated graded is the relative tensor product $E^0 B \otimes_{E^0 A} E^0 C$ of graded algebras, where $E^0 A$ denotes the associated graded algebra of the filtered algebra A. Our goal in this section is to show that this is possible, provided we allow ourselves to take cofibrant replacements of these algebras in a suitable model category. We will do this by considering filtered objects, and in particular filtered modules over a filtered algebra, from an abstract perspective.

Let $\mathbf{N}$ denote the partially ordered set of natural numbers $0, 1, \dots,$ considered as a category. If $\mathbf{C}$ is a category, we write $\mathrm{Seq}(\mathbf{C})$ for the category $\mathrm{Fun}(\mathbf{N}, \mathbf{C})$ of sequences of morphisms in $\mathbf{C}$. A filtered object of $\mathbf{C}$, if $\mathbf{C}$ is for example the category of chain complexes of abelian groups, can then be thought of as a certain kind of object of $\mathrm{Seq}(\mathbf{C})$.

Addition of natural numbers is a symmetric monoidal structure on $\mathbf{N}$, so if $\mathbf{C}$ is a category with finite colimits and a symmetric monoidal structure that commutes with finite colimits in each variable (for short, $\mathbf{C}$ is a symmetric monoidal category *compatible with finite colimits*) we can equip $\mathrm{Seq}(\mathbf{C})$ with the Day convolution tensor product. This has as unit the constant sequence

$$I \to I \to \cdots$$

with value the unit I in $\mathbf{C}$, and if A and B are sequences in $\mathbf{C}$ their tensor product $A \otimes B$ is given by

$$(A \otimes B)_n = \operatorname*{colim}_{i+j \leq n} A_i \otimes B_j.$$

Remark 3.21 A simple cofinality argument shows that this colimit is isomorphic to the iterated pushout

$$A_n \otimes B_0 \amalg_{A_{n-1}\otimes B_0} A_{n-1} \otimes B_1 \amalg_{A_{n-2}\otimes B_1} \cdots \amalg_{A_0 \otimes B_{n-1}} A_0 \otimes B_n,$$

which we can also describe as the coequalizer of the two obvious maps

$$\coprod_{s+t=n-1} A_s \otimes B_t \rightrightarrows \coprod_{i+j=n} A_i \otimes B_j.$$

In other words, $(A \otimes B)_n$ is the quotient of $\coprod_{i+j=n} A_i \otimes B_j$ where we identify the images of $A_s \otimes B_t$ with $s+t=n-1$ in $A_{s+1} \otimes B_t$ and $A_s \otimes B_{t+1}$.

If A is an algebra object in $\mathrm{Seq}(\mathbf{C})$, the Day convolution on $\mathrm{Seq}(\mathbf{C})$ induces a relative tensor product on the category $\mathrm{Mod}_A(\mathrm{Seq}(\mathbf{C}))$ of A–modules, given by the (reflexive) coequalizer

$$M \otimes A \otimes N \rightrightarrows M \otimes N \to M \otimes_A N,$$

where M and N are A–modules in $\mathrm{Seq}(\mathbf{C})$. If $\mathbf{C}$ is, for instance, chain complexes, then a filtered algebra A in $\mathbf{C}$ is in particular an algebra object of $\mathrm{Seq}(\mathbf{C})$, and filtered A–modules M and N are also modules for A in $\mathrm{Seq}(\mathbf{C})$. The tensor product of A–modules then yields an object $M \otimes_A N$ in $\mathrm{Seq}(\mathbf{C})$, but in general this need not be a filtered object of $\mathbf{C}$, as the maps in this sequence need no longer be monomorphisms. However, we can use a model structure on $\mathbf{C}$ to deal with this: If $\mathbf{C}$ is a combinatorial model category, we can equip $\mathrm{Seq}(\mathbf{C})$ with the projective model structure. A cofibrant object in $\mathrm{Seq}(\mathbf{C})$ is then a sequence

$$A_0 \to A_1 \to \cdots$$

where the objects A_i are all cofibrant, and the morphisms $A_i \to A_{i+1}$ are all cofibrations. If cofibrations in $\mathbf{C}$ are monomorphisms, as they are for chain complexes or simplicial algebras, then a cofibrant object of $\mathrm{Seq}(\mathbf{C})$ is thus in particular a filtered object.

The Day convolution tensor product interacts well with this model structure:

Proposition 3.22 (Isaacson) *Let $\mathbf{C}$ be a symmetric monoidal combinatorial model category that satisfies the monoid axiom. Then $\mathrm{Seq}(\mathbf{C})$ is also a symmetric monoidal combinatorial model category with respect to the Day convolution and satisfies the monoid axiom.*

Proof This is a special case of [16, Proposition 8.4]. □

We can now apply results of Schwede and Shipley to get the following:

Corollary 3.23 *Let $\mathbf{C}$ be a symmetric monoidal combinatorial model category that satisfies the monoid axiom, and suppose A is a commutative algebra object in $\mathrm{Seq}(\mathbf{C})$. Then:*

(i) *The category $\mathrm{Alg}(\mathrm{Seq}\,\mathbf{C})$ of associative algebra objects of $\mathrm{Seq}(\mathbf{C})$ is a combinatorial model category. The forgetful functor to $\mathrm{Seq}(\mathbf{C})$ creates weak equivalences and fibrations, and the free-forgetful adjunction*

$$\mathrm{Seq}(\mathbf{C}) \rightleftarrows \mathrm{Alg}(\mathrm{Seq}(\mathbf{C}))$$

is a Quillen adjunction.

(ii) *If the unit of $\mathbf{C}$ is cofibrant, then the forgetful functor $\mathrm{Alg}(\mathrm{Seq}(\mathbf{C})) \to \mathrm{Seq}(\mathbf{C})$ preserves cofibrant object.*

(iii) *The category $\mathrm{Mod}_A(\mathrm{Seq}(\mathbf{C}))$ of A–modules is a symmetric monoidal combinatorial model category satisfying the monoid axiom. The forgetful functor to $\mathrm{Seq}(\mathbf{C})$ creates weak equivalences and fibrations, and the free-forgetful adjunction*

$$F_A\colon \mathrm{Seq}(\mathbf{C}) \rightleftarrows \mathrm{Mod}_A(\mathrm{Seq}(\mathbf{C})) :U_A$$

is a Quillen adjunction.

(iv) *If the underlying object of A is cofibrant in $\mathrm{Seq}(\mathbf{C})$ then the forgetful functor U_A also preserves cofibrations.*

Proof (i), (ii) and (iii) follow from [27, Theorem 4.1], and (iv) is an easy consequence of the construction of the model structure using [27, Lemma 2.3]: If I is a set of generating cofibrations in $\mathrm{Seq}(\mathbf{C})$, then $F_A(I)$ is a set of generating cofibrations in A–modules. The triple $U_A F_A$ is $A \otimes -$, which is a left Quillen functor if A is cofibrant in $\mathrm{Seq}(\mathbf{C})$. Thus U_A takes the generating cofibrations to cofibrations. But U_A also preserves colimits, so as any cofibration is a transfinite composite of pushouts of generating cofibrations this means it preserves all cofibrations. □

Corollary 3.24 *Let $\mathbf{C}$ be a symmetric monoidal combinatorial model category that satisfies the monoid axiom, and suppose A is a commutative algebra object in $\mathrm{Seq}(\mathbf{C})$ whose underlying object in $\mathrm{Seq}(\mathbf{C})$ is cofibrant. If M and N are cofibrant A–modules, then $M \otimes_A N$ is a cofibrant object of $\mathrm{Seq}(\mathbf{C})$.*

This is the result we need to make our spectral sequence: if A is a suitable filtered algebra in, say, chain complexes, and M and N are filtered A–modules, we can take cofibrant replacements for them in the model structure on $\mathrm{Mod}_A(\mathrm{Seq}(\mathbf{C}))$ to get a cofibrant relative tensor product over A, which is in particular a filtered object and so gives a spectral sequence.

Next we want to analyze the associated graded object of such a relative tensor product, which will allow us to describe the E^1–page of our spectral sequence:

Definition 3.25 Let $\mathbf{C}$ be a category with finite colimits and a zero object 0. Denote the product $\prod_{i=0}^{\infty} \mathbf{C}$ by $\mathrm{Gr}(\mathbf{C})$ and write $\mathrm{Triv}\colon \mathrm{Gr}(\mathbf{C}) \to \mathrm{Seq}(\mathbf{C})$ for the functor that sends $(X_i)_{i\in\mathbb{N}}$ to the sequence

$$X_0 \xrightarrow{0} X_1 \xrightarrow{0} \cdots .$$

This has a left adjoint $E^0\colon \mathrm{Seq}(\mathbf{C}) \to \mathrm{Gr}(\mathbf{C})$, the *associated graded* functor. We have $(E^0A)_0 = A_0$ and $(E^0A)_n$ for $n > 0$ is the quotient A_n/A_{n-1}, ie the pushout

$$\begin{tikzcd} A_{n-1} \arrow[r] \arrow[d] & A_n \arrow[d] \\ 0 \arrow[r] & A_n/A_{n-1}. \end{tikzcd}$$

If $\mathbf{C}$ has a symmetric monoidal structure, then we can equip $\mathrm{Gr}(\mathbf{C})$ with a graded tensor product (another Day convolution), given by

$$(X \otimes Y)_n = \coprod_{i+j=n} X_i \otimes Y_j .$$

The unit is $(I, 0, 0, \ldots)$.

Proposition 3.26 *Let $\mathbf{C}$ be a symmetric monoidal category compatible with finite colimits that has a zero object. Then the functor $E^0\colon \mathrm{Seq}(\mathbf{C}) \to \mathrm{Gr}(\mathbf{C})$ is symmetric monoidal.*

Proof E^0 clearly preserves the unit, so it suffices to show that there is a natural isomorphism $E^0M \otimes E^0N \xrightarrow{\simeq} E^0(M \otimes N)$.

By definition, $E^0_n(M \otimes N)$ is the cofiber of $(M \otimes N)_{n-1} \to (M \otimes N)_n$. For $n \in \mathbf{N}$, let $(\mathbf{N} \times \mathbf{N})_{\leq n}$ denote the full subcategory of $\mathbf{N} \times \mathbf{N}$ spanned by the objects (i, j) with $i + j \leq n$; if $M \boxtimes N$ denotes the composite functor

$$\mathbf{N} \times \mathbf{N} \xrightarrow{M\times N} \mathbf{C} \times \mathbf{C} \xrightarrow{\otimes} \mathbf{C},$$

then $(M \otimes N)_n$ is by definition given by the colimit of $M \boxtimes N$ restricted to $(\mathbf{N}\times\mathbf{N})_{\leq n}$. Let α denote the inclusion $(\mathbf{N} \times \mathbf{N})_{\leq (n-1)} \hookrightarrow (\mathbf{N} \times \mathbf{N})_{\leq n}$; then $(M \otimes N)_{n-1}$ is isomorphic to the colimit of the left Kan extension $\alpha_!(M \boxtimes N)|_{(\mathbf{N}\times\mathbf{N})_{\leq(n-1)}}$. Thinking of 0 as the constant diagram of shape $(\mathbf{N}\times\mathbf{N})_{\leq n}$ with value 0, we can write $E^0_n(M\otimes N)$

as the pushout of two maps between colimits of diagrams of the same shape. Moreover, these maps arise from natural transformations, so since colimits commute we can identify $E_n^0(M \otimes N)$ with the colimit of the functor $\beta\colon (\mathbf{N}\times\mathbf{N})_{\le n} \to \mathbf{C}$ that assigns to (i,j) the cofiber of the map

$$\phi_{i,j}\colon (\alpha_!(M \boxtimes N)|_{(\mathbf{N}\times\mathbf{N})_{\le(n-1)}})(i,j) \to (M \boxtimes N)(i,j).$$

If $i+j<n$ then, since $(\mathbf{N}\times\mathbf{N})_{\le(n-1)}$ is a full subcategory of $(\mathbf{N}\times\mathbf{N})_{\le n}$, the map $\phi_{i,j}$ is an isomorphism, so $\beta(i,j)\cong 0$. It follows that the colimit of β is just the coproduct $\coprod_{i+j=n}\beta(i,j)$, and it remains to show that $\beta(i,j)$ is isomorphic to $E_i^0 M \otimes E_j^0 N$.

If $i+j=n$, let $(\mathbf{N}\times\mathbf{N})_{<(i,j)}$ be the full subcategory of $\mathbf{N}\times\mathbf{N}$ spanned by the objects (x,y) with $x\le i$ and $y \le j$, except for (i,j). Then by definition we have that $\alpha_!((M\boxtimes N)|_{(\mathbf{N}\times\mathbf{N})_{\le(n-1)}})(i,j)$ is the colimit of $M\boxtimes N$ restricted to $(\mathbf{N}\times\mathbf{N})_{<(i,j)}$.

Write $(\mathbf{N}\times\mathbf{N})^0_{<(i,j)}$ for the full subcategory

$$(i-1,j) \leftarrow (i-1,j-1) \rightarrow (i,j-1)$$

of $(\mathbf{N}\times\mathbf{N})_{<(i,j)}$. We claim the inclusion $(\mathbf{N}\times\mathbf{N})^0_{<(i,j)} \hookrightarrow (\mathbf{N}\times\mathbf{N})_{<(i,j)}$ is cofinal, and so gives an isomorphism of colimits. By [20, Theorem IX.3.1], to see this it suffices to show that the categories

$$((\mathbf{N}\times\mathbf{N})^0_{<(i,j)})_{(x,y)/} = (\mathbf{N}\times\mathbf{N})^0_{<(i,j)} \times_{(\mathbf{N}\times\mathbf{N})_{<(i,j)}} ((\mathbf{N}\times\mathbf{N})_{<(i,j)})_{(x,y)/}$$

are nonempty and connected. But this category is either all of $(\mathbf{N}\times\mathbf{N})^0_{<(i,j)}$ if $x\le i-1$ and $y\le j-1$, or the single object $(i,j-1)$ if $x=i$, or the single object $(i-1,j)$ if $y=j$; these are certainly all nonempty and connected. We may thus identify $\alpha_!((M\boxtimes N)|_{(\mathbf{N}\times\mathbf{N})_{\le(n-1)}})(i,j)$ with the pushout $M_i\otimes N_{j-1} \amalg_{M_{i-1}\otimes N_{j-1}} M_{i-1}\otimes N_j$ and $\beta(i,j)$ with the total cofiber of the square

$$\begin{tikzcd} M_{i-1}\otimes N_{j-1} \arrow[r] \arrow[d] & M_{i-1}\otimes N_j \arrow[d] \\ M_i \otimes N_{j-1} \arrow[r] & M_i \otimes N_j. \end{tikzcd}$$

The cofibers of the columns here are $E_i^0 M\otimes N_{j-1}$ and $E_i^0 M \otimes N_j$, since the tensor product preserves colimits in each variable, and so the total cofiber $\beta(i,j)$ is isomorphic to the cofiber of the map $E_i^0 M\otimes N_{j-1} \to E_i^0 M\otimes N_j$, which is $E_i^0 M\otimes E_j^0 N$, as required. □

Corollary 3.27 *Suppose A is a commutative algebra object of $\mathrm{Seq}(\mathbf{C})$, where $\mathbf{C}$ is as above. Then the adjunction $E^0 \dashv \mathrm{Triv}$ induces an adjunction*

$$E^0\colon \mathrm{Mod}_A(\mathrm{Seq}(\mathbf{C})) \rightleftarrows \mathrm{Mod}_{E^0A}(\mathrm{Gr}(\mathbf{C})) :\mathrm{Triv}$$

such that E^0 is symmetric monoidal.

This is immediate from Proposition 3.26 and the following easy formal observation:

Lemma 3.28 *Let $\mathbf{C}$ and $\mathbf{D}$ be symmetric monoidal categories, and suppose*

$$F\colon \mathbf{C} \rightleftarrows \mathbf{D} :G$$

is an adjunction such that F is symmetric monoidal. If A is a commutative algebra object of $\mathbf{C}$, this induces an adjunction

$$F_A\colon \mathrm{Mod}_A(\mathbf{C}) \rightleftarrows \mathrm{Mod}_{FA}(\mathbf{D}) :G_A$$

such that F_A is symmetric monoidal.

This allows us to identify the associated graded of a relative tensor product:

Corollary 3.29 *Let $\mathbf{C}$ be a symmetric monoidal category compatible with finite colimits that has a zero object. Suppose A is a commutative algebra object in $\mathrm{Seq}(\mathbf{C})$ and that M and N are A–modules. Then there is a natural isomorphism*

$$E^0(M \otimes_A N) \cong E^0M \otimes_{E^0A} E^0N.$$

Finally, we check that the colimit of a relative tensor product is the expected one:

Proposition 3.30 *Suppose $\mathbf{C}$ is a symmetric monoidal category compatible with small colimits. Then the colimit functor $\mathrm{Seq}(\mathbf{C}) \to \mathbf{C}$ is symmetric monoidal.*

Proof The unit for the tensor product on $\mathrm{Seq}(\mathbf{C})$ is the constant sequence with value I, the unit for the tensor product on $\mathbf{C}$. Thus colim preserves the unit. It remains to show that the natural map $\mathrm{colim}_n(A \otimes B)_n \to \mathrm{colim}_n A_n \otimes \mathrm{colim}_n B_n$ is an isomorphism. But the object $\mathrm{colim}_n(A \otimes B)_n$ is clearly the colimit over $(i,j) \in \mathbf{N} \times \mathbf{N}$ of $A_i \otimes B_j$. Since the tensor product on $\mathbf{C}$ preserves colimits in each variable, this colimit is indeed equivalent to $(\mathrm{colim}_{i \in \mathbf{N}} A_i) \otimes (\mathrm{colim}_{i \in \mathbf{N}} B_i)$. □

Applying Lemma 3.28, we get:

Corollary 3.31 *Let* $\mathbf{C}$ *be as above. Suppose* A *is a commutative algebra object in* $\mathrm{Seq}(\mathbf{C})$ *with colimit* $\bar{A}$. *Then the colimit-constant adjunction induces an adjunction*

$$\mathrm{colim}\colon \mathrm{Mod}_A(\mathrm{Seq}(\mathbf{C})) \rightleftarrows \mathrm{Mod}_{\bar{A}}(\mathbf{C}) :\mathrm{const}$$

where the left adjoint is symmetric monoidal.

Corollary 3.32 *Suppose* A *is a commutative algebra object in* $\mathrm{Seq}(\mathbf{C})$ *with colimit* $\bar{A}$, *and* M *and* N *are* A*–modules with colimits* $\overline{M}$ *and* $\overline{N}$. *Then* $\mathrm{colim}\, M \otimes_A N$ *is naturally isomorphic to* $\overline{M} \otimes_{\bar{A}} \overline{N}$.

3.4 A Serre spectral sequence for simplicial commutative algebras

In this subsection we construct a multiplicative "Serre spectral sequence" for the homotopy groups of the cofiber of a cofibration of simplicial commutative algebras. We derive this by studying a spectral sequence for filtered modules over a filtered differential graded algebra. Our spectral sequence has the same form as one constructed by Quillen [25], but his construction does not give the multiplicative structure.

Remark 3.33 We will implicitly assume that all filtrations we consider are nonnegatively graded and *exhaustive*, in the sense that if $F_0 A \subseteq F_1 A \subseteq \cdots$ is a filtration of A, then A is the union of the subobjects $F_i A$.

Proposition 3.34 *Suppose* A *is a filtered commutative differential graded* k*–algebra, nonnegatively graded, where* k *is a field, and* B *and* C *are filtered* A*–modules, also nonnegatively graded.*

(i) *If* B *and* C *are cofibrant in the model structure on* A*–modules in sequences of maps of chain complexes of Corollary 3.23, then the tensor product* $B \otimes_A C$ *has a canonical filtration with associated graded*
$$E^0_*(B \otimes_A C) \cong E^0_* B \otimes_{E^0_* A} E^0_* C.$$

(ii) *Suppose* B *and* C *are in addition filtered* A*–algebras. Then the filtration of (i) makes* $B \otimes_A C$ *a filtered algebra, so the associated spectral sequence is multiplicative.*

Proof Since k is a field, every k–module is projective, hence in the projective model structure on the category $\mathrm{Ch}_k^{\geq 0}$ of nonnegatively graded chain complexes of k–modules every object is cofibrant. Thus in the projective model structure on $\mathrm{Seq}(\mathrm{Ch}_k^{\geq 0})$ the cofibrant objects are precisely those that are sequences of monomorphisms, ie those that correspond to filtered chain complexes. Part (i) then follows from Corollaries 3.24, 3.29 and 3.32 applied to the projective model structure on chain complexes of k–modules.

If B and C are filtered A–algebras, then we may regard them as associative algebra objects in the category $\mathrm{Mod}_A(\mathrm{Seq}(\mathrm{Ch}_k^{\geq 0}))$. Their relative tensor product is then also an associative algebra object in this category, and by (i) its underlying object in $\mathrm{Seq}(\mathrm{Ch}_k^{\geq 0})$ corresponds to a filtered chain complex. Thus $B \otimes_A C$ is a filtered algebra, and so yields a multiplicative spectral sequence. □

Proposition 3.35 *Suppose A is a commutative differential graded k–algebra and B and C are A–modules, all nonnegatively graded. Filter A and B by degree, and give C the trivial filtration with $F_pC = C$ for all $p \geq 0$. Let B' and C' be cofibrant replacements of B and C as A–modules in* $\mathrm{Seq}(\mathrm{Ch}_k^{\geq 0})$*. Then in the spectral sequence associated to the induced filtration on $B' \otimes_A C'$ we have:*

(i) $E^1_{s,t} = (B' \otimes_A \pi_{t-s} C)_s$, *where A acts on $\pi_* C$ via the map $A \to \pi_0 A$.*

(ii) $E^2_{s,t} = \pi_s(B' \otimes_A \pi_{t-s} C)$.

Proof The graded tensor product $E^0_* B' \otimes_{E^0_* A} E^0_* C'$ has in degree (s,t) the coequalizer of

$$\bigoplus_{\substack{i+j+k=s \\ \rho+\sigma+\tau=t}} E^0_i B'_\rho \otimes E^0_j A_\sigma \otimes E^0_k C'_\tau \rightrightarrows \bigoplus_{\substack{m+n=s \\ \alpha+\beta=t}} E^0_m B'_\alpha \otimes E^0_n C'_\beta.$$

In our case $E^0_l B'_\gamma$ and $E^0_l A_\gamma$ are zero unless $l = \gamma$, and $E^0_l C'_\gamma$ is zero unless $l = 0$, so this is the coequalizer of

$$\bigoplus_{i+j=s} B'_i \otimes A_j \otimes C'_{t-s} \rightrightarrows B'_s \otimes C'_{t-s}.$$

Now observe that the map $A_j \otimes C'_{t-s} \to C'_{t-s}$ is zero unless $j = 0$, since A_j is in filtration j and so the product must lie in $(E^0_j C')_{t-s} = 0$. Thus we can describe this coequalizer as killing all elements of the form $a \cdot b$ with $a \in A$ and $b \in B'$, giving $(B' \otimes_A \pi_0 A)_s \otimes_{\pi_0 A} C'_{t-s} \cong (B' \otimes_{\pi_0 A} C'_{t-s})_s$.

The differential in $B' \otimes_A C'$ satisfies the Leibniz rule, so if $b \otimes c$ is in filtration s then $d(b \otimes c) = db \otimes c + b \otimes dc$. Here db is in lower filtration than b, since it is in lower degree, hence d_0 comes from the differential in C'. Thus

$$E^1_{s,t} \cong (B' \otimes_{\pi_0 A} \pi_{t-s} C)_s.$$

Similarly, the next differential d_1 comes from the differential in B, giving

$$E^2_{s,t} \cong \pi_s(B' \otimes_{\pi_0 A} \pi_{t-s} C).$$

□

The Dold–Kan correspondence extends to a Quillen equivalence of model categories (see [28, Section 4]) between simplicial modules and chain complexes, where the weak equivalences in the two categories are the π_*–isomorphisms and the quasi-isomorphisms, respectively. Using these model structures we can define *derived tensor products* as follows:

Definition 3.36 If A is a simplicial graded $\mathbb{F}_2$–algebra, M is a right A–module, and N is a left A–module, then the *derived tensor product* $M \otimes_A^{\mathbb{L}} N$ is the homotopy colimit of the simplicial diagram given by the bar construction, $M \otimes A^{\otimes \bullet} \otimes N$. Similarly, if A is an algebra in chain complexes of graded $\mathbb{F}_2$–vector spaces, M is a right A–module, and N is a left A–module, we define a derived tensor product $M \otimes_A^{\mathbb{L}} N$ as the analogous homotopy colimit.

Remark 3.37 In the simplicial case, the homotopy colimit is given by the diagonal of the bar construction.

Remark 3.38 If M is a cofibrant A–module, then for any N the derived tensor product $M \otimes_A^{\mathbb{L}} N$ is equivalent to the ordinary tensor product $M \otimes_A N$.

Lemma 3.39 *Let A be a simplicial graded algebra, X a right A–module, and Y a left A–module. There is a natural quasi-isomorphism*

$$N(X) \otimes_{N(A)}^{\mathbb{L}} N(Y) \to N(X \otimes_A^{\mathbb{L}} Y).$$

Proof There is a natural quasi-isomorphism $N(U) \otimes N(V) \to N(U \otimes V)$ for all simplicial abelian groups U and V. Thus there is a natural transformation of simplicial diagrams $N(X \otimes A^{\otimes \bullet} \otimes Y) \to NX \otimes (NA)^{\otimes \bullet} \otimes NY$ that is a quasi-isomorphism levelwise. This implies that the induced map on homotopy colimits is also a quasi-isomorphism. □

Corollary 3.40 *Suppose given simplicial augmented graded $\mathbb{F}_2$–algebras A, B and C, and maps $A \to B$ and $A \to C$. Then there is a multiplicative spectral sequence*

$$E^2_{s,t} = \pi_s(B \otimes_A^{\mathbb{L}} \pi_{t-s}(C)) \Rightarrow \pi_t(B \otimes_A^{\mathbb{L}} C).$$

Proof Let P be a cofibrant replacement for NB as an NA–module in $\mathrm{Seq}(\mathrm{Ch}_k^{\geq 0})$. Then by Proposition 3.35 we have a spectral sequence

$$E^2_{s,t} = \pi_s(P \otimes_{NA} \pi_{t-s} NC) \Rightarrow \pi_{t-s}(P \otimes_{NA} NC),$$

which is multiplicative by Proposition 3.34. Since taking the colimit of a sequence is a left Quillen functor, P is also a cofibrant replacement for NB as an NA–module, so we can write this as

$$E^2_{s,t} = \pi_s(NB \otimes^{\mathbb{L}}_{NA} \pi_{t-s} NC) \Rightarrow \pi_{t-s}(NB \otimes^{\mathbb{L}}_{NA} NC).$$

By Lemma 3.39 we have a natural quasi-isomorphism $NB \otimes^{\mathbb{L}}_{NA} NC \to N(B \otimes^{\mathbb{L}}_A C)$ and, since $\pi_{t-s} NC \cong \pi_{t-s} C$ is concentrated in a single degree, a natural quasi-isomorphism $NB \otimes^{\mathbb{L}}_{NA} \pi_{t-s} NC \to N(B \otimes^{\mathbb{L}}_A \pi_{t-s} C)$. Thus we have a natural isomorphism

$$E^2_{s,t} \cong \pi_s(B \otimes^{\mathbb{L}}_A \pi_{t-s} C) \Rightarrow \pi_{t-s}(B \otimes^{\mathbb{L}}_A C). \qquad \square$$

As observed by Turner [32, Proof of Lemma 3.1], this spectral sequence can be used to get a "Serre spectral sequence" for cofibration sequences of simplicial commutative algebras:

Corollary 3.41 ("Serre spectral sequence") *Suppose $f\colon A \to B$ is a cofibration of simplicial augmented graded commutative $\mathbb{F}_2$–algebras with cofiber C and $\pi_0 A = \mathbb{F}_2$. Then there is a multiplicative spectral sequence*

$$\pi_s(C) \otimes_{\mathbb{F}_2} \pi_{t-s}(A) \Rightarrow \pi_t(B).$$

Proof By Corollary 3.40 there is a multiplicative spectral sequence

$$E^2_{s,t} = \pi_s(B \otimes^{\mathbb{L}}_A \pi_{t-s} A) \Rightarrow \pi_t(B).$$

By definition $C \cong B \otimes_A \mathbb{F}_2$, and so

$$C \otimes_{\mathbb{F}_2} \pi_t A \cong (B \otimes_A \mathbb{F}_2) \otimes_{\mathbb{F}_2} \pi_t A \cong B \otimes_A \pi_t A,$$

which is isomorphic to $B \otimes^{\mathbb{L}}_A \pi_t A$ since $A \to B$ is a cofibration and the model structure on simplicial commutative algebras is left proper by Theorem 3.4. Since $\mathbb{F}_2$ is a field we have

$$\pi_s(C \otimes_{\mathbb{F}_2} \pi_{t-s} A) \cong \pi_s C \otimes_{\mathbb{F}_2} \pi_{t-s} A,$$

and so we can rewrite the E^2–term of the spectral sequence as

$$E^2_{s,t} \cong \pi_s(C) \otimes_{\mathbb{F}_2} \pi_{t-s}(A). \qquad \square$$

4 Derived functors of U

Our goal in this section is to compute $\pi_* U(M)$ where M is a simplicial unstable A–module. As a simplicial commutative algebra, $U(M)$ depends only on the top nonzero Steenrod operations in M; in Section 4.1 we consider graded vector spaces equipped with only these operations, which we call *restricted vector spaces*, and observe that a simplicial restricted vector space decomposes up to weak equivalence as a coproduct of simple pieces. In Section 4.2 we compute the derived functors of U for these simpler objects, which gives a description of $\pi_* U(M)$ as a graded commutative algebra with higher divided square operations. Using this we then give a more functorial description of $\pi_* U(M)$ in Section 4.3, which in particular lets us identify the action of the Steenrod operations.

4.1 Restricted vector spaces

In this subsection we define restricted vector spaces and make some observations about their structure; in particular, we show that a chain complex of restricted vector spaces always decomposes up to quasi-isomorphism as a direct sum of certain very simple complexes.

Definition 4.1 A *restricted vector space* (over $\mathbb{F}_2$) is a nonnegatively graded vector space V equipped with linear maps $\phi_i\colon V^i \to V^{2i}$ for all i, called the *restriction maps* of V, such that $\phi_0\colon V^0 \to V^0$ is the identity. A homomorphism of restricted vector spaces $f\colon V \to W$ is a homomorphism of graded vector spaces such that $\phi_i f^i = f^{2i}\phi_i$ for all i. We write Restr for the category of restricted vector spaces and restricted vector space homomorphisms. This is an abelian category.

Definition 4.2 For $n \le 0$, let $F(n)$ be the free restricted vector space with one generator ι_n in degree n. Thus $F(n)^{2^r n} = \mathbb{F}_2$ with $\phi_{2^r n} = \mathrm{id}$ and $F(n)^i = 0$ otherwise; in particular $F(0)$ is just $\mathbb{F}_2$ in degree 0.

For $k, n > 0$, let $T(n,k)$ be the nilpotent restricted vector space with one generator $\iota_{n,k}$ in degree n subject to $\phi^k \iota_{n,k} = 0$; that is, $T(n,k) = F(n)/\phi^k$. Thus $T(n,k)^{2^r n} = \mathbb{F}_2$ for $r = 0, \dots, k$ with $\phi_{2^r n} = \mathrm{id}$ for $r = 0, \dots, k-1$, and $T(n,k)^i$ is 0 otherwise.

Definition 4.3 Let V be a restricted vector space. A *basis* S of V consists of sets S^i of elements of V^i such that S^0 is a basis for V^0 and if $i = 2^r p$ with p odd, then the set $(S^{2^r p} \cup \phi(S^{2^{r-1} p}) \cup \dots \cup \phi^r(S^p)) \setminus \{0\}$ is a basis for V^i.

Remark 4.4 It is clear that any restricted vector space has a basis, since we can inductively choose complements of $\phi(V_i)$ in V_{2i}. Equivalently, any restricted vector space decomposes as a direct sum of copies of $F(n)$ and $T(n,k)$.

Definition 4.5 Let $C(q)$ be the nonnegatively graded chain complex of restricted vector spaces

$$\cdots \to 0 \to 0 \to F(q),$$

with $F(q)$ in degree 0, and let $C(q,k)$ be the chain complex

$$\cdots \to 0 \to F(2^k q) \hookrightarrow F(q),$$

with $F(q)$ in degree 0 and $F(2^k q)$ in degree 1.

Given a chain complex C, we denote by $C[n]$ the suspended chain complex with $C[n]_i = C_{i-n}$. Then clearly

$$\mathrm{H}_*(C(q)[n]) \cong \begin{cases} F(q) & \text{if } * = n, \\ 0 & \text{if } * \neq n, \end{cases}$$

$$\mathrm{H}_*(C(q,k)[n]) \cong \begin{cases} T(q,k) & \text{if } * = n, \\ 0 & \text{if } * \neq n. \end{cases}$$

Proposition 4.6 *Any chain complex of restricted vector spaces is quasi-isomorphic to a direct sum of copies of $C(q)[n]$ and $C(q,k)[n]$.*

Proof Let (V_*, d) be a chain complex of restricted vector spaces. Pick a basis S_i of $\mathrm{H}_i(V_*)$. For $v \in S_i^q$ define W_v to be $F(q)[i]$ if $\phi^r v$ is never zero, and $C(q,k)[i]$ if $\phi^k v = 0$ but $\phi^r v \neq 0$ for $r < k$. Let $\hat{v}$ be a lift of v to V_i; in the first case $\hat{v}$ defines a map $\psi_v \colon W_v \to V_\bullet$. In the second case, since $\phi^k(v) = 0$ we can pick $\hat{w} \in V_{i+1}$ such that $d(\hat{w}) = \phi^k(\hat{v})$; then $\hat{v}$ and $\hat{w}$ define a map ψ_v from W_v to $V_\bullet$. Let $W := \bigoplus_{v \in S} W_v$ and let $\psi \colon W \to V$ be $\bigoplus_{v \in S} \psi_v$. Then ψ is a quasi-isomorphism, since it is clear that on homology ψ_v induces the inclusion in $\mathrm{H}_i(V_\bullet, d)$ of the subspace generated by v. □

By the Dold–Kan correspondence the category $\mathrm{Ch}(\mathrm{Restr})_{\geq 0}$ of nonnegatively graded chain complexes of restricted vector spaces is equivalent to the category of simplicial restricted vector spaces. Let's write $K[n,q]$ and $K[n,q,k]$ for the simplicial objects corresponding to $F(q)[n]$ and $C(q,k)[n]$, respectively, under this equivalence; then Proposition 4.6 corresponds to:

Corollary 4.7 *Any simplicial restricted vector space is weakly homotopy equivalent to a coproduct of copies of $K[n,q]$ and $K[n,q,k]$.*

4.2 Computation of the derived functors of U

In this subsection we will prove the main technical result of this paper: we compute the homotopy groups of the free unstable A–algebra on a simplicial unstable A–module. As an algebra, $U(V)$ depends only on V as a restricted vector space: it is the "enveloping algebra" of V, given by the free graded commutative algebra on V subject to the relation $x^2 = \phi(x)$, ie $S(V)/(x^2 = \phi(x))$, where $S(V)$ is the graded symmetric algebra on V. If V is a simplicial restricted vector space we may ask about $\pi_*(UV)$. It does not depend functorially on $\pi_*(V)$; we do not have Dold's Theorem 3.15 working for us. We will describe $\pi_*(UV)$ in terms of $\pi_*(V)$, but not functorially. Our description will use the functor $\mathfrak{S}$, given to us by Dold's theorem, such that $\pi_*(SV) = \mathfrak{S}(\pi_*(V))$. It is described in detail above, in Theorem 3.17. We will also use the "loops" functor Ω and its first derived functor Ω_1, defined by the exact sequence

$$0 \to \Sigma\Omega_1 V \to \Phi V \xrightarrow{\phi} V \to \Sigma\Omega V \to 0.$$

where Φ denotes the "doubling" functor, $(\Phi V)_n^{2q} = V_n^q$ and $(\Phi V)_n^{2q+1} = 0$.

Here is the result:

Theorem 4.8 *If V is a simplicial restricted vector space, then there is a (noncanonical) isomorphism*

$$\pi_* UV \cong U(\pi_0 V)[0] \otimes \mathfrak{S}(\Sigma\Omega\pi_{*>0} V) \otimes \mathfrak{S}((\Sigma\Omega_1\pi_{*>0} V)[1]),$$

where $[1]$ denotes a shift by 1 in the simplicial degree. (By noncanonical *we mean that the isomorphism depends on a choice of basis of $\pi_* V$.)*

By Corollary 4.7 we know that any simplicial restricted vector space is weakly equivalent to a coproduct of copies of $K[n,q]$ and $K[n,q,k]$. To show that this carries over to a decomposition of $U(M)$ up to weak equivalence, we observe U preserves weak equivalences and colimits:

Proposition 4.9 *U, considered as a functor from restricted vector spaces to graded commutative $\mathbb{F}_2$–algebras, preserves colimits and weak equivalences.*

Proof We first show that U preserves colimits. Let U' denote U, regarded as a functor from restricted vector spaces to augmented graded commutative $\mathbb{F}_2$–algebras. The forgetful functor from augmented algebras to algebras preserves colimits, so it suffices to show that U' preserves colimits. But this is clear since U' has a right adjoint, namely the augmentation ideal functor for augmented graded commutative

$\mathbb{F}_2$–algebras, regarded as a functor to restricted vector spaces with restriction maps given by squaring.

To see that U preserves weak equivalences, consider the word-length filtration on $U(V)$ for V a simplicial restricted vector space. This gives rise to a spectral sequence of the form

$$\pi_t E_s(V) = \mathfrak{E}_s(\pi_* V)_t \Rightarrow \pi_t U(V),$$

where $\mathfrak{E}$ is as in Section 3.2 Moreover, since the filtration is natural in V, so is the spectral sequence. Thus a weak equivalence $f\colon V \to W$ of simplicial restricted vector spaces induces a morphism of spectral sequences that gives an isomorphism on the E^1–page, since this only depends on the homotopy of the simplicial restricted vector space. This implies that the map is an isomorphism of spectral sequences and hence, as these spectral sequences converge, it follows that $U(f)$ is a weak equivalence of simplicial graded commutative algebras. □

Combining this with Corollary 4.7 we see that $U(M)$, for any simplicial restricted vector space M, is weakly equivalent to a tensor product of copies of $U(K[n,q])$ and $U(K[n,q,k])$. It thus suffices to prove Theorem 4.8 in these two cases. We begin with the easiest case, namely $\pi_* U(K[n,q])$ for $q > 0$. For this we need to recall the explicit form of the Dold–Kan construction:

Definition 4.10 Let $\mathbf{C}$ be an abelian category. The *Dold–Kan construction*

$$K\colon \mathrm{Ch}(\mathbf{C})_{\geq 0} \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{C})$$

sends a nonnegatively graded chain complex A to the simplicial object $K(A)$ defined as follows: We set

$$K(A)_n = \bigoplus_{\alpha\colon [n] \twoheadrightarrow [k]} A_k,$$

where the coproduct is over surjective maps out of $[n]$. Then a map $K(A)_n \to K(A)_m$ is described by a "matrix" of maps $f_{\alpha,\beta}\colon A_k \to A_l$ from the component corresponding to $\alpha\colon [n] \twoheadrightarrow [k]$ to the component corresponding to $\beta\colon [m] \twoheadrightarrow [l]$. To define the map $\phi^*\colon K(A)_n \to K(A)_m$ corresponding to $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$ we take this to be given by

$$f_{\alpha,\beta} := \begin{cases} \mathrm{id} & \text{if } l = k \text{ and } \beta = \alpha\phi, \\ d & \text{if } l = k-1 \text{ and } d^0\beta = \alpha\phi, \\ 0 & \text{otherwise.} \end{cases}$$

The Dold–Kan correspondence (see Dold [7], Dold and Puppe [8], Kan [17]) is then that the functor K is an equivalence of categories, with inverse the normalized chain complex functor.

Lemma 4.11 *Let $\mathbb{F}_2[n,q]$ denote the chain complex of graded vector spaces that is 0 except in the degree n, where it is $\mathbb{F}_2[q]$ (the graded vector space with $\mathbb{F}_2$ in degree q and 0 elsewhere). Then for $q>0$ we have $UK[n,q] \cong S(K\mathbb{F}_2[n,q])$, where $K\mathbb{F}_2[n,q]$ is the Dold–Kan construction for graded vector spaces applied to $\mathbb{F}_2[n,q]$. In particular, we have an isomorphism*

$$\pi_* UK[n,q] \cong \mathfrak{S}\mathbb{F}_2[n,q]$$

for all n and $q>0$.

Proof From the definition of the Dold–Kan functor K we have

$$K[n,q]_i = \bigoplus_{[i]\twoheadrightarrow[n]} F(q)$$

and so for $q>0$ we have

$$UK[n,q]_i \cong \bigotimes_{[i]\twoheadrightarrow[n]} UF(q) \cong \bigotimes_{[i]\twoheadrightarrow[n]} S\mathbb{F}_2[q] \cong S\Bigl(\bigoplus_{[i]\twoheadrightarrow[n]} \mathbb{F}_2[q]\Bigr) \cong S(K\mathbb{F}_2[n,q])_i.$$

Moreover, the simplicial structure maps in $UK[n,q]$ and $SK\mathbb{F}_2[n,q]$ are also clearly the same (on "components" they are either the identity or zero), so these simplicial graded vector spaces are isomorphic. □

For the case $q=0$ the algebra $UF(0)$ is not a symmetric algebra, since we impose the relation $x^2=x$ on the generator x: it is a *Boolean algebra*. If V is a vector space, we write $b(V)$ for the free Boolean algebra $s(V)/(x^2=x)$ on V (where $s(V)$ is the ungraded symmetric algebra on V). By Theorem 3.15, there is a functor $\mathfrak{b}\colon \mathrm{grVect}\to\mathrm{grVect}$ such that $\pi_* b(V) = \mathfrak{b}(\pi_* V)$ for V a simplicial vector space.

Lemma 4.12 *Suppose V is a graded $\mathbb{F}_2$–vector space. Then*

$$\mathfrak{b}(V)_* = \begin{cases} b(V_0) & \text{if } * = 0,\\ 0 & \text{otherwise.}\end{cases}$$

Proof Suppose V is a simplicial vector space. Then Theorem 3.9 implies that for any element a in degree $n>0$ in the chain complex associated to $b(V)$ such that $da=0$, there exists an element $\delta_1 a$ in degree $n+1$ such that $d\delta_1 a = \phi(a) = a$. Thus $\pi_* b(V) = 0$ for $*>0$. □

Lemma 4.13 *For any n we have an isomorphism $UK[n,0]\cong b(K\mathbb{F}_2[n])[0]$, the simplicial graded vector space with $b(K\mathbb{F}_2[n])$ in degree 0, and so*

$$\pi_* UK[n,0] = \begin{cases} (b(\mathbb{F}_2))[0] & \text{if } n=0,\\ b(0)[0]\cong \mathbb{F}_2[0] & \text{if } n\neq 0.\end{cases}$$

Proof As in the proof of Lemma 4.11 we have

$$UK[n,0]_i \cong \bigotimes_{[i]\twoheadrightarrow[n]} UF(0) \cong \bigotimes_{[i]\twoheadrightarrow[n]} b(\mathbb{F}_2)[0] \cong b\Bigg(\bigoplus_{[i]\twoheadrightarrow[n]} \mathbb{F}_2\Bigg)[0] \cong b(K\mathbb{F}_2[n])[0]_i,$$

and the simplicial structure maps are again the same. □

Now we consider the consider the more complicated case, namely $UK[n,q,k]$. There is a cofibration sequence

$$F(q)[n] \to C(q,k)[n] \to F(2^k q)[n+1]$$

of chain complexes of restricted vector spaces. By the Dold–Kan correspondence this gives a cofibration sequence

$$K[n,q] \to K[n,q,k] \to K[n+1,2^k q]$$

of simplicial restricted vector spaces, and so a cofibration sequence

$$U(K[n,q]) \to U(K[n,q,k]) \to U(K[n+1,2^k q])$$

of simplicial commutative $\mathbb{F}_2$–algebras by Proposition 4.9. We want to apply the Serre spectral sequence of Corollary 3.41 to this cofibration sequence to compute $\pi_* U(K[n,q,k])$; to do this we first observe that the map $U(K[n,q]) \to U(K[n,q,k])$ is a cofibration of simplicial graded commutative algebras:

Lemma 4.14 *The map $U(K[n,q]) \to U(K[n,q,k])$ is almost-free in the sense of Definition 3.5, and so is a cofibration of simplicial graded commutative algebras.*

Proof We have

$$K[n,q,k]_i = \bigoplus_{[i]\twoheadrightarrow[n]} F(q) \oplus \bigoplus_{[i]\twoheadrightarrow[n+1]} F(2^k q)$$

and $UK[n,q,k]_i \cong UK[n,q]_i \otimes S(V_i)$ where $V_i = \bigoplus_{[i]\twoheadrightarrow[n+1]} \mathbb{F}_2[2^k q]$. It is clear that the simplicial structure maps ψ^* take $UK[q,n]_*$ to itself, and are induced by maps between the V_n except when ψ is such that for some β and γ we have $\beta\psi = d^0\gamma$, since this is the case when the differential in the chain complex occurs in the definition of ψ^* for $K[n,q,k]$.

But the map $\beta\colon [i] \to [n+1]$ is surjective and order-preserving, so it must send 0 to 0. Thus $\beta\psi$ will hit 0 in $[n+1]$ for all β if ψ hits 0 in $[i]$, in which case $\beta\psi$ cannot be of the form $d^0\gamma$. This is clearly the case for the degeneracies $s^j\colon [i+1] \to [i]$ for all j (as they are surjective) and the face maps $d^j\colon [i-1] \to [i]$ for $j \neq 0$. Thus the structure maps of $U(K[n,q,k])$ corresponding to all degeneracies and all face maps other than d_0 are induced from maps between the V_i. □

Proposition 4.15 *The Serre spectral sequence for the cofibration sequence*

$$U(K[n,q]) \xrightarrow{\alpha} U(K[n,q,k]) \xrightarrow{\beta} U(K[n+1,2^k q])$$

collapses at the E^2–page.

Proof By Corollary 3.41 the spectral sequence in question is a multiplicative spectral sequence of the form

$$E^2_{s,t} = \pi_s(UK[n+1,2^k q]) \otimes \pi_{t-s}(UK[n,q]) \Rightarrow \pi_t(UK[n,q,k]),$$

with differentials $d^r\colon E^r_{s,t} \to E^r_{s-r,t-1}$. Write ι to denote the fundamental class in $\pi_n K[n,q]{=}E^2_{0,n}$ and κ for the fundamental class in $\pi_{n+1} K[n{+}1,2^k q]{=}E^2_{n+1,n+1}$. Since the spectral sequence is multiplicative, it suffices to show that there are no nonzero differentials on the classes $\delta_I\iota$ and $\delta_I\kappa$. Clearly there are no possible nonzero differentials on $\delta_I\iota$, and the only differential on κ that hits a nonzero group is d^{n+1}, but $d^{n+1}\kappa$ cannot be ι since they differ in internal grading. Moreover, the groups that might support differentials hitting κ are all zero, so κ must survive to E^∞.

There is an obvious map of cofibration sequences from

$$U(K[n,q]) \to U(K[n,q,k]) \to U(K[n+1,2^k q])$$

to

$$\mathbb{F}_2[0] \to U(K[n+1,2^k q]) \xrightarrow{\mathrm{id}} U(K[n+1,2^k q]),$$

and the map $\mathbb{F}_2[0] \to U(K[n+1,2^k q])$ is a cofibration since $U(K[n+1,2^k q])$ is free. Thus we get a morphism of spectral sequences, given on the E^2–page by projection to $\pi_* U(K[n+1,2^k q])$ and on the E^∞–page by $\pi_*\beta$. This means that $\pi_*\beta$ must send κ to the fundamental class κ' in $\pi_* U(K[n+1,2^k q])$, and so for any admissible sequence I the class $\delta_I\kappa$ is mapped to $\delta_I\kappa'$, which is nonzero. This implies that $\delta_I\kappa$ must also survive to E^∞ for all I. By multiplicativity, this means the spectral sequence has no nonzero differentials, ie it collapses on the E^2–page. □

Corollary 4.16 *There is an isomorphism*

$$\pi_* U(K[n,q,k]) \cong \mathfrak{S}\mathbb{F}_2[n,q] \otimes \mathfrak{S}\mathbb{F}_2[n+1,2^k q]$$

of algebras over the triple $\mathfrak{S}$.

Proof The map $\pi_*\beta\colon \pi_*UK[n,q,k]\to\pi_*UK[n+1,2^kq]$ is surjective, as we saw in the proof of Proposition 4.15. Since $\pi_*UK[n+1,2^kq]$ is free, choosing a preimage of the generator gives a map $\pi_*UK[n+1,2^kq]\to\pi_*UK[n,q,k]$ of $\mathfrak{S}$–algebras. Since the tensor product is the coproduct, we get a map

$$\pi_*UK[n,q]\otimes\pi_*UK[n+1,2^kq]\to\pi_*UK[n,q,k]$$

of $\mathfrak{S}$–algebras. Filter the left-hand side by degree and the right-hand side by the filtration from the Serre spectral sequence. The collapse of this spectral sequence implies that this gives an isomorphism of the graded objects associated to the filtration, hence this map is an isomorphism of bigraded vector spaces and so also an isomorphism of $\mathfrak{S}$–algebras. □

Combining Lemmas 4.11 and 4.13 and Corollary 4.16 now completes the proof of Theorem 4.8. Applying this to the E_2–term of our spectral sequence, we deduce the following:

Corollary 4.17

(i) *If E is a connected spectrum of finite type, the E_2–term of the spectral sequence for $\mathrm{H}^*\Omega^\infty E$ is of the form*

$$E_2\cong UD(\mathrm{H}^*E)[0]\otimes\mathfrak{S}(\Sigma\Omega\mathbb{L}_{*>0}D(\mathrm{H}^*E))\otimes\mathfrak{S}(\Sigma\Omega_1\mathbb{L}_{*>0}D(\mathrm{H}^*E)[1]).$$

(ii) *If in addition the top squares in $\mathbb{L}_*D(\mathrm{H}^*E)$ are all zero for $*>0$, then the E_2–term is given by*

$$E_2\cong UD(\mathrm{H}^*E)[0]\otimes\mathfrak{E}(\mathbb{L}_{*>0}D(\mathrm{H}^*E)).$$

4.3 A functorial description of the derived functors

The description of π_*UM for a simplicial restricted vector space we obtained above is compatible with the products and δ–operations. However, in the case of interest, M is the underlying simplicial restricted vector space of a simplicial unstable A–module; this means that there are also Steenrod operations on π_*UM. We will now give a more functorial description of π_*UM that is also compatible with these operations.

If M is a simplicial unstable A–module, we have a natural transformation $M\to\Sigma\Omega M$, which induces a map $\pi_*UM\to\pi_*U\Sigma\Omega M$. Since the top squares in $\Sigma\Omega M$ are all zero, as a simplicial commutative algebra $U\Sigma\Omega M$ is isomorphic to $E\Sigma\Omega M$, and hence $\pi_*U\Sigma\Omega M$ is isomorphic to $\mathfrak{E}(\pi_*\Sigma\Omega M)$. We can also easily describe the action of the Steenrod operations here, using the following observation of Dwyer:

Proposition 4.18 [9, Proposition 2.7] *Let R be a simplicial unstable algebra over the Steenrod algebra; then $\pi_* R$ supports both higher divided squares and Steenrod operations. These are related as follows:*

$$\mathrm{Sq}^k\,\delta_i = \begin{cases} 0 & \text{for } k \text{ odd,} \\ \delta_i\,\mathrm{Sq}^{k/2} & \text{for } k \text{ even,} \end{cases}$$

for $i \geq 2$. Moreover, if all squares in R are zero, then the same is true for $i = 1$.

Proof Write $\mathrm{Sq} := \mathrm{Sq}^0 + \mathrm{Sq}^1 + \cdots$. By the Cartan formula, $\mathrm{Sq}\colon R \to R$ is an algebra homomorphism. Since the operation δ_i is natural, this means $\delta_i\,\mathrm{Sq} = \mathrm{Sq}\,\delta_i$ in $\pi_* R$. Considering the homogeneous parts in each internal degree on both sides gives the result. □

We first consider the case where M is a simplicial unstable A–module such that $\pi_0 M = 0$:

Proposition 4.19 *Suppose M is a levelwise projective simplicial unstable A–module such that $\pi_0 M = 0$. Then there is a natural isomorphism of commutative bigraded $\mathbb{F}_2$–algebras*

$$\pi_* UM \to \mathfrak{S}(\pi_* \Sigma\Omega M),$$

compatible with Steenrod operations and higher divided squares.

Proof Observe that if N is a simplicial unstable A–module such that $\pi_0 N = 0$ and the top squares in N vanish (such as $N = \Sigma\Omega M$), then we have an isomorphism $\mathfrak{E}(\pi_* N) \cong \mathfrak{S}(\pi_* N) \otimes \mathfrak{S}(\delta_1 \pi_* N)$, compatible with the Steenrod operations. Moreover, the inclusion $\mathfrak{S}(\pi_* N) \hookrightarrow \mathfrak{E}(\pi_* N)$ and retraction $\mathfrak{E}(\pi_* N) \to \mathfrak{S}(\pi_* N)$ are compatible with Steenrod operations. Taking $N = \Sigma\Omega M$ we thus have a natural map of graded algebras

$$\pi_* UM \to \mathfrak{E}(\pi_* \Sigma\Omega M) \to \mathfrak{S}(\pi_* \Sigma\Omega M),$$

compatible with Steenrod squares and δ–operations. We will show that this map is an isomorphism.

M is weakly equivalent to a coproduct of copies of $K[n,q]$ and $K[n,q,k]$, by Corollary 4.7. We know U preserves weak equivalences and colimits by Proposition 4.9, and Ω preserves coproducts and weak equivalences between levelwise projective objects. Thus it suffices to prove the result when M is $K[n,q]$ or $K[n,q,k]$.

In the first case, $\Sigma\Omega K[n,q]$ is $K(\mathbb{F}_2[n,q])$, and by Lemma 4.11 we know that $UK[n,q]$ is $SK(\mathbb{F}_2[n,q])$ for $n > 0$, so the map $UK[n,q] \to U\Sigma\Omega K[n,q]$ is the natural map $SK(\mathbb{F}_2[n,q]) \to EK(\mathbb{F}_2[n,q])$. On homotopy this is just the inclusion of the factor $\mathfrak{S}(\mathbb{F}_2[n,q])$.

When $n=0$, we have $\Sigma\Omega K[n,0] \simeq 0$, so

$$\pi_* UK[n,0] \to \pi_* U\Sigma\Omega K[n,0] \to \mathfrak{S}(\pi_*\Sigma\Omega K[n,0])$$

is the identity map on $\mathbb{F}_2[0]$ by Lemma 4.13.

For $UK[n,q,k]$ we consider the extension sequence

$$UK[n,q] \to UK[n,q,k] \to UK[n+1,2^k q].$$

On homotopy, this leads to a commutative diagram

$$\begin{tikzcd}
\pi_* UK[n,q] \arrow[r] \arrow[d] & \mathfrak{S}(\pi_*\Sigma\Omega K[n,q]) \arrow[d] \\
\pi_* UK[n,q,k] \arrow[r] \arrow[d] & \mathfrak{S}(\pi_*\Sigma\Omega K[n,q,k]) \arrow[d] \\
\pi_* UK[n+1,2^k q] \arrow[r] & \mathfrak{S}(\pi_*\Sigma\Omega K[n+1,2^k q]).
\end{tikzcd}$$

Here we have already shown that the top and bottom horizontal morphism are isomorphisms. But the chain complex $\Sigma\Omega C(q,k)$ is clearly $\mathbb{F}_2[q] \oplus \Sigma\mathbb{F}_2[2^k q]$, so the right vertical maps are a split extension sequence. Moreover, from the proof of Corollary 4.16 we know that the lower left vertical map is surjective, and that choosing a preimage of the generator gives an isomorphism

$$\mathfrak{S}(\pi_*\Sigma\Omega K[n,q]) \otimes \mathfrak{S}(\pi_*\Sigma\Omega K[n+1,2^k q]) \xrightarrow{\sim} \pi_* UK[n,q,k].$$

The composite of this with the map $\pi_* UK[n,q,k] \to \mathfrak{S}(\pi_*\Sigma\Omega K[n,q,k])$ is also an isomorphism (since it is determined by where it sends the generators). Thus by the 2-out-of-3 property the middle horizontal map here must also be an isomorphism, which completes the proof. □

For a general simplicial unstable A–module M we have a projection $M \to \pi_0 M[0]$. Writing $M_{>0}$ for the fiber of this map, we have a pushout square

$$\begin{tikzcd}
M_{>0} \arrow[r] \arrow[d] & M \arrow[d] \\
0 \arrow[r] & M_0
\end{tikzcd}$$

where M_0 is weakly equivalent to $\pi_0 M[0]$ (as can be seen from the long exact sequence in homotopy groups). Thus we have a pushout diagram

$$\begin{tikzcd} UM_{>0} \arrow[r] \arrow[d] & UM \arrow[d] \\ \mathbb{F}_2 \arrow[r] & UM_0 \end{tikzcd}$$

of simplicial unstable A–algebras. On homotopy we thus have maps

$$\pi_* UM_{>0} \to \pi_* UM \to U\pi_0 M[0],$$

where the second map is an isomorphism on π_0. We thus have a canonical map $U\pi_0 M[0] \to \pi_* UM$, and since the tensor product is the coproduct here we get a map

$$\pi_* UM_{>0} \otimes U\pi_0 M[0] \to \pi_* UM,$$

compatible with all the operations in play. Moreover, this is an isomorphism; as usual, this follows from considering the case where M is $K[n,q]$ or $K[n,q,k]$.

Theorem 4.20 *Suppose M is a simplicial unstable A–module, and let $M' \to M$ be a weak equivalence where M' is levelwise projective. Then we have a natural isomorphism of commutative bigraded $\mathbb{F}_2$–algebras*

$$\mathfrak{S}(\pi_{*>0}\Sigma\Omega M') \otimes U\pi_0 M[0] \xrightarrow{\sim} \pi_* UM,$$

compatible with Steenrod operations and higher divided squares. Moreover, there is a short exact sequence

$$0 \to \Sigma\Omega_1 \pi_{*>0} M[1] \to \pi_{*>0}\Sigma\Omega M' \to \Sigma\Omega\pi_{*>0} M \to 0$$

of graded unstable A–modules.

Proof Since U preserves weak equivalences by Proposition 4.9, $\pi_* UM' \cong \pi_* UM$, so the desired isomorphism follows from Proposition 4.19. The short exact sequence is a consequence of the hyperhomology spectral sequence

$$\Sigma\Omega_s \pi_t M \Rightarrow \pi_{s+t}\Sigma\Omega M$$

(see for example [14]), which collapses. □

Corollary 4.21 *Let X be a connected spectrum of finite type. In the infinite loops spectral sequence for X the E_2–term is isomorphic to*

$$\mathfrak{S}(\mathbb{L}_{*>0}(\Sigma\Omega D)(\mathrm{H}^* X)) \otimes UD(\mathrm{H}^* X),$$

compatibly with products, Steenrod operations, and higher divided squares. Moreover, *there is a short exact sequence*

$$0 \to \Sigma\Omega_1 \mathbb{L}_{*>0} D(\mathrm{H}^* X)[1] \to \mathbb{L}_{*>0} \Sigma\Omega D(\mathrm{H}^* X) \to \Sigma\Omega \mathbb{L}_{*>0} D(\mathrm{H}^* X) \to 0$$

of graded unstable A–modules.

Proof Recall the E_2–term is obtained by applying UD to a free simplicial resolution of $\mathrm{H}^* X$ as an A–module. Applying D to this free resolution gives a levelwise free simplicial unstable A–module, so this is immediate from Theorem 4.20. □

5 Examples

Corollary 4.17 reduces the analysis of the E_2–term of our spectral sequence to the computation of the derived functors of D. In this section we will apply results about these functors from the literature to describe the spectral sequence in two simple examples.

5.1 Eilenberg–Mac Lane spectra

The spectral sequence is clearly trivial for Eilenberg–Mac Lane spectra having the form $\Sigma^k \mathrm{H}\mathbb{F}_2$. For a slightly less trivial example, consider the Eilenberg–Mac Lane spectra $\Sigma^k \mathrm{H}\mathbb{Z}$ and $\Sigma^k \mathrm{H}\mathbb{Z}/2^n$, where k must be positive for our convergence result to apply. The mod-2 cohomology of the spectrum $\mathrm{H}\mathbb{Z}$, originally computed by Serre [29], is the A–module A/Sq^1. Since $\operatorname{Sq}^1 \operatorname{Sq}^1 = 0$, this has a simple free resolution, namely

$$\cdots \to \Sigma^2 A \xrightarrow{\cdot \operatorname{Sq}^1} \Sigma A \xrightarrow{\cdot \operatorname{Sq}^1} A.$$

From this we see that, writing $F(n) = D(\Sigma^n A)$ for the free unstable A–module on a generator in degree n, the derived functors $\mathbb{L}_* D(\mathrm{H}^* \Sigma^k \mathrm{H}\mathbb{Z})$ are given by the cohomology of the complex

$$\cdots \to F(k+2) \xrightarrow{\cdot \operatorname{Sq}^1} F(k+1) \xrightarrow{\cdot \operatorname{Sq}^1} F(k).$$

But it is easy to see that this complex is exact for $k > 0$, and so $\mathbb{L}_* D(\mathrm{H}^* \Sigma^k \mathrm{H}\mathbb{Z})$ is 0 for $* > 0$. It follows that our spectral sequence has only a single column, and so collapses to give

$$\mathrm{H}^*(K(\mathbb{Z}, k)) = \mathrm{H}^*(\Omega^\infty \Sigma^k \mathrm{H}\mathbb{Z}) \cong U(F(k)/\operatorname{Sq}^1).$$

Similarly, the spectrum $\Sigma^k \mathrm{H}\mathbb{Z}/2^n$ has cohomology $A/\operatorname{Sq}^1 \oplus \Sigma A/\operatorname{Sq}^1$, so again D has no derived functors and $\mathrm{H}^*(K(\mathbb{Z}/2^n, k)) \cong U(F(k)/\operatorname{Sq}^1) \otimes U(F(k+1)/\operatorname{Sq}^1)$. Of course, these results agree with Serre's computations in [29].

5.2 Suspension spectra

In this subsection we consider the spectral sequence for infinite loop spaces of the form $\Omega^\infty\Sigma^\infty X$, where X is a connected space. We will show, by a dimension-counting argument, that the spectral sequence collapses in this case.

The cohomology $\mathrm{H}^*(\Sigma^\infty X) \cong \widetilde{\mathrm{H}}^* X$ is an unstable A–module. Lannes and Zarati [19] computed the derived functors $\mathbb{L}_*(D)(M)$ for an unstable A–module M. An alternative computation (in the dual, homological, case), using a chain complex originally due to Singer [30], has been given by Kuhn and McCarty [18], and we will use their formulation of the result. Before stating this, we must introduce some notation:

Definition 5.1 Let M be an A–module. Let $\mathcal{R}_s M$ be the quotient of the graded $\mathbb{F}_2$–vector space generated by symbols $Q^I x$ in degree $|x| + i_1 + \cdots + i_s$, where $x \in M$ and $I = (i_1, \ldots, i_s)$, by the instability and Adém relations for the Dyer–Lashof algebra as well as linearity relations ($Q^I(x+y) = Q^I x + Q^I y$). This becomes an A–module via the (dual) Nishida relation

$$\mathrm{Sq}^i\, Q^j x = \sum_k \binom{j-k}{i-2k} Q^{i+j-k}\, \mathrm{Sq}^k\, x.$$

Let $d\colon \mathcal{R}_s(\Sigma M) \to \mathcal{R}_{s-1}(M)$ be defined by $d(Q^I \sigma x) = Q^{i_1,\ldots,i_{s-1}}(\mathrm{Sq}^{i_s+1}\, x)$; this is a map of A–modules. Writing $R_s M := \Sigma \mathcal{R}_s \Sigma^{s-1}$ we can think of d as a map $R_s M \to R_{s-1} M$.

The result, in the form given by [18, Theorems 4.22 and 4.34], is then:

Theorem 5.2 (Singer, Lannes and Zarati, Kuhn and McCarty) *Let M be an A–module.*

(i) *The sequence*

$$\cdots \to R_s M \xrightarrow{d} R_{s-1} M \to \cdots \to R_0 M$$

is a chain complex, and $\mathrm{H}_*(R_* M)$ is naturally isomorphic to $\mathbb{L}_*(D)(M)$.

(ii) *If M is an unstable A–module, then the differential in $R_* M$ is zero and thus $\mathbb{L}_* D(M) \cong R_* M$.*

(iii) *If M is an unstable A–module, then $\mathbb{L}_s D(M)$ is an s–fold suspension of an unstable module.*

By (iii), it follows that for M unstable all the top squares in $\mathbb{L}_s D(M)$ are zero for $s > 0$. By Corollary 4.17 we therefore have an isomorphism

$$\mathbb{L}_*(UD)(M) \cong U(M)[0] \otimes \mathfrak{E}(\mathbb{L}_{*>0} D(M)).$$

Thus if E is a spectrum such that $\mathrm{H}^* E$ is an unstable A–module, in the E^2–term of our spectral sequence for $\mathrm{H}^*(\Omega^\infty E)$ an element $v \in \mathrm{H}^k E$ gives:

- $\sigma Q^I \sigma^{s-1} v$ in degree $(-s, k + |I| + s)$, where $I = (i_1, \ldots, i_s)$ is an allowable sequence, ie $i_t \le 2 i_{t+1}$, and $i_1 > i_2 + \cdots + i_s + |v| + s - 1$ (for brevity we'll denote this element by $\overline{Q}^I v$);
- $\delta_J \overline{Q}^I v$ in degree $(-s - |J|, 2^l(k + |I| + s))$, for J an admissible sequence of length l.

The E_2–page of the spectral sequence is an exterior algebra on these generators.

Now suppose X is a connected space of finite type; then $\mathrm{H}^* X \cong (\mathrm{H}_*(X))^\vee$. In this case the spectral sequence for $\Sigma^\infty X$ converges by Theorem 2.1. We wish to compare the E_2–page to the known cohomology $\mathrm{H}^*(QX) \cong (\mathrm{H}_*(QX))^\vee$. Recall that the homology $\mathrm{H}_*(QX)$ can be described in terms of the Dyer–Lashof operations Q^j:

Theorem 5.3 (May [6]) *If X is a space, the homology $\mathrm{H}_*(QX)$ is a polynomial algebra on generators $Q^J v$ where v ranges over a basis of $\mathrm{H}_*(X)$, and $J = (j_1, \ldots, j_s)$ is an* allowable *sequence, meaning $j_t \le 2 j_{t+1}$ for all t, and $j_1 > j_2 + \ldots + j_s + |v|$. The element $Q^J v$ is in degree $|v| + |J|$.*

To see that the spectral sequence must collapse, it suffices to prove the following:

Proposition 5.4 *There is a grading-preserving bijection between the exterior algebra generators $\overline{Q}^I x$, $\delta_J \overline{Q}^I x$ of the E_2–page and the Dyer–Lashof operations $Q^K x$, together with their powers $(Q^K x)^{2^r}$, for each x in a basis for the reduced cohomology of X.*

Proof Write $k := |x|$. Observe that for any allowable sequence I with

$$i_1 > i_2 + \cdots + i_s + k + s - 1,$$

the total degree of $\overline{Q}^I x$ is the same as the degree of $Q^I x$. However, there are more nonzero Dyer–Lashof operations on x than those given by these sequences: we are missing those where

$$i_2 + \cdots + i_s + k < i_1 \le i_2 + \cdots + i_s + |x| + s - 1.$$

To relate these to the E_2–term, we change the indexing of the Dyer–Lashof operations: If $J = (j_1, \dots, j_s)$ is an allowable sequence, then for $Q^J v$ to be nonzero (and not a square) in $\mathrm{H}_*(QX)$ we must have, for positive integers $l_1, \dots, l_s$,

$$j_i = k + j_s + j_{s-1} + \cdots + j_{i+1} + l_i.$$

The allowability condition, expressed in terms of the l_i, says that $l_i \leq l_{i+1}$. Thus there exist nonnegative integers $a_1, \dots, a_s$ (with $a_1 > 0$) such that $l_{i+1} = l_i + a_{i+1}$ (and $l_1 = a_1$). In terms of the a_i the element $Q^J x$ has degree

$$2^s k + \sum_{j=1}^{s} \sum_{r=1}^{j} 2^{j-1} a_r = 2^s k + \sum_{r=1}^{s} \left(\sum_{j=r}^{s} 2^{j-1} \right) a_r = 2^s k + \sum_{r=1}^{s} (2^s - 2^{r-1}) a_r.$$

Let's write $q^{a_1,\dots,a_s} x$ for the element $Q^J x$ with J of this form. We also extend the notation by writing $q^{0,a_1,\dots,a_s} x$ for $(q^{a_1,\dots,a_s} x)^2$, etc.

Defining $\overline{q}^{a_1,\dots,a_s} x$ similarly, we see that $\overline{q}^{a_1,\dots,a_s} x$ and $q^{a_1+(s-1),a_2,\dots,a_s} x$ have the same degree in $\mathrm{H}_*(QX)$.

Now suppose δ_I is an admissible sequence of δ–operations of length l. Then there exist nonnegative integers r_t such that $i_l = r_l \geq 1$ and $i_t = 2i_{t+1} + r_t$ for $t < l$; in terms of the r_t the admissibility criterion says that $r_1 + \cdots + r_l \leq s$, and $|I| = \sum_i (2^i - 1) r_i$. Then the total degree of $\delta_I \overline{q}^{a_1,\dots,a_s} x$ is the same as the degree of $q^K x$, where

$$K = \left(s - \sum r_t, r_1, r_2, \dots, r_{l-1}, a_1 + r_l - 1, a_2, \dots, a_s \right).$$

To see that this gives a bijection between the generators, we describe its inverse: For $q^{b_1,\dots,b_\sigma} x$ in $H_*(QX)$, let L be the unique integer with $0 \leq L < \sigma$ such that

$$b_1 + \cdots + b_L + L < \sigma \leq b_1 + \cdots + b_{L+1} + L + 1.$$

Then we define

$$\begin{aligned} s &:= \sigma - L, \\ r_L &:= s - b_1 - \cdots - b_L - L \quad \text{and} \quad r_t := b_{t+1} \text{ for } t = 1, \dots, L-1, \\ a_1 &:= b_{L+1} - r_L + 1 \quad \text{and} \quad a_i := b_{L+i} \text{ for } i = 2, \dots, s. \end{aligned}$$

Then

$$(b_1, \dots, b_\sigma) = \left(s - L - \sum_{t=1}^{l} r_t, r_1, \dots, r_{L-1}, a_1 + r_L - 1, a_2, \dots, a_s \right),$$

so $q^{b_1,\dots,b_\sigma}$ corresponds to $\delta_I \overline{q}^{a_1,\dots,a_s} x$ where $I = (i_1, \dots, i_L)$ is the admissible sequence determined by the r_t, ie with $i_t := \sum_{j=t}^{L} 2^{j-t} r_j$. □

Corollary 5.5 *For X a connected space of finite type, the spectral sequence*

$$\mathbb{L}_*(UD)(\mathrm{H}^*X) \Rightarrow \mathrm{H}^*(QX)$$

collapses at the E_2–page.

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Homotopy groups of diagonal complements

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For X a connected finite simplicial complex we consider $\Delta^d(X,n)$, the space of configurations of n ordered points of X such that no $d+1$ of them are equal, and $B^d(X,n)$, the analogous space of configurations of unordered points. These reduce to the standard configuration spaces of distinct points when $d=1$. We describe the homotopy groups of $\Delta^d(X,n)$ (resp. $B^d(X,n)$) in terms of the homotopy (resp. homology) groups of X through a range which is generally sharp. It is noteworthy that the fundamental group of the configuration space $B^d(X,n)$ abelianizes as soon as we allow points to collide, ie $d\geq 2$.

55Q52; 55P10

In memory of Abbas Bahri so greatly missed

1 Introduction

Let X be a topological space and $\Delta_{d+1}(X,n)\subset X^n$ the union of the $(d+1)^{\text{st}}$ diagonal arrangement in X^n, that is,

$$\Delta_{d+1}(X,n)=\big\{(x_1,\dots,x_n)\in X^n \mid x_{i_0}=x_{i_1}=\dots=x_{i_d} \text{ for some sequence } 1\leq i_0<\dots<i_d\leq n\big\}.$$

Its complement in X^n is the *configuration space of no* $d+1$ *equal points in* X, which is written

$$\Delta^d(X,n)=X^n-\Delta_{d+1}(X,n).$$

This is the space of ordered tuples of n points in X with the multiplicity of each entry in the tuple *at most* d (hence the notation Δ^d as opposed to Δ_d for *at least* d). It is useful to think of these tuples as configurations of n ordered points in X with the property that d of the points can collide but not $d+1$. The symmetric group $\mathfrak{S}_n$ acts on $\Delta^d(X,n)$, and the quotient is denoted by $B^d(X,n)$.

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We have increasing filtrations

$$F(X,n) := \Delta^1(X,n) \subset \Delta^2(X,n) \subset \cdots \subset \Delta^n(X,n) = X^n, \tag{1}$$
$$B(X,n) := B^1(X,n) \subset B^2(X,n) \subset \cdots \subset B^n(X,n) = \mathrm{SP}^n X,$$

with $\mathrm{SP}^n X := X^n/\mathfrak{S}_n$ being the n^{th} symmetric product. Here we have written $F(X,n)$ and $B(X,n)$ for the standard configuration spaces of ordered (resp. unordered) pairwise distinct points of cardinality n. Various other notations for $F(X,n)$ in the literature include $C_n(X)$, $\mathrm{Conf}_n(X)$, etc, while $B(X,n)$ is sometimes written $\mathrm{Braid}(X,n)$ in the geometric topology literature; reminiscent of the fact that its fundamental group is the so-called n^{th} braid group of X.

In some exceptional cases, the spaces $\Delta^d(X,n)$ and $B^d(X,n)$ can be empty (if, for example, X is a point and $d < n$), but otherwise they have a rich and interesting geometry; see Kallel and Taamallah [18]. An early appearance of $\Delta^d(X,n)$ is in paper of Cohen and Lusk [8] in connection with Borsuk–Ulam type results while more recent applications to the *colored Tverberg theorem for manifolds* appear in Blagojević, Matschke and Ziegler [4]. In the case V is a vector space, the spaces $\Delta^d(V,n)$ are subspace complements dubbed *non-*$(d+1)$*–equal arrangements* in Björner and Welker [3], and their homology is made explicit in Dobrinskaya and Turchin [9] as an algebra over the little disks operad, with interesting applications to the spaces of non-d–equal immersions. In the case $X = \mathbb{C}$, the spaces $B^d(\mathbb{C},n)$ are intimately related to spaces of based holomorphic maps from the Riemann sphere into complex projective space $\mathbb{P}^d$; see Guest, Kozlowski and Yamaguchi [12] and Kallel [16]. In all cases, these spaces seem to have been studied so far exclusively for when X is a manifold. One of our objectives in this paper is to give some sharp results on the homology and homotopy groups of the non-d–equal configurations of X when X is a more general polyhedral space.

Throughout this paper, a space X is a finite simplicial complex, that is, the realization of a finite abstract simplicial complex. Unless specified, all spaces are connected.

Theorem 1.1 *Let X be a connected finite simplicial complex that is not a point, and $d,n \geq 2$. Then*

$$\pi_i(B^d(X,n)) \cong \pi_i(\mathrm{SP}^n(X)) \quad \text{for } 0 \leq i \leq 2d-2.$$

In particular $\pi_1(B^d(X,n)) \cong H_1(X;\mathbb{Z})$ when $d \geq 2$, $n \geq 2$. Moreover, if X is simply connected, $2 \leq d \leq n$, then

$$\pi_i(B^d(X,n)) \cong \widetilde{H}_i(X;\mathbb{Z}) \quad \text{for } 0 \leq i \leq 2d-2,$$

where $\widetilde{H}(-;\mathbb{Z})$ is reduced integral homology.

The bound $2d-2$ in the theorem is sharp as is illustrated by the case where X a Euclidean space; see Section 4. Note that the special case of the fundamental group says that allowing a single collision is enough to abelianize the fundamental group. This can be expected since collisions kill the braiding; see Section 8.

The homotopy groups of $\Delta^d(X,n)$ turn out to depend on local connectivity properties of the space. We say X has *local homotopical dimension* r if for any $x \in X$ and any neighborhood U of x, there is an open neighborhood $V \subset U$ of x such that $V - \{x\}$ is r–connected; see Definition 7.1.

Theorem 1.2 *Let X be a locally finite simplicial complex with local homotopical dimension $r \geq 0, d \geq 1$. Then*

$$\pi_i(\Delta^d(X,n)) \cong \pi_i(X)^n \quad \text{for } i \leq rd + 2d - 2.$$

Remark 1.3 If d is at least n, both spaces are equal $\Delta^d(X,n) = X^n$ and all homotopy groups agree. When $d < n$ this bound is in general optimal as can be seen in the case of manifolds. For example, $\mathbb{R}^2$ has local homotopical dimension 0 and $\Delta^d(\mathbb{R}^2, d+1) \simeq S^{2d-1}$ is precisely $2d{-}2$–connected.

Remark 1.4 For a polyhedral pair (X, Y), the *homotopical depth* of Y in X is set to be n if the pair $(X, X \setminus Y)$ is n–connected; see Eyral [10]. Theorem 1.2 is saying that the homotopical depth of the diagonal arrangement $\Delta_{d+1}(X,n)$ in X^n is at least $rd + 2d - 2$. This appears to be the first complete such calculation for this kind of arrangements of subspaces.

To prove both of these theorems, we use a localization principle for homotopy groups, Theorem 4.2, relating the local connectivities of pairs $(V, V \setminus Y)$ to the global connectivity of $(X, X \setminus Y)$ for closed $Y \subset X$ and V local neighborhoods in a cover. In both cases the proof reduces to studying the case of V being the union of various simplices joining along a simplex. For Theorem 1.2, the argument amounts to giving a homotopical decomposition of $\Delta^d(V,n)$ when V is such a union. We recall that by a homotopical decomposition of a space X we mean a *diagram* $\mathcal{D}\colon I \to \mathrm{Top}$; ie a functor from a small category I to the category of topological spaces and continuous maps, so that the map $\operatorname{hocolim}_I \mathcal{D} \to \operatorname{colim}_I \mathcal{D} \cong X$ is a weak equivalence; see Section 7. Our decomposition extends similar results of Sun [25]. Since we are able to control the connectivity of each space making up the diagram, we are able to derive our bound.

Theorem 1.1 on the other hand relies on a different argument. First we treat the case of a manifold based on the idea of scanning maps. The general case appeals to a theorem of Smale [24] relating the connectivity of a map to that of its preimages.

Since Smale's theorem works for proper maps, a technical issue we have to deal with is the construction in Section 5 of a $\mathfrak{S}_n$–equivariant simplicial complex which is a deformation retract of $\Delta^d(X,n)$ for X again a finite complex. As pointed out by the referee, similar techniques are in Björner et al [2, chapter 4] and have been applied to hyperplane arrangements by Blagojević and Ziegler [5], for example (see references therein). Section 5 is of independent interest and has relevance to more recent constructions of CW-retracts for configuration spaces; see Tamaki [26].

The first section of the paper discusses motivational examples and general connectivity results. The second section discusses the special case of graphs. Proposition 3.1 gives a simplified and then expanded version of a useful theorem of Morton, which is used to give an amusing description of the homotopy type of the configuration space of two points on a wedge of circles in Proposition 3.4.

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2 Preliminaries

We start with some classical examples of diagonal arrangements and their complements. The extreme cases $d=1$ and $d=n-1$ are most encountered in the literature. The case $\Delta^1(X,n)=F(X,n)$ corresponds to the configuration space of pairwise distinct points

$$F(X,n)=\{(x_1,\dots,x_n)\in X^n \mid x_i\neq x_j \quad \text{for } i\neq j\}.$$

The action of $\mathfrak{S}_n$ on $F(X,n)$ is free and we have a regular covering $F(X,n)\to B(X,n)$. If X is a manifold of dimension greater than 2, then $\pi_1(F(X,n))\cong\pi_1(X^n)$ by a codimension argument (see Proposition 2.5), while $\pi_1(B(X,n))$ is a wreath product $\pi_1(X)\wr\mathfrak{S}_n$; this is standard, but a leisurely exposition can be found in [15].

Example 2.1 When $d=n-1$, $B^{n-1}(X,n)$ is the complement in $\mathrm{SP}^n(X)$ of the diagonal embedding $\Delta\colon X\hookrightarrow \mathrm{SP}^n X$, $x\mapsto[x,\dots,x]$. When $X=\mathbb{C}$, the elementary symmetric functions give a diffeomorphism $\mathrm{SP}^n(\mathbb{C})\cong\mathbb{C}^n$ and the image of $\Delta(\mathbb{C})$ corresponds under this diffeomorphism to the rational normal curve V diffeomorphic to the Veronese embedding $x\mapsto(x,x^2,\dots,x^n)$. One can check that

$$B^{n-1}(\mathbb{C},n)\cong\mathrm{SP}^n(\mathbb{C})-V\simeq S^{2n-3}.$$

A short proof of this equivalence is given in [12, Lemma 2.7], while another quick argument would be to use simple connectivity of $B^{n-1}(\mathbb{C},n)$ and Alexander duality. In general, for $\mathbb{R}^k$, $k \geq 2$, $\Delta^{n-1}(\mathbb{R}^k,n) = (\mathbb{R}^k)^n - \Delta$ is the complement of the thin diagonal, and this deforms onto the orthogonal complement of the diagonal $\Delta = \{(x,\ldots,x)\}$ minus the origin, so that $\Delta^{n-1}(\mathbb{R}^k,n)$ is, up to homotopy, the unit sphere S^{nk-k-1} in $\{(x_1,\ldots,x_n)\in(\mathbb{R}^k)^n \mid \sum x_i = 0\} = \Delta^{\perp}$. This deformation can be made equivariant with respect to the permutation action of $\mathfrak{S}_n$ so that the $\mathfrak{S}_n$–quotient is $B^{n-1}(\mathbb{R}^k,n)$. We show below that this space is simply connected as soon as n is at least 3 (in fact it is $2n-4$–connected; Lemma 4.12).

Lemma 2.2 *If S is the unit sphere in $H = \{(v_1,\ldots,v_n) \in (\mathbb{R}^k)^n \mid \sum v_i = 0\}$, and if $\mathfrak{S}_n$ acts on H, and hence on S, by permutation of coordinates, then the quotient $Q_{n,k} := S/\mathfrak{S}_n$ is simply connected whenever $nk-k-1 \geq 2$.*

Proof We use the following useful main result of Armstrong [1]: let G be a discontinuous group of homeomorphisms of a path connected, simply connected, locally compact metric space X, and let H be the normal subgroup of G generated by those elements that have fixed points; then the fundamental group of the orbit space X/G is isomorphic to the factor group G/H. We apply this result to $G = \mathfrak{S}_n$ and $X = S$, which is simply connected. The point is that when $n \geq 3$, the fixed points of the permutation action are of the form $(v_1,\ldots,v_n)$ with $v_i = v_j$ for some $i < j$, which means that all transpositions are in H and hence $G = H$. □

The argument of Armstong used in the proof of Lemma 2.2 implies that if $\Delta^d(X,n)$ is simply connected, then $\pi_1(B^d(X,n))$ is the quotient of $\mathfrak{S}_n$ by the normal subgroup generated by elements having fixed points, and this subgroup is the entire group if $d \geq 2$. This establishes a useful conclusion.

Corollary 2.3 *If $\Delta^d(X,n)$ is simply connected, then so is $B^d(X,n)$ if $d \geq 2$.*

The following result, valid for smooth manifolds, is a special case of Theorem 1.1.

Proposition 2.4 *When $X = M$ is a closed smooth, ie C^∞, manifold,* $\dim M \geq 2$*, and $n \geq 3$, then $\pi_1(B^{n-1}(M,n))$ is isomorphic to $H_1(M;\mathbb{Z})$.*

Proof A tubular neighborhood of the diagonal copy of M in $\mathrm{SP}^n M$ can be identified with the total space of the following subbundle. Let $TM^{\oplus n}$ be the n–fold Whitney sum of the tangent bundle TM of M, $\dim M = m$, and let η be the subbundle with fiber $H = \{(v_1,\ldots,v_n) \mid \sum v_i = 0\}$. The total space of this subbundle is homeomorphic to a neighborhood of diagonal M in $M^{\times n}$. Now $\mathfrak{S}_n$ acts on this bundle fiberwise (linearly on each fiber) and the fiberwise quotient ζ has fiber $H/\mathfrak{S}_n$ which can be identified with the cone $c(S^{(n-1)m-1}/\mathfrak{S}_n)$, where $\dim M = m$ and $S^{(n-1)m-1}$ is the unit sphere in H. According to [18, Proposition 4.1], for a smooth closed manifold M, a neighborhood deformation retract V of the diagonal M in $\mathrm{SP}^n M$ is homeomorphic to the total space of ζ. The fiberwise apexes of the fiberwise cone give the zero section of this bundle. The complement of this section is $S(M)$ which is, up to fiberwise equivalence, a bundle over M with fiber $S^{(n-1)m-1}/\mathfrak{S}_n$. By construction we have the homotopy pushout

$$\begin{tikzcd} S(M) \arrow[r, "\pi"] \arrow[d] & M \arrow[d, "\Delta"] \\ B^{n-1}(M,n) \arrow[r, "\iota"] & \mathrm{SP}^n M \end{tikzcd}$$

If $n=2$, $m\geq 2$, $S(M)$ is the projectivized tangent bundle with fiber $S^{m-1}/\mathbb{Z}_2 = \mathbb{R}P^{m-1}$. When $n\geq 3$ and $m\geq 2$, $S(M)$ has simply connected fiber (Lemma 2.2) so that π induces an isomorphism on fundamental groups, and by the van Kampen theorem, ι induces an isomorphism on π_1 as well; ie $\pi_1(B^{n-1}(M,n))\cong\pi_1(\mathrm{SP}^n M)\cong H_1(M;\mathbb{Z})$ for $n\geq 3$. $\square$

To complete this section, we state a well-known result which will be seen in Section 4 as a special manifestation of the localization principle.

Proposition 2.5 *If $S=\bigcup S_j$ is a finite union of submanifolds of a smooth manifold M, closed with real codimension $d\geq 2$, then the inclusion $M-S\hookrightarrow M$ induces an isomorphism on homotopy groups π_i for $0\leq i\leq d-2$, and an epimorphism on π_{d-1}.*

A proof of the above proposition, using standard transversality arguments, can be found, for example, in [14, Lemma 5.3]. This proposition is not true if the ambient space is not a manifold. For example, $B(\mathbb{R}^m,2)$ is the complement of the diagonal in $\mathrm{SP}^2(\mathbb{R}^m)$ and we have the homotopy equivalence $B(\mathbb{R}^m,2)\simeq\mathbb{R}P^{m-1}$ so that $\pi_1(B(\mathbb{R}^m,2))\cong\mathbb{Z}_2$ no matter the codimension of the diagonal $m\geq 3$.

As a consequence we have the following precursor of Theorem 1.1.

Corollary 2.6 *If X is a topological surface and $d\geq 2$, then $\pi_1(B^d(X,n))\cong H_1(X,\mathbb{Z})$.*

Proof The real plane $\mathbb{R}^2$ has the special property that $\mathrm{SP}^n(\mathbb{R}^2)$ is diffeomorphic to $\mathbb{R}^{2n}$. This implies right away that when S is a topological surface, $\mathrm{SP}^n(S)$ is a manifold of dimension $2n$, and that $B_{d+1}(X,n)$ is the union of submanifolds of dimension at most $2(n-d)=2n-2d$. This means that $B^d(S,n)=\mathrm{SP}^n(S)-B_{d+1}(S,n)$ is the complement of a finite union of submanifolds of codimension at least $2d>2$. By Proposition 2.5, $\pi_1(B^d(S,n))\cong\pi_1(\mathrm{SP}^n S)$ and this is again $H_1(S,\mathbb{Z})$ for $n>1$. $\square$

3 The case of the circle

Write $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, ie the unit circle in the complex plane. There is a map $B^d(S^1, n) \to S^1$ which multiplies the points of a configuration in S^1. This map is well defined since S^1 is abelian. This map turns out to have contractible fibers so that in particular $B^d(S^1, n) \simeq S^1$; see Proposition 3.2.

Let $\Delta_{n-1} = \{(s_1, \dots, s_n) \mid 0 \le s_i \le 1, \sum s_i = 1\}$ be the $n-1$–dimensional simplex and write $\Delta_{n-1}(d)$ the partial compactification of the open simplex $\mathring{\Delta}_{n-1}$, where we allow at most d *consecutive* s_i to be zero (using cyclic ordering, ie s_n and s_1 are consecutive to each other). In particular $\Delta_n(1) = \mathring{\Delta}_{n-1}$. We will write $\mathbb{Z}_n$ for the cyclic group of order n. Using a similar action as in [6, page 407], we have the following.

Proposition 3.1 *Let $\mathbb{Z}_n$ with multiplicative generator τ act on $S^1 \times \Delta_{n-1}(d)$ via*

$$\tau(e^{i\theta}, s_1, \dots, s_n) = (e^{i\theta + i2\pi s_1}, s_2, \dots, s_n, s_1).$$

Then the quotient by the action, written $S^1 \ltimes_{\mathbb{Z}_n} \Delta_{n-1}(d)$, is homeomorphic to $B^d(S^1, n)$. When $d = 1$, there is a $\mathfrak{S}_n$–equivariant homeomorphism

$$F(S^1, n) \cong (S^1 \times \mathring{\Delta}_{n-1}) \times_{\mathbb{Z}_n} \mathfrak{S}_n.$$

Proof The cyclic group appears for a simple reason: any configuration $(x_1, \dots, x_n)$ can be brought into a unique counterclockwise configuration up to cyclic permutation. More precisely let $(x_1, \dots, x_n) \in \Delta^d(S^1, n)$. Then there is a permutation $\sigma \in \mathfrak{S}_n$ bringing this configuration to a counterclockwise ordering $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Let s_i be the arc distance (divided by 2π) measured counterclockwise between $x_{\sigma(i)}$ and $x_{\sigma(i+1)}$. When $x_i \neq x_j$ for $i \neq j$, the choice of σ is unique up to cyclic permutation and there is a well-defined map

$$\begin{gathered} F(S^1, n) \to (S^1 \times \mathring{\Delta}_{n-1}) \times_{\mathbb{Z}_n} \mathfrak{S}_n, \\ (x_1, \dots, x_n) \mapsto [(x_{\sigma(1)}, (s_1, \dots, s_n)); \sigma], \end{gathered}$$

which is a homeomorphism. Here $(s_1, \dots, s_n)$ is in the open simplex $\mathring{\Delta}_{n-1}$ if and only if none of the s_i are zero. When there is collision, ie $d > 1$, then the choice of σ, up to cyclic permutation, is not unique anymore, but there is a map at the level of unordered configuration spaces

$$\begin{gathered} B^d(S^1, n) \to S^1 \ltimes_{\mathbb{Z}_n} \Delta_{n-1}(d), \\ [x_1, \dots, x_n] \mapsto [x_{\sigma(1)}; (s_1, \dots, s_n)], \end{gathered}$$

where σ again is any permutation bringing $(x_1, \dots, x_n)$ into cyclic ordering.

This map is independent of the choice of σ and it is a homeomorphism with inverse

$$[x_{\sigma(1)};(s_1,\dots,s_n)]\mapsto[x_{\sigma(1)},x_{\sigma(1)}e^{i2\pi s_1},x_{\sigma(1)}e^{i2\pi(s_1+s_2)},\dots,x_{\sigma(1)}e^{i2\pi(s_1+\dots+s_{n-1})}].$$

Note that when $x_i=x_{i+1}$ in the cyclic ordering, $s_i=0$, so the faces of Δ_{n-1} where the s_i vanish (consecutively) correspond to when points come together. □

Proposition 3.2 *Identify $S^1=[0,1]/_{0\sim 1}$. Then addition*

$$m\colon B^d(S^1,n)\to S^1,\quad m([x_1,\dots,x_n])=x_1+x_2+\dots+x_n$$

is a bundle map with fiber $\Delta_{n-1}(d)$. In particular m is a homotopy equivalence.

Proof The composite

$$\rho\colon S^1\times_{\mathbb{Z}_n}\Delta_{n-1}(d)\to B^d(S^1,n)\xrightarrow{m}S^1$$

sends $(x,(s_1,\dots,s_n))$ to $nx+(n-1)s_1+(n-2)s_2+\dots+s_{n-1}$. This map is well defined on orbits since $\rho(x+s_1,(s_2,\dots,s_n,s_1))=\rho(x,(s_1,\dots,s_n))$. The preimage of a point $y\in S^1$ under m are all unordered tuples $[x_1,\dots,x_n]$ such that $x_1+x_2+\dots+x_n=y \mod \mathbb{Z}$. All preimages are homeomorphic and we can choose $y=0$. The preimage $\rho^{-1}(0)$ consists of all classes $[x,(s_1,\dots,s_n)]$ such that

$$(n-1)s_1+(n-2)s_2+\dots+s_{n-1}+nx \mod \mathbb{Z}.$$

We wish to show this is a copy of $\Delta_{n-1}(d)$. Consider the map $\phi\colon\Delta_{n-1}(0)\to\rho^{-1}(0)$ defined as follows. Given $(s_1,\dots,s_n)$, $\sum s_i=1$, let

$$m_s=\frac{-1}{n}((n-1)s_1+(n-2)s_2+\dots+s_{n-1})$$

brought modulo $\mathbb{Z}$ to the interval $[0,1]$ and define

$$\phi\colon(s_1,\dots,s_n)\mapsto[m_s,(s_1,\dots,s_n)]\in S^1\times_{\mathbb{Z}_n}\Delta_{n-1}(d).$$

This map is well defined and continuous. It is surjective by construction. It is also injective for the following reason. If $s=(s_1,\dots,s_n)$ and $s'=(s'_1,\dots,s'_n)$ map to the same point under ϕ, they must be the same up to cyclic permutation. Let's assume $s'=(s_{k+1},\dots,s_n,s_1,s_2,\dots,s_k)$, $0<k<n$ $(s_0=s_n)$. A quick computation shows that

$$m_{s'}=m_s+s_1+\dots+s_k-k/n.$$

But in $S^1\times_{\mathbb{Z}_n}\Delta_{n-1}(d)$, $[m_{s'},(s'_1,\dots,s'_n)]=[m_s-k/n,(s_1,\dots,s_n)]$ so that $\phi(s')$ can never be $\phi(s)$ unless $k=0$ or $s_i=s'_i=1/n$. In both cases $s=s'$. This proves the injectivity and hence that ϕ is a homeomorphism. It remains to check that ρ is a bundle map and this is left as an exercise. □

Remark 3.3 (Morton) When $d=1$, $m\colon \mathrm{SP}^n(S^1)\to S^1$ is an $n-1$–disk bundle that is trivial if and only if n is odd. The open disk bundle is $B(S^1,n)$ and its sphere bundle is $B_2(S^1,n)$.

3A Wedges of circles

As discussed, $B^d(S^1,n)\simeq S^1$. The situation gets more complicated quickly for other graphs. The following is a neat little application of our constructions for the case $d=1$.

Proposition 3.4 *$B(\bigvee^k S^1,2)$ is homotopy equivalent to $\bigvee^{\frac{3}{2}k(k-1)+1} S^1$.*

Proof Let's first understand the $k=2$ case.

We will write $B(S^1\vee S^1,2)$ as the union of three subspaces:

$$X_1=\{[(x,*),(y,*)]\mid x\neq y\},\quad X_2=\{[(*,x),(*,y)]\mid x\neq y\},$$
$$X_3=\{[(x,*),(*,y)]\mid (x,y)\neq(*,*)\}.$$

We have that

$$X_1\cong B(S^1,2),\quad X_2\cong B(S^1,2),\quad X_3\cong (S^1\times S^1)^*,$$

where $(S^1\times S^1)^*$ means the punctured torus $S^1\times S^1-\{(*,*)\}$. Notice that $X_1\cap X_2=\varnothing$ while $X_1\cap X_3=\{(x,*),(*,*),x\neq *\}\cong (S^1)^*$ are punctured circles hence contractible intervals. The punctured torus X_3 deformation retracts onto a wedge $S^1\vee S^1$. During this deformation both punctured circles corresponding to the intersection with X_1 and X_2 retract onto the wedgepoint. After the retraction we obtain a wedge $S^1\vee S^1\vee Y_1\vee Y_2$ where each $Y_i=X_i/\sim$ is the open Möbius band $X_i=S^1\times]0,1[$ with an interval $*\times]0,1[$ retracted to a point. Therefore $Y_i\simeq S^1$ and the claim follows in this case.

For the general case of a bouquet of k–circles, $k>2$, we write an element from the i^{th} leaf as x^i. Then $B(\bigvee^k S^1,2)$ becomes the union of subspaces

$$X_{i,j}:=\{[(x^i,*),(y^j,*)]\mid x^i\neq y^j \text{ if } i=j\},$$
$$X_i^j:=\{[(x^i,*),(*,y^j)]\mid (x^i,y^j)\neq(*,*) \text{ if } i=j\},$$
$$X^{i,j}:=\{[(*,x^i),(*,y^j)]\mid x^i\neq y^j \quad \text{if } i=j\},$$

over all $k\geq i\geq j\geq 1$. As before $X_{i,i}=B(S^1,2)$ is the open Möbius band. For $i>j$, $X_{i,i}$ and $X_{j,j}$ are disjoint. Also and as is clear, $X_i^j\cap X_r^s=\varnothing$ if $\{i,j\}\neq\{r,s\}$. Each union $B_{i,j}:=X_{i,j}\cup X_{i,i}\cup X_i^j$ is the subconfiguration space of 2 points on the i^{th} and j^{th} leaves and hence is, up to homotopy, a wedge of 4 circles. The homotopy

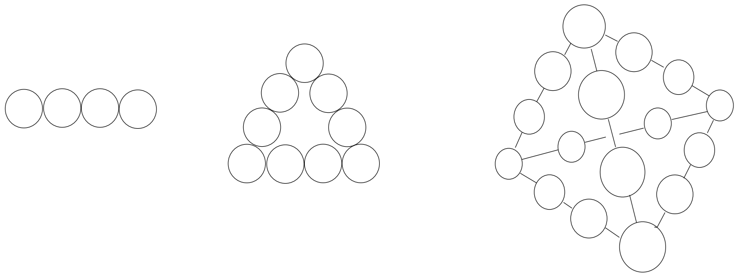


Figure 1: An intermediate homotopy type of $B(\bigvee^k S^1, 2)$ for $k = 2$, 3 and 4, respectively. These are strings of circles making up a necklace in the shape of a $k-1$–dimensional simplex.

deforming each $X_{i,i}$ to S^1 is the same if performed in $B_{i,j}$ or $B_{i,k}$. This is to say that the homotopies deforming the $B_{i,j}$ to a wedge of 4 circles are compatible and we obtain a deformation retract of $B(\bigvee^k S^1, 2)$ which looks like a *necklace* of circles tied in the shape of the $k-1$–dimensional simplex. This is depicted in Figure 1 for $k = 2, 3$ and 4.

The homotopy type of this space is not hard to work out: it is a wedge of all those circles appearing in the necklace with another wedge of circles describing the homotopy type of the 1–skeleton of Δ_{k-1}. In the necklace there is one circle for each vertex of the $k-1$–simplex and two circles for each edge, this gives a total of k^2 circles. On the other hand the one-skeleton of the $k-1$–simplex, denoted by $\Delta_{k-1}^{(1)}$, is homotopy equivalent to $\bigvee^N S^1$ where $N = \frac{1}{2}k(k-3)+1$ circles. Indeed the Euler characteristic

$$\chi(\Delta_{k-1}^{(1)}) = \#\text{edges} - \#\text{vertices} = \tfrac{1}{2}k(k-1) - k = \tfrac{1}{2}k(k-3),$$

and this must be $\chi(\bigvee^N S^1) = N - 1$. Putting this together yields

$$B\Big(\bigvee^k S^1, 2\Big) \simeq \bigvee^{k^2} S^1 \vee \bigvee^{\frac{1}{2}k(k-3)+1} S^1 \simeq \bigvee^{\frac{3}{2}k(k-1)+1} S^1,$$

and the proof is complete. □

Remark 3.5 The first homology group of $B(\Gamma, n)$ for graphs has been worked out in [21]. Their method uses discrete Morse theory. In particular one can deduce from [21, Theorem 3.16] that $H_1(B(\bigvee^k S^1, 2)) = \mathbb{Z}^{1+3k(k-1)/2}$ in full agreement with our Proposition 3.4 (in their theorem one uses that the braid index is 2, $N_1 = 2k(k-1) - \frac{1}{2}k(k-1) - (k-1)$ and the first Betti number of the graph is of course k). In the case of trees T, the homology groups of the unordered configuration space $B(T, n)$ are torsion free and their ranks computed by Farley; references and details are in [21].

4 The localization principle and the case of manifolds

Our main approach is to find conditions on X so that the inclusion $B^d(X,n) \hookrightarrow \mathrm{SP}^n(X)$ induces an isomorphism on some homotopy groups through a range. We start with a preliminary lemma. We say a space X is *locally punctured connected* if for every $x \in X$ and neighborhood U of x, there is an open V, $x \in V \subset U$ such that $V - \{x\}$ is connected.

Lemma 4.1 *Let X be path-connected, locally contractible, and not a point. If $d \geq 2$, then both $\Delta^d(X,n)$ and $B^d(X,n)$ are connected. If, furthermore, X is locally punctured connected, then both $\Delta^d(X,n)$ and $B^d(X,n)$ are connected for all $d \geq 1$.*

Proof For both claims, it suffices to show that $\Delta^d(X,n)$ is connected. We need to join $(x_1,\dots,x_n)$ to $(y_1,\dots,y_n)$ by a path, for any two choices of tuples in $\Delta^d(X,n)$. By deforming locally, we can arrange that the x_i and the y_j are all pairwise distinct. Now X is path-connected so there is a path γ_i from x_i to y_i. Via γ_1 we construct a path in $\Delta^d(X,n)$ from $(x_1,x_2,\dots,x_n)$ to $(y_1,x_2,\dots,x_n)$ by putting $\gamma_1(t)$ in the first coordinate. At any given time $t \in [0,1]$, $\gamma_1(t)$ can only coincide with one x_i at a time, and hence this path is well defined in $\Delta^d(X,n)$ if $d \geq 2$. Construct next the path from $(y_1,x_2,\dots,x_n)$ to $(y_1,y_2,x_3,\dots,x_n)$ by putting $\gamma_2(t)$ in the second coordinate. This is again a well-defined path in $\Delta^d(X,n)$. We can continue this process. The composition $\gamma_n \circ \cdots \circ \gamma_1$ is a path in $\Delta^d(X,n)$ from $(x_1,\dots,x_n)$ to $(y_1,\dots,y_n)$.

To establish the second claim, we proceed by induction on $n \geq d$. For $n = d$, $\Delta^d(X,d) = X^d$ and there is nothing to prove. For $n > d$, consider the projection

$$\Delta^d(X,n) \to \Delta^d(X,n-1)$$

that omits the last coordinate. The preimage of a tuple $(x_1,\dots,x_{n-1})$ is $X - \{x_{i_1},\dots,x_{i_j}\}$ if x_{i_r} repeats d–times in the tuple. Since X is locally punctured connected, this preimage is connected. Since the base space of this projection is also connected by inductive hypothesis, it follows that the total space is connected, as desired. □

For the higher homotopy groups, the starting point is the following principle, which relates the local connectivity properties of a space to its global properties. All spaces appearing below are connected. The following result is in [20, Theorem 1.4].

Theorem 4.2 (localization) *Let X be a Hausdorff topological space and Y be a closed subset of X. If for every point $y \in Y$, and every neighborhood $U \subset X$ of y, there is an open $V \subset U$ containing y such that the pair $(V, V \setminus Y)$ is k–connected, $k \geq 0$, then the pair $(X, X \setminus Y)$ is k–connected.*

We recall what it means for the pair (X, A) to be k–connected or that $\pi_i(X, A) = 0$ for all $i \le k$ [13, Chapter 4]. If $k \ge 1$, this means that every map $(I^r, \partial I^r) \to (X, A)$ from the closed cube I^r, $1 \le r \le k$, is homotopic (relative its boundary) to a map $I^r \to A$. For any $x \in X$, $(X, \{x\})$ is 1–connected if and only if X is simply connected. Being 0–connected, or equivalently writing $\pi_0(X, A) = 0$, means in our terms that X and A are connected and that any point in X is connected by a path to a point in A. Note that in the theorem, if either $V \backslash Y$ or $X \backslash Y$ is not connected, then the theorem fails.

Example 4.3 $X = \mathbb{R}^3$ and L a line in $\mathbb{R}^3$. The pair $(X, X \setminus L)$ is 1–connected but not 2–connected. Indeed take a square which is intersected transversally through its interior by L. That square cannot be deformed away from L with the boundary being kept fixed.

The following is a consequence of Theorem 4.2. We say that a closed subset Y in X is *tame* if there is a neighborhood N of Y such that N deformation retracts onto Y and $X \setminus Y$ deformation retracts onto $X \setminus N$. Submanifolds are tame and so are subcomplexes of simplicial complexes; see Proposition 5.1.

Corollary 4.4 *Let Y be a tame subspace of X and suppose for every $y \in Y$ and neighborhood U of y in X, there is a contractible neighborhood $V \subset U$, such that $Y \cap V$ is tame in V and $V \setminus Y$ is k–connected, $k \ge 0$. Then $\pi_i(X) \cong \pi_i(X \setminus Y)$ for $i \le k$.*

Proof The point is that when Y is tame in X, Theorem 4.2 implies that the induced map $\pi_k(X \backslash Y) \to \pi_k(X)$ is surjective, and $\pi_i(X \backslash Y) \to \pi_i(X)$ is an isomorphism for $i \le k-1$. Let's show that for (V, y) as in the statement of the theorem, the pair $(V, V \setminus Y)$ is $k+1$–connected. Since $V \cap Y$ is tame in V, choose a neighborhood N of Y in V that deformation retracts onto Y and such that $V \setminus N$ deformation retracts onto $V \setminus Y$. We can replace, up to homotopy, the pair $(V, V \setminus Y)$ by $(V, V \setminus N)$, where now $V \setminus N$ is closed in V. We can apply the long exact sequence in homotopy of the pair $(V, V \setminus N)$

$$\to \pi_{k+1} V \to \pi_{k+1}(V, V \setminus N) \xrightarrow{\partial} \pi_k(V \setminus N) \to \cdots$$
$$\to \pi_1(V, V \setminus N) \xrightarrow{\partial} \pi_0(V \setminus N) \to \pi_0(V).$$

Since for $i \le k$, $\pi_i(V \backslash N) = 0 = \pi_{i+1}(V)$, we see that $\pi_i(V, V \backslash Y) \cong \pi_i(V, V \backslash N) = 0$ for $i \le k+1$. From Theorem 4.2 it follows that $(X, X \setminus Y)$ is $k+1$–connected. The same argument as above with the long exact sequence of the pair (X, Y) with Y tame in X shows that $\pi_i(X) \cong \pi_i(X \setminus Y)$ for $i \le k$. □

Remark 4.5 In the case of a submanifold S in M of codimension d, a neighborhood of a point deformation retracts onto a sphere S^{d-1}, which is $d-2$–connected. By the previous corollary this gives that M is weakly equivalent to $M-S$ up to dimension $d-2$ (Proposition 2.5). A similar argument applies when $S=\bigcup S_j$ is the union of submanifolds intersecting transversally.

The following key lemma shows how we can apply the above results to diagonal arrangements.

Lemma 4.6 *Let X be a finite simplicial complex such that for every $x \in X$ and neighborhood U of x, there is a subneighborhood V containing x such that $\Delta^d(V,k)$ (resp. $B^d(V,k)$) is r–connected for any $k \geq 1$. Then $\pi_i(\Delta^d(X,n)) \cong \pi_i(X^n)$ (resp. $\pi_i(\mathrm{SP}^n X) \cong \pi_i(B^d(X,n))$) for $i \leq r$.*

Proof We have to estimate the connectivity of the pair $(X^n, \Delta^d(X,n)) = (X^n, X^n - \Delta_{d+1}(X,n))$ (resp. that of $(\mathrm{SP}^n M, \mathrm{SP}^n(M) - B_{d+1}(M,n))$. Note that $\Delta_{d+1}(M,n)$ (resp. $B_{d+1}(M,n)$) is tame in M^n (resp. $\mathrm{SP}^n M$) according to Section 5. One can check they verify the hypothesis of Corollary 4.4. In the ordered case, choose a point in $\Delta_{d+1}(X,n)$ which, after permutation, can be brought to the form

$$(\underbrace{x_1,\dots,x_1}_{i_1},\underbrace{x_2,\dots,x_2}_{i_2},\dots,\underbrace{x_r,\dots,x_r}_{i_r}), \tag{2}$$

with $x_i \neq x_j$ if $i \neq j$, $\sum i_\alpha = n$ and $i_1 > d$. A neighborhood W of this point in X^n is homeomorphic to $V_1^{i_1} \times \cdots \times V_r^{i_r}$, where V_i is a contractible neighborhood of x_i in X, and the V_i are pairwise disjoint. Clearly

$$W - \Delta_{d+1}(X,n) \cong \Delta^d(V_1,i_1) \times \cdots \times \Delta^d(V_r,i_r).$$

By hypothesis we can assume all the $\Delta^d(V_i,i_j)$ to be r–connected so that $W - \Delta_{d+1}(X,n)$ is also r–connected and hence, by Corollary 4.4, $\pi_i(\Delta^d(X,n)) = \pi_i(X^n - \Delta_{d+1}(X,n)) \cong \pi_i(X^n)$ for $i \leq r$.

A similar proof holds in the unordered case. Given a point in $B_{d+1}(M,n) \subset \mathrm{SP}^n(M)$ as in (2), a small contractible neighborhood of it in $\mathrm{SP}^n M$ is

$$U \cong \mathrm{SP}^{i_1}(V_1) \times \mathrm{SP}^{i_2}(V_2) \times \cdots \times \mathrm{SP}^{i_r}(V_r),$$

the V_i are pairwise distinct, and

$$B^d(U,n) = U - B_{d+1}(X,n) \cong B^d(V_1,i_1) \times B^d(V_2,i_2) \times \cdots \times B^d(V_r,i_r). \tag{3}$$

If we choose each V_j so that $B^d(V_j,i_j)$ is r–connected (hypothesis), the complement $B^d(U,n)$ will also be r–connected and the claim follows again from Corollary 4.4. □

In the case of a manifold, we can already make the following easy conclusions.

Corollary 4.7 *Let M be a manifold of dimension $m\geq 1$.*

(i) *If $md\geq 3$, then $\pi_i(\Delta^d(M,n))\cong \pi_i(M)^n$ for $i\leq md-2$.*

(ii) *If $m\geq 2$ and $d\geq 2$, then $\pi_1(B^d(M,n))\cong H_1(M,\mathbb{Z})$.*

Proof Every point of M has a neighborhood homeomorphic to $\mathbb{R}^m$. The fat diagonal $\Delta_{d+1}(\mathbb{R}^m,n)$ in $(\mathbb{R}^m)^n$ has codimension $mn-m(n-d)=md\geq 3$, so its complement $\Delta^d(\mathbb{R}^m,n)$ is $md-2$–connected (Proposition 2.5). Now apply Lemma 4.6 to get (i). On the other hand $\Delta^d(\mathbb{R}^m,k)$ is simply connected if $d\geq 2$ and $m\geq 2$, so by Armstrong's result (Corollary 2.3), $\pi_1(B^d(\mathbb{R}^m,k))$ is also trivial and (ii) follows. □

Remark 4.8 As we pointed out, Corollary 4.7(i) is not true for $md=2$ as illustrated by $F(\mathbb{R}^2,2)\simeq S^1$. This corollary is a special case of Theorem 1.1. Also let's point out that $\Delta^d(\mathbb{R}^m,n)$ has torsion free homology starting with spherical classes in $dm-1$ as already indicated, and all homology classes are represented by products of spheres [9].

We now derive Theorem 1.1 when X is a manifold. Again X is r–connected if $\pi_i(X)=0$ for $0\leq i\leq r$.

Lemma 4.9 *Let $\Omega^m_*(-)$ denote a connected component of the loop space $\Omega^m(-)$, $m\geq 1$ and $d\geq 1$. Then $\Omega^m_*\,\mathrm{SP}^d S^m$ is $2d-2$–connected.*

Proof Let's review the simplest cases. The case $d=1$ is obvious since $\Omega^m S^m$ breaks down into components indexed by the integers, and each component is 0–connected but not 1–connected since $\pi_1(\Omega^m_* S^m)\cong\pi_{m+1}(S^m)$ is $\mathbb{Z}$ if $m=2$ and $\mathbb{Z}_2$ if $m\geq 3$. When $m=1$, $\mathrm{SP}^d S^1\simeq S^1$ so that $\Omega_* S^1$ is contractible and hence certainly $2d-2$–connected for any d. When $m=2$, $\mathrm{SP}^d S^2\cong\mathbb{P}^d$ is complex projective space and

$$\Omega^2\,\mathrm{SP}^d S^2=\Omega^2\mathbb{P}^d\cong\mathbb{Z}\times\Omega^2 S^{2d+1}.$$

Each component is a copy of $\Omega^2 S^{2d+1}$, which is $2d-2$–connected, and the bound is sharp.

In general we invoke [18, Theorem 5.9] which states that for r–connected X, $r\geq 1$,

$$\pi_i(\mathrm{SP}^n X)\cong \widetilde{H}_i(X;\mathbb{Z}),\quad 0\leq i\leq r+2n-1. \tag{4}$$

This gives that for $i\geq 1$ and $m\geq 2$,

$$\pi_i(\Omega^m_*\,\mathrm{SP}^d S^m)\cong\pi_{i+m}(\mathrm{SP}^d S^m)\cong H_{i+m}(S^m)=0,$$
$$i+m\leq (m-1)+2d-1=m+2d-2.$$

This gives $i\leq 2d-2$ and a lower bound for the connectivity is $2d-2$. □

Proposition 4.10 *Assume $m \geq 2, n \geq d \geq 1$. Then $B^d(\mathbb{R}^m, n)$ is $2d-2$–connected. Moreover if X is a 1–connected manifold and $n \geq 2$, then $\pi_i(B^d(X,n)) \cong \widetilde{H}_i(X;\mathbb{Z})$ for $0 \leq i \leq 2d-2$.*

Proof This relies on results from [16; 18]. The case $d = 1$ being trivial, we assume $d \geq 2$. Consider the sequence of embeddings

$$\tau_n\colon B^d(\mathbb{R}^m, n) \hookrightarrow B^d(\mathbb{R}^m, n+1), \quad [x_1, \dots, x_n] \mapsto [x_1, \dots, x_n, |x_1| + \cdots + |x_n| + 1]. \tag{5}$$

The direct limit is $B^d(\mathbb{R}^m, \infty)$, and it is shown in [16] that there is a scanning map

$$\tau\colon B^d(\mathbb{R}^m, \infty) \to \Omega^m_* \operatorname{SP}^d S^m$$

that induces a homology isomorphism. Since both spaces are simply connected when $d \geq 2$ (Corollary 2.3 and Lemma 4.9) and have the homotopy type of CW complexes, the map τ is a homotopy equivalence. Moreover, the maps τ_n in (5) induce homology embeddings according to [27, Chapitre 3]. Iterating, we get homology embeddings

$$H_*(B^d(\mathbb{R}^m, d+1)) \hookrightarrow H_*(B^d(\mathbb{R}^m, n) \hookrightarrow H_*(B^d(\mathbb{R}^m, \infty)) \cong H_*(\Omega^m_* \operatorname{SP}^d S^m).$$

By Lemma 4.9 the groups on the extreme right are trivial for $* \leq 2d-2$. This gives that $H_*(B^d(\mathbb{R}^m, n)) = 0$ for $n \geq d$ and $* \leq 2d-2$. Since the space is simply connected, it is $2d-2$–connected as well. It then follows by Lemma 4.6 that $\pi_i(B^d(X,n)) \cong \pi_i(\operatorname{SP}^n(X))$ for $i \leq 2d-2$. This proves the main statement. In the case X is r–connected with $r \geq 1$, it follows by the inequality in (4), since $2d-2 \leq r + 2n - 1$, that $\pi_i(B^d(X,n)) \cong \pi_i(\operatorname{SP}^n(X)) \cong \widetilde{H}_i(X;\mathbb{Z})$ in the range of dimensions $0 \leq i \leq 2d-2$. □

Example 4.11 Consider the case $B^{n-1}(S^2, n)$, $n \geq 3$. Since $\operatorname{SP}^n(S^2) \cong \mathbb{P}^n$ is a $2n$–dimensional manifold, by Proposition 2.5, $\pi_i(B^{n-1}(S^2,n)) \cong \pi_i(\mathbb{P}^n)$ for $1 \leq i \leq 2(n-1)-2 = 2n-4$. On the other hand, from the Hopf fibration, $\pi_i(\mathbb{P}^n) \cong \pi_i(S^{2n+1})$ for $i > 2$ and $\pi_2(\mathbb{P}^n) = \mathbb{Z}$. This shows precisely that $\pi_i(B^{n-1}(S^2,n)) \cong H_i(S^2, \mathbb{Z})$ for $1 \leq i \leq 2n-4$, as expected.

The claim that $B^d(\mathbb{R}^k, n)$ is $2d-2$–connected has an nice alternative proof in the case $d = n-1$.

Lemma 4.12 *$B^{n-1}(\mathbb{R}^k, n)$ is $2n-4$–connected, $n \geq 2, k \geq 1$.*

Proof The case $k = 1$ is trivial. We let $k \geq 2$ and invoke some main results from [17; 18]. Let S be the unit sphere as in Lemma 2.2 and let $Q_{n,k}$ be its quotient under the

$\mathfrak{S}_n$–action. We have already indicated that $Q_{n,k} \simeq B^{n-1}(\mathbb{R}^k, n)$. On the other hand, according to [17, Theorems 1.1, 1.3 and 1.5],

$$\Sigma^{k+1} Q_{n,k} \simeq \overline{\mathrm{SP}}^n(S^k), \tag{6}$$

where Σ means suspension and $\overline{\mathrm{SP}}^n(Y)$ means the *symmetric smash* $Y^{\wedge(n)}/\mathfrak{S}_n$, which is also the cofiber of the embedding of $\mathrm{SP}^{n-1} Y$ into $\mathrm{SP}^n Y$ induced by adjoining a basepoint to an unordered tuple $[x_1, \dots, x_{n-1}]$. It is shown [17, Theorems 1.2 and 1.3] that if X is r–connected, then $\overline{\mathrm{SP}}^n(\Sigma X)$ is $2n+r-1$–connected. This gives that $\overline{\mathrm{SP}}^n(S^k) = \overline{\mathrm{SP}}^n(\Sigma S^{k-1})$ is $2n+k-3$–connected, and hence so is $\Sigma^{k+1} Q_{n,k}$ by (6). Since in this range $Q_{n,k}$ is already simply connected, it must therefore be $2n-4$–connected. □

Remark 4.13 That the connectivity bound in the above theorem doesn't depend on k is not surprising. Indeed when $n = 2$, $B(\mathbb{R}^k, 2) \simeq \mathbb{R}P^{k-1}$ and this is never 1–connected no matter what k is.

5 An equivariant deformation retract of diagonal complements

Let X_* be an abstract simplicial complex and $|X_*|$ its geometric realization. Let A_* be a subcomplex of X_*. We say a subcomplex A_* of X_* is *full* if every simplex of X_* whose vertices are in A_* is itself in A_*. The following fundamental result (called the retraction lemma in [5]) can be found in Munkres' book [22, Lemma 70.1].

Proposition 5.1 *Let A_* be a full subcomplex of the finite simplicial complex X_*. Let C_* consist of all simplices of X_* that are disjoint from A_*. Then $|A_*|$ is a deformation retract of $|X_*| - |C_*|$, and $|C_*|$ is a deformation retract of $|X_*| - |A_*|$.*

The argument of proof is short but instrumental to extract useful properties of this compactification. We review this argument. The fact that A_* is full says that C_* is also full, and that simplices of X_* consist of simplices in C_*, simplices in A_* and simplices of the form

$$\sigma * \tau, \quad \sigma \in A_*, \quad \tau \in C_*,$$

where $\sigma * \tau$ is the join of both simplices. Figure 2 illustrates the situation when X_* is the full simplex Δ_3 on 4 vertices v_0, v_1, v_2, v_3, $A_* = [v_0 v_1]$ and $C_* = [v_2 v_3]$.

The deformation of $|X_*| - |A_*|$ onto $|C_*|$ is as in the figure. It starts at a point $tx + \sum_{i\in I} s_i v_i$, with v_i vertices in C_*, $i \in I$, $t + \sum s_i = 1$, $t \neq 1$, and ends at the point $\sum (t_j / \sum s_i) v_j$, $j \in I$.

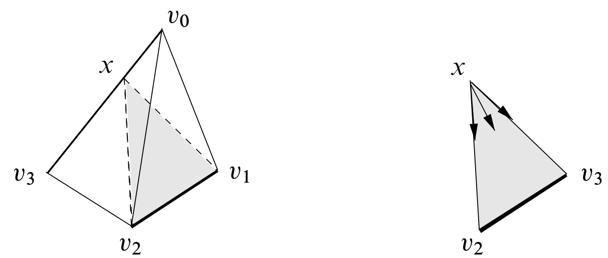


Figure 2: Munkres' deformation along the join (right) after deleting the apex x

Two important consequences are in order:

- If A_* is full, $|X_*| - |A_*|$ deformation retracts onto the *largest* subcomplex that does not meet $|A_*|$. Note that if A_* is not full, then its first barycentric subdivision $\mathrm{Sd}\, A_*$ is always full in $\mathrm{Sd}\, X_*$. The barycentric subdivision comes with a natural ordering on vertices.
- The deformation retraction illustrated in Figure 2 has the property that if it starts in a simplex of X_* it will stay in that simplex (and deforms onto a face of it).

For ease we will write X for either X_* or its realization. The context will be clear.

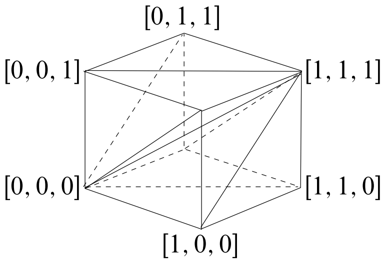
Munkres' observation nicely applies to the diagonal arrangements. Given X an ordered simplicial complex, X^n can be given naturally a structure of a simplicial complex such that the various diagonals are subcomplexes; see [23, Section 1], and also the proof of Lemma 5.2 below. We can then apply Proposition 5.1 to the configuration space $X^n - \Delta_{d+1}(X,n)$. Among all diagonal arrangements, only the thin diagonal $\Delta_n(X,n)$ is full. We therefore have to pass to a barycentric subdivision. Let $\mathrm{Sd}(X^n)$ be the barycentric subdivision of X^n_*. This restricts to $\mathrm{Sd}(\Delta_{d+1}(X,n))$.

Lemma 5.2 *There is an $\mathfrak{S}_n$–equivariant deformation retraction of $\Delta^d(X,n)$ onto the largest subcomplex $W^d(X,n)$ not intersecting $|\mathrm{Sd}(X^n)| - |\mathrm{Sd}(\Delta_{d+1}(X,n))|$.*

Proof That the complement deformation retracts onto $W^d(X,n)$ is a direct consequence of Proposition 5.1 as applied to the pair $(\mathrm{Sd}(X^n), \mathrm{Sd}(\Delta_{d+1}(X,n)))$ with $\mathrm{Sd}(\Delta_{d+1}(X,n))$ being full. We need check this deformation is equivariant under the symmetric group action. Recall that the simplicial decomposition of X^n is made out as follows, where X of course is an *ordered* simplicial complex [23]. A vertex of X^n is of the form $(v_1, \dots, v_n)$ where v_i is a vertex of X. Different $(q+1)$–vertices

$$w_0 = (v_{01}, \dots, v_{0n}), \quad w_1 = (v_{11}, \dots, v_{1n}), \quad \dots, \quad w_q = (v_{q1}, \dots, v_{qn}),$$

form a q–dimensional simplex if and only if for each $k = 1, 2, \dots, n$, $(q+1)$–vertices v_{0k}, v_{1k}, $\dots$, v_{qk} are contained in a simplex of X and $v_{0k} \le v_{1k} \le \dots \le v_{qk}$; see Figure 3 for the decomposition of X^3 in the case $X = [0,1]$ with vertices $[0] \le [1]$.



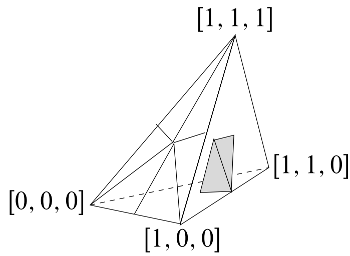


Figure 3: Left: simplicial decomposition for $[0,1]^3$ with 8 vertices, 19 edges, 18 triangular faces and 6 tetrahedral faces. Note that $\Delta_2([0,1],3)$ is not full and we need to pass to a barycentric subdivision. Right: the configuration space $|\mathrm{Sd}(X^3)| - |\mathrm{Sd}(\Delta_2(X,3)|$ deformation retracts onto the subcomplex $W^2([0,1],3)$ made out of 6 contractible connected components. The figure shows one such component in one tetrahedral face.

Note that, as asserted, $\Delta_2([0,1],3)$ is not full, as the 2–simplex (bottom) ($[0,0,0]$, $[1,0,0]$, $[1,1,0]$) has all three vertices in $\Delta_2(X,3)$ but is not itself a simplex of $\Delta_2(X,3)$.

Generally a vertex is in $\Delta_{d+1}(X,n)$ if and only if it is of the form $(v_1,\dots,v_n)$ for some vertices $v_1,\dots,v_n$ of X with $v_{i_0} = \cdots = v_{i_d}$ for some choice of sequence $i_0 < i_1 < \cdots < i_d$. Obviously every permutation acting on X^n permutes vertices of X^n_* and the order between them so it must take simplices to simplices. The action is simplicial and the quotient space $\mathrm{SP}^n(X)$ inherits a cellular decomposition. Moreover, the action remains simplicial after passing to a barycentric subdivision. Indeed since any new introduced vertex is of the form $\frac{1}{k}\sum v_i$, it is sent by $\sigma \in \mathfrak{S}_n$ to $\frac{1}{k}\sum \sigma(v_i)$, which is the barycenter of $(\sigma(v_1),\dots,\sigma(v_k))$.

After one subdivision, a simplicial neighborhood of $\mathrm{Sd}(\Delta_{d+1}(X,n))$ consists of all simplices of $\mathrm{Sd}(X^n)$ having at least one vertex of the form $(v_1,\dots,v_n)$ with $v_{i_0} = \cdots = v_{i_d}$ for some sequence $i_0 < i_1 < \cdots < i_d$. This simplicial neighborhood is therefore $\mathfrak{S}_n$–invariant and its complement $W^d(X,n)$ is invariant, as well. Clearly the permutation action on X^n commutes with Munkres' deformation since it takes combinations $\sum t_i v_i$ to $\sum t_i \sigma(v_i)$ (see Figure 2). It therefore descends to a deformation retraction of $B^d(X,n)$ onto $W^d(X,n)/\mathfrak{S}_n =: \mathcal{W}^d(X,n)$. □

Corollary 5.3 *For a finite simplicial complex X, the $\mathfrak{S}_n$–quotient $\mathcal{W}^d(X,n)$ of $W^d(X,n)$ is a compact deformation retract of $B^d(X,n)$.*

We need one more observation.

Lemma 5.4 *Let A be a subcomplex of X. The deformation retraction of $|\mathrm{Sd}(X^n)| - |\mathrm{Sd}(\Delta_{d+1}(X,n))|$ onto its compactified space $W^d(X,n)$ restricts to a deformation retraction of $|\mathrm{Sd}(A^n)| - |\mathrm{Sd}(\Delta_{d+1}(A,n))|$ onto $W^d(A,n)$.*

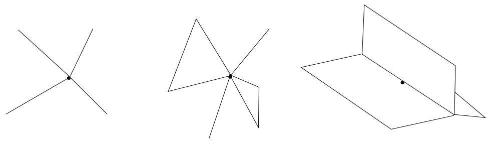


Figure 4: Contractible neighborhoods of the dotted point in simplicial X

Proof Since A is a subcomplex of X, $\Delta_{d+1}(A,n)$ is a subcomplex of $\Delta_{d+1}(X,n)$ and $Sd(A^n)$ is a subcomplex of $Sd(X^n)$. Both $Sd(\Delta_{d+1}(X,n))$ and $Sd(\Delta_{d+1}(A,n))$ are full subcomplexes. The assertion now follows from the fact that if the deformation retraction starts in a simplex of $Sd(X^n)$; in particular in $Sd(A^n)$, it will stay in that simplex. □

6 Proof Theorem 1.1

We appeal to the following useful theorem of Steve Smale which is a generalization of classical results of Begle and Vietoris. A similar statement for maps between simplicial complexes can be deduced from work of Farjoun [11, Corollary 9.B.3, page 163].

Theorem 6.1 [24] *Let X and Y be connected, locally compact, separable metric spaces, and let X be locally contractible. Let f be a mapping of X into Y for which f^{-1} carries compact sets into compact sets. If, for each $y \in Y$, $f^{-1}(y)$ is locally contractible and r–connected, $r \geq 0$, then the induced homomorphism $\pi_k(X) \to \pi_k(Y)$ is an isomorphism for $0 \leq k \leq r$, and is onto for $k = r+1$.*

Theorem 6.1 uses maps that are proper and preimages that are at least connected. Maps between configuration spaces obtained by projections are seldom proper. Combining the above theorem with Section 5 yields, however, the following main result.

Theorem 6.2 *Let X be a connected finite simplicial complex with at least two vertices, $d \geq 2, n \geq 2$. Then*

$$\pi_i(B^d(X,n)) \cong \pi_i(\mathrm{SP}^n(X)), \quad 0 \leq i \leq 2d-2.$$

Proof The starting point is Lemma 4.6 where it suffices to show that $\pi_i(B^d(V,n)) = 0$ for $i \leq 2d-2$ for V a small contractible neighborhood of a point in X. A neighborhood V of $x \in X$ is one of three types; either (i) Euclidean space, (ii) halfspace or (iii) it is a union of such halfspaces along a shared boundary. See Figure 4.

We claim that in all cases, $B^d(V,n)$ is $2d-2$–connected.

In the case that x is an interior point of a simplex that is not a face of a larger simplex, it has a neighborhood $V \cong \mathbb{R}^m$ with $m \geq 1$. When $m = 1$, $B^d(\mathbb{R}^1, n)$ is contractible. When $m \geq 2$, $B^d(\mathbb{R}^m, n)$ is $2d-2$–connected according to Proposition 4.10.

If x belongs to a boundary face, then V is homeomorphic to halfspace H (with boundary). This halfspace can be isotoped into its interior $\mathring{H}$ so we have a map $B^d(H, n) \to B^d(\mathring{H}, n)$ obtained from a deformation retraction (setting $t = 1$). Since $B^d(\mathring{H}, n)$ is $2d-2$–connected, as seen earlier, it follows immediately that $B^d(H, n)$ has the same connectivity (at least).

In the third and final case, x lies in the intersection of two or more simplices of X as in Figure 4. Let V be a contractible neighborhood made out of simplices which meet along a simplex A. Let Γ be a simplex in V of dimension m. Of course A is in the boundary of Γ. Let

$$B^d(\Gamma, A, n) = \coprod_{0 \leq k \leq n} B^d_A(\Gamma, k)/\sim,$$

where $B^0_A(\Gamma, k) = *$ is a given point in A and $B^d_A(\Gamma, k) = B^d(\Gamma, k) \cup \mathrm{SP}^k(A)$, ie the only points that can repeat more than d times in Γ are those that are in A. The equivalence relation $\sim$ is such that $x \sim *$ if $x \in A$ and $[x_1, \ldots, x_i, \ldots, x_k] \sim [x_1, \ldots, \hat{x}_i, \ldots, x_k]$ if $x_i \in A$. Here, as customary, $\hat{x}_i$ means the i^{th} entry is suppressed. We have a projection

$$\lambda\colon B^d(V, n) \to B^d(\Gamma, A, n), \tag{7}$$

which sends a tuple $[x_1, \ldots, x_n]$ to the new tuple obtained by replacing all $x_i \notin \Gamma$ by $*$. One can view λ as a projection of $[x_1, \ldots, x_n]$ to the subtuple made up of those entries $x_i \in \Gamma$. This map is continuous by the very nature of the construction $B^d(\Gamma, A, n)$, ie any entry x_i that exits or enters into Γ must pass through A. The base space $B^d(\Gamma, A, n)$ is contractible since there is a deformation retraction of Γ onto A which extends to $B^d(\Gamma, A, n)$.

Next write an element in $B^d(\Gamma, A, n)$ as an equivalence class $[[x_1, \ldots, x_k]]$ with $x_i \in \Gamma - A$ and some $k \leq n$. The preimage $\lambda^{-1}[[x_1, \ldots, x_k]]$ consists of all possible unordered n–tuples containing $x_1, x_2, \ldots, x_k$ with remaining entries $y_1, \ldots, y_{n-k}$ such that $[y_1, \ldots, y_{n-k}] \in B^d((V - \Gamma) \cup A, n-k)$. This preimage is a copy of $B^d((V - \Gamma) \cup A, n-k)$. By induction on the number of simplices of V, we can assume that $B^d((V - \Gamma) \cup A, n-k)$ is $2d-2$–connected (the case of a single simplex has been discussed at the beginning of the proof). The map $\lambda\colon B^d(V, n) \to B^d(\Gamma, A, n)$ has then a contractible base and preimages that are $2d-2$–connected. We wish to show that the total space is $2d-2$–connected. We cannot use the Smale–Vietoris theorem (Theorem 6.1) directly since λ is not proper. To get around this, we pass to the compactified versions and show that λ can be deformed to a proper map. Let $\mathcal{W}^d(V,n)$ be the compact deformation retract of $B^d(X,n)$ discussed in Corollary 5.3. The restriction $\tilde{\lambda}$ of λ to $\mathcal{W}^d(V,n)$ maps onto $\mathcal{W}^d(\Gamma, A, n) \subset B^d(\Gamma, A, n)$ and we have the diagram

$$\begin{tikzcd}
\mathcal{W}^d(V,n) \arrow[r] \arrow[d, "\tilde{\lambda}"] & B^d(V,n) \arrow[d, "\lambda"] \\
\mathcal{W}^d(\Gamma, A, n) \arrow[r] & B^d(\Gamma, A, n)
\end{tikzcd}$$

where the horizontal maps are inclusions and deformation retractions. This last statement follows from the fact that the deformation retraction of $B^d(V,n)$ onto $\mathcal{W}^d(V,n)$ descends to a deformation retraction of $B^d(\Gamma, A, n)$ onto $\mathcal{W}^d(\Gamma, A, n)$ as a consequence of Lemma 5.4. Thus given a configuration $\zeta = [[x_1, \dots, x_k]] \in \mathcal{W}^d(\Gamma, A, n)$, $k \le n$, $x_i \notin A$, we can consider its preimage $\tilde{\lambda}^{-1}(\zeta)$ in $\mathcal{W}^d(V,n)$ and its preimage $\lambda^{-1}(\zeta)$ in $B^d(V,n)$. Then $\lambda^{-1}(\zeta)$ deformation retracts onto $\tilde{\lambda}^{-1}(\zeta)$. Here $\tilde{\lambda}^{-1}(\zeta) \subset \lambda^{-1}(\zeta) = B^d((V-\Gamma)\cup A, n-k)$. Since $\lambda^{-1}(\zeta)$ is $2d-2$–connected, this shows that $\tilde{\lambda}^{-1}(\zeta)$ is also $2d-2$–connected. The map $\tilde{\lambda}$ is now proper, being a map between compact spaces. Moreover, both total and base spaces are connected by Lemma 4.1. We can invoke Theorem 6.1 to conclude that the total space $\mathcal{W}^d(V,n)$ and hence $B^d(V,n)$ are $2d-2$–connected as desired. $\square$

7 Proof of Theorem 1.2

Our objective is to find conditions on X so that the inclusion $\Delta^d(X,n) \hookrightarrow X^n$ induces an isomorphism on some homotopy groups through a range (the homotopical depth). The proof given in the unordered case $B^d(X,n)$ in Section 6 fails here because the analogue of (7) is now a map $\Delta^d(V,n) \to \Delta^d(\Gamma, A, n)$ which has *disconnected* fibers, so Smale's theorem doesn't automatically apply. In fact we need an entirely new approach.

First some definitions.

Definition 7.1

- If $x \in X$ and U is a neighborhood of x, then we call V a *subneighborhood* (of x in U) if V is open and $x \in V \subset U$.
- A space X is *locally contractible* if for any $x \in X$ and any neighborhood U of x, there is a subneighborhood V which deformation retracts onto x.
- A space X has *local homotopical dimension* k if, for x, U as above, there is a subneighborhood V such that $V - \{x\}$ is k–connected. For instance, being *locally punctured connected* means having 0 local homotopical dimension. A manifold of dimension m has local homotopical dimension $m-2$ but not $m-1$.

If X is a simplicial complex, we call a *chamber* of X any simplex that is not contained in another simplex as a face. Obviously if X has local homotopical dimension r, then chambers must have dimensions at least $r+2$. We call a simplex a *shared face* if it is shared by two chambers or more. This shared face doesn't need to be of codimension 1. In Figure 4, the complex on the far right is made out of three chambers (of dimension 2) joining along a shared edge. A shared face $A=\Gamma_1\cap\cdots\cap\Gamma_k$ is called *essential* if $X=\Gamma_1\cup\cdots\cup\Gamma_k$ is not a cell, ie homeomorphic to a ball or to a halfball. This rules out cases like X being a regular polygon triangulated so that the origin $A=o$ is the common vertex of all triangles. A neighborhood $V\setminus\{o\}$ is, up to homotopy, a circle in this case, so that o behaves like an interior point of a chamber (and is inessential).

Lemma 7.2 *A finite simplicial complex X has local homotopical dimension r if and only if all chambers are of dimension at least $r+2$ and all essential shared faces are of dimension at least $r+1$.*

Proof It suffices to consider points $x\in X$ that are either in the interior of a chamber or in the interior of a shared face. In the case that x is in the interior of a chamber, $V\cong\mathbb{R}^m$ so m (the dimension of the chamber) must be at least $r+2$ (Proposition 4.10). On the other hand, if x lies in the interior of a shared face A, a small neighborhood V of x is the union of chambers $\Gamma_1\cup\cdots\cup\Gamma_q$ joining along A, with $q\geq 2$. If A is inessential, then a neighborhood V of $x\in A$ is either a ball or a halfball of dimension at least $r+2$. Suppose x to be essential and let $s=\dim A$. Then $V-\{x\}\simeq\bigvee S^s$ is a bouquet (this holds even if $s=0$ and A is vertex). Since this neighborhood must be r–connected, s must be at least $r+1$. □

The following is our main statement. Here we assume $d<n$; otherwise $\Delta^d(X,n)=X^n$ and there is nothing to prove.

Theorem 7.3 *Let X be a locally finite polyhedral space with local homotopical dimension r, $r\geq 0$, and let $1\leq d<n$. Then*

$$\pi_i(\Delta^d(X,n))\cong\pi_i(X)^n\quad \textit{for } i\leq rd+2d-2.$$

Proof The starting point is Lemma 4.6. As in the proof of Lemma 7.2, a contractible neighborhood V of $x\in X$ is one of three types: (i) $V\cong\mathbb{R}^m$ with $m\geq r+2$, (ii) halfspace H of dimension $m\geq r+2$ or (iii) it is a union of such halfspaces along a shared face of dimension at least $r+1$. We must show that $\Delta^d(V,n)$ is $dr+2d-2$ connected.

In the case $V\cong\mathbb{R}^m$, we know by Corollary 4.7 that $\Delta^d(\mathbb{R}^m,n)$ is $dm-2$–connected, and that $dm-2=d(r+2)-2=dr+2d-2$ as claimed. If V is homeomorphic to halfspace H (with boundary L), V is the inverse limit of a nested sequence of spaces $H_i \supset H$ such that $H_i \cong \mathbb{R}^m$ for all i, it follows that $\Delta^d(H,n)$ is the inverse limit of the spaces $\Delta^d(H_i,n)$, which are $dr+2d-2$–connected, and it has this same connectivity. We are left with the case that V is the union of simplices (chambers) $\Gamma_1 \cup \cdots \cup \Gamma_q$ joining along an essential face A. We can assume without loss of generality that any two faces join along A, ie $\Gamma_i \cap \Gamma_j = A$. Luckily the structure of this neighborhood V is sufficiently nice to allow us to give a decomposition of $\Delta^d(V,n)$ as the colimit of an explicit diagram.

We start by observing that each configuration of n points of V gives rise to a tuple of integers $(k_1,\dots,k_q)$, $k_1+\cdots+k_q \geq n$, where k_i denotes the number of points of the configuration inside the face Γ_i. Obviously these k_i–configurations can overlap when points of the configuration fall in A. Keeping track of the various overlaps can be expressed in terms of a poset of intersections. More precisely, set the index set

$$I = \{1,2,\dots,q\}^n = \{(i_1,\dots,i_n) \mid i_j \in \{1,2,\dots,q\}\}.$$

We can cover $\Delta^d(V,n)$ by the closed sets $U_{(i_1,\dots,i_q)}$, $(i_1,\dots,i_q) \in I$, where

$$U_{(i_1,\dots,i_q)} = \{(x_1,\dots,x_n) \in \Delta^d(V,n) \mid x_j \in \Gamma_{i_j}, i_j \in \{1,2,\dots,q\}\}.$$

Let $\mathcal{D}$ be the intersection poset P_U associated to the cover U_I of $\Delta^d(V,n)$, also referred to as *subspace diagram*. It is clear by construction that $\operatorname{colim}\mathcal{D}$ is precisely $\Delta^d(V,n)$. Here's how this poset diagram looks for $k=2$ and $d=1$, ie for the configuration space $F(X\cup_A Y,2)$; see [25, Theorem 2.0.17]:

$$\begin{tikzcd}[column sep=tiny]
 & X\times Y-\Delta A & & F(Y,2) & & Y\times X-\Delta A & & F(X,2) & \\
X\times A-\Delta A \arrow[ur] & & A\times Y-\Delta A \arrow[ul] \arrow[ur] & & Y\times A-\Delta A \arrow[ul] \arrow[ur] & & A\times X-\Delta A \arrow[ul] \arrow[ur] & & X\times A-\Delta A \arrow[ul] \\
 & & & & F(A,2) \arrow[ullll] \arrow[ull] \arrow[u] \arrow[urr] \arrow[urrrr] & & & &
\end{tikzcd} \tag{8}$$

(the spaces on the extreme right and left are being identified).

Going back to the general diagram $\mathcal{D}$, since all inclusions are closed cofibrations (this is standard to check [25]), we have

$$\Delta^d(V,n) = \operatorname{colim}(\mathcal{D}) \simeq \operatorname{hocolim}(\mathcal{D}).$$

In fact the canonical map from the homotopy colimit of a sequence of inclusions of T1 topological spaces to the actual colimit is a weak equivalence; see [7]. The connectivity of this (sequential) homotopy colimit $\Delta^d(V,n)$ is at least the least connectivity of the spaces making up the diagram. If we set $\Gamma_0 = A$, these spaces are of the form $\Gamma_{i_1}\times\cdots\times\Gamma_{i_n}\cap\Delta^d(V,n)$ (we refer to these subspaces as the *constituent subspaces* of the diagram). Each of these constituent subspaces is quite manageable and we can apply the localization principle to it. Indeed $\Gamma_{i_1}\times\cdots\times\Gamma_{i_n}\cap\Delta^d(V,n)$ is the complement in $\Gamma_{i_1}\times\cdots\times\Gamma_{i_n}$ of subspaces of certain codimensions. The smallest codimension is attained by $\Delta_{d+1}(A,n)$ in A^n, that is, for $\Delta^d(A,n)$. If $s=\dim A$, then this codimension is ds. It follows that the smallest connectivity among the constituent subspaces is $ds-2\geq d(r+1)-2 = dr+d-2$. As pointed out, the connectivity of $\Delta^d(V,n)$ (as a homotopy colimit) must be at least the connectivity of $\Delta^d(A,n)$, which is $dr+d-2$. This is not quite the connectivity we seek and we must improve it by d.

To do so observe that there is associated to the poset P_U of the cover a natural filtration whose j^{th} space is $\mathcal{F}_j = \operatorname{colim} P_j$, where P_j is the poset consisting of

$$\begin{gathered}\Gamma_{i_1}\times\cdots\times\Gamma_{i_n}\cap\Delta^d(V,n),\quad i_{k_1}=\cdots=i_{k_s}=0\quad\text{for } s\geq n-j,\\ \text{and some subset } \{k_1,\dots,k_s\}\subset\{1,\dots,n\}\end{gathered}$$

with $i_s\in\{0,1,\dots,q\}$ and $\Gamma_0 = A$, as pointed out. In other words, $\mathcal{F}_j$ is the subspace where at most j of the entries can be outside of A. We have the series of inclusions

$$\mathcal{F}_0 = \Delta^d(A,n)\subset\mathcal{F}_1\subset\cdots\subset\mathcal{F}_n = \Delta^d(V,n).$$

If we organize our poset vertically as in (8), then $\mathcal{F}_j$ is the pushout of the first $j+1$ rows from the bottom.

For example, $F(\Gamma_1\cup_A\Gamma_2,2)$ (the case depicted in diagram (8) with $X=\Gamma_1, Y=\Gamma_2$), there are three filtration terms starting with $\mathcal{F}_0 = F(A,2)$, the colimit $\mathcal{F}_1$ of the first two rows and $\mathcal{F}_2$ being the whole colimit. The special case of $F(\mathbb{R}^2,2)=\Delta^1(\mathbb{R}^2,2)$ is enlightening ($d=1$, $n=2$), where here we write $\mathbb{R}^2=\Gamma_1\cup_A\Gamma_2$ with the Γ_i being two halfplanes joining along $A\cong\mathbb{R}$. The first filtration term is $\mathcal{F}_0 = F(\mathbb{R},2)\simeq S^0$. The next filtration term is

$$\mathcal{F}_1 = (\Gamma_1\times A\cup A\times\Gamma_1\cup\Gamma_2\times A\cup A\times\Gamma_2)\cap\Delta^1(\mathbb{R}^2,2).$$

Each term $(\Gamma_i\times A\cup A\times\Gamma_j)\cap\Delta^1(\mathbb{R}^2,2)=\Gamma_i\times A\cup A\times\Gamma_i-\operatorname{diag}(A)$ deformation retracts onto a circle so $\mathcal{F}_1$ is the union of two circles along an S^0, ie $\mathcal{F}_1\simeq S^1\vee S^1\vee S^1$. Finally $\mathcal{F}_2\simeq F(\mathbb{R}^2,2)\simeq S^1$. The connectivity changes going from $\mathcal{F}_0$ to $\mathcal{F}_1$, and remains stable afterwards.

Let's organize into a row R_k the constituent subspaces $\Gamma_{i_1}\times\cdots\times\Gamma_{i_n}\cap\Delta^d(V,n)$ where precisely k of the Γ_{i_j} are not equal to $A=\Gamma_0$. One point we will capitalize on is that in the range $0\leq k\leq d$, $\Gamma_{i_1}\times\cdots\times\Gamma_{i_n}\cap\Delta^d(V,n)$ is the complement in

$\Gamma_{i_1} \times \cdots \times \Gamma_{i_n}$ of tuples with $d+1$–diagonal elements lying only in A. At the first stage, all components of R_1 intersect along $\mathcal{F}_0 = \Delta^d(A,n)$.

If $n = d+1$, the situation is very clear. Here $\Delta^d(A, d+1) = A^{d+1} - \mathrm{diag}(A) \simeq S^{ds-1}$, and all constituent subspaces for $1 \leq k \leq d$ are of the form

$$\Gamma_{i_1} \times \cdots \times \Gamma_{i_{d+1}} \cap \Delta^d(V, d+1) = \Gamma_{i_1} \times \cdots \times \Gamma_{i_{d+1}} - \mathrm{diag}(A),$$

thus they are contractible since they are the complement of a closed subspace in the boundary of a cube. This means that going up the filtration, we are suspending in various ways the spherical class, as in the example discussed earlier, and the connectivity in homology is going up by one at every step.

For more general n, the constituent subspaces are not, in general, contractible but we have the following useful lemma.

Lemma 7.4 *The inclusion $\mathcal{F}_{k-1} \hookrightarrow \mathcal{F}_k$ is null-homotopic for $k \leq d+1$.*

Proof We need some notation. We introduce $\mathcal{F}_k(n)$ for the filtration terms of $\Delta^d(V,n)$ (we added the index n to the previous notation). We also introduce $\mathcal{F}_{k,j}(n)$ for the subspace of all configurations $(x_1, \dots, x_n) \in \mathcal{F}_k(n)$ where x_j can be in all of H. We have that $\mathcal{F}_k(n) = \bigcup_{1 \leq j \leq n} \mathcal{F}_{k,j}(n)$. There is an inclusion

$$\mathcal{F}_{k-1}(n) \hookrightarrow \mathcal{F}_{k,n}(n) \subset \mathcal{F}_k(n).$$

On the other hand there are various embeddings of $\mathcal{F}_j(n)$ into $\mathcal{F}_j(n+1)$ one of which is given by

$$(x_1, \dots, x_n) \mapsto (\phi_1(x_1), \dots, \phi_1(x_n), p_n), \tag{9}$$

where ϕ_t is any isotopy of the halfspace H extending an isotopy of A onto its halfspace $(a_1, \dots, a_s)$, $a_1 < 0$, and $p_n = (n, 0, \dots, 0) \in A \cong \mathbb{R}^s$. The first observation is that the inclusion $\mathcal{F}_{k-1}(n) \hookrightarrow \mathcal{F}_{k,n}(n)$ is homotopic to the composite

$$\mathcal{F}_{k-1}(n) \to \mathcal{F}_{k-1}(n-1) \hookrightarrow \mathcal{F}_{k-1}(n) \hookrightarrow \mathcal{F}_{k,n}(n),$$

where the first map is projection discarding the last configuration, and the middle map is the inclusion (9). The idea here is that the last coordinate $x_n \in H$ can be moved in H away from A, all configurations are then mapped by ϕ_t, and after that the last coordinate is brought down to p_n. Note that the last configuration can move in H without constraint since $k \leq d$. Next we factor the composite above $\mathcal{F}_{k-1}(n-1) \hookrightarrow \mathcal{F}_k(n)$ through $\mathcal{F}_{k-1}(n-1) \hookrightarrow \mathcal{F}_{k,n-1}(n)$ and reiterate this construction to factor the map up to homotopy, this time through $\mathcal{F}_{k-1}(n-2)$, etc. At the end, the map $\mathcal{F}_{k-1}(n) \hookrightarrow \mathcal{F}_k(n)$ factors through $\mathcal{F}_{k-1}(k-1)$ which is contractible. □

Going back to our colimit diagram, the constituent subspaces for the k^{th} row R_k are $V_I := \Gamma_{i_1}\times\cdots\times\Gamma_{i_n}\cap\Delta^d(V,n)$, where $I=(i_1,i_2,\dots,i_n)$ is an ordered tuple with $n-k$ entry 0. In the range $k<n-d$, the smallest connectivity of a constituent subspace is $ds-2$; this is because the smallest codimension strata we are removing from $\Gamma_{i_1}\times\cdots\times\Gamma_{i_n}$ to obtain V_I have the codimension of the thin diagonal in A^{d+1} and this is ds. In the range $n-d\le k<n$, this minimal codimension starts jumping by one unit going from row to row. More precisely the connectivity of the constituent subspaces of R_k in the indicated range is at least $ds-2+k-(n-d-1)$. This minimal connectivity remains the same from R_{n-1} to R_n (no jump there). There are therefore precisely d jumps. At the level of filtrations now, $\mathcal{F}_k=R_k\cup\mathcal{F}_{k-1}$ and we have a pushout diagram where we are gluing $ds-2+k-(n-d-1)$–connected spaces intersecting along $ds-2+k-(n-d-1)-1$–connected spaces. Using the Mayer–Vietoris sequence, and inducting on the sequences I, we see immediately that the homological connectivity (in short H_*–connectivity) of the pushout $\mathcal{F}_k$ must be at least $ds-2+k-(n-d-1)$, for $n-d\le k<n$. The H_*–connectivity of $\mathcal{F}_n$ is, as we pointed out, that of $\mathcal{F}_{n-1}$, which is thus at least $ds-2+d$. Since $s\ge r+1$ (Lemma 7.2), this H_*–connectivity is at least $dr+2d-2$–connected.

Finally to get the connectivity, we need argue that $\mathcal{F}_n$ is simply connected. In fact $\mathcal{F}_k$ becomes 1–connected as soon as $k\ge 1$. To see this, we go back to the colimit diagram (8) where the smallest connectivity of the constituent subspaces V_I is $d(r+1)-2$. When this is larger than 1, each V_I is simply connected, and so is the colimit, and the theorem holds. Now some V_I fail to be simply connected when $d(r+1)\le 2$, that is, when (i) $r=0, d=1$, or (ii) $r=1=d$, or (iii) $r=0, d=2$. In the first case, the theorem is equivalent to saying that $F(X,n)$ is connected if X is locally punctured connected. This is precisely Lemma 4.1 so this case is settled. In case (ii), we are looking at $\Delta^d(A,n)=\Delta^1(\mathbb{R}^2,n)=F(\mathbb{R}^2,n)$ as the bottom space of our colimit diagram. This is of course not simply connected, but the map $\pi_1(\mathcal{F}_0)\to\pi_1(\mathcal{F}_1)$ is the trivial map since it is induced from a null homotopic map (Lemma 7.4), so that $\mathcal{F}_1$, and inductively $\mathcal{F}_k$, are simply connected by the van Kampen theorem. The remaining case (iii) occurs when $\Delta^d(A,n)=\Delta^2(\mathbb{R},n)$. The fundamental group of this space is discussed in Example 8.1. Here too the fundamental group trivializes from $\mathcal{F}_1$ onwards so that $\mathcal{F}_n=\Delta^d(V,n)$ is simply connected. □

7A The homology of the filtration terms

This subsection is of independent interest and gives a description of the homology of the filtration terms. This is sketchy but details can be filled in. First of all, there is a nice way to see that the inclusion $\mathcal{F}_0\to\mathcal{F}_1$ induces the trivial map in homology without resorting to Lemma 7.4. Here $\mathcal{F}_0 = \Delta^d(A,n)$ has torsion free homology admitting a basis realized by products of spheres [9]. We need to understand how these homology classes occur. There is a spherical class in

$$\Delta^d(A, d+1) = A^{d+1} - \mathrm{diag}(A) \simeq S^{ds-1},$$

where $s = \dim A$. Now $\Delta^d(A, d+1)$ embeds in $\Delta^d(A,n)$ in many ways as in (9) (recall that $d < n$). This embedding has a retract so induces a monomorphism in homology. The image of the spherical class in this case is denoted $\{x_1, \dots, x_{d+1}\}$. The various other embeddings, obtained by choosing another subset of indices $\{i_1, \dots, i_{d+1}\} \subset \{1, 2, \dots, n\}$, give rise to spherical homology classes $\{x_{i_1}, \dots, x_{i_{d+1}}\}$. These classes generate the homology of $\Delta^d(A,n)$ in a very precise sense. There is an action of the operad $\{D^s(k)\}_{k\ge 0}$ of little s–dimensional disks on $\bigcup_{n\ge 1}\Delta^d(A,n)$, where $s = \dim A$ and $D^s(k)$ is the space of k pairwise disjoint open disks in the unit disk of dimension s (to keep with the terminology the word "disk" is used instead of "ball"). The action of $D^s(2) \simeq S^{s-1}$ is given as follows:

$$D^s(2) \times \Delta^d(A, n_1) \times \Delta^d(A, n_2) \to \Delta^d(A, n_1 + n_2),$$

and yields a bracket operation in homology:

$$[-,-]\colon H_p(\Delta^d(A,n_1)) \otimes H_q(\Delta^d(A,n_2)) \to H_{p+q+s-1}(\Delta^d(A, n_1+n_2)).$$

The product map in homology is given by the action of $H_0(D^s(2))$ and is the induced map in homology of the concatenation of two configurations after placing the first one in a disk of radius $\frac{1}{2}$ centered at $\left(-\frac{1}{2}, 0, \dots, 0\right)$ and the other in another disk of the same radius centered at $\left(\frac{1}{2}, 0, \dots, 0\right)$. One main theorem of [9] reads as follows. The bracket of two cycles is important to understand and can be described as follows. Given a cycle (or chain) c in $\Delta^d(A,n)$, we say we *localize* it in a disk D^s if we choose a homeomorphism (which can be made canonical) between $A \cong \mathbb{R}^s$ and D, and take the image of c in $\Delta^d(D,n)$ via this homeomorphism. We obtain the bracket $[\alpha_1, \alpha_2]$ by localizing the cycles respectively in two disjoints disks D_1 and D_2 and taking the new cycle obtained by rotating D_1 around D_2 (or D_2 around D_1, up to sign) in $\mathbb{R}^s$.

Theorem 7.5 [9, Proposition 3.9] *The homology of $\Delta^d(A,n)$ is torsion free, generated additively by products of iterated brackets where each factor is either x_i or an iterated bracket of the form*

$$[\cdots[[B_1, B_2], B_3], \dots, B_\ell], \quad \ell \ge 1,$$

where each B_s is of the form

$$B_s = [\cdots[[\{x_{j_{1,s}}, x_{j_{2,s}}, \dots, x_{j_{d+1,s}}\}, x_{i_{1,s}}], x_{i_{2,s}}], \dots, x_{i_{\ell_s,s}}]$$

(further conditions are stated on indices to get a basis).

Let's argue, for example, that $\{x_1,\dots,x_{d+1}\}$ maps to zero in the homology of the next filtration term. Consider the following diagram of inclusions:

$$\begin{tikzcd}
\Delta^d(A,d+1) \arrow[r] \arrow[d, "\iota"] & \Delta^d(A,n)=\mathcal{F}_0 \arrow[d] \\
(\Gamma_1\times A^d)\cap\Delta^d(V,n) \arrow[r] & \mathcal{F}_1
\end{tikzcd}$$

The bottom space $(\Gamma_1\times A^d)\cap\Delta^d(V,n)=\Gamma_1\times A^d-\mathrm{diag}(A)$ is contractible since we are removing a subspace from the boundary of $\Gamma_1\times A^d$. The map ι is trivial and the commutativity of the diagram shows that $\{x_1,\dots,x_{d+1}\}$ maps trivially in $\mathcal{F}_1$. A class of the form $[\{x_1,\dots,x_{d+1}\},x_{d+2}]$ dies in $\mathcal{F}_1$, for example, since this class can be represented by the composite

$$S^{ds-1}\times S^{d-1}\to A^{d+2}-\mathrm{sing}\hookrightarrow A^{d+1}\times H-\mathrm{sing}\hookrightarrow\mathcal{F}_1. \tag{10}$$

The first map is obtained from the operadic action. Here the factor S^{d-1} is the locus of x_{d+2} rotating in some sphere in $\mathbb{R}^d$, so when x_{d+2} is allowed to be in H, this sphere is coned off and the composite of the first two maps in (10) is trivial on the top homology class which by definition is $[\{x_1,\dots,x_{d+1}\},x_{d+2}]$. A similar argument applies to show that the image of B_s as in the notation of Theorem 7.5 is trivial in $\mathcal{F}_1$. For the image of the bracket $[B_s,B_t]$, one can argue similarly. One constructs this class by localizing B_s and B_t in distinct disks D_1 and D_2, and rotating one disk around the other. But the class B_s is the boundary of a chain in $H'\times D_1^{n-1}\cup D_1\times H'\times D^{n-2}\cup\cdots\cup D^{n-1}\times H'\subset\mathcal{F}_1$, where H' is the part of H with boundary D_1. This means that $[B_s,B_t]$ must map to zero in $H_*(\mathcal{F}_1)$. It remains to be shown that the image of a product is trivial, but this is immediate.

Note that there are many ways a given class $[\cdots[[B_1,B_2],B_3],\dots,B_\ell]$ can die in $\mathcal{F}_1$, and so in $\mathcal{F}_1$ we obtain suspension classes one degree higher. This describes the homology of $\mathcal{F}_1$ and clearly it is one degree more connected than $\mathcal{F}_0$.

8 Fundamental groups

In this final section we take a more pedestrian look at the isomorphism $\pi_1(B^d(X,n))\cong H_1(X,\mathbb{Z})$ for $d\ge 2$. This is expressed in terms of braids. As before X is a simplicial complex. Note that loops in $\mathrm{SP}^n X$, based at a basepoint of the form $[*,\dots,*]$, say, lift to X^n under the quotient projection; see [18, Section 5] for example:

$$\begin{tikzcd}
 & X^n \arrow[d] \\
S^1 \arrow[r] \arrow[ur, dotted] & \mathrm{SP}^n X
\end{tikzcd}$$

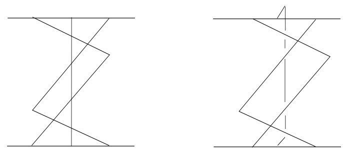


Figure 5: The left braid cannot be trivialized in $\Delta^2(I,3)$

This says that a homotopy class of a loop $\gamma\colon S^1 \to \mathrm{SP}^n X$ based at $[*,\dots,*]$ can be represented by a tuple $[\gamma_1,\dots,\gamma_n]$, where $\gamma_i\colon S^1 \to X$ is a loop in X. Moreover and by the simplicial approximation theorem, any loop in X^n deforms into an n–tuple of simplicial loops in X so that γ can be represented by an unordered tuple of simplicial loops in $\mathrm{SP}^n(X)$ for some simplicial decomposition.

We can try to describe loops in $\Delta^d(X,n)$ and $B^d(X,n)$ in the same way but both spaces are not simplicial complexes in general, only of the homotopy type of one. However, after passing to a barycentric subdivision, $B^d(X,n) = \mathrm{SP}^n X - B_{d+1}(X,n)$ deformation retracts onto a cellular complex $\mathcal{W}^d(X,n)$ (Lemma 5.2). A loop $S^1 \to B^d(X,n)$ deforms into a loop into $\mathcal{W}^d(X,n)$ which is cellular. Therefore and without loss of generality, we can represent a loop $\gamma\colon S^1 \to B^d(X,n)$ within its homotopy class by a tuple of paths $t \mapsto [\gamma_1(t),\dots,\gamma_n(t)]$, with γ_i a simplicial path in X (not necessarily a closed loop) and $t \in [0,1]$. This is a *braid with n–strands*. These paths or strands at any time t do not intersect in more than d points, and $[\gamma_1(0),\dots,\gamma_n(0)] = [\gamma_1(1),\dots,\gamma_n(1)]$. This is similar for loops into $\Delta^d(X,n)$.

As a first example, consider $X = I$: the unit interval. By codimension argument, $\Delta^d(I,n)$ is simply connected if $d \geq 3$, so the only interesting case is when $d=2$ and we are removing from I^n codimension 2 subspaces corresponding to when $x_i = x_j = x_k$. According to Example 2.1, $\Delta^2(I,3) \simeq S^1$ and $\pi_1(\Delta^2(I,3)) \cong \mathbb{Z}$. An element in the fundamental group can be represented by a braid with 3–strands embedded in $I \times I$, not all of which can pass by the same point at the same time. A nontrivial element is depicted in the left-hand side of Figure 5. This braid cannot be trivialized in $\Delta^2(I,3)$, but it is amusing to try. By moving the strands around while keeping their endpoints fixed, there is no way we can separate them without going through a triple point.

Example 8.1 For $n \geq 3$, the fundamental group of $\Delta^2(I,n)$ has been analyzed by Khovanov [19]. There he shows that $\Delta^2(I,n)$ is a $K(\pi,1)$ and then gives a presentation for π. This presentation is given as follows. Define the right-angled Coxeter group TW_n to be the group generated by the simple transpositions $s_i = (i,i+1), i \in [n-1]$, subject to the relations

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \text{ if } |i-j| > 1.$$

Define $\phi\colon TW_n \to \mathfrak{S}_n$ by $\phi(s_i) = s_i$ for all $i \in [n-1]$. Then $\pi_1(\Delta^2(I,n)) \cong \ker\phi$.

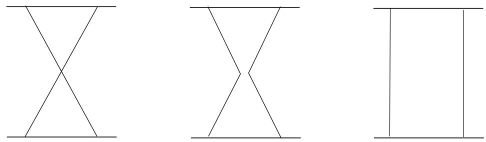


Figure 6: Resolving the intersection points

In the unordered case it is possible to kill the braiding by *interchanging* strands. Represent an element of $\pi_1(B^d(X,n))$ by a braid with n–strands embedded in $X \times I$. Suppose we have two intersecting strands. There is a way to resolve the intersection points, illustrated in Figure 6. The figure depicts a loop $f(t) = [f_1(t), f_2(t)]$ with two strands crossing for some $s \in [0,1]$. Define $\tilde{f} = [\tilde{f_1}, \tilde{f_2}]$ to be such that $\tilde{f_i}(t) = f_i(t)$ if $t \leq s$, and $\tilde{f_1}(t) = f_2(t)$, $\tilde{f_2}(t) = f_1(t)$ if $t \geq s$. These give two representations of the same loop in $\Omega B^d(X,n)$ for $d \geq 2$. The difference, however, is that after changing f by $\tilde{f}$, by a small homotopy we can now separate the strands of $\tilde{f}$ so that no intersection occurs. This also explains why the fundamental group must be abelian [18].

For example, using this resolution of intersections, we can immediately trivialize the braid in $\Omega B^2(\mathbb{R},3)$ depicted in Figure 5 (left). This is no surprise since $B^2(\mathbb{R},3) \simeq B^2(I,3)$ is contractible and is identified with the 3–simplex with one edge removed.

The resolution of intersections when applied to loops in ΩV, with V a tree, implies that we have a surjection $\pi_1(B(V,n)) \to \pi_1(B^d(V,n))$. Since $B^d(V,n)$ is connected for $d \geq 2$ (Lemma 4.1), pick the basepoint in this fundamental group to be $[x_1, \dots, x_n]$ with $x_i \neq x_j, i \neq j$, and write a braid $\gamma(t) = [\gamma_1(t), \dots, \gamma_n(t)]$. As discussed, we can assume the γ_i to be *nonintersecting* strands. Since V is one dimensional, necessarily $\gamma_i(0) = \gamma_i(1) = x_i$, so all strands must start and finish at the same point. Each γ_i can be homotoped to the constant strand at x_i, without further intersections, and the loop we started out with is trivial up to homotopy. The above discussion allows us to give a streamlined proof of the following proposition which we have already obtained as a corollary to Theorem 1.1.

Proposition 8.2 *If X is a connected simplicial complex which is not reduced to a point, $n \geq 2, d \geq 2$, then there is an isomorphism $\pi_1(B^d(X,n)) \cong H_1(X;\mathbb{Z})$.*

Proof We need show that the inclusion $B^d(X,n) \hookrightarrow \mathrm{SP}^n X$ induces an isomorphism on fundamental group if $d > 1$. If we invoke Lemma 4.6 as before, this boils down to showing that for V a contractible neighborhood in X, $B^d(V,n)$ is simply connected whenever $d \geq 2$ and for any $n \geq 1$. If V is a contractible neighborhood in simplicial X (as in the proof of Theorem 6.2), any element in $\pi_1(B^d(V,n))$ can be represented by a braid and by resolving the intersection points. This braid can be homotoped to the trivial braid. $\square$

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E_n–cohomology with coefficients as functor cohomology

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Building on work of Livernet and Richter, we prove that E_n–homology and E_n–cohomology of a commutative algebra with coefficients in a symmetric bimodule can be interpreted as functor homology and cohomology. Furthermore, we show that the associated Yoneda algebra is trivial.

13D03, 18G15, 55P48

1 Introduction

The little n–cubes operad was introduced to study n–fold loop spaces (see Boardman and Vogt [2] and May [13]). An E_n–operad is a Σ_*–cofibrant operad weakly equivalent to the operad formed by the singular chains on the little n–cubes operad, and algebras over such an operad are called E_n–algebras. Those are A_∞–algebras which are in addition commutative up to higher homotopies of a certain level depending on n. For a Σ_*–cofibrant operad one can define a suitable notion of homology and cohomology of algebras over this operad as a derived functor. For E_1–algebras this operadic notion of homology coincides with Hochschild homology. For E_∞–algebras one retrieves Γ–homology as defined by Robinson; see Robinson and Whitehouse [17]. In general, for a commutative algebra viewed as an E_n–algebra, E_n–homology can be seen to coincide with higher order Hochschild homology as defined in Pirashvili [14]; see Ginot, Tradler and Zeinalian [8] and Ziegenhagen [19].

Many notions of homology can be expressed as functor homology. The case of Hochschild homology and cyclic homology has been studied by Richter and Pirashvili in [16]. The same authors give a functor homology interpretation of Γ–homology in [15]. In [10], Hoffbeck and Vespa show that Leibniz homology of Lie algebras is functor homology. A more general approach to functor homology for algebras over an operad and their operadic homology is discussed in [6] by Fresse.

For the case of E_n–homology, functor homology interpretations of E_n–homology have been given by Livernet and Richter in [11] and Fresse in [4]. Both articles are exclusively concerned with the case of trivial coefficients. As proved in [5], E_n–homology with trivial coefficients coincides up to a suspension with the homology of a

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generalized iterated bar construction. Muriel Livernet and Birgit Richter use this in [11] to prove that E_n–homology of a commutative algebra with trivial coefficients can be interpreted as functor homology over a category of trees denoted by Epi_n. Fresse shows in [4] that this result can be extended to arbitrary E_n–algebras.

Recent work by Fresse and the author shows that E_n–homology and E_n–cohomology of a commutative algebra with coefficients in a symmetric bimodule can also be calculated via the iterated bar construction; see Fresse and Ziegenhagen [7]. We show in this article that the functor homology interpretation of Livernet and Richter can be extended to the case with coefficients and also holds for cohomology. More precisely, we introduce a category Epi_n^+ of trees extending the category Epi_n and a functor $b\colon {\mathrm{Epi}_n^+}^{\mathrm{op}} \to k$–mod, where k is any commutative unital ring. Then to a commutative nonunital k–algebra A and a symmetric A–bimodule M we associate Loday functors $\mathcal{L}(A;M)\colon \mathrm{Epi}_n^+ \to k$–mod and $\mathcal{L}^c(A;M)\colon {\mathrm{Epi}_n^+}^{\mathrm{op}} \to k$–mod and prove the following theorem:

Theorem 1.1 *We have an isomorphism*

$$H_*^{E_n}(A;M) \cong \mathrm{Tor}_*^{\mathrm{Epi}_n^+}(b, \mathcal{L}(A;M)),$$

and, if k is self-injective, an isomorphism

$$H^*_{E_n}(A;M) \cong \mathrm{Ext}^*_{{\mathrm{Epi}_n^+}^{\mathrm{op}}}(b, \mathcal{L}^c(A;M)).$$

This implies that there is an action on E_n–cohomology by the corresponding Yoneda algebra. We show that this algebra is trivial.

Outline We give an overview of the constructions of [11] in Section 2. In Section 3 we recall how to calculate E_n–homology and –cohomology of commutative algebras with coefficients in a symmetric bimodule via the iterated bar construction. To do this one introduces a twisting differential. In Section 4 we enlarge the category defined by Livernet and Richter to incorporate this twisting differential. We define E_n–homology and –cohomology for functors from this category to k–modules. Finally we show that there are Loday functors linking these notions to the usual notion of E_n–homology and –cohomology. We prove our main theorem in Section 5. In Section 6 we recall the definition of the Yoneda pairing and show that the Yoneda algebra is trivial.

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Conventions In the following we assume that $1 \leqslant n < \infty$. Let k be a commutative unital ring. We denote by A a commutative nonunital k–algebra and by M a symmetric A–bimodule. We often view A and M as differential graded k–modules concentrated in degree zero. Let $A_+ = A \oplus k$ be the unital augmented algebra obtained by adjoining a unit to A. We denote by $sc \in \Sigma C$ the element defined by $c \in C$ in the suspension of a graded k–module C. The k–module $k[X]$ is the free k–module generated by a set X. For $l \geqslant 0$ we denote by $[l]$ the set $[l] = \{0, \dots, l\}$.

2 The category Epi_n encoding the n–fold bar complex

In [5] Fresse proves that E_n–homology of E_n–algebras with trivial coefficients can be computed via the iterated bar complex. Livernet and Richter use this in [11] to give an interpretation of E_n–homology of commutative algebras with trivial coefficients as functor homology. They encode the information necessary to define an iterated bar complex in a category Epi_n of trees. We recall the construction of this category.

Definition 2.1 Let C be a differential graded nonunital algebra. The bar complex $B(C)$ is the differential graded k–module given by

$$B(C) = (\overline{T}^c(\Sigma C), \partial_B),$$

where $\overline{T}^c(\Sigma C)$ denotes the reduced tensor coalgebra on ΣC equipped with the differential induced by the differential of C. The twisting cochain ∂_B is defined by

$$\partial_B([c_1|\cdots|c_l]) = \sum_{i=1}^{l-1}(-1)^{i-1}[c_1|\cdots|c_i c_{i+1}|\cdots|c_l].$$

Here we use the classical bar notation and denote $sc_1 \otimes \cdots \otimes sc_l \in (\Sigma C)^{\otimes l}$ by $[c_1|\cdots|c_l]$. If C is commutative, the shuffle product

$$\mathrm{sh}\colon B(C) \otimes B(C) \to B(C)$$

is defined by

$$\mathrm{sh}([c_1|\cdots|c_j] \otimes [c_{j+1}|\cdots|c_{j+l}]) = \sum_{\sigma \in \mathrm{sh}(j,l)} \pm [c_{\sigma^{-1}(1)}|\cdots|c_{\sigma^{-1}(j+l)}],$$

with $\mathrm{sh}(j,l) \subset \Sigma_{j+l}$ the set of (j,l)–shuffles. For homogeneous elements $c_1, \dots, c_{j+l}$ the summand $[c_{\sigma^{-1}(1)}|\cdots|c_{\sigma^{-1}(j+l)}]$ is decorated by the graded signature $(-1)^\epsilon$, with

$$\epsilon = \prod_{\substack{i<l \\ \sigma(i)>\sigma(l)}} (|c_i|+1)(|c_l|+1).$$

The shuffle product makes $B(C)$ a commutative differential graded k–algebra.

We can iterate this construction and form the n–fold bar complex $B^n(A)$. The results in [5] for E_n–algebras imply that for any k–projective commutative nonunital k–algebra A we have

$$H_*^{E_n}(A;k) = H_*(\Sigma^{-n} B^n(A)).$$

Elements in the n–fold bar construction $B^n(A)$ correspond to sums of planar fully grown trees with leaves labelled by elements in A; see [3]. We fix some terminology concerning trees.

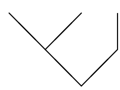
Definition 2.2 A planar fully grown n–level tree t is a sequence

$$t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$$

of order-preserving surjections. The element $i \in [r_j]$ is called the i^{th} vertex of the j^{th} level. The elements in $[r_n]$ are also called leaves. The degree of a tree t is given by the number of its edges, ie by

$$d(t) = \sum_{j=1}^{n} (r_j + 1).$$

For example, the 2–level tree



is given by the sequence $[2] \xrightarrow{f_2} [1]$ with $f_2(0) = f_2(1) = 0$, $f_2(2) = 1$.

Definition 2.3 For a given vertex $i \in [r_j]$ the subtree $t_{j,i}$ is the $(n-j)$–level subtree of t given by

$$t_{j,i} = [|f_n^{-1} \cdots f_{j+1}^{-1}(i)|-1] \xrightarrow{g_n} [|f_{n-1}^{-1} \cdots f_{j+1}^{-1}(i)|-1] \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_{j+2}} [|f_{j+1}^{-1}(i)|-1],$$

with g_l the map making the diagram

$$\begin{tikzcd}
{[|f_l^{-1} \cdots f_{j+1}^{-1}(i)| - 1]} \arrow[r, "g_l"] \arrow[d, "\cong"] & {[|f_{l-1}^{-1} \cdots f_{j+1}^{-1}(i)| - 1]} \arrow[d, "\cong"] \\
f_l^{-1} \cdots f_{j+1}^{-1}(i) \arrow[r, "f_l"] & f_{l-1}^{-1} \cdots f_{j+1}^{-1}(i)
\end{tikzcd}$$

commute. Here the vertical maps are the unique order-preserving bijections.

Definition 2.4 [11, Definition 3.1] The category Epi_n has as objects planar fully grown trees with n levels. A morphism from

$$[r_n] \xrightarrow{f_n^r} \cdots \xrightarrow{f_2^r} [r_1] \quad \text{to} \quad [s_n] \xrightarrow{f_n^s} \cdots \xrightarrow{f_2^s} [s_1]$$

consists of surjections $h_i\colon [r_i] \to [s_i]$, $1 \leqslant i \leqslant n$, such that the diagram

$$\begin{tikzcd} {[r_n]} \arrow[r, "f_n^r"] \arrow[d, "h_n"] & {[r_{n-1}]} \arrow[r, "f_{n-1}^r"] \arrow[d, "h_{n-1}"] & \cdots \arrow[r, "f_2^r"] & {[r_1]} \arrow[d, "h_1"] \\ {[s_n]} \arrow[r, "f_n^s"] & {[s_{n-1}]} \arrow[r, "f_{n-1}^s"] & \cdots \arrow[r, "f_2^s"] & {[s_1]} \end{tikzcd}$$

commutes and such that h_i is order-preserving on the fibres $(f_i^r)^{-1}(l)$ of f_i^r for all $l \in [r_{i-1}]$. For $i = 1$ we require that the map h_1 is order-preserving on $[r_1]$. The composite of two morphisms $(g_n, \dots, g_1)\colon t^q \to t^r$ and $(h_n, \dots, h_1)\colon t^r \to t^s$ is given by $(h_n g_n, \dots, h_1 g_1)$.

Observe that since A is concentrated in degree zero, the degree of a labelled tree viewed as an element in $B^n(A)$ is given by the number of edges of the tree. Lemma 3.5 in [11] says that the maps in Epi_n decreasing the number of edges by one are exactly the summands of the differential of $B^n(A)$. This motivates the following definition.

Definition 2.5 [11, Definition 3.7] Let $F\colon \mathrm{Epi}_n \to k$–mod be a covariant functor. Let $\widetilde{C}^{E_n}(F)$ be the $(\mathbb{N} \cup \{0\})^n$–graded k–module with

$$\widetilde{C}^{E_n}_{(r_n,\dots,r_1)}(F) = \bigoplus F(t),$$

where the sum is indexed over all trees

$$t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1].$$

Let $d_i\colon [r_n] \to [r_n - 1]$ denote the order-preserving surjection which maps i and $i+1$ to i. For $1 \leqslant j \leqslant n$ let $\tilde{\partial}_j\colon \widetilde{C}^{E_n} \to \widetilde{C}^{E_n}$ be the following map lowering the j^{th} degree by one:

- For $j = n$ define $\tilde{\partial}_j$ restricted to $F(t)$ as
$$\sum_{\substack{0 \leqslant i < r_n \\ f_n(i) = f_n(i+1)}} (-1)^{s_{n,i}} F(d_i, \mathrm{id}_{[r_{n-1}]}, \dots, \mathrm{id}_{[r_1]}).$$
- Let $1 \leqslant j < n$, $0 \leqslant i < r_j$ and $\sigma \in \mathrm{sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))$. Let $h = h_{i,\sigma}$ be the unique morphism of trees, exhibited in [11, Lemma 3.5], such that

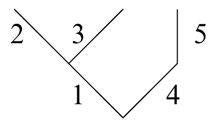
$h_j = d_i\colon [r_j] \to [r_j - 1]$, $h_l = \mathrm{id}$ for $l < j$, and h_{j+1} restricted to $f_{j+1}^{-1}(\{i, i+1\})$ acts like σ. Then $\widetilde{\partial}_j$ is the map whose restriction to $F(t)$ equals

$$\sum_{\substack{0 \leqslant i < r_j \\ f_j(i) = f_j(i+1)}} \sum_{\sigma \in \mathrm{sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))} \epsilon(\sigma; t_{j,i}, t_{j,i+1})(-1)^{s_{j,i}} F(h_{i,\sigma}).$$

The signs arise from switching the degree -1 map d_i with suspensions, as well as from the graded signature of the permutation σ in the cases $j < n$. More precisely, we number the edges in the tree t from bottom to top and from left to right. For example, the 2–level tree

$$[2] \xrightarrow{f_2} [1] \quad \text{with } f_2(0) = f_2(1) = 0 \text{ and } f_2(2) = 1$$

is decorated as indicated in the following picture:



Then for $j < n$ we acquire a sign $(-1)^{s_{j,i}}$, where $s_{j,i}$ is the number of the rightmost top edge of the $(n-j)$–level subtree $t_{j,i}$ of t. For $j = n$ set $s_{n,i}$ to be the label of the edge whose leaf is the i^{th} leaf for $0 \leqslant i \leqslant n$.

For $j < n$ the map $F(h_{i,\sigma})$ is not only decorated by $(-1)^{s_{j,i}}$ but also by a sign associated to $\sigma \in \mathrm{sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))$: Let $t_1, \dots, t_a$ be the $(n-j-1)$–level subtrees of t above the j–level vertex i, ie the $(n-j-1)$–level subtrees forming $t_{j,i}$. Similarly let $t_{a+1}, \dots, t_{a+b}$ denote the $(n-j-1)$–level subtrees above $i+1$. Then σ determines a shuffle of $\{t_1, \dots, t_a\}$ and $\{t_{a+1}, \dots, t_{a+b}\}$. The sign $\epsilon(\sigma; t_{j,i}, t_{j,i+1})$ picks up a factor $(-1)^{(d(t_x)+1)(d(t_y)+1)}$ whenever $x < y$ and $\sigma(x) > \sigma(y)$.

Lemma 2.6 *For any functor $F\colon \mathrm{Epi}_n \to k$–mod the $(\mathbb{N} \cup \{0\})^n$–graded module $\widetilde{C}^{E_n}(F)$ together with $\widetilde{\partial}_1, \dots, \widetilde{\partial}_n$ forms a multicomplex, which we again denote by $\widetilde{C}^{E_n}(F)$.*

Definition 2.7 [11, Definition 3.7] The homology

$$H_*^{E_n}(F) = H_*(\mathrm{Tot}(\widetilde{C}^{E_n}(F)))$$

of the total complex associated to $\widetilde{C}^{E_n}(F)$ is called the E_n–homology of $F\colon \mathrm{Epi}_n \to k$–mod.

Livernet and Richter show that there is a Loday functor

$$\mathcal{L}(A;k)\colon \mathrm{Epi}_n \to k\text{–mod}$$

associated to every nonunital commutative algebra A such that

$$H_*^{E_n}(\mathcal{L}(A;k)) = H_*^{E_n}(A;k)$$

whenever A is k–projective. They then prove that E_n–homology of functors is indeed functor homology:

Theorem 2.8 [11, Theorem 4.1] *Let* $\widetilde{b}\colon \mathrm{Epi}_n^{\mathrm{op}} \to k\text{–mod}$ *be the functor given by*

$$\widetilde{b}(t) = \begin{cases} k & \text{if } t = [0] \to \cdots \to [0], \\ 0 & \text{otherwise.} \end{cases}$$

Then for $F\colon \mathrm{Epi}_n \to k\text{–mod}$ *we have*

$$H_*^{E_n}(F) = \mathrm{Tor}_*^{\mathrm{Epi}_n}(\widetilde{b}, F).$$

3 E_n–homology with coefficients via the iterated bar complex

Recent work by Fresse and the author (see [7]) shows that, at least for a commutative nonunital k–algebra A and a symmetric A–bimodule M, the iterated bar complex can also be used to calculate E_n–homology and –cohomology with coefficients. In order to incorporate the action of A on M one has to add a twisting cochain

$$\delta\colon A_+ \otimes B^n(A) \to A_+ \otimes B^n(A)$$

to the complex $A_+ \otimes B^n(A)$.

Definition 3.1 Given an n–level tree $t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$ and $a_0, \dots, a_{r_n} \in A$, let $t(a_0, \dots, a_{r_n})$ denote the element in $B^n(A)$ defined by t with leaves labelled by $a_0, \dots, a_{r_n}$. The twisting morphism $\delta\colon A_+ \otimes B^n(A) \to A_+ \otimes B^n(A)$ is given by

$$\begin{aligned}\delta(a \otimes t(a_0, \dots, a_{r_n})) = &\sum_{\substack{0 \leqslant l \leqslant r_{n-1} \\ |f_n^{-1}(l)|>1 \\ x=\min f_n^{-1}(l)}} (-1)^{s_{n,x}-1} a a_x \otimes (t \setminus x)(a_0, \dots, \widehat{a}_x, \dots, a_{r_n}) \\ &+ \sum_{\substack{0 \leqslant l \leqslant r_{n-1} \\ |f_n^{-1}(l)|>1 \\ y=\max f_n^{-1}(l)}} (-1)^{s_{n,y}} a_y a \otimes (t \setminus y)(a_0, \dots, \widehat{a}_y, \dots a_{r_n})\end{aligned}$$

for $a \in A_+$. Here for $s \in [r_n]$ such that s is not the only element in the corresponding 1–fibre of t containing s, ie in the 1–fibre $f_n^{-1}(u)$ with $f_n(s) = u$, we let $t \setminus s$ be the tree obtained by deleting the leaf s. To be more precise,

$$t \setminus s = \ [r_n - 1] \xrightarrow{f_n'} [r_{n-1}] \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} [r_1]$$

with

$$f_n'(x) = \begin{cases} f_n(x), & x < s, \\ f_n(x+1), & x \geqslant s. \end{cases}$$

The sign $(-1)^{s_{n,i}}$ is as in Definition 2.5.

Remark 3.2 (a) Intuitively the map δ deletes leaves and acts with the corresponding label on the coefficient module A_+. The leaves which are deleted are either on the left or on the right of a 1–fibre of the tree. For $n = 1$ compare this to the complex calculating Hochschild homology $\mathrm{HH}(A; A_+)$: the standard differential maps $a \otimes a_0 \otimes \cdots \otimes a_l \in A_+ \otimes A^{\otimes l+1}$ to

$$aa_0 \otimes a_1 \otimes \cdots \otimes a_l + (-1)^{l+1} a_l a \otimes a_0 \otimes \cdots \otimes a_{l-1} \\ + \sum_{i=0}^{l-1} (-1)^{i+1} a \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_l.$$

The first two summands correspond to the twist δ, while the other summands correspond to ∂_B.

(b) In the definition of the map δ we only consider 1–fibres of cardinality at least two. If we wanted to take 1–fibres of cardinality one into account, we would add two summands for each such fibre: Both summands would replace t by a tree obtained by deleting the 1–fibre and then deleting further edges to obtain a fully grown tree again. One summand would multiply $a \in A_+$ from the right by the label a_x of the leaf x corresponding to the deleted fibre, the other summand would multiply by a_x from the left. Note that these summands are not of the appropriate degree, since we delete more than one edge. However, the two terms just described cancel each other out anyway, because for commutative A multiplying $a \in A_+$ with $a_x \in A$ from the left equals multiplying a with a_x from the right.

In Section 4 we will define E_n–homology and E_n–cohomology of functors defined on a category which extends the category Epi_n. The following theorem will allow us to argue in Remark 4.8 and Remark 4.10 that E_n–homology and E_n–cohomology of functors encompass E_n–homology and E_n–cohomology of commutative algebras with coefficients in a symmetric bimodule.

Theorem 3.3 [7] *For a commutative k–projective nonunital k–algebra A and a symmetric A–bimodule M we have*

$$H_*^{E_n}(A; M) = H_*(M \otimes_{A_+} (A_+ \otimes \Sigma^{-n} B^n(A), \delta))$$

and

$$H^*_{E_n}(A; M) = H^*(\mathrm{Hom}_{A_+}((A_+ \otimes \Sigma^{-n} B^n(A), \delta), M)).$$

4 The category Epi_n^+ encoding the n–fold bar complex with coefficients

We would like to establish a functor homology interpretation for E_n–homology of a commutative algebra A with coefficients in a symmetric A–bimodule M as well as for E_n–cohomology. To model E_n–homology with coefficients as functor homology we have to enlarge the category Epi_n to incorporate the summands of the twisting cochain δ.

Definition 4.1 The objects of the category Epi_n^+ are given by planar fully grown trees with n levels (see Definition 2.2). A morphism from

$$t^r = [r_n] \xrightarrow{f_n^r} \cdots \xrightarrow{f_2^r} [r_1] \quad \text{to} \quad t^s = [s_n] \xrightarrow{f_n^s} \cdots \xrightarrow{f_2^s} [s_1]$$

is represented by a sequence of maps $(h_n, \dots, h_1)$, where:

- For $i = 2, \dots, n-1$, the map $h_i\colon [r_i] \to [s_i]$ is a surjection which is order-preserving on the fibres $(f_i^r)^{-1}(l)$ for all $l \in [r_{i-1}]$. For $i = 1$ we require $h_1\colon [r_1] \to [s_1]$ to be order-preserving.
- The map
$$h_n\colon [r_n] \to [s_n]_+ := [s_n] \sqcup \{+\}$$
has $[s_n]$ in its image. We also require that the restriction of h_n to $h_n^{-1}([s_n])$ is order-preserving on the fibres of f_n^r. Furthermore, the intersection of $h_n^{-1}([s_n])$ with a fibre $(f_n^r)^{-1}(l)$ must be a (potentially empty) interval for all $l \in [r_{n-1}]$, ie of the form $\{a, a+1, \dots, a+b\}$ with $b \geqslant -1$.
- The diagram
$$\begin{tikzcd} h_n^{-1}([s_n]) \arrow[r, "f_n^r"] \arrow[d, "h_n"] & {[r_{n-1}]} \arrow[r] \arrow[d, "h_{n-1}"] & \cdots \arrow[r] & {[r_2]} \arrow[r, "f_1^r"] \arrow[d, "h_2"] & {[r_1]} \arrow[d, "h_1"] \\ {[s_n]} \arrow[r, "f_n^s"] & {[s_{n-1}]} \arrow[r] & \cdots \arrow[r] & {[s_2]} \arrow[r, "f_1^s"] & {[s_1]} \end{tikzcd}$$
commutes.

Finally, we identify certain morphisms by imposing the following equivalence relation on the set of morphisms from t^r to t^s: we identify morphisms h and h' if

- ${h_n}^{-1}(+) = {h'_n}^{-1}(+)$, and
- for all $1 \leqslant i \leqslant n$, the restrictions of h_i and h'_i to $f^r_{i+1} \dots f^r_n([r_n] \setminus h^{-1}_n(+))$ coincide.

The composition of two morphisms $(g_n, \dots, g_1)\colon t^q \to t^r$ and $(h_n, \dots, h_1)\colon t^r \to t^s$ is defined by composing componentwise and sending $+$ to $+$, ie

$$(h_n, \dots, h_1) \circ (g_n, \dots, g_1) := ((hg)_n, h_{n-1}g_{n-1}, \dots, h_1 g_1)$$

with

$$(hg)_n(x) = \begin{cases} + & \text{if } g_n(x) = +, \\ h_n g_n(x) & \text{otherwise.} \end{cases}$$

A straightforward calculation shows that composition in Epi_n^+ is well defined and associative.

Remark 4.2 (a) It is clear that Epi_n is a subcategory of Epi_n^+ and that both categories share the same objects. Let $\delta_i\colon [r_n] \to [r_n - 1]_+$ be the map

$$\delta_i(x) = \begin{cases} x & \text{if } x < i, \\ + & \text{if } x = i, \\ x-1 & \text{if } x > i. \end{cases}$$

Given a tree $t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$ such that i is the minimal or maximal element of a fibre $f_n^{-1}(l)$ containing at least two elements, let $\widehat{f_n}$ be given by

$$\widehat{f_n}(x) = \begin{cases} f_n(x) & \text{if } x < i, \\ f_n(x+1) & \text{if } x \geqslant i. \end{cases}$$

Let $t' = [r_n - 1] \xrightarrow{\widehat{f_n}} [r_{n-1}] \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} [r_1]$. Then, intuitively, the category Epi_n^+ is built from Epi_n by adding morphisms of the form $(\delta_i, \mathrm{id}, \dots \mathrm{id})\colon t \to t'$. The requirement that the elements of a fibre of f_n that are not mapped to $+$ form an interval reflects the fact that we have only added morphisms of the aforementioned kind.

(b) We only added morphisms $(\delta_i, \mathrm{id}, \dots, \mathrm{id})\colon t \to t'$ such that i is not the only element in the corresponding 1–fibre of t. Nevertheless, it is possible to map 1–fibres of cardinality one to $+$ by first applying maps which merge edges in lower levels. For example, the map

$$(h_2, h_1)\colon (\,[1] \xrightarrow{\mathrm{id}} [1]\,) \to (\,[0] \xrightarrow{\mathrm{id}} [0]\,)$$

with $h_2(0)=+$, $h_2(1)=0$ and $h_1(0)=h_1(1)=0$ arises as the composite $(h_2'',h_1'')\circ(h_2',h_1')$ of the maps

$$(h_2',h_1')\colon (\,[1]\xrightarrow{\mathrm{id}}[1]\,)\to(\,[1]\xrightarrow{0,1\mapsto 0}[0]\,),\quad h_2'=\mathrm{id},\quad h_1'(0)=h_1'(1)=0$$

and

$$(h_2'',h_1'')\colon (\,[1]\xrightarrow{0,1\mapsto 0}[0]\,)\to(\,[0]\xrightarrow{\mathrm{id}}[0]\,),\quad h_2''=\delta_0,\quad h_1'=\mathrm{id}.$$

(c) The motivation for defining Epi_n^+ is to model the complex calculating E_n–homology of A with coefficients in M. Hence imposing the above equivalence relation on the set of morphisms is necessary: it should not matter what precisely happens to a subtree of a tree t if all its leaves get mapped to $+$, ie in which order and on what side of an element we act on with a family of elements of A.

After defining the category Epi_n^+ which also models the summands of the twisting cochain δ, we can proceed to define E_n–homology of a functor.

Definition 4.3 Let $F\colon \mathrm{Epi}_n^+\to k$–mod be a functor. As in Definition 2.5 set

$$C^{E_n}_{r_n,\dots,r_1}(F):=\bigoplus F(t),$$

where the sum is indexed over all trees

$$t=[r_n]\to\cdots\to[r_1].$$

Define maps $\partial_j\colon C^{E_n}_{r_n,\dots,r_j,\dots,r_1}(F)\to C^{E_n}_{r_n,\dots,r_j-1,\dots,r_1}(F)$ lowering the j^{th} degree by one by

$$\partial_j=\widetilde{\partial}_j\quad\text{for } i<n\qquad\text{and}\qquad \partial_n=\widetilde{\partial}_n+\delta_{\min}+\delta_{\max},$$

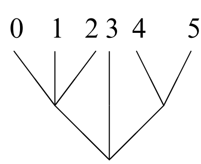
with

$$\delta_{\min}=\sum_{\substack{0\leqslant l\leqslant r_{n-1}\\ |f_n^{-1}(l)|>1}}(-1)^{s_{n,\min f_n^{-1}(l)}-1}F(\delta_{\min f_n^{-1}(l)},\mathrm{id},\dots,\mathrm{id}),$$

$$\delta_{\max}=\sum_{\substack{0\leqslant l\leqslant r_{n-1}\\ |f_n^{-1}(l)|>1}}(-1)^{s_{n,\max f_n^{-1}(l)}}F(\delta_{\max f_n^{-1}(l)},\mathrm{id},\dots,\mathrm{id}).$$

The integers $s_{n,i}$ are as in Definition 2.5.

Example 4.4 Let t be the 2–level tree



Then

$$\delta_{\min} = (-1)^1 F(\delta_0, \mathrm{id}) + (-1)^7 F(\delta_4, \mathrm{id}),$$
$$\delta_{\max} = (-1)^4 F(\delta_2, \mathrm{id}) + (-1)^9 F(\delta_5, \mathrm{id}).$$

We already know from [11, Lemma 3.8] that $(C^{E_n}, \widetilde{\partial}_1, \dots, \widetilde{\partial}_n)$ is a multicomplex. Hence it suffices to prove the following lemma, which can be done via a tedious but straightforward calculation; see [19, Lemma 4.14].

Lemma 4.5 *Let* $F\colon \mathrm{Epi}_n^+ \to k$–mod. *The maps defined above satisfy the identities*

$$(\delta_{\min} + \delta_{\max})\partial_j + \partial_j(\delta_{\min} + \delta_{\max}) = 0 \quad \text{for all } j < n,$$
$$(\delta_{\min} + \delta_{\max})^2 + \widetilde{\partial}_n(\delta_{\min} + \delta_{\max}) + \widetilde{\partial}_n(\delta_{\min} + \delta_{\max}) = 0.$$

Hence $C^{E_n}(F)$ *is a multicomplex.*

Definition 4.6 Let $F\colon \mathrm{Epi}_n^+ \to k$–mod be a functor. The E_n–homology of F is

$$H_*^{E_n}(F) = H_*(\mathrm{Tot}(C^{E_n}(F))).$$

Remark 4.7 Given a functor $\widetilde{F}\colon \mathrm{Epi}_n \to k$–mod, we can extend $\widetilde{F}$ to a functor $F\colon \mathrm{Epi}_n^+ \to k$–mod by setting $F(h) = 0$ for every morphism $h\colon t^r \to t^s$ in Epi_n^+ such that $h([r_n]) \cap \{+\} \neq \varnothing$. With these definitions $H^{E_n}(F)$ coincides with the E_n–homology of $\widetilde{F}$ as defined in Definition 2.7. In this sense the definition of E_n–homology we just gave extends the definition given in [11, Definition 3.7].

We are specifically interested in calculating E_n–homology of commutative algebras, which is the E_n–homology of the following functors.

Remark 4.8 The Loday functor $\mathcal{L}(A; M)\colon \mathrm{Epi}_n^+ \to k$–mod is the following functor: For a given tree $t = [r_n] \to \cdots \to [r_1]$ set

$$\mathcal{L}(A; M)(t) = M \otimes A^{\otimes r_n + 1}.$$

If $(h_n, \dots, h_1)\colon t^r \to t^s$ is a morphism, define

$$\mathcal{L}(A; M)(h_n, \dots, h_1)\colon M \otimes A^{\otimes r_n + 1} \to M \otimes A^{\otimes s_n + 1}$$

by

$$m \otimes a_0 \otimes \cdots \otimes a_{r_n} \mapsto \Bigg(m \cdot \prod_{\substack{i \in [r_n] \\ h_n(i) = +}} a_i\Bigg) \otimes \Bigg(\prod_{\substack{i \in [r_n] \\ h_n(i) = 0}} a_i\Bigg) \otimes \cdots \otimes \Bigg(\prod_{\substack{i \in [r_n] \\ h_n(i) = s_n}} a_i\Bigg).$$

Then

$$\mathrm{Tot}(C^{E_n}(\mathcal{L}(A;M))) = \Sigma^{-n}(M \otimes_{A_+} (A_+ \otimes B^n(A), \delta)).$$

In particular, by Theorem 3.3 we have

$$H_*^{E_n}(\mathcal{L}(A;M)) = H_*^{E_n}(A;M)$$

if A is k–projective. Note that $\mathcal{L}(A;k)$ agrees with the extension of the Loday functor defined by Livernet and Richter in [11, Definition 3.1] to Epi_n^+.

We now consider E_n–cohomology. The definition of E_n–cohomology is dual to the definition of E_n–homology.

Definition 4.9 Let $G\colon \mathrm{Epi}_n^{+\,\mathrm{op}} \to k$–mod be a functor. The E_n–cohomology of G is defined as

$$H^*_{E_n}(G) = H_*(\mathrm{Tot}(C_{E_n}(G))),$$

with the multicomplex $C_{E_n}(G)$ defined as follows. We set

$$C_{E_n}^{r_n,\dots,r_1}(G) = \bigoplus G(t),$$

where the sum is indexed over trees

$$t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1].$$

The differentials

$$\partial_j\colon C_{E_n}^{r_n,\dots,r_j,\dots,r_1}(G) \to C_{E_n}^{r_n,\dots r_j+1,\dots,r_1}(G)$$

raise the j^{th} degree by one. For $j = n$ define ∂_n restricted to $G(t)$ as

$$\sum_{\substack{0\leqslant i<r_n\\ f_n(i)=f_n(i+1)}} (-1)^{s_{n,i}} G(d_i, \mathrm{id}, \dots, \mathrm{id}) + \sum_{\substack{0\leqslant l\leqslant r_{n-1}\\ |f_n^{-1}(l)|>1}} (-1)^{s_{n,\min f_n^{-1}(l)}-1} G(\delta_{\min f_n^{-1}(l)}, \mathrm{id}, \dots, \mathrm{id}) + \sum_{\substack{0\leqslant l\leqslant r_{n-1}\\ |f_n^{-1}(l)|>1}} (-1)^{s_{n,\max f_n^{-1}(l)}} G(\delta_{\max f_n^{-1}(l)}, \mathrm{id}, \dots, \mathrm{id}).$$

For $1 \leqslant j < n$ the map ∂_j restricted to $G(t)$ is given by

$$\sum_{\substack{0\leqslant i<r_j\\ f_j(i)=f_j(i+1)}} \sum_{\sigma\in \mathrm{sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))} \epsilon(\sigma; t_{j,i}, t_{j,i+1})(-1)^{s_{j,i}} G(h_{i,\sigma}).$$

Here $h = h_{i,\sigma}$ again denotes the unique morphism of trees, exhibited in [11, Lemma 3.5], such that $h_j = d_i\colon [r_j] \to [r_j - 1]$, $h_l = \mathrm{id}$ for $l < j$, and h_{j+1} restricted to $f_{j+1}^{-1}(\{i, i+1\})$ acts like σ.

As was the case for E_n–homology, this definition generalizes E_n–cohomology of commutative algebras with coefficients in a symmetric bimodule:

Remark 4.10 Define $\mathcal{L}^c(A; M)\colon {\mathrm{Epi}_n^+}^{\mathrm{op}} \to k$–mod on $t = [r_n] \to \cdots \to [r_1]$ by

$$\mathcal{L}^c(A; M)(t) = \mathrm{Hom}_k(A^{\otimes r_n + 1}, M).$$

If $(h_n, \dots, h_1)$ is a morphism from t^r to t^s, define

$$\mathcal{L}^c(A; M)(h_n, \dots, h_1)\colon \mathrm{Hom}_k(A^{\otimes s_n + 1}, M) \to \mathrm{Hom}_k(A^{\otimes r_n + 1}, M)$$

by

$$(\mathcal{L}^c(A; M)(h_n, \dots, h_1)(f))(a_0 \otimes \cdots \otimes a_{r_n}) \\ = \Bigg(\prod_{\substack{i \in [r_n] \\ h_n(i) = +}} a_i\Bigg) \cdot f\Bigg(\Bigg(\prod_{\substack{i \in [r_n] \\ h_n(i) = 0}} a_i\Bigg) \otimes \cdots \otimes \Bigg(\prod_{\substack{i \in [r_n] \\ h_n(i) = s_n}} a_i\Bigg)\Bigg).$$

Then $\mathrm{Tot}(C_{E_n}(\mathcal{L}^c(A; M)))$ coincides with the complex computing E_n–cohomology of A with coefficients in M. Theorem 3.3 hence yields that

$$H^*_{E_n}(\mathcal{L}^c(A; M)) = H^*_{E_n}(A; M)$$

if A is k–projective.

5 E_n–cohomology as functor cohomology

In [11, Theorem 4.1] Livernet and Richter show that E_n–homology with trivial coefficients can be interpreted as functor homology. We now extend this result to E_n–homology and E_n–cohomology with arbitrary coefficients. As in [11], we prove that E_n–homology coincides with functor homology by using the axiomatic characterizations of Tor and Ext. For a background on functor homology we refer the reader to [16]. We first show that certain projective functors are acyclic. Recall that for a small category $\mathcal{C}$ a functor $F\colon \mathcal{C} \to k$–mod is called projective if it has the usual lifting property with respect to objectwise surjective natural transformations. For $t \in \mathrm{Epi}_n^+$ define projective functors P_t and P^t by

$$P_t = k[\mathrm{Epi}_n^+(t, -)]\colon \mathrm{Epi}_n^+ \to k\text{–mod} \quad \text{and} \quad P^t = k[\mathrm{Epi}_n^+(-, t)]\colon {\mathrm{Epi}_n^+}^{\mathrm{op}} \to k\text{–mod}.$$

In the proof of the following lemma, we will consider trees obtained by restricting a given tree to certain leaves.

Definition 5.1 Let $t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$ be a tree. For fixed $I \subset [r_n]$ set $r_i^I = |f_{i+1}\cdots f_n(I)| - 1$. Define a tree t^I as the upper row in

$$\begin{tikzcd}
{[r_n^I]} \arrow[r, "f_n^I"] \arrow[d] & {[r_{n-1}^I]} \arrow[r, "f_{n-1}^I"] \arrow[d] & \cdots \arrow[r, "f_2^I"] & {[r_1^I]} \arrow[d] \\
I \arrow[r, "f_n"] & f_n(I) \arrow[r, "f_{n-1}"] & \cdots \arrow[r, "f_2"] & f_2\cdots f_n(I)
\end{tikzcd}$$

Here the vertical morphisms are determined by requiring that they are bijective and order-preserving, while the maps f_n^I are defined by requiring that all squares commute. Intuitively t^I is the subtree of t given by restricting t to edges connecting leaves labelled by I with the root (the bottom vertex of the tree t).

Lemma 5.2 *Let $t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$ be a tree. Let $I \subset [r_n]$ be a set such that $I \cap f_n^{-1}(i)$ is a (possibly empty) interval for all $i \in [r_{n-1}]$. Then we can define a morphism*

$$h^I = (h_n^I, \dots, h_1^I)\colon t \to t^I$$

in Epi_n^+ as follows: The map h_n^I maps all $x \in [r_n] \setminus I$ to $+$ and is an order-preserving bijection restricted to I. For $i < n$ we require that h_i^I restricted to $f_{i+1}\cdots f_n(I)$ is the order-preserving bijection to $[r_i^I]$ and that h_i^I be order-preserving on the whole set $[r_i]$.

Proof Recall that a morphism in Epi_n^+ is an equivalence class with respect to the equivalence relation introduced in Definition 4.1. Since $I = [r_n] \setminus (h_n^I)^{-1}(+)$ the above requirements uniquely determine h^I up to equivalence. The maps h_i^I assemble to a morphism in Epi_n^+ since they are chosen to be order-preserving and the squares

$$\begin{tikzcd}
f_{i+1}\cdots f_n(I) \arrow[r, "f_i"] \arrow[d, "h_i^I"] & f_i\cdots f_n(I) \arrow[d, "h_{i-1}^I"] \\
{[r_i^I]} \arrow[r, "f_i^I"] & {[r_{i-1}^I]}
\end{tikzcd}$$

commute by definition of f_i^I. Furthermore $(h_n^I)^{-1}(+) \cap f_n^{-1}(i) = I \cap f_n^{-1}(i)$ is an interval. $\square$

Now we are in the position to compute the E_n–homology of the representable projectives.

Lemma 5.3 *Fix a tree* $t=[r_n]\xrightarrow{f_n}\cdots\xrightarrow{f_2}[r_1]$. *Then*

$$H_*^{E_n}(P_t)=\begin{cases}0 & \text{if } *>0,\\ \bigoplus_{i\in[r_n]}k & \text{if } *=0.\end{cases}$$

Proof Set $C:=\mathrm{Tot}(C^{E_n}(P_t))$. We define an ascending filtration by subcomplexes of C by

$$F^pC_{s_n,\dots,s_1}:=\bigoplus k\big[\{(h_n,\dots,h_1)\in P_t(t^s):|h_n^{-1}([s_n])|\leqslant p+1\}\big],$$

where the sum is indexed over trees

$$t^s=[s_n]\xrightarrow{f_n^s}\cdots\xrightarrow{f_2^s}[s_1].$$

Hence F^pC is generated by morphisms that map at least r_n-p leaves to $+$. This yields a first quadrant spectral sequence

$$E^1_{p,q}=H_{p+q}(F^pC/F^{p-1}C)\implies H_{p+q}(C).$$

The quotient $F^pC/F^{p-1}C$ can be identified with the free k–module generated by morphisms $(h_n,\dots,h_1)\in k[\mathrm{Epi}_n^+(t,t^s)]$ with $|h_n^{-1}([s_n])|=p+1$. The differentials $\delta_{\min}$ and $\delta_{\max}$ vanish on this quotient. The remaining summands of ∂_n and the differentials $\partial_{n-1},\dots,\partial_1$ do not change the number of leaves that get mapped to $+$. We conclude that $F^pC/F^{p-1}C$ is isomorphic to D as a complex, where

$$D_{s_n,\dots,s_1}=\bigoplus k\big[\{(h_n,\dots,h_1)\in P_t(t^s):|h_n^{-1}([s_n])|=p+1\}\big]$$

with differentials $\partial_1,\dots,\partial_{n-1}$ and $\widehat{\partial}_n=\partial_n-\delta_{\min}-\delta_{\max}$, and where the sum is indexed over trees t^s as above. The complex D can be decomposed further: The remaining differentials do not only respect the number of deleted leaves but also the set of deleted leaves itself. Hence D is the direct sum of subcomplexes D^I with

$$D^I_{s_n,\dots,s_1}=\bigoplus k\big[\{(h_n,\dots,h_1)\in P_t(t^s):h_n^{-1}([s_n])=I\}\big]$$

such that I is a subset of $[r_n]$ of cardinality $p+1$, and the sum is over trees t^s as above. Notice that the differentials of D and D^I look like the differentials used in Definition 2.5 to define E_n–homology of functors from Epi_n to k–mod. We will show that D^I in fact can be identified with the complex associated to such a functor. More precisely, D^I is the complex computing E_n–homology of the representable functor $k[\mathrm{Epi}_n(t^I,-)]\colon \mathrm{Epi}_n \to k$–mod: Denote by $h^I\colon t \to t^I$ the morphism defined in Lemma 5.2. We define

$$\Psi\colon \widetilde{C}^{E_n}(\mathrm{Epi}_n(t^I,-)) \to D^I$$

by mapping $j \in \mathrm{Epi}_n(t^I,t^s)$ to $\Psi(j) = j \circ h^I$. Since j does not delete any leaves this yields an element of D^I. We define an inverse Φ to Ψ by mapping $h \in D^I$ to the composite of the columns in

$$\begin{tikzcd}
{[r_n^I]} \arrow[r, "f_n^I"] \arrow[d] & {[r_{n-1}^I]} \arrow[r, "f_{n-1}^I"] \arrow[d] & \cdots \arrow[r, "f_2^I"] & {[r_1^I]} \arrow[d] \\
I \arrow[r, "f_n"] \arrow[d, "h_n"] & f_n(I) \arrow[r, "f_{n-1}"] \arrow[d, "h_{n-1}"] & \cdots \arrow[r, "f_2"] & f_2 \cdots f_n(I) \arrow[d, "h_1"] \\
{[s_n]} \arrow[r, "g_n"] & {[s_{n-1}]} \arrow[r, "g_{n-1}"] & \cdots \arrow[r, "g_2"] & {[s_1]}
\end{tikzcd}$$

Here the upper vertical maps are order-preserving bijections. We see that $\Phi(h)_i$ only depends on $h_{i|f_{i+1}\cdots f_n(I)}$, ie Φ is well defined on equivalence classes. It is obvious that each $\Phi(h)_i$ is surjective and that the usual requirements on commutativity are satisfied. Consider a fibre $(f_i^I)^{-1}(l)$: The map $\Phi(h)_i$ first sends it order-preservingly and surjectively to $f_{i+1}\cdots f_n(I) \cap f_i^{-1}(l') \subset [r_i]$, where l' denotes the image of l under the map $[r_{i-1}^I] \to f_i \cdots f_n(I)$. Since h_i preserves the order on fibres of f_i we see that $\Phi(h)_i$ is order-preserving on the fibres of f_i^I. Hence Φ is indeed a map from D^I to $\widetilde{C}^{E_n}(\mathrm{Epi}_n(t^I,-))$.

Finally, we note that obviously $\Phi \circ \Psi$ is the identity. To show that Ψ is a left inverse for Φ one writes down $(\Psi\circ\Phi)(h)$ for a given h and uses that $((\Psi\circ\Phi)(h))_i$ only needs to coincide with h_i on $f_{i+1}\cdots f_n(I)$. The maps Φ and Ψ commute with composition, hence also with applying the differentials. Since the signs in the differentials applied to a morphism h are determined by the target tree t^s of h, there is no trouble with signs either. Hence we have constructed an isomorphism

$$D^I \cong \widetilde{C}^{E_n}(\mathrm{Epi}_n(t^I,-))$$

of complexes.

We know from [11, Section 4] that $H_*(\mathrm{Tot}(\widetilde{C}^{E_n}(\mathrm{Epi}_n(t^I,-)))) = 0$ for $* > 0$ and that

$$H_0(\mathrm{Tot}(\widetilde{C}^{E_n}(\mathrm{Epi}_n(t^I,-)))) = \begin{cases} k & \text{if } t^I = [0] \to [0] \to \cdots \to [0], \\ 0 & \text{otherwise.} \end{cases}$$

Since $t^I = [0] \to [0] \to \cdots \to [0]$ implies $p+1 = |I| = 1$, we see that the E^1–term of our spectral sequence is

$$E^1_{p,q} = H_{p+q}(F^pC/F^{p-1}C) = \begin{cases} \bigoplus_{i\in[r_n]} k & \text{if } p = q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The spectral sequence collapses and the claim follows. □

Having proved that $H^{E_n}_*(P_t)$ is acyclic we can use the axiomatic description of Tor (see eg [9, Chapter 2]).

Theorem 5.4 *Denote by* $b\colon \mathrm{Epi}_n^{+\mathrm{op}} \to k\text{–mod}$ *the functor given by the cokernel of*

$$(\delta_0, \mathrm{id}, \dots, \mathrm{id})_* - (d_0, \mathrm{id}, \dots, \mathrm{id})_* + (\delta_1, \mathrm{id}, \dots, \mathrm{id})_*\colon P^{[1]\to[0]\to\cdots\to[0]} \to P^{[0]\to\cdots\to[0]}.$$

Then for any $F\colon \mathrm{Epi}_n^+ \to k\text{–mod}$ *we have*

$$H^{E_n}_*(F) \cong \mathrm{Tor}^{\mathrm{Epi}_n^+}_*(b, F),$$

and this isomorphism is natural in F.

Proof A short exact sequence $0 \to F \to G \to H \to 0$ of functors yields a short exact sequence of chain complexes

$$0 \to \mathrm{Tot}(C^{E_n}(F)) \to \mathrm{Tot}(C^{E_n}(G)) \to \mathrm{Tot}(C^{E_n}(H)) \to 0.$$

This in turn gives rise to a long exact sequence on homology. We already showed that $H^{E_n}_*(P_t)$ is zero in positive degrees. Every projective functor from Epi_n^+ to k–mod receives a surjection from a sum of functors of the form of P_t. It hence is a direct summand of this sum. Therefore $H^{E_n}_*(P)$ vanishes in positive degrees for all projective functors P. Finally, the zeroth E_n–homology of a functor F is given by the cokernel of

$$(-1)^{n-1}F(\delta_0, \mathrm{id}, \dots, \mathrm{id}) + (-1)^n F(d_0, \mathrm{id}, \dots, \mathrm{id}) + (-1)^{n+1}F(\delta_1, \mathrm{id}, \dots, \mathrm{id}).$$

Using the natural isomorphism $P^t \otimes_{\mathrm{Epi}_n^+} F \cong F(t)$ of k–modules and that tensor products are right exact, one sees that this coincides with $b \otimes_{\mathrm{Epi}_n^+} F$. □

Every functor $F\colon \mathrm{Epi}_n^+ \to k\text{–mod}$ gives rise to a functor $F^*\colon \mathrm{Epi}_n^{+\mathrm{op}} \to k\text{–mod}$, its dual, by setting $F^*(t) = \mathrm{Hom}_k(F(t), k)$. Since we just proved that E_n–homology of projective functors vanishes, we can relate E_n–homology with E_n–cohomology via the following spectral sequence.

Proposition 5.5 (see eg [18, Theorem 10.49]) *If $F(t)$ is k–free for every $t \in \mathrm{Epi}_n^+$, there is a first quadrant spectral sequence*

$$E^2_{p,q} = \mathrm{Ext}^q_k(H_p^{E_n}(F), k) \implies H^{p+q}_{E_n}(F^*).$$

In particular, whenever k is injective as a k–module, E_n–homology of F and E_n–cohomology of its dual are dual to each other.

Examples of commutative self-injective rings include fields, group algebras of finite commutative groups over a self-injective ring, quotients R/I of a principal ideal domain R with $I \neq 0$, and commutative Frobenius rings [1, Chapter 5, Section 18]. The product of self-injective rings is again self-injective.

Theorem 5.6 *Suppose that k is injective as a k–module and let $G\colon \mathrm{Epi}_n^{+\,\mathrm{op}} \to k$–mod be a functor. Then there is an isomorphism*

$$H^*_{E_n}(G) \cong \mathrm{Ext}^*_{\mathrm{Epi}_n^{+\,\mathrm{op}}}(b, G).$$

This isomorphism is natural in G.

Proof That $H^*_{E_n}$ maps short exact sequences to long exact sequences follows as in the homological case. Since the projective functor P_t is finitely generated and k–free, the functor P_t^* is injective. The universal coefficient spectral sequence (Proposition 5.5) yields that these modules are acyclic. But then all other injective modules are acyclic too, since they are direct summands of products of these. Finally, let $G\colon \mathrm{Epi}_n^{+\,\mathrm{op}} \to k$–mod be an arbitrary functor. Then the zeroth E_n–cohomology of G is by definition the kernel of

$$(-1)^{n-1} G(\delta_0, \mathrm{id}, \dots, \mathrm{id}) + (-1)^n G(d_0, \mathrm{id}, \dots, \mathrm{id}) + (-1)^{n+1} G(\delta_1, \mathrm{id}, \dots, \mathrm{id}).$$

The Yoneda lemma and the left exactness of $\mathrm{Nat}_{\mathrm{Epi}_n^{+\,\mathrm{op}}}(-, G)$ yield that this kernel results from applying $\mathrm{Nat}_{\mathrm{Epi}_n^{+\,\mathrm{op}}}(-, G)$ to b. □

6 Functor cohomology and cohomology operations

We recall the definition of the Yoneda pairing on Ext. The Yoneda pairing is usually defined in the context of modules over a ring (see eg [12, Chapter III, Sections 5–6]). But it is well known to be easily generalized to suitable abelian categories with enough projectives and injectives. We assume that k is self-injective in this section.

Definition 6.1 Let F, G and H be functors from $\mathrm{Epi}_n^{+\mathrm{op}}$ to k–mod. Let P_F denote a projective resolution of F and I_H an injective resolution of H. There is a pairing

$$\mu\colon \mathrm{Ext}^*_{\mathrm{Epi}_n^{+\mathrm{op}}}(G,H) \otimes \mathrm{Ext}^*_{\mathrm{Epi}_n^{+\mathrm{op}}}(F,G) \to \mathrm{Ext}^*_{\mathrm{Epi}_n^{+\mathrm{op}}}(F,H),$$

defined as the composite

$$\begin{array}{c}
\mathrm{Ext}^m_{\mathrm{Epi}_n^{+\mathrm{op}}}(G,H) \otimes \mathrm{Ext}^n_{\mathrm{Epi}_n^{+\mathrm{op}}}(F,G) \\
\| \\
H_m(\mathrm{Nat}_{\mathrm{Epi}_n^{+\mathrm{op}}}(G, I_H)) \otimes H_n(\mathrm{Nat}_{\mathrm{Epi}_n^{+\mathrm{op}}}(P_F, G)) \\
\downarrow \\
H_{n+m}\big(\mathrm{Nat}_{\mathrm{Epi}_n^{+\mathrm{op}}}(G, I_H) \otimes \mathrm{Nat}_{\mathrm{Epi}_n^{+\mathrm{op}}}(P_F, G)\big) \\
\downarrow \\
H_{n+m}(\mathrm{Nat}_{\mathrm{Epi}_n^{+\mathrm{op}}}(P_F, I_H)) = \mathrm{Ext}^{n+m}_{\mathrm{Epi}_n^{+\mathrm{op}}}(F,H).
\end{array}$$

Here the second map is induced by composing natural transformations. This associative pairing is called the Yoneda pairing.

In particular, there is a natural action of

$$\mathrm{Ext}^*_{\mathrm{Epi}_n^{+\mathrm{op}}}(b,b) = H^*_{E_n}(b)$$

on E_n–cohomology. One could hope to find cohomology operations via this action. For example, if the characteristic of k is a prime p, Hochschild cohomology $\mathrm{HH}^*(A; A_+)$ is a p–restricted Gerstenhaber algebra, ie the Lie algebra structure on $\Sigma^{-1}\,\mathrm{HH}^*(A; A_+)$ comes with a restriction. We will determine $H^*_{E_n}(b)$ to see whether we can find new or old cohomology operations using the Yoneda pairing. For the remainder of this section we will denote $b\colon \mathrm{Epi}_n^{+\mathrm{op}} \to k$–mod by b_n since we will have to consider trees of varying levels. Since we are going to work homologically we make b_n^*, the dual of b_n, explicit. Intuitively, b_n^* is the functor assigning to a tree its set of leaves.

Proposition 6.2 *The functor b_n^* assigns $k\langle [r_n]\rangle = k[\{0,\dots,r_n\}]$ to a given tree $t = [r_n] \to \cdots \to [r_1]$. Denoting the generators of $k\langle [r_n]\rangle$ by $\alpha_0,\dots,\alpha_{r_n}$, it induces the maps*

$$b_n^*(\tau_n,\dots,\tau_{j+1},d_i,\mathrm{id},\dots,\mathrm{id})\colon k\langle [r_n]\rangle \to k\langle [r_n]\rangle, \quad \alpha_m \mapsto \alpha_{\tau_n^{-1}(m)}$$

for suitable $\tau_{j+1} \in \Sigma_{[r_{j+1}]}, \dots, \tau_n \in \Sigma_{[r_n]}$ *as in* [11, Lemma 3.5],

$$b_n^*(d_i, \mathrm{id}, \dots, \mathrm{id})\colon k\langle[r_n+1]\rangle \to k\langle[r_n]\rangle, \quad \alpha_m \mapsto \begin{cases} \alpha_m & \text{if } m \leqslant i, \\ \alpha_{m-1} & \text{if } m > i, \end{cases}$$

$$b_n^*(\delta_i, \mathrm{id}, \dots, \mathrm{id})\colon k\langle[r_n+1]\rangle \to k\langle[r_n]\rangle, \quad \alpha_m \mapsto \begin{cases} \alpha_m & \text{if } m < i, \\ 0 & \text{if } m = i, \\ \alpha_{m-1} & \text{if } m > i. \end{cases}$$

We will show that b_n^* is indeed acyclic with respect to E_n–homology. The case $n = 1$ can be easily calculated:

Proposition 6.3 *For* $n = 1$ *we have*

$$H^r_{E_1}(b_1) \cong H_r^{E_1}(b_1^*) = 0$$

for $r > 0$ *and*

$$H^0_{E_1}(b_1) \cong H_0^{E_1}(b_1^*) = k.$$

For $n > 1$ we derive the acyclicity of b_n^* from the case $n = 1$. For this we need the following lemma. Recall that the differential ∂_n is induced by morphisms which act on the top level of a given tree. Intuitively, the following lemma states that ∂_n can be split into parts that correspond to morphisms acting on the different fibres.

Lemma 6.4 *Let* $F\colon \mathrm{Epi}_n^+ \to k$–mod *be a functor and* $r_1, \dots, r_{n-1} \geqslant 0$. *Consider the* $r_{n-1}+1$*–fold multicomplex*

$$M_{x_0,\dots,x_{r_{n-1}}}(F) = \bigoplus_{\substack{t=[x_0+\dots+x_{r_{n-1}}]\xrightarrow{f_n}\cdots\xrightarrow{f_2}[r_1] \\ |f_n^{-1}(0)|=x_0+1,\, |f_n^{-1}(i)|=x_i \text{ for all } 1\leqslant i\leqslant r_{n-1}}} F(t)$$

where the i^{th} *differential* d^i *of the multicomplex is the part of* ∂_n *induced by morphisms operating on the fibre* $f_n^{-1}(i)$. *Then*

$$\mathrm{Tot}(M) \cong \Sigma^{-r_1-\dots-r_{n-1}}(C^{E_n}_{(*,r_{n-1},\dots,r_1)}(F), \partial_n).$$

Furthermore, we can split M *into submulticomplexes corresponding to the underlying* $(n-1)$*–level tree* T*: Let* $t^{x_0+1,x_1,\dots,x_{r_{n-1}}}$ *be the tree extending* T *with top-level fibres of cardinality* $x_0+1, x_1, \dots, x_{r_{n-1}}$. *Let*

$$M^T_{x_1,\dots,x_{r_{n-1}}} = F(t^{x_0+1,x_1,\dots,x_{r_{n-1}}}).$$

Then

$$M_{*,\dots,*}(F) = \bigoplus_{T=[r_{n-1}]\to\cdots\to[r_1]} (M^T_{*,\dots,*}, d^0, \dots, d^{r_{n-1}}).$$

Proof The differential ∂_n is the sum of the maps d^i for $0 \leqslant i \leqslant r_{n-1}$, each of them leaving all 1–fibres except for $f_n^{-1}(i)$ unchanged. Two such differentials d^i and d^j commute except for their signs: Since d^i deletes or merges edges left of $f_n^{-1}(j)$ for $i < j$, we find that $d^i d^j = -d^j d^i$. Hence it is clear that up to a shift we can interpret $C^{E_n}_{(*,r_{n-1},\dots,r_1)}(F)$ as a total complex as above. All the differentials d^i leave the lower levels of a tree t as they were. Hence the splitting above holds, allowing us to consider one $(n-1)$–tree shape at a time. □

Theorem 6.5 *For all $n \geqslant 0$ we have*

$$H_s^{E_n}(b_n^*) = \begin{cases} k & \text{if } s = 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Proof We will prove that

$$H_*(C^{E_n}_{(*,r_{n-1},\dots,r_1)}(b_n^*), \partial_n) = 0$$

except when $r_{n-1} = 0$. Note that if $r_{n-1} = 0$ this forces $r_{n-2}, \dots, r_1 = 0$, and

$$(C^{E_n}_{(*,0,\dots,0)}(b_n^*), \partial_n) \cong C_*^{E_1}(b_1^*).$$

By Proposition 6.3 and Lemma 6.4 this gives rise to a copy of k in $H_0^{E_n}(b_n^*)$.

Now fix $r_{n-1} \geqslant 1$, $r_{n-2}, \dots, r_1 \geqslant 0$. Let $T = [r_{n-1}] \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} [r_1]$ be an $(n-1)$–level tree. Consider the corresponding summand M^T of the multicomplex $M(b_n^*)$ discussed in Lemma 6.4. According to the lemma it suffices to show that the homology of the total complex associated to M^T is trivial for all trees T as above. Let us start by calculating the homology of M^T in the zeroth direction, ie for each given $x_1, \dots, x_{r_{n-1}} \geqslant 1$ we consider the complex

$$(M^T_{*,x_1,\dots,x_{r_{n-1}}}, d^0) = \Bigg(\bigoplus_{\substack{t=[*+x_1+\dots+x_{r_{n-1}}] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1] \\ |f_n^{-1}(0)|=*+1,\, |f_n^{-1}(i)|=x_i}} b_n^*(t), d^0 \Bigg).$$

Since we fixed T, for each p there is exactly one tree $t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$ with $|f_n^{-1}(0)| = p+1$ and $|f_n^{-1}(i)| = x_i$ for $1 \leqslant i \leqslant r_{n-1}$. Let $q = r_n - p$. The differential d^0 maps $\alpha_j \in b_n^*(t) = k\langle \alpha_0, \dots, \alpha_{p+q} \rangle$ to

$$(-1)^{n-1} b_n^*(\delta_0, \mathrm{id}, \dots, \mathrm{id})(\alpha_j) + \sum_{i=0}^{p-1} (-1)^{n+i} b_n^*(d_i, \mathrm{id}, \dots, \mathrm{id})(\alpha_j) \\ + (-1)^{n+p} b_n^*(\delta_p, \mathrm{id}, \dots, \mathrm{id})(\alpha_j).$$

Thus for $j \leqslant p$ the element $d^0(\alpha_j)$ coincides up to a sign $(-1)^{n-1}$ with the image of $\alpha_j \in b_1^*([p])$ under the differential d_{E_1} of $C_*^{E_1}(b_1^*)$. If $j > p$ all the induced morphisms are the identity. Hence $(M^T_{*,x_1,\dots,x_{r_{n-1}}}, d^0)$ is isomorphic to

$$\cdots \xrightarrow{d_{E_1}\oplus 0} b_1^*([3]) \oplus k^q \xrightarrow{d_{E_1}\oplus \mathrm{id}} b_1^*([2]) \oplus k^q \xrightarrow{d_{E_1}\oplus 0} b_1^*([1]) \oplus k^q \xrightarrow{d_{E_1}\oplus \mathrm{id}} b_1^*([0]) \oplus k^q$$

and $H_p(M^T_{*,x_1,\dots,x_{r_{n-1}}}, d^0)$ is concentrated in degree $p = 0$, where it is k. We showed in Proposition 6.3 that $H_0^{E_1}(b_1^*) = b_1^*([0])$. Hence a cycle in $H_0(M^T_{*,x_1,\dots,x_{r_{n-1}}}, d^0)$ is given by $\alpha_0 \in b_n^*(t^{1,x_1,\dots,x_{r_{n-1}}})$, where $t^{1,x_1,\dots,x_{r_{n-1}}}$ is the tree which extends T with top-level fibres of cardinality $1, x_1, \dots, x_{r_{n-1}}$.

We now determine how d^1 acts on these cycles. The differential d^1 is induced by morphisms acting on leaves in the second-to-left top-level fibre. All of these morphisms leave the leftmost leaf invariant and therefore each of the induced maps sends α_0 to α_0. Hence for fixed $x_2, \dots, x_{r_{n-1}} \geqslant 1$ the chain complex $(H_0(M^T_{*,*,x_2\dots,x_{r_{n-1}}}, d^0), d^1)$ is one-dimensional on the generator α_0 in each degree r with differential

$$d^1(\alpha_0) = (-1)^{2n-1} \sum_{i=0}^{r+1} (-1)^i \alpha_0.$$

We see that the homology of $(H_0(M^T_{*,*,x_2\dots,x_{r_{n-1}}}, d^0), d^1)$ vanishes completely and the homology of the total complex of M^T is zero. Hence $(C^{E_n}_{(*,r_{n-1},\dots,r_1)}(b_n^*), \partial_n)$ has trivial homology as well, whenever $r_{n-1} \geqslant 1$. $\square$

Corollary 6.6 *No nontrivial cohomology operations arise on E_n–cohomology via the Yoneda pairing defined in Definition 6.1.*

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The η–inverted $\mathbb{R}$–motivic sphere

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We use an Adams spectral sequence to calculate the $\mathbb{R}$–motivic stable homotopy groups after inverting η. The first step is to apply a Bockstein spectral sequence in order to obtain h_1–inverted $\mathbb{R}$–motivic Ext groups, which serve as the input to the η–inverted $\mathbb{R}$–motivic Adams spectral sequence. The second step is to analyze Adams differentials. The final answer is that the Milnor–Witt $(4k-1)$–stem has order 2^{u+1}, where u is the 2–adic valuation of $4k$. This answer is reminiscent of the classical image of J. We also explore some of the Toda bracket structure of the η–inverted $\mathbb{R}$–motivic stable homotopy groups.

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1 Introduction

The first exotic property of motivic stable homotopy groups is that the Hopf map η is not nilpotent. This means that inverting η can be useful for understanding the global structure of motivic stable homotopy groups.

In Andrews and Miller [3] and Guillou and Isaksen [5], the η–inverted $\mathbb{C}$–motivic 2–completed stable homotopy groups $\hat{\pi}^{\mathbb{C}}_{*,*}[\eta^{-1}]$ were explicitly computed to be

$$\mathbb{F}_2[\eta^{\pm 1}][\mu, \varepsilon]/\varepsilon^2.$$

This result naturally suggests that one should study the structure of η–inverted motivic stable homotopy groups over other fields.

In the present article, we consider the η–inverted $\mathbb{R}$–motivic 2–completed stable homotopy groups $\hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}]$. Our main tool is the motivic Adams spectral sequence, which takes the form

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}] \Longrightarrow \hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}].$$

Here $\mathcal{A}^{\mathbb{R}}$ is the $\mathbb{R}$–motivic Steenrod algebra, and $\mathbb{M}_2^{\mathbb{R}}$ is the motivic $\mathbb{F}_2$–cohomology of $\mathbb{R}$. We will exhaustively compute this spectral sequence.

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We begin with computing the Adams E_2–page $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}]$ using the ρ–Bockstein spectral sequence; see Hill [6] and Dugger and Isaksen [4]. This spectral sequence takes the form

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho][h_1^{-1}] \Longrightarrow \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}],$$

where $\mathcal{A}^{\mathbb{C}}$ is the $\mathbb{C}$–motivic Steenrod algebra and $\mathbb{M}_2^{\mathbb{C}}$ is the motivic $\mathbb{F}_2$–cohomology of $\mathbb{C}$.

The input to the ρ–Bockstein spectral sequence is completely known from Guillou and Isaksen [5]. In order to deduce differentials, one first observes, as in Dugger and Isaksen [4], that the groups

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[\rho^{-1}, h_1^{-1}]$$

with ρ and h_1 both inverted are easy to describe. Then there is only one pattern of ρ–Bockstein differentials that is consistent with this ρ–inverted calculation.

Having obtained the Adams E_2–page $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}]$, the next step is to compute Adams differentials. The extension of scalars functor from $\mathbb{R}$–motivic homotopy theory to $\mathbb{C}$–motivic homotopy theory induces a map

$$\begin{tikzcd}
\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}] \arrow[r, Rightarrow] \arrow[d] & \hat{\pi}_{*,*}^{\mathbb{R}}[\eta^{-1}] \arrow[d] \\
\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[h_1^{-1}] \arrow[r, Rightarrow] & \hat{\pi}_{*,*}^{\mathbb{C}}[\eta^{-1}]
\end{tikzcd}$$

of Adams spectral sequences. The bottom Adams spectral sequence is completely understood; see Andrews and Miller [3] and Guillou and Isaksen [5]. The Adams d_2 differentials in the top spectral sequence can then be deduced by the comparison map.

This leads to a complete description of the h_1–inverted $\mathbb{R}$–motivic Adams E_3–page. Over $\mathbb{C}$, it turns out that the h_1–inverted Adams spectral sequence collapses at this point. However, over $\mathbb{R}$, there are higher differentials that we deduce from manipulations with Massey products and Toda brackets.

In the end, we obtain an explicit description of the h_1–inverted $\mathbb{R}$–motivic Adams E_∞–page, from which we can read off the η–inverted stable motivic homotopy groups over $\mathbb{R}$.

In order to state the result, we need a bit of terminology. Because η belongs to $\hat{\pi}_{1,1}^{\mathbb{R}}$, it makes sense to use a grading that is invariant under multiplication by η. The Milnor–Witt n–stem is the direct sum $\Pi_n = \bigoplus_p \hat{\pi}_{p+n,p}^{\mathbb{R}}$. Then multiplication by η is an endomorphism of the Milnor–Witt n–stem.

Theorem 1.1 (1) *The η–inverted Milnor–Witt 0–stem $\Pi_0[\eta^{-1}]$ is $\mathbb{Z}_2[\eta^{\pm 1}]$, where $\mathbb{Z}_2$ is the ring of 2–adic integers.*

(2) *If $k > 1$, then the η–inverted Milnor–Witt $(4k-1)$–stem $\Pi_{4k-1}[\eta^{-1}]$ is isomorphic to $\mathbb{Z}/2^{u+1}[\eta^{\pm 1}]$ as a module over $\mathbb{Z}_2[\eta^{\pm 1}]$, where u is the 2–adic valuation of $4k$.*

(3) *The η–inverted Milnor–Witt n–stem $\Pi_n[\eta^{-1}]$ is zero otherwise.*

For degree reasons, the product structure on $\hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}]$ is very simple. However, there are many interesting Toda brackets. We explore much of the 3–fold Toda bracket structure in this article. In particular, we will show that all of $\hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}]$ can be constructed inductively via Toda brackets, starting from just 2 and the generator of the Milnor–Witt 3–stem.

Theorem 1.1 gives a familiar answer. These groups have the same order as the classical image of J. For example, Π_3 consists of elements of order 8, which is the same as the order of the image of J in the classical 3–stem. Similarly, Π_7 consists of elements of order 16, which is the same as the order of the image of J in the classical 7–stem. One might expect a geometric proof that directly compares the classical image of J spectrum with the η–inverted $\mathbb{R}$–motivic sphere. However, higher structure in the form of Toda brackets suggests that such a direct proof is not possible.

We also observe that our calculations are reminiscent of the classical Adams spectral sequence for v_1–periodic homotopy at odd primes, as carried out in Andrews [2]. We are not aware of a structural reason why the calculations are so similar.

The calculation of the η–inverted $\mathbb{R}$–motivic homotopy groups leads to questions about η–inverted motivic homotopy groups over other fields. We leave it to the reader to speculate on the behavior of these η–inverted groups over other fields.

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2 Preliminaries

2.1 Notation

We continue with notation from [4] as follows:

(1) $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$ is the motivic cohomology of $\mathbb{C}$ with $\mathbb{F}_2$ coefficients, where τ has bidegree $(0,1)$.

(2) $\mathbb{M}_2^{\mathbb{R}} = \mathbb{F}_2[\tau,\rho]$ is the motivic cohomology of $\mathbb{R}$ with $\mathbb{F}_2$ coefficients, where τ and ρ have bidegrees $(0,1)$ and $(1,1)$, respectively.

(3) $\mathcal{A}^{\mathrm{cl}}$ is the classical mod 2 Steenrod algebra.

(4) $\mathcal{A}^{\mathbb{C}}$ is the mod 2 motivic Steenrod algebra over $\mathbb{C}$.

(5) $\mathcal{A}^{\mathbb{R}}$ is the mod 2 motivic Steenrod algebra over $\mathbb{R}$.

(6) $\mathrm{Ext}_{\mathrm{cl}}$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$.

(7) $\mathrm{Ext}_{\mathbb{C}}$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$.

(8) $\mathrm{Ext}_{\mathbb{R}}$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$.

(9) $\hat{\pi}^{\mathbb{C}}_{*,*}$ is the motivic stable homotopy ring of the 2–completed motivic sphere spectrum over $\mathbb{C}$.

(10) $\hat{\pi}^{\mathbb{R}}_{*,*}$ is the motivic stable homotopy ring of the 2–completed motivic sphere spectrum over $\mathbb{R}$.

(11) Π_n is the Milnor–Witt n–stem $\bigoplus_p \hat{\pi}^{\mathbb{R}}_{p+n,p}$.

(12) $\mathcal{R} = \mathbb{F}_2[\rho, h_1^{\pm 1}]$.

(13) The symbols v_1^4 and P are used interchangeably for the Adams periodicity operator.

2.2 Grading conventions

We follow [7] in grading Ext according to (s, f, w), where:

(1) f is the Adams filtration, ie the homological degree.

(2) $s + f$ is the internal degree, ie that corresponding to the first coordinate in the bidegree of the Steenrod algebra.

(3) s is the stem, ie the internal degree minus the Adams filtration.

(4) w is the weight.

Following this grading convention, the elements τ and ρ, as elements of $\mathrm{Ext}_{\mathbb{R}}$, have degrees $(0, 0, -1)$ and $(-1, 0, -1)$ respectively.

We will consider the groups $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ in which h_1 has been inverted. The degree of h_1 is $(1, 1, 1)$. As in [5], for this purpose it is convenient to introduce the following gradings whose values are zero for h_1:

(5) $mw = s - w$ is the Milnor–Witt degree.

(6) $c = s + f - 2w$ is the Chow degree.

In order to avoid notational clutter, we will often drop h_1 from the notation. Since h_1 is a unit, no information is lost by doing this. The correct powers of h_1 can always be recovered by checking degrees.

For example, in Lemma 3.1 below, we claim that there is a differential $d_3^\rho(v_1^4) = \rho^3 v_2$ in the ρ–Bockstein spectral sequence. Strictly speaking, this formula is nonsensical because $d_3^\rho(v_1^4)$ has Adams filtration 5 while v_2 has Adams filtration 1. The correct full formula is $d_3^\rho(v_1^4) = \rho^3 h_1^4 v_2$.

If we are to ignore multiples of h_1, we must rely on gradings that take value 0 on h_1. This explains our preference for Milnor–Witt degree mw and Chow degree c.

3 The ρ–Bockstein spectral sequence

Recall [6; 4] that the ρ–Bockstein spectral sequence takes the form

$$\mathrm{Ext}_{\mathbb{C}}[\rho] \Longrightarrow \mathrm{Ext}_{\mathbb{R}}.$$

After inverting h_1, by [5, Theorem 1.1] this takes the form

$$\mathcal{R}[v_1^4, v_2, v_3, \ldots] \Longrightarrow \mathrm{Ext}_{\mathbb{R}}[h_1^{-1}],$$

where $\mathcal{R} = \mathbb{F}_2[\rho, h_1^{\pm 1}]$. Table 1 lists the generators of the Bockstein E_1–page.

(mw, c)	generator
$(0, 1)$	ρ
$(4, 4)$	v_1^4
$(3, 1)$	v_2
$(7, 1)$	v_3
$(15, 1)$	v_4
$(2^n - 1, 1)$	v_n

Table 1: Bockstein E_1–page generators

Lemma 3.1 *In the ρ–Bockstein spectral sequence, there are differentials*

$$d^\rho_{2^n-1}(v_1^{2^n}) = \rho^{2^n-1} v_n \quad \text{for } n \geq 2.$$

All other nonzero differentials follow from the Leibniz rule.

The first few examples of these differentials are $d_3(v_1^4) = \rho^3 v_2$, $d_7(v_1^8) = \rho^7 v_3$ and $d_{15}(v_1^{16}) = \rho^{15} v_4$.

Proof Inverting ρ induces a map

$$\begin{tikzcd} \mathrm{Ext}_{\mathbb{C}}[h_1^{-1}][\rho] \arrow[r, Rightarrow, "\rho\text{–Bss}"] \arrow[d, "\rho\text{–inv}"'] & \mathrm{Ext}_{\mathbb{R}}[h_1^{-1}] \arrow[d, "\rho\text{–inv}"] \\ \mathrm{Ext}_{\mathbb{C}}[h_1^{-1}][\rho^{\pm 1}] \arrow[r, Rightarrow, "\rho\text{–Bss}"] & \mathrm{Ext}_{\mathbb{R}}[h_1^{-1}, \rho^{-1}] \end{tikzcd}$$

of ρ–Bockstein spectral sequences. We will establish differentials in the ρ–inverted spectral sequence. The map of spectral sequences then implies that the same differentials occur when ρ is not inverted.

Recall [4, Theorem 4.1] there is an isomorphism $\mathrm{Ext}_{\mathrm{cl}}[\rho^{\pm 1}] \cong \mathrm{Ext}_{\mathbb{R}}[\rho^{-1}]$ sending the classical element h_0 to the motivic element h_1. Using also that $\mathrm{Ext}_{\mathrm{cl}}[h_0^{-1}] = \mathbb{F}_2[h_0^{\pm 1}]$, it follows $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}, \rho^{-1}]$ is isomorphic to $\mathcal{R}[\rho^{-1}]$. Then the ρ–inverted ρ–Bockstein spectral sequence takes the form

$$\mathcal{R}[\rho^{-1}][v_1^4, v_2, v_3, \dots] \overset{\rho\text{–Bss}}{\Longrightarrow} \mathcal{R}[\rho^{-1}].$$

Because the target of the ρ–inverted spectral sequence is very small, essentially everything must either support a differential or be hit by a differential.

The ρ–Bockstein differentials have degree $(-1, 0)$ with respect to the grading (mw, c) used in Table 1. The elements $\rho^k v_2$ cannot support differentials because there are no elements in the Milnor–Witt 2–stem. The only possibility is that after inverting ρ, there is a ρ–Bockstein differential $d_3(v_1^4) = \rho^3 v_2$.

Then the ρ–inverted E_4–page is $\mathcal{R}[v_1^8, v_3, v_4, \dots]$. The elements $\rho^k v_3$ cannot support differentials because the ρ–inverted E_4–page has no elements in the Milnor–Witt 6–stem. The only possibility is that after inverting ρ, there is a ρ–Bockstein differential $d_7(v_1^8) = \rho^7 v_3$.

In general, the ρ–inverted E_{2^n-1}–page is $\mathcal{R}[v_1^{2^n}, v_n, v_{n+1}, \dots]$. The elements $\rho^k v_n$ cannot support differentials because the ρ–inverted E_{2^n-1}–page has no elements in the Milnor–Witt (2^n-2)–stem. The only possibility is that after inverting ρ, there is a ρ–Bockstein differential $d_{2^n-1}(v_1^{2^n}) = \rho^{2^n-1} v_n$. □

The ρ–Bockstein E_∞–page can be directly computed from the Leibniz rule and the differentials in Lemma 3.1. For example, $d_3(v_1^4) = \rho^3 v_2$, so $d_3(v_1^{4+8k}) = \rho^3 v_1^{8k} v_2$. This establishes the relation $\rho^3 v_1^{8k} v_2 = 0$.

To ease the notation in Proposition 3.2, we write P rather than v_1^4.

Proposition 3.2 *The ρ–Bockstein E_∞–page is the $\mathcal{R}$–algebra on the generators $P^{2^{n-1}k} v_n$ for $n \geq 2$ and $k \geq 0$ (see Table 2), subject to the relations*

$$\rho^{2^n-1} P^{2^{n-1}k} v_n = 0$$

for $n \geq 2$ and $k \geq 0$, and

$$P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m + P^{2^{n-1}(k+2^{m-n}j)} v_n \cdot v_m = 0$$

for $m \geq n \geq 2$, $k \geq 0$ and $j \geq 0$.

(mw, c)	generator	ρ–torsion
$(0,1)$	ρ	∞
$(0,0)$	h_1	∞
$(3,1)+k(8,8)$	$P^{2k}v_2$	3
$(7,1)+k(16,16)$	$P^{4k}v_3$	7
$(15,1)+k(32,32)$	$P^{8k}v_4$	15
$(2^n-1,1)+k(2^{n+1},2^{n+1})$	$P^{2^{n-1}k}v_n$	2^n-1

Table 2: Bockstein E_∞–page generators

Remark 3.3 In practice, the relations mean that every P can be shifted onto the v_n with minimal n in any monomial. Thus an $\mathcal{R}$–module basis is given by monomials of the form $P^{2^{n-1}k} v_n \cdot v_{m_1} \cdots v_{m_a}$, where $n \leq m_1 \leq \cdots \leq m_a$. For example,

$$P^2 v_2 \cdot P^4 v_2 = P^6 v_2 \cdot v_2, \quad P^4 v_2 \cdot P^8 v_3 = P^{12} v_2 \cdot v_3, \quad P^4 v_3 \cdot P^{48} v_5 = P^{52} v_3 \cdot v_5.$$

4 The Adams E_2–page

Having obtained the ρ–Bockstein E_∞–page in Section 3, our next task is to consider hidden extensions in $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$. We will show that there are no hidden relations. This will require some careful analysis of degrees, as well as some manipulations with Massey products.

The ρ–Bockstein E_∞–page is an associated graded object of $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$. Elements of the E_∞–page only determine elements of $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$ up to higher filtration. Therefore, we must be careful about choosing specific generators of $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$.

We will show in Lemma 4.1 that $P^{2^{n-1}k} v_n$ detects a unique element of $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$. Therefore, we may unambiguously use the same notation $P^{2^{n-1}k} v_n$ for an element of $\mathrm{Ext}_\mathbb{R}[h_1^{-1}]$.

In general, the ρ–Bockstein spectral sequence does not allow for hidden extensions by ρ. More precisely, if x is an element of the ρ–Bockstein E_∞–page such that $\rho^k x = 0$, then x detects an element of $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ that is also annihilated by ρ^k. Beware that x might detect more than one element of $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$, and some such elements might not be annihilated by ρ^k. Nevertheless, there is always at least one element that is annihilated by ρ^k.

For example, the relation $\rho^{2^{n-1}} P^{2^{n-1}k} v_n = 0$ in the ρ–Bockstein E_∞–page lifts to give the same relation in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.

Lemma 4.1 *For each $n \geq 2$ and $k \geq 0$, the element $P^{2^{n-1}k} v_n$ of the Bockstein E_∞–page detects a unique element of* $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.

Proof We need to show that in the ρ–Bockstein E_∞–page, $P^{2^{n-1}k} v_n$ does not share bidegree with an element of higher filtration.

First suppose that $P^{2^{n-1}k} v_n$ has the same bidegree as $\rho^b P^{2^{m-1}j} v_m$. Then

$$(2^n - 1, 1) + k(2^{n+1}, 2^{n+1}) = (2^m - 1, 1) + j(2^{m+1}, 2^{m+1}) + b(0, 1).$$

Considering only the Milnor–Witt degree, we have

$$2^n(2k+1) = 2^m(2j+1).$$

Therefore, $n = m$ and $k = j$, so $b = 0$.

Suppose that $P^{2^{n-1}k} v_n$ shares bidegree with some element x. By Remark 3.3, we may assume that x is of the form $\rho^b P^{2^{m_1-1}j} v_{m_1} \cdot v_{m_2} \cdots v_{m_a}$, where $m_1 \leq m_2 \leq \cdots \leq m_a$. Since $\rho^{2^{m_1-1}} P^{2^{m-1}j} v_{m_1} = 0$, we may also assume that $b \leq 2^{m_1} - 2$. Because of the previous paragraph, we may assume that $a \geq 2$. We wish to show that $b = 0$.

We first show that $n \geq m_a$. Let $u(x)$ be the difference $mw - c$. We have that $u(P^{2^{m_1-1}j} v_{m_1}) = 2^{m_1} - 2$ and $u(\rho) = -1$. Since $b \leq 2^{m_1} - 2$, it follows that $u(\rho^b P^{2^{m_1-1}j} v_{m_1}) \geq 0$. Thus

$$2^n - 2 = u(P^{2^{n-1}k} v_n) = u(\rho^b P^{2^{m_1-1}j} v_{m_1}) + u(v_{m_2} \cdots v_{m_a}) \geq u(v_{m_a}) = 2^{m_a} - 2,$$

so that $n \geq m_a$.

Now consider the Milnor–Witt and Chow degrees modulo 4. We have

$$(-1, 1) \equiv (-a, a+b) \pmod 4,$$

so $a \equiv 1 \pmod 4$ and $b \equiv 0 \pmod 4$. Thus either $b = 0$, which was what we wanted to show, or $b \geq 4$.

We may now assume that $b \geq 4$. Since $\rho^4 P^{2j} v_2 = 0$, we must have $m_1 \geq 3$, so that all m_i, and also n, are at least 3.

Next, consider degrees modulo 8. Comparing degrees gives

$$(-1,1) \equiv (-a, a+b) \pmod 8.$$

Thus $b \equiv 0 \pmod 8$, so that $b \geq 8$. Since $\rho^8 P^{4j} v_3 = 0$, we must have $j_1 \geq 4$, and therefore n and all other j_i are also at least 4. This argument can be continued to establish that b and n must be arbitrarily large under the assumption that $b > 0$. $\square$

Lemma 4.2 *For each $n \geq 2$ and $k \geq 0$, the element $P^{2^{n-1}k} v_n \cdot v_n$ of the Bockstein E_∞–page detects a unique element of $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.*

Proof The Milnor–Witt degree of $P^{2^{n-1}k} v_n \cdot v_n$ is even, while the Milnor–Witt degree of $\rho^b P^{2^{m-1}j} v_m$ is odd. Therefore, these elements cannot share bidegree.

Now suppose that the element $P^{2^{n-1}k} v_n \cdot v_n$ has the same bidegree as the element $\rho^b P^{2^{m_1-1}j} v_{m_1} \cdot v_{m_2} \cdots v_{m_a}$, with $m_1 \leq m_2 \leq \cdots \leq m_a$, $b \leq 2^{m_1} - 2$ and $a \geq 2$. The rest of the proof is essentially the same as the proof of Lemma 4.1. Consider $u = mw - c$ to get that $n \geq m_a$. Then consider congruences $(-2,2) \equiv (-a, a+b)$ modulo higher and higher powers of 2 to obtain that $b = 0$. $\square$

Remark 4.3 The obvious generalization of Lemma 4.2 to elements of the form $P^{2^{n-1}k} v_n \cdot v_m$ is false. For example, $P^2 v_2 \cdot v_5$ has the same degree as $\rho^4 v_3^6$.

Remark 4.4 Lemmas 4.1 and 4.2 are equivalent to the claim that there are no ρ multiples in the ρ–Bockstein E_∞–page in the same bidegrees as either $P^{2^{n-1}k} v_n$ or $P^{2^{n-1}k} v_n \cdot v_n$. This implies that there are also no ρ multiples in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ that share bidegree with these elements; we will need this fact later.

Lemma 4.5 *$\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ is zero when the Milnor–Witt stem mw and the Chow degree c are both equal to $2i$ with $i \geq 1$.*

Proof Under the condition $mw = c = 2i$, inspection of Table 1 shows the ρ–Bockstein E_1–page consists of products of elements of the form v_1^4 or $\rho^{2^n+2^m-4} v_n v_m$. In the E_∞–page, $\rho^{2^n+2^m-4} v_n v_m = 0$ since $\rho^{2^n-1} v_n = 0$. Also, v_1^{4k} supports a differential for all $k \geq 0$. $\square$

Lemma 4.6 *For each $n \geq 2$, $k \geq 0$ and $m > n$, we have a Massey product*

$$P^{2^{n-1}k+2^{m-2}} v_n = \langle \rho^{2^m-2^n} v_m, \rho^{2^n-1}, P^{2^{n-1}k} v_n \rangle$$

in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ with no indeterminacy.

Proof The Bockstein differential $d^{\rho}_{2^m-1}(P^{2^{m-2}}) = \rho^{2^m-1} v_m$ and May's convergence theorem [8, Theorem 4.1] imply that the Massey product is detected by $P^{2^{n-1}k+2^{m-2}} v_n$ in the ρ–Bockstein E_∞–page. There are no crossing Bockstein differentials as all classes are in nonnegative ρ–filtration. Lemma 4.1 says that this ρ–Bockstein E_∞–page element detects a unique element of $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.

The indeterminacy of the bracket is generated by products of the form $\rho^{2^m-2^n} v_m \cdot x$ and $y \cdot P^{2^{n-1}k} v_n$, where x and y have appropriate bidegrees. We showed in Lemma 4.5 that 0 is the only possibility for x or y. □

Remark 4.7 Lemma 4.6 gives many different Massey products for the same element. For example,

$$P^8 v_2 = \langle \rho^4 v_3, \rho^3, P^6 v_2 \rangle = \langle \rho^{12} v_4, \rho^3, P^4 v_2 \rangle = \langle \rho^{28} v_5, \rho^3, v_2 \rangle.$$

Lemma 4.8 *For $m > n \geq 2$, there is a Massey product*

$$P^{2^{n-1}k+2^{m-2}} v_n = \langle P^{2^{n-1}k} v_n, \rho^{2^m-2} v_m, \rho \rangle$$

in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ with no indeterminacy.

Proof The Massey product formula follows from the Bockstein differential

$$d^{\rho}_{2^m-1}(P^{2^{m-2}}) = \rho^{2^m-1} v_m$$

and May's convergence theorem [8, Theorem 4.1]. There are no crossing Bockstein differentials as all classes are in nonnegative ρ–filtration. As in the proof of Lemma 4.6, we need Lemma 4.1 to tell us that the element $P^{2^{n-1}k+2^{m-2}} v_n$ of the ρ–Bockstein E_∞–page detects a unique element of $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.

The indeterminacy of the bracket is generated by products of the form $P^{2^{n-1}k} v_n \cdot x$ and $y \cdot \rho$. We showed in Lemma 4.5 that 0 is the only possibility for x. We observed in Remark 4.4 that $y \cdot \rho$ must be zero because there are no multiples of ρ in the appropriate bidegree. □

The relations in the Bockstein E_∞–page given in Proposition 3.2 may lift to $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ with additional terms that are multiples of ρ. In other words, there may be hidden relations in the Bockstein spectral sequence. For example, for degree reasons it is possible that $P^2 v_2 \cdot P^{16} v_5 + P^{18} v_2 \cdot v_5$ equals $\rho^4 P^{16} v_3 \cdot v_3^5$. Proposition 4.9 shows that there are no such hidden terms in the relations in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$.

Proposition 4.9 *There are no hidden relations in the Bockstein spectral sequence.*

Proof The relation $\rho^{2^{n-1}} P^{2^{n-1}k} v_n = 0$ in the ρ–Bockstein E_∞–page lifts to give the same relation in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$, as we observed in the discussion preceding Lemma 4.1. Therefore, we need only compute the products $P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m$ in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ for $m \geq n$.

Lemma 4.6 implies that $P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m$ equals

$$P^{2^{n-1}k} v_n \langle \rho^{2^m} v_{m+1}, \rho^{2^m-1}, P^{2^{m-1}(j-1)} v_m \rangle.$$

Shuffle to obtain

$$\langle P^{2^{n-1}k} v_n, \rho^{2^m} v_{m+1}, \rho^{2^m-1} \rangle P^{2^{m-1}(j-1)} v_m.$$

This expression is contained in

$$\langle P^{2^{n-1}k} v_n, \rho^{2^{m+1}-2} v_{m+1}, \rho \rangle P^{2^{m-1}(j-1)} v_m,$$

which equals $P^{2^{n-1}k+2^{m-1}} v_n \cdot P^{2^{m-1}(j-1)} v_m$ by Lemma 4.8.

By induction, $P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m$ equals $P^{2^{n-1}(k+2^{m-n}j)} v_n \cdot v_m$. □

Theorem 4.10 $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ *is the $\mathcal{R}$–algebra on the generators $P^{2^{n-1}k} v_n$ for $n \geq 2$ and $k \geq 0$ (see Table 2), subject to the relations*

$$\rho^{2^{n-1}} P^{2^{n-1}k} v_n = 0$$

for $n \geq 2$ and $k \geq 0$, and

$$P^{2^{n-1}k} v_n \cdot P^{2^{m-1}j} v_m + P^{2^{n-1}(k+2^{m-n}j)} v_n \cdot v_m = 0$$

for $m \geq n \geq 2$, $k \geq 0$ and $j \geq 0$.

Proof This follows immediately from Propositions 3.2 and 4.9. □

Remark 4.11 Analogously to Remark 3.3, an $\mathcal{R}$–module basis for $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$ is given by monomials of the form $P^{2^{n-1}k} v_n \cdot v_{m_1} \cdots v_{m_a}$, where $n \leq m_1 \leq \cdots \leq m_a$.

5 Adams differentials

Before computing with the h_1–inverted $\mathbb{R}$–motivic Adams spectral sequence, we will consider convergence. A priori, there could be an infinite family of homotopy classes linked together by infinitely many hidden η multiplications. These classes would not be detected in $\mathrm{Ext}_{\mathbb{R}}[h_1^{-1}]$. Lemma 5.1 implies that this cannot occur for degree reasons.

Lemma 5.1 *Let $m > 0$ be a fixed Milnor–Witt stem. There exists a constant A such that $\mathrm{Ext}_{\mathbb{R}}^{(s,f,w)}$ vanishes when $s - w = m$, s is nonzero, $f > A$ and $f > s + 1$.*

Lemma 5.1 can be restated in the following more casual form: within a fixed Milnor–Witt stem, there exists a horizontal line and a line of slope 1 such that $\mathrm{Ext}_{\mathbb{R}}$ vanishes in the region above both lines, except in the 0–stem. Figure 1 depicts the shape of the vanishing region.

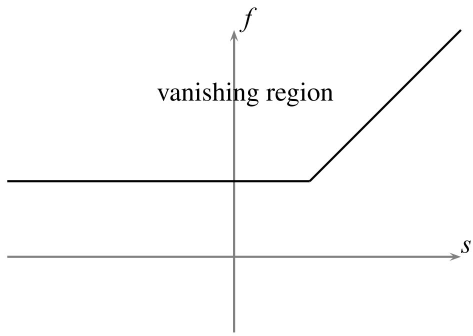

Figure 1: The vanishing region in a Milnor–Witt stem

Proof This argument occurs in $\mathrm{Ext}_{\mathbb{R}}$, where h_1 has not been inverted.

As explained in [4, Theorem 4.1], the elements in the m–stem of the classical Ext groups $\mathrm{Ext}_{\mathrm{cl}}$ correspond to elements of $\mathrm{Ext}_{\mathbb{R}}$ in the Milnor–Witt m–stem that remain nonzero after ρ is inverted, ie that support infinitely many multiplications by ρ. Each stem of $\mathrm{Ext}_{\mathrm{cl}}$ is finite except for the 0–stem. For $m > 0$, choose A to be larger than the Adams filtrations of all of the elements in the m–stem of $\mathrm{Ext}_{\mathrm{cl}}$. Then A is larger than the Adams filtrations of every element of $\mathrm{Ext}_{\mathbb{R}}$ in the Milnor–Witt m–stem that remain nonzero after ρ is inverted.

Let x be a nonzero element of $\mathrm{Ext}_{\mathbb{R}}^{(s,f,w)}$ such that $s - w = m$, $f > A$ and $f > s + 1$. We will show that s must equal zero.

The choice of A guarantees that x is annihilated by some positive power of ρ. Suppose that $\rho^k x = 0$ but $\rho^{k-1} x$ is nonzero, for some $k > 0$. Then there must be a differential in the ρ–Bockstein spectral sequence of the form $d_k(y) = \rho^k x$, where y is an element of $\mathrm{Ext}_{\mathbb{C}}$ in degree $(s - k + 1, f - 1, w - k)$.

The argument from [1] establishes a vanishing line of slope 1 in the nonzero stems of $\mathrm{Ext}_{\mathbb{C}}$. The conditions $f > s+1$ and $k > 0$ imply that the element y lies strictly above this vanishing line, so it must be of the form $\tau^a h_0^b$ with $b \geq 1$. The only ρ–Bockstein differentials on such classes are $d_1(\tau^{2c+1} h_0^b) = \rho\tau^{2c} h_0^{b+1}$, which implies that x must be of the form $\tau^{2c} h_0^b$. This shows that $s = 0$. □

The h_1–inverted motivic Adams spectral sequence over $\mathbb{C}$ was studied in [5; 3]. It takes the form

$$\mathbb{F}_2[h_1^{\pm 1}, P, v_2, v_3, \dots] \Longrightarrow \hat{\pi}^{\mathbb{C}}_{*,*}[\eta^{-1}],$$

where $\hat{\pi}^{\mathbb{C}}_{*,*}[\eta^{-1}]$ is the η–inverted motivic stable homotopy ring of the 2–completed motivic sphere spectrum over $\mathbb{C}$. This spectral sequence has differentials

$$d_2(P^k v_n) = P^k v_{n-1}^2$$

for all $k \geq 0$ and all $n \geq 3$. As usual, we omit any powers of h_1.

Lemma 5.2 *In the h_1–inverted $\mathbb{R}$–motivic Adams spectral sequence, there are differentials*

$$d_2(P^{2^{n-1}k} v_n) = P^{2^{n-1}k} v_{n-1}^2$$

for all $k \geq 0$ and all $n \geq 3$.

Proof There is an extension of scalars functor from $\mathbb{R}$–motivic homotopy theory to $\mathbb{C}$–motivic homotopy theory. This functor induces a map

$$\begin{array}{ccc} \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})[h_1^{-1}] & \Longrightarrow & \hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}] \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[h_1^{-1}] & \Longrightarrow & \hat{\pi}^{\mathbb{C}}_{*,*}[\eta^{-1}] \end{array}$$

from the $\mathbb{R}$–motivic Adams spectral sequence to the $\mathbb{C}$–motivic Adams spectral sequence. This map takes ρ to zero.

The above map of spectral sequences implies that the $\mathbb{R}$–motivic Adams differential $d_2(P^{2^{n-1}k} v_n)$ equals $P^{2^{n-1}k} v_{n-1}^2$ plus terms that are divisible by ρ. Lemma 4.2 implies that there are no possible additional terms in the relevant bidegree. □

Our next task is to completely describe the Adams E_3–page. First, we explore some elements that survive to the E_3–page. We will consider these elements more carefully in Proposition 5.4.

Despite the differential $d_2(P^{4k} v_3) = P^{4k} v_2^2$, the element $\rho^3 P^{4k} v_3$ survives to the E_3–page because $\rho^3 P^{4k} v_2^2$ is zero. Similarly, $\rho^{2^{n-1}-1} P^{2^{n-1}k} v_n$ survives to the E_3–page.

The element $P^2 v_2^2$ looks like it should be hit by an Adams d_2 differential on $P^2 v_3$. However, $P^2 v_3$ did not survive the ρ–Bockstein spectral sequence. Therefore, there is nothing to hit $P^2 v_2^2$ and it survives to the Adams E_3–page. The same observation applies to the elements $P^{2^{n-1}(2j+1)} v_n^2$.

We record the following simple computation, as we will employ it several times.

Lemma 5.3 *Let S be an $\mathbb{F}_2$–algebra. Let $B = S[w_1, w_2, \dots]$ be a polynomial ring in infinitely many variables, and define a differential on B by $\partial(w_n) = w_{n-1}^2$ for $n \geq 2$. Then $\mathrm{H}^*(B, \partial) \cong S[w_1]/w_1^2$.*

In fact, we will use a slight generalization of Lemma 5.3 in which $\partial(w_n)$ is equal to $u_n w_{n-1}^2$, where u_n is a unit in S. This generalization implies, for example, that the h_1–inverted $\mathbb{C}$–motivic Adams E_3–page is $\mathbb{F}_2[h_1^{\pm 1}, P, v_2]/v_2^2$.

Proposition 5.4 *The h_1–inverted $\mathbb{R}$–motivic Adams E_3–page is free as an $\mathcal{R}$–module on the generators listed in Table 3 for $n \geq 2$, $k \geq 0$ and $j \geq 0$. Almost all products of these generators are zero, except that*

$$P^{4k} v_2 \cdot P^{4j+2} v_2 = P^{4k+4j+2} v_2^2$$

and for $n \geq 3$,

$$\rho^{2^{n-1}-1} P^{2^{n-1}\cdot 2k} v_n \cdot \rho^{2^{n-1}-1} P^{2^{n-1}(2j+1)} v_n = \rho^{2^n-2} P^{2^{n-1}(2k+2j+1)} v_n^2.$$

(mw, c)	generator	ρ–torsion
$(0,0)$	1	∞
$(3,1)+k(8,8)$	$P^{2k}v_2$	3
$(7,4)+k(16,16)$	$\rho^3 P^{4k} v_3$	4
$(15,8)+k(32,32)$	$\rho^7 P^{8k} v_4$	8
$(2^n-1, 2^{n-1})+k(2^{n+1}, 2^{n+1})$	$\rho^{2^{n-1}-1} P^{2^{n-1}k} v_n$	2^{n-1}
$(6,2)+(2j+1)(8,8)$	$P^{2(2j+1)} v_2^2$	3
$(14,2)+(2j+1)(16,16)$	$P^{4(2j+1)} v_3^2$	7
$(30,2)+(2j+1)(32,32)$	$P^{8(2j+1)} v_4^2$	15
$(2^{n+1}-2, 2)+(2j+1)(2^{n+1}, 2^{n+1})$	$P^{2^{n-1}(2j+1)} v_n^2$	2^n-1

Table 3: $\mathcal{R}$–module generators for the Adams E_3–page

Remark 5.5 The relations in Proposition 5.4 are just the ones that are obvious from the notation. For example,

$$v_2 \cdot P^2 v_2 = P^2 v_2^2, \qquad \rho^3 P^4 v_3 \cdot \rho^3 P^8 v_3 = \rho^6 P^{12} v_3^2.$$

Proof of Proposition 5.4 Let $\mathrm{Ext}\langle k, b\rangle$ be the $\mathbb{F}_2[h_1^{\pm 1}]$–submodule of the h_1–inverted $\mathbb{R}$–motivic Adams E_2–page on generators of the form $\rho^b P^k v_{m_1} v_{m_2} \cdots v_{m_a}$ such that $m_1 \leq m_2 \leq \cdots \leq m_a$. Note that $b \leq 2^{m_1} - 2$ in this situation, since $\rho^{2^{m_1}-1} P^k v_{m_1} = 0$.

Also, k must be a multiple of 2^{m_1-1}. By Lemma 5.2 and the fact that ρ is a permanent cycle, each $\mathrm{Ext}\langle k,b\rangle$ is a differential graded submodule. Thus it suffices to compute the cohomology of each $\mathrm{Ext}\langle k,b\rangle$.

We start with $\mathrm{Ext}\langle 0,b\rangle$, which is equal to $\rho^b\cdot\mathbb{F}_2[h_1^{\pm1},v_m,v_{m+1},\dots]$ as a differential graded $\mathbb{F}_2[h_1^{\pm1}]$–module, where m is the smallest integer such that $b\le 2^m-2$. Now Lemma 5.3 implies that $H^*(\mathrm{Ext}\langle 0,b\rangle,d_2)$ is a free $\mathbb{F}_2[h_1^{\pm1}]$–module on two generators ρ^b and $\rho^b v_m$.

So far, we have demonstrated that the powers of ρ and the elements

$$v_2,\quad \rho v_2,\quad \rho^2 v_2,\quad \rho^3 v_3,\dots,\rho^6 v_3,\quad \rho^7 v_4,\dots$$

are present in the h_1–inverted $\mathbb{R}$–motivic Adams E_3–page.

The module $\mathrm{Ext}\langle k,b\rangle$ is zero when k is odd.

Now assume that k is equal to 2 modulo 4. If $b\le 2$, then $\mathrm{Ext}\langle k,b\rangle$ is equal to $\rho^b P^k v_2\cdot\mathbb{F}_2[h_1^{\pm1},v_2,v_3,\dots]$ as a differential graded $\mathbb{F}_2[h_1^{\pm1}]$–module. Lemma 5.3 implies that $H^*(\mathrm{Ext}\langle k,b\rangle,d_2)$ is a free $\mathbb{F}_2[h_1^{\pm1}]$–module on two generators $\rho^b P^k v_2$ and $\rho^b P^k v_2^2$. If $b\ge 3$, then $\mathrm{Ext}\langle k,b\rangle$ is zero because $\rho^3 P^k v_2=0$.

We have now shown that the elements

$$P^k v_2,\quad \rho P^k v_2,\quad \rho^2 P^k v_2,\quad P^k v_2^2,\quad \rho P^k v_2^2,\quad \rho^2 P^k v_2^2$$

are present in the h_1–inverted $\mathbb{R}$–motivic Adams E_3–page for all k congruent to 2 modulo 4.

Next assume that k is equal to 4 modulo 8. If $b\le 2$, then $\mathrm{Ext}\langle k,b\rangle$ is the free $\mathbb{F}_2[h_1^{\pm1}]$–module on generators $\rho^b P^k v_{m_1}\cdots v_{m_a}$ such that m_1 equals 2 or 3, and $m_1\le\cdots\le m_a$. There is a short exact sequence

$$0\to\mathrm{Ext}\langle k,b\rangle\to\rho^b P^k\cdot\mathbb{F}_2[h_1^{\pm1},v_2,v_3,\dots]\to\rho^b P^k\cdot\mathbb{F}_2[h_1^{\pm1},v_4,v_5,\dots]\to 0,$$

where the differential is defined on the second and third terms in the obvious way. By Lemma 5.3, the homology of the middle term has two generators $\rho^b P^k$ and $\rho^b P^k v_2$, while the homology of the right term has two generators $\rho^b P^k$ and $\rho^b P^k v_4$. Analysis of the long exact sequence in homology shows that $H^*(\mathrm{Ext}\langle k,b\rangle,d_2)$ has two generators $\rho^b P^k v_2$ and $\rho^b P^k v_3^2$.

Now assume that $3\le b\le 6$. Since $\rho^b P^k v_2=0$, we get that $\mathrm{Ext}\langle k,b\rangle$ is equal to $\rho^b P^k v_3\cdot\mathbb{F}_2[h_1^{\pm1},v_3,v_4,\dots]$. Lemma 5.3 implies that $H^*(\mathrm{Ext}\langle k,b\rangle,d_2)$ is a free $\mathbb{F}_2[h_1^{\pm1}]$–module on two generators $\rho^b P^k v_3$ and $\rho^b P^k v_3^2$.

Finally, if $b \geq 7$, then $\mathrm{Ext}\langle k,b\rangle$ is zero because $\rho^7 P^k v_2 = 0$ and $\rho^7 P^k v_3 = 0$. This finishes the argument when k is equal to 4 modulo 8, and we have shown that $\mathrm{Ext}_{\mathbb{R}}[h_1^{\pm 1}]$ contains the elements

$$\begin{gathered} P^k v_2,\ \rho P^k v_2,\ \rho^2 P^k v_2, \\ \rho^3 P^k v_3, \dots, \rho^6 P^k v_3, \\ P^k v_3^2,\ \rho P^k v_3^2, \dots, \rho^6 P^k v_3^2. \end{gathered}$$

Analysis of the other cases is the same as the argument for $k \equiv 4$ modulo 8. The details depend on the value of k modulo 2^i and inequalities of the form $2^j - 1 \leq b \leq 2^{j+1} - 2$. In each case there is a short exact sequence of differential graded modules whose first term is $\mathrm{Ext}\langle k,b\rangle$ and whose other two terms have homology that is computed by Lemma 5.3. □

We have now calculated the h_1–inverted $\mathbb{R}$–motivic E_3–page. This E_3–page is displayed in Figure 2. Beware that the grading on this chart is not the same as in a standard Adams chart. The Milnor–Witt stem $mw = s - w$ is plotted on the horizontal axis, while the Chow degree $c = s + f - 2w$ is plotted on the vertical axis. As a result, an Adams d_r differential has slope $-r+1$, rather than slope $-r$. Vertical lines in Figure 2 represent multiplications by ρ.

Our next goal is to establish the Adams d_3 differentials. Inspection of Figure 2 reveals that the only possible nonzero d_3 differentials might be supported on elements of the form $\rho^b P^{2^{n-1}k} v_n$ for $n \geq 4$. In fact, these differentials all occur, as indicated in Figure 2 by lines that go left one unit and up two units. We will establish these d_3 differentials by first proving a homotopy relation in Lemma 5.6.

Lemma 5.6 *For each $n \geq 2$ and $j \geq 0$, the element $P^{2^{n-1}(2j+1)} v_n^2$ is a permanent cycle that detects a ρ–divisible element of the η–inverted $\mathbb{R}$–motivic homotopy groups.*

Proof Inspection of Figure 2 shows that $P^{2^{n-1}(2j+1)} v_n^2$ cannot support a differential. Lemma 4.8 implies that

$$P^{2^{n-1}(2j+1)} v_n^2 \in \langle \rho, \rho^{2^{n+1}-2} v_{n+1}, P^{2^n j} v_n^2 \rangle \quad \text{in } \mathrm{Ext}_{\mathbb{R}}[h_1^{-1}].$$

In fact, the Massey product has no indeterminacy because of Remark 4.4 and Lemma 4.5.

We will now apply Moss's convergence theorem [10, Theorem 1.2] to this Massey product. There is an Adams differential $d_2(P^{2^n j} v_{n+1}) = P^{2^n j} v_n^2$, so $P^{2^n j} v_n^2$ detects the homotopy element 0. By inspection of Figure 2, $\rho^{2^{n+1}-2} v_{n+1}$ is a permanent

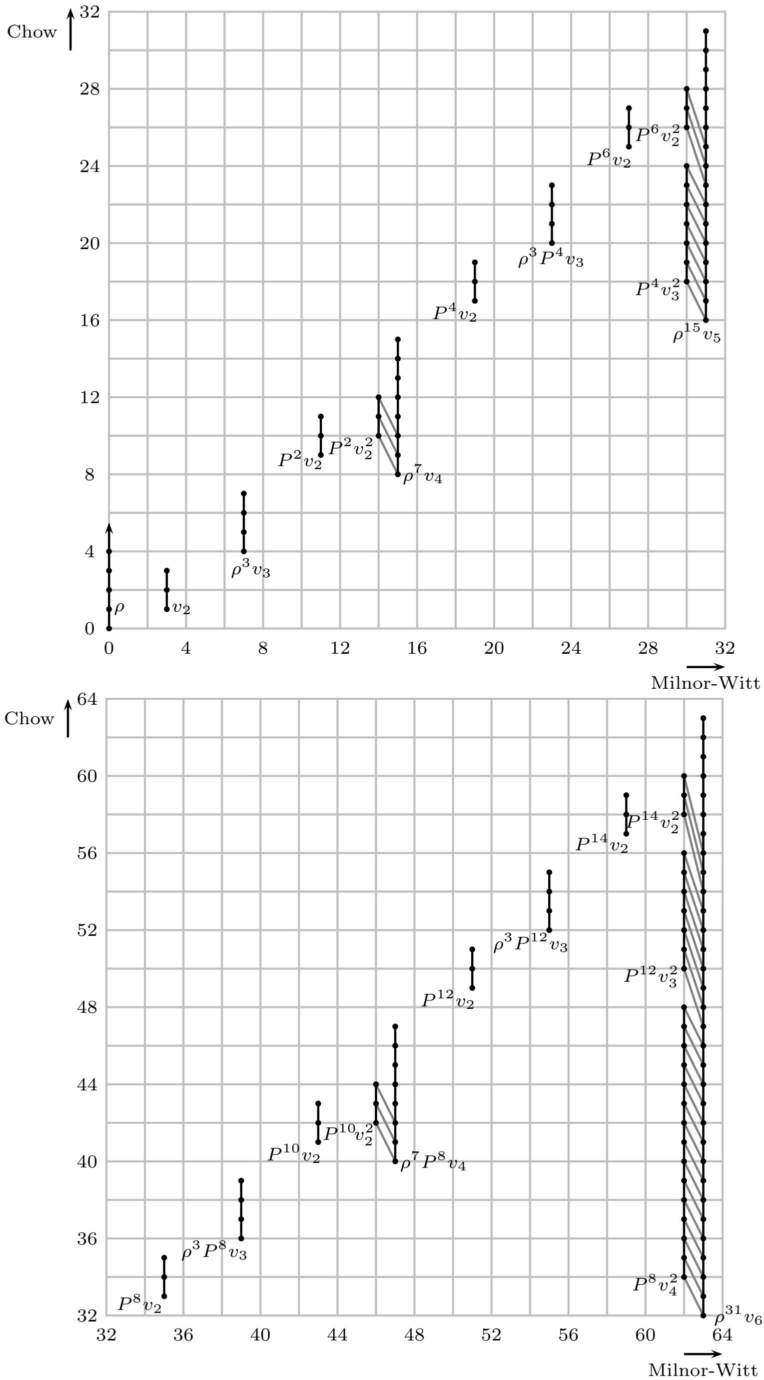

Figure 2: The η–inverted $\mathbb{R}$–motivic Adams E_3–page

cycle; let α be a homotopy element detected by it. Moreover, $\rho\alpha$ is zero in homotopy because there are no classes in higher filtration that could detect it.

Moss's convergence theorem says that the Toda bracket $\langle \rho, \alpha, 0\rangle$ contains an element that is detected by $P^{2^{n-1}(2j+1)} v_n^2$. This Toda bracket consists entirely of multiples of ρ. $\square$

Lemma 5.7 *$d_3(\rho^{2^{n-1}-1} P^{2^{n-1}k} v_n) = P^{2^{n-3}+2^{n-1}k} v_{n-2}^2$ for $n \geq 4$.*

Proof Lemma 5.6 shows that $P^{2^{n-3}+2^{n-1}k} v_{n-2}^2$ detects a class that is divisible by ρ. By inspection of Figure 2, there are no classes in lower filtration. Therefore, $P^{2^{n-3}+2^{n-1}k} v_{n-2}^2$ must detect zero, ie must be hit by a differential. It is apparent from Figure 2 that there is only one possible differential. $\square$

Lemma 5.8 describes the higher Adams differentials.

Lemma 5.8 *For $n \geq r+1$ and $r \geq 3$,*

$$d_r(\rho^{2^n-2^{n-r+2}-r+2} P^{2^{n-1}k} v_n) = P^{2^{n-1}k+2^{n-2}-2^{n-r}} v_{n-r+1}^2.$$

Proof The proof is essentially the same as the proof of Lemma 5.7. In the Milnor–Witt stem congruent to 2 modulo 4, Lemma 5.6 implies that every homotopy element is divisible by ρ. This implies that they must all be hit by differentials. Figure 2 indicates that there is just one possible pattern of differentials. $\square$

From Lemma 5.8, it is straightforward to derive the h_1–inverted Adams E_∞–page, as shown in Figure 3.

Proposition 5.9 *The h_1–inverted Adams E_∞–page is the $\mathcal{R}$–module on generators given in Table 4 for $n \geq 2$.*

(mw, c)	generator	ρ–torsion
$(0, 0)$	1	∞
$(3, 1) + k(8, 8)$	$P^{2k} v_2$	3
$(7, 4) + k(16, 16)$	$\rho^3 P^{4k} v_3$	4
$(15, 11) + k(32, 32)$	$\rho^{10} P^{8k} v_4$	5
$(2^n-1, 2^n-n-1) + k(2^{n+1}, 2^{n+1})$	$\rho^{2^n-n-2} P^{2^{n-1}k} v_n$	$n+1$

Table 4: $\mathcal{R}$–module generators for the Adams E_∞–page

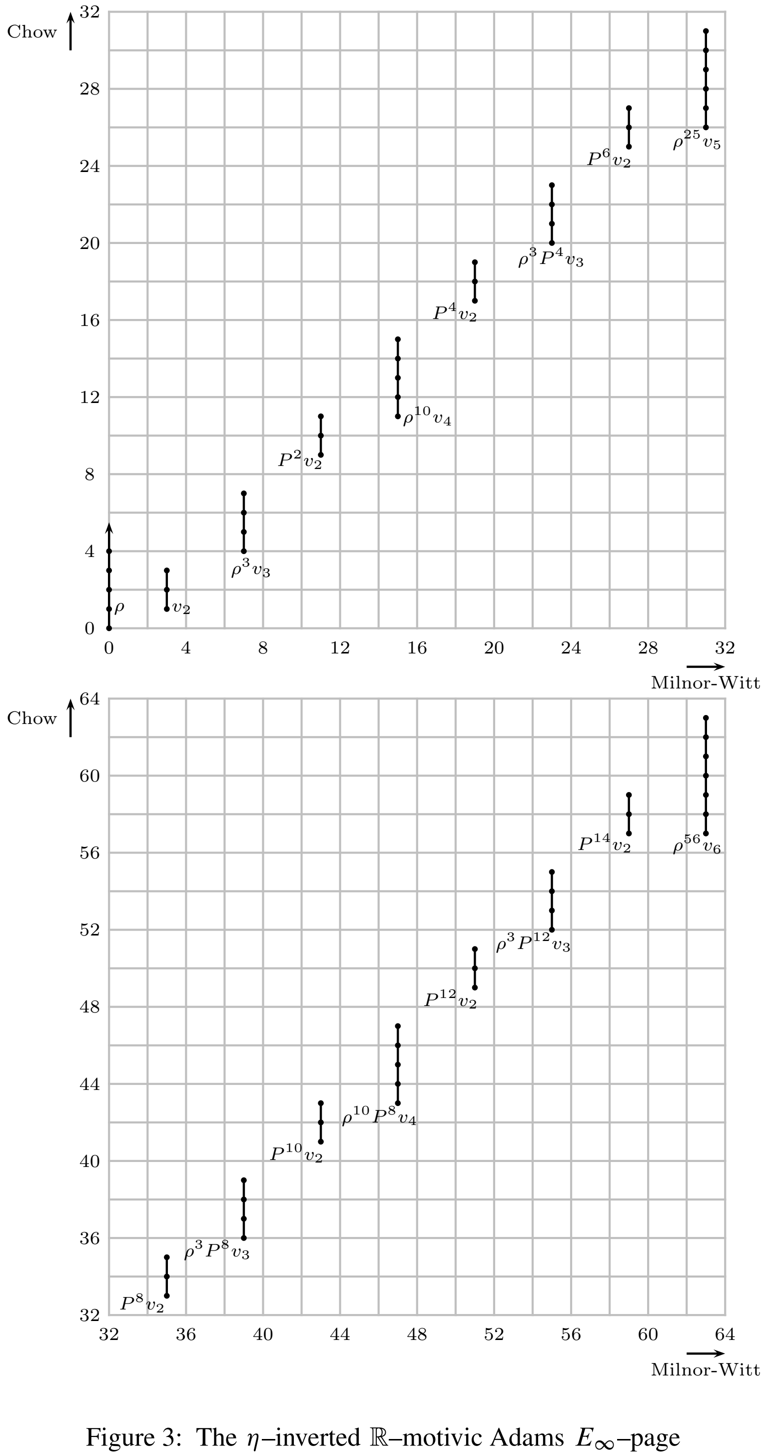

Figure 3: The η–inverted $\mathbb{R}$–motivic Adams E_∞–page

6 η–inverted homotopy groups

From the h_1–inverted Adams E_∞–page, it is a short step to the η–inverted stable homotopy ring. First we must choose generators. Recall that Π_n is the Milnor–Witt n–stem $\bigoplus_p \hat{\pi}^{\mathbb{R}}_{p+n,p}$.

Definition 6.1 For k nonnegative and n at least 2, let $P^{2^{n-1}k}\lambda_n$ be an element of $\Pi_{2^{n+1}k+2^n-1}[\eta^{-1}]$ that is detected by $\rho^{2^n-n-2}P^{2^{n-1}k}v_n$.

There are choices in these definitions, which are measured by Adams E_∞–page elements in higher filtration. For example, there are four possible choices for λ_2 because of the presence of ρv_2 and $\rho^2 v_2$ in higher filtration.

Theorem 6.2 *The η–inverted $\mathbb{R}$–motivic stable homotopy ring, as a $\mathbb{Z}_2[\eta^{\pm 1}]$–module, is generated by 1 and $P^{2^{n-1}k}\lambda_n$ for $n \geq 2$ and $k \geq 0$. The generator $P^{2^{n-1}k}\lambda_n$ lies in $\Pi_{2^{n+1}k+2^n-1}[\eta^{-1}]$ and is annihilated by 2^{n+1}. All products are zero, except for those involving 2 or η.*

Proof In the η–inverted stable homotopy ring, ρ and 2 differ by a unit because $\rho\eta^2 = -2\eta$; see [9]. Therefore, the ρ–torsion information given in Proposition 5.9 translates to 2–torsion information in homotopy.

Except for 1, all $\mathbb{Z}_2[\eta^{\pm 1}]$–module generators lie in Milnor–Witt stems that are congruent to 3 modulo 4. Therefore, such generators must multiply to zero. □

Table 5 lists all generators through the Milnor–Witt 63–stem. The table also identifies Toda brackets that contain each generator. These Toda brackets are computed in Section 7.

Table 5 also reveals a pattern that matches the classical image of J.

Corollary 6.3 *If $k > 1$, then $\Pi_{4k-1}[\eta^{-1}]$ is isomorphic to $\mathbb{Z}/2^{u+1}[\eta^{\pm 1}]$ as a module over $\mathbb{Z}_2[\eta^{\pm 1}]$, where u is the 2–adic valuation of $4k$.*

7 Toda brackets

Even though its primary multiplicative structure is uninteresting, the η–inverted $\mathbb{R}$–motivic stable homotopy ring has rich higher structure in the form of Toda brackets. We will explore some of the 3–fold Toda bracket structure. In particular, we will show that all of the generators can be inductively constructed via Toda brackets, starting

mw	E_∞	$\hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}]$	2^k–torsion	bracket	indeterminacy
0	1	1	∞		
3	v_2	λ_2	3		
7	$\rho^3 v_3$	λ_3	4	$\langle 2^3, \lambda_2, \lambda_2\rangle$	$2^3\lambda_3$
11	$P^2 v_2$	$P^2\lambda_2$	3	$\langle 2^4, \lambda_3, \lambda_2\rangle$	
15	$\rho^{10} v_4$	λ_4	5	$\langle 2^3, \lambda_2, P^2\lambda_2\rangle$	$2^3\lambda_4$
19	$P^4 v_2$	$P^4\lambda_2$	3	$\langle 2^5, \lambda_4, \lambda_2\rangle$	
23	$\rho^3 P^4 v_3$	$P^4\lambda_3$	4	$\langle 2^5, \lambda_4, \lambda_3\rangle$	
27	$P^6 v_2$	$P^6\lambda_2$	3	$\langle 2^5, \lambda_4, P^2\lambda_2\rangle$	
31	$\rho^{25} v_5$	λ_5	6	$\langle 2^3, \lambda_2, P^6\lambda_2\rangle$	$2^3\lambda_5$
35	$P^8 v_2$	$P^8\lambda_2$	3	$\langle 2^6, \lambda_5, \lambda_2\rangle$	
39	$\rho^3 P^8 v_3$	$P^8\lambda_3$	4	$\langle 2^6, \lambda_5, \lambda_3\rangle$	
43	$P^{10} v_2$	$P^{10}\lambda_2$	3	$\langle 2^6, \lambda_5, P^2\lambda_2\rangle$	
47	$\rho^{10} P^8 v_4$	$P^8\lambda_4$	5	$\langle 2^6, \lambda_5, \lambda_4\rangle$	
51	$P^{12} v_2$	$P^{12}\lambda_2$	3	$\langle 2^6, \lambda_5, P^4\lambda_2\rangle$	
55	$\rho^3 P^{12} v_3$	$P^{12}\lambda_3$	4	$\langle 2^6, \lambda_5, P^4\lambda_3\rangle$	
59	$P^{14} v_2$	$P^{14}\lambda_2$	3	$\langle 2^6, \lambda_5, P^6\lambda_2\rangle$	
63	$\rho^{56} v_6$	λ_6	7	$\langle 2^3, \lambda_2, P^{14}\lambda_2\rangle$	$2^3\lambda_6$

Table 5: $\mathbb{Z}_2[\eta^{\pm 1}]$–module generators for $\hat{\pi}^{\mathbb{R}}_{*,*}[\eta^{-1}]$

from just 2 and λ_2. Table 5 lists one possible Toda bracket decomposition for each generator of Π_n for all n less than or equal to 63.

We observed in the proof of Theorem 6.2 that the element ρ of the Adams E_∞–page detects the element 2 of the η–inverted stable homotopy ring. We will use this fact frequently in the following results.

Lemma 7.1 *The Toda bracket $\langle 2^3, \lambda_2, \lambda_2\rangle$ contains an element detected by $\rho^3 v_3$ in Π_7, and its indeterminacy is detected by $\rho^6 v_3$.*

Proof Moss's convergence theorem [10, Theorem 1.2] and the differential $d_2(v_3) = v_2^2$ show that $\langle 2^3, \lambda_2, \lambda_2\rangle$ is detected by $\rho^3 v_3$.

The indeterminacy follows from the facts that there are no multiples of λ_2 and that there is a unique multiple of 2^3 in Π_7. □

Remark 7.2 The proof of Lemma 7.1 applies just as well to show that $\langle 2^4, \lambda_3, \lambda_3 \rangle$ is detected by $\rho^{10} v_4$ in Π_{15}. In higher stems, the analogous brackets do not produce generators. For example, the Massey product $\langle \rho^5, \rho^{10} v_4, \rho^{10} v_4 \rangle$ is already defined in Ext, which implies that the corresponding Toda bracket must be detected in filtration least 27. However, $\rho^{25} v_5$ detects the generator of Π_{31}, and it lies in filtration 26.

Lemma 7.3 *For n at least 2, the Toda bracket $\langle 2^3, \lambda_2, P^{2^{n-1}-2} \lambda_2 \rangle$ is detected by the class $\rho^{2^{n+1}-n-3} v_{n+1}$. The indeterminacy in this Toda bracket is generated by $2^3 \lambda_{n+1}$.*

Proof This follows from Moss's convergence theorem [10, Theorem 1.2], together with the Adams differential $d_n(\rho^{2^{n+1}-n-6} v_{n+1}) = P^{2^{n-1}-2} v_2^2$. □

Lemma 7.4 *For n at least 2, the Toda bracket $\langle 2^{n+2}, \lambda_{n+1}, P^{2^{n-1}-2} \lambda_2 \rangle$ is detected by $P^{2^n-2} v_2$. The Toda bracket has no indeterminacy.*

Proof Lemma 4.8 implies that there is a Massey product

$$P^{2^n-2} v_2 = \langle \rho^{n+2}, \rho^{2^{n+1}-n-3} v_{n+1}, P^{2^{n-1}-2} v_2 \rangle,$$

with no indeterminacy. Moss's convergence theorem [10, Theorem 1.2] establishes the desired result. □

Lemma 7.5 *If $m > n \geq 2$, then the Toda bracket $\langle 2^{m+1}, \lambda_m, P^{2^{n-1}k} \lambda_n \rangle$ is detected by $\rho^{2^n-n-2} P^{2^{m-2}+2^{n-1}k} v_n$. The Toda bracket has no indeterminacy.*

Proof Lemma 4.8 implies that there is a Massey product

$$\rho^{2^n-n-2} P^{2^{m-2}+2^{n-1}k} v_n = \langle \rho^{m+1}, \rho^{2^m-m-2} v_m, \rho^{2^n-n-2} P^{2^{n-1}k} v_n \rangle.$$

Moss's convergence theorem [10, Theorem 1.2] establishes the desired result. □

Proposition 7.6 *Every generator $P^{2^{n-1}k} \lambda_n$ of the η–inverted $\mathbb{R}$–motivic stable homotopy ring can be constructed via iterated 3–fold Toda brackets starting from 2 and λ_2.*

Proof Lemmas 7.3 and 7.4 alternately show that the generators λ_n and the generators $P^{2^n-2} \lambda_2$ can be constructed via iterated 3–fold Toda brackets starting from 2 and λ_2. Then Lemma 7.5 shows that any $P^{2^{n-1}k} \lambda_n$ can be constructed. □

Example 7.7 Suppose we wish to find a Toda bracket decomposition for $P^{40} \lambda_3$. Since $40 = 2^{7-2} + 2^{3-1} \cdot 4$, we can apply Lemma 7.5 with $m = 7$, $n = 3$ and $k = 4$ to conclude that $P^{40} \lambda_3$ is detected by the Toda bracket $\langle 2^8, \lambda_7, P^8 \lambda_3 \rangle$.

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Characteristic classes of fiber bundles

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In this paper, we construct new characteristic classes of fiber bundles via flat connections with values in infinite-dimensional Lie algebras of derivations. In fact, choosing a fiberwise metric, we construct a chain map to the de Rham complex on the base space, and show that the induced map on cohomology groups is independent of the choice of metric. Moreover, we show that, applied to a surface bundle, our construction gives Morita–Miller–Mumford classes.

57R20; 55R40

1 Introduction

The purpose of this paper is to construct characteristic classes of fiber bundles which are not necessarily principal bundles whose structure group is a finite-dimensional Lie group. The difficulty is that a diffeomorphism group, which is considered as the structure group for a general fiber bundle, is huge compared to a finite-dimensional Lie group.

An idea to overcome the difficulty is a "linearization" which means to replace the diffeomorphism group $\mathrm{Diff}(X)$ for the fiber X with the automorphism group of the tensor algebra of the first homology group $H = H_1(X)$. A main tool is the Maurer–Cartan form of the space of expansions which was originally considered by Kawazumi [4; 5] in the case of free groups.

Diagrammatically, the construction is as follows:

$$\begin{tikzcd} \text{fiber bundle} \arrow[r, "\text{linearization}"] \arrow[dr, dotted] & \text{flat } \mathrm{Aut}(\widehat{L}H/I\text{–connection}) \arrow[d, "\text{Chern–Simons construction}"] \\ & \text{characteristic class} \end{tikzcd}$$

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In this paper, for any fiber bundle $E \to B$ whose structure group satisfies a certain condition, choosing a fiberwise metric, we construct a chain map from the Chevalley–Eilenberg complex to the de Rham complex on the base space B via a flat $\mathrm{Aut}(\widehat{L}H/I)$–connection, and show that the induced map on cohomology groups is independent of the choice. We show that our construction gives Morita–Miller–Mumford classes if applied to a closed surface bundle. The similar construction of Morita–Miller–Mumford classes of 1–punctured surface bundles was previously given in [5].

Our construction is algebraic, and is closely related to the formal geometry of Kontsevich [7; 8] (see also Penkava and Schwarz [13]). For example, we have a description of our characteristic classes in terms of a graph complex with general valency when the Lie algebra is free. We intend to get such a graph complex in general cases, which is hopefully closely related to a combinatorial model of unstable homology of mapping class groups as in Godin [3].

It is interesting to compare our construction to a different approach on diffeomorphism groups with noncommutative geometry in Lott [9].

The paper is organized as follows: in Section 2, we introduce tools for construction of characteristic classes. The notions defined in Sections 2A and 2B are used in Section 3, and Johnson maps defined in Section 2C are used in Section 4. In Sections 3A and 3B, we construct characteristic classes under two different conditions of fiber bundles. In Section 3C, we clarify the relation between characteristic classes constructed in Sections 3A and 3B. In Section 3D, we describe these characteristic classes by Lie algebra cohomologies of the Lie algebra of derivations. In Section 4, we prove that our characteristic classes of a closed surface bundle give Morita–Miller–Mumford classes.

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2 Preliminaries

For a finitely generated group π, we set $H = \pi^{\mathrm{ab}} \otimes \mathbb{R}$. We consider the completed tensor Hopf algebra $\widehat{T}H$ and the completed free Lie algebra $\widehat{L}H$ generated by H. The Lie algebra $\widehat{L}H$ can be regarded as the primitive part of the completed Hopf algebra $\widehat{T}H$. (For completed Hopf algebras, see Quillen [14].) Let J be the augmented ideal of $\widehat{T}H$, which consists of all elements whose constant term is zero. The algebra $\widehat{T}H$ is described by the projective limit of finite-dimensional vector spaces

$$\widehat{T}H = \prod_{n=0}^{\infty} H^{\otimes n} = \varprojlim_{n} \widehat{T}H/J^n,$$

so it is endowed with the limit topology. We define $L_{\geq k} := \widehat{L}H \cap J^k$ and $L_k := \widehat{L}H \cap H^{\otimes k}$.

A closed two-sided Lie ideal I contained in $L_{\geq 2}$ is called a *decomposable ideal* of $\widehat{L}H$. We identify I with the closed two-sided ideal generated by I in $\widehat{T}H$. Then $\widehat{T}H/I$ is a completed Hopf algebra.

We denote the completion of the group Hopf algebra $\mathbb{R}\pi$ of π by $\widehat{\mathbb{R}\pi}$.

2A Automorphisms and derivations

In this section, we will define infinite-dimensional Lie groups satisfying the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Inn}(\widehat{L}H/I) & \longrightarrow & \mathrm{IAut}(\widehat{L}H/I) & \longrightarrow & \mathrm{IOut}(\widehat{L}H/I) & \longrightarrow & 1 \\
 & & {\scriptstyle \text{surj.}}\big\uparrow & & {\scriptstyle \text{surj.}}\big\uparrow & & {\scriptstyle \text{surj.}}\big\uparrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\widehat{L}H) & \longrightarrow & \mathrm{IAut}_I(\widehat{L}H) & \longrightarrow & \mathrm{IOut}_I(\widehat{L}H) & \longrightarrow & 1
\end{array}$$

and corresponding Lie algebras

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{IDer}(\widehat{L}H/I) & \longrightarrow & \mathrm{Der}^+(\widehat{L}H/I) & \longrightarrow & \mathrm{ODer}^+(\widehat{L}H/I) & \longrightarrow & 0 \\
 & & {\scriptstyle \text{surj.}}\big\uparrow & & {\scriptstyle \text{surj.}}\big\uparrow & & {\scriptstyle \text{surj.}}\big\uparrow & & \\
0 & \longrightarrow & \mathrm{IDer}(\widehat{L}H) & \longrightarrow & \mathrm{Der}^+_I(\widehat{L}H) & \longrightarrow & \mathrm{ODer}^+_I(\widehat{L}H) & \longrightarrow & 0
\end{array}$$

for a decomposable ideal I. Here the rows are exact.

We define the group of *positive automorphisms* of $\widehat{L}H/I$ by

$$\mathrm{IAut}(\widehat{L}H/I) := \{f \in \mathrm{Aut}(\widehat{L}H/I) : f(x) = x + L_{\geq 2}/I\},$$

where $\mathrm{Aut}(\widehat{L}H/I)$ is the group of completed Lie algebra automorphisms of $\widehat{L}H/I$. It is a projective limit of finite-dimensional Lie groups, and has the Lie algebra

$$\mathrm{Der}^+(\widehat{L}H/I) := \{X \in \mathrm{Der}(\widehat{L}H/I) : X(H) \subset L_{\geq 2}/I\},$$

where $\mathrm{Der}(\widehat{L}H/I)$ is the Lie algebra consisting of continuous Lie algebra derivations on $\widehat{L}H/I$. We call an element of $\mathrm{Der}^+(\widehat{L}H/I)$ a *positive derivation* on $\widehat{L}H/I$.

Hopf algebra automorphisms and Hopf derivations of $\widehat{T}H/I$ are determined by their restrictions to the primitive part $\widehat{L}H/I$, so $\mathrm{IAut}(\widehat{L}H/I)$ and $\mathrm{Der}^+(\widehat{L}H/I)$ can be regarded as a subgroup of completed Hopf algebra automorphisms of $\widehat{T}H/I$ and a Lie subalgebra of continuous Hopf derivations on $\widehat{T}H/I$, respectively. Here Hopf derivation means derivation and coderivation.

An inner automorphism of $\widehat{L}H/I$ is a Hopf algebra automorphism $\iota_a\colon \widehat{L}H/I \to \widehat{L}H/I$ defined for all x in $\widehat{L}H/I$ by

$$\iota_a(x) := axa^{-1},$$

where a is a given fixed group-like element of $\widehat{T}H/I$. Then ι_a is a positive automorphism of $\widehat{L}H/I$. We denote the normal subgroup of $\mathrm{IAut}(\widehat{L}H/I)$ consisting of inner automorphisms by $\mathrm{Inn}(\widehat{L}H/I)$. It has the Lie algebra

$$\mathrm{IDer}(\widehat{L}H/I) := \{\mathrm{ad}(a) \in \mathrm{Der}^+(\widehat{L}H/I) : a \in \widehat{L}H/I\},$$

which is the Lie algebra of inner derivations on $\widehat{L}H/I$. We also define the quotient group

$$\mathrm{IOut}(\widehat{L}H/I) := \mathrm{IAut}(\widehat{L}H/I)/\,\mathrm{Inn}(\widehat{L}H/I),$$

and its Lie algebra,

$$\mathrm{ODer}^+(\widehat{L}H/I) := \mathrm{Der}^+(\widehat{L}H/I)/\,\mathrm{IDer}(\widehat{L}H/I).$$

We also use the groups of positive automorphisms and outer automorphisms of $\widehat{L}H$ which preserve a decomposable ideal I:

$$\mathrm{IAut}_I(\widehat{L}H) := \{f \in \mathrm{IAut}(\widehat{L}H) : f(I) = I\}, \quad \mathrm{IOut}_I(\widehat{L}H) := \mathrm{IAut}_I(\widehat{L}H)/\,\mathrm{Inn}(\widehat{L}H).$$

These Lie algebras are the Lie algebras of positive derivations and outer derivations on $\widehat{L}H$ which preserve I:

$$\mathrm{Der}_I^+(\widehat{L}H) := \{X \in \mathrm{Der}^+(\widehat{L}H) : X(I) \subset I\}, \quad \mathrm{ODer}_I^+(\widehat{L}H) := \mathrm{Der}_I^+(\widehat{L}H)/\mathrm{IDer}(\widehat{L}H),$$

respectively.

If I is a homogeneous ideal, the Lie algebra of derivations on $\widehat{L}H/I$ has the natural grading.

Definition Let H be a vector space and I a homogeneous decomposable ideal of $\widehat{L}H$. The degree k component of $\mathrm{Der}(\widehat{L}H/I)$ is defined by

$$\mathrm{Der}^k(LH/I) := \{X \in \mathrm{Der}^+(\widehat{L}H/I) : X(H) \subset L_{k+1}/(L_{k+1} \cap I)\}.$$

Then we have the decomposition

$$\mathrm{Der}^+(\widehat{L}H/I) = \prod_{k=1}^{\infty} \mathrm{Der}^k(LH/I).$$

In the same way, $\mathrm{IDer}^k(LH/I)$ and $\mathrm{ODer}^k(LH/I)$ are defined.

2B Space of expansions

In this section, we define the spaces containing Chen expansions of a fundamental group. These spaces play an important role in our construction of characteristic classes in Section 3. It is a generalization of the spaces of Magnus expansions of free groups in [5].

Definition For a decomposable ideal I of $\widehat{L}H$, an I–*expansion* of π is a completed Hopf algebra isomorphism $\theta\colon \widehat{\mathbb{R}\pi} \to \widehat{T}H/I$ satisfying

$$\theta(g) = 1 + [g] + J^2/I$$

for any $g \in \pi$. (For such a type of isomorphism, see [10].) The space of I–expansions is denoted by $\Theta(\pi, I)$.

The group $\mathrm{IAut}(\widehat{L}H/I)$ acts on $\Theta(\pi, I)$ as follows:

$$f \cdot \theta := f \circ \theta$$

for $f \in \mathrm{IAut}(\widehat{L}H/I)$ and $\theta \in \Theta(\pi, I)$. This action is free and transitive because $\theta \circ \theta_0^{-1} \in \mathrm{IAut}(\widehat{L}H/I)$ and

$$(\theta \circ \theta_0^{-1}) \cdot \theta_0 = \theta$$

for all θ, $\theta_0 \in \Theta(\pi, I)$. We describe the space of conjugacy classes of I–expansions by $\bar{\Theta}(\pi, I) := \mathrm{Inn}(\widehat{L}H/I)\backslash\Theta(\pi, I)$. The group $\mathrm{IOut}(\widehat{L}H/I)$ acts on $\bar{\Theta}(\pi, I)$ freely and transitively. (We denote the conjugacy class of $\theta \in \Theta(\pi, I)$ by $[\theta]$.)

We consider the space $\bar{\Theta}(\pi)$ of conjugacy classes of all possible expansions of π. It is parametrized by

$$\mathrm{IOut}(\widehat{L}H) \times_{\mathrm{IOut}_{I_0}(\widehat{L}H)} \bar{\Theta}(\pi, I_0)$$

fixing a decomposable ideal I_0 which satisfies $\Theta(\pi, I_0) \neq \varnothing$. We consider $\mathrm{IOut}_I(\widehat{L}H)$ acting on $\bar{\Theta}(\pi, I)$ through the natural homomorphism $\mathrm{IOut}_I(\widehat{L}H) \to \mathrm{IOut}(\widehat{L}H/I)$. The outer automorphism group $\mathrm{Out}(\pi)$ of π acts on $\bar{\Theta}(\pi)$ by

$$\varphi \cdot [\theta] = [\,|\varphi| \circ \theta \circ \varphi^{-1}]$$

for $\varphi \in \mathrm{Out}(\pi)$, $\theta \in \bar{\Theta}(\pi)$. Here $|\varphi|$ means the Hopf algebra isomorphism between quotient Hopf algebras of the tensor Hopf algebra $\widehat{T}H$ induced by $\varphi \in \mathrm{Out}(\pi)$.

The space $\mathcal{I}(\pi)$ of all ideals I satisfying $\bar{\Theta}(\pi, I) \neq \varnothing$ is parametrized by

$$\mathrm{IOut}(\widehat{L}H)/\, \mathrm{IOut}_{I_0}(\widehat{L}H) = \mathrm{IAut}(\widehat{L}H)/\, \mathrm{IAut}_{I_0}(\widehat{L}H)$$

fixing a decomposable ideal I_0 which satisfies $\bar{\Theta}(\pi, I_0) \neq \varnothing$. The group

$$\mathrm{GL}(\pi) := \mathrm{Im}(\mathrm{Aut}(\pi) \to \mathrm{GL}(H)) = \mathrm{Im}(\mathrm{Out}(\pi) \to \mathrm{GL}(H))$$

acts on $\mathcal{I}(\pi)$ by

$$\varphi \cdot I = \varphi(I)$$

for $\varphi \in \mathrm{GL}(\pi)$, $I \in \mathcal{I}(\pi)$. Through the natural homomorphism $\mathrm{Out}(\pi) \to \mathrm{GL}(\pi)$, the natural map $\bar{\Theta}(\pi) \to \mathcal{I}(\pi)$ is regarded as an $\mathrm{Out}(\pi)$–equivariant $\bar{\Theta}(\pi, I_0)$–bundle.

2C Generalization of a Johnson map

We define a generalization of a Johnson map for free groups by Kawazumi [4]. The results in this section shall be used only in Section 4.

Proposition 2.1 *Let π be a finitely generated group. If π has a homogeneous decomposable ideal, ie there exists a homogeneous decomposable ideal I such that $\Theta(\pi, I) \neq \varnothing$, then such an ideal is unique. We denote it by I_π.*

Proof Suppose a homogeneous decomposable ideal I' also satisfies $\Theta(\pi, I') \neq \varnothing$. Take $\theta \in \Theta(\pi, I)$ and $\theta' \in \Theta(\pi, I')$ and set $f := \theta' \circ \theta^{-1}$. Then

$$f(x) = x + J^2/I'$$

for $x \in H$. From the equation

$$0 = f(y) = y + J^{k+1}/(J^{k+1} \cap I')$$

for $y \in H^{\otimes k} \cap I$, we get $y \in I'$ by assumption that I' is homogeneous. Since I is also homogeneous, we obtain $I \subset I'$. In the same way, we can prove $I \supset I'$. Therefore we have $I = I'$. □

Definition Let π be a finitely generated group which has a homogeneous decomposable ideal I_π. For $\theta \in \Theta(\pi, I_\pi)$, we define the *Johnson map* $\tau^\theta\colon \mathrm{Aut}(\pi) \to \mathrm{IAut}(\widehat{L}H/I_\pi)$ associated to θ by

$$\tau^\theta(\varphi) = \theta \circ \varphi \circ \theta^{-1} \circ |\varphi|^{-1}.$$

Note that uniqueness of I_π gives $|\varphi|(I_\pi) = I_\pi$. By the decomposition

$$\mathrm{IAut}(\widehat{L}H/I_\pi) \subset \prod_{p=1}^{\infty} \mathrm{Hom}(H, L_{p+1}/(L_{p+1} \cap I_\pi)),$$

we denote the $\mathrm{Hom}(H, L_{p+1}/(L_{p+1} \cap I_\pi))$ component of τ^θ by τ^θ_p.

For a group-like element $a \in \widehat{T}H/I_\pi$, the equation

$$\tau^{\iota_a\theta}(\varphi) = \iota_a\theta\varphi\theta^{-1}\iota_a^{-1}|\varphi|^{-1} = \iota_a\tau^\theta(\varphi)\iota^{-1}_{|\varphi|(a)}$$

holds. Then we can define $\tau^{[\theta]}\colon \mathrm{Out}(\pi) \to \mathrm{IOut}(\widehat{L}H/I_\pi)$ for $[\theta] \in \bar{\Theta}(\pi, I_\pi)$. We can also obtain $\tau_1^{[\theta]}$ for $[\theta] \in \bar{\Theta}(\pi, I_\pi)$.

Definition Given a filtered algebra A with decreasing filtration $\{F_k(A)\}_{k=0}^\infty$,

$$\mathrm{gr}(A) := \bigoplus_{k=0}^{\infty} F_k(A)/F_{k+1}(A)$$

is a graded algebra by the multiplication induced by the multiplication of A. When A is the completion of a graded algebra V, then $\pi(\widehat{V}) = V$ holds.

When G is a group with decreasing central filtration $\{F_k(G)\}_{k=0}^\infty$, a similar thing happens. Then

$$\mathrm{gr}(G) := \bigoplus_{k=0}^{\infty} F_k(G)/F_{k+1}(G)$$

is a graded Lie algebra by the Lie bracket induced by the commutator of G.

Propositions 2.2, 2.3, 2.4 and 2.5 are straightforward generalizations of those in [4].

Proposition 2.2 *For an I_π–expansion $\theta \in \Theta(\pi, I_\pi)$, the map $\mathrm{gr}(\theta)\colon \mathrm{gr}(\pi) \otimes \mathbb{R} \to \mathrm{gr}(\widehat{T}H/I_\pi) = TH/I_\pi$ induced by θ is the natural identification $\mathrm{gr}(\pi) \otimes \mathbb{R} \to LH/I_\pi$. Specifically, $\mathrm{gr}(\theta)$ does not depend on the choice of I_π–expansion θ. (Here the group π and the algebra $\widehat{T}H$ are filtered by the lower central series $\{\Gamma_k\pi\}_{k=1}^\infty$ and the power series $\{J^k\}_{k=1}^\infty$ of the augmented ideal J, respectively.)*

Proof Decompose θ into the sum

$$\theta = \sum_{k=0}^{\infty} \theta_k,$$

with respect to grading. For a positive integer k, $x \in \Gamma_{k-1}\pi$ and $y \in \pi$,

$$\theta([x,y]) = 1 + \theta_{k-1}(x)[y] - [y]\theta_{k-1}(x) + J^{k+1}/(J^{k+1} \cap I_\pi).$$

Therefore $\theta_k([x,y]) = [\theta_{k-1}(x), [y]]$ holds. Thus the result follows by induction. □

Let π be a finitely generated group. The *Andreadakis filtration* of the automorphism group $\mathrm{Aut}(\pi)$ of π is defined by

$$\mathcal{A}_\pi(k) := \mathrm{Ker}(\mathrm{Aut}(\pi) \to \mathrm{Aut}(\pi/\Gamma_{k+1}\pi)).$$

Note $\mathcal{A}_\pi(1) = \mathrm{IAut}(\pi)$. The k^{th} Johnson homomorphism

$$\tau_k\colon \mathcal{A}_\pi(k)/\mathcal{A}_\pi(k+1) \to \mathrm{Hom}(\pi^{\mathrm{ab}}, \mathrm{gr}^{k+1}(\pi))$$

is defined as follows: For $\varphi \in \mathcal{A}_\pi(k)$ and $x \in \pi$, we set $s_\varphi(x) := x^{-1}\varphi(x) \in \Gamma_{k+1}\pi$. If $x \in \Gamma_2\pi$, then $s_\varphi(x) \in \Gamma_{k+2}\pi$ holds. Thus, we can define

$$\tau_k(\varphi)([x]) = [s_\varphi(x)] \in \Gamma_{k+1}\pi/\Gamma_{k+2}\pi.$$

In addition, the fact $\tau_k(\varphi) = 0$ for $\varphi \in \mathcal{A}_\pi(k+1)$ induces the homomorphism

$$\tau_k\colon \mathcal{A}_\pi(k)/\mathcal{A}_\pi(k+1) \to \mathrm{Hom}(\pi^{\mathrm{ab}}, \mathrm{gr}^{k+1}(\pi)).$$

The direct sum of these maps with respect to k defines the Lie algebra homomorphism

$$\tau\colon \mathrm{gr}^+(\mathrm{Aut}(\pi)) \to \mathrm{Der}^+(\mathrm{gr}(\pi))$$

(see Satoh [15] for details). Here we denote the positive degree part of $\mathrm{gr}(G)$ by $\mathrm{gr}^+(G)$ for a group G with central filtration. By definition, an inner automorphism is mapped to an inner derivation by this map. Then the Lie algebra homomorphism

$$\bar{\tau}\colon \mathrm{gr}^+(\mathrm{Out}(\pi)) \to \mathrm{ODer}^+(\mathrm{gr}(\pi))$$

is induced by τ. Here $\mathrm{Out}(\pi)$ is filtered by

$$\mathcal{O}_\pi(k) := \mathrm{Ker}(\mathrm{Out}(\pi) \to \mathrm{Out}(\pi/\Gamma_{k+1}\pi)).$$

Proposition 2.3 *For $\theta \in \Theta(\pi, I_\pi)$ and $m \geq 1$, the map induced by τ^θ,*

$$\mathrm{gr}^m(\tau^\theta)\colon \mathrm{gr}^m(\mathrm{Aut}(\pi)) \to \mathrm{gr}^m(\mathrm{Aut}(\widehat{L}H/I_\pi)) = \mathrm{Der}^m(LH/I_\pi),$$

is equal to the m^{th} Johnson homomorphism τ_m through the natural identification $\mathrm{gr}(\pi)\otimes\mathbb{R} \to LH/I_\pi$. Specifically, $\mathrm{gr}^+(\tau^\theta)$ does not depend on the choice of I_π–expansion θ. Here the group $\mathrm{Aut}(\pi)$ is filtered by $\{\mathcal{A}_\pi(k)\}_{k=0}^\infty$ and the group $\mathrm{Aut}(\widehat{L}H/I_\pi)$ is filtered by $\{\mathrm{Aut}^{\geq k}(\widehat{L}H/I_\pi)\}_{k=0}^\infty$, where

$$\mathrm{Aut}^{\geq k}(\widehat{L}H/I_\pi) = \bigl\{f \in \mathrm{Aut}(\widehat{L}H/I_\pi) : f(x) = x + L_{\geq k+1}/(L_{\geq k+1}\cap I_\pi) \text{ for } x \in H\bigr\}.$$

Proof Suppose $\tau_k^\theta(\varphi) = \tau_k(\varphi)$ for $\varphi \in \mathcal{A}_\pi(k)$ and $1 \leq k < m$. Then $\tau_k^\theta(\varphi) = 0$ holds for $\varphi \in \mathcal{A}_\pi(m)$ and $1 \leq k < m$. From

$$\tau^\theta(\varphi)([x]) = \theta\varphi\theta^{-1}([x]) = \theta(\varphi(x)) + \tau_m^\theta(\varphi)([x])$$

for $x \in \pi$, we obtain

$$\tau_m(\varphi)([x]) = \theta_{m+1}(s_\varphi(x)) \equiv \theta(s_\varphi(x)) - 1 = \theta(x)^{-1}\theta(\varphi(x)) - 1 \equiv \tau_m^\theta(\varphi)([x])$$

modulo $L_{\geq m+2}/(L_{\geq m+2} \cap I_\pi)$. Then we have $\tau_m^\theta(\varphi) = \tau_m(\varphi)$ for all $m \geq 1$ and $\varphi \in \mathcal{A}_\pi(m)$ inductively. □

Proposition 2.4 *The first Johnson map τ_1^θ is a cocycle of* $\mathrm{Aut}(\pi)$ *with coefficient* $\mathrm{Der}^1(LH/I_\pi)$*, and the cohomology class* $[\tau_1^\theta] \in H^1(\mathrm{Aut}(\pi); \mathrm{Der}^1(LH/I_\pi))$ *does not depend on the choice of I_π–expansion θ.*

Proof The cocycle condition of τ_1^θ is written by

$$\tau_1^\theta(\varphi\psi) = \tau_1^\theta(\varphi) + |\varphi|\tau_1^\theta(\psi)$$

for every $\varphi, \psi \in \mathrm{Aut}(\pi)$. This follows from the formula

$$\tau^\theta(\varphi\psi) = \tau^\theta(\varphi)|\varphi|\tau^\theta(\psi)|\varphi|^{-1}.$$

Independence of its cohomology class is proved as follows. Because $\theta_2'(x) = \theta_2(x)$ for $x \in \Gamma_2\pi$ by Proposition 2.2, we can define $F \in \mathrm{Der}^1(LH/I_\pi)$ by

$$F([x]) := \theta_2'(x) - \theta_2(x)$$

for $[x] \in H$. By the formula

$$\tau_1^\theta(\varphi)([x]) = \theta_2(x) - |\varphi|\theta_2(\varphi^{-1}(x))$$

for $[x] \in H$ and $\varphi \in \mathrm{Aut}(\pi)$, we have

$$\tau_1^\theta - \tau_1^{\theta'} = dF.$$

Thus $[\tau_1^{\theta'}] = [\tau_1^\theta] \in H^1(\mathrm{Aut}(\pi); \mathrm{Der}^1(LH/I_\pi))$. □

Similar results for the outer automorphism group also hold:

Proposition 2.5 *For $m \geq 1$, the map induced by τ^θ,*

$$\mathrm{gr}^m(\tau^\theta)\colon \mathrm{gr}^m(\mathrm{Out}(\pi)) \to \mathrm{gr}^m(\mathrm{Out}(\widehat{L}H/I_\pi)) = \mathrm{ODer}^m(LH/I_\pi)$$

for $\theta \in \bar{\Theta}(\pi, I_\pi)$, is identified with the m^{th} Johnson homomorphism $\bar{\tau}_m$ of $\mathrm{Out}(\pi)$*. Here the group* $\mathrm{Out}(\pi)$ *is filtered by $\{\mathcal{O}_\pi(k)\}_{k=0}^\infty$, and the group* $\mathrm{Out}(\widehat{L}H/I_\pi)$ *is filtered by* $\{\mathrm{Out}^{\geq k}(\widehat{L}H/I_\pi)\}_{k=0}^\infty$*, where*

$$\mathrm{Out}^{\geq k}(\widehat{L}H/I_\pi) = \mathrm{Im}\big(\mathrm{Aut}^{\geq k}(\widehat{L}H/I_\pi) \to \mathrm{Out}(\widehat{L}H/I_\pi)\big).$$

Also, the first Johnson map τ_1^θ *is a cocycle of* $\mathrm{Out}(\pi)$ *with coefficient* $\mathrm{ODer}^1(LH/I_\pi)$, *and the cohomology class*

$$[\tau_1^\theta] \in H^1(\mathrm{Out}(\pi); \mathrm{ODer}^1(LH/I_\pi))$$

does not depend on the choice of $\theta \in \bar{\Theta}(\pi, I_\pi)$.

3 Characteristic classes

For an oriented closed manifold X, we set the fundamental group $\pi = \pi_1(X)$ and the first homology group $H = H_1(X; \mathbb{R})$. For the purpose, we recall a result of K-T Chen.

Definition [1; 2] Let X be a manifold. We denote the suspension of $H_+(X;\mathbb{R}) := \bigoplus_{p>0} H_p(X;\mathbb{R})$ by $H_+ := H_+(X;\mathbb{R})[1]$. The completed tensor algebra $T := \widehat{T}H_+$ of the suspension H_+ is a (completion of a) graded algebra. A pair (ω, δ) satisfying the following conditions is a *formal homology connection*:

(i) A T–coefficient differential form $\omega \in A^*(X) \otimes T$ is described by

$$\omega = \sum_{k=1}^{\infty} \sum_{i_1,\dots,i_k} \omega_{i_1 \cdots i_k} X_{i_1} \cdots X_{i_k},$$

where $X_1, \dots, X_n$ is a homogeneous basis of H_+ and the differential form $\omega_{i_1 \cdots i_k} \in A^*(X)$ is a $(\deg X_{i_1} + \cdots + \deg X_{i_k} + 1)$–form and they satisfy

$$\int_{X_p} \omega_p = 1.$$

(ii) A linear map $\delta \colon T \to T$ is a differential with degree -1 of the graded algebra T such that

$$\delta(H_+) \subset \prod_{q=2}^{\infty} H_+^{\otimes q}.$$

(iii) The flatness condition $\delta\omega + d\omega = \epsilon(\omega) \wedge \omega$ holds, where $\epsilon \colon A^*(X) \to A^*(X)$ is defined by $\epsilon(\alpha) = (-1)^p \alpha$ for $\alpha \in A^p(X)$, and δ and ϵ are extended onto $A^*(X) \otimes T$.

Let (ω, δ) be a formal homology connection of X. Then we can obtain the chain map $C_*(\Omega X; \mathbb{R}) \to T$ from the cubical chain complex of the loop space ΩX of X to the algebra T defined by the iterated integral

$$\sigma \mapsto \sum_{n=0}^{\infty} \int_\sigma \underbrace{\omega \cdots \omega}_{n}.$$

Furthermore, by the result of Chen, the homomorphism $H_0(\Omega X;\mathbb{R}) \to H_0(T,\delta)$ induces a Hopf algebra isomorphism $\theta_\omega\colon \widehat{\mathbb{R}\pi} \simeq \widehat{T}H/I_\omega$ where $I_\omega := \delta(H_2(X;\mathbb{R})[1])$. This is called a Chen expansion. In our notation, I_ω is a decomposable ideal and θ_ω is an I_ω–expansion.

Theorem 3.1 [1; 2] *Let (X,g) be an oriented closed Riemannian manifold and*

$$A^*(X) = \mathcal{H}_g \oplus dA^*(X) \oplus d_g^* A^*(X)$$

be the Hodge decomposition of (X,g). Here $\mathcal{H}_g$ is the space of harmonic forms and $d_g^*\colon A^*(X) \to A^*(X)$ is the adjoint operator of d with respect to g. Then there exists a unique formal homology connection (ω_g,δ_g) such that

$$\omega_g = \sum_{i=1}^m \omega_i X_i + \sum_{p\geq 2} \sum_{i_1,\dots,i_p} \omega_{i_1\cdots i_p} X_{i_1}\cdots X_{i_p},$$

where $X_1,\dots,X_m$ is a basis of H_+, $\omega_i \in \mathcal{H}_g$ and $\omega_{i_1\cdots i_p} \in d_g^* A^*(X)$.

We denote the group of diffeomorphisms of X preserving the orientation by $\mathrm{Diff}_+(X)$. For a Riemannian metric g on X and $\varphi \in \mathrm{Diff}_+(X)$, we define the metric $\varphi_* g$ on X by

$$(\varphi_* g)(u,v) := g(\varphi^* u, \varphi^* v)$$

for cotangent vectors $u,v \in T_x^* X$ and $x \in X$. Then, since

$$\mathcal{H}_{\varphi_* g} = (\varphi^*)^{-1}(\mathcal{H}_g), \quad d^*_{\varphi_* g} A^*(X) = (\varphi^*)^{-1}(d_g^* A^*(X))$$

for $\varphi \in \mathrm{Diff}_+(X)$, we have

$$(\omega_{\varphi_* g}, \delta_{\varphi_* g}) = \big(((\varphi^*)^{-1} \otimes |\varphi|)(\omega_g), |\varphi| \circ \delta_g \circ |\varphi|^{-1}\big).$$

Here $|\varphi|$ means the algebra isomorphism $\widehat{T}H \to \widehat{T}H$ induced by $\varphi \in \mathrm{Diff}_+(X)$. Thus the corresponding Chen expansions and ideals satisfy

$$\theta_{\omega_{\varphi_* g}} = |\varphi| \circ \theta_{\omega_g} \circ \varphi_*^{-1}\colon \widehat{\mathbb{R}\pi_1}(X,\varphi(*)) \to \widehat{T}H/|\varphi|(I_{\omega_g}), \quad I_{\omega_{\varphi_* g}} = |\varphi|(I_{\omega_g}) \tag{3-1}$$

for a diffeomorphism φ of X preserving the base point $*$. Here we denote the homomorphism $\widehat{T}H/I_{\omega_g} \to \widehat{T}H/|\varphi|(I_{\omega_g})$ induced by $|\varphi|\colon \widehat{T}H \to \widehat{T}H$ by the same symbol $|\varphi|$.

Based on these considerations, we describe characteristic classes of fiber bundles using expansion spaces as follows: let $E \to B$ be an oriented fiber bundle whose fiber X is an oriented closed manifold and set $\pi = \pi_1(X)$. We choose a fiberwise metric $g_{E/B}$ on $E \to B$. Take an open covering $\{U_i\}_i$ of B and local trivializations

$\varphi_i\colon \pi^{-1}(U_i) \simeq U_i \times X$ of E. Then $g_{E/B}$ defines fiberwise metrics g_i of trivial bundles $U_i \times X \to U_i$. The map $\theta_i\colon U_i \to \bar{\Theta}(\pi)$ can be defined by $x \mapsto [\theta_{\omega_{g_i(x)}}]$. For any differential form $\alpha \in A^*(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)}$,

$$\theta_i^*\alpha = \theta_j^*\alpha$$

holds on $U_i \cap U_j$ because the correspondence from metrics to Chen expansions is $\mathrm{Diff}_+(X)$–equivariant from the equations (3-1). (It also means $\theta_i^*\alpha$ is independent of the choice of local trivializations.) Thus the family $\{\theta_i^*\alpha\}_i$ of differential forms on the open covering defines the differential form $\theta^*\alpha$ on B by gluing. The correspondence $\alpha \mapsto \theta^*\alpha$ is a chain map $A^*(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \to A^*(B)$. Given two distinct open coverings, we can prove that the maps $A^*(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \to A^*(B)$ for two coverings are equal by taking a refinement of these open coverings.

Since any two fiberwise metrics g_0, g_1 of a fiber bundle can be connected by a segment $(1-t)g_0 + tg_1$, the chain maps $A^*(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \to A^*(B)$ for distinct metrics are chain homotopy equivalent. Therefore the induced homomorphism

$$\Phi_E\colon H^*_{\mathrm{DR}}(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \to H^*_{\mathrm{DR}}(B)$$

does not depend on the choice of fiberwise metric. So the following theorem holds:

Theorem 3.2 *For an oriented fiber bundle $E \to B$ whose fiber is X, the map $A^*(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \to A^*(B)$ constructed above is a chain map, and the induced map Φ_E on cohomology groups is independent of the choice of fiberwise metric.*

Remark 3.3 Let $\mathcal{G}$ be the structure group of $E \to B$ and set $\widetilde{S} := \mathrm{Im}(\mathcal{G} \to \mathrm{Out}(\pi))$. We obtain the map

$$H^*_{\mathrm{DR}}(\bar{\Theta}(\pi))^{\widetilde{S}} \to H^*_{\mathrm{DR}}(\bar{\Theta}(\pi))^{\mathrm{Out}(\pi)} \xrightarrow{\Phi_E} H^*_{\mathrm{DR}}(B).$$

We also denote this map by Φ_E.

3A Homologically trivial bundles

In this section we assume that a fiber bundle $E \to B$ with fiber X has the structure group

$$\mathcal{T}(X) := \mathrm{Ker}(\mathrm{Diff}_+(X) \to \mathrm{GL}(H)).$$

Fix a decomposable ideal I_0 which satisfies $\Theta(\pi, I_0) \neq \varnothing$. We choose a fiberwise metric g of $E \to B$. Since the structure group of $E \to B$ is $\mathcal{T}(X)$, the correspondence of ideals by g gives a map $q\colon B \to \mathcal{I}(\pi)$. Because the topological group $\mathrm{IOut}_{I_0}(\widehat{L}H)$ is contractible, the pullback $q^*\,\mathrm{IOut}(\widehat{L}H) \to B$ of the principal $\mathrm{IOut}_{I_0}(\widehat{L}H)$–bundle

$\mathrm{IOut}(\widehat{L}H) \to \mathcal{I}(\pi)$ is trivial. Taking a trivialization of the principal bundle, we get the $\mathrm{IOut}(\pi)$–equivariant map

$$s\colon q^*\bar{\Theta}(\pi) = q^*\,\mathrm{IOut}(\widehat{L}H) \times_{\mathrm{IOut}_{I_0}(\widehat{L}H)} \bar{\Theta}(\pi, I_0) \simeq B \times \bar{\Theta}(\pi, I_0) \to \bar{\Theta}(\pi, I_0).$$

Thus we can obtain the characteristic map

$$H^*_{\mathrm{DR}}(\bar{\Theta}(\pi, I_0))^{\mathrm{IOut}(\pi)} \xrightarrow{s^*} H^*_{\mathrm{DR}}(q^*\bar{\Theta}(\pi))^{\mathrm{IOut}(\pi)} \xrightarrow{\Phi_E} H^*_{\mathrm{DR}}(B),$$

where $\mathrm{IOut}(\pi) := \mathrm{Ker}(\mathrm{Out}(\pi) \to \mathrm{GL}(H))$. This map does not depend on the choice of fiberwise metric and trivialization. (Independence from the choice of trivialization of $q^*\,\mathrm{IOut}(\widehat{L}H) \to B$ comes from contractibility of $\mathrm{IOut}_{I_0}(\widehat{L}H)$.)

Remark 3.4 We remark that the map Φ_E of Theorem 3.2 and Remark 3.3 factors the natural maps

$$H^*_{\mathrm{DR}}(\bar{\Theta}(\pi))^{\mathrm{IOut}(\pi)} \to H^*_{\mathrm{DR}}(q^*\bar{\Theta}(\pi))^{\mathrm{IOut}(\pi)} \to H^*_{\mathrm{DR}}(B)$$

by construction. The map $H^*_{\mathrm{DR}}(q^*\bar{\Theta}(\pi))^{\mathrm{IOut}(\pi)} \to H^*_{\mathrm{DR}}(B)$ is also regarded as Φ_E.

3B The fiber which has a homogeneous ideal and a splitting

If the fundamental group $\pi = \pi_1(X)$ of fiber X satisfies some conditions, we can obtain stronger results. In this section we assume that the fundamental group $\pi = \pi_1(X)$ of fiber X has the homogeneous ideal I_π and there exists a splitting

$$\mathrm{Der}^+(\widehat{L}H) = V \oplus \mathrm{Der}^+_{I_\pi}(\widehat{L}H)$$

as graded S–vector spaces. Here a group S is the image of a structure group of $E \to B$ in $\mathrm{GL}(\pi)$.

Example 1 If π is a free group, we have $I_\pi = 0$. So π clearly satisfies the condition above for any $S \subset \mathrm{GL}(\pi)$. For example, the fundamental group of a manifold whose second Betti number is zero is a free group.

Example 2 We consider the case of $X = \Sigma_g$, an oriented surface of genus $g \geq 1$. Then $\pi = \pi_1(X)$ has the homogeneous ideal $I_\pi = (\omega)$. Here we denote the intersection form of Σ_g by

$$\omega = \sum_{i=1}^{g} [X_i, X_{i+g}] \in L_2,$$

where $X_1, \dots, X_{2g}$ is a symplectic basis of H. We set

$$S = \mathrm{Im}\big(\mathrm{Diff}_+(\Sigma_g) \to \mathrm{GL}(\pi)\big) = \mathrm{Sp}\big(H_1(\Sigma_g; \mathbb{Z}), \omega\big) \subset \mathrm{Sp}(H, \omega).$$

Since all finite-dimensional representations of $\mathrm{Sp}(H,\omega)$ are completely reducible, π satisfies the splitting condition.

Lemma 3.5 *The* $\mathrm{GL}(\pi)$*–equivariant principal* $\mathrm{IAut}_{I_\pi}(\hat{L}H)$*–bundle* $\mathrm{IAut}(\hat{L}H) \to \mathcal{I}(\pi)$ *is* S*–equivariant trivial.*

Proof Let t be a real number. Then for any $X \in \mathrm{Der}^k(LH)$, we define the map $w_t\colon \mathrm{Der}^+(\hat{L}H) \to \mathrm{Der}^+(\hat{L}H)$ by $w_t(X) = t^k X$. If $X \in \mathrm{Der}^k(LH)$ and $Y \in \mathrm{Der}^l(LH)$, we have

$$[w_t(X), w_t(Y)] = [t^k X, t^l Y] = t^{k+l}[X,Y] = w_t([X,Y]).$$

So w_t is a Lie algebra homomorphism. We define $F_t\colon \mathrm{IAut}(\hat{L}H) \to \mathrm{IAut}(\hat{L}H)$ by

$$F_t(\exp X) = \exp w_t(X).$$

Since w_t is a Lie algebra homomorphism, F_t is a group homomorphism, and F_t is commutative with the action of $\mathrm{GL}(H)$ on $\mathrm{IAut}(\hat{L}H)$ since w_t preserves the degree of derivations. Furthermore, since I_π is a homogeneous ideal, F_t can be restricted to the $\mathrm{GL}(\pi)$–equivariant map $\mathrm{IAut}_{I_\pi}(\hat{L}H) \to \mathrm{IAut}_{I_\pi}(\hat{L}H)$ and induces $\mathcal{I}(\pi) \to \mathcal{I}(\pi)$.

We set $\mathrm{IAut}^{\le k}(\hat{L}H) := \mathrm{IAut}(\hat{L}H/L_{\ge k+1})$. We also write the corresponding Lie algebras in the same way. Since

$$\mathrm{IAut}(\hat{L}H) = \varprojlim_k \mathrm{IAut}^{\le k}(\hat{L}H),$$

we have that

$$\mathcal{I}(\pi) = \varprojlim_k \mathrm{IAut}^{\le k}(\hat{L}H)/\,\mathrm{IAut}^{\le k}_{I_\pi}(\hat{L}H).$$

We set $\mathcal{I}(\pi)_k := \mathrm{IAut}^{\le k}(\hat{L}H)/\,\mathrm{IAut}^{\le k}_{I_\pi}(\hat{L}H)$.

By assumption of a splitting

$$\mathrm{Der}^+(\hat{L}H) = V \oplus \mathrm{Der}^+_{I_\pi}(\hat{L}H),$$

a splitting

$$\mathrm{Der}^{\le k}(\hat{L}H) = V^{\le k} \oplus \mathrm{Der}^{\le k}_{I_\pi}(\hat{L}H)$$

is induced. Here the superscript $\le k$ of graded vector spaces means the part with degree $\le k$.

The map $\mathrm{Der}^{\le k}(\hat{L}H) \to \mathrm{IAut}^{\le k}(\hat{L}H)$ defined by

$$X + Y \mapsto \exp X \cdot \exp Y$$

is a local diffeomorphism at 0. Therefore so is the map $c\colon V^{\le k} \to \mathcal{I}(\pi)_k$ induced by the map above. It gives a diffeomorphism $c\colon U \to O$ of a neighborhood U of 0 and a neighborhood O of $[\mathrm{id}]$. Then, for all $t > 0$, the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "c"] \arrow[d, "w_t"] & O \arrow[d, "F_t"] \\ w_t(U) \arrow[r, "c"] & F_t(O) \end{tikzcd}$$

Therefore, the restriction $c\colon w_t(U) \to F_t(U)$ is also a diffeomorphism. Thus,

$$c\colon V^{\le k} = \bigcup_t w_t(U) \to \bigcup_t F_t(O) = \mathcal{I}(\pi)_k$$

is a global diffeomorphism. The inverse limit

$$V \to \mathcal{I}(\pi)$$

of the maps is also a diffeomorphism. We obtain a section

$$\mathcal{I}(\pi) \simeq V \xrightarrow{\exp} \mathrm{IAut}(\widehat{L}H)$$

of the S–equivariant principal $\mathrm{IAut}_{I_\pi}(\widehat{L}H)$–bundle $\mathrm{IAut}(\widehat{L}H) \to \mathcal{I}(\pi)$. □

Let $\widetilde{S}$ be the image of a structure group of $E \to B$ in $\mathrm{Out}(\pi)$. From Lemma 3.5, the $\mathrm{GL}(\pi)$–equivariant principal $\mathrm{IOut}_{I_\pi}(\widehat{L}H)$–bundle $\mathrm{IOut}(\widehat{L}H) \to \mathcal{I}(\pi)$ is S–equivariantly trivial. Then there exists an $\widetilde{S}$–equivariant diffeomorphism

$$\begin{aligned} \bar{\Theta}(\pi) &= \mathrm{IOut}(\widehat{L}H) \times_{\mathrm{IOut}_{I_\pi}(\widehat{L}H)} \bar{\Theta}(\pi, I_\pi) \\ &\simeq (\mathcal{I}(\pi) \times \mathrm{IOut}_{I_\pi}(\widehat{L}H)) \times_{\mathrm{IOut}_{I_\pi}(\widehat{L}H)} \bar{\Theta}(\pi, I_\pi) = \mathcal{I}(\pi) \times \bar{\Theta}(\pi, I_\pi). \end{aligned}$$

Since the map $F_t\colon \mathcal{I}(\pi) \to \mathcal{I}(\pi)$ defined in the proof of the theorem above is a $\mathrm{GL}(\pi)$–equivariant homotopy, $\mathcal{I}(\pi)$ is $\mathrm{GL}(\pi)$–contractible. Thus we have:

Theorem 3.6 *The space $\bar{\Theta}(\pi)$ is $\widetilde{S}$–equivariant homotopic to $\bar{\Theta}(\pi, I_\pi)$.*

Thus the characteristic map Φ_E of $E \to B$ is described by

$$H^*_{\mathrm{DR}}(\bar{\Theta}(\pi, I_\pi))^{\widetilde{S}} = H^*_{\mathrm{DR}}(\bar{\Theta}(\pi))^{\widetilde{S}} \to H^*_{\mathrm{DR}}(B).$$

3C Relation between two constructions

Let π be a finitely generated group which has a homogeneous ideal I_π. We consider the case of $S=1$, $\widetilde{S}=\mathrm{IOut}(\pi)$, $I_0=I_\pi$ in Sections 3A and 3B. We choose a trivialization of $q^*\,\mathrm{IOut}(\widehat{L}H)\to B$ which was induced by a trivialization of $\mathrm{IOut}(\widehat{L}H)\to\mathcal{I}(\pi)$ in Section 2B. Then the diagram

$$\begin{tikzcd} q^*\bar{\Theta}(\pi) \arrow[r, "\sim"] \arrow[d] & B\times\bar{\Theta}(\pi,I_\pi) \arrow[r] \arrow[d] & \bar{\Theta}(\pi,I_\pi) \\ \bar{\Theta}(\pi) \arrow[r, "\sim"] & \mathcal{I}(\pi)\times\bar{\Theta}(\pi,I_\pi) \arrow[ur, "\text{h.e.}"'] & \end{tikzcd}$$

commutes. So taking $\mathrm{IOut}(\pi)$–invariant de Rham cohomologies, we obtain the diagram

$$\begin{tikzcd} H^*_{\mathrm{DR}}(\bar{\Theta}(\pi,I_\pi))^{\mathrm{IOut}(\pi)} \arrow[r] \arrow[dr] & H^*_{\mathrm{DR}}(q^*\bar{\Theta}(\pi))^{\mathrm{IOut}(\pi)} \arrow[r] & H^*_{\mathrm{DR}}(B) \\ & H^*_{\mathrm{DR}}(\bar{\Theta}(\pi))^{\mathrm{IOut}(\pi)} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Thus the characteristic maps obtained in two ways are equal in the common case.

Remark 3.7 Using the construction in this section, we can obtain characteristic maps of a fiber bundle whose structure group is a subgroup of $\mathrm{Diff}_+(X,*)$ by replacing Out with Aut, $\bar{\Theta}$ with Θ, ODer^+ with Der^+, and so on.

3D Lie algebra cohomology of derivations

We construct invariant differential forms on $\bar{\Theta}(\pi,I)$ from a flat connection on $\bar{\Theta}(\pi,I)$ and Chevalley–Eilenberg cochains.

Since $\mathrm{IOut}(\widehat{L}H/I)$ acts freely and transitively on $\bar{\Theta}(\pi,I)$, we get the $\mathrm{ODer}^+(\widehat{L}H/I)$–coefficient differential form $\eta\in A^1(\bar{\Theta}(\pi,I);\mathrm{ODer}^+(\widehat{L}H/I))$ by pulling back the right-invariant Maurer–Cartan form of $\mathrm{IOut}(\widehat{L}H/I)$. The form η satisfies the equation

$$d\eta+\tfrac{1}{2}[\eta,\eta]=0.$$

Lemma 3.8 *Let $\varphi\in\mathrm{Out}(\pi)$ be an automorphism of π which satisfies $|\varphi|(I)=I$. The Maurer–Cartan form η on $\bar{\Theta}(\pi,I)$ satisfies*

$$\varphi^*\eta=\mathrm{Ad}(|\varphi|)\eta.$$

Proof For $\theta_0 \in \bar{\Theta}(\pi, I)$, we will denote $q_{\theta_0}\colon \bar{\Theta}(\pi, I) \to \mathrm{IOut}(\hat{L}H/I)$ by $\theta \mapsto \theta \circ \theta_0^{-1}$ and the Maurer–Cartan form on $\mathrm{IOut}(\hat{L}H/I)$ by

$$\tilde{\eta} \in A^*(\mathrm{IOut}(\hat{L}H/I); \mathrm{ODer}^+(\hat{L}H/I)).$$

Then $\eta = q_{\theta_0}^* \tilde{\eta}$ and it is independent of the choice of θ_0. Since $q_{\theta_0} \circ \varphi = \mathrm{Ad}(|\varphi|) \circ q_{\varphi\cdot\theta_0}$,

$$\varphi^* \eta = (q_{\theta_0} \circ \varphi)^* \tilde{\eta} = q_{\varphi\cdot\theta_0}^* \,\mathrm{Ad}(|\varphi|)^* \tilde{\eta} = \mathrm{Ad}(|\varphi|) q_{\varphi\cdot\theta_0}^* \tilde{\eta} = \mathrm{Ad}(|\varphi|)\eta.$$

Thus we complete the proof. □

From Lemma 3.8, the map $C^*(\mathrm{ODer}^+(\hat{L}H/I)) \to A^*(\bar{\Theta}(\pi, I))^{\mathrm{IOut}(\pi)}$ can be defined by

$$c \mapsto c(\eta^m)$$

for any $c \in C^m(\mathrm{ODer}^+(\hat{L}H/I))^{\mathrm{GL}(\pi)}$. Here the power η^m of η is defined by the product

$$\begin{aligned}(A^*(X) \otimes \mathrm{ODer}^+(\hat{L}H/I))^{\otimes m} \\ = A^*(X)^{\otimes m} \otimes \mathrm{ODer}^+(\hat{L}H/I)^{\otimes m} \to A^*(X) \otimes \mathrm{ODer}^+(\hat{L}H/I)^{\otimes m}\end{aligned}$$

which is the wedge product with respect to $A^*(X)$–components. We set

$$\eta = \sum \eta_\lambda D_\lambda,$$

using a (topological) basis $\{D_\lambda\}_{\lambda\in\Lambda}$ of $\mathrm{ODer}^+(\hat{L}H/I)$ and a well-ordered set Λ. Then we can write

$$\eta^m = \sum_{\lambda_1 < \cdots < \lambda_m} \eta_{\lambda_1} \wedge \cdots \wedge \eta_{\lambda_m} D_{\lambda_1} \wedge \cdots \wedge D_{\lambda_m} \in A^*(X) \otimes \bigwedge^m \mathrm{ODer}^+(\hat{L}H/I).$$

Thus we can define the above linear map $C^*(\mathrm{ODer}^+(\hat{L}H/I)) \to A^*(\bar{\Theta}(\pi, I))^{\mathrm{IOut}(\pi)}$. This map is a chain map. In fact, we have

$$\begin{aligned} d(c(\eta^m)) &= \sum_{s=1}^m (-1)^{s-1} c(\underbrace{\eta \cdots \overset{s}{d\eta} \cdots \eta}_{m}) \\ &= \sum_{s=1}^m \frac{(-1)^s}{2} c(\underbrace{\eta \cdots \overset{s}{[\eta, \eta]} \cdots \eta}_{m}) \\ &= (dc)(\eta^{m+1}). \end{aligned}$$

Combining it with the result in Section 3A, we obtain the following:

Theorem 3.9 *Let $E \to B$ be a fiber bundle whose fiber is X and structure group is $\mathcal{T}(X)$. Fix a decomposable ideal satisfying $\bar{\Theta}(\pi, I_0) \neq \varnothing$. Then the map*

$$H^*(\mathrm{ODer}^+(\hat{L}H/I_0)) \to H^*_{\mathrm{DR}}(\bar{\Theta}(\pi, I_0))^{\mathrm{IOut}(\pi)} \to H^*_{\mathrm{DR}}(B)$$

is an invariant of an oriented fiber bundle $E \to B$.

If $I_0 = I_\pi$ is the homogeneous ideal, the group $\mathrm{GL}(\pi)$ acts on $\mathrm{ODer}^+(\hat{L}H/I_\pi)$ by the adjoint action. So we define by $C^*(\mathrm{ODer}^+(\hat{L}H/I_\pi))^{\mathrm{GL}(\pi)}$ the chain complex of $\mathrm{GL}(\pi)$–invariant Chevalley–Eilenberg cochains of $\mathrm{ODer}^+(\hat{L}H/I_\pi)$ with respect to the action. In the same way as the construction above, we obtain the chain map

$$C^*(\mathrm{ODer}^+(\hat{L}H/I_\pi))^{\mathrm{GL}(\pi)} \to A^*(\bar{\Theta}(\pi, I_\pi))^{\mathrm{Aut}(\pi)}.$$

We have the following theorem using the construction in Section 3B:

Theorem 3.10 *Let $E \to B$ be a fiber bundle whose fiber is X and structure group is a subgroup $\mathcal{G}$ of the diffeomorphism group $\mathrm{Diff}_+(X)$. We set $\pi = \pi_1(X)$, $\tilde{S} = \mathrm{Im}(\mathcal{G} \to \mathrm{Out}(\pi))$, and $S = \mathrm{Im}(\mathcal{G} \to \mathrm{GL}(\pi))$. We suppose the group π has the homogeneous ideal I_π and there exists a splitting*

$$\mathrm{Der}^+(\hat{L}H) = V \oplus \mathrm{Der}^+_{I_\pi}(\hat{L}H)$$

as graded S–vector spaces. Then the map

$$H^*(\mathrm{ODer}^+(\hat{L}H/I_\pi))^S \to H^*_{\mathrm{DR}}(\bar{\Theta}(\pi, I_\pi))^{\tilde{S}} \to H^*_{\mathrm{DR}}(B)$$

is an invariant of an oriented fiber bundle $E \to B$.

4 Closed surface bundles

Let $\mathcal{M}_g$ be the mapping class group of an oriented closed surface Σ_g of genus $g \geq 2$ and let $\mathcal{T}_g$ be the Teichmüller space of genus g, which is the space Met_g of metrics which have constant curvature -1 on Σ_g modulo $\mathrm{Diff}_0(\Sigma_g)$. The quotient orbifold $\mathbb{M}_g := \mathcal{T}_g/\mathcal{M}_g$ is called the moduli space of genus g. Since $\mathcal{T}_g$ is contractible, we have the canonical isomorphism

$$H^*(\mathbb{M}_g; \mathbb{R}) \simeq H^*(\mathcal{M}_g; \mathbb{R}).$$

We denote the first real homology group $H_1(\Sigma_g; \mathbb{R})$ of Σ_g by H and the intersection form on Σ_g by ω. The alternating form ω can be regarded as an element of L_2 by the identity

$$\omega = \sum_{i=1}^{g} [X_i, X_{i+g}],$$

where $X_1, \dots, X_{2g}$ is any symplectic basis of H.

Lemma 4.1 [12] *There exist identifications*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{IDer}^1(LH/(\omega)) \arrow[r] \arrow[d, equal] & \mathrm{Der}^1(LH/(\omega)) \arrow[r] \arrow[d, equal] & \mathrm{ODer}^1(LH/(\omega)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H \arrow[r, "\cdot\wedge\omega"] & \Lambda^3 H \arrow[r] & \Lambda^3 H/H \arrow[r] & 0
\end{tikzcd}$$

According to [6], all Morita–Miller–Mumford classes of $\mathcal{M}_g$ are constructed from the twisted cohomology class $[\tilde{k}] \in H^1(\mathcal{M}_g; \Lambda^3 H/H)$, which is the cohomology class of a crossed homomorphism $\tilde{k}\colon \mathcal{M}_g \to \Lambda^3 H/H$ defined in [11], through the map $\alpha^{\mathrm{KM}}\colon \mathrm{Hom}(\Lambda^*(\Lambda^3 H/H), \mathbb{R})^{\mathrm{Sp}(H)} \to H^*(\mathcal{M}_g; \mathbb{R})$ defined by

$$c \mapsto c([\tilde{k}]^m)$$

for $c \in \mathrm{Hom}(\Lambda^m(\Lambda^3 H/H), \mathbb{R})^{\mathrm{Sp}(H)}$. Here the power is related to the cup product. We shall describe a relation between the map α_{KM} and our construction.

Since every hyperbolic metric on Σ_g admits a Kähler structure, the corresponding ideal of a hyperbolic metric is (ω). So we can define the map $\theta_C\colon \mathcal{T}_g \to \bar{\Theta}(\pi, (\omega))$ by giving Chen expansions corresponding to hyperbolic metrics. It can be constructed by the argument which is an analogue for the case of 1–punctured surfaces in [5]. Pulling back the flat connection $\eta \in A^1\big(\bar{\Theta}(\pi, (\omega)); \mathrm{ODer}^+(\hat{L}H/(\omega))\big)$ defined by the action of $\mathrm{IOut}(\hat{L}H/(\omega))$ on $\bar{\Theta}(\pi, (\omega))$, we obtain the flat connection

$$\theta_C^* \eta \in A^1\big(\mathbb{M}_g; \mathcal{T}_g \times_{\mathcal{M}_g} \mathrm{ODer}^+(\hat{L}H/(\omega))\big).$$

Since

$$\theta_C^*\colon A^*\big(\bar{\Theta}(\pi, (\omega))\big)^{\mathcal{M}_g} \to A^*(\mathcal{T}_g)^{\mathcal{M}_g} = A^*(\mathbb{M}_g),$$

we obtain

$$\theta_C^*(c(\eta_1^m)) = c([\theta_C^* \eta_1]^m) = (-1)^m c([\tilde{k}]^m) = (-1)^m \alpha^{\mathrm{KM}}(c) =: \bar{\alpha}^{\mathrm{KM}}(c)$$

for $c \in \mathrm{Hom}(\Lambda^*(\Lambda^3 H/H), \mathbb{R})^{\mathrm{Sp}(H)} \subset Z^*(\mathrm{ODer}^+(\hat{L}H/(\omega)))^{\mathrm{Sp}(H)}$. Here we use

$$-[\theta_C^* \eta_1] = [\tilde{k}] \in H^1(\mathcal{M}_g; \Lambda^3 H/H) \simeq H^1(\mathbb{M}_g; \mathcal{T}_g \times_{\mathcal{M}_g} \Lambda^3 H/H),$$

where η_1 is the $\mathrm{ODer}^1(\hat{L}H/(\omega))$–component of η. It is proved as follows: The crossed homomorphism τ_1^θ for any $\theta \in \bar{\Theta}(\pi, (\omega))$ is an extension of the first Johnson homomorphism $\tau_1\colon \mathcal{I}_g \to \Lambda^3 H/H$ from Proposition 2.5. The equation

$$[\tilde{k}] = [\tau_1^\theta] \in H^1(\mathcal{M}_g; \Lambda^3 H/H)$$

follows from the result of [11] (see [4]). The relation between the holonomy τ^θ and the iterated integral of the flat connection η gives

$$\tau^\theta(\varphi)^{-1} = \sum_{n=0}^{\infty} \int_\theta^{\varphi\cdot\theta} \underbrace{\eta\cdots\eta}_{n}$$

for $\varphi \in \mathcal{M}_g$ and $\theta \in \bar{\Theta}(\pi,(\omega))$. Therefore the correspondence follows from the equation

$$\tau_1^{\theta_C(x)}(\varphi) = -\int_{\theta_C(x)}^{\varphi\cdot\theta_C(x)} \eta_1 = -\int_x^{\varphi_* x} \theta_C^* \eta_1$$

for $\varphi \in \mathcal{M}_g$ and a fixed point $x \in \mathcal{T}_g$. So we obtain $[\widetilde{k}] = [\tau_1^{\theta_C(x)}] = -[\theta_C^*\eta_1]$.

Thus we get the theorem:

Theorem 4.2 *The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Hom}(\Lambda^*(\Lambda^3 H/H),\mathbb{R})^{\mathrm{Sp}(H)} \arrow[r] \arrow[drr, "\bar{\alpha}^{KM}"'] & H^*(\mathrm{ODer}^+(\hat{\mathcal{L}}H/(\omega)))^{\mathrm{Sp}(H)} \arrow[r, "\theta_C^*"] & H^*_{\mathrm{DR}}(\mathbb{M}_g) \arrow[d, equal] \\
& & H^*(\mathcal{M}_g;\mathbb{R})
\end{tikzcd}$$

The map θ_C^* can be interpreted from the viewpoint of our characteristic map as follows. We consider the oriented Σ_g–bundle

$$\mathrm{Met}_g \times_{\mathrm{Diff}_+(\Sigma_g)} \Sigma_g \to \mathbb{M}_g.$$

We note that $\mathrm{Met}_g \times_{\mathrm{Diff}_+(\Sigma_g)} \Sigma_g \to \mathbb{M}_g$ is not a fiber bundle exactly since the fiber on $[x] \in \mathbb{M}_g$, $x \in \mathrm{Met}_g$, is isomorphic to the global quotient orbifold $\Sigma_g/\,\mathrm{Isom}(\Sigma_g,x)$, where $\mathrm{Isom}(\Sigma_g,x)$ is the isometry group of Riemann surface (Σ_g,x). However our construction of the characteristic map works as well as fiber bundles. We give the tautological metric μ to this bundle, ie the metric μ_c on the fiber of c represents the class c for all $c \in \mathbb{M}_g$.

The chain map constructed from the Chen expansion of μ in the manner of Section 3B is equal to $\theta_C^*\colon A^*\big(\bar{\Theta}(\pi,(\omega))\big)^{\mathcal{M}_g} \to A^*(\mathbb{M}_g)$.

Theorem 4.3 *The homomorphism* $\theta_C^*\colon H^*\big(\mathrm{ODer}^+(\hat{\mathcal{L}}H/(\omega))\big)^{\mathrm{Sp}(H)} \to H^*_{\mathrm{DR}}(\mathbb{M}_g)$ *is the characteristic map of the fiber bundle* $\mathrm{Met}_g \times_{\mathrm{Diff}_+(\Sigma_g)} \Sigma_g \to \mathbb{M}_g$ *constructed in Section 3B.*

The theorem above means our characteristic maps are nontrivial.

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The simple loop conjecture for 3–manifolds modeled on Sol

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The simple loop conjecture for 3–manifolds states that every 2–sided immersion of a closed surface into a 3–manifold is either injective on fundamental groups or admits a compression. This can be viewed as a generalization of the loop theorem to immersed surfaces. We prove the conjecture in the case that the target 3–manifold admits a geometric structure modeled on Sol.

57M35; 57M50

1 Introduction

The simple loop conjecture for 3–manifolds is as follows.

Conjecture [7, Problem 3.96] *Let Σ be a closed surface and let M be a closed 3–manifold. If $F\colon \Sigma \to M$ is a 2–sided immersion for which the induced map $F_*\colon \pi_1\Sigma \to \pi_1 M$ is not injective, then there is an essential simple loop in Σ that represents an element of the kernel of F_*.*

When the map F is an embedding, this follows from the loop theorem of Papakyriakopoulos (see, for instance, Hempel [6]).

The simple loop conjecture is known to hold when the target 3–manifold is a Seifert fibered 3–manifold or a graph 3–manifold, by the work of Hass [4] and Rubinstein and Wang [11],[1] respectively. An analogous result for maps between surfaces is due to Gabai [3].

The goal of this paper is the following result.

Theorem 1 *The simple loop conjecture holds when the target 3–manifold admits a geometric structure modeled on* Sol.

[1] It is unclear whether the techniques of [11] apply to Sol manifolds, though they seem to be implicitly ruling them out (see, for instance, [11, Lemma 1.0.2]). At any rate, the techniques in this paper offer a substantially different approach to the problem.

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If M is a 3–manifold that is finitely covered by a torus bundle over S^1, then M admits a geometric structure modeled on one of Euclidean 3–space, Nil, or Sol. Since all compact Euclidean and Nil manifolds are Seifert fibered (see [12]), we obtain the following corollary.

Corollary 2 *The simple loop conjecture holds when the target 3–manifold is finitely covered by a torus bundle over S^1.*

This document is organized as follows. In Section 2 we give some definitions and notation for the objects that will be studied. Section 3 contains a brief survey of which compact 3–manifolds admit geometric structures modeled on Sol. This entails a refinement of a classification given by Scott in [12], and reduces the problem at hand to studying maps from closed surfaces into certain kinds of torus bundles over S^1 and orientable torus semi-bundles. In Sections 4 and 5 we give proofs of the simple loop conjecture for these two types of 3–manifold, respectively. We conclude in Section 6 with some remarks regarding how the results presented here relate to a group-theoretic formulation of the simple loop conjecture, and we show that it fails to hold when the target group is metabelian.

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2 Definitions

If M is a connected manifold, the *orientation character* of M is a homomorphism $\rho_M\colon \pi_1 M \to \mathbb{Z}/2$ whose value on $b \in \pi_1 M$ is nontrivial if and only if some (and hence any) loop in M representing b is orientation reversing. (Equivalently, $\rho_M(b)$ is nontrivial if and only if b acts on the universal cover of M by an orientation reversing homeomorphism.) A manifold is orientable if and only if its orientation character is trivial.

If M and N are connected manifolds with orientation characters ρ_M and ρ_N, a map $F\colon M \to N$ is called 2*–sided* if $\rho_N \circ F_* = \rho_M$. Otherwise F is 1*–sided*. Hence F is 2–sided if and only if it takes orientation preserving loops in M to orientation preserving loops in N, and likewise for orientation reversing loops. There are other (equivalent) definitions of 2–sidedness for immersions of manifolds, but since most of the arguments in this paper involve the fundamental groups of the manifolds in question, the given definition will be more useful.

We will call a loop in a manifold M *essential* if it is neither nullhomotopic nor homotopic into the boundary of M. Loops that are not essential will be called *inessential*.

For a space X, we write $|X|$ to denote the number of connected components of X. For a compact surface Σ with $L \subset \Sigma$ an embedded closed 1–manifold, we will write $\Sigma \setminus\!\!\setminus L$ to denote the metric completion of $\Sigma \setminus L$ (with respect to some choice of complete metric on Σ). Thus $\Sigma \setminus\!\!\setminus L$ is the space obtained by gluing copies of S^1 onto the open ends of $\Sigma \setminus L$.

We refer the reader to [12] for an explanation of what it means for a manifold to admit a geometric structure, as well as some basic facts about the Euclidean, Nil, and Sol geometries. In particular, we will need the following two results.

Theorem 3 [12, Theorem 5.2] *If M is a closed 3–manifold which admits a geometric structure modeled on one of the eight geometries, then the geometry involved is unique.*

Corollary 4 (see [12, Theorem 5.3(ii)]) *If M is a closed 3–manifold that admits a Seifert fibering, then M does not admit a geometric structure modeled on* Sol.

2.1 Torus bundles and semi-bundles

By *torus bundle* we mean a fiber bundle over S^1 whose fibers are tori. This can also be viewed as a quotient $T \times I/((p,0) \sim (\phi(p),1))$, where T is a torus and $\phi\colon T \to T$ is a homeomorphism.

For each $i \in \{1,2\}$, let N_i be either a twisted I–bundle over a torus or a Klein bottle, so that $\partial N_i \cong T$. A *torus semi-bundle* $M = N_1 \cup_\phi N_2$ is obtained by gluing N_1 and N_2 by a homeomorphism $\phi\colon \partial N_1 \to \partial N_2$. Such a 3–manifold is orientable if and only if both N_1 and N_2 are twisted I–bundles over a Klein bottle.

If M is a torus semi-bundle, at times we will refer to the *middle torus* of M, which is the image of ∂N_1 and ∂N_2 after gluing. We will also make use of maps $\rho_i\colon \pi_1 N_i \to \mathbb{Z}/2$, which are the quotients of $\pi_1 N_i$ by the index two subgroup corresponding to the double covers of N_i by the product $T \times I$. (This is sometimes called the *monodromy* of the I–bundle N_i.) Notice that for $b \in \pi_1 N_i$, $\rho_i(b)$ is trivial if and only if b is represented by a loop that is homotopic into ∂N_i. Furthermore, when N_i is a twisted I–bundle over a torus (and is therefore nonorientable), ρ_i coincides with the orientation character of N_i.

If M is a torus semi-bundle, then there is a double cover of M that is the union of the two $T \times I$ double covers of N_1 and N_2 along their boundaries (via some homeomorphism of the torus). This is a torus bundle over a circle, and is in turn covered by $T\times\mathbb{R}$ with deck group $\mathbb{Z}$. Hence M is covered by $T\times\mathbb{R}$ with deck group the *infinite dihedral group* $D=\langle g_1,g_2 \mid g_1^2=g_2^2=1\rangle$. The induced action on $\mathbb{R}$ is the usual discrete action of D on $\mathbb{R}$, where g_1 and g_2 act by reflections about 0 and 1, respectively. The projection $T\times\mathbb{R}\to\mathbb{R}$ therefore induces a projection $M\to I(2,2)$, where $I(2,2)$ is a 1–dimensional orbifold called the *mirrored interval*. (See [1] for definitions and notation.) It follows that M can be viewed as an *orbifold fiber bundle* over $I(2,2)$. The generic fibers of this bundle are 2–sided tori in M, and the fibers over the mirrored points are the 1–sided tori or Klein bottles of M.

3 Classification of compact 3–manifolds modeled on Sol

In [12], Scott gives the following classification of closed 3–manifolds modeled on Sol. (Note that a homeomorphism $\phi\colon T\to T$ of a torus is called *hyperbolic* if ϕ_* acts on $H_1(T;\mathbb{Z})$ with $\mathrm{tr}(T)^2>4$.)

Theorem 5 [12, Theorem 5.3(i)] *Let M be a closed 3–manifold. Then M possesses a geometric structure modeled on* Sol *if and only if M is finitely covered by a torus bundle over S^1 with hyperbolic monodromy. In particular, M itself is either a bundle over S^1 with fiber the torus or Klein bottle or is the union of two twisted I–bundles over the torus or Klein bottle.*

We refine this classification as follows.

Theorem 6 *Let M be a closed 3–manifold. Then M possesses a geometric structure modeled on* Sol *if and only if one of the following holds:*

(1) *M is a torus bundle over S^1 with hyperbolic monodromy.*

(2) *M is an orientable torus semi-bundle with gluing map (in canonical coordinates) given by $\left(\begin{smallmatrix} r & s \\ t & u\end{smallmatrix}\right)$, where $rstu\neq 0$.*

The notion of *canonical coordinates* on the middle torus of a torus semi-bundle is explained in the definition that precedes Proposition 1.5 of [14].

Proof It is shown in [14] that an orientable torus semi-bundle admits a Sol structure if and only if its gluing map is of the form stated above. Hence to complete the proof we must show that the other types of 3–manifolds mentioned in Scott's classification do *not* admit geometric structures modeled on Sol.

Case 1 (M is a Klein bottle bundle over S^1) Let

$$B = \langle a, b \mid aba^{-1}b = 1\rangle$$

be the fundamental group of a Klein bottle, and let $A = \langle a^2, b\rangle \approx \mathbb{Z} \oplus \mathbb{Z}$ be the normal subgroup of B corresponding to the double cover of the Klein bottle by a torus. The fundamental group of M has the form

$$\pi_1 M = \langle B, t \mid txt^{-1} = \phi(x),\ \forall x \in B\rangle$$

for some automorphism ϕ of B coming from a homeomorphism of the Klein bottle.

We now show that every such automorphism of B preserves the subgroup A. We first observe that every element of B can be written uniquely as $a^i b^j$ for $i, j \in \mathbb{Z}$. Since ϕ must preserve the commutator subgroup $[B, B] = \langle b^2\rangle$, we have $\phi(b^2) = b^{\pm 2}$, and a short computation shows that in fact $\phi(b) = b^{\pm 1}$. It follows that $\phi(a) = a^i b^j$, where $i, j \in \mathbb{Z}$ and i is odd, since otherwise ϕ has image in the proper subgroup A. We have

$$\phi(a^2) = (a^i b^j)(a^i b^j) = (a^i a^i)(b^{-j} b^j) = a^{2i},$$

and similarly $\phi^{-1}(a^2) = a^{2i'}$ for some $i' \in \mathbb{Z}$. From $a^2 = \phi^{-1}(\phi(a^2)) = a^{2i\cdot i'}$ we find that $i \cdot i' = 1$, and so $i = \pm 1$. In summary, $\phi(b) = b^{\pm 1}$ and $\phi(a^2) = a^{\pm 2}$, so ϕ preserves the subgroup A.

We therefore conclude that $\pi_1 M$ contains an index 2 subgroup of the form

$$H = \langle A, t \mid txt^{-1} = \phi|_A(x),\ \forall x \in A\rangle.$$

Let $\widehat{M}$ be the double cover of M corresponding to H, which is a torus bundle over S^1 with monodromy $\phi|_A$. By the argument in the previous paragraph, there is a choice of basis for A so that

$$\phi|_A = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Therefore $\phi|_A$ corresponds to a periodic homeomorphism of the torus, and so $\widehat{M}$ admits a Euclidean structure by [12, Theorem 5.5]. It follows that M does not admit a Sol structure, for if it did the structure could be lifted to a Sol structure on $\widehat{M}$, which would violate Theorem 3.

Case 2 (M is a Klein bottle semi-bundle) Then M is double covered by a Klein bottle bundle over S^1 and therefore has a degree 4 cover that is a torus bundle over S^1 that admits a Euclidean structure. As in the previous case, M does not admit a Sol structure.

Case 3 (M is a nonorientable torus semi-bundle) Then M is the union of two twisted I–bundles N_1 and N_2 over a torus or Klein bottle, at least one of which (say N_1) is an I–bundle over a torus. We will show that M admits a Seifert fibering, and therefore does not admit a Sol structure by Corollary 4.

Choose an arbitrary Seifert fibration for N_2; up to isomorphism there are precisely two of these when N_2 is an I–bundle over a Klein bottle (see [5], for instance) and infinitely many when N_2 is an I–bundle over a torus, as we will show.

If T is a torus, then for any $p/q \in \mathbb{Q} \cup \{\infty\}$, T can be foliated by p/q–curves. This foliation extends to the product Seifert fibration of $T \times I$ by p/q–curves in each torus $T \times \{t\}$. Finally, since the covering involution corresponding to the cover $T \times I \to N_1$ preserves the fibration on $T \times I$, it descends to a Seifert fibration of N_1 so that ∂N_1 is foliated by p/q curves. Note that this is the one of the "generalized" Seifert fibrations as defined in [12], as the critical fibers are not isolated. In fact, the one-sided torus in N_1 forms a subsurface of critical fibers.

It follows that a Seifert fibration on M can be constructed by choosing a Seifert fibration on N_1 so that the foliation of the boundary agrees with the image of the foliation of ∂N_2 under the gluing map. □

4 Torus bundles

The first of the two main theorems that will imply Theorem 1 is the following.

Theorem 7 *If M is a torus bundle, then the simple loop conjecture holds for M.*

In fact, a slightly stronger result holds for most surfaces.

Theorem 8 *Let Σ be a closed surface and let M be a torus bundle. If $\chi(\Sigma)$ is even and negative and $F\colon \Sigma \to M$ is a 2–sided map, then there is an essential simple loop in Σ that represents an element of* $\ker F_*$. *If $\chi(\Sigma)$ is odd then there is no 2–sided map $\Sigma \to M$.*

After we prove Theorem 8, to complete the proof of Theorem 7 it will remain to handle the two cases where $\chi(\Sigma) = 0$. The simple loop conjecture is known to hold for maps $\Sigma \to M$ where Σ is a torus and M is any 3–manifold [4, Section 4.4], and Proposition 11 will deal with the case in which Σ is a Klein bottle.

Let L be a (not necessarily connected) 1–submanifold of a surface Σ and let α be an arc in Σ with endpoints on L and interior disjoint from L. Then *surgery* of L along α entails fattening α to a strip $I \times I$ with $L \cap (I \times I) = \partial I \times I$, deleting the interior of $\partial I \times I$ from L, and gluing in $I \times \partial I$ to L. Notice that if α is an arc between two

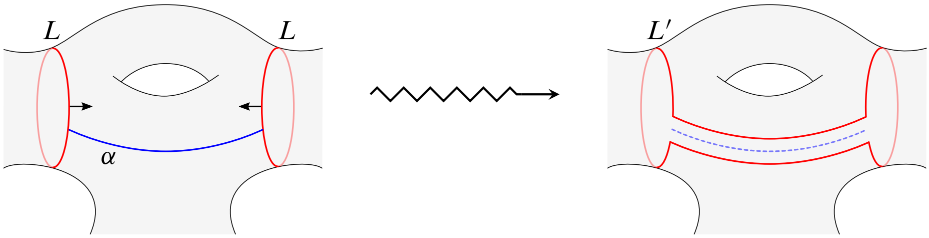

Figure 1: Surgery along α reduces the number of components of L by one

distinct components of L, then the result of surgery along α is to connect the two components of L by a bridge, as shown in Figure 1.

The following can be established by a standard homotopy argument.

Lemma 9 *Let Σ be a (not necessarily closed) surface, let J denote the open interval $(0,1)$, and let $H\colon \Sigma \to J$ be a map that is transverse to a point $r \in J$. If α is an arc that connects two components of $L = H^{-1}(r)$ whose interior is disjoint from L, then H can be homotoped in a neighborhood of α so that the preimage of r changes by surgery along α.*

Lemma 10 *Let Σ be a closed surface, let $G\colon \Sigma \to S^1$ be a π_1–surjective map, and choose $q \in S^1$. Then G can be homotoped so that the preimage $L = G^{-1}(q)$ is an essential 2–sided simple loop in Σ.*

Proof Choose G within its homotopy class so that q is a regular value of G and $L = G^{-1}(q)$ is a collection of disjoint simple loops in Σ with a *minimal* number of components. Observe that L is 2–sided but may not be connected. We shall show that the minimality assumption on L along with the assumption that G is π_1–surjective forces L to be connected.

Choose a co-orientation of $q \in S^1$ and pull it back to a co-orientation of L in Σ. We summarize this data by drawing a single arrow orthogonal to each component of L that indicates to which side of each component the co-orientation is pointing, as demonstrated in Figures 1 and 2. When we cut Σ along L to obtain $\Sigma \setminus\!\setminus L$, we label the boundary components of the resulting surface with the co-orientations of the components of the L that the boundary components correspond to.

We can homotope G to reduce the number of components of L whenever a component Σ_0 of $\Sigma \setminus\!\setminus L$ has two boundary loops that are either *both co-oriented into* or *both co-oriented out of* Σ_0. This happens, for instance, whenever Σ_0 has three or more boundary components. Start by choosing a simple arc $\alpha \subset \Sigma_0$ connecting the two boundary components of Σ_0 with coherent co-orientations, so that $G(\alpha)$ is a null-homotopic loop in S^1 based at q. If U is a small neighborhood of α in Σ, then we can homotope G with support in U so that $G|_U$ is not surjective. Hence $G|_U$ has image in a subset of S^1 homeomorphic to $J = (0, 1)$, and so we may apply Lemma 9 to $G|_U$ to obtain a further homotopy of G supported in U. This has the effect of surgering L along α, which reduces of the number of components of L by one as shown in Figure 1.

Another reduction of L is possible if some component Σ_0 of $\Sigma \setminus\setminus L$ has only one boundary component. In this case, we homotope G by sending all of Σ_0 past q; this homotopy can be taken to be the identity outside of any neighborhood of Σ_0. If L' is the preimage of q after the homotopy, then L' consists of the same loops as L except for the loop that formed the boundary of Σ_0, which has been eliminated.

It follows that if G is chosen to minimize the number of components of L, then every component Σ_0 of $\Sigma \setminus\setminus L$ has exactly two boundary components: one co-oriented into Σ_0 and the other co-oriented out of Σ_0, as shown in Figure 2.

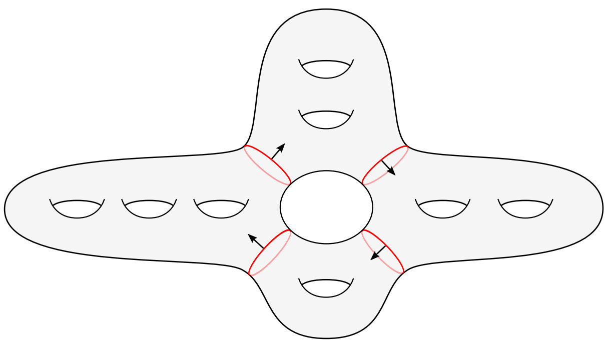

Figure 2: If L has more than one component, then no loop in Σ can have a signed intersection of ± 1 with L

We now observe that the homomorphism $G_*\colon \pi_1\Sigma \to \pi_1 S^1 \approx \mathbb{Z}$ is given by signed intersection with L, where the sign measures whether a loop in Σ agrees with the co-orientation of L. From the construction of the co-orientation we see that G_* must have image $|L|\mathbb{Z} \le \mathbb{Z}$. Since G_* is surjective, we have $|L| = 1$, and so L is connected. This completes the proof. $\square$

Proof of Theorem 8 Let $P\colon M \to S^1$ denote the bundle projection of M, and let $G = P \circ F\colon \Sigma \to S^1$.

Case 1 (the map G is π_1–surjective) Applying Lemma 10 to G, we may homotope G so that the preimage of a point $q \in S^1$ is a 2–sided simple loop $L \subset \Sigma$ for which any loop in $\Sigma \setminus\!\!\setminus L$ has inessential image under G. Since we have that $G(\Sigma \setminus\!\!\setminus L) \subset \{S^1 \setminus q\}$, we may use the homotopy lifting property of the fiber bundle $M \to S^1$ to homotope F so that $F(\Sigma \setminus\!\!\setminus L) \subset M \setminus M_q$, where M_q is the fiber of M lying above q.

Since $M \setminus M_q$ is homeomorphic to $T \times I$ and is therefore orientable, it follows from the 2–sidedness of F that $\Sigma \setminus\!\!\setminus L$ must be orientable. Therefore $\Sigma \setminus\!\!\setminus L$ is an orientable compact surface with two boundary components, and so $\chi(\Sigma \setminus\!\!\setminus L) = \chi(\Sigma)$ must be even. This proves the claim that there is no 2–sided map $\Sigma \to M$ when $\chi(\Sigma)$ is odd.

We may now suppose that $\chi(\Sigma) = 2-2g$, where $g \geq 2$ is an integer. Then $\chi(\Sigma \setminus\!\!\setminus L) = 2-2g$, so $\Sigma \setminus\!\!\setminus L$ is the connect sum of a twice-punctured sphere with $g-1$ tori. It follows that there is an embedded punctured torus Σ_0 in $\Sigma \setminus\!\!\setminus L$. The boundary loop β of Σ_0 is a separating simple loop in Σ whose corresponding element in $\pi_1\Sigma$ is the commutator of the elements represented by loops γ and δ, as shown in Figure 3.

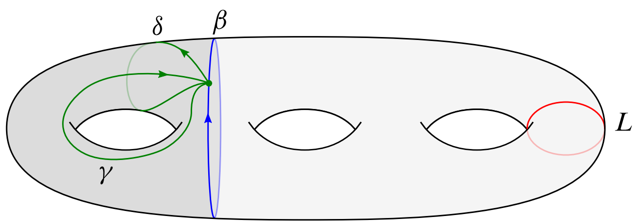

Figure 3: The simple loop β in $\ker F_*$ is the boundary of the punctured torus $\Sigma_0 \subset \Sigma$

The loops β, γ and δ all have image in $M \setminus M_q$, and since $M \setminus M_q$ has abelian fundamental group it follows that $F_*[\beta]$ is trivial in $\pi_1 M$. Thus β is the desired essential simple loop in the kernel of F_*. (A similar argument shows that any essential separating loop in $\Sigma \setminus\!\!\setminus L$ must represent an element of $\ker F_*$.)

Case 2 (the map G is not π_1–surjective) In this case, either G_* is the zero map or it has image $n\mathbb{Z} \leq \mathbb{Z} \approx \pi_1 S^1$ for some $n \neq 0, \pm 1$.

If G_* is the zero map, then G is homotopic to a constant map, and the homotopy can be lifted to a homotopy of F so that the resulting image of Σ is contained in a torus fiber M_p of M. Since M_p is an orientable 2–sided submanifold of M, by the 2–sidedness of F we have that Σ is orientable, and so $\chi(\Sigma)$ cannot be odd. If $\chi(\Sigma) \leq -2$ then there is an essential separating loop in Σ, and we argue as above that such a loop represents an element of $\ker F_*$.

If instead G_* has image a finite-index subgroup $n\mathbb{Z} \leq \mathbb{Z}$, then $p_*^{-1}(n\mathbb{Z})$ is a proper finite-index subgroup of $\pi_1 M$ and F lifts to the corresponding cover $\widetilde{M} \to M$. Since $\widetilde{M}$ must also be a torus bundle over a circle and the projection $\widetilde{M} \to M$ is π_1–injective, we may replace M by $\widetilde{M}$ and F by its lift and appeal to Case 1. □

The following result will complete the proof of Theorem 7.

Proposition 11 *Let K be a Klein bottle and let G be an infinite torsion-free group. If $f\colon \pi_1 K \to G$ is a homomorphism with nontrivial kernel, then there is an essential simple loop in K that represents an element of* $\ker f$.

Proof We proceed by reducing to the case in which f has image an infinite cyclic subgroup of G. Write the fundamental group of K as

$$\pi_1 K = \langle a, b \mid aba^{-1}b = 1\rangle,$$

and let $H = \langle a^2, b\rangle \leq \pi_1 K$ be the index 2 subgroup of $\pi_1 K$ corresponding to the double cover of K by a torus. The kernel of $f|_H$ must be nontrivial: for if $x \in \ker f_*$ is not the identity then $x^2 \in H \cap \ker f_*$ is also not the identity. Hence $f|_H$ is a non-injective map from a rank 2 free-abelian group to a torsion-free group, and so the image of $f|_H$ is either trivial or infinite cyclic. If $f(H) = 1$, then since $f(a)^2 = f(a^2) = 1$ and M is torsion-free, $f(a)$ must be trivial. In this case f is the trivial map and we're done. If $f(H)$ is infinite cyclic, then $f(\pi_1 K)$ is a virtually infinite cyclic torsion-free group, and so must be infinite cyclic; see, for instance, [13, Theorem 5.12].

Therefore we may replace f by a surjective map $f'\colon \pi_1 K \to \mathbb{Z}$. Since S^1 is a $K(\mathbb{Z}, 1)$, there is a map $F\colon K \to S^1$ with $F_* = f'$, and so Lemma 10 can be applied to obtain an essential 2–sided simple loop $L \subset K$ such that every loop in $K \setminus L$ has inessential image in S^1. Hence we see that $K \setminus L$ is an annulus, the core of which is an essential simple loop in K that represents an element of $\ker f'$, and hence of $\ker f$. □

5 Torus semi-bundles

The following theorem, together with Theorem 7, will establish Theorem 1.

Theorem 12 *If M is an orientable torus semi-bundle that admits a geometric structure modeled on* Sol*, then the simple loop conjecture holds for M.*

As in the torus bundle case, we have a slightly stronger statement for maps from surfaces of sufficiently large genus into orientable torus semi-bundles.

Theorem 13 *Let Σ be a closed surface and let M be an orientable torus semi-bundle. If $\chi(\Sigma) < -2$ and $F\colon \Sigma \to M$ is a 2–sided map, then there is an essential simple loop in Σ that represents an element of* $\ker F_*$.

To prove the theorem, we will employ the following two lemmas, which allow us to homotope maps from surfaces to torus semi-bundles into a simplified position.

Lemma 14 *Let M be an orientable torus semi-bundle with middle torus $S \subset M$, let Σ be a (not necessarily closed) surface, and let $F\colon \Sigma \to M$ be a map that is transverse to S. Suppose that $\alpha \subset \Sigma$ is a simple arc that connects two distinct components of $L = F^{-1}(S)$ whose interior is disjoint from L, and that $F(\alpha)$ is homotopic (rel endpoints) into S. Then F can be homotoped in a neighborhood of α so that the preimage of S changes by surgery along α.*

Proof Let U be a tubular neighborhood of α in Σ that does not intersect any components of L except the two that are connected by α. Since $F(\alpha)$ is homotopic into S, after possibly shrinking U we can homotope F with support in U so that $F|_U$ has image that does not intersect either of the 1–sided surfaces that are the zero sections of the twisted I–bundles that were used to construct M.

It follows that $F|_U$ has image in a subset of M that is homeomorphic to $T \times J$, where T is a torus and $J = (0, 1)$. Let $P\colon T \times J \to J$ denote the projection onto the second factor, and let $r \in J$ be the image of S. Then $P \circ F|_U\colon U \to J$ satisfies the assumptions of Lemma 9, so we may apply it to obtain a homotopy of $P \circ F|_U$ after which L has been surgered along α. Since $T \times J \to J$ is a fiber bundle, we can lift the homotopy of $P \circ F|_U$ to a homotopy of $F|_U$, and from that we obtain a homotopy of F supported in U, as desired. □

Lemma 15 *Let M be an orientable torus semi-bundle with middle torus $S \subset M$, let Σ be a closed surface with $\chi(\Sigma) < 0$, and let $F\colon \Sigma \to M$ be a (2–sided) map that injects on simple loops (that is, there are no elements represented by simple loops in the kernel of F_*). Then F can be homotoped so that $L = F^{-1}(S)$ is either empty or is a collection of parallel 2–sided separating essential simple loops in Σ.*

Figure 4 shows a typical picture of $L \subset \Sigma$ when $L \neq \varnothing$.

Proof In the notation of Section 2.1, let $M = N_1 \cup_\phi N_2$ with monodromies

$$\rho_i\colon \pi_1 N_i \to \mathbb{Z}/2.$$

Choose F within its homotopy class so that F is transverse to S and so that $L = F^{-1}(S)$ is a *minimal* collection of 2–sided simple loops in Σ.

Step 1 First, suppose that some component Σ_0 of $\Sigma \setminus\setminus L$ has three or more boundary components. Let C_1, C_2, C_3 be three of the boundary components of Σ_0. (Since S

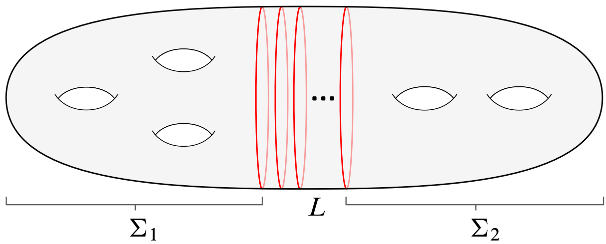

Figure 4: The multicurve L is a collection of parallel loops separating Σ into a collection of annuli along with two punctured surfaces, Σ_1 and Σ_2

separates M, no two of the C_i correspond to the same component of L.) Choose a basepoint $q \in S$; after a homotopy of F supported in a tubular neighborhood of the C_i, we may assume that each C_i contains a point p_i for which $F(p_i) = q$. In Σ_0 choose simple arcs α from p_1 to p_2, α' from p_2 to p_3, and α'' from p_1 to p_3 such that α'' is path-homotopic to the concatenation of α and α', as shown in Figure 5.

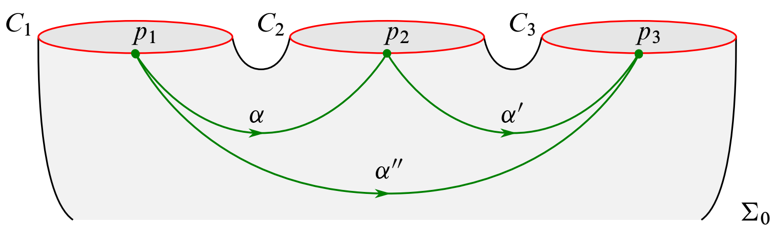

Figure 5: The arcs α, α', and α'' joining the boundary components of Σ_0

By construction, each of $F(\alpha)$, $F(\alpha')$ and $F(\alpha'')$ are loops in M based at q, and without loss of generality all three lie in N_1. It follows that $\rho_1[F(\alpha)]$, $\rho_1[F(\alpha')]$ and $\rho_1[F(\alpha'')]$ are elements in $\mathbb{Z}/2$ with $\rho_1[F(\alpha)] + \rho_1[F(\alpha')] = \rho_1[F(\alpha'')]$, and so one of the three elements must be trivial in $\mathbb{Z}/2$. Hence one of the arcs (say α) in Σ_0 has image under F that is homotopic into $\partial N_1 = S$, and so by Lemma 14 we can homotope F so that the result on L is surgery along α, which reduces the number of components of L.

Step 2 Next, suppose that some component Σ_0 of $\Sigma \setminus\setminus L$ has two boundary components and is not an annulus. As in the previous step, we can homotope F in a neighborhood of $\partial\Sigma_0$ so that each boundary component has a point p_i $(i = 1, 2)$ that maps to the basepoint $q \in S$. Without loss of generality we assume that $F(\Sigma_0) \subset N_1$. There are two cases to consider.

Case 2A (there is a simple loop $\alpha \subset \Sigma_0$ based at p_1 with $\rho_1[F(\alpha)]$ nontrivial in $\mathbb{Z}/2$) Homotope α in Σ_0 so that α becomes the concatenation of two simple arcs α' and α'' from p_1 to p_2, as shown in Figure 6.

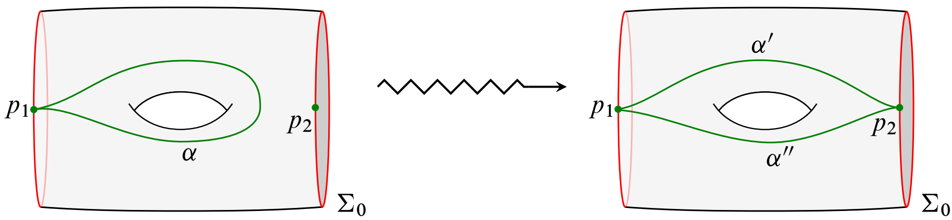

Figure 6: Pulling α towards p_2 and viewing it as two arcs

It follows that $F(\alpha')$ and $F(\alpha'')$ are loops in N_1 based at q, and since $\rho_1[F(\alpha')] + \rho_1[F(\alpha'')] = \rho_1[F(\alpha)]$ is nontrivial in $\mathbb{Z}/2$, one of $\rho_1[F(\alpha')]$ and $\rho_1[F(\alpha'')]$ must be trivial. As before, an arc with trivial image can be used (Lemma 14) to homotope F and surger L, which reduces the number of components of L by one.

Case 2B (for every simple loop $\alpha \subset \Sigma_0$ based at p_1, $\rho_1[F(\alpha)]$ is trivial) Since we assumed Σ_0 is not an annulus, it is a twice-punctured orientable surface of genus greater than 0. It follows that we can find two simple loops γ and δ in Σ_0 whose commutator in $\pi_1\Sigma_0$ is represented by a simple loop β; see Figure 7.

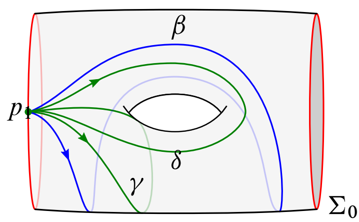

Figure 7: The simple loop β represents the commutator of $[\gamma]$ and $[\delta]$

Since $[\beta], [\gamma], [\delta] \in \pi_1\Sigma_0$, all have trivial image under $\rho_1 \circ F_*$, $\rho_1[F(\beta)]$, $\rho_1[F(\gamma)]$, and $\rho_1[F(\delta)]$ must lie in the subgroup of $\pi_1 N_1$ corresponding to the boundary S. But since $\pi_1 S$ is abelian, the commutator $F_*[\beta]$ is trivial. This contradicts the assumption that F injects on simple loops, and so it is impossible that $\rho_1 \circ F_*$ is trivial on every simple loop in Σ_0.

We conclude that the number of components of L can be reduced whenever some component of $\Sigma \setminus\!\setminus L$ has exactly two boundary components and is not an annulus.

Step 3 It follows from the previous two steps that if F is chosen in its homotopy class so that L has a minimal number of components, then L is either empty or every component of $\Sigma \setminus\!\!\setminus L$ is either an annulus or a surface with exactly one boundary component. The assumption that $\chi(\Sigma) < 0$ rules out the possibility that *every* component of $\Sigma \setminus\!\!\setminus L$ is an annulus, and so Σ consists of two punctured orientable surfaces connected by some number of annuli. □

Proof of Theorem 13 Let Σ be a closed surface with $\chi(\Sigma) < -2$, let $M = N_1 \cup_\phi N_2$ be a torus semi-bundle, and let $F \colon \Sigma \to M$ be a 2–sided map. By Lemma 15, we may assume that F has been homotoped so that $L = F^{-1}(S)$ is either empty or is a collection of parallel curves as in Figure 4. (According to the lemma, if this is not possible then we can already find a simple loop in $\ker F_*$.)

If $L = \varnothing$ then without loss of generality F has image in N_1, which is homotopy equivalent to a Klein bottle. Since $\pi_1 N_1$ does not contain the fundamental group of any surface of negative Euler characteristic, the induced map $\pi_1 \Sigma \to \pi_1 N_1$ has nontrivial kernel. Using Gabai's result [3], we conclude that there is a simple loop in the kernel of F_*.

We now consider the case in which $L \neq \varnothing$. If Σ_1 and Σ_2 are the two non-annular subsurfaces of Σ as shown in Figure 4, then

$$\chi(\Sigma_1) + \chi(\Sigma_2) = \chi(\Sigma).$$

It follows that either $\chi(\Sigma_1) < -1$ or $\chi(\Sigma_2) < -1$.

Without loss of generality, we will henceforth assume that $\chi(\Sigma_1) < -1$ and that $F(\Sigma_1) \subset N_1$.

If $f = \rho_1 \circ (F|_{\Sigma_1})_* \colon \pi_1(\Sigma_1) \to \mathbb{Z}/2$, then since F sends $\partial\Sigma_1$ (which is a component of L) into S, we have $f[\partial\Sigma_1] = 0$. It follows that f represents a class in $H^1(\Sigma_1, \partial\Sigma_1; \mathbb{Z}/2)$. If f represents the trivial class, then all of $F(\Sigma_1)$ is homotopic into S, and we can homotope F to send all of Σ_1 past S and reduce the number of components of L, contradicting the assumption that F has already been homotoped to minimize the number of components. Therefore f is nontrivial in $H^1(\Sigma_1, \partial\Sigma_1; \mathbb{Z}/2)$, and so by Lefschetz duality, there is a nontrivial homology class $f_* \in H_1(\Sigma_1; \mathbb{Z}/2)$ for which the value of f on any loop α based on $\partial\Sigma_1$ is given by the signed intersection (mod 2) of α with any 1–chain representing f_*.

Let ℓ be a simple loop in Σ_1 that represents f_*. (A simple loop representative exists by [10].) Since f_* is nontrivial, ℓ is essential and every loop in $\Sigma_1 \setminus \ell$ is in the kernel of f and therefore has image in N_1 that is homotopic into S. The fact that $\chi(\Sigma_1) < -1$ implies that $\Sigma_1 \setminus\!\!\setminus \ell$ is homeomorphic to a closed surface of genus at least one with three open discs removed. As in the proof of Theorem 8, we can find an embedded punctured torus P in $\Sigma_1 \setminus\setminus \ell$ whose boundary β represents the commutator of simple loops γ and δ contained in P. Since $[\beta]$, $[\gamma]$ and $[\delta]$ all have image under F_* in the abelian subgroup $\pi_1 S \leq \pi_1 M$, we conclude that β is the desired simple loop representing an element of $\ker F_*$. □

With Proposition 11 and the proof of the simple loop conjecture when the domain is a torus given in [4], we will complete the proof of Theorem 12 with the following special case.

Lemma 16 *Let Σ denote the closed orientable surface with $\chi(\Sigma) = -2$. If M is an orientable torus semi-bundle and $F\colon \Sigma \to M$ is a (2–sided) map, then either there is an essential simple loop in* $\ker F_*$ *or M does not admit a geometric structure modeled on* Sol.

Proof By Lemma 15, we can homotope F so that the preimage $L = F^{-1}(S)$ of the middle torus of M is a minimal collection of parallel curves in Σ as in Figure 4. As in the proof of Theorem 13 we may also assume that $L \neq \varnothing$, so L separates Σ into punctured tori Σ_1 and Σ_2 along with a collection of $n = |L| - 1$ annuli.

Case $n = 0$ In this case, L is connected and separates Σ into punctured tori Σ_1 and Σ_2. We can write the fundamental group of Σ as

$$\pi_1 \Sigma = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1] = [a_2, b_2] \rangle,$$

where a_i and b_i are the generators of the fundamental group of Σ_i. The fundamental group of M has presentation

$$\pi_1 M = \langle x_1, y_1, x_2, y_2 \mid x_i y_i x_i^{-1} y_i = 1, x_1^2 = x_2^{2r} y_2^t, y_1 = x_2^{2s} y_2^u \rangle,$$

where x_i and y_i are the generators of the fundamental group of the twisted I–bundle over a Klein bottle N_i, and M has been constructed by gluing N_1 to N_2 via a homeomorphism $\partial N_1 \to \partial N_2$ whose matrix is

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

with respect to the bases $\langle x_i^2, y_i \rangle$ of the fundamental groups of the boundaries of the N_i. By the definition of L we see that F restricts to a proper map of Σ_i into N_i, and so $F_*(a_i)$ and $F_*(b_i)$ must lie in $\langle x_i, y_i \rangle$ for $i = 1, 2$. The subgroup $\langle x_i, y_i \rangle$ of $\pi_1 M$ is isomorphic to the fundamental group of a Klein bottle, and its commutator subgroup is infinite cyclic with generator y_i^2. Hence the commutators $[a_i, b_i]$ are mapped to even powers of y_i, and from the relation in $\pi_1\Sigma$ we obtain an equation

$$y_1^{2k_1} = y_2^{2k_2}$$

for some integers k_1 and k_2. Applying the rightmost relation of the presentation of $\pi_1 M$ given above, we have

$$x_2^{4sk_1} y_2^{2uk_1} = y_2^{2k_2}.$$

Since this is an equation in $\langle x_2^2, y_2\rangle \approx \mathbb{Z}\oplus\mathbb{Z}$, we can conclude that $4sk_1 = 0$, and so either $k_1 = 0$ or $s = 0$. If $k_1 = 0$, it follows that the curve L (which represents the elements $[a_1, b_1]$ and $[a_2, b_2]$ in $\pi_1\Sigma$) has image $y_1^{2k_1} = 1$, so L is an essential simple loop in the kernel of F_*. If $s = 0$, then by Theorem 6 it follows that M does not admit a geometric structure modeled on Sol.

Case $n > 0$ In this case, L has multiple components; we will show that F can be lifted to a torus semi-bundle cover of M in which the preimage of the middle torus is connected, thereby reducing to the case in which $n = 0$. Choose points $p_0, \dots, p_n$ on the $n+1$ components of L, and let $\alpha \subset \Sigma$ be a simple arc with end points at p_0 and p_n whose intersection with L is the points p_i. For $i = 0, \dots, n-1$ let α_i denote the segment of α between p_i and p_{i+1}, as shown in Figure 8.

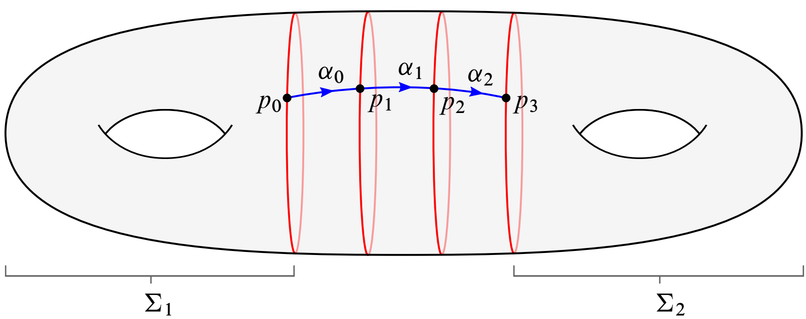

Figure 8: The arc α connecting the points p_i in the case $n = 3$

By adjusting F by a homotopy that preserves L, we may assume that $F(p_i) = q$ for some basepoint $q \in S \subset M$, and so $F(\alpha_i)$ is a loop in M based at q representing an element $w_i \in \pi_1 M$.

In the notation of the previous case, we assume that $F_*(a_1)$ and $F_*(b_1)$ lie in the subgroup $\langle x_1, y_1\rangle \le \pi_1 M$, and by the definition of L we have that $w_i \in \langle x_{j_i}, y_{j_i}\rangle$, where $j_i = 1$ if i is odd and $j_i = 2$ if i is even. We may also assume that $w_i \notin \langle x_{j_i}^2, y_{j_i}\rangle$, for if $w_i \in \langle x_{j_i}^2, y_{j_i}\rangle$ then α_i is a proper simple arc in a component $\Sigma \setminus\!\!\setminus L$ with image homotopic into S, and we can reduce the number of components of L, which contradicts the minimality assumption. If $w = w_0 \cdots w_{n-1}$, then we have

$$F_*(\pi_1 \Sigma) \le \langle x_1, y_1, w x_k w^{-1}, w y_k w^{-1} \rangle,$$

where $k = 1$ if n is odd and $k = 2$ if n is even.

If $D = \langle g_1, g_2 \mid g_1^2 = g_2^2 = 1 \rangle$ denotes the infinite dihedral group, then there is a homomorphism $f\colon \pi_1 M \to D$ given by $x_i \mapsto g_i$ and $y_i \mapsto 1$ for $i = 1, 2$. The cover of M corresponding to $\ker f$ is $T \times \mathbb{R}$ with deck group D, as described in Section 2.1. For each $i = 0, \dots, n-1$, since $w_i \notin \langle x_{j_i}^2, y_{j_i} \rangle$ we have $f(w_i) = g_{j_i}$, and it follows that $f(w)$ is a reduced word in D of length n starting with g_2. The image of $\pi_1 \Sigma$ under the composition $f \circ F_*$ is the subgroup

$$H = \langle g_1, f(w) g_k f(w)^{-1} \rangle \le D,$$

which itself is isomorphic to the infinite dihedral group. Let $\widehat{M}$ be the quotient of $S \times \mathbb{R}$ by H, which is another torus semi-bundle that is the cover of M corresponding to the subgroup $f^{-1}(H)$. Then $\widehat{M}$ contains $n+1$ tori $S_0, \dots, S_n$ that are lifts of S, and the result of splitting $\widehat{M}$ along these tori is n products $T \times I$ (each of which double-covers N_1 or N_2) along with two twisted I–bundles over a Klein bottle (each of which projects to N_1 or N_2 by a homeomorphism). The S_i are parallel and one can show that $\widehat{F}^{-1}(S_i)$ is connected for $i = 0, \dots, n$, where $\widehat{F}\colon \Sigma \to \widehat{M}$ is the lift of F to $\widehat{M}$. Hence we can take any of the S_i to be the "middle torus" of $\widehat{M}$.

Therefore we may apply the argument of the first case of this proof to $\widehat{F}$ to find either an essential simple loop in $\ker \widehat{F}_*$ or that $\widehat{M}$ is Seifert fibered. In the former case, an essential simple loop in $\ker \widehat{F}_*$ is also an essential simple loop in $\ker F_*$. In the latter, if $\widehat{M}$ is Seifert fibered then it carries a Euclidean or Nil structure, and therefore so does M. It follows that M is Seifert fibered as well. □

6 The simple loop conjecture for metabelian groups

An *orientation character* on a group G is a homomorphism $\rho_G\colon G \to \mathbb{Z}/2$, and an *oriented group* is a pair (G, ρ_G) where ρ_G is an orientation on G. When G is the fundamental group of a manifold M, we take ρ_G to be the orientation character ρ_M defined in Section 2. Similarly, one can say what it means for a homomorphism between two oriented groups to be 2*–sided*. It then seems natural to ask if the following generalization of the simple loop conjecture holds for a fixed oriented group G.

Statement *Let Σ be a closed surface and let (G, ρ_G) be an oriented group. If $f\colon \pi_1 \Sigma \to G$ is a 2–sided homomorphism that is not injective, then there is an essential simple loop in Σ that represents an element of the kernel of f.*

When G is the fundamental group of an aspherical 3–manifold this is equivalent to the simple loop conjecture for 3–manifolds. This statement is known to be false when $G = \mathrm{PSL}(2,\mathbb{C})$ by work of Cooper and Manning [2] and when $G = \mathrm{PSL}(2,\mathbb{R})$ by work of Mann [9]. (In both cases, G carries the trivial orientation character as it is identified with the groups of orientation-preserving isometries of hyperbolic 3– and 2–space, respectively.)

A group is called *metabelian* if it fits into a short exact sequence of the form

$$1 \to A \to G \to B \to 1,$$

where A and B are abelian groups. For example, the fundamental groups of the torus bundles treated in Section 4 are metabelian with $A = \mathbb{Z}\oplus\mathbb{Z}$ and $B = \mathbb{Z}$. One might be led to ask if the group-theoretic version of the simple loop conjecture holds for metabelian groups, and if a technique similar to that of Section 4 can be used to prove it. We provide the following result in this direction.

Theorem 17 *Let (G, ρ_G) be an oriented group that fits into an exact sequence of the form*

$$1 \to A \to G \to \mathbb{Z} \to 1,$$

where A is abelian, and suppose that $A \le \ker\rho_G$. If Σ is a closed surface of genus at least two, then the group-theoretic version of the simple loop conjecture holds for Σ and G.

Proof This is a group-theoretic analogue to the proof of Theorem 8. Let $p\colon G \to \mathbb{Z}$ denote the projection map in the short exact sequence. For a surface Σ and a 2–sided homomorphism $f\colon \pi_1\Sigma \to G$, we may assume that f is surjective. For if not, then either $f(\pi_1\Sigma)$ lies in A and any separating simple loop in Σ represents an element of $\ker f$, or $p\circ f$ has nontrivial image and we replace G by $f(\pi_1\Sigma)$, ρ_G by $(\rho_G)|_{f(\pi_1\Sigma)}$, A by $A\cap f(\pi_1\Sigma)$, and $\mathbb{Z}$ by $(p\circ f)(\pi_1\Sigma)\approx\mathbb{Z}$.

There is a map $\Sigma \to S^1$ whose induced homomorphism on fundamental groups is $p\circ f$, and by applying Lemma 10 to this map we find a simple nonseparating loop $L\subset\Sigma$ such that every element of $\pi_1(\Sigma \mathbin{\backslash\backslash} L) \le \pi_1\Sigma$ is contained in $\ker(p\circ f)$. By exactness, $f(\pi_1(\Sigma \mathbin{\backslash\backslash} L))$ is contained in A, and the assumptions that f is 2–sided and that $A\le\ker\rho_G$ imply that $\Sigma \mathbin{\backslash\backslash} L$ must be orientable.

As shown in the proof of Theorem 8 there are essential simple loops β, γ, and δ in Σ representing elements of $\ker(p\circ f)$ and with $[\beta]$ equal to the commutator of $[\gamma]$ and $[\delta]$. By exactness, $f[\beta]$, $f[\gamma]$ and $f[\delta]$ are contained in A, and since A is abelian we have that $f[\gamma]$ is trivial. □

We conclude by showing that, despite the previous result, the group-theoretic simple loop conjecture does not hold for *all* torsion-free metabelian groups. This is a torsion-free version of a finite example due to Casson [8, Section 2].

Example 18 Let Σ be a surface of genus $g \geq 2$. We will give a topological construction of the quotient of $\pi_1\Sigma$ by its second derived subgroup, which is sometimes called the *metabelianization* of $\pi_1\Sigma$. From the construction we will see that the kernel of $\pi_1\Sigma \to G$ does not contain any elements represented by simple loops in Σ.

First, let $B = H_1(\Sigma)$ (with $\mathbb{Z}$ coefficients understood), let $f_1\colon \pi_1\Sigma \to B$ be the abelianization map, and let $K_1 = \ker f_1$. Let $P\colon \widehat{\Sigma} \to \Sigma$ be the cover of Σ corresponding to K_1. Next, let $f_2\colon \pi_1\widehat{\Sigma} \to H_1(\widehat{\Sigma})$ be the analogous natural map for $\widehat{\Sigma}$, and let $K_2 = \ker f_2$. We have $K_2 \leq \pi_1\widehat{\Sigma} \approx K_1 \leq \pi_1\Sigma$, and so we identify K_2 with its image under P_* and consider it a subgroup of $\pi_1\Sigma$.

Observe that K_1 does not contain any element of $\pi_1\Sigma$ represented by a nonseparating simple loop in Σ, but does contain every element represented by a separating simple loop in Σ. Hence every separating simple loop in Σ lifts to $\widehat{\Sigma}$; we now show that every such loop lifts to a *nonseparating* simple loop in $\widehat{\Sigma}$.

We first observe that $B \approx \mathbb{Z}^{2g}$ is a one-ended group. Since B acts properly on $\widehat{\Sigma}$ with compact quotient Σ, it follows that $\widehat{\Sigma}$ is a one-ended space. Any inessential separating simple loop in $\widehat{\Sigma}$ must therefore separate $\widehat{\Sigma}$ into a compact piece and a noncompact piece. Hence if β is a simple separating loop in Σ for which some (and hence any) lift $\widehat{\beta}$ of β separates $\widehat{\Sigma}$, then $\widehat{\beta}$ cuts off a compact subsurface $\widehat{\Sigma}_{\widehat{\beta}} \subset \widehat{\Sigma}$. If $\widehat{\beta}'$ is another lift of β, then $\widehat{\beta}$ and $\widehat{\beta}'$ are disjoint, and the regularity of the cover $\widehat{\Sigma} \to \Sigma$ implies that there is a deck transformation of $\widehat{\Sigma}$ that takes $\widehat{\beta}'$ to $\widehat{\beta}$. This deck transformation must take $\widehat{\Sigma}_{\widehat{\beta}'}$ homeomorphically onto $\widehat{\Sigma}_{\widehat{\beta}}$. If one of these subsurfaces is contained in the other (say we have $\widehat{\Sigma}_{\widehat{\beta}'} \subset \widehat{\Sigma}_{\widehat{\beta}}$), then $\widehat{\beta}$ and $\widehat{\beta}'$ must be parallel. However, this is impossible: for by choosing hyperbolic metrics on Σ and $\widehat{\Sigma}$ so that the covering action is by isometries, and choosing β, $\widehat{\beta}$ and $\widehat{\beta}'$ to be the unique geodesics in their homotopy classes, we see that if $\widehat{\beta}$ and $\widehat{\beta}'$ are parallel then they are not distinct lifts of β.

It follows that the subsurfaces $\widehat{\Sigma}_{\widehat{\beta}}$ (as $\widehat{\beta}$ ranges over the lifts of β) must be disjoint. In particular, each such subsurface does not contain any lifts of β in its interior. Thus the covering map $\widehat{\Sigma} \to \Sigma$ restricts to a cover of a component of $\Sigma \setminus \beta$ by $\widehat{\Sigma}_{\widehat{\beta}}$, and since $\widehat{\beta}$ projects to β via a homeomorphism, the restricted cover is a homeomorphism. However, this is impossible, as $\widehat{\Sigma}_{\widehat{\beta}}$ is not a disk and so must contain a nonseparating simple loop, and this nonseparating loop is a lift of its image under the covering projection. We have already observed that such loops do not lift from Σ to $\widehat{\Sigma}$, and so from this contradiction we conclude that $\widehat{\beta}$ (and hence every lift of β to $\widehat{\Sigma}$) must be nonseparating.

It follows that K_2 does not contain *any* elements represented by simple loops of Σ, since the nonseparating simple loops in Σ are homologically nontrivial, and the separating simple loops of Σ lift to homologically nontrivial loops in $\widehat{\Sigma}$. Hence if we let $G = \pi_1\Sigma/K_2$ and let $f\colon \pi_1\Sigma \to G$ be the quotient map, then f is a noninjective map with no elements represented by essential simple loops in its kernel. If $A = \pi_1\widehat{\Sigma}/K_2 \approx H_1(\widehat{\Sigma})$, then A is abelian and we have

$$G/A = (\pi_1\Sigma/K_2)/(\pi_1\widehat{\Sigma}/K_2) \approx \pi_1\Sigma/\pi_1\widehat{\Sigma} \approx \pi_1\Sigma/K_1 \approx H_1(\Sigma),$$

which is also abelian. Thus we see that G is metabelian, for it fits into the short exact sequence

$$1 \to H_1(\widehat{\Sigma}) \to G \to H_1(\Sigma) \to 1,$$

and so we have constructed the desired group G and map $f\colon \pi_1\Sigma \to G$.

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