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Strong Heegaard diagrams and strong L–spaces

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We study a class of 3–manifolds called strong L–spaces, which by definition admit a certain type of Heegaard diagram that is particularly simple from the perspective of Heegaard Floer homology. We provide evidence for the possibility that every strong L–space is the branched double cover of an alternating link in the three-sphere. For example, we establish this fact for a strong L–space admitting a strong Heegaard diagram of genus 2 via an explicit classification. We also show that there exist finitely many strong L–spaces with bounded order of first homology; for instance, through order eight, they are connected sums of lens spaces. The methods are topological and graph-theoretic. We discuss many related results and questions.

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1 Introduction

The purpose of this paper is to study a family of 3–manifolds called *strong L–spaces*. These manifolds are defined by a combinatorial condition on Heegaard diagrams, and they arise naturally in the context of Heegaard Floer homology.

In its simplest form, the Heegaard Floer homology of a closed, oriented 3–manifold Y is a finitely generated abelian group $\widehat{\mathrm{HF}}(Y)$, defined as follows. We present Y by means of a Heegaard diagram H, consisting of a closed, oriented surface S of genus g and two disjoint unions $\alpha = \alpha_1 \cup \cdots \cup \alpha_g$ and $\beta = \beta_1 \cup \cdots \cup \beta_g$ of embedded circles in S, each of which spans a g–dimensional subspace of $H_1(S;\mathbb{Z})$ and which intersect each other transversally. To such a diagram, we associate a chain complex $\widehat{\mathrm{CF}}(H)$, which is freely generated by the set $\mathfrak{S}(H)$ of unordered g–tuples of points in S with one point on each α circle and one point on each β circle. By adapting the machinery of Lagrangian Floer homology, Ozsváth and Szabó [34] define a differential ∂ on $\widehat{\mathrm{CF}}(H)$ that also depends on some additional choices of analytic data. They prove that the homology $H_*(\widehat{\mathrm{CF}}(H), \partial)$ depends only on the 3–manifold Y and not on the specific choice of Heegaard diagram or analytic data. This homology group is denoted $\widehat{\mathrm{HF}}(Y)$.

Define the *determinant* $\det(Y)$ of a 3–manifold Y to be the order of $H_1(Y;\mathbb{Z})$ if this group is finite (ie when Y is a rational homology sphere) and 0 otherwise. With

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respect to a natural $\mathbb{Z}/2$–grading, Ozsváth and Szabó [33, Proposition 5.1] showed that the Euler characteristic of $\widehat{\mathrm{CF}}(Y)$ is equal to $\det(Y)$. As a result, for any Heegaard diagram H presenting Y, we have

$$|\mathfrak{S}(H)| = \operatorname{rank} \widehat{\mathrm{CF}}(H) \geq \operatorname{rank} \widehat{\mathrm{HF}}(Y) \geq \det(Y). \tag{1}$$

If $\det(Y) \neq 0$ and $\widehat{\mathrm{HF}}(Y)$ is free abelian of rank $\det(Y)$, then Y is called an *L–space*. In view of (1), such manifolds can be seen as having the simplest possible Heegaard Floer homology.[1] Examples of L–spaces include S^3, lens spaces (whence the name), all manifolds with finite fundamental group (Ozsváth and Szabó [35]), and branched double covers of nonsplit alternating (or, more generally, quasi-alternating) knots and links in S^3 (Ozsváth and Szabó [36]). The classification of L–spaces is one of the major outstanding questions in Heegaard Floer theory. For instance, it is conjectured that a rational homology sphere Y is an L–space if and only if $\pi_1(Y)$ is not left-orderable. This conjecture is known to hold (at least mod torsion) for many classes of manifolds, including all geometric, non-hyperbolic manifolds; see Boyer, Gordon and Watson [3].[2]

The minimum size of $\mathfrak{S}(H)$, ranging over all Heegaard diagrams H for a rational homology sphere Y, may be viewed as a measure of the topological complexity of Y. This quantity is called the *simultaneous trajectory number* of Y and is denoted $M(Y)$; see Ozsváth and Szabó [33, Section 1.2]. Both $\det(Y)$ and $\operatorname{rank}(\widehat{\mathrm{HF}}(Y))$ provide lower bounds on $M(Y)$ by (1). The second author and Lewallen introduced the following definition:

Definition 1.1 [21] A closed, oriented 3–manifold Y is called a *strong L–space* if $M(Y) = \det(Y)$. A Heegaard diagram H for Y for which $|\mathfrak{S}(H)| = \det(Y)$ is called a *strong Heegaard diagram*.

By (1), a strong L–space is an L–space. In view of the conjecture mentioned above, the second author and Lewallen [21] proved that the fundamental group of a strong L–space is not left-orderable. As a simple direct consequence, a strong L–space cannot admit an $\mathbb{R}$–covered taut foliation. In fact, Ozsváth and Szabó [31] established the much deeper result that an L–space does not admit any coorientable taut foliation; see also Kazez and Roberts [19].

[1] Some authors define Y to be an L–space under the weaker condition that $\operatorname{rank} \widehat{\mathrm{HF}}(Y) = \det(Y)$, which can be easier to verify. We refer to such a Y as an *L–space mod torsion*. In fact, there is no known example of a rational homology sphere Y for which $\widehat{\mathrm{HF}}(Y)$ contains torsion; the two definitions may be equivalent.

[2] Subsequent to the submission of this paper, Hanselman, Rasmussen, Rasmussen, and Watson [14; 15; 40] proved that the conjecture holds for all graph manifolds.

Our motivating problem is to describe the homeomorphism types of strong L–spaces and strong Heegaard diagrams. As we shall see, the condition of being a strong L–space is quite restrictive, more so than being an L–space. For instance, the only strong L–space with determinant 1 is S^3, so the Poincaré homology sphere is an L–space but not a strong L–space [33; 21].

The standard genus-1 Heegaard diagram for a lens space $L(p,q)$ (consisting of a single α curve and a single β curve on a torus, intersecting p times) is clearly a strong diagram, so lens spaces are strong L–spaces. A broader source of examples derives from work of the first author [11, Corollary 3.5], who showed that the double cover of S^3 branched along a nonsplit alternating link is a strong L–space. Note that these spaces subsume lens spaces, which are branched double covers of two-bridge links, although the strong diagrams for these spaces in [11] do not have genus 1. Although we can generate many families of strong Heegaard diagrams, we were unable to produce any new examples of strong L–spaces besides the ones just mentioned. Indeed, our main results support an affirmative answer to the following question:

Question 1.2 Is every strong L–space the branched double cover of an alternating link in S^3?

As a first approach to Question 1.2, recall that the determinant of a link $L \subset S^3$ is the absolute value of its (single-variable) Alexander polynomial evaluated at -1: $\det(L) = |\Delta_L(-1)|$. Let $\Sigma(L)$ denote the double cover of S^3 branched over L. Then $\det(L) = \det(\Sigma(L))$, in part justifying our terminology. A classical theorem of Bankwitz [1] and Crowell [7] asserts that there exist finitely many alternating links of bounded determinant. Therefore, an affirmative answer to Question 1.2 would imply the same about strong L–spaces. We deduce this fact directly as a corollary of the following result, which we prove using topological and graph-theoretic methods:

Theorem 1.3 *There exist finitely many rational homology spheres with bounded simultaneous trajectory number.*

Corollary 1.4 *There exist finitely many strong L–spaces with bounded determinant.* □

By contrast, there exist infinitely many irreducible L–spaces with the same determinant. Reducible examples are easy to exhibit, since Heegaard Floer homology satisfies a Künneth principle for connected sums. Thus, for example, the connected sum of any L–space with arbitrarily many copies of the Poincaré sphere is an L–space with the same determinant. For irreducible examples, the Seifert fibered spaces of type $(2,2,n)$

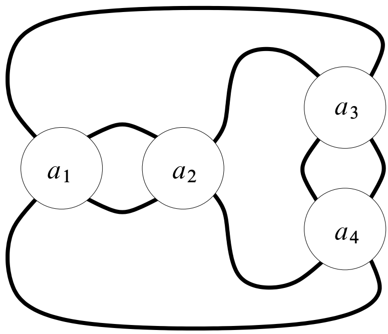

Figure 1: Diagram of the link $L(a_1, a_2, a_3, a_4)$, with a_1, a_2, a_3, $a_4 \in \mathbb{Q} \cup \{1/0\}$

all have determinant 4 and finite fundamental group, so they are L–spaces. (It is unknown whether there exist infinitely many irreducible L–spaces with determinant less than 4.) Additionally, the first author and Watson [12] gave an infinite family of hyperbolic manifolds with determinant 25 that are L–spaces mod torsion.

Using similar techniques to those in the proof of Theorem 1.3, we prove:

Theorem 1.5 *If Y is a strong L–space with $\det(Y) \leq 8$, then Y is the branched double cover of an alternating link. Specifically, Y is a connected sum of lens spaces.*

In another direction towards Question 1.2, we describe all strong L–spaces that admit strong Heegaard diagrams of genus 2. For $a_1, a_2, a_3, a_4 \in \mathbb{Q} \cup \{1/0\}$, let $L(a_1, a_2, a_3, a_4)$ denote the link presented by the diagram in Figure 1, where the four balls are filled in with the rational tangles specified by a_1, a_2, a_3 and a_4 (see Section 5.2). Note that the diagram is alternating if and only if all a_i have the same sign, where $0/1$ and $1/0$ are considered to have both signs. Setting $a_1 = a_2 = a_3 = a_4 = \pm 1$ results in the minimal diagram D of the figure-eight knot, so the diagram can be regarded as substituting arbitrary rational tangles for the crossings in D. The branched double cover $\Sigma(L(a_1, a_2, a_3, a_4))$ is either

(1) a connected sum of one or two genus-1 manifolds;

(2) a small Seifert fibered space; or

(3) a graph manifold whose JSJ decomposition consists of two Seifert fibered spaces over D^2 with two exceptional fibers.

Theorem 1.6 *Let Y be a strong L–space that admits a strong Heegaard diagram of genus 2. Then $Y \cong \Sigma(L(a_1, a_2, a_3, a_4))$ for some $a_1, \ldots, a_4 \in \mathbb{Q} \cup \{1/0\}$ of the same sign. In particular, Y is the branched double cover of an alternating link in S^3.*

Usui [43; 44] has obtained similar results.

A key ingredient for proving Theorems 1.3 and 1.6 is Proposition 3.1, which implies that every strong L–space admits a Heegaard diagram H that is both strong and 1–*extendible*. In such a diagram, the signs with which the α and β curves may intersect are highly constrained: all points of intersection between any two curves have the same sign, and the associated intersection matrix is a Pólya matrix (see Section 2). Theorems 1.3 and 1.5 then follow from topological and graph-theoretic arguments, and their proofs appear in Section 4. To prove Theorem 1.6, we show in Section 5 that every strong, 1–extendible Heegaard diagram of genus 2 has a standard form, which coincides precisely with a particular class of Heegaard diagrams for $\Sigma(L(a_1, a_2, a_3, a_4))$ for $a_1, \dots, a_4$ of the same sign.

In Section 6, we prove some results concerning Floer simple knots in strong L–spaces that admit genus-2 strong diagrams. The existence of such knots has applications to Dehn surgery and minimal genus problems. In Section 7, we discuss the connections between strong L–spaces and other notions pertaining to Heegaard splittings. We close in Section 8 with several questions motivated by the present work.

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2 Preliminaries

For an oriented, properly embedded curve α in a surface (possibly with boundary), let $-\alpha$ denote the same curve with the opposite orientation, and, for any integer n, let $n\alpha$ denote a multicurve obtained by taking $|n|$ parallel copies of $\operatorname{sgn}(n)\alpha$. For oriented multicurves α and β that meet transversally, let $\alpha + \beta$ denote an oriented multicurve obtained from $\alpha \cup \beta$ by forming the oriented resolution at every point of $\alpha \cap \beta$. Note that $n\alpha$ and $\alpha + \beta$ specify unique multicurves up to isotopy. For oriented multicurves α and β in an oriented surface S that meet transversally, write $\alpha \cdot \beta$ for their algebraic intersection number and $|\alpha \cap \beta|$ for their geometric intersection number. The multicurves α and β intersect *coherently* if all intersection points have the same sign, ie if $|\alpha \cdot \beta| = |\alpha \cap \beta|$.

A Heegaard diagram H consists of a surface S, a basepoint z, two collections of attaching curves $\alpha = \alpha_1 \cup \dots \cup \alpha_g$ and $\beta = \beta_1 \cup \dots \cup \beta_g$ specifying a pair of handlebodies, and a choice o of orientation on S and each α and β curve:

$$(S, z, \alpha, \beta, o).$$

Henceforth, we generally suppress reference to the basepoint z and orientation o, and simply refer to the diagram as (S,α,β).

Let $\mathrm{Sym}^g(S)$ denote the g^{th} symmetric power of S, the quotient of $S^{\times g}$ by the action of the symmetric group on g letters. It is a $2g$–dimensional manifold. Let $\mathbb{T}_\alpha$ (resp. $\mathbb{T}_\beta$) be the image in $\mathrm{Sym}^g(S)$ of $\alpha_1\times\cdots\times\alpha_g$ (resp. $\mathbb{T}_\beta$); this is an embedded g–dimensional torus. The tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ intersect transversally in a finite number of points; let $\mathfrak{S}(H)=\mathbb{T}_\alpha\cap\mathbb{T}_\beta$. A point of $\mathfrak{S}(H)$ is a tuple $\mathbf{x}=(x_1,\dots,x_g)$, where $x_i\in\alpha_i\cap\beta_{\sigma_{\mathbf{x}}(i)}$, for some permutation $\sigma_{\mathbf{x}}$. Elements of $\mathfrak{S}(H)$ are *generators* of the group $\widehat{\mathrm{CF}}(H)$ discussed above.

For each $x\in\alpha_i\cap\beta_j$, let $\eta(x)\in\{\pm1\}$ denote the local sign of intersection of α_i with β_j at x. For $\mathbf{x}=(x_1,\dots,x_g)\in\mathfrak{S}(H)$, the local sign of intersection of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ is given by

$$\eta(\mathbf{x})=\mathrm{sign}(\sigma_{\mathbf{x}})\prod_{i=1}^{g}\eta(x_i),$$

where $\mathrm{sign}(\sigma_{\mathbf{x}})\in\{\pm1\}$ is the signature of the permutation $\sigma_{\mathbf{x}}$. Note that changing the orientation of S or a single α or β curve negates $\eta(\mathbf{x})$ for all $\mathbf{x}\in\mathfrak{S}(H)$.

The *intersection matrix* $M(H)$ is the $g\times g$ matrix of integers whose (i,j) entry is

$$m_{i,j}=\alpha_i\cdot\beta_j=\sum_{x\in\alpha_i\cap\beta_j}\eta(x).$$

Since $M(H)$ is a presentation matrix for $H_1(Y)$, we have $\det(Y)=|\det(M(H))|$, which explains our choice of terminology. The permutation expansion of the determinant gives

$$\begin{aligned}\det(M(H))&=\sum_\sigma\mathrm{sign}(\sigma)m_{1,\sigma(1)}\cdots m_{g,\sigma(g)}\\&=\sum_\sigma\mathrm{sign}(\sigma)\sum_{\substack{\{(x_1,\dots,x_g)\mid\\ x_i\in\alpha_i\cap\beta_{\sigma(i)}\}}}\prod_{i=1}^{g}\eta(x_i)\\&=\sum_{\mathbf{x}\in\mathfrak{S}(H)}\eta(\mathbf{x}).\end{aligned}$$

In particular, we see that

$$|\mathfrak{S}(H)|\geq\det(Y),$$

as noted in Section 1. The following two lemmas are immediate:

Lemma 2.1 *H is a strong Heegaard diagram if and only if all generators in $\mathfrak{S}(H)$ have the same sign.* □

Lemma 2.2 *If H is a strong Heegaard diagram, and some generator in $\mathfrak{S}(H)$ includes a point of $\alpha_i \cap \beta_j$, then α_i and β_j intersect coherently.* □

We call a Heegaard diagram H *coherent* if all pairs of α and β curves intersect coherently. As we shall see, the fact that (many of) the curves in a strong Heegaard diagram intersect coherently will be vital to our classification results. However, a strong diagram need not be coherent. For instance, if H is a genus-2 Heegaard diagram in which the intersections $\alpha_1 \cap \beta_1$ and $\alpha_2 \cap \beta_2$ are coherent and nonempty, while $\alpha_1 \cap \beta_2 = \varnothing$, then H is strong irrespective of the signs of points in $\alpha_2 \cap \beta_1$, since these points are not included in elements of $\mathfrak{S}(H)$. Such a diagram can be obtained by taking the connected sum of two standard Heegaard diagrams for lens spaces and isotoping the α curve of one summand so that it intersects the β curve of the other summand. However, Proposition 3.1 below will enable us to restrict our attention to coherent Heegaard diagrams.

Given a Heegaard diagram H, the *intersection graph* $G(H)$ is the bipartite graph with vertex set $A \sqcup B$, where $A = \{a_1, \dots, a_g\}$ and $B = \{b_1, \dots, b_g\}$, and for which the set of edges joining a_i and b_j is $\{e_x \mid x \in \alpha_i \cap \beta_j\}$. Note that $\mathfrak{S}(H)$ is in natural one-to-one correspondence with the set of perfect matchings of $G(H)$. The *degree* of a vertex v in a graph G is the number of edges in G with an endpoint at v. Write $\delta(G)$ for the minimum vertex degree in G, and write $\delta(H) = \delta(G(H))$. A graph G is *k–extendible* if any k–tuple of disjoint edges extends to a perfect matching of G. A Heegaard diagram H is *k–extendible* if its intersection graph $G(H)$ is. In particular, H is 1–extendible if every point $x \in \alpha \cap \beta$ is contained in a generator $\mathbf{x} \in \mathfrak{S}(H)$. As an immediate consequence of the previous two lemmas, we have:

Lemma 2.3 *A strong, 1–extendible Heegaard diagram is coherent.* □

For a strong diagram H, the combinatorics of $M(H)$ and $G(H)$ are expressed by the closely related concepts of Pólya matrices and Pfaffian orientations, respectively. A *Pólya matrix* is a $g \times g$ square matrix $M = (m_{i,j})$ for which all nonzero terms in the expansion

$$\det(M) = \sum_\sigma \operatorname{sign}(\sigma) m_{1,\sigma(1)} \cdots m_{g,\sigma(g)}$$

come with the same sign. Equivalently, if we write $|M|$ for the matrix $(|m_{i,j}|)$ and

$$\operatorname{per}(N) = \sum_\sigma n_{1,\sigma(1)} \cdots n_{g,\sigma(g)}$$

for the *permanent* of a $g \times g$ square matrix $N = (n_{i,j})$, then M is a Pólya matrix if

$$|\det(M)| = \operatorname{per}(|M|).$$

We refer to Vazirani and Yannakakis [45] for the definition of a Pfaffian orientation.

Proposition 2.4 *The following are equivalent conditions on a coherent Heegaard diagram H:*

(1) *H is strong;*

(2) *$M(H)$ is a Pólya matrix; and*

(3) *$G(H)$ has a Pfaffian orientation.*

Proof The equivalence of (1) and (2) is the effective content of Lemma 2.1. The equivalence of (2) and (3) follows from [45]. □

The sign pattern of the entries of a Pólya matrix is highly constrained. This fact plays a key role in the proof that the fundamental group of a strong L–space is not left-orderable [21]. Pólya matrices obey a deep structure theorem due independently to McCuaig [25] and Robertson, Seymour and Thomas [41]. We apply a result of their work in Section 7.3.

3 Extendibility

The purpose of this section is to show that every rational homology sphere admits a 1–extendible Heegaard diagram attaining its simultaneous trajectory number. In particular, every strong L–space admits a 1–extendible, strong Heegaard diagram. This result is very useful in the subsequent sections. The technical statement is as follows:

Proposition 3.1 *Let H be a doubly pointed Heegaard diagram for a rational homology sphere. Then there exists a sequence of handleslides and isotopies in the complement of the basepoints that transforms H into a 1–extendible Heegaard diagram H' such that $|\mathfrak{S}(H)| = |\mathfrak{S}(H')|$. In particular, if H is strong, then so is H'.*

Corollary 3.2 *A rational homology sphere Y admits a 1–extendible Heegaard diagram H for which $|\mathfrak{S}(H)| = M(Y)$.* □

To prove Proposition 3.1, we begin by establishing a simple criterion to recognize that a given Heegaard diagram can be converted into a reducible one by a sequence of isotopies and handleslides. In order to state it in a sharp form, we require a little notation and background concerning Heegaard diagrams and presentations of the fundamental group.

Given any free group $F^z = \langle z_1, \dots, z_g \rangle$ and a value $1 \le k \le g$, we obtain groups

$$F^z_{\le k} = \langle z_1, \dots, z_k \rangle, \quad F^z_k = \langle z_k \rangle \quad \text{and} \quad F^z_{\ge k} = \langle z_k, \dots, z_g \rangle,$$

and projections from F^z to each. For any word $w \in F^z$, let

$$w_{\leq k} = w_{<k+1}, \quad w_k \quad \text{and} \quad w_{\geq k} = w_{>k-1}$$

denote the respective images of w under these projections; each of them is a subword of w. In a similar spirit, for a collection of curves $\gamma_1, \dots, \gamma_g$, we define

$$\gamma_{\leq k} = \gamma_{<k+1} = \gamma_1 \cup \cdots \cup \gamma_k \quad \text{and} \quad \gamma_{>k} = \gamma_{k+1} \cup \cdots \cup \gamma_g.$$

Let $H = (S, \alpha, \beta)$ denote a Heegaard diagram that presents a 3–manifold Y. Form the free group $F^x = \langle x_1, \dots, x_g \rangle$. Choose a point $p_i \in \alpha_i$ disjoint from β and traverse a full loop around α_i starting at p_i. For each intersection point between α_i and β_j with sign ϵ, record the term x_j^ϵ. The product of these terms, in order from left to right, yields a word $w_\beta(\alpha_i) \in F_x$. Observe that a different choice of p_i will result in the same word up to cyclic equivalence. As a result, we regard $w_\beta(\alpha_i)$ and its subwords up to cyclic equivalence. The group $\pi_1(Y)$ admits the presentation $F^x / \langle w_\beta(\alpha_1), \dots, w_\beta(\alpha_g) \rangle$. In an analogous manner, we define the free group $F^y = \langle y_1, \dots, y_g \rangle$, associate a cyclic word $w_\alpha(\beta_j)$ in $y_1, \dots, y_g$ to each curve β_j, and obtain the presentation $F^y / \langle w_\alpha(\beta_1), \dots, w_\alpha(\beta_g) \rangle$ for $\pi_1(Y)$.

Lemma 3.3 *Suppose that $H = (S, \alpha, \beta, z_1, z_2)$ is a doubly pointed genus-g Heegaard diagram for a rational homology sphere Y and, for some $0 < k < g$,*

$$\alpha_{\leq k} \cap \beta_{>k} = \varnothing.$$

Then it is possible to perform handleslides and isotopies in the complement of z_1 and z_2 to produce a Heegaard diagram $H' = (S, \alpha', \beta', z_1, z_2)$ such that

- $|\mathfrak{S}(H)| = |\mathfrak{S}(H')|$,
- $w_{\beta'}(\alpha'_i) = w_\beta(\alpha_i)_{\leq k}$ *and* $w_{\alpha'}(\beta'_j) = w_\alpha(\beta_j)_{\leq k}$ *for all* $i, j \leq k$, *and*
- $w_{\beta'}(\alpha'_i) = w_\beta(\alpha_i)_{>k}$ *and* $w_{\alpha'}(\beta'_j) = w_\alpha(\beta_j)_{>k}$ *for all* $i, j > k$.

In particular, $\alpha'_{\leq k} \cap \beta'_{>k} = \alpha'_{>k} \cap \beta_{\leq k} = \varnothing$, while there is an identification $\alpha'_i \cap \beta'_j = \alpha_i \cap \beta_j$ preserving local signs of intersection when i, $j \leq k$ or i, $j > k$. Thus, H' is reducible with summands $H_1 = (S_{\leq k}, \alpha'_{\leq k}, \beta'_{\leq k})$ and $H_2 = (S_{>k}, \alpha'_{>k}, \beta'_{>k})$. If H is strong, then so are H_1 and H_2.

Remark 3.4 Some restriction on Y or H is necessary in order to guarantee the conclusion of Lemma 3.3. For example, take $Y = S^1 \times S^2$ (so $b_1(Y) > 0$), consider its standard genus-1 Heegaard diagram with no intersection points, and stabilize the diagram once. Label the curves in the resulting diagram so that $\alpha_1 \cap \beta_2 = \varnothing$. The conclusion of Lemma 3.3 in this case (with $k = 1$) would produce a genus-2 Heegaard

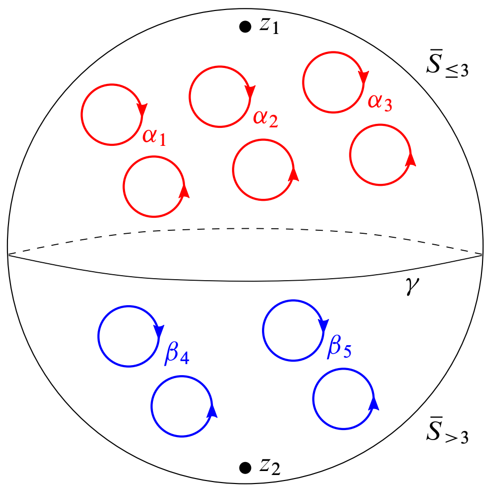

Figure 2: A cut-open Heegaard diagram as in the proof of Lemma 3.3, with $k=3$ and $g=5$

diagram in which each pair of curves is disjoint, but such a diagram must present $\#^2(S^1\times S^2)\neq Y$, a contradiction.

Proof of Lemma 3.3 Recall that

$$H_1(Y;\mathbb{Z})\cong H_1(S;\mathbb{Z})/\langle[\alpha_1],\dots,[\alpha_g],[\beta_1],\dots,[\beta_g]\rangle.$$

Since Y is a rational homology sphere, the classes $[\alpha_1],\dots,[\alpha_g]$, $[\beta_1],\dots,[\beta_g]$ are linearly independent in $H_1(S;\mathbb{Z})$. In particular, $\alpha_1,\dots,\alpha_k$, $\beta_{k+1},\dots,\beta_g$ are g disjoint curves representing linearly independent classes in $H_1(S;\mathbb{Z})$. It follows that $S\smallsetminus(\alpha_{\le k}\cup\beta_{>k})$ is a $2g$–punctured sphere. As a result, there exists an unoriented curve $\gamma\subset S$, unique up to isotopy, with a regular neighborhood $N(\gamma)$ that separates S into subsurfaces $\bar{S}_{\le k}$ and $\bar{S}_{>k}$ such that $\{z_1\}\cup\alpha_{\le k}\subset\bar{S}_{\le k}$ and $\{z_2\}\cup\beta_{>k}\subset\bar{S}_{>k}$. See Figure 2.

The space $N(\gamma)\cup\bar{S}_{>k}\smallsetminus\beta_{>k}$ is a $2(g-k)$–punctured disk, and $N(\gamma)\cap\bar{S}_{\le k}$ is a collar neighborhood of its boundary. We radially isotope $\beta_{\le k}\cap(N(\gamma)\cup\bar{S}_{>k})$ away from z_2 and into $N(\gamma)\cap\bar{S}_{\le k}$, permitting arcs to pass over punctures in the process. Passing an arc over a puncture corresponds to a handleslide in S, so we may interpret this process as a sequence of isotopies and handleslides of $\beta_{\le k}$ over $\beta_{>k}$ in S which is the identity outside of $N(\gamma)\cup\bar{S}_{>k}\smallsetminus\{z_2\}$. The resulting collection of curves $\beta'_{\le k}$ supported in $\bar{S}_{\le k}$ has the property that $w_\alpha(\beta'_j)_{\le k}=w_\alpha(\beta_j)_{\le k}$ for all $j\le k$, since $\alpha_{\le k}\cap\beta_{>k}=\varnothing$ and the handleslides do not change the intersections of $\beta_{\le k}$ with $\alpha_{\le k}$.

Similarly, we isotope and handleslide the curves $\alpha_{>k}$ over $\alpha_{\le k}$ in $N(\gamma) \cup \bar{S}_{\le k} \setminus \{z_1\}$ to get curves $\alpha'_{>k}$ supported in $\bar{S}_{>k}$ and such that $w_{\beta'}(\alpha'_i)_{>k} = w_{\beta'}(\alpha_i)_{>k}$ for all i. Lastly, we define $\alpha'_i = \alpha_i$ for $i = 1, \dots, k$ and $\beta'_j = \beta_j$ for $j = k+1, \dots, g$.

The preceding remarks and the fact that $\beta'_{\le k} \cap \alpha'_{\ge k} = \varnothing$ yield the stated conclusions about $w(\alpha'_i)$ and $w(\beta'_j)$. Capping off $\bar{S}_{\le k}$ and $\bar{S}_{>k}$ along γ results in the surfaces $S_{\le k}$ and $S_{>k}$ required by the lemma. Finally, note that there exists a natural bijection between $\mathfrak{S}(H)$ and $\mathfrak{S}(H')$, since a generator in $\mathfrak{S}(H)$ cannot use any element of $\alpha_{>k} \cap \beta_{\le k}$. In particular, if H is strong, then so is H'. □

Proof of Proposition 3.1 We proceed by induction on $g(H)$. For $g(H) = 1$ the assertion is trivial, so we proceed to the induction step. By a theorem of Hetyei, a bipartite graph G with bipartition (A, B) is 1–extendible if and only if for every subset $T \subset A$, the set of neighbors of T in B has cardinality at least $\max\{|B|, |A|+1\}$ [17; 23, Theorem 4.1.1]. Since $G(H)$ contains a perfect matching, it is 0–extendible. Assume that it is not 1–extendible. Then there exists some proper, nonempty subset $T \subset A$ such that T has exactly $|T|$ neighbors in B. In other words, there exist some k α circles that are disjoint from some $g-k$ β circles, where $1 \le k < g$. Thus, Lemma 3.3 implies that there exists a sequence of handleslides and isotopies which transform H into a reducible diagram H', and which is strong provided that H is. By induction on the genera of the summands of H', the statement of the proposition follows. □

A simple induction using Lemma 3.3 establishes the following corollary:

Corollary 3.5 *If a strong diagram has an upper triangular intersection matrix, then it presents a connected sum of lens spaces.* □

4 Finiteness results

In this section, we prove Theorem 1.3, which asserts that there exist finitely many rational homology spheres with bounded simultaneous trajectory number, and Theorem 1.5, which classifies the strong L–spaces with determinant up to 8.

Lemma 4.1 *Let H be a Heegaard diagram for a rational homology sphere. Suppose that H contains an attaching curve that has $m \ge 1$ intersection points with another attaching curve and at most one other intersection point. Then there exists a sequence of isotopies and handleslides converting H into a 1–extendible Heegaard diagram $H_1 \# H_2$ with $g(H_1) = 1$, $|\mathfrak{S}(H_1)| = m$ and $|\mathfrak{S}(H)| = |\mathfrak{S}(H_1 \# H_2)|$. If H is strong, then so are H_1 and H_2.*

Proof Without loss of generality, assume that α_1 has $m \geq 1$ intersection points with β_1. If α_1 meets only β_1, then the result follows directly from Lemma 3.3 and induction on the genus, as in the proof of Proposition 3.1. Suppose instead that α_1 meets another curve β_2. Label the intersection points along α_1 consecutively by $p_1, \dots, p_m \in \beta_1$ and $q \in \beta_2$. Perform m consecutive handleslides of β_1 over β_2, guided along the oriented segment of α_1 from p_i to q for $i = m, \dots, 1$ in turn. Let β_1' denote the resulting curve and H' the resulting Heegaard diagram. Observe that α_1 has a single intersection point in H'. Applying Lemma 3.3 and induction on the genus to H' establishes the existence of handleslides and isotopies converting H' into a 1–extendible Heegaard diagram of the required form with the property that $|\mathfrak{S}(H')| = |\mathfrak{S}(H_1 \# H_2)|$.

It remains to establish that $|\mathfrak{S}(H')| = |\mathfrak{S}(H)|$. To do so, we exhibit a bijection between the perfect matchings in $G(H')$ and $G(H)$. Let $G' \subset G(H)$ denote the subgraph obtained by removing the m edges between a_1 and b_1. Observe that $G(H')$ is constructed from G' by inserting m parallel edges between a_i and b_1 for each edge between a_i and b_2 in $G(H)$. Thus, G' is a common subgraph of $G(H)$ and $G(H')$. In particular, we obtain a trivial bijection between the perfect matchings of $G(H)$ and $G(H')$ contained in this common subgraph. Consider another perfect matching in $G(H)$. Then it uses the k^{th} edge between a_1 and b_1 for some $1 \leq k \leq m$. It also has an edge e from a_i to b_2 for some $i > 1$. We construct a perfect matching in $G(H')$ by removing these two edges, putting in the edge from a_1 to b_2, and putting in the k^{th} new edge from a_i to b_1 that corresponds to e. It is clear that this construction sets up a bijection between the perfect matchings in $G(H)$ and $G(H')$ not contained in G', and so completes the required bijection between perfect matchings in $G(H)$ and $G(H')$. □

For the remainder of this section, let H_0 denote the standard genus-1 Heegaard diagram of $\mathbb{RP}^3$. Recall that the minimum vertex degree of a graph G is denoted $\delta(G)$, and $\delta(H) = \delta(G(H))$ for a Heegaard diagram H.

Lemma 4.2 *Let H be a Heegaard diagram for a rational homology sphere Y. Then there exists a sequence of isotopies, handleslides, and destabilizations converting H into a 1–extendible Heegaard diagram $H' = (\#^n H_0) \# H''$ such that $|\mathfrak{S}(H)| = |\mathfrak{S}(H')|$ and $\delta(H'') \geq 3$. If H is strong, then so are H' and H''.* □

Proof We proceed by induction on the genus g of H. The result is true if $g = 0$, so suppose that $g > 0$ and the result holds for Heegaard diagrams of genus less than g. By Proposition 3.1, we may assume that H is 1–extendible. If $\delta(H) \geq 3$, then the desired result follows at once. Otherwise, $\delta(H) \leq 2$. Apply Lemma 4.1 to an attaching curve in H that contains at most two intersection points. In the composite Heegaard diagram $H_1 \# H_2$ guaranteed by Lemma 4.1, H_1 is the standard genus-1 diagram for S^3 or $\mathbb{R}\mathrm{P}^3$. Destabilize H_1 in the former case, and apply the induction hypothesis to H_2 to complete the induction step. □

Lemma 4.3 *There exist finitely many 1–extendible bipartite graphs G with $\delta(G) \geq 3$ and a bounded number of perfect matchings.*

Proof Fix a natural number d and suppose that G is a 1–extendible bipartite graph that contains d or fewer perfect matchings. Let n denote the number of vertices and m the number of edges of G. Then [23, Theorem 7.6.2] establishes that

$$d \geq m - n + 2. \tag{2}$$

In addition, $m \geq \frac{3}{2}n$, since $\delta(G) \geq 3$. Applying (2), we obtain

$$n \leq 2(d-2) \quad \text{and} \quad m \leq d + n - 2 \leq 3(d-2).$$

Since both of m and n are bounded in terms of d, G is one of finitely many graphs. □

Lemma 4.4 *For any graph G, there exist finitely many Heegaard diagrams H with $G(H) = G$.*

Proof We may assume that G has a bipartition $V(G) = A \sqcup B$ with $|A| = |B| = g$. Label the edges of G by $e_1, \dots, e_k$. Let $\alpha_1, \dots, \alpha_g$, $\beta_1, \dots, \beta_g$ be oriented copies of S^1, and let $N(\alpha_i)$ (resp. $N(\beta_i)$) be the total space of a trivial I–bundle over α_i (resp. b_i). Up to homeomorphism, there are finitely many ways to choose points $x_1, \dots, x_k$, $y_1, \dots, y_k$, where if e_k is an edge connecting a_{i_k} and b_{j_k} then $x_k \in \alpha_{i_k}$ and $y_k \in \beta_{j_k}$. Given such a choice and a choice of $(\epsilon_1, \dots, \epsilon_k) \in \{\pm 1\}^k$, form a possibly disconnected, oriented surface-with-boundary N by plumbing together the annuli $N(\alpha_i)$ and $N(\beta_j)$ so that x_k and y_k are identified and α_{i_k} and β_{j_k} meet with sign ϵ_k at that point. There are finitely many ways to glue surfaces-with-boundary to N along their common boundaries to obtain a closed, oriented surface S of genus g. Thus, up to homeomorphism, there are finitely many tuples (S, α, β), where S is a closed, oriented surface of genus g and α and β are g–tuples of curves in S whose intersection graph is given by G, and in particular finitely many such Heegaard diagrams. □

Proof of Theorem 1.3 Fix a natural number d and suppose that Y is a rational homology sphere with $M(Y) \leq d$. By Lemma 4.2, Y has a 1–extendible Heegaard diagram $H = (\#^n H_0) \# H'$ with $M(Y) = |\mathfrak{S}(H)| = 2^n |\mathfrak{S}(H')|$ and $\delta(H') \geq 3$. Thus,

$n \leq \log_2 d$ and $G(H')$ contains at most d perfect matchings. By Lemma 4.3, there exist finitely many possibilities for $G(H')$, so by Lemma 4.4, there exist finitely many possibilities in turn for H'. Therefore, there exist finitely many possibilities for H and hence for Y, as required. □

We now turn to the proof of Theorem 1.5, which asserts that every strong L–space with determinant ≤ 8 is the branched double cover of an alternating link in S^3. In the proof, we apply an estimate on the number of perfect matchings in a cubic bipartite graph due to Voorhoeve [48]. A graph is *cubic* if every vertex has degree 3. The proof of [23, Theorem 8.1.7] establishes the following version of Voorhoeve's result:

Lemma 4.5 *Define $h\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ recursively by $h(1) = 2$ and $h(g) = \lceil \frac{4}{3} h(g-1) \rceil$, and define $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ by $f(g) = \lceil \frac{3}{2} h(g) \rceil$. Then a cubic bipartite graph on $2g$ vertices contains at least $f(g)$ perfect matchings. In particular, a cubic bipartite graph with 8 or more vertices contains at least $f(4) = 9$ perfect matchings.* □

Proof of Theorem 1.5 Choose a 1–extendible, strong Heegaard diagram H of minimum genus g presenting Y. If $g = 1$, then Y is the branched double cover of an alternating two-bridge link. If $g = 2$, then the result follows from Theorem 1.6 (proven in Section 5 and independent of this result). If there exists a sequence of isotopies and handleslides converting H into a connected sum of strong Heegaard diagrams $H_1 \# H_2$, then, by induction on g, H_i presents $\Sigma(L_i)$ for some nonsplit alternating link L_i, and then $Y \cong \Sigma(L_1 \# L_2)$ exhibits Y in the stated form.

Thus, unless the desired conclusion holds, we may assume henceforth that $g \geq 3$ and that the hypothesis of Lemma 4.1 does not hold; in particular, every vertex of $G(H)$ has degree at least 3, and no vertex of degree 3 is incident with parallel edges (two edges with the same pair of endpoints). We seek a contradiction to these conditions under the assumption that $\det(Y) \leq 8$.

First, suppose that $g = 3$. Since $M(H)$ is a Pólya matrix, it contains at least one zero entry, as noted in Section 2. Since the condition of Lemma 4.1 does not hold, no row or column of $M(H)$ contains more than one zero, and if a row or column contains a zero, then its nonzero entries are at least two in absolute value. It follows that up to permuting its rows and columns, $M(H)$ *dominates* one of the following three matrices, in the sense that the absolute values of its entries bound from above those of the corresponding matrix:

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

If a Pólya matrix P dominates a nonnegative matrix N, then $|\det(P)| = \mathrm{per}(|P|) \geq \mathrm{per}(N)$. It follows that

$$|\mathfrak{S}(H)| = |\det(M(H))| \geq \min\{16, 20, 16\} = 16,$$

a contradiction.

Next, suppose that $g = 4$. We argue that $G(H)$ contains a cubic subgraph on 8 vertices, which by Lemma 4.5 is a contradiction. Inequality (2) implies that $G(H)$ contains at most 14 edges, and by hypothesis it contains at least 12 edges. If it has 12 edges, then $G(H)$ is the desired subgraph. If it has 13 edges, then there exists a unique vertex of degree 4 in each of A and B. Since no vertex of degree 3 is incident with parallel edges, it follows that there exists an edge e between the vertices of degree 4, and then $G(H) \smallsetminus e$ is the desired subgraph. If it has 14 edges, then the degree sequences of vertices in A and in B belong to $\{(3,3,3,5), (3,3,4,4)\}$. Again, since no vertex of degree 3 is incident with parallel edges, every vertex of degree 4 or more is incident with another such vertex; furthermore, if there are two vertices of degree 5, then there are parallel edges between them. In any case, we can locate edges e and f such that $G(H) \smallsetminus \{e, f\}$ is the desired subgraph.

Lastly, suppose that $g \geq 5$. $G(H)$ cannot contain a cubic subgraph on $2g$ vertices by Lemma 4.5, because then it would contain more than 8 perfect matchings, a contradiction. Thus, $G(H)$ contains at least $3g + 1$ edges. Since $|\mathfrak{S}(H)| \leq 8$, inequality (2) implies that $g \leq 5$. Consequently, $g = 5$, there are vertices $a \in A$ and $b \in B$ of degree four, and all other vertices have degree three and are not incident with parallel edges. If there exists an edge $e = (a, b)$, then $G(H) \smallsetminus e$ is a cubic subgraph on 10 vertices, a contradiction. Therefore, a and b are nonadjacent and $G(H)$ has no parallel edges. Thus, $G(H) \smallsetminus \{a, b\}$ is a 2–regular bipartite graph. It is easy to see in this case that any two-edge matching that uses vertices a and b extends to a perfect matching of $G(H)$. However, there are 16 such two-edge matchings, whereas $G(H)$ has at most 8 perfect matchings, a contradiction. This concludes the proof that $Y \cong \Sigma(L)$ for some alternating link $L \subset S^3$.

Finally, the determinant of an alternating link is greater than or equal to its crossing number, with equality only for $(2, n)$–torus links [7]. It follows that L has at most seven crossings or is the $(2, 8)$–torus link.

The knot tables indicate that all prime, alternating links through seven crossings with determinant ≤ 8 are two-bridge links. Therefore, L is a connected sum of two-bridge links, and $\Sigma(L)$ is a connected sum of lens spaces. $\square$

5 Strong diagrams of genus 2

The purpose of this section is to prove Theorem 1.6, describing all strong L–spaces admitting strong Heegaard diagrams of genus 2.

5.1 Coherent multicurves in an annulus

We begin with some technical but elementary statements concerning curves in an annulus that will enable us to recognize certain standard configurations within a strong Heegaard diagram.

Construction 5.1 Fix orientations on the circle S^1 and the interval I. Let $A = S^1 \times I$ denote the annulus, equipped with the product orientation. For integers p_1, q_1, p_2 and q_2, let $\boldsymbol{\gamma}(p_1, q_1, p_2, q_2)$ denote the pair of oriented multicurves $(\gamma_{p_1,q_1}, \gamma_{p_2,q_2})$ obtained as follows: choose distinct $s_1 \neq s_2 \in S^1$ and $t_1 < t_2 \in I$, and set

$$\gamma_{p_i,q_i} = p_i(S^1 \times \{t_i\}) + q_i(\{s_i\} \times I),$$

where we do not allow the parallel copies of $\pm S^1 \times \{t_1\}$ (resp. $\pm\{s_1\} \times I$) to overlap or interleave with the parallel copies of $\pm S^1 \times \{t_2\}$ (resp. $\pm\{s_2\} \times I$), and we require γ_{p_1,q_1} and γ_{p_2,q_2} to meet transversally. (See Figure 3.)

In Construction 5.1, we have

$$\gamma_{p_1,q_1} \cdot \gamma_{p_2,q_2} = p_1 q_2 - q_1 p_2 \quad \text{and} \quad |\gamma_{p_1,q_1} \cap \gamma_{p_2,q_2}| = |p_1 q_2| + |q_1 p_2|.$$

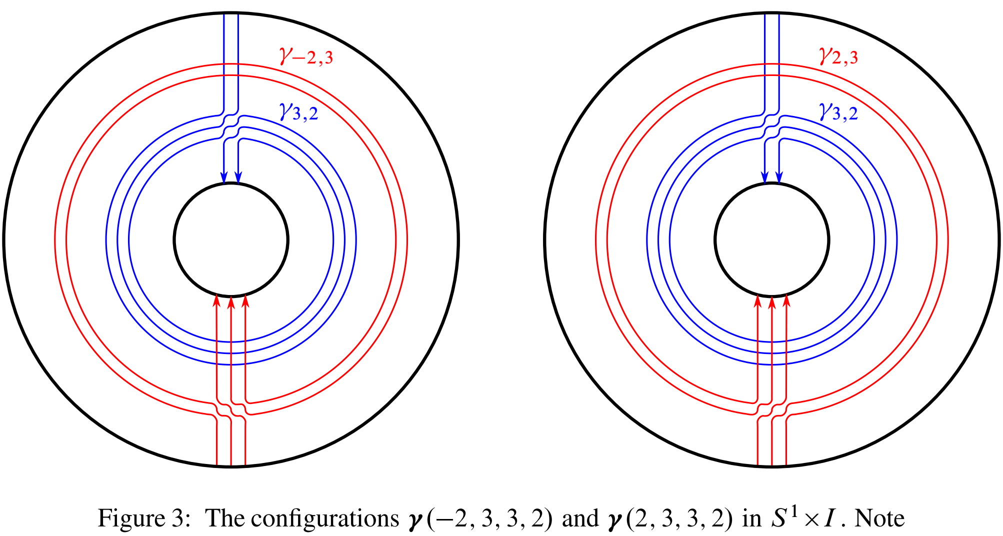

Figure 3: The configurations $\boldsymbol{\gamma}(-2, 3, 3, 2)$ and $\boldsymbol{\gamma}(2, 3, 3, 2)$ in $S^1 \times I$. Note that the former is in minimal position while the latter is not.

In particular, the pair $(\gamma_{p_1,q_1}, \gamma_{p_2,q_2})$ is in minimal position except when $p_1 q_2$ and $q_1 p_2$ are either both positive or both negative.

The ambient isotopy class (fixing a point on each component of ∂A) of each multicurve γ_{p_i,q_i} depends only on (p_i, q_i), but the full configuration $\boldsymbol{\gamma}(p_1, q_1, p_2, q_2)$ depends a priori on more than just the isotopy classes of the individual multicurves. However, the following lemma says that when the pair $(\gamma_{p_1,q_1}, \gamma_{p_2,q_2})$ is in minimal position, the configuration is uniquely characterized up to homeomorphism.

Lemma 5.2 *Let $r_1, \dots, r_k$, $s_1, \dots, s_l$ be points in S^1, ordered cyclically according to the standard orientation of S^1. Suppose that we are given properly embedded, oriented multicurves $\alpha = \alpha_1 \cup \dots \cup \alpha_k$ and $\beta = \beta_1 \cup \dots \cup \beta_l$ in A satisfying the following properties:*

- *For some fixed $[a] \in \mathbb{Z}/k$, for each $i \in \{1, \dots, k\}$, α_i is a path from $(r_i, 0)$ to $(r_{i+a}, 1)$ (indices modulo k), and any two paths α_i and $\alpha_{i'}$ are disjoint.*
- *For some fixed $[b] \in \mathbb{Z}/l$, for each $j \in \{1, \dots, l\}$, β_j is a path from $(s_j, 0)$ to $(s_{j+b}, 1)$ (indices modulo l), and any two paths β_j and $\beta_{j'}$ are disjoint.*
- *The multicurves α and β intersect transversally and coherently.*

Then for some integers a and b restricting to the given classes $[a]$ and $[b]$ and satisfying $|al - bk| = |\alpha \cap \beta|$, there is a homeomorphism of A taking (α, β) to $\boldsymbol{\gamma}(a, k, b, l)$.

Proof For concreteness, assume that the intersection points in $\alpha_i \cap \beta_j$ all have positive sign; the other case is analogous.

Let a be the representative of $[a]$ in $\{0, \dots, k-1\}$. We may identify A with the quotient of the rectangle $R = [0, k] \times [0, 1]$ by the relation $(0, y) \sim (k, y)$ for all $y \in [0, 1]$, such that α_i is the image of $\{i\} \times [0, 1]$; denote the projection map $\pi\colon R \to A$. For $i = 1, \dots, k$, note that $\pi(i, 0) = (r_i, 0)$, while $\pi(i, 1) = (r_{i+a}, 1)$ (indices modulo k). Therefore, β is the image of a collection of oriented arcs $\widetilde{\beta} \subset R$ with endpoints on

$$([0,1] \times \{0\}) \cup ([k-a, k-a+1] \times \{1\}) \cup (\{0, k\} \times [0, 1]).$$

For $i = 1, \dots, k$, let R_i denote the square $[i-1, i] \times [0, 1]$. Since α meets β positively, any segment of $\widetilde{\beta} \cap R_i$ (with its inherited orientation) must enter R_i through $\{i\} \times [0, 1]$ (or through $[0, 1] \times \{0\}$ if $i = 1$) and exit through $\{i-1\} \times [0, 1]$ (or through $[a-1, a] \times \{1\}$ if $i = a$). After an ambient isotopy of R_i, we may arrange that $\widetilde{\beta} \cap R_i$ is a union of line segments of negative slope. Next, after an ambient isotopy of R that leaves the first coordinate fixed and reparametrizes the second coordinate respecting $\sim$, we may arrange that $\widetilde{\beta}$ is union of oriented line segments of negative slope beginning on

$$([0,1] \times \{0\}) \cup (\{k\} \times [0, 1])$$

and ending on

$$([k-a, k-a+1]\times\{1\}) \cup (\{0\}\times[0,1]).$$

Let $m = |\widetilde{\beta} \cap (\{0\}\times[0,1])| = |\widetilde{\beta}\cap(\{k\}\times[0,1])|$. Let c be the number of segments of $\widetilde{\beta}$ that begin on $\{k\}\times[0,1]$ and end on $[k-a, k-a+1]\times\{1\}$. After another ambient isotopy of R that leaves the first coordinate fixed and reparametrizes the second coordinate respecting $\sim$, we may arrange that $\widetilde{\beta}$ is a union of line segments of negative slope as above, with the additional properties that

- $m-c$ segments end on $\{0\}\times\left[0,\frac{1}{2}\right]$, and c segments end on $\{0\}\times\left[\frac{1}{2},1\right]$;
- $m-c$ segments begin on $\{k\}\times\left[0,\frac{1}{2}\right]$, and c segments end on $\{k\}\times\left[\frac{1}{2},1\right]$; and
- $\widetilde{\beta}\cap\left([0,k]\times\left\{\frac{1}{2}\right\}\right)$ consists of l points, all in the interior of R_1.

We may now reparametrize $R/{\sim}$ by the transformation

$$f(x,y) = \begin{cases} (x,y) & \text{if } y\le \frac{1}{2},\\ (x+2ay-a, y) & \text{if } y \ge \frac{1}{2}. \end{cases}$$

After this reparametrization, $\beta\cap\left([0,k]\times\left[\frac{1}{2},1\right]\right)/{\sim}$ can be arranged to be vertical, while $\beta\cap\left([0,k]\times\left[\frac{1}{2},1\right]\right)/{\sim}$ winds with positive slope. From here, it is not hard to identify (α,β) with $\boldsymbol{\gamma}(a,k,c-m,l)$. □

Construction 5.3 Returning to the notation in Construction 5.1, suppose that

$$\gcd(p_1,q_1) = \gcd(p_2,q_2) = 1.$$

Decompose S^1 as a union of two arcs $e_1\cup e_2$ so that $\partial\gamma_{p_i,q_i}\subset e_i\times\{0,1\}$. Let R_1 and R_2 be 1–handles (ie copies of $I\times I$), and let N be the oriented surface obtained by attaching R_1 and R_2 to A along $e_1\times\{0,1\}$ and $e_2\times\{0,1\}$, respectively. For $i=1,2$, by attaching q_i parallel cores of R_i to γ_{p_i,q_i}, we obtain a closed curve η_{p_i/q_i}. We may canonically orient these curves by requiring q_1, $q_2\ge 0$ and taking the induced orientation.

5.2 Conventions for rational tangles

We briefly review some basic facts and conventions about rational tangles and their branched double covers; for further information, see Gordon [9, Section 4] and Cromwell [6, Chapter 8].

Consider the sphere $S^2 = \partial B^3$, and fix an equatorial $S^1\subset S^2$. Let

$$Q = \{\mathrm{NE}, \mathrm{NW}, \mathrm{SW}, \mathrm{SE}\}$$

be a set of four points on this S^1, ordered cyclically. By a slight abuse of notation, we shall denote by μ either the (unoriented) arc of S^1 joining SW and NW or the arc joining SE and NE, and denote by λ either the arc joining NW and NE or the arc joining SW and SE. For $p/q \in \mathbb{Q} \cup \{1/0\}$, let $R(p/q)$ denote the p/q rational tangle in B^3, with endpoints on Q; our convention is that the components of $R(1/0)$ (resp. $R(0/1)$) are pushoffs of the two choices of μ (resp. λ). Note that $R(p/q)$ can be represented with an alternating diagram such that the first crossing encountered when entering from SW or NE is an undercrossing, and the first crossing encountered when entering from NW or SE is an overcrossing, precisely when $p/q \geq 0$.

The branched double cover $\Sigma(S^2, Q)$ is a torus, and $\Sigma(B^3, R(p/q))$ is a solid torus. Let $\tilde{\mu}$ (resp. $\tilde{\lambda}$) be the preimage of μ (resp. λ), which up to isotopy does not depend on the choice above. We orient $\tilde{\mu}$ and $\tilde{\lambda}$ such that $\tilde{\mu} \cdot \tilde{\lambda} = -1$ when $\Sigma(S^2, Q)$ is oriented as the boundary of $\Sigma(B^3, R(p/q))$. As explained in [9, Section 4], for any $p/q \in \mathbb{Q} \cup \{1/0\}$, the curve $p\tilde{\mu} + q\tilde{\lambda}$ bounds a compressing disk in $\Sigma(B^3, R(p/q))$.

5.3 A construction of genus-2 Heegaard diagrams

As before, for a_1, a_2, a_3, $a_4 \in \mathbb{Q} \cup \{1/0\}$, let $L(a_1, a_2, a_3, a_4)$ denote the link given by the diagram in Figure 1, where the four balls are filled in with rational tangles following the conventions described above. If a_1, a_2, a_3 and a_4 have the same sign, then the diagram obtained in this manner (using alternating diagrams for the rational tangles) is alternating. We now describe a construction of a Heegaard diagram for $\Sigma(L(a_1, a_2, a_3, a_4))$.

Construction 5.4 For $a_i = p_i/q_i \in \mathbb{Q} \cup \{1/0\}$, for $i = 1, \dots, 4$, let $H(a_1, a_2, a_3, a_4)$ denote the Heegaard diagram shown in Figure 4. Formally, we take two copies of the configuration from Construction 5.3, denoted by $(N, \eta_{-q_1/p_1}, \eta_{q_2/p_2})$ and $(N', \eta_{p_3/q_3}, \eta_{-p_4/q_4})$, and form the union $S = (N \cup N')/\sim$, where $R_1 \subset N$ is identified with $R_2 \subset N'$ and $R_2 \subset N$ is identified with $R_1 \subset N'$, each via a $90°$ rotation of $I \times I$; we define

$$\alpha_1 = \eta_{-q_1/p_1} \subset N, \quad \beta_1 = \eta_{q_2/p_2} \subset N,$$
$$\alpha_2 = \eta_{p_3/q_3} \subset N', \quad \beta_2 = \eta_{-p_4/q_4} \subset N'.$$

Observe that

$$M(H(a_1, a_2, a_3, a_4)) = \begin{pmatrix} -p_1q_2 - q_1p_2 & -q_1p_4 \\ -q_2p_3 & p_3q_4 + q_3p_4 \end{pmatrix},$$

so

$$\det M(a_1, a_2, a_3, a_4) = -p_1q_2q_3p_4 - q_1p_2q_3p_4 - q_1q_2p_3p_4,$$

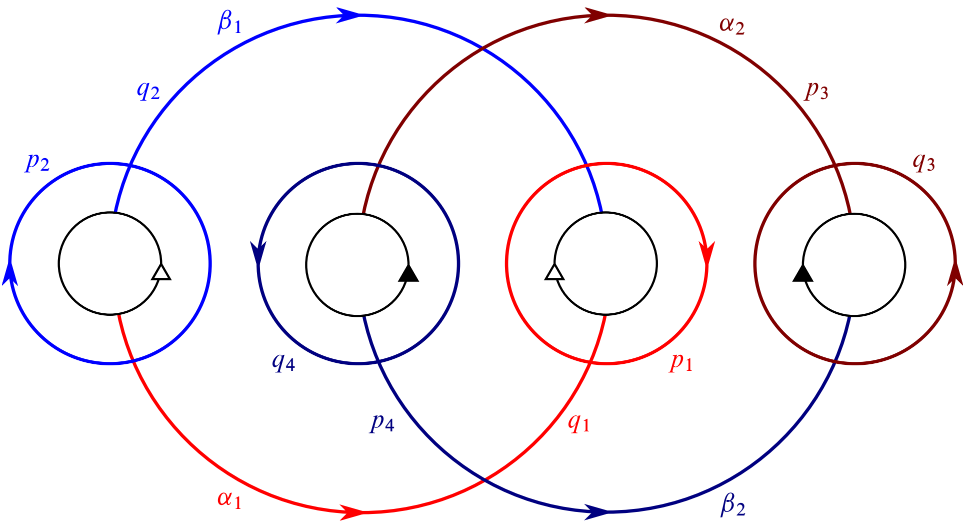

Figure 4: Template for the Heegaard diagram $H(a_1, a_2, a_3, a_4)$, with $a_i = p_i/q_i \in \mathbb{Q} \cup \{1/0\}$ for $i = 1, \dots, 4$

while

$$|\mathfrak{S}(H(a_1, a_2, a_3, a_4))| = |p_1 q_2 q_3 p_4| + |q_1 p_2 q_3 p_4| + |q_1 q_2 p_3 p_4|.$$

It is easy to see that $H(a_1, a_2, a_3, a_4)$ is strong if and only if a_1, a_2, a_3 and a_4 all have the same sign. Additionally, note that when $a_1 = 1/0$ and $a_3 = 0/1$ or when $a_2 = 1/0$ and $a_4 = 0/1$, $H(a_1, a_2, a_3, a_4)$ is a connected sum of genus-1 strong diagrams.

Proposition 5.5 *For a_1, a_2, a_3, $a_4 \in \mathbb{Q} \cup \{1/0\}$, $H(a_1, a_2, a_3, a_4)$ presents the manifold $\Sigma(L(a_1, a_2, a_3, a_4))$.*

Proof Figure 5 depicts the link $L = L(a_1, a_2, a_3, a_4)$ along with some additional decoration that we will use in order to produce a Heegaard decomposition of $\Sigma(L)$. We will then show that $H(a_1, a_2, a_3, a_4)$ presents this decomposition. The red curve C in the projection plane is the cross-section of a sphere P that meets L in six points $x_1, \dots, x_6$. The regions cut out by C in the projection plane are the cross-sections of the balls cut out by P in S^3. The two additional arcs in blue and yellow are cross-sections of equatorial disks D_1 and D_2 for these balls. The space $S^3 \smallsetminus (P \cup D_1 \cup D_2)$ consists of four open balls, whose closures we denote by $A_1, \dots, A_4$; note that $(A_i, A_i \cap L)$ is the rational tangle $(B^3, R(a_i))$. Orient P as the boundary of $A_1 \cup A_3$. The double cover $V_i = \Sigma(A_i, A_i \cap L)$ is then a solid torus, and $\Sigma(L) = V_1 \cup \dots \cup V_4$.

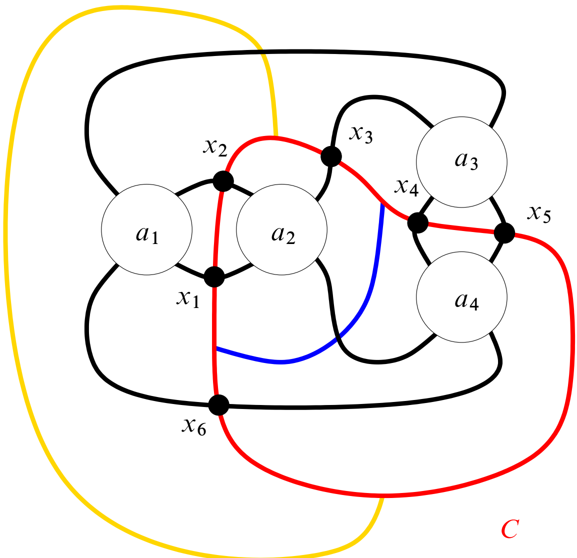

Figure 5: Decomposing $(S^3, L(a_1, a_2, a_3, a_4))$

Since $A_1 \cap A_3 = D_1$, $A_2 \cap A_4 = D_2$, and D_1 and D_2 each meet L in a single point, it follows that $U_1 = \Sigma(A_1 \cup A_3, (A_1 \cup A_3) \cap L)$ and $U_2 = \Sigma(A_2 \cup A_4, (A_2 \cup A_4) \cap L)$ are each genus-2 handlebodies. Specifically, $U_1 = V_1 \natural V_3$ and $U_2 = V_2 \natural V_4$, where $\natural$ denotes the boundary connected sum. Thus, $U_1 \cup_S U_2$ is a genus-2 Heegaard decomposition of $\Sigma(L)$, where $S = \partial U_1 = -\partial U_2 = \Sigma(P, P \cap L)$.

For $i = 1, \dots, 6$, let C_i denote the arc of C that runs clockwise from x_i to x_{i+1} in Figure 5 (subscripts mod 6). Let $N_i \subset P$ denote a small regular neighborhood of C_i and $N = N_1 \cup \cdots \cup N_6$. Let $\tilde{x}_i$, γ_i and γ denote the respective preimages of x_i, C_i and C in $S \subset \Sigma(L)$ for $i = 1, \dots, 6$. Then $\widetilde{N}_i = \Sigma(N_i, \{x_i, x_{i+1}\})$ is an annulus with core γ_i, and the union $\widetilde{N} = \Sigma(N, \{x_1, \dots, x_6\})$ is a cyclic plumbing of these annuli. We may orient the curves γ_i so that $\gamma_i \cdot \gamma_{i+1} = 1$ for $i = 1, \dots, 6$. Now $S = \Sigma(P, \{x_1, \dots, x_6\})$ results from gluing A to the double cover of $P \smallsetminus N$, which consists of four disks; it follows that $\gamma \subset S$ appears as shown in Figure 6.

Define

$$\mu_1 = C_1, \quad \mu_2 = C_1, \quad \mu_3 = C_3, \quad \mu_4 = C_5,$$
$$\lambda_1 = C_6, \quad \lambda_2 = C_2, \quad \lambda_3 = C_4, \quad \lambda_4 = C_4,$$

and observe that μ_i and λ_i are framing arcs for the rational tangle $(A_i, L \cap A_i)$, as described above. Let $\tilde{\mu}_i$, $\tilde{\lambda}_i \subset S \cap \partial V_i$ denote the preimages of these curves

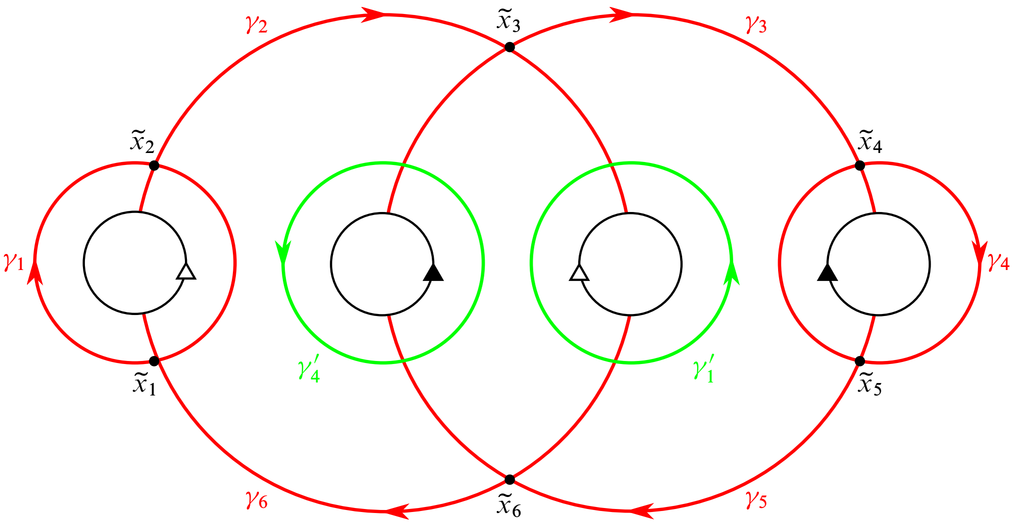

Figure 6: The surface $S = \Sigma(P, \{x_1, \ldots, x_6\})$ along with the branch points $\tilde{x}_1, \ldots, \tilde{x}_6$, the 1–complex γ in red, and the curves γ_1' and γ_4' in green. The curve γ_i consists of the pair of arcs of γ with endpoints $\tilde{x}_i$ and $\tilde{x}_{i+1}$. The curves γ_1' and γ_4' are parallel copies of γ_1 and γ_4, respectively.

$(i = 1, \ldots, 4)$, oriented as follows:

$$\begin{aligned} \tilde{\mu}_1 &= \gamma_1, & \tilde{\mu}_2 &= \gamma_1, & \tilde{\mu}_3 &= \gamma_3, & \tilde{\mu}_4 &= -\gamma_5, \\ \tilde{\lambda}_1 &= \gamma_6, & \tilde{\lambda}_2 &= \gamma_2, & \tilde{\lambda}_3 &= -\gamma_4, & \tilde{\lambda}_4 &= \gamma_4. \end{aligned}$$

With respect to the orientation of S as ∂U_1, $\tilde{\mu}_i \cdot \tilde{\lambda}_i = (-1)^i$, $i = 1, \ldots, 4$. Since the orientation of S agrees with the boundary orientation of ∂V_i exactly when $i = 1$ or 3, we see that $\tilde{\mu}_i \cdot \tilde{\lambda}_i = -1$ with respect to the boundary orientation of V_i for all $i = 1, \ldots, 4$. Thus, it follows that a curve of type $p_i \tilde{\mu}_i + q_i \tilde{\lambda}_i$ bounds a compressing disk for V_i.

Let γ_1' and γ_4' be parallel copies of γ_1 and γ_4, as shown in Figure 6, and define

$$\begin{aligned} \alpha_1 &= p_1 \gamma_1' + q_1 \gamma_6, & \beta_1 &= p_2 \gamma_1 + q_2 \gamma_2, \\ \alpha_2 &= p_3 \gamma_3 - q_3 \gamma_4, & \beta_2 &= -p_4 \gamma_5 + q_4 \gamma_4'. \end{aligned}$$

Then α_1 and α_2 are disjoint compressing disks for U_1, and β_1 and β_2 are disjoint compressing disks for U_2, so $(S, \alpha_1 \cup \alpha_2, \beta_1 \cup \beta_2)$ presents $\Sigma(L)$. By construction,

$$(S, \alpha_1 \cup \alpha_2, \beta_1 \cup \beta_2) = H(a_1, a_2, a_3, a_4),$$

as required. □

5.4 Strong, 1–extendible Heegaard diagrams of genus 2

We will now show that any strong L–space admitting a genus-2 strong Heegaard diagram can be represented by a Heegaard diagram of the form in Construction 5.4. The main technical result of this section is:

Proposition 5.6 *Suppose H is a 1–extendible, irreducible, strong Heegaard diagram of genus 2 such that $|\alpha_i \cap \beta_j| \geq 2$ for each i and j. Then there exist simple closed curves δ_1 and δ_2 on S such that:*

- *Each δ_i divides S into a union $S = T_i \cup U_i$ of two genus-1 surfaces.*
- *δ_1 separates α_1 from α_2, and δ_2 separates β_1 from β_2.*
- *$|\delta_1 \cap \delta_2| = 4$.*
- *$T_1 \cap T_2$ and $U_1 \cap U_2$ are quadrilaterals whose boundaries each consist of two arcs of δ_1 and two arcs of δ_2, and the intersection of either of these regions with any α or β circle consists of arcs that connect opposite sides of the quadrilateral.*
- *$T_1 \cap U_2$ and $U_1 \cap T_2$ are annuli whose boundary components each consist of an arc of δ_1 and an arc of δ_2, and the intersection of either of these regions with any α or β circle consists of arcs that connect opposite boundary components.*

Before proving Proposition 5.6, we show how it implies Theorem 1.6.

Proof of Theorem 1.6 Let Y be a strong L–space with a genus-2 strong Heegaard diagram H. By Proposition 3.1, we may assume that H is 1–extendible. If H is reducible, then Y is a connected sum of lens spaces. Furthermore, if any α circle meets a β circle in a single point, then we may destabilize to obtain a genus-1 diagram, so Y is a lens space. Thus, we may assume that $|\alpha_i \cap \beta_j| \geq 2$ for each i and j.

Form the decompositions $S = T_1 \cup U_1 = T_2 \cup U_2$ as in Proposition 5.6, and assume that the curves are labeled such that $\alpha_1 \subset T_1$, $\alpha_2 \subset U_1$, $\beta_1 \subset U_2$ and $\beta_2 \subset T_2$. The intersection of $\alpha_1 \cup \beta_1$ with the annulus $A_1 = T_1 \cap U_2$ satisfies the hypotheses of Lemma 5.2, so this intersection can be identified with a standard configuration of the form $(A, \boldsymbol{\gamma}(p,q,r,s))$, as described in Construction 5.1, for some integers p, q, r and s. By attaching the quadrilaterals $T_1 \cap T_2$ and $U_1 \cap U_2$, we see that $T_1 \cup U_2$ is a linear plumbing of three annuli, denoted N, and we can then identify $(T_1 \cup U_2, \alpha_1, \beta_1)$ with the configuration $(N, \eta_{p/q}, \eta_{r/s})$ from Construction 5.1. In a similar manner, $(T_2 \cup U_1, \alpha_2, \beta_2)$ can be identified with $(N', \eta_{p'/q'}, \eta_{r'/s'})$, where N' is homeomorphic to N. This identifies H with the description of $H(-q/p, s/r, p'/q', -r'/s')$ as given in Construction 5.4, so $Y \cong \Sigma(L(-q/p, s/r, p'/q', -r'/s'))$. Moreover, since H is a strong diagram, we deduce that $-q/p$, s/r, p'/q' and $-r'/s'$ all have the same sign, and therefore $L(-q/p, s/r, p'/q', -r'/s')$ is an alternating link. $\square$

Proof of Proposition 5.6 To begin, assume that the curves are labeled so that $|\alpha_1 \cap \beta_1|$ is minimal among the four pairs of curves. Choose orientations so that α_1 intersects both β_1 and β_2 positively, and α_2 intersects β_1 positively and β_2 negatively. Let $v = |\alpha_1 \cap \beta_1|$.

Let $R_1, \dots, R_n$ denote the regions of $S - \alpha_1 - \beta_1$. Let e denote the Euler measure (see eg [37, Lemma 6.2]). Then $-2 = e(S) = \sum e(R_i)$. Since all intersection points in $\alpha_1 \cap \beta_1$ have the same sign, as we traverse a boundary component of any R_i, the arcs of α_1 (resp. β_1) must alternate in orientation. In particular, the number of arcs in each boundary component of each R_i is a multiple of 4. It follows that $e(R_i) \le 0$ for all i. Thus, each R_i may be a disk with 4, 8, or 12 sides ($e = 0, -1, -2$, respectively); an annulus with two 4–sided boundary components ($e = -2$); or a genus-1 surface with one 4–sided boundary component ($e = -2$). Consequently, $\{R_1, \dots, R_n\}$ consists of a number of 4–sided disks (which we will refer to as rectangles), along with either

(1) two 8–sided disks;

(2) a 12–sided disk;

(3) an annulus; or

(4) a once-punctured torus.

Let N denote a regular neighborhood of $\alpha_1 \cup \beta_1$; note that $\chi(N) = -v$. Note that $-2 = \chi(S) = \chi(N) + \sum_{i=1}^n \chi(R_i)$, so $\sum_{i=1}^n \chi(R_i) = v - 2$. Thus, the number of rectangles equals $v-4$ in case (1), $v-3$ in case (2), $v-2$ in case (3) and $v-1$ in case (4). We may assume that the nonrectangular region(s) are labeled R_1 (and R_2, in case (1)).

If R_i is a rectangle, then any segment of α_2 or β_2 that meets R_i must enter and exit R_i on opposite sides. Furthermore, R_i cannot meet both α_2 and β_2, since otherwise there would be a rectangle whose four sides are arcs of α_1, β_1, α_2 and β_2, which is prohibited in a strong diagram. Thus, all of the points in $\alpha_2 \cap \beta_2$ (which is a nonempty intersection) must lie in the nonrectangular region(s) of $S - \alpha_1 - \beta_2$. We consider the four cases enumerated above.

Case (1) (two octagons) Without loss of generality, suppose that α_2 and β_2 intersect inside the octagonal region R_1. Label the edges of R_1 consecutively by $e_0, \dots, e_7$, so that e_0, e_2, e_4 and e_6 are arcs of α_1 and e_1, e_3, e_5 and e_7 are arcs of β_1. Let a and b be the components of $\alpha_2 \cap R_i$ and $\beta_2 \cap R_1$, respectively, containing some point of $\alpha_2 \cap \beta_2$, and assume that b intersects e_0. Since all points of $\alpha_2 \cap \beta_0$ have the same sign, the other endpoint of b must be on either e_2 or e_6. However, then a must intersect either e_1 or e_7, respectively, and we obtain a rectangle whose four sides are arcs of α_1, β_1, α_2 and β_2, which is prohibited. Thus, case (1) cannot occur.

Case (2) (one dodecagon) Suppose that α_2 and β_2 intersect inside the dodecagon R_1. Label the edges of R_1 clockwise by $e_0, \dots, e_{11}$, so that odd indices correspond to arcs of α_1 and even indices correspond to arcs of α_2. We consider these indices as elements of $\mathbb{Z}/12$.

The edges e_i inherit orientations from the orientations on α_1 and β_1. Since α_1 and β_1 intersect positively, we may assume that e_0, e_3, e_4, e_7, e_8 and e_{11} are oriented clockwise (ie opposite to the boundary orientation on ∂R_1), and the remaining edges are oriented counterclockwise. By our orientation conventions, segments of α_2 may enter R_1 through e_3, e_7 and e_{11} and exit through e_1, e_5 and e_9; segments of β_2 may enter through e_2, e_6 and e_{10} and exit through e_0, e_4 and e_8.

Lemma 5.7 *For $i \in \mathbb{Z}/12$, if γ_i is an arc that begins in R_1, exits through e_i, and proceeds, passing through rectangular regions of $S - \alpha_1 - \beta_1$ without turning left or right, until it reenters R_1 through some e_j, then $j = i + 6$. (That is, γ_i reenters R_1 through the edge directly opposite the one through which it exited.)*

Proof For notational simplicity, we consider the case where $i = 0$; the others proceed similarly.

Because the arc γ_i runs parallel to β_1 (up to isotopy), it intersects α_1 positively when it reenters R_1 through e_j. This implies that $j = 2$, 6 or 10.

Let β_1' and β_1'' be parallel copies of β_1 that run just to the left and right of β_1, respectively. Note that $\beta_1' \cap R_1$ consists of arcs e_1', e_5' and e_9' that run parallel to e_1, e_5 and e_9, respectively, where e_k' runs from a point in the interior of e_{k+1} to a point in the interior of e_{k-1}. Up to isotopy, we may take γ_0 to be a segment of β_1' that begins on e_1' and ends on e_1', e_5' or e_9', depending on whether $j = 2$, 6 or 10, respectively. If $j = 2$, then $\gamma_0 \cup e_1'$ is an embedded circle that runs parallel to β_1 on the left, meaning that it is actually equal to all of β_1', a contradiction. A similar argument using β_1'' excludes the $j = 10$ case. Thus, we conclude that $j = 6$. □

For $i \in \mathbb{Z}/12$, let $p_i = |\gamma_i \cap (\alpha_1 \cup \beta_1)|$, so that γ_i passes through $p_i - 1$ rectangular regions. Clearly, $p_i = p_{i+6}$. Since every rectangle meets a unique γ_i with $i \in \{0, 2, 4\}$ and a unique γ_j with $j \in \{1, 3, 5\}$, we have

$$p_0 + p_2 + p_4 = p_1 + p_3 + p_5 = v. \tag{3}$$

Let a_0 and b_0 be the components of $\alpha_2 \cap R_i$ and $\beta_2 \cap R_1$, respectively, containing some point of $\alpha_2 \cap \beta_2$. Up to relabeling, we may assume that b_0 exits R_1 through e_0.

Lemma 5.8 *If a and b are any components of $\alpha_2 \cap R_i$ and $\beta_2 \cap R_1$, respectively, that intersect in R_1, then a enters R_1 through e_3 and exits through e_9, and b enters through e_6 and exits through e_0.*

Proof Suppose b enters R_1 through e_i, where $i \in \{2, 6, 10\}$. Then a cannot intersect $e_{i\pm 1}$, since that would produce a prohibited rectangle. Therefore, b must exit R_1 through e_{i+6}, or else a would be forced to intersect $e_{i\pm 1}$. Symmetrically, a must enter R_1 through r_j, where $j \in \{3, 7, 11\}$, and exit through r_{j+6}. It follows that $j = i + 3$. Furthermore, if a' and b' are another pair of components that intersect, then a' and b' must enter and exit through the same sides as a and b, respectively, since otherwise a and a' (resp. b and b') would intersect, violating the embeddedness of α_2 (resp. β_2). Since we know that b_0 exits through e_0, we conclude that the same is true for any b. □

It follows from the previous two lemmas that for some nonnegative integers x, y, z, x', y' and z', with $y, y' \geq 1$, we have:

- $\alpha_2 \cap R_1$ consists of x arcs from e_3 to e_5, x arcs from e_{11} to e_9, y arcs from e_3 to e_9, z arcs from e_3 to e_1, and z arcs from e_7 to e_9.
- $\beta_2 \cap R_1$ consists of x' arcs from e_6 to e_8, x' arcs from e_2 to e_0, y' arcs from e_6 to e_0, z' arcs from e_6 to e_4, and z' arcs from e_{10} to e_0.

Therefore, we have

$$|\alpha_2 \cap \beta_1| = zp_1 + (x+y+z)p_3 + xp_5,$$
$$|\alpha_1 \cap \beta_2| = (x'+y'+z')p_0 + x'p_2 + z'p_4,$$
$$|\alpha_2 \cap \beta_2| = yy'.$$

We make the following three observations:

(1) At least one of x or z is zero, since otherwise every rectangular region of $S \setminus (\alpha_1 \cup \alpha_2)$ would meet α_2, and therefore β_2 would be constrained to lie in R_1, a contradiction. Similarly, at least one of x' or z' is zero.

(2) At least one of x or z' is zero, since otherwise the arcs of α_2 from e_3 to e_5 would intersect the arcs of β_2 from e_6 to e_4, violating Lemma 5.8. Similarly, at least one of x' or z is zero.

(3) If $x = z = 0$, then α_2 consists entirely of y parallel copies of a circle that runs from e_6 to e_0 in R_1 and then from e_0 back to e_6 outside of R_1. Since α_2 is a single circle, we then have $y = 1$. By the minimality of $|\alpha_1 \cap \beta_1|$,

$$p_3 = |\alpha_2 \cap \beta_1| \geq |\alpha_1 \cap \beta_1| = p_1 + p_3 + p_5,$$

a contradiction. Thus, x and z are not both zero. Similarly, x' and z' are not both zero.

Without loss of generality, let us assume that $x=0$, $z\neq 0$, $x'=0$ and $z'\neq 0$; the opposite case is handled symmetrically. We can choose a pair of embedded octagons $D_1, D_2 \subset R_1$ such that

- D_1 contains $\alpha_2 \cap R_1$ and has edges on e_1, e_3, e_7 and e_9;
- D_2 contains $\beta_2 \cap R_1$ and has edges on e_0, e_4, e_6 and e_{10};
- $D_1 \cap D_2$ is a quadrilateral that contains $\alpha_2 \cap \beta_2$.

(See Figure 7.) We now define T_1 (resp. T_2) to be the union of D_1 (resp. D_2) with strips that are tubular neighborhoods of γ_1 and γ_3 (resp. γ_0 and γ_4); this is a genus-1 surface with one boundary component that contains α_2 (resp. β_2) in its interior but avoids α_1 (resp. β_1). The strips attached to D_1 cannot intersect the strips attached to D_2, since they contain segments of α_2 and β_2, respectively; thus,

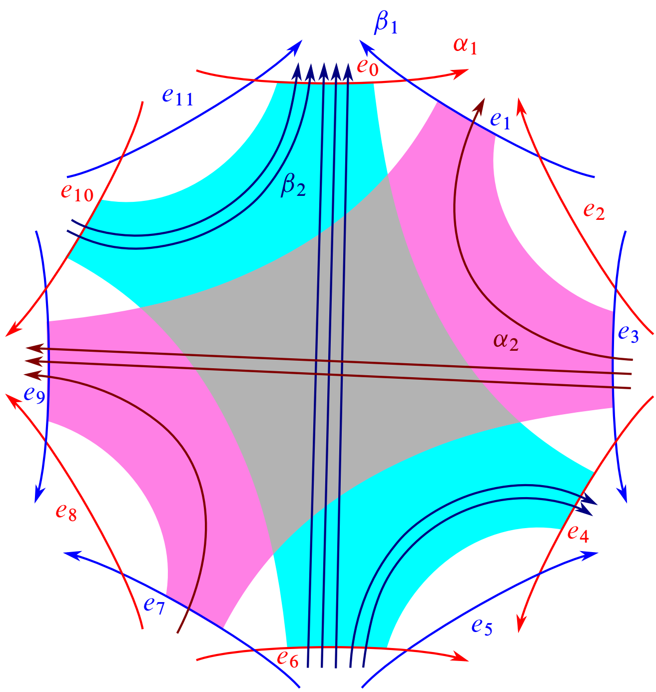

Figure 7: A dodecagon region of $S \setminus (\alpha_1 \cup \beta_1)$. The octagons D_1 and D_2 found in case (2) of the proof of Proposition 5.6 are shown in pink and light blue, respectively, overlapping in the gray region.

$T_1 \cap T_2 = D_1 \cap D_2$, as required. Defining $U_1 = \overline{S \smallsetminus T_1}$ and $U_2 = \overline{S \smallsetminus T_2}$, it is easy to see that $T_1 \cap U_2$ and $T_2 \cap U_1$ are annuli and that $U_1 \cap U_2$ is a quadrilateral, as required.

Case (3) (one annulus) Label the edges of R_1 by $e_0, \dots, e_7$, where:

- e_0, e_2, e_4 and e_6 are segments of α_1, and e_1, e_3, e_5 and e_7 are segments of β_1.
- The edges on one boundary component of R_1 are e_0, e_1, e_2 and e_3, ordered according to the boundary orientation. The edges on the other component are e_4, e_5, e_6 and e_7, ordered likewise.
- The orientations on e_2, e_3, e_4 and e_5 agree with the boundary orientation, while the orientations on e_0, e_1, e_6 and e_7 disagree with the boundary orientation.

An argument just like Lemma 5.7 shows:

Lemma 5.9 *For $i \in \mathbb{Z}/8$, if γ_i is an arc that begins in R_1, exits through e_i, and proceeds, passing through rectangular regions of $S - \alpha_1 - \beta_1$ without turning left or right, until it reenters R_1 through some e_j, then $j = i + 4$.* □

According to our orientation conventions, segments of α_2 may enter R_1 through e_1 or e_7 and exit through e_3 or e_5, and segments of β_2 may enter through e_2 or e_4 and exit through e_0 and e_6.

Lemma 5.10 *Every component of $\alpha_2 \cap R_1$ enters R_1 through the same edge (say e_i, where $i \in \{1, 7\}$) and exits through e_{i+4}. Likewise, every component of $\beta_2 \cap R_1$ enters R_1 through the same edge (say e_j, where $j \in \{2, 4\}$) and exits through e_{j+4}.*

Proof If there is some component a of $\alpha_2 \cap R_1$ that intersects e_i, and some component a' (possibly the same as a) that intersects $e_{i\pm 2}$, then by Lemma 5.9, α_2 meets every rectangular region of $S - \alpha_1 - \beta_1$, which implies that α_2 does not meet any rectangular region. However, this implies that $\beta_2 \cap \alpha_1 = \varnothing$, a contradiction. An analogous argument applies for β_2. □

Assume for concreteness that segments of α_2 enter R_1 through e_1 and exit through e_5, and segments of β_2 enter R_1 through e_4 and exit through e_0. (The other cases are handled symmetrically.) Lemma 5.10 together implies that the multicurves $\alpha_2 \cap R_1$ and $-\beta_2 \cap R_1$ satisfy the hypotheses of Lemma 5.2; therefore, $(R_1, R_1 \cap (\alpha_2 \cup \beta_2))$ may be identified with the standard configuration given in Construction 5.1. As shown in Figure 8, there are embedded annuli N_1, $N_2 \subset R_1$ such that

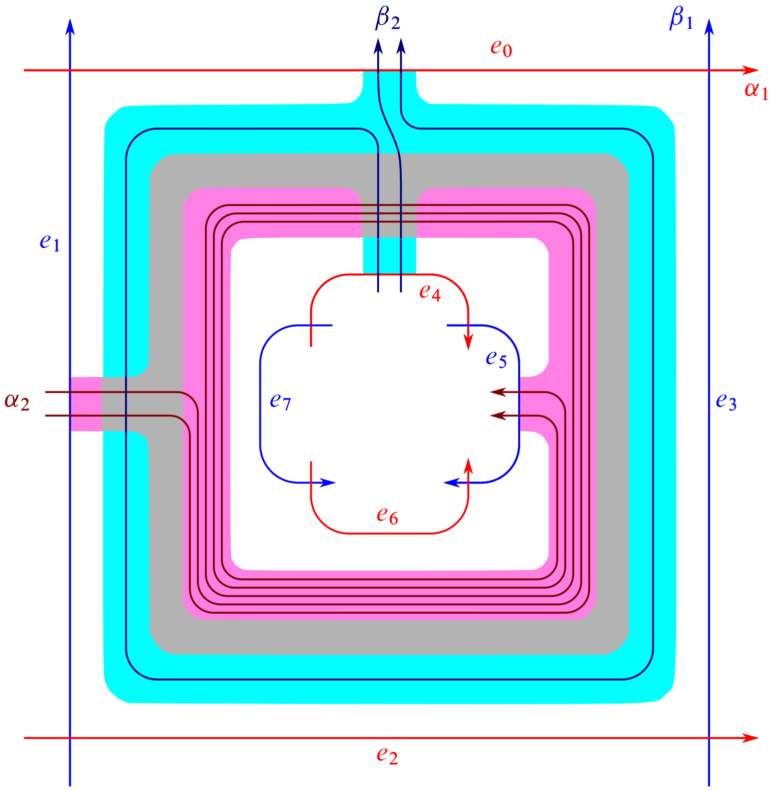

Figure 8: An annulus region of $S \setminus (\alpha_1 \cup \beta_1)$. The annuli N_1 and N_2 found in case (3) of the proof of Proposition 5.6 are shown in pink and light blue, respectively, overlapping in the gray region.

- N_1 contains $\alpha_2 \cap R_1$, and $\partial N_1 \cap \partial R_1$ consists of proper subarcs of e_1 and e_5;
- N_2 contains $\beta_2 \cap R_1$, and $\partial N_2 \cap \partial R_1$ consists of proper subarcs of e_0 and e_4;
- $N_1 \cap N_2$ is an annulus, and $|\partial N_1 \cap \partial N_2| = 4$.

Let T_1 (resp. U_2) be the union of N_1 (resp. N_2) with a 1–handle that follows γ_1 (resp. γ_0); this is then a genus-1 surface with one boundary component that contains α_2 (resp. β_2) in its interior but avoids α_1 (resp. β_1). Moreover, $T_1 \cap U_2$ is an annulus whose boundary components each consist of an arc of $\delta_1 = \partial T_1$ and an arc of $\delta_2 = \partial U_2$. If we define $U_1 = \overline{S \setminus T_1}$ and $T_2 = \overline{S \setminus U_2}$, it is easy to verify that the remaining intersections are as required.

Case (4) (one punctured torus) Label the edges of R_1 by e_0, e_1, e_2 and e_3, so that e_0 and e_2 are segments of α_1, and e_1 and e_3 are segments of β_1. In this case, a path γ_i that exits R_1 through e_i will proceed through all $v-1$ rectangular regions before reentering through e_{i+2} (indices modulo 4). Thus, we see that α_2 and β_2 cannot both exit R_1, or else they would be forced to intersect in a rectangular region of $S-\alpha_1-\alpha_2$. Hence, either $\alpha_1 \cap \beta_2 = \varnothing$ or $\alpha_2 \cap \beta_1 = \varnothing$, violating our assumption that $|\alpha_i \cap \beta_j| \geq 2$. Thus, case (4) does not arise.

This concludes the proof of Proposition 5.6. □

Corollary 5.11 *The minimum genus of a strong Heegaard diagram for a strong L–space can exceed its Heegaard genus.*

This corollary is proven by either of the following examples:

Example 5.12 Let L denote the Borromean rings, which is a prime, three-component, alternating link of bridge number 3. Therefore, $Y = \Sigma(L)$ is an irreducible strong L–space of Heegaard genus 2. Suppose Y can represented by a strong Heegaard diagram H of genus 2, which may be assumed to be 1–extendible by Proposition 3.1. Since Y is irreducible, $M(H)$ does not contain a 0 entry. Since $M(H)$ is a 2×2 presentation matrix for $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4$, every entry in $M(H)$ is a nonzero multiple of 4. However, since $M(H)$ is a Pólya matrix, we must then have

$$|\det(M(H))| \geq 4 \cdot 4 + 4 \cdot 4 > 16,$$

a contradiction. Thus, any strong Heegaard diagram for Y has genus greater than 2.

More generally, it is straightforward to find prime, alternating 3–bridge diagrams without nontrivial Conway circles. By the solution of the Tait flyping conjecture in this case [26; 42; 10], no such link is isotopic to a link of the form $L(a_1, a_2, a_3, a_4)$ with all a_i the same sign. Therefore, its branched double cover is a strong L–space of Heegaard genus 2, but it does not possess a strong Heegaard diagram of genus 2.

Example 5.13 Using the program SnapPy, Nathan Dunfield has identified 316 strong L–spaces among the 11,031 small hyperbolic manifolds in the Hodgson–Weeks census. Most of these manifolds have Heegaard genus 2, and all have Heegaard genus at most 3. Theorem 1.6 implies that these manifolds cannot admit strong diagrams of genus 2. In fact, Dunfield was only able to find strong Heegaard diagrams of genus 4 through 7 for any of these manifolds. Moreover, all of Dunfield's examples can be realized as Dehn fillings of the minimally twisted 5–chain link, so they are branched double covers of links obtained from filling rational tangles into the pentacle graph, and all of these links are in fact alternating. (See [8; 24].)

Corollary 5.11 motivates the following questions:

Question 5.14 Can the difference between the Heegaard genus of a strong L–space Y and the minimal genus of a strong Heegaard diagram for Y be arbitrarily large?

Question 5.15 Is every strong L–space admitting a strong Heegaard diagram of genus 3 a Seifert fibered space or graph manifold?

6 Floer simple knots

Given a Heegaard diagram $H = (S, \alpha, \beta)$ for a 3–manifold Y, a pair of basepoints w $z \in S \smallsetminus (\alpha \cup \beta)$ determines an oriented knot $K \subset Y$, as explained in [32]. Briefly, we let γ_α (resp. γ_β) be an arc from w to z in the complement of the α (resp. β) circles, pushed into the α (resp. β) handlebody, and we set $K = \gamma_\alpha \cup -\gamma_\beta$. The pair of basepoints induce a filtration on the complex $\widehat{\mathrm{CF}}(H)$ whose associated graded complex is denoted $\widehat{\mathrm{CFK}}(H)$, and the homology of $\widehat{\mathrm{CFK}}(H)$ is an invariant called the *knot Floer homology* of K, denoted $\widehat{\mathrm{HFK}}(Y, K)$ [32; 38].

A knot K in an L–space Y is called *Floer simple* if $\widehat{\mathrm{HFK}}(Y, K) \cong \widehat{\mathrm{HF}}(Y)$. The classification of Floer simple knots is of great interest, in part because any Floer simple knot minimizes rational genus in its homology class [28]. In particular, the only nullhomologous knot in Y that is Floer simple is the unknot. Note that there exist homology classes in certain L–spaces that do not contain any Floer simple knot [22, Proposition 7.4].

Strong Heegaard diagrams provide a natural source of Floer simple knots. Given a strong Heegaard diagram H for a strong L–space Y, a knot K is called *simple with respect to* H if it can be represented by a pair of basepoints in H; it follows that K is Floer simple. Each homology class in a lens space contains a unique simple knot (with respect to the genus-1 strong Heegaard diagram). One formulation of the famous Berge conjecture asserts that any knot in a lens space with an integer-slope surgery to S^3 must be a simple knot [2]. This would follow from the conjecture that every Floer simple knot in a lens space is simple [16; 39].

Thus, topological considerations motivate the investigation of the existence and uniqueness of simple knots in strong L–spaces, as in the following proposition:

Proposition 6.1 *If Y is presented by a strong diagram H of genus 2, then every homology class in $H_1(Y; \mathbb{Z})$ is represented by a simple knot with respect to H.* □

Proposition 6.1 follows at once from the following theorem:

Theorem 6.2 *Given a genus-2 Heegaard diagram $H = (S, \alpha, \beta)$ of a rational homology sphere Y and any homology class $x \in H_1(Y; \mathbb{Z})$, there exists a pair of basepoints $z_1, z_2 \in S$ such that the doubly pointed Heegaard diagram $(S, \alpha, \beta, z_1, z_2)$ represents a knot $K \subset Y$ with $[K] = x$.*

Theorem 6.2 does not generalize to diagrams of higher genus. As an example, take a genus-3 Heegaard diagram H for $Y = \#^3 L(p,q)$ by forming the connected sum of three genus-1 strong Heegaard diagrams for its individual summands. Then H contains $3p-2$ regions, so the knots in Y represented by pairs of basepoints in H represent at most $(3p-2)^2$ homology classes, whereas $H_1(Y;\mathbb{Z})$ has order p^3.

Proof of Theorem 6.2 First we recall how to determine the homology class represented by a knot $K \subset Y$ presented by a doubly pointed Heegaard diagram $H = (S, \alpha, \beta, z_1, z_2)$. The i^{th} column of the intersection matrix $M(H')$ is the vector of intersection numbers

$$v_i = (\alpha_1 \cdot \beta_i, \dots, \alpha_g \cdot \beta_i) \in \mathbb{Z}^g,$$

and we have an isomorphism

$$H_1(Y;\mathbb{Z}) \cong \operatorname{coker}(M(H')) \cong \mathbb{Z}^g / \langle v_1, \dots, v_g \rangle.$$

Choose an arc $\gamma \subset S \smallsetminus \beta$ transverse to the α curves that begins at z_1 and ends at z_2, and form the vector $v(\gamma) = (\alpha_1 \cdot \gamma, \dots, \alpha_g \cdot \gamma) \in \mathbb{Z}^g$. Then, under the above isomorphism, the class $[K]$ maps to the image of v in $\mathbb{Z}^g / \langle v_1, \dots, v_g \rangle$.

Now we return to the case at hand. Choose a region of H whose boundary contains arcs of both β_1 and β_2, and place a basepoint z_0 in it. For $i = 1, 2$, traverse a parallel copy of β_i' of β_i based at z_0 and place a basepoint on β_i' in each region it enters. To each subarc $\gamma \subset \beta_i'$ that starts at z_0 and ends at one of these basepoints, associate the vector $v(\gamma) = (\alpha_1 \cdot \gamma, \alpha_2 \cdot \gamma)$. The collection of these vectors gives a sequence S_i beginning with $(0,0)$ and ending with $v_i = (\alpha_1 \cdot \beta_i, \alpha_2 \cdot \beta_i)$, in which each pair of consecutive terms differs by $(\pm 1, 0)$ or $(0, \pm 1)$.

We claim that $S_1 - S_2 = \{s_1 - s_2 \mid s_i \in S_i\}$ contains a vector in each equivalence class of $\mathbb{Z}^2 / \langle v_1, v_2 \rangle$. Assuming this, choose $x \in \mathbb{Z}^2/\langle v_1, v_2 \rangle \cong H_1(Y;\mathbb{Z})$ and represent $-x$ by $s_1 - s_2 \in S_1 - S_2$. Choose $\gamma_i \subset \beta_i'$ with $v(\gamma_i) = s_i$ and write $\partial v_i = z_i - z_0$ for $i = 1, 2$. Then the concatenation of $-\gamma_1$ and γ_2 is an arc $\gamma \subset S \smallsetminus \beta$ from z_1 to z_2 with $v(\gamma) = -s_1 + s_2$. It follows that the knot represented by the doubly pointed Heegaard diagram $(S, \alpha, \beta, z_1, z_2)$ represents the class x. Since x was arbitrary, the theorem follows from the asserted claim.

Now we establish the claim. Let $\Gamma_i \subset \mathbb{R}^2$ denote the rectilinear curve from $(0,0)$ to v_i that interpolates the sequence S_i. Regard Γ_i as the image of a map $F_i \colon I \to \mathbb{R}^2$.

It follows that F_i is homotopic rel endpoints to the linear map from $(0,0)$ to v_i, and it descends to a based map $f_i\colon S^1 \to \mathbb{R}^2/\langle v_1, v_2\rangle$. Since Y is a rational homology sphere, it follows that v_1 and v_2 are linearly independent, $\mathbb{R}^2/\langle v_1, v_2\rangle \cong T^2$ and $\{[f_1],[f_2]\}$ is a basis for $H_1(T^2)$. It follows that the map $f\colon S^1 \times S^1 \to T^2$ given by $f(x,y) = f_1(x) - f_2(y)$ induces an isomorphism on H_1. In particular, f has degree one, so it surjects. On the other hand, the image of f is the image of the set $\Gamma_1 - \Gamma_2 = \{\gamma_1 - \gamma_2 \mid \gamma_i \in \Gamma_i\}$ in $\mathbb{R}^2/\langle v_1, v_2\rangle$. We conclude that $\Gamma_1 - \Gamma_2$ contains a vector in each equivalence class of $\mathbb{R}^2/\langle v_1, v_2\rangle$ and hence each class of $\mathbb{Z}^2/\langle v_1, v_2\rangle$.

To complete the argument, we show that $S_1 - S_2 = (\Gamma_1 - \Gamma_2) \cap \mathbb{Z}^2$. Thus, suppose that $\gamma_1 - \gamma_2 \in \mathbb{Z}^2$ with $\gamma_i \in \Gamma_i$. Then γ_i lies on the unit segment (vertical or horizontal) between two consecutive points in S_i. If γ_1 lies on a horizontal segment, then its y–coordinate is integral. Since $\gamma_1 - \gamma_2 \in \mathbb{Z}^2$, γ_2 has an integral y–coordinate, so it lies on a horizontal segment, too. In this case, we take $s_i \in S_i$ to be the left endpoint of the horizontal segment containing γ_i, and $s_1 - s_2 = \gamma_1 - \gamma_2$. Similarly, if the γ_i both lie on vertical segments, then the top endpoints furnish points $s_i \in S_i$ such that $s_1 - s_2 = \gamma_1 - \gamma_2$. In summary, we recover the claim asserting that $S_1 - S_2$ contains a vector in each equivalence class of $\mathbb{Z}^2/\langle v_1, v_2\rangle$, and the theorem follows as described. □

Remark 6.3 With more effort, it appears possible to show that homologous simple knots with respect to a given genus-2 strong Heegaard diagram are in fact isotopic. This is trivially the case with genus-1 strong diagrams. We do not know whether there exist two strong Heegaard diagrams H_1 and H_2 for a space, and knots K_1 and K_2 that are simple with respect to H_1 and H_2, respectively, such that $[K_1] = [K_2]$ but $K_1 \neq K_2$.

Proposition 6.1 has an interesting application concerning nonorientable surfaces in 4–manifolds. If Y is a rational homology sphere with $H_2(Y;\mathbb{Z}/2) \neq 0$, then any class $x \in H_2(Y;\mathbb{Z}/2)$ can be represented by an closed, embedded, nonorientable surface of some genus. The minimal genus of such a surface is a measure of the topological complexity of Y, first studied by Bredon and Wood [4]. It is natural to ask whether the genus can be lowered by adding a dimension, ie whether the minimal genus of a surface in Y representing x is the same as the minimal genus of a surface in $Y \times I$ representing the corresponding homology class. For the strong L–spaces considered in Section 5, the answer to this question is negative:

Corollary 6.4 *Let Y be a strong L–space admitting a strong Heegaard diagram of genus 2, and let W be any smooth homology cobordism from Y to itself (eg $W = Y \times I$). For any nonzero homology class $x \in H_2(W;\mathbb{Z}/2)$, the minimal genus of a smoothly embedded, closed, connected, nonorientable surface in W representing*

x is the same as the minimal genus of such a surface in Y itself representing the corresponding element $\overline{x} \in H_2(Y;\mathbb{Z}/2)$. Furthermore, this minimal genus is equal to

$$\max_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} 2\big(d(Y,\mathfrak{s}) - d\big(Y, \mathfrak{s} + \mathrm{PD}(\beta(\overline{x}))\big)\big), \tag{4}$$

where $\beta\colon H_2(Y;\mathbb{Z}/2) \to H_1(Y;\mathbb{Z})$ is the Bockstein homomorphism.

Here $\mathrm{Spin}^c(Y)$ is the set of spin^c structures on Y and $d(Y,\mathfrak{s})$ denotes the *correction term* of the spin^c structure $\mathfrak{s}$, a rational number derived from the Heegaard Floer homology of Y [30]. When Y is an L–space, $d(Y,\mathfrak{s})$ equals the absolute rational grading of the summand $\widehat{\mathrm{HF}}(Y,\mathfrak{s}) \subset \widehat{\mathrm{HF}}(Y)$. Additionally, the correction terms of a strong L–space can be readily computed (up to an overall shift) from a strong Heegaard diagram H using a formula of Lee and Lipshitz [20] for the grading difference between two generators of $\widehat{\mathrm{CF}}(H)$. This information is enough to determine the quantity (4).

Proof of Corollary 6.4 Work of Ni and Wu [28] and of Ruberman, Strle and the second author [22] shows that (4) is a lower bound on the minimal genus in both Y and W (for any rational homology sphere Y), and that both of these bounds are sharp when Y is an L–space and $\beta(\overline{x})$ is represented by a Floer simple knot. □

Question 6.5 If Y is a strong L–space, is every homology class of order 2 represented by a simple knot with respect to some strong Heegaard diagram for Y?

An affirmative answer would imply that the conclusion of Corollary 6.4 applies to every strong L–space.

7 Waves, antiwaves, and weak reducibility

7.1 Waves

A *wave* in a Heegaard diagram H is an arc γ that is properly embedded in a region R of H and whose endpoints lie on the interiors of distinct arcs of ∂R and on the same α or β curve, such that the local signs of intersection at the two endpoints of γ are opposite. Waves were introduced by Volodin, Kuznetsov and Fomenko [47], who used them to formulate an algorithm that they conjectured would detect whether or not a given Heegaard diagram presents S^3. Their key observation was the following:

Lemma 7.1 [47, Theorem 4.3.1] *Suppose that γ is a wave in a Heegaard diagram $H = (S,\alpha,\beta)$ with endpoints on the curve α_i. Let a_1 and a_2 denote the two arcs into*

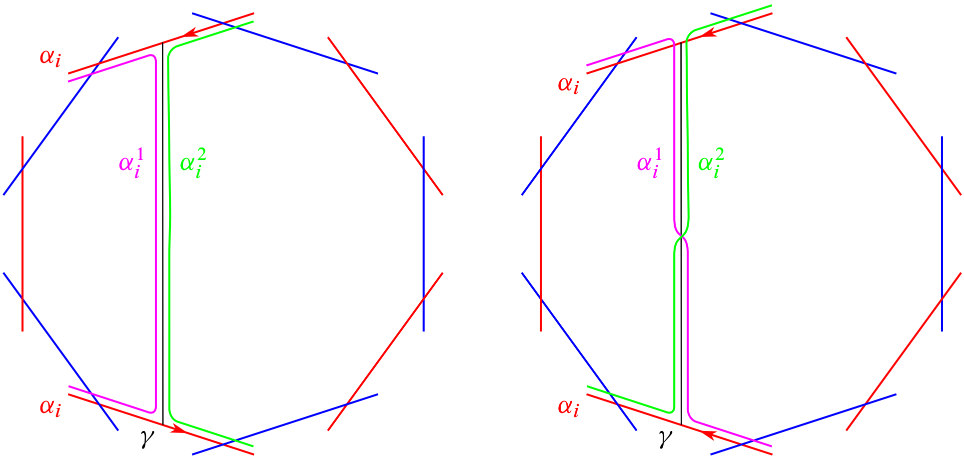

Figure 9: A wave (left) and an antiwave (right), along with the curves α_i^1 and α_i^2, perturbed to be transverse to α_i. For the wave, the three curves α_i, α_i^1 and α_i^2 are pairwise disjoint and cobound a pair of pants; for the antiwave, each pair of curves meets in a point.

which $\partial\gamma$ splits α_i, and set $\alpha_i^1 = a_1 \cup \gamma$ and $\alpha_i^2 = a_2 \cup \gamma$.[3] *Then it is possible to replace α_i with exactly one of α_i^1 and α_i^2 to obtain a Heegaard diagram $H' = (S, \alpha', \beta)$ which presents the same manifold as H. An analogous statement holds with the roles of α and β exchanged.* □

Given an arbitrary Heegaard diagram H, iterating the *wave move* described in Lemma 7.1 yields a Heegaard diagram for Y containing no waves. Volodin, Kuznetsov, and Fomenko conjectured that the only waveless diagram for S^3 in any given genus g is the standard one. This conjecture was subsequently shown to be true for Heegaard diagrams of genus 2 [18] but false for diagrams of higher genus [27; 29; 46].

Proposition 7.2 *A 1–extendible, strong Heegaard diagram H for a strong L–space Y does not contain a wave.*

Proof Without loss of generality, suppose to the contrary that H contains a wave γ with endpoints on the curve α_i. Consider the curves α_i^1 and α_i^2 defined in Lemma 7.1. Observe that $\alpha_i \cap \beta$ is the disjoint union of the nonempty sets $\alpha_i^1 \cap \beta$ and $\alpha_i^2 \cap \beta$. Since H is 1–extendible, $\mathfrak{S}(H)$ is a nontrivial disjoint union $\mathfrak{S}_1 \sqcup \mathfrak{S}_2$, where $\mathfrak{S}_j$ consists of the generators that use a point in α_i^j. Let H' denote the Heegaard diagram

[3] Observe that the curves α_i^1 and α_i^2 can be isotoped slightly to cobound a pair of pants in S with α_i.

for Y guaranteed by Lemma 7.1 with, say, $\alpha_i' = \alpha_i^j$. Note that H' is strong, and $\mathfrak{S}(H') = \mathfrak{S}_j$. Then

$$\det(Y) = |\mathfrak{S}(H')| = |\mathfrak{S}_j| < |\mathfrak{S}(H)| = \det(Y),$$

a contradiction. □

7.2 Antiwaves and formal L–spaces

The set $\mathcal{S}$ of strong L–spaces constitutes a class of 3–manifolds for which it is easy to certify the property that they are L–spaces. The set of formal L–spaces constitutes another such class. This family is well-known to experts but has not previously appeared in the literature. To define it, recall that a *triad* is a triple of closed, oriented 3–manifolds that are obtained by Dehn filling a compact manifold with torus boundary along a triple of curves at pairwise distance (ie geometric intersection number) one. A simple and useful way to recognize a triad is from a triple of Heegaard diagrams $H_i = (S, \alpha_0 \cup \alpha_i, \beta)$ for $i = 1, 2, 3$, where α_1, α_2 and α_3 are three curves on S at pairwise distance one. If Y_i denotes the manifold presented by H_i, then (Y_1, Y_2, Y_3) forms a triad. In general, if (Y_1, Y_2, Y_3) denotes a triad of rational homology spheres, then an elementary calculation in homology shows that they may be permuted so that $\det(Y_1) + \det(Y_2) = \det(Y_3)$.

Definition 7.3 The set of *formal L–spaces* $\mathcal{F}$ is the smallest set of manifolds such that

- $S^3 \in \mathcal{F}$; and
- if (Y_1, Y_2, Y_3) is a triad with Y_1, $Y_2 \in \mathcal{F}$ and $\det(Y_1) + \det(Y_2) = \det(Y_3)$, then $Y_3 \in \mathcal{F}$.

An elementary application of the surgery triangle in Heegaard Floer homology shows that every manifold in $\mathcal{F}$ is an L–space. Compare Definition 7.3 to the definition of a quasi-alternating link [36, Section 2]. In particular, the branched double cover of a quasi-alternating link is a formal L–space. Both $\mathcal{S}$ and $\mathcal{F}$ are proper subsets of the set of all L–spaces, as the Poincaré homology sphere is in the complement of both.

Proposition 7.4 *There exists a formal L–space that is not a strong L–space; thus, $\mathcal{F} \not\subset \mathcal{S}$.*

Proof The pretzel link $L = P(2, 2, -3)$ is a nonalternating, quasi-alternating link with determinant 8. Its branched double cover $Y = \Sigma(L)$ is the small Seifert fibered space $\Sigma(2, 2, -3)$, and it belongs to $\mathcal{F}$, as remarked above. Since $\det(Y) = 8$ and Y is not a connected sum of lens spaces, Theorem 1.5 shows that Y is not a strong L–space. Thus, $Y \in \mathcal{F} - \mathcal{S}$. □

Question 7.5 Is every strong L–space a formal L–space? That is, do we have $\mathcal{S} \subset \mathcal{F}$?

Note that an affirmative answer to Question 1.2 would imply an affirmative answer to Question 7.5. To approach Question 7.5, we introduce the notion of an *antiwave* in a Heegaard diagram H. This is an arc γ that is properly embedded in a region R of H and whose endpoints lie on the interiors of distinct arcs of ∂R and on the same α or β curve, such that the local signs of intersection at the two endpoints of γ are the same. The relevance of antiwaves is given by the following:

Proposition 7.6 *If a 1–extendible strong Heegaard diagram H for a strong L–space Y contains an antiwave, then Y fits into a triad with strong L–spaces Y_1 and Y_2 with* $\det(Y) = \det(Y_1) + \det(Y_2)$.

Thus, one might try to answer Question 7.5 by showing that every strong L–space admits a strong, 1–extendible Heegaard diagram that contains an antiwave.

Proof of Proposition 7.6 Without loss of generality, assume that H contains an antiwave γ with endpoints on α_i. As in the case of a wave, let a_1 and a_2 denote the two arcs of $\alpha_i \smallsetminus \partial\gamma$, and let $\alpha_i^j = a_j \cup \gamma$ for $j = 1, 2$, with orientation induced from that of α_i. We may perturb α_i^1 and α_i^2 so that they meet α_i and each other transversally, with each pair of curves meeting in a single point. Just as in the proof of Proposition 7.2, there is a natural identification of $\alpha_i \cap \beta$ with the disjoint union $(\alpha_i^1 \cap \beta) \sqcup (\alpha_i^2 \cap \beta)$, leading to a nontrivial decomposition $\mathfrak{S}(H) = \mathfrak{S}_1 \sqcup \mathfrak{S}_2$.

For $j = 1, 2$, let $H_j = (S, \alpha^j, \beta)$ be the diagram obtained by replacing α_i with the perturbed copy of α_i^j. Note that $\mathfrak{S}(H_j)$ is identified with $\mathfrak{S}_j$, preserving all signs of intersection. We must verify that H_j is a Heegaard diagram, ie that the curves in α^j are linearly independent in $H_1(S;\mathbb{Z})$. Since all points in $\mathfrak{S}(H)$ have the same sign, the intersection matrix $M(H_j)$ has

$$|\det(M(H_j))| = |\mathfrak{S}(H_j)| = |\mathfrak{S}_j| \neq 0.$$

Therefore, the curves in α^j must be linearly independent, as required. Let Y_j be the 3–manifold presented by H_j; then Y_j is a strong L–space. By construction, (Y_1, Y_2, Y) forms a triad. Moreover,

$$\det(Y) = |\mathfrak{S}(H)| = |\mathfrak{S}_1| + |\mathfrak{S}_2| = |\mathfrak{S}(H_1)| + |\mathfrak{S}(H_2)| = \det(Y_1) + \det(Y_2),$$

as required. $\square$

7.3 Weak reducibility

A Heegaard splitting is called *weakly reducible* if there exist disjoint compressing disks in the two handlebodies into which the Heegaard surface decomposes the 3–manifold. This notion was introduced by Casson and Gordon, who showed that given a weakly reducible Heegaard splitting of a manifold, either the splitting is reducible or else the manifold contains an incompressible surface of positive genus [5, Theorem 3]. The following result is elementary.

Proposition 7.7 *A strong Heegaard diagram of genus $g \geq 3$ describes a weakly reducible Heegaard splitting.*

Proof Given a strong Heegaard diagram, convert it into a 1–extendible strong Heegaard diagram H for the same Heegaard splitting by Proposition 3.1. Then $M = M(H)$ is a Pólya matrix and H is coherent. The proposition then follows from the assertion that a $g \times g$ Pólya matrix M contains a zero entry for $g \geq 3$, which we now establish. Let

$$N = \begin{pmatrix} b & o & t \\ c & u & d \\ r & a & g \end{pmatrix}$$

denote the top-left 3×3 minor of M and m the product of its remaining diagonal entries. Since M is a Pólya matrix, either $m = 0$ or else all nonzero terms in the expansion

$$\det(N) = bug + cat + rod - bad - cog - rut$$

have the same sign, since each of these terms' products with m contributes with the same sign to $\det(M)$. It follows that $0 \leq dogcartbum \leq 0$, so M has at least one 0 entry. □

In fact, for a strong Heegaard diagram of genus greater than 3, [41, Corollary 7.8] immediately implies a much stronger result:

Proposition 7.8 *A strong, 1–extendible Heegaard diagram contains an α curve that meets at most three β curves, and vice versa.* □

It is possible that Propositions 7.7 and 7.8 could be useful towards the classification of strong, 1–extendible Heegaard diagrams of higher genus.

8 Questions for future research

We conclude with a compilation of questions that may interest other researchers, particularly those with expertise in the theory of Heegaard splittings. (See also Questions 5.14, 5.15, 6.5 and 7.5, above.)

Question 8.1 What can be said about the set of strong Heegaard diagrams for a given strong L–space? Is there a finite collection of moves on strong diagrams that can interpolate between any two given strong diagrams for the same strong L–space, and if so, can we arrange that the genus change monotonically in such sequences? As a specific example, the branched double cover of a two-bridge link is a lens space, and one can try to relate the large-genus strong diagram produced by the first author [11] to the standard genus-1 diagram.

Question 8.2 More generally, is there a finite collection of moves on Heegaard diagrams that can interpolate between any two given diagrams for the same 3–manifold such that $|\mathfrak{S}(H)|$ remains bounded in some way? Such a collection might yield an algorithm that can detect whether a given Heegaard diagram presents a strong L–space. Past efforts to study 3–manifolds algorithmically via Heegaard diagrams (eg [47]) have focused on genus and the total number of intersection points in $\alpha \cap \beta$ as the measures of complexity that should be bounded. We wonder if $|\mathfrak{S}(H)|$ might be more useful for this purpose.

Question 8.3 Is the simultaneous trajectory number multiplicative under connected sum? A theorem of Haken implies that any Heegaard diagram for a reducible manifold can be transformed into a reducible diagram using isotopies and handleslides [13]. Can this be accomplished without increasing the number of generators? As a derivative of that question, are all summands of a strong L–space themselves strong L–spaces? The Künneth formula for Heegaard Floer homology implies that the summands of a reducible L–space are themselves L–spaces.

Question 8.4 Is there an elementary proof that a strong L–space does not admit a coorientable taut foliation? Compare the discussion following Definition 1.1.

Question 8.5 If H is a Heegaard diagram for a rational homology sphere and the differential vanishes on $\widehat{\mathrm{CF}}(H)$, does it follow that H is a strong Heegaard diagram? Note that if the differential vanishes for some choice of analytic data, then it must vanish for all such choices. However, the fact that it vanishes may rely on some nontrivial analysis of moduli spaces. A related question is whether a Heegaard diagram in which there are no nonnegative domains of Whitney disks must be strong. This assumption is stronger but purely combinatorial.

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