An invariant of rational homology 3–spheres via vector fields

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We give an alternative construction of the Kontsevich–Kuperberg–Thurston invariant for rational homology 3–spheres. This construction is a generalization of the original construction of the Kontsevich–Kuperberg–Thurston invariant. As an application, we give a Morse homotopy theoretic description of the Kontsevich–Kuperberg–Thurston invariant (close to a description by Watanabe).

1 Introduction

M Kontsevich [6], S Axelrod and I M Singer [2] proposed the Chern–Simons perturbation theory and gave a topological invariant of 3–manifolds with a flat connection. Based on Kontsevich's work, G Kuperberg and D Thurston [7] constructed a topological invariant of rational homology 3–spheres. They showed that this invariant is a universal finite type invariant for integral homology 3–spheres by showing surgery formulas. C Lescop obtained surgery formulas of other types in [9] and [10]. Lescop revisited the Kontsevich–Kuperberg–Thurston construction more precisely and gave a more direct proof of well-definedness of this invariant in [8].

K Fukaya [3] constructed a topological invariant of 3–manifolds with two local coefficients using Morse functions. Fukaya's invariant is closely related to the 2–loop term (ie the graph θ term) of the Chern–Simons perturbation theory. M Futaki [4] pointed out that Fukaya's invariant depends on the choice of Morse functions.

T Watanabe [14] gave an invariant of integral homology 3–spheres without local coefficients using Morse functions. He also investigated higher loop graphs (and broken graphs), and then he defined a topological invariant of homology 3–spheres corresponding to higher order terms of the Chern–Simons perturbation theory. In this paper, we extend his construction to any rational homology 3–sphere. The construction of the 2–loop term of Watanabe's invariant is related to the construction of a Morse propagator constructed by Lescop [11].

Fukaya's construction is inspired by the construction of the 2–loop term of the Chern–Simons perturbation theory, and he conjectured in [3, Section 8] that his invariant is

related to the 2–loop term of the Chern–Simons perturbation theory. Watanabe also conjectured in [14, Conjecture 2.14] that his invariant is related to Axelrod and Singer's invariant [2] or Kontsevich's invariant [6].

Let Y be a rational homology 3–sphere with a basepoint $\infty \in Y$. The Kontsevich–Kuperberg–Thurston invariant and Watanabe's Morse homotopy invariant are defined by using extra pieces of information from Y: a framing of $T(Y \setminus \infty)$ and a collection of Morse functions on Y, respectively.

In this paper, we give an alternative construction of the Kontsevich–Kuperberg–Thurston invariant by using a collection of appropriate 2–forms on the unit sphere bundle of the tangent bundle of $Y \setminus \infty$. In particular, we give a construction of such a collection of 2–forms by using a collection of vector fields on $Y \setminus \infty$. We prove that it is possible to regard the constructions of the Kontsevich–Kuperberg–Thurston invariant and Watanabe's Morse homotopy invariant as special cases of our construction. In fact, a framing gives us a nonvanishing vector field and a Morse function gives us a gradient vector field.

We remark that our construction of the principal term of the degree-1 part is related to Lescop's invariant [12] for rational homology 3–spheres with nonvanishing vector fields.

The organization of this paper is as follows. In Section 2, we prepare some notation. In Section 3, we review the Kontsevich–Kuperberg–Thurston invariant. In Section 4, we give an alternative construction of this invariant, and use it to give a Morse homotopy theoretic description of the invariant. In Section 5, we first review some lemmas from [8] used in Section 3 and Section 4, and then we prove the main theorems. In the Appendix, we give a more direct proof of Proposition 5.3 in the case $n = 1$.

2 Notation and some remarks

In this article, all manifolds are smooth and oriented. Homology and cohomology are with rational coefficients. Let c be a $\mathbb{Q}$–linear sum of finitely many maps from compact

k–dimensional manifolds with corners to a topological space X. We consider c as a k–chain of X via appropriate (not unique) triangulations of each k–manifold.

Let Y be a submanifold of a manifold X. Let $c = \sum_i a_i(f_i \colon \Sigma_i \to X)$ be a chain of X, where $f_i \colon \Sigma_i \to X$ are smooth maps from compact manifolds with corners and a_i are rational numbers. If f_i is transverse to Y for each i, then we say that c is transverse to Y.

Let us denote by $\underline{\mathbb{R}}^k$ the trivial vector bundle over an appropriate base space with rank $k \in \mathbb{N}$. For a real vector space X, we define $S(X)$ by

$$S(X) = (X \setminus \{0\})/\mathbb{R}_+.$$

We denote $S(X)$ by SX. For a real vector bundle $E \to B$ over a manifold B, we denote by SE or $S(E)$ the fiber bundle where the fiber over $x \in B$ is SE_x. When E has an Euclidean metric, SE is the unit sphere bundle of E.

When B is a submanifold of a manifold A, we denote by $B\ell(A, B)$ the manifold given by a real blowing up of A along B. Namely $B\ell(A, B) = (A \setminus B) \cup S\nu_B$, where ν_B is the normal bundle of $B \subset A$ (see [7] for more details about real blowing up). Note that if a submanifold $C \subset A$ is transverse to B, then $B\ell(C, C \cap B)$ is a properly embedded submanifold of $B\ell(A, B)$.

Let us denote by $\Delta \subset A^n$ the fat diagonal of the nth power of a manifold A: namely, $\Delta = \{(x_1, \ldots, x_n) \in A^n \mid \#\{x_1, \ldots, x_n\} < n\}$.

Conventions on orientations

Boundaries are oriented by the outward normal first convention. Products are oriented by the order of the factors. Let $y \in B$ be a regular point of a smooth map $f \colon A \to B$ between smooth manifolds A and B. Let us orient $f^{-1}(y)$ by the following rules: $T_x f^{-1}(y) \oplus f^* T_{f(x)} B = T_x A$, for any $x \in f^{-1}(y)$ where $f^* \colon T_{f(x)} B \to T_x A$ is a linear map satisfying $f_* \circ f^* = \mathrm{id}_{T_{f(x)} B}$. We denote by $-X$ the orientation reversed manifold of an oriented manifold X.

Let X be a 3–dimensional real vector space. Let V be a finite set. Define $\check{S}_V(X)$ to be the set of injective maps from V to X up to translations and dilations. Set $\underline{k} = \{1, \ldots, k\}$. We denote $\check{S}_{\underline{k}}(X)$ by $\check{S}_k(X)$. Note that $\check{S}_2(X) = S(X)$. For an $\mathbb{R}^3$ vector bundle $E \to B$, we denote by $\check{S}_V(E) \to B$ the fiber bundle where the fiber over $x \in B$ is $\check{S}_V(E_x)$.
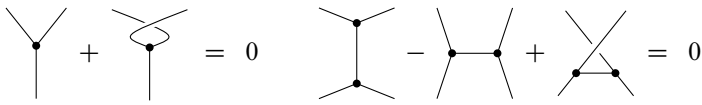
Figure 1: The relations AS (left) and IHX (right). Here the orientation of each vertex is given by counterclockwise order of the half edges.

3   The Kontsevich–Kuperberg–Thurston invariant

In this section, we introduce some notation about configuration spaces and Jacobi diagrams. Then we review the construction of the Kontsevich–Kuperberg–Thurston invariant. Most of this section follows [8].

3.1   Jacobi diagrams

This subsection follows [8, Sections 1.3, 2.3]. A *Jacobi diagram* of degree n is defined to be a trivalent graph with $2n$ vertices and $3n$ edges without simple loops. For a Jacobi diagram $\overline{\Gamma}$, we denote by $H(\overline{\Gamma})$, $E(\overline{\Gamma})$ and $V(\overline{\Gamma})$ the set of half edges, the set of edges and the set of vertices, respectively. An *orientation* of a vertex of $\overline{\Gamma}$ is a cyclic order of the three half-edges that meet at the vertex. A Jacobi diagram is *oriented* if all its vertices are oriented. Let $\mathbb{R}\langle$degree-n oriented Jacobi diagrams$\rangle$ be the real vector space freely generated by all degree-n oriented Jacobi diagrams. Let

$$\mathcal{A}_n(\varnothing) = \mathbb{R}\langle\text{degree } n \text{ oriented Jacobi diagrams}\rangle/\text{AS, IHX,}$$

where the relations AS and IHX are locally represented as in Figure 1. Let

$$\mathcal{E}_n = \{\Gamma = (\overline{\Gamma}, \varphi_E, \varphi_V, \text{ori}_E)\}$$

up to isomorphism. Here $\overline{\Gamma}$ is a connected Jacobi diagram of degree n, $\varphi_E\colon E(\overline{\Gamma}) \cong \{1, 2, \ldots, 3n\}$ and $\varphi_V\colon V(\overline{\Gamma}) \cong \{1, 2, \ldots, 2n\}$ are labels of edges and vertices respectively, and ori_E is a collection of orientations for all edges. For example, $\#\mathcal{E}_1 = 8$. These data and an orientation of $\overline{\Gamma}$ induce two orientations of $H(\overline{\Gamma})$. Here an orientation of a finite set is an order of all elements of the set up to even permutations. The first one is the *edge-orientation* induced by φ_E and ori_E. The edge-orientation is induced by the following order of $H(f)$: the first half edge is the initial half-edge of $\varphi_E^{-1}(1)$ with respect to ori_E, the second half edge is the terminal half-edge of $\varphi_E^{-1}(1)$, the third half edge is the initial half-edge of $\varphi_E^{-1}(2)$, and so on. The second one is the *vertex-orientation* induced by φ_V and orientation of $\overline{\Gamma}$. The vertex-orientation is induced by the following order of $H(f)$: the first three half edges meet at $\varphi_V^{-1}(1)$ with an order with respect to the orientation of $\overline{\Gamma}$, the second three half edges meet

at $\varphi_V^{-1}(2)$, and so on. We choose the orientation of $\overline{\Gamma}$ so that the edge-orientation coincides with the vertex-orientation. Let us denote by $[\Gamma] \in \mathcal{A}_n(\varnothing)$ the oriented Jacobi diagram given by Γ in such a way.

Remark 3.1 The notation $\mathcal{A}_{2n,3n}$ used in [14] coincides with the notation $\mathcal{A}_n(\varnothing)$ used in [8] as $\mathbb{R}$–vector spaces.

3.2 The configuration space $\check{C}_{2n}(Y)$

This subsection follows [8, Sections 1.1, 1.2, 2.1].

Let Y be a homology 3–sphere with a basepoint ∞. Let $N(\infty; Y)$ be a regular neighborhood (that is diffeomorphic to an open ball) of ∞ in Y, and let $N(\infty; S^3)$ be a regular neighborhood of ∞ in $S^3 = \mathbb{R}^3 \cup \infty$. We fix an orientation-preserving diffeomorphism $\varphi^\infty \colon (N(\infty; Y), \infty) \cong (N(\infty; S^3), \infty)$ between $N(\infty; Y)$ and $N(\infty; S^3)$. We use φ^∞ to identify $N(\infty; Y)$ with $N(\infty; S^3)$. Let B_Y denote $Y \setminus N(\infty; Y)$.

Let

$$\check{C}_{2n}(Y) = (Y \setminus \infty)^{2n} \setminus \Delta = \{\{1, \dots, 2n\} \hookrightarrow Y \setminus \infty\}.$$

Let $C_2(Y)$ be given by real blowing up of Y^2 along $\infty^2 = (\infty, \infty)$, $\overline{\infty \times (Y \setminus \infty)}$, $\overline{(Y \setminus \infty) \times \infty}$ and $\overline{\Delta \setminus \infty^2}$ in turn, where the space $\overline{\infty \times (Y \setminus \infty)}$ is the closure of $\infty \times (Y \setminus \infty)$ in $B\ell(Y^2, \infty^2)$. Let us denote by $q \colon C_2(Y) \to Y^2$ the composition of the blowing down maps. Then

$$\partial C_2(Y) = ST_\infty Y \times (Y \setminus \infty) \cup (Y \setminus \infty) \times ST_\infty Y \cup S\nu_{\Delta(Y \setminus \infty)} \cup q^{-1}(\infty^2).$$

The isomorphism

$$(TY^2 \supset)\nu_{\Delta(Y \setminus \infty)} \to T(Y \setminus \infty), \quad (-u, u) \mapsto u,$$

identifies $\nu_{\Delta(Y \setminus \infty)}$ with $T(Y \setminus \infty)$. Then we identify $S\nu_{\Delta(Y \setminus \infty)}$ with $STY|_{Y \setminus \infty}$.

Let p_1 and p_2 denote the projections

$$p_1 \colon (\partial C_2(Y) \supset)ST_\infty Y \times (Y \setminus \infty) \to ST_\infty Y = ST_\infty S^3 = S^2,$$
$$p_2 \colon (\partial C_2(Y) \supset)(Y \setminus \infty) \times ST_\infty Y \to ST_\infty Y = ST_\infty S^3 = S^2,$$

where $ST_\infty Y$ and $ST_\infty S^3$ are identified by φ^∞. We denote by $\iota_{S^2} \colon S^2 \to S^2$ the involution induced by $\times(-1) \colon \mathbb{R}^3 \to \mathbb{R}^3$.

From the definition of the compactification $C_2(S^3)$, it is possible to extend the map

$$\text{int } C_2(S^3) = (\mathbb{R}^3 \times \mathbb{R}^3) \setminus \Delta \to S^2, \quad (x, y) \mapsto (y - x)/\|y - x\|$$

to $C_2(S^3)$ (see [8, Lemma 1.1] for more details). Let $p_c\colon C_2(S^3) \to S^2$ denote this map. Since $q^{-1}(N(\infty; Y)^2) \subset C_2(Y)$ is identified with $q^{-1}(N(\infty; S^3)^2) \subset C_2(S^3)$ by φ^∞, we get a map $\partial C_2(Y) \cap q^{-1}(N(\infty; Y)^2) \xrightarrow{p_c} S^2$. Since $p_1, \iota_{S^2} \circ p_2$ and p_c are compatible on the boundary, as in the case $Y = S^3$, these maps define the map

$$p_Y\colon \partial C_2(Y) \setminus STB_Y \to S^2.$$

For each $\Gamma = (\bar{\Gamma}, \varphi_E, \varphi_V, \mathrm{ori}_E) \in \mathcal{E}_n$ and for each $\varphi_E^{-1}(i) \in E(\bar{\Gamma})$, let $s(i), t(i) \in \{1, \dots, 2n\}$ denote the labels of the initial vertex and the terminal vertex of $\varphi_E^{-1}(i)$, respectively. The embedding $\{1, 2\} \cong \{s(i), t(i)\} \hookrightarrow \{1, \dots, 2n\}$ induces the projection

$$P_i(\Gamma) = \pi_{\check{C}_{2n}(Y)}\colon \check{C}_{2n}(Y) \to \check{C}_2(Y) \, (\subset C_2(Y)).$$

We define a map $P_i(\Gamma)\colon \check{S}_{2n}(E) \to SE$ in the same manner for an oriented $\mathbb{R}^3$ bundle $E \to B$. See [8, Section 2.3] for more details.

3.3 Admissible forms and propagators

Definition 3.2 (admissible form) A closed 2–form $\omega \in \Omega^2(\partial C_2(Y))$ of $\partial C_2(Y)$ is an *admissible form with respect to* ω_{S^2} if the restriction of ω to $\partial C_2(Y) \setminus STB_Y$ is $p_Y^* \omega_{S^2}$, where ω_{S^2} is a closed 2–form of S^2 with total volume 1.

Let ω be an admissible form. Since Y is a rational homology 3–sphere, we can check that $H^2(C_2(Y), \partial C_2(Y); \mathbb{R}) \cong H_4(C_2(Y); \mathbb{R}) = 0$ and $H^3(C_2(Y), \partial C_2(Y); \mathbb{R}) \cong H_3(C_2(Y); \mathbb{R}) = 0$ as in the case $Y = S^3$ (see [8, Lemma 2.1] for more details). Then the restriction $H^2(C_2(Y); \mathbb{R}) \to H^2(\partial C_2(Y); \mathbb{R})$ is an isomorphism. Thus there is a closed 2–form ω_C on $C_2(Y)$ satisfying $\omega_C|_{\partial C_2(Y)} = \omega$.

Definition 3.3 (propagator) We call ω_C a *propagator with respect to* ω.

Let $\{\omega_i\}_{i \in \underline{3n}}$ be a collection of admissible forms.

Definition 3.4 Let $\omega_{i,C}$ be a propagator with respect to ω_i. Then we define

$$z_n((\omega_i)_{i \in \underline{3n}}) = \sum_{\Gamma \in \mathcal{E}_n} \left(\int_{\check{C}_{2n}(Y)} \bigwedge_{i \in \underline{3n}} P_i(\Gamma)^* \omega_{i,C} \right) [\Gamma] \in \mathcal{A}_n(\varnothing).$$

We will see that this definition makes sense and is consistent in Lemma 3.5. For an $\mathbb{R}^3$–bundle E over a manifold Z with corners and a collection $(\eta_i)_{i \in \underline{3n}}$ of forms over $S_2(E)$ satisfying $\sum_{i=1}^{3n} \deg(\eta_i) = 6n - 4 + \dim Z$, define $X_n(E; (\eta_i)_{\underline{3n}}) \in \mathcal{A}_n(\varnothing)$ as

$$X_n(E; (\eta_i)_{\underline{3n}}) = \sum_{\Gamma \in \mathcal{E}_n} \left(\int_{\check{S}_{2n}(E)} \bigwedge_{i \in \underline{3n}} P_i(\Gamma)^* \eta_i \right) [\Gamma].$$

From the work of Lescop, especially in [8, Section 2.4], the following lemma holds.

Lemma 3.5 (1) *The integrals in Definition 3.4 are convergent.*

(2) *$z_n((\omega_i)_{i\in\underline{3n}})$ is independent of the choice of propagators $(\omega_{i,C})_{i\in\underline{3n}}$.*

(3) *For any other collection $(\omega_i')_{i\in\underline{3n}}$ of admissible forms, there exist closed 2–forms $\Omega_i = \Omega(\omega_i, \omega_i')$ on $[0,1] \times STB_Y$ and closed 2–forms $\Omega_{S^2,i}$ on $[0,1] \times S^2$ satisfying the following conditions:*

    (a) $\Omega_i|_{\{0\}\times STB_Y} = \omega_i|_{STB_Y}$,

    (b) $\Omega_i|_{\{1\}\times STB_Y} = \omega_i'|_{STB_Y}$,

    (c) $\Omega_i|_{[0,1]\times\partial STB_Y} = (\mathrm{id} \times p_Y)^*\Omega_{S^2,i}$,

    (d) $\Omega_{S^2,i}|_{\{0\}\times S^2} = \omega_{S^2,i}$ *and*

    (e) $\Omega_{S^2,i}|_{\{1\}\times S^2} = \omega_{S^2,i}'$.

    Here $\omega_i|_{\partial C_2(Y)\setminus STB_Y} = p_Y^*\omega_{S^2,i}$ and $\omega_i'|_{\partial C_2(Y)\setminus STB_Y} = p_Y^*\omega_{S^2,i}'$. For these forms, $z_n((\omega_i')_{i\in\underline{3n}}) - z_n((\omega_i)_{i\in\underline{3n}}) = X_n([0,1] \times TB_Y; (\Omega_i)_{i\in\underline{3n}})$.

The proof of this lemma will be outlined in Section 5.1.

3.4 Review of the Kontsevich–Kuperberg–Thurston invariant

In this section, we review the construction of the Kontsevich–Kuperberg–Thurston invariant z_n^{KKT} for rational homology 3–spheres with a little modification.

For a framing τ' of TY, we denote by $\sigma_Y(\tau') \in \mathbb{Z}$ the signature defect of τ' defined as follows: Take a compact 4–manifold X_0 bounded by Y. Let $p_1(\tau'; X_0) \in H^4(X_0, Y; \mathbb{Z})$ be the relative Pontrjagin class. By the Hirzebruch signature theorem, the difference $\sigma_Y(\tau') = \int_{X_0} p_1(\tau'; X_0) - 3\operatorname{Sign} X_0$ is independent of the choice of X_0. See [5] or [1] for more details. Let τ_{S^3} be a framing[1] of TS^3 satisfying the following two conditions:

- $\sigma_{S^3}(\tau_{S^3}) = 2$,

- $\tau_{S^3}|_{S^3\setminus N'(\infty;S^3)} = \tau_{\mathbb{R}^3}|_{S^3\setminus N'(\infty;S^3)}$, where $\tau_{\mathbb{R}^3}\colon T\mathbb{R}^3 \cong \underline{\mathbb{R}^3}$ is the standard framing of $T\mathbb{R}^3$.

Here $N'(\infty; S^3)$ is a neighborhood of ∞ smaller than $N(\infty; S^3)$; that is, $\infty \in N'(\infty; S^3) \subset N(\infty; S^3)$.

Remark 3.6 There is no special meaning in the number 2 in the first condition above. The anomaly term $\widetilde{z}_n^{\mathrm{anomaly}}(\vec{\gamma})$ would be independent of the choice of τ_{S^3} even if $\sigma_{S^3}(\tau_{S^3})$ were not 2. We remark that there is no framing τ on S^3 such that $\sigma_{S^3}(\tau) = 0$.

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[1]There is such a framing. For example, the Lie framing $\tau_{\mathrm{SU}(2)}$ of the tangent bundle of $S^3 = \mathrm{SU}(2)$ corresponding to left multiplication satisfies $\sigma_{S^3}(\tau_{\mathrm{SU}(2)}) = 2$. See [5] for more details. We can get τ_{S^3} by modifying $\tau_{\mathrm{SU}(2)}$.

Let $\tau_Y \colon T(Y \setminus \infty) \cong \underline{\mathbb{R}}^3$ be a framing satisfying $\tau_Y|_{N(\infty;Y) \setminus \infty} = \tau_{\mathbb{R}^3}$, where $\underline{\mathbb{R}}^3$ is the trivial $\mathbb{R}^3$–bundle over $N(\infty;Y) \setminus \infty$. We call such a framing *an admissible framing*. We have that $\tau_Y|_{Y \setminus N(\infty;Y)} \cup \tau_{S^3}|_{N(\infty;S^3)}$ is a framing of TY by the above conditions on τ_Y. We define

$$\sigma_{Y \setminus \infty}(\tau_Y) = \sigma_Y(\tau_Y|_{Y \setminus N(\infty;Y)} \cup \tau_{S^3}|_{N(\infty;S^3)}) - \sigma_{S^3}(\tau_{S^3})$$
$$= \sigma_Y(\tau_Y|_{Y \setminus N(\infty;Y)} \cup \tau_{S^3}|_{N(\infty;S^3)}) - 2,$$

and call it the *signature defect* of τ_Y. For example, $\sigma_{\mathbb{R}^3}(\tau_{\mathbb{R}^3}) = 0$.

The composition of the framing τ_Y and the projection map $\underline{\mathbb{R}}^3 \to \mathbb{R}^3$ induces the map $p_\Delta(\tau_Y) \colon STB_Y \to S^2$. Thanks to the conditions we imposed on τ_Y, the maps $p_\Delta(\tau_Y)$ and $p_Y \colon \partial C_2(Y) \setminus STB_Y \to S^2$ are compatible. So we get the map

$$p(\tau_Y) = p_Y \cup p_\Delta(\tau_Y) \colon \partial C_2(Y) \to S^2.$$

Therefore, we have an admissible form $\omega_{\tau_Y} = p(\tau_Y)^* \omega_{S^2}$ for any closed 2–form ω_{S^2} of S^2 with total volume 1.

Definition 3.7 We define $z_n^{\mathrm{KKT}}(Y; \tau_Y) = z_n((\omega_{\tau_Y})_{i \in \underline{3n}})$.

Lemma 3.8 [8, Theorem 1.9, Proposition 2.11] *For any other framing τ_Y', there exist constants $\delta_n \in \mathcal{A}_n(\varnothing)$ such that*

$$z_n^{\mathrm{KKT}}(Y; \tau_Y') - z_n^{\mathrm{KKT}}(Y; \tau_Y) = -\tfrac{1}{4}\big(\sigma_{Y \setminus \infty}(\tau_Y') - \sigma_{Y \setminus \infty}(\tau_Y)\big)\delta_n.$$

The central definition that we study here generalizes the definitions that are given in [7] by Kuperberg and Thurston and in [8] by Lescop.

Definition 3.9 Let $(\omega_i)_{i \in \underline{3n}}$ be a collection of admissible forms and τ_Y be an admissible framing. We define

$$z_n^{\mathrm{KKT}}(Y) = z_n((\omega_i)_{i \in \underline{3n}}) - \widetilde{z}_n^{\mathrm{anomaly}}((\omega_i)_{i \in \underline{3n}}) \in \mathcal{A}_n(\varnothing),$$

where

$$\widetilde{z}_n^{\mathrm{anomaly}}((\omega_i)_{i \in \underline{3n}}) = X_n\big([0,1] \times TB_Y; (\Omega_i(\omega_{\tau_Y}, \omega_i))_{i \in \underline{3n}}\big) - \tfrac{1}{4}\sigma_{Y \setminus \infty}(\tau_Y)\delta_n.$$

Lemmas 3.5 and 3.8 imply that Definition 3.9 is consistent.

We will give an alternative description of $\widetilde{z}_n^{\mathrm{anomaly}}((\omega_i)_{i \in \underline{3n}})$ without framings in Section 4.1. We remark that δ_n is given by an explicit formula in [8, Proposition 2.10]. Lemma 3.5(3) implies that $\widetilde{z}_n^{\mathrm{anomaly}}((\omega_i)_{i \in \underline{3n}})$ is independent of the choice of τ_Y. Thus $\widetilde{z}_n^{\mathrm{anomaly}}((\omega_i)_{i \in \underline{3n}})$ is consistent.

Remark 3.10 The universal finite type invariant Z_n^{KKT} described in [8] is equal to the degree-n part of

$$\exp\left(\sum_n \frac{1}{2^{3n}(3n)!(2n)!} z_n^{\mathrm{KKT}}\right).$$

See the discussion before Lemma 2.12 in [8] for more details.

4 Alternative description of z_n^{KKT}

This section is the main part of this paper. In Section 4.1, we give an alternative description of the anomaly term. In Section 4.2, we give a construction of an admissible form by using a vector field. In Section 4.3, we give an explicit propagator by using a Morse function. This construction is close to T Watanabe's Morse homotopy invariant.

4.1 Alternative description of the anomaly term $\widetilde{z}_n^{\mathrm{anomaly}}((\omega_i)_{i \in \underline{3n}})$

The idea of this alternative construction is based on the construction of the anomaly term of Watanabe's invariant in [14]. We note that Watanabe defined his invariant only for integral homology 3–spheres.

Let Y be a closed 3–manifold with a basepoint $\infty \in Y$. Let ω_i be an admissible form with respect to a closed 2–form $\omega_{S^2,i}$ of S^2. Let τ_{S^3} be a framing as in Section 3.4. The framing τ_{S^3} induces the map $p(\tau_{S^3})\colon STS^3 \to S^2$.

Let X be a compact connected 4–manifold with $\partial X = Y$ and $\chi(X) = 0$. For example, we can take $X = (T^4 \# \mathbb{C}P^2) \setminus B^4$ when $Y = S^3$. Since $\chi(X) = 0$, it is possible to extend TY to an oriented $\mathbb{R}^3$–subbundle of TX. We denote by $T^v X$ such an extended bundle. Let $F_X \to ST^v X$ be the tangent bundle along the fiber of the sphere bundle $ST^v X \to X$. Let W_i be a closed 2–form on $ST^v X$ satisfying the following conditions:

- $2W_i$ represents the Euler class $e(F_X)$ of $F_X \to ST^v X$,
- $W_i|_{STY} = \omega_i|_{STB_Y} \cup p(\tau_{S^3})^* \omega_{S^2,i}|_{STN(\infty;Y)}$.

Theorem 4.1 $\widetilde{z}_n^{\mathrm{anomaly}}((\omega_i)_{i \in \underline{3n}}) = \frac{3}{4}\delta_n \operatorname{Sign} X + X_n(T^v X; (W_i)_{i \in \underline{3n}}) + \frac{1}{2}\delta_n \in \mathcal{A}_n(\varnothing)$.

For any closed connected 4–manifold X' with $\chi(X') = 0$, we can define $T^v X'$ and $X(T^v X'; (W_i')_{i \in \underline{3n}})$ for any closed 2–form W_i' of $ST^v X'$ which represents $\frac{1}{2} e(F_{X'})$. The well-definedness of the right hand side of Theorem 4.1 will be a direct consequence of the following theorem.

Theorem 4.2 *For any closed connected 4–manifold X' with $\chi(X') = 0$,*

$$-\tfrac{3}{4}\delta_n \operatorname{Sign} X' = X_n(T^v X'; (W_i')_{i \in \underline{3n}}).$$

Theorem 4.1 and 4.2 will be proved in Section 5.

4.2 An admissible form given by a collection of vector fields

In this section, we give a construction of an admissible form by using a smooth vector field. We will use these admissible forms in the next subsection.

For $a \in S^2 \subset \mathbb{R}^3$, the map $q_a \colon \mathbb{R}^3 \to \mathbb{R}$ is defined by $q_a(x) = \langle x, a \rangle$, where $\langle \, , \, \rangle$ is the standard inner product on $\mathbb{R}^3$.

Definition 4.3 A smooth vector field γ on $Y \setminus \infty$ is an *admissible vector field (with respect to a)* if the following conditions hold:

- $\gamma|_{N(\infty;Y)\setminus\infty} = a \, (= \operatorname{grad} q_a|_{N(\infty;S^3)\setminus\infty})$,
- γ is transverse to the zero section in $T(Y \setminus \infty)$.

Example 4.4 We give two important examples of admissible vector fields with respect to a:

(1) Let $\tau_{\mathbb{R}^3} \colon T\mathbb{R}^3 \cong \underline{\mathbb{R}^3}$ be the standard framing of $T\mathbb{R}^3$. We regard $a \in \mathbb{R}^3$ as a constant section of the trivial bundle $\underline{\mathbb{R}^3}$. If $\tau \colon T(Y \setminus \infty) \cong \underline{\mathbb{R}^3}$ is an admissible framing, the pullback vector field $\tau^* a$ is an admissible vector field with respect to a.

(2) For a Morse function $f \colon Y \setminus \infty \to \mathbb{R}$ such that $f|_{N(\infty;Y)\setminus\infty} = q_a|_{N(\infty;S)\setminus\infty}$, $\operatorname{grad} f$ is an admissible vector field with respect to a.

The following lemma plays an important role in this construction. For an admissible vector field γ, let

$$\overline{c}_\gamma = \overline{\{[\gamma(x)] \in ST_x Y \mid x \in Y \setminus (\infty \cup \gamma^{-1}(0))\}} \subset ST(Y \setminus \infty),$$

where $[\gamma(x)] \in ST_x Y$ is the class of $\gamma(x) \in T_x Y$ up to dilation. Here we choose the orientation of $\overline{c}_\gamma$ such that the restriction to $\overline{c}_\gamma$ of the projection $STY \to Y$ is orientation preserving.

Lemma 4.5 $c_0(\gamma) = \overline{c}_\gamma \cup \overline{c}_{-\gamma}$ *is a submanifold of $ST(Y \setminus \infty)$ without boundary.*

Proof It is sufficient to check this claim near $\gamma^{-1}(0)$. Let $x \in \gamma^{-1}(0)$. Identify a neighborhood of x with an open neighborhood U of 0 in $\mathbb{R}^3$ by a chart that maps x to 0. Then $\gamma(u)$ reads $(u \in U, \Psi(u) \in T_u\mathbb{R}^3 = \mathbb{R}^3)$ in such a chart where $\Psi(0) = 0$ and $d_0\Psi \in GL(T_0\mathbb{R}^3) = GL(\mathbb{R}^3)$. Under this chart,

$$c_\gamma \cap STY|_U = \left\{ \left(tv, \frac{\Psi(tv)}{\|\Psi(tv)\|} \right) \,\Big|\, t \in (0, \infty),\, v \in S^2,\, tv \in U \right\}.$$

Set $\psi_t(v) = \Psi(tv)/\|\Psi(tv)\|$ when $t \in (0, \infty)$ and $\psi_0(v) = d_0\Psi(v)/\|d_0\Psi(v)\|$ for $v \in S^2$. Thus $\psi_0(v)$ is the limit of $\psi_t(v) = \Psi(tv)/\big(t\|(1/t)\Psi(tv)\|\big)$ where t approaches 0. Then

$$\overline{c}_\gamma \cap STY|_U = \{(tv, \psi_t(v)) \mid t \in [0, \infty),\, v \in S^2,\, tv \in U\},$$

$$\overline{c}_{-\gamma} \cap STY|_U = \{(tv, -\psi_t(v)) \mid t \in [0, \infty),\, v \in S^2,\, tv \in U\}.$$

Let $\varepsilon_0 \in \{1, -1\}$ be the sign of the Jacobian of ψ_0; namely $\varepsilon_0 = 1$ if and only if the Jacobian of ψ_0 is positive. Fix $\varepsilon > 0$ small enough so that for any $t \in [0, \varepsilon)$, we have $tS^2 \subset U$ and ψ_t is a diffeomorphism of S^2. Let $B(\varepsilon) = \{x \in \mathbb{R}^3 \mid \|x\| < \varepsilon\}$. We have

$$c_0(\gamma) \cap STY|_{B(\varepsilon)}$$
$$= \varepsilon_0 \{(t\psi_t^{-1}(w), w) \mid t \in [0, \varepsilon),\, w \in S^2\} \cup -\varepsilon_0 \{(t\psi_t^{-1}(-w), w) \mid t \in [0, \varepsilon),\, w \in S^2\}.$$

Let $\Phi(t, w) = \big(|t|\psi_{|t|}^{-1}((t/|t|)w), w\big)$ if $t \neq 0$ and $\Phi(0, w) = (0, w)$. Then Φ is a smooth embedding. Furthermore, $\Phi|_{(0,\varepsilon) \times S^2}$ (resp. $\Phi|_{(-\varepsilon,0) \times S^2}$) is an embedding into c_γ (resp. $c_{-\gamma}$) that preserves the orientation if and only if ψ_0 does, so that

$$c_0(\gamma) \cap STY|_{B(\varepsilon)} = \{\Phi(t, w) \mid t \in (-\varepsilon, \varepsilon),\, w \in S^2\}.$$

Then $c_0(\gamma)$ is an oriented topological manifold that reads as the image of the embedding Φ near $STY|_x$. $\qquad\square$

We define

$$c(\gamma) = p_Y^{-1}(a) \cup p_Y^{-1}(-a) \cup (c_0(\gamma) \cap STB_Y) \subset \partial C_2(Y).$$

By the definition of admissible vector fields and by Lemma 4.5, $c(\gamma)$ is a closed 3–manifold. Therefore, $c(\gamma)$ determines a 3–cycle of $\partial C_2(Y)$.

Let $\omega_{S^2}^a$ be an antisymmetric closed 2–form on S^2 such that $\omega_{S^2}^a$ represents $\frac{1}{2}$ times the Poincaré dual of $[a] + [-a]$ and the support of $\omega_{S^2}^a$ is concentrated near $\{a, -a\}$. Let $\omega(\gamma)$ be an admissible form on $\partial C_2(Y)$ satisfying the following conditions:[2]

---

[2]All admissible forms satisfy the first condition because $H^2(\partial C_2(Y); \mathbb{R}) \cong \mathbb{R}$.

- $2\omega(\gamma)$ represents the Poincaré dual of $[c(\gamma)]$,
- the support of $\omega(\gamma)$ is concentrated near $c(\gamma)$, and
- $\omega(\gamma)|_{\partial C_2(Y)\setminus STB_Y} = p_Y^* \omega_{S^2}^a$; namely $\omega(\gamma)$ is an admissible form.

4.3 Morse homotopy theoretic description of z_n^{KKT}

In this subsection, we give a Morse homotopy theoretic description of z_n^{KKT} by using a collection of $3n$ Morse functions. This description is deeply inspired by the work of Watanabe in [14].

We first prepare some notations about Morse functions. Let Y be a closed 3–dimensional manifold with a basepoint $\infty \in Y$. Fix a point $a \in S^2$.

Definition 4.6 A Morse function $f: Y \setminus \infty \to \mathbb{R}$ is an *admissible Morse function with respect to a* if it satisfies the following conditions:

- $f|_{N(\infty;Y)\setminus\infty} = q_a|_{N(\infty;S^3)\setminus\infty}$, and
- f has no critical point of index 0 or 3.

Let f be a Morse function on $Y \setminus \infty$ (or Y) and ξ a metric on Y satisfying the Morse–Smale condition. Let $\mathrm{grad}\, f$ be the gradient vector field of f corresponding to ξ. Let $\mathrm{Crit}(f)$ denote the set of all critical points of f. Let $\{\Phi_f^t\}_{t\in\mathbb{R}}: Y \setminus \infty \to Y \setminus \infty$ be the 1–parameter group of diffeomorphisms associated to $\mathrm{grad}\, f$. We denote by

$$\mathcal{A}_p = \left\{x \in Y \mid \lim_{t\to-\infty} \Phi_f^t(x) = p\right\} \quad \text{and} \quad \mathcal{D}_p = \left\{x \in Y \mid \lim_{t\to\infty} \Phi_f^t(x) = p\right\}$$

the ascending manifold and the descending manifold at $p \in \mathrm{Crit}(f)$, respectively.

Let us orient ascending manifolds and descending manifolds by imposing the condition $T_p\mathcal{A}_p \oplus T_p\mathcal{D}_p \cong T_pY$ for any $p \in \mathrm{Crit}(f)$. Let $p, q \in \mathrm{Crit}(f)$ be critical points of indices 2 and 1, respectively. By the Morse–Smale condition, $\mathcal{D}_p \cap \mathcal{A}_q$ is a 1–manifold. Let us orient $\mathcal{D}_p \cap \mathcal{A}_q$ by the following rule:

We fix a Riemannian metric on Y. The normal bundle $N\mathcal{D}_p$ of $\mathcal{D}_p$ and the normal bundle $N\mathcal{A}_q$ of $\mathcal{A}_q$ are oriented by

$$T\mathcal{D}_p \oplus N\mathcal{D}_p = TY|_{\mathcal{D}_p} \quad \text{and} \quad T\mathcal{A}_q \oplus N\mathcal{A}_q = TY|_{\mathcal{A}_q}.$$

Let $n(\mathcal{D}_p)$ and $n(\mathcal{A}_q)$ be positively oriented normal vector fields of $\mathcal{D}_p$ and $\mathcal{A}_q$, respectively. We orient the normal bundle $N(\mathcal{D}_p \cap \mathcal{A}_q)$ of $\mathcal{D}_p \cap \mathcal{A}_q$ by using the frame $(n(\mathcal{A}_q)|_{\mathcal{D}_p\cap\mathcal{A}_q}, n(\mathcal{D}_p)|_{\mathcal{D}_p\cap\mathcal{A}_q})$. Then we orient $\mathcal{D}_p \cap \mathcal{A}_q$ by

$$T(\mathcal{D}_p \cap \mathcal{A}_q) \oplus N(\mathcal{D}_p \cap \mathcal{A}_q) = TY|_{\mathcal{D}_p\cap\mathcal{A}_q}.$$
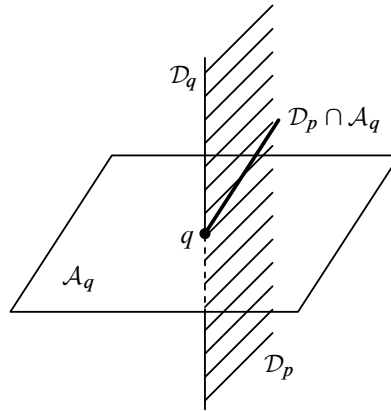
Figure 2: The orientation of $\mathcal{D}_p \cap \mathcal{A}_q$

We give a sign ± 1 to each connected component of $\mathcal{D}_p \cap \mathcal{A}_q$. A component has sign $+1$ if the orientation of this component is from q to p. See Figure 2.

We assume f is an admissible Morse function. Let $\mathrm{Crit}(f) = \{p_1, \ldots, p_k, q_1, \ldots, q_k\}$ be the set of critical points of f where $\mathrm{ind}(p_i) = 2$ and $\mathrm{ind}(q_i) = 1$. Let

$$0 \to C_2(Y \setminus \infty; f) \xrightarrow{\hat{\partial}} C_1(Y \setminus \infty; f) \to 0$$

be the Morse complex of f with rational coefficients. Here the boundary map is defined as $\hat{\partial}([p_i]) = \sum_j \hat{\partial}_{ij}[q_j]$, where $\hat{\partial}_{ij}$ is the sum of the signs of all the connected components of $\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}$. Let

$$\hat{g} \colon C_1(Y \setminus \infty; f) \to C_2(Y \setminus \infty; f), \quad \hat{g}([q_j]) = \sum_j \hat{g}_{ji}[p_i],$$

be the inverse map of $\hat{\partial}$. (In [14], $\hat{g}$ is called a combinatorial propagator.)

We now construct $\mathcal{M}(f)$, which is a weighted sum of (noncompact) 4–manifolds in $Y^2 \setminus \Delta$. Let $M_{\to}(f)$ be the 4–manifold which is the image of the inclusion map

$$\varphi_f \colon (Y \setminus \infty) \times (0, \infty) \hookrightarrow (Y \setminus \infty) \times (Y \setminus \infty), \quad (x, t) \mapsto (x, \Phi_f^t(x)).$$

We choose the orientation of $M_{\to}(f)$ so that the map φ_f preserves orientations. Define

$$\mathcal{M}(f) = M_{\to}(f) + \sum_{i,j} \hat{g}_{ji}(\mathcal{D}_{p_i} \times \mathcal{A}_{q_j}) \setminus \Delta.$$

We remark that the orientation of $\mathcal{M}(f)$ does not depend on the choice of orientations of $\mathcal{D}_{p_i}$ and $\mathcal{A}_{q_j}$.

Let $a_1, \ldots, a_{3n} \in S^2 \subset \mathbb{R}^3$ be points such that any three distinct points are linearly independent in $\mathbb{R}^3$. Let $f_1, \ldots, f_{3n}$ be a collection of sufficiently generic admissible Morse functions with respect to $a_1, \ldots, a_{3n}$. Take metrics $\xi_1, \ldots, \xi_{3n}$ on Y and use ξ_i for grad f_i.

Set $\mathcal{M}(\pm f_i) = \mathcal{M}(f_i) + \mathcal{M}(-f_i)$. For generic ξ_i, f_i and a_i, the weighted sums of manifolds

$$P_1(\Gamma)|_{(Y\backslash\infty)^{2n}\backslash\Delta}^{-1}(\mathcal{M}(\pm f_1)), \ldots, P_{3n}(\Gamma)|_{(Y\backslash\infty)^{2n}\backslash\Delta}^{-1}(\mathcal{M}(\pm f_{3n}))$$

transversally intersect at finitely many points for any $\Gamma \in \mathcal{E}_n$. We can prove this fact in a similar manner as in the proofs of Proposition 2.4 and Lemma 4.2 in [14].

Theorem 4.7 *For generic $f_1, \ldots, f_{3n}$ and $\xi_1, \ldots, \xi_{3n}$,*

$$z_n^{\mathrm{KKT}}(Y)$$

$$= \sum_{\Gamma \in \mathcal{E}_n} \frac{1}{2^{3n}} \#\left(\bigcap_{i=1}^{3n} P_i(\Gamma)|_{(Y\backslash\infty)^{2n}\backslash\Delta}^{-1}(\mathcal{M}(\pm f_i)) \right)[\Gamma] - \widetilde{z}_n^{\mathrm{anomaly}}\big((\omega(\mathrm{grad}\, f_i))_{i\in\underline{3n}}\big).$$

Here $\omega(\mathrm{grad}\, f_i)$ is an admissible form corresponding to the admissible vector field grad f_i.

5 Proofs

In this section we prove Theorems 4.1, 4.2 and 4.7.

5.1 Needed lemmas from [8]

We first give the outline of the proof of Lemma 3.5. See [8, Section 2.4] for more details.

Outline of the proof of Lemma 3.5

(1) See [8, Proposition 2.5].

(2) See [8, Proposition 2.9].

(3) Fix $j \in \underline{3n}$. It is enough to study the case where $\omega_i = \omega_i'$ for any $i \neq j$. Since $[\omega_j'] = [\omega_j] \in H^2(\partial C_2(Y); \mathbb{R})$ and $[\omega_{S^2,j}'] = [\omega_{S^2,j}] \in H^2(S^2; \mathbb{R})$, there exist closed 2–forms Ω_j^0 on $[0, 1] \times \partial C_2(Y)$ and $\Omega_{S^2,j}$ on $[0, 1] \times S^2$ such that

- $\Omega_j^0|_{\{0\}\times\partial C_2(Y)} = \omega_j$,
- $\Omega_j^0|_{\{1\}\times\partial C_2(Y)} = \omega_j'$,
- $\Omega_{S^2,j}|_{\{0\}\times S^2} = \omega_{S^2,j}$ and
- $\Omega_{S^2,j}|_{\{1\}\times S^2} = \omega_{S^2,j}'$.

More precisely, by the same argument of Lemma 2.15 in [8], we can take Ω_j^0 so that $\Omega_j^0|_{[0,1]\times(\partial C_2(Y)\setminus STB_Y)} = p_Y^*\Omega_{S^2,j}$. Let $\Omega_i^0 = \pi^*\omega_i$ for any $i \neq j$, where $\pi\colon [0,1]\times\partial C_2(Y) \to \partial C_2(Y)$ is the projection. Similarly, let $\Omega_{S^2,i} = \pi^*\omega_{S^2,i}$ for $i \neq j$, where $\pi\colon [0,1]\times S^2 \to S^2$ is the projection. Set $\Omega_i = \Omega_i^0|_{[0,1]\times STB_Y}$.

On the other hand, there is a natural compactification $C_{2n}(Y)$ of $\check{C}_{2n}(Y)$; this is a compact $6n$–dimensional manifold with corners, and there is a natural extension to $C_{2n}(Y)$ of $P_i(\Gamma)\colon \check{C}_{2n}(Y) \to C_2(Y)$; see [8, Section 3] for more details. By Stokes' theorem,

$$
0 = \sum_{\Gamma\in\mathcal{E}_n} \int_{[0,1]\times C_{2n}(Y)} d\left(\bigwedge_i (\mathrm{id}\times P_i(\Gamma))^*\Omega_i^0\right)[\Gamma]
$$

$$
= z_n((\omega_i')_{i\in\underline{3n}}) - z_n((\omega_i)_{i\in\underline{3n}}) - \sum_{\Gamma\in\mathcal{E}_n} \int_{[0,1]\times\partial C_{2n}(Y)} \left(\bigwedge_i (\mathrm{id}\times P_i(\Gamma))^*\Omega_i^0\right)[\Gamma].
$$

Lemmas 2.17, 2,18, 2.19, 2.20 and 2.21 of [8] imply that

$$
\sum_{\Gamma\in\mathcal{E}_n} \int_{[0,1]\times\partial C_{2n}(Y)} \left(\bigwedge_i (\mathrm{id}\times P_i(\Gamma))^*\Omega_i^0\right)[\Gamma]
$$

$$
= \sum_{\Gamma\in\mathcal{E}_n} \int_{[0,1]\times\check{S}_{2n}(TB_Y)} \left(\bigwedge_i (\mathrm{id}\times P_i(\Gamma))^*\Omega_i\right)[\Gamma]
$$

$$
= X_n([0,1]\times TB_Y; (\Omega_i)_{i\in\underline{3n}}). \qquad\square
$$

Two lemmas To prove Theorem 4.1 and Theorem 4.2, we prepare two lemmas. These lemmas are essentially given by C Lescop in [8].

Lemma 5.1 *Let $E \to Z$ be an oriented $\mathbb{R}^3$–bundle over a manifold Z with corners and let $(\eta_i)_{i\in\underline{3n}}$ be a collection of differential forms (not necessarily closed) over SE such that $\sum_{i=1}^{3n} \deg(\eta_i) = 6n - 4 + \dim Z - 1$. Then*

$$
X_n(E|_{\partial Z}; (\eta_i|_{S(E|_{\partial Z})})_{i\in\underline{3n}}) = \sum_{k=1}^{3n} (-1)^{\sum_{j=1}^{k-1}\deg(\eta_j)} X_n\left(E; ((\eta_i)_{i\in\underline{3n}})\left(\frac{d\eta_k}{\eta_k}\right)\right),
$$

where $((\eta_i)_{i\in\underline{3n}})(d\eta_k/\eta_k)$ is the collection of forms obtained from $(\eta_i)_{i\in\underline{3n}}$ by replacing η_k with $d\eta_k$.

Outline of the proof There is a natural compactification $S_{2n}(E)$ of $\check{S}_{2n}(E)$; this is a compact $(6n-4+\dim Z)$–dimensional manifold with corners, and there is a natural

extension of $P_i(\Gamma)\colon \check{S}_{2n}(E) \to SE$ to $S_{2n}(E)$; see [8, Section 3] for details. By Stokes' theorem,

$$\sum_i^{3n}(-1)^{\sum_{j=1}^{k-1}\deg(\eta_j)}X_n\bigg(E;((\eta_i)_{i\in\underline{3n}})\bigg(\frac{d\eta_k}{\eta_k}\bigg)\bigg)$$

$$= \sum_{\Gamma\in\mathcal{E}_n}\int_{S_{2n}(E)}d\bigg(\bigwedge_i P_i(\Gamma)^*\eta_i\bigg)[\Gamma]$$

$$= \sum_{\Gamma\in\mathcal{E}_n}\int_{\partial S_{2n}(E)}\bigg(\bigwedge_i P_i(\Gamma)^*\eta_i\bigg)[\Gamma]$$

$$= \sum_{\Gamma\in\mathcal{E}_n}\int_{\check{S}_{2n}(E|_{\partial Z})}\bigg(\bigwedge_i P_i(\Gamma)^*\eta_i|_{S(E|_{\partial Z})}\bigg)[\Gamma]$$

$$\qquad + \sum_{\Gamma\in\mathcal{E}_n}\int_{\partial S_{2n}(E)\backslash\check{S}_{2n}(E|_{\partial Z})}\bigg(\bigwedge_i P_i(\Gamma)^*\eta_i\bigg)[\Gamma].$$

Lemmas 2,18, 2.19, 2.20 and 2.21 of [8] imply that

$$\sum_{\Gamma\in\mathcal{E}_n}\int_{\partial S_{2n}(E)\backslash\check{S}_{2n}(E|_{\partial Z})}\bigg(\bigwedge_i P_i(\Gamma)^*\eta_i\bigg)[\Gamma] = 0.$$

The lemma is proved.   □

Lemma 5.2 Let $E \to Z$ and $E_0 \to Z_0$ be oriented $\mathbb{R}^3$–bundles over manifolds with corners Z and Z_0. A bundle map $f\colon E \to E_0$ induces a map $p(f)\colon SE \to SE_0$. Let $\eta_i = p(f)^*\eta_i^0$ for a closed form η_i^0 on SE_0 for $i \in \underline{3n}$ satisfying $\sum_{i=1}^{3n}\deg(\eta_i) = 6n-4+\dim Z$. If $\dim Z_0 < \dim Z$, then $X_n(E;(\eta_i)_{i\in\underline{3n}}) = 0$.

Proof We have

$$X_n(E;(\eta_i)_{i\in\underline{3n}}) = \sum_{\Gamma\in\mathcal{E}_n}\int_{\check{S}_{2n}(E)}\bigg(\bigwedge_i P_i(\Gamma)^*p(f)^*\eta_i^0\bigg)[\Gamma].$$

The map

$$\check{S}_{2n}(E) \xrightarrow{\prod_i P_i(\Gamma)} (SE)^{3n} \xrightarrow{p(f)^{3n}} (SE_0)^{3n}$$

factors through $\check{S}_{2n}(E_0)$. Hence we have

$$\bigg(p(f)^{3n}\circ\prod_i P_i(\Gamma)\bigg)^*\eta_i^0 \in \mathrm{Im}\big(\Omega^{6n-4+\dim Z}(\check{S}_{2n}(E_0)) \to \Omega^{6n-4+\dim Z}(\check{S}_{2n}(E))\big).$$

Since

$$\dim\check{S}_{2n}(E_0) = 6n-4+\dim Z_0 < 6n-4+\dim Z,$$

we have $\Omega^{6n-4+\dim Z}(\check{S}_{2n}(E_0)) = 0$. Then for any Γ, we have

$$\int_{\check{S}_{2n}(E)} \bigwedge_i P_i(\Gamma)^* p(f)^* \eta_i^0 = 0. \qquad \square$$

5.2 Proofs of Theorems 4.1 and 4.2

These proofs consist of three steps.

Step 1 (well-definedness of the alternative description of the anomaly term) Here we prove the following proposition.

Proposition 5.3 *There exist constants $\mu_n \in \mathcal{A}_n(\varnothing)$ such that for any closed oriented 4–manifold X' where the Euler characteristic of each component of X' is zero, for any $\mathbb{R}^3$–subbundle $T^v X'$ of TX' and for any closed 2–form W_i' of $T^v X'$ as in Section 4.1,*

$$\mu_n \operatorname{Sign} X' = X_n(T^v X'; (W_i')_{i \in \underline{3n}}).$$

To prove this proposition, we prepare a lemma.

We assume that $\operatorname{Sign} X' = 0$. Let $v_{X'}$ be the unit vector field of TX' such that $T^v X'$ is the normal bundle of $v_{X'}$ in TX'.

Lemma 5.4 *There exists a compact 5–manifold Z and there exists an oriented 3–subbundle $T^v Z$ of TZ such that*

- *$\partial Z = X' \sqcup X'$, and*
- *$T^v Z|_{X'} = T^v X'$.*

Proof Since $\operatorname{Sign} X' = 0$, there exists a compact connected 5–manifold Z_0 such that $\partial Z_0 = X'$. Let $v_{Z_0} \in \Gamma T Z_0|_{X'}$ be the outward unit vector field of $X' = \partial Z_0$. By performing surgery along the knots generating $H_1(Z_0; \mathbb{Z}/2)$ if necessary, we may assume that $H_1(Z_0; \mathbb{Z}/2) \cong H^4(Z_0; \partial Z_0; \mathbb{Z}/2) = 0$. Thus the primary obstruction to extending the 2–framing $(v_{Z_0}, v_{X'})$ of $T Z_0|_{X'}$ into Z_0 is in $H^5(Z_0, \partial Z_0; \pi_4(V_{5,2}))$, where $V_{5,2}$ is the real Stiefel manifold. It is known that $\pi_4(V_{5,2}) = \mathbb{Z}/2$; see [15, Theorem 10]. Then this primary obstruction is in $H^5(Z_0, \partial Z_0; \mathbb{Z}/2)$. Let $Z = Z_0 \# Z_0$. The connected sum $Z_0 \# Z_0$ is given by $(Z_0 \setminus B^5) \cup [0,1] \times S^4 \cup (Z_0 \setminus B^5)$, where B_5 is a small ball in Z_0. We can extend the 2–framing $(v_{Z_0} \sqcup v_{Z_0}, v_{X'} \sqcup v_{X'})$ of $T Z|_{X' \sqcup X'}$ to $Z_0 \setminus B_5 \sqcup Z_0 \setminus B_5$. Then the obstruction to extending this 2–framing to TZ is concentrated in $[0,1] \times S^4$. Since $H^5(S^4; \pi_4(V_{5,2})) = \mathbb{Z}/2$, there are

two 2–framings on S^4 up to homotopy. So the homotopy class of the 2–framing on $\{0\} \times S^4$ coincides with that of the 2–framing on $\{1\} \times S^4$. Thus we can extend this 2–framing to $[0, 1] \times S^4$. Then we can also extend it to Z. So we can take $T^v Z$ as the orthonormal bundle of the $\mathbb{R}^2$–subbundle spanned by the extended 2–framing. $\square$

Let $F_Z \to ST^v Z$ be the tangent bundle along the fiber of the sphere bundle $ST^v Z \to Z$. Let $\widetilde{W}_i$ be a closed 2–form on $ST^v Z$ that represents half of the Euler class $e(F_Z)$ of F_Z and satisfies $\widetilde{W}_i|_{ST^v X} = W_i$. By Lemma 5.1, we have

$$X_n(T^v Z|_{\partial Z}; (W_i)_{i \in \underline{3n}}) = \sum_{k=1}^{3n} X_n\left(T^v Z; ((\widetilde{W}_i)_{i \in \underline{3n}})\left(\frac{d\widetilde{W}_k}{\widetilde{W}_k}\right)\right)$$
$$= 0.$$

This lemma implies that for any closed 4–manifold X' satisfying $\chi(X') = 0$, the number

$$X_n(ST^v X'; (W'_i)_{i \in \underline{3n}})$$

is a cobordism invariant. In fact, for any closed 4–manifolds X'_1 and X'_2 satisfying $\chi(X'_1) = \chi(X'_2) = 0$ and $\operatorname{Sign} X'_1 = \operatorname{Sign} X'_2$ and 2–forms $W'_{1,i}$ and $W'_{2,i}$ as above,

$$X_n\big(ST^v X'_1; (W'_{1,i})_{i \in \underline{3n}}\big) - X_n\big(ST^v X'_2; (W'_{2,i})_{i \in \underline{3n}}\big)$$
$$= X_n\big(ST^v(X'_1 \sqcup -X'_2); (W'_{1,i} \sqcup W'_{2,i})_{i \in \underline{3n}}\big)$$
$$= 0$$

by the above lemma.

We define a constant $\mu_n \in \mathcal{A}_n(\varnothing)$ as follows:

$$\mu_n = \tfrac{1}{2} X_n\big(ST^v(\mathbb{C}P^2 \# \mathbb{C}P^2 \# T^4 \# T^4); (W_i)_{i \in \underline{3n}}\big).$$

Proof of Proposition 5.3 Since the cobordism group of closed 4–manifolds is $\mathbb{Z}$ and $\operatorname{Sign}(\mathbb{C}P^2 \# \mathbb{C}P^2 \# T^4 \# T^4) = 2$, we have

$$\mu_n \operatorname{Sign} X' = X_n(ST^v X'; (W'_i)_{i \in \underline{3n}})$$

for any closed 4–manifold X' with $\chi(X') = 0$. $\square$

Let Y be a closed 3–manifold with a basepoint $\infty \in Y$. Let X be a compact connected 4–manifold with $\partial X = Y$ and $\chi(X) = 0$. We take a collection of closed 2–forms $(W_i)_{i \in \underline{3n}}$ on $ST^v X$ as in Section 4.1.

Corollary 5.5 $-\mu_n \operatorname{Sign} X + X_n(T^v X; (W_i)_{i \in \underline{3n}})$ *is independent of the choice of X and W_i as above.*

Proof Let $\bar{X}$ be an alternative choice for X. We take an $\mathbb{R}^3$–bundle $T^v\bar{X}$ and a collection of 2–forms $(\bar{W}_i)_{i\in\underline{3n}}$ as above. Let $\iota\colon -\bar{X}\to\bar{X}$ be the identity map (but orientation reversing). Since the $\mathbb{R}^3$–bundles T^vX and $\iota^*T^v\bar{X}$ are compatible on Y, we have an $\mathbb{R}^3$–subbundle $T^vX\cup\iota^*T^v\bar{X}$ of $T(X\cup(-\bar{X}))$. We have that $S\iota^*T^v\bar{X}$ coincides with $ST^v\bar{X}$ as a manifold, but it has the opposite orientation. Then the $\bar{W}_i$ are closed 2–forms on $S\iota^*T^v\bar{X}$ such that $\bar{W}_i|_{ST(-Y)} = W_i|_{STY}$. Therefore,

$$\mu_n(\operatorname{Sign} X - \operatorname{Sign}\bar{X}) = X_n(T^vX\cup\iota^*T^v\bar{X}; (W_i\cup\bar{W}_i)_{i\in\underline{3n}})$$
$$= X_n(T^vX; (W_i)_{i\in\underline{3n}}) + X_n(\iota^*T^v\bar{X}; (\bar{W}_i)_{i\in\underline{3n}})$$
$$= X_n(T^vX; (W_i)_{i\in\underline{3n}}) - X_n(T^v\bar{X}; (\bar{W}_i)_{i\in\underline{3n}}). \qquad\square$$

Step 2 $(z_n((\omega_i)_{i\in\underline{3n}})+\mu_n\operatorname{Sign}X-X_n(T^vX; (W_i)_{i\in\underline{3n}})$ is a topological invariant of Y)
Let Y be a rational homology 3–sphere and X a compact connected 4–manifold with $\partial X = Y$ and $\chi(X) = 0$. Let τ_Y be an admissible framing of $T(Y\setminus\infty)$. Let $\tau_Y' = \tau_Y|_{B_Y}\cup\tau_{S^3}|_{N(\infty;S^3)}$. So τ_Y' is a framing of TY. Take a 2–form $W(\tau_Y,\omega_{S^2,i})$ on X as in Section 4.1 such that $W(\tau_Y,\omega_{S^2,i})|_{STY} = p(\tau_Y')^*\omega_{S^2,i}$ for a volume form $\omega_{S^2,i}$ on S^2.

Lemma 5.6 *We have that $X_n(T^vX; (W(\tau_Y,\omega_{S^2,i}))_{i\in\underline{3n}})$ is independent of the choice of $\omega_{S^2,1},\ldots,\omega_{S^2,3n}$.*

Proof Let $\omega_{S^2,i}'$ be an alternative choice for $\omega_{S^2,i}$. Since we have $[\omega_{S^2,i}'] = [\omega_{S^2,i}]\in H^2(S^2;\mathbb{R})$, there exists a closed 2–form $\Omega_{S^2,i}$ on $[0,1]\times S^2$ that satisfies $\Omega_{S^2,i}|_{\{0\}\times S^2} = \omega_{S^2,i}$ and $\Omega_{S^2,i}|_{\{1\}\times S^2} = \omega_{S^2,i}'$.

Let $[0,1]\times STY\subset ST^vX$ be the collar of ST^vX such that $\{0\}\times STY = \partial ST^vX$. We take $W(\tau_Y,\omega_{S^2,i})|_{[0,1]\times STY} = (\operatorname{id}\times p(\tau_Y))^*\Omega_{S^2,i}$. Thus $W(\tau_Y,\omega_{S^2,i})|_{\{1\}\times STY} = p(\tau_Y')^*\omega_{S^2,i}'$. Then we have

$$X_n\big(T^vX; (W(\tau_Y,\omega_{S^2,i}))_{i\in\underline{3n}}\big) - X_n\big(T^vX; (W(\tau_Y,\omega_{S^2,i}'))_{i\in\underline{3n}}\big)$$
$$= \sum_{\Gamma\in\mathcal{E}_n}\int_{[0,1]\times\check{S}_{2n}(TY)}\bigwedge_i P_i(\Gamma)^*W(\tau_Y,\omega_{S^2,i})[\Gamma]$$
$$= \sum_{\Gamma\in\mathcal{E}_n}\int_{[0,1]\times\check{S}_{2n}(TY)}\bigwedge_i P_i(\Gamma)^*(\operatorname{id}\times p(\tau_Y))^*\Omega_{S^2,i}[\Gamma].$$

In Lemma 5.2, we take

$$Z = [0,1]\times Y, \qquad E = [0,1]\times TY, \quad f = \operatorname{id}\times(p\circ\tau_Y),$$
$$Z_0 = [0,1]\times\{\mathrm{pt}\}, \quad E_0 = [0,1]\times\mathbb{R}^3, \quad \eta_i = \Omega_{S^2,i}.$$

Here $p\colon \underline{\mathbb{R}}^3 \to \mathbb{R}^3$ is the standard projection. For any $\Gamma \in \mathcal{E}_n$, we have

$$\int_{[0,1]\times\check{S}_{2n}(TY)} \bigwedge_i P_i(\Gamma)^*(\mathrm{id}\times p(\tau_Y))^*\Omega_{S^2,i} = 0.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Thanks to Corollary 5.5 and Lemma 5.6,

$$-\mu_n \operatorname{Sign} X_{S^3} + X_n\big(T^v S^3; (W(\tau_{\mathbb{R}^3}, \omega_{S^2,i}))_{i\in\underline{3n}}\big)$$

is independent of the choice of a 4–manifold X_{S^3} bounded by S^3 and of the forms $\omega_{S^2,1}, \ldots, \omega_{S^2,3n}$. We define

$$c_n = -\mu_n \operatorname{Sign} X_{S^3} + X_n(T^v S^3; (W(\tau_{\mathbb{R}^3}, \omega_{S^2,i}))_{i\in\underline{3n}}) \in \mathcal{A}_n(\varnothing).$$

Let $I_n((\omega_i)_{i\in\underline{3n}}) = -\mu_n \operatorname{Sign} X + X_n(T^v X; (W_i)_{i\in\underline{3n}}) - c_n$.

Proposition 5.7 $z_n((\omega_i)_{i\in\underline{3n}}) - I_n((\omega_i)_{i\in\underline{3n}})$ *is a topological invariant of* Y.

Proof Let $\omega'_{S^2,i}$ and ω'_i be alternative choices for $\omega_{S^2,i}$ and ω_i, respectively. We take a corresponding 2–form W'_i in $ST^v X$.

By Lemma 3.5, there exist closed 2–forms Ω_i on $[0,1]\times STB_Y$ such that

$$z_n((\omega'_i)_{i\in\underline{3n}}) - z_n((\omega_i)_{i\in\underline{3n}}) = X_n([0,1]\times TB_Y; (\Omega_i)_{i\in\underline{3n}}).$$

Let $\widetilde{X} = X \cup_{\partial X=\{0\}\times Y} [0,1]\times Y \cup_{\{1\}\times Y=\partial X} -X$. Then

$$\widetilde{\Omega}_i = W_i \cup (\Omega_i \cup (\mathrm{id}\times p(\tau_{S^3})|_{N(\infty;Y)})^*\Omega_{S^2,i}) \cup W'_i$$

is a closed 2–form on $ST^v\widetilde{X} = ST^v X \cup [0,1]\times STY \cup -ST^v X$. Thanks to Proposition 5.3, $X_n(T^v\widetilde{X}; (\widetilde{\Omega}_i)_{i\in\underline{3n}}) = \mu_n \operatorname{Sign} \widetilde{X} = 0$. Thus

$I_n((\omega'_i)_{i\in\underline{3n}}) - I_n((\omega_i)_{i\in\underline{3n}})$

$$\begin{aligned}
&= X_n(T^v X; (W'_i)_{i\in\underline{3n}}) - X_n(T^v X; (W_i)_{i\in\underline{3n}}) \\
&= -X_n(T^v\widetilde{X}; (\widetilde{\Omega}_i)_{i\in\underline{3n}}) \\
&\quad + X_n([0,1]\times TY; (\Omega_i \cup ((\mathrm{id}\times p(\tau_{S^3})|_{N(\infty;Y)})^*\Omega_{S^2,i})_{i\in\underline{3n}}) \\
&= X_n([0,1]\times TY; (\Omega_i \cup (\mathrm{id}\times p(\tau_{S^3})|_{N(\infty;Y)})^*\Omega_{S^2,i})_{i\in\underline{3n}}) \\
&= X_n([0,1]\times TB_Y; (\Omega_i)_{i\in\underline{3n}}) \\
&\quad + X_n([0,1]\times TN(\infty;Y); ((\mathrm{id}\times p(\tau_{S^3})|_{N(\infty;Y)})^*\Omega_{S^2,i})_{i\in\underline{3n}}).
\end{aligned}$$

Thanks to Lemma 5.2, $X_n([0,1] \times TN(\infty; Y); ((\mathrm{id} \times p(\tau_{S^3}))^* \Omega_{S^2, i})_{i \in \underline{3n}}) = 0$. Thus $I_n((\omega_i')_{i \in \underline{3n}}) - I_n((\omega_i)_{i \in \underline{3n}}) = X_n([0,1] \times TB_Y; (\Omega_i)_{i \in \underline{3n}})$. Therefore,

$$z_n((\omega_i)_{i \in \underline{3n}}) - I_n((\omega_i)_{i \in \underline{3n}}) - (z_n((\omega_i')_{i \in \underline{3n}}) - I_n((\omega_i')_{i \in \underline{3n}})) = 0. \qquad \square$$

Step 3 (proof of $I_n((\omega_i)_{i \in \underline{3n}}) = \widetilde{z}_n^{\mathrm{anomaly}}((\omega_i)_{i \in \underline{3n}})$) Recall that $\widetilde{z}_n^{\mathrm{anomaly}}((\omega_i)_{i \in \underline{3n}})$ is defined in Definition 3.9. We next prove $I_n((\omega_i)_{i \in \underline{3n}}) = \widetilde{z}_n^{\mathrm{anomaly}}((\omega_i)_{i \in \underline{3n}})$ and determine constants μ_n and c_n.

Let Y be a closed 3–manifold (which may not be a rational homology 3–sphere) with a basepoint ∞. We denote $I_n(\tau_Y) = I_n((\omega_{\tau_Y})_{i \in \underline{3n}})$ for an admissible framing τ_Y of $T(Y \setminus \infty)$.

Since Definition 3.9 and Proposition 5.7 imply that $I_n((\omega_i)_{i \in \underline{3n}}) - \widetilde{z}_n^{\mathrm{anomaly}}((\omega_i)_{i \in \underline{3n}})$ does not depend on the ω_i, we only need to show that

$$I_n(\tau_Y) = -\tfrac{1}{4} \sigma_{Y \setminus \infty}(\tau_Y) \delta_n$$

for some admissible framing τ_Y of $T(Y \setminus \infty)$.

Lemma 5.8 $\qquad\qquad I_n(\tau_{\mathbb{R}^3}) = -\tfrac{1}{4} \sigma_{\mathbb{R}^3}(\tau_{\mathbb{R}^3}) \delta_n = 0.$

Proof Because the definition of c_n, we have $I_n((\omega_{\tau_{\mathbb{R}^3}})_{i \in \underline{3n}}) = 0$. $\qquad\qquad\square$

Since $z_n^{\mathrm{KKT}}(S^3; \tau) - I_n((\omega_\tau)_{i \in \underline{3n}})$ is independent of the choice of an admissible framing τ of $T\mathbb{R}^3$, we have the following corollary.

Corollary 5.9 *For any admissible framing τ of $T\mathbb{R}^3$, the equation $I_n((\omega_\tau)_{i \in \underline{3n}}) = -\tfrac{1}{4} \sigma_{\mathbb{R}^3}(\tau) \delta_n$ holds.*

We introduce a cobordism group $\Omega_3^{\mathrm{Sign}=0}$, generated by all framed closed 3–manifolds (Y, τ) and divided by a cobordism relation $\sim$: $(Y, \tau) \sim \varnothing$ if and only if there exists a pair (X, τ_X) such that

- Sign $X = 0$, and
- τ_X is a 3–framing of TX such that $\tau_X|_Y = \tau$.

For a framed 3–manifold (Y, τ), set $J_n(\tau) = -\mu_n \operatorname{Sign} X + X_n(T^v X; (W_\tau)_{i \in \underline{3n}})$. Here X and $T^v X$ are same as above, and W_τ is a closed 2–form of $ST^v X$ satisfying $[W_\tau] = \tfrac{1}{2} e(F_X)$ and $W_\tau|_{STY} = p(\tau)^* \omega_{S^2}$. Then $I_n(\tau_Y) = J_n(\tau_Y') - c_n$. Recall that $\sigma_{Y \setminus \infty}(\tau_Y) = \sigma_Y(\tau_Y') - 2$.

Lemma 5.10 *Both $J_n(\bullet)$ and $-\tfrac{1}{4} \sigma_\bullet(\bullet) \delta_n$ factor through $\Omega_3^{\mathrm{Sign}=0}$.*

Proof By definition of the signature defect, $\sigma_\bullet(\bullet)$ is obviously a cobordism invariant.

We show that $J_n(\bullet)$ is a cobordism invariant. Let (X, τ_X) be a cobordism between a framed 3–manifold $(Y, \tau_X|_Y)$ and $\varnothing$. Let $T^v X$ be the $\mathbb{R}^3$–subbundle of TX spanned by τ_X. Then τ_X induces the map $p(\tau_X)\colon ST^v X \to S^2$. Take $W_{\tau_X} = p(\tau_X)^* \omega_{S^2}$. Then we have

$$J_n(\tau_X|_Y) = -\mu_n \operatorname{Sign} X + X_n(T^v X; (W_{\tau_X})_{i \in \underline{3n}})$$

$$= \sum_{\Gamma \in \mathcal{E}_n} \int_{\check{S}_{2n}(T^v X)} \bigwedge_i P_i(\Gamma)^* p(\tau_X)^* \omega_{S^2}[\Gamma].$$

Lemma 5.2 implies that the above integral vanishes for any $\Gamma \in \mathcal{E}_n$. Therefore, $J_n(\tau_X|_Y) = 0$. $\square$

Lemma 5.11 (1) $\Omega_3^{\operatorname{Sign}=0} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$ and (S^3, τ_{S^3}) *is a generator.*

(2) $c_n = -\frac{1}{2}\delta_n$.

(3) *For any framed closed 3–manifold* (Y, τ), $J_n(\tau) = -\frac{1}{4}\sigma_Y(\tau)\delta_n$.

(4) $\mu_n = -\frac{3}{4}\delta_n$.

Proof (1) Let (Y, τ) be a framed closed 3–manifold. We use the same symbol (Y, τ) for its cobordism class. We give the spin structure on Y using τ. Since the 3–dimensional spin cobordism group is equal to zero, there exists a spin 4–manifold X such that $\partial X = Y$. Let $k = \chi(X)$. We may assume that $k \geq 0$, by replacing X by $X \# n(S^2 \times S^2)$ for a sufficiently large integer n if necessary. Let X_0 be the spin 4–manifold obtained by removing k disjoint 4–balls; ie $X_0 = X \setminus k B^4$. Then $\chi(X_0) = 0$ and $\partial X_0 = Y \sqcup k(-S^3)$. Here $-S^3$ is S^3 with the opposite orientation. We denote by $-S_i^3$ the ith $(-S^3)$–boundary of X_0. Then $\partial X_0 = Y \sqcup -S_1^3 \sqcup \cdots \sqcup -S_k^3$.

By obstruction theory, there is a 3–framing τ_{X_0} of TX_0 that satisfies $\tau_{X_0}|_Y = \tau$, and $\tau_{X_0}|_{-S_i^3}$ is a framing of $T(-S_i^3)$ for each i; see [5] for more details. Let $\tau_i = -(\tau_{X_0}|_{-S_i^3})$. Thus $(Y, \tau) = \sum_{i=1}^k (S^3, \tau_i)$ in $\Omega_3^{\operatorname{Sign}=0}$.

Let τ^s be any framing of TS^3. It is known that $\sigma_{S^3}(\tau^s) \in 4\mathbb{Z} + 2$; see [5]. Let $X_1 = (S^4 \setminus (|\sigma_{S^3}(\tau^s)| + 2)B^4) \# \frac{1}{2}|\sigma_{S^3}(\tau^s)|(S^2 \times S^2)$. Then X_1 is a spin 4–manifold satisfying $\operatorname{Sign}(X_1) = 0$, $\chi(X_1) = 0$ and $\partial X_1 = (S^3 \sqcup S^3) \sqcup |\sigma(\tau^s)|(-S^3)$. By obstruction theory, there exists a 3–framing τ_{X_1} of TX such that $\tau_{X_1}|_{S^3} = \tau^s$ and $\tau_{X_1}|_{-S^3} = \tau_{S^3}^+$ or $\tau_{X_1}|_{-S^3} = \tau_{S^3}^-$, where $\tau_{S^3}^\pm$ is a framing of TS^3 that satisfies $\sigma_{S^3}(\tau_{S^3}^\pm) = \pm 2$, respectively. Thus $2(Y, \tau) = n_+(S^3, \tau_{S^3}^+) + n_-(S^3, \tau_{S^3}^-)$ in $\Omega_3^{\operatorname{Sign}=0}$ for some nonnegative integers n_+ and n_-.

The spin 4–manifold $[0, 1] \times S^3$ is a cobordism between $(S^3, \tau_{S^3}^+) \sqcup (S^3, \tau_{S^3}^-)$ and $\varnothing$. Therefore, we have $2(Y, \tau) = (n_+ - n_-)(S^3, \tau_{S^3}^+)$ in $\Omega_3^{\operatorname{Sign}=0}$.

(2)   Since $0 = I_n(\tau_{\mathbb{R}^3}) = J_n(\tau_{S^3}) - c_n$, we have $J_n(\tau_{S^3}) = c_n$. Let $\tau_{S^3}^-$ be a framing of TS^3 satisfying $\sigma_{S^3}(\tau_{S^3}^-) = -2$. Since $J_n(\bullet)$ is a cobordism invariant, $J_n(\tau_{S^3}^-) = -J_n(\tau_{S^3}) = -c_n$. Thanks to Corollary 5.9,

$$-2c_n = J_n(\tau_{S^3}^-) - c_n = -\tfrac{1}{4}\sigma_{S^3}(\tau_{S^3}^-)\delta_n + \tfrac{1}{2}\delta_n = \delta_n.$$

Then we have $c_n = -\tfrac{1}{2}\delta_n$.

(3) $J_n(\tau_{S^3}) = c_n = -\tfrac{1}{2}\delta_n = -\tfrac{1}{4}\sigma_{S^3}(\tau_{S^3})\delta_n$ and τ_{S^3} is a generator.

(4)   Recall that $-K3$ is a spin 4–manifold with $\chi(-K3) = 24$ and $\mathrm{Sign}(-K3) = 16$. Let $X_a = (-K_3) \# 11T^4 \setminus (B^4 \sqcup B^4)$. Then X_a is a spin 4–manifold satisfying $\chi(X_a) = 0$ and $\mathrm{Sign}(X_a) = 16$. It is possible to deal with $\partial X_a = S^3 \sqcup -S^3$. There is a 3–framing τ_{X_a} of TX_a such that $\tau_{X_a}|_{\partial X_a}$ is a framing of $T(\partial X_a)$. The 3–framing τ_{X_a} spans the $\mathbb{R}^3$–subbundle $T^v X_a$ of TX_a. Let $W = p(\tau_{X_a})^* \omega_{S^2}$ be the closed 2–form on $ST^v X_a$. Thanks to Lemma 5.2, $X_n(T^v X_a; (W)_{i \in \underline{3n}}) = 0$. Then we have

$$\begin{aligned} J_n(\tau_{X_a}|_{\partial X_a}) &= -\mu_n \, \mathrm{Sign}\, X_a + X_n(T^v X_a; (W)_{i \in \underline{3n}}) \\ &= -16\mu_n. \end{aligned}$$

By (3) and the definition of the signature defect, we have

$$J_n(\tau_{X_a}|_{\partial X_a}) = -\tfrac{1}{4}\sigma_{\partial X_a}(\tau_{X_a}|_{\partial X_a})\delta_n = \tfrac{3}{4}\,\mathrm{Sign}\,X_a\delta_n = 12\delta_n.$$

Therefore $\mu_n = -\tfrac{3}{4}\delta_n$. $\qquad\qquad\square$

Corollary 5.12   *For any admissible framing τ_Y of $T(Y \setminus \infty)$,*

$$I_n(\tau_Y) = -\tfrac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n.$$

Proof $I_n(\tau_Y) = J_n(\tau_Y') - c_n = -\tfrac{1}{4}\sigma_Y(\tau_Y')\delta_n + \tfrac{1}{2}\delta_n = -\tfrac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n.$ $\qquad\square$

5.3   Proof of Theorem 4.7

Let f be an admissible Morse function with respect to $a \in S^2$. Let $\mathrm{Crit}(f) = \{p_1, \ldots, p_k, q_1, \ldots, q_k\}$ where $\mathrm{ind}(p_i) = 2$ and $\mathrm{ind}(q_i) = 1$. Let

$$\hat{\partial} \colon C_2(Y \setminus \infty; f) \to C_1(Y \setminus \infty; f), \quad \hat{\partial}[p_i] = \sum_j \hat{\partial}_{ij}[q_j],$$

be the boundary map of the Morse complex.

We first give a compactification $\mathcal{M}_S(\pm f)$ of $\mathcal{M}(f) \cup \mathcal{M}(-f)$ and then show that $\mathcal{M}_S(\pm f)$ is a 4–cycle in $(C_2(Y), \partial C_2(Y))$. Recall that the 4–manifold $M_\to(f)$ is the image of the inclusion map

$$\varphi_f \colon (Y \setminus \infty) \times (0, \infty) \hookrightarrow (Y \setminus \infty) \times (Y \setminus \infty), \quad (x, t) \mapsto (x, \Phi_f^t(x)),$$

and the map φ_f preserves orientations. Let $\bar{M}_\rightarrow(f)$ be the closure of $M_\rightarrow(f)$ in $(Y \setminus \infty)^2 \setminus \Delta$. With our orientation conventions, the following two lemmas hold.

Lemma 5.13 [11, Lemma 4.3] [14, Proposition 3.4] $\bar{M}_\rightarrow(f)$ *is a manifold with corners satisfying the following conditions:*

(1) $\mathrm{int}\, \bar{M}_\rightarrow(f) = M_\rightarrow(f)$, *and*

(2) $\partial \bar{M}_\rightarrow(f) = -\sum_i \mathcal{D}_{p_i} \times \mathcal{A}_{p_i} - \sum_j \mathcal{D}_{q_j} \times \mathcal{A}_{q_j}$.

Lemma 5.14 [14, Proposition 3.17] *There are manifolds with corners* $\bar{\mathcal{A}}_{q_j}$ *and* $\bar{\mathcal{D}}_{p_i}$ *satisfying the following conditions:*

(1) $\mathrm{int}\, \bar{\mathcal{A}}_{q_j} = \mathcal{A}_{q_j}$, $\mathrm{int}\, \bar{\mathcal{D}}_{p_i} = \mathcal{D}_{p_i}$,

(2) $\partial \bar{\mathcal{A}}_{q_j} = -\sum_i \hat{\partial}_{ij} \mathcal{A}_{p_i}$, $\partial \bar{\mathcal{D}}_{p_i} = -\sum_j \hat{\partial}_{ij} \mathcal{D}_{q_j}$.

We denote by $\bar{\mathcal{A}}_{p_i} \to Y$ the extension of $\mathcal{A}_{p_i} \to Y$ to $\bar{\mathcal{A}}_{p_i}$ and we write $\bar{\mathcal{A}}_{p_i}$ instead of $\bar{\mathcal{A}}_{p_i} \to Y$. We also define $\bar{\mathcal{D}}_{p_i}$, $\bar{\mathcal{A}}_{q_j}$, and so on.

Definition 5.15 (1) $\mathcal{M}_S(f)$ is the closure, in $C_2(Y)$, of

$$\bar{M}_\rightarrow(f) + \sum_{i,j} \hat{g}_{ji}\big((\bar{\mathcal{D}}_{p_i} \times \bar{\mathcal{A}}_{q_j}) \cap (Y \setminus \infty)^2 \setminus \Delta\big).$$

(2) $\mathcal{M}_S(\pm f) = \mathcal{M}_S(f) + \mathcal{M}_S(-f)$.

Thanks to the Morse–Smale condition, $\mathcal{M}_S(f)$ and $\mathcal{M}_S(\pm f)$ are 4–chains of $C_2(Y)$.

Lemma 5.16 $\mathcal{M}_S(f)$, $\mathcal{M}_S(-f)$ *and* $\mathcal{M}_S(\pm f)$ *are 4–cycles in* $(C_2(Y), \partial C_2(Y))$.

Proof Since $\mathrm{Im}(\partial(\bar{\mathcal{D}}_{p_i} \times \bar{\mathcal{A}}_{q_j}) \to Y^2) = -\sum_k \hat{\partial}_{kj} \bar{\mathcal{D}}_{p_i} \times \bar{\mathcal{A}}_{p_k} - \sum_k \hat{\partial}_{ik} \bar{\mathcal{D}}_{q_k} \times \bar{\mathcal{A}}_{q_j}$,

$$\mathrm{Im}\bigg(\sum_{i,j} \hat{g}_{ji} \partial(\bar{\mathcal{D}}_{p_i} \times \bar{\mathcal{A}}_{q_j} \to Y^2)\bigg) = -\sum_{i,j,k} \hat{g}_{ji} \hat{\partial}_{kj} \bar{\mathcal{D}}_{p_i} \times \bar{\mathcal{A}}_{p_k} - \sum_{i,j,k} \hat{g}_{ji} \hat{\partial}_{ik} \bar{\mathcal{D}}_{q_k} \times \bar{\mathcal{A}}_{q_j}$$

$$= -\sum_{i,j,k} \delta_{ki} \bar{\mathcal{D}}_{p_i} \times \bar{\mathcal{A}}_{p_k} - \sum_{i,j,k} \delta_{jk} \bar{\mathcal{D}}_{q_k} \times \bar{\mathcal{A}}_{q_j}$$

$$= -\sum_i \bar{\mathcal{D}}_{p_i} \times \bar{\mathcal{A}}_{p_i} - \sum_j \bar{\mathcal{D}}_{q_j} \times \bar{\mathcal{A}}_{q_j}$$

$$= -\partial \bar{M}_\rightarrow(f).$$

Therefore, $\partial \mathcal{M}_S(f) \setminus \partial C_2(Y) = \varnothing$. $\qquad\square$
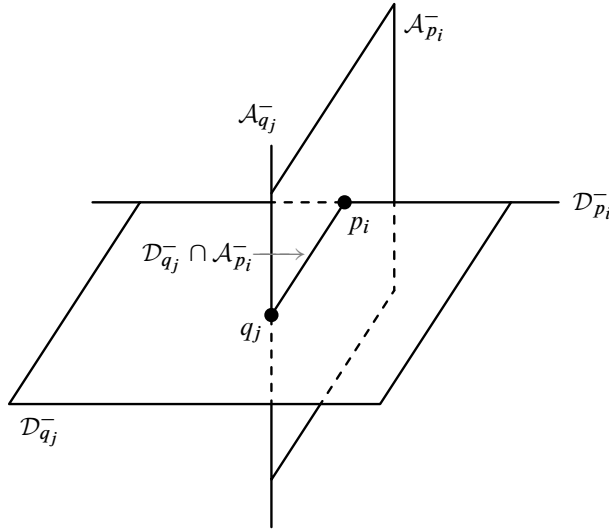
Figure 3: The orientation of $\mathcal{D}_{q_j}^- \cap \mathcal{A}_{p_i}^-$

The critical points of the Morse function $(-f \colon Y \setminus \infty \to \mathbb{R})$ form the set $\mathrm{Crit}(-f) = \{q_1, \ldots, q_k, p_1, \ldots, p_k\}$, where $\mathrm{ind}(q_i) = 2$ and $\mathrm{ind}(p_i) = 1$. We remark that the ascending manifold $\mathcal{A}_{p_i}^-$ of p_i corresponding to $-f$ is $\mathcal{D}_{p_i}$, and the descending manifold $\mathcal{D}_{q_j}^-$ of q_j corresponding to $-f$ is $\mathcal{A}_{q_j}$. We orient $\mathcal{A}_{p_i}^-$ as $\mathcal{D}_{p_i}$ and orient $\mathcal{D}_{q_j}^-$ as $\mathcal{A}_{q_j}$.

Lemma 5.17 *With our conventions, the orientations of $\mathcal{D}_{q_j}^- \cap \mathcal{A}_{p_i}^-$ and $\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}$ are opposite.*

Proof Let $n(\mathcal{D}_{p_i})$ and $n(\mathcal{A}_{q_j})$ be positively oriented normal vector fields of $\mathcal{D}_{p_i}$ and $\mathcal{A}_{q_j}$, respectively. Then $n(\mathcal{D}_{p_i})$ and $n(\mathcal{A}_{q_j})$ are positively oriented normal vector fields of $\mathcal{A}_{p_i}^-$ and $\mathcal{D}_{q_j}^-$, respectively. Since the orientation of the normal bundle $N(\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j})$ induced by the frame $(n(\mathcal{A}_{q_j})|_{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}}\, n(\mathcal{D}_{p_i})|_{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}})$ is opposite to that of the normal bundle $N(\mathcal{D}_{q_j}^- \cap \mathcal{A}_{p_i}^-)$ of the frame $(n(\mathcal{D}_{p_i})|_{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}}, n(\mathcal{A}_{q_j})|_{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}})$, the orientation of $\mathcal{D}_{q_j}^- \cap \mathcal{A}_{p_i}^-$ is opposite to that of $\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}$. (See Figure 3.) □

By this lemma, the sign of each component of $\mathcal{D}_{q_j}^- \cap \mathcal{A}_{p_i}^-$ coincides with the sign of the corresponding component in $\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}$. So the boundary map $\widehat{\partial}_{\overline{ji}}$ (resp. $\widehat{g}_{ij}^-$) corresponding to $-f$ is equal to $\widehat{\partial}_{ij}$ (resp. $\widehat{g}_{ji}$) corresponding to f. Thus $\mathcal{M}_S(-f)$ is equal to the closure of $\overline{M}_{\to}(-f) + \sum_{i,j} \widehat{g}_{ji}((\overline{A}_{q_j} \times \overline{D}_{p_i}) \cap (Y \setminus \infty)^2 \setminus \Delta)$ with our orientation conventions.

Furthermore, we have the following description.

Lemma 5.18 $\partial \mathcal{M}_S(\pm f) \cap STB_Y = \overline{\{[\mathrm{grad}_x\, f], [-\,\mathrm{grad}_x\, f] \mid x \in B_Y \setminus \mathrm{Crit}(f)\}}$.

Proof Note that $(\overline{\mathcal{D}}_{p_i} \times \overline{\mathcal{A}}_{q_j}) \cap \Delta = \overline{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}}$. There exists $\varepsilon \in \{-1, 1\}$ such that the oriented boundary of the closure of $(\mathcal{D}_{p_i} \times \mathcal{A}_{q_j}) \cap ((Y \setminus \infty)^2 \setminus \Delta)$ meets STB_Y as $\varepsilon \pi^{-1}(\overline{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}})$, where $\pi^{-1}(\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j})$ is oriented as the product of $\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}$ by the fiber of STB_Y, which has a canonical orientation. Here $\pi \colon STY \to Y$ is the projection. By definition of $\mathcal{M}_S(f)$, we have

$$\partial \mathcal{M}_S(f) \cap STB_Y = \overline{\{[\mathrm{grad}_x\, f] \mid x \in B_Y \setminus \mathrm{Crit}(f)\}} + \varepsilon \sum_{i,j} \widehat{g}_{ji} \pi^{-1}(\overline{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}}).$$

We recall that $\mathcal{A}_{p_i}^- = \mathcal{D}_{p_i}$ and $\mathcal{D}_{q_j}^- = \mathcal{A}_{q_j}$. The involution $T \colon Y^2 \to Y^2, (x, y) \to (y, x)$ induces maps $T' \colon (\mathcal{D}_{p_i} \times \mathcal{A}_{q_j}) \cap ((Y \setminus \infty)^2 \setminus \Delta) \to (\mathcal{A}_{q_j} \times \mathcal{D}_{p_i}) \cap ((Y \setminus \infty)^2 \setminus \Delta)$ and $T_{C_2(Y)} \colon C_2(Y) \to C_2(Y)$. With our orientation conventions, T' is orientation-preserving so that $T_{C_2(Y)}$ is orientation-preserving from $\varepsilon \pi^{-1}(\overline{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}})$ to the boundary part $\pm \pi^{-1}(\overline{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}})$ of the closure of $(\mathcal{A}_{q_j} \times \mathcal{D}_{p_i}) \cap ((Y \setminus \infty)^2 \setminus \Delta)$, which therefore reads $-\varepsilon \pi^{-1}(\overline{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}})$ since $T_{C_2(Y)}$ reverses the orientation of the fiber of STB_Y. Then we have

$$\partial \mathcal{M}_S(\pm f) \cap STB_Y$$
$$= \overline{\{[\mathrm{grad}_x\, f], [-\,\mathrm{grad}_x\, f]\}} + \varepsilon \sum_{i,j} \widehat{g}_{ji} \pi^{-1}(\overline{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}}) - \varepsilon \sum_{i,j} \widehat{g}_{ji} \pi^{-1}(\overline{\mathcal{D}_{p_i} \cap \mathcal{A}_{q_j}})$$
$$= \overline{\{[\mathrm{grad}_x\, f], [-\,\mathrm{grad}_x\, f]\}}. \qquad \square$$

Lemma 5.19 $\partial \mathcal{M}_S(\pm f) = c(\mathrm{grad}\, f)$ for any admissible Morse function f.

Proof Since $\mathrm{grad}\, f|_{N(\infty; Y)} = a$, if $(x, u) \in \partial \mathcal{M}_S(\pm f) \cap ((Y \setminus \infty) \times ST_\infty Y)$, then $u \in \{a, -a\}$. On the other hand, $\partial \mathcal{M}_S(\pm f) \cap (\{x\} \times ST_\infty Y) = \{(x, a), (x, -a)\}$ as sets for any $x \notin \mathrm{Crit}(f)$. Since $\partial \mathcal{M}_S(\pm f)$ is a 3–cycle, we have

$$\partial \mathcal{M}_S(f) \cap ((Y \setminus \infty) \times ST_\infty Y) = -(Y \setminus \infty) \times \{a, -a\}$$

with our orientation conventions. We have $\mathcal{M}_S(\pm f) \setminus STB_Y = p_Y^{-1}(\{a, -a\})$ by a similar argument. Due to this fact and Lemma 5.18, the proof is complete. $\qquad \square$

We follow the notation $a_1, \ldots, a_{3n}, f_1, \ldots, f_{3n}$ as in Section 4.3. Let $C_{2n}(Y)$ be the compactification of $\check{C}_{2n}(Y)$ given in [8, Section 3]. In the following proposition, the notion "generic $f_1, \ldots, f_{3n}$" means that $\partial C_{2n}(Y) \cap (\bigcap_i P_i(\Gamma)^{-1} \mathcal{M}_S(\pm f_i)) = \varnothing$ for any $\Gamma \in \mathcal{E}_n$.

Remark 5.20 We can show that $\partial C_{2n}(Y) \cap \left(\bigcap_i P_i(\Gamma)^{-1} \mathcal{M}_S(\pm f_i) \right) = \varnothing$ for generic $f_1, \ldots, f_{3n}$ by an argument similar to Lemma 4.2 in [14].

Proof of Theorem 4.7 We have that grad $f_1, \ldots,$ grad f_{3n} are admissible vector fields. We take a propagator $\omega_C(\mathrm{grad}\, f_i)$ whose support is concentrated near $\mathcal{M}_S(\pm f_i)$ for each $i \in \underline{3n}$. Since $f_1, \ldots, f_{3n}$ are generic, it is possible to choose these propagators $\omega_C(\mathrm{grad}\, f_1), \ldots, \omega_C(\mathrm{grad}\, f_{3n})$ satisfying $\bigwedge_i P_i(\Gamma)^* \omega(\mathrm{grad}\, f_i) \equiv 0$ near the boundary of $C_{2n}(Y)$ for any $\Gamma \in \mathcal{E}_n$. Then

$$\#\left(\bigcap_i P_i(\Gamma)^{-1} \mathcal{M}(\pm f_i) \right) = \#\left(\bigcap_i P_i(\Gamma)^{-1} \mathcal{M}_S(\pm f_i) \right)$$

for any $\Gamma \in \mathcal{E}_n$. By intersection theory, we have

$$\int_{C_{2n}(Y)} \bigwedge_i P_i(\Gamma)^* \omega_C(\mathrm{grad}\, f_i) = \frac{1}{2^{3n}} \#\left(\bigcap_i P_i(\Gamma)^{-1} \mathcal{M}_S(\pm f_i) \right)$$

for any $\Gamma \in \mathcal{E}_n$. Therefore,

$$z_n^{\mathrm{KKT}}(Y) = z_n((\omega(\mathrm{grad}\, f_i))_{i \in \underline{3n}}) - \widetilde{z}_n^{\mathrm{anomaly}}((\omega(\mathrm{grad}\, f_i))_{i \in \underline{3n}})$$

$$= \sum_{\Gamma \in \mathcal{E}_n} \left(\int_{C_{2n}(Y)} \bigwedge_i P_i(\Gamma)^* \omega_C(\mathrm{grad}\, f_i) \right) [\Gamma] - \widetilde{z}_n^{\mathrm{anomaly}}((\omega(\mathrm{grad}\, f_i))_{i \in \underline{3n}})$$

$$= \sum_{\Gamma \in \mathcal{E}_n} \frac{1}{2^{3n}} \#\left(\bigcap_i P_i(\Gamma)^{-1} \mathcal{M}_S(\pm f_i) \right) [\Gamma] - \widetilde{z}_n^{\mathrm{anomaly}}((\omega(\mathrm{grad}\, f_i))_{i \in \underline{3n}})$$

$$= \sum_{\Gamma \in \mathcal{E}_n} \frac{1}{2^{3n}} \#\left(\bigcap_i P_i(\Gamma)^{-1} \mathcal{M}(\pm f_i) \right) [\Gamma] - \widetilde{z}_n^{\mathrm{anomaly}}((\omega(\mathrm{grad}\, f_i))_{i \in \underline{3n}}). \quad \square$$

Appendix: Another proof of Proposition 5.3 when $n = 1$

In this section, we give a more direct proof of Proposition 5.3 when $n = 1$. Note that $\mathcal{A}_1(\varnothing) = \mathbb{Q}[\theta]$ and $\#\mathcal{E}_1 = 8$. We first prepare some notation. Let $\pi_1 \colon F_{X'} \to ST^v X'$ be the tangent bundle along the fiber of $\pi_2 \colon ST^v X' \to X'$.

Let $e(F_{X'}) \in H^2(ST^v X')$ be the Euler class of $F_{X'}$, and let $p_1(F_{X'}) \in H^4(ST^v X')$ be the first Pontrjagin class of $F_{X'}$. By a standard argument, the Chern–Weil theory

for example, we have $p_1(F_{X'}) = e(F_{X'})^2$; eg see Corollary 15.8 of [13]. Then

$$X_1\big(T^v X'; (W_i')_{i \in \underline{3n}}\big) = 8 \bigg(\int_{ST^v X'} W_1' \wedge W_2' \wedge W_3' \bigg)[\theta]$$

$$= \int_{ST^v X'} e(F_{X'})^3 [\theta]$$

$$= \int_{ST^v X'} e(F_{X'}) p_1(F_{X'}) [\theta]$$

$$\overset{(*)}{=} \int_{ST^v X'} e(F_{X'}) \pi_2^* p_1(TX') [\theta]$$

$$= 2 \int_{X'} p_1(TX') [\theta]$$

$$= 6[\theta] \operatorname{Sign} X'.$$

Equation $(*)$ is given by $\underline{\mathbb{R}} \oplus F_{X'} = \pi^* T^v X'$ and $\underline{\mathbb{R}} \oplus T^v X' = TX'$. Then we have $X_1(T^v X'; (W_i')_{i \in \underline{3n}}) = 6[\theta] \operatorname{Sign} X'$.

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