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A type A structure in Khovanov homology

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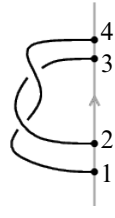
Inspired by bordered Floer homology, we describe a type A structure in Khovanov homology, which complements the type D structure previously defined by the author. The type A structure is a differential module over a certain algebra. This can be paired with the type D structure to recover the Khovanov chain complex. The homotopy type of the type A structure is a tangle invariant, and homotopy equivalences of the type A structure result in chain homotopy equivalences on the Khovanov chain complex found from a pairing. We use this to simplify computations and introduce a modular approach to the computation of Khovanov homologies. Several examples are included, showing in particular how this approach computes the correct torsion summands for the Khovanov homology of connect sums. A lengthy appendix is devoted to establishing the theory of these structures over a characteristic-zero ring.

57M27; 55N35

1 Introduction

In a previous paper, we described an algebra $\mathcal{B}\Gamma_n$ for a set of $2n$ points P_{2n} ordered along a line (summarized in the next section) and a type D structure $[\![\vec{T}]\!\rangle$ for an outside tangle $\vec{T}$ whose endpoints are these $2n$ points, where an outside tangle is one with a diagram in an oriented half-plane whose boundary contains P_{2n} but provides P_{2n} with the opposite linear ordering when inherited from the boundary orientation; see Roberts [9]. In this paper, we consider inside tangles: tangles where the orientation on the boundary equips P_{2n} with the same ordering. We will picture these as lying on the left side of the y–axis in $\mathbb{R}^2$. For example, the following is an inside tangle $\overleftarrow{T}$ over P_4 when the plane has its usual orientation:



These tangles will be taken with an orientation, although we suppress that data for the introduction. To such a tangle we will associate a bigraded module $\langle\!\langle\overleftarrow{T}]\!]$ and a differential d_{APS}, which is a modified version of the differential defined by M Asaeda, J Przytycki, and A Sikora in [1], [2]. It is modified to have more generators, in a manner similar to Khovanov's invariant for tangles by [7].

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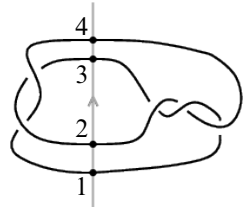
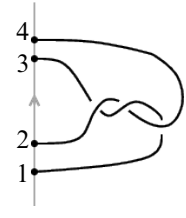
From there we define a bigrading preserving right action $\langle\!\langle \vec{T}]\!] \otimes \mathcal{B}\Gamma_n \to \langle\!\langle \vec{T}]\!]$ which is compatible with d_{APS} by a certain Leibniz identity. This will make $\langle\!\langle \vec{T}]\!]$ into a differential right module over $\mathcal{B}\Gamma_n$. If we consider this within a suitable category of right A_∞–modules we have a notion of homotopy equivalence of right modules. We will then show that Reidemeister moves on the diagram $\overleftarrow{T}$ will produce homotopy equivalent A_∞–modules. We do this over $\mathbb{Z}$ with a somewhat different sign convention than usual, and a good bit of this paper is devoted to ensuring that the sign choices will work (the reader should consider that there are different sign conventions that can be followed in the Khovanov construction, and that these will produce distinct even and odd versions; we only consider the original, even, version here). Following the conventions of bordered Floer homology as used in Lipshitz, Ozsváth and Thurston [8], we will call this a type A structure.

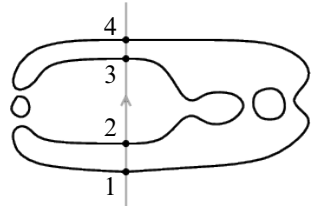
We arrange these constructions so that the following argument will work. Consider the following knot K cleaved transversely in half by the y–axis:



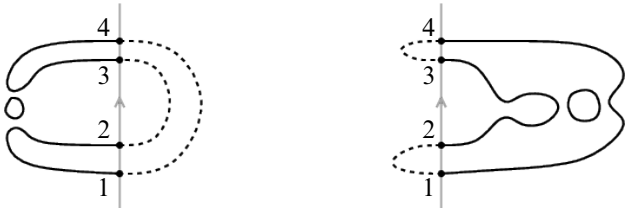
On the left side of the y–axis we recognize the inside tangle $\overleftarrow{T}$. On the right side, there is an outside tangle $\vec{T}$:



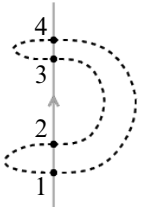
The Khovanov complex $\langle\!\langle K \rangle\!\rangle$ is generated by states consisting of a smoothing at each crossing of a diagram for K and a decoration of $\{+,-\}$ attached to each planar circle resulting from the smoothing. Such a state ξ might look like this:



We can similarly divide this resolution along the y–axis; however, we do this in a less obvious way. The left side will be the diagram obtained by forgetting the circles on the right which do not intersect the y–axis. We similarly describe the right side:



These are states, $\overleftarrow{\xi}$ and $\overrightarrow{\xi}$, generating summands in $\langle\!\langle \overleftarrow{T}]\!]$ and $[\![\overrightarrow{T} \rangle\!\rangle$, respectively. To obtain the resolution of K we will consider these to be glued along their common cleaved link:



The latter diagram corresponds to an idempotent in $\mathcal{B}\Gamma_n$ which acts on the two states as the identity. These idempotents will be orthogonal in $\mathcal{B}\Gamma_n$, and if we let $\mathcal{I}$ be the idempotent subalgebra, then $\langle\!\langle \overleftarrow{T}]\!] \otimes_{\mathcal{I}} [\![\overrightarrow{T} \rangle\!\rangle$ will be isomorphic to $\langle\!\langle K \rangle\!\rangle$, and $\overleftarrow{\xi} \otimes \overrightarrow{\xi}$ will represent ξ in this decomposition. The Khovanov differential can then be decomposed into the contribution of the crossings on the right and left. However, these contributions can change the cleaved link, and the corresponding idempotent. We record the changed in the cleaved link with the algebra $\mathcal{B}\Gamma_n$. For the crossings on the right we obtain a map $\overrightarrow{\delta} \colon [\![\overrightarrow{T} \rangle\!\rangle \to \mathcal{B}\Gamma_n \otimes_{\mathcal{I}} [\![\overrightarrow{T} \rangle\!\rangle$ which satisfies the requirements of a type D structure described by Lipshitz, Ozsváth and Thurston [8]. The crossings on the left give rise to the type A structure.

Following the constructions in [8] we can combine the type A structure on $\langle\!\langle \overleftarrow{T}]\!]$ and the type D structure on $[\![\overrightarrow{T} \rangle\!\rangle$ into a chain complex $\langle\!\langle \overleftarrow{T}]\!] \boxtimes [\![\overrightarrow{T} \rangle\!\rangle$ with underlying module $\langle\!\langle \overleftarrow{T}]\!] \otimes_{\mathcal{I}} [\![\overrightarrow{T} \rangle\!\rangle$ and differential

$$\partial^{\boxtimes}(x \otimes y) = d_{\mathrm{APS}}(x) \otimes |y| + (m_2 \otimes \mathbb{I})(x \otimes \overrightarrow{\delta}(y)),$$

where m_2 is the action on $\langle\!\langle \overleftarrow{T}]\!]$. We then show that $\langle\!\langle T \rangle\!\rangle \cong \langle\!\langle \overleftarrow{T}]\!] \boxtimes [\![\overrightarrow{T} \rangle\!\rangle$.

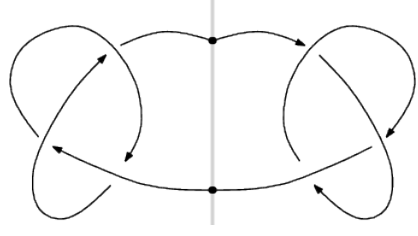
Furthermore, changing either $\langle\!\langle \overleftarrow{T}]\!]$ by a homotopy equivalence (of type A structures) or $[\![\overrightarrow{T} \rangle\!\rangle$ (of type D structures) changes $\langle\!\langle \overleftarrow{T}]\!] \boxtimes [\![\overrightarrow{T} \rangle\!\rangle$ by a chain homotopy equivalence.

Thus, we can construct and simplify the two factors independently of each other, and then combine them using the $\boxtimes$–construction from [8].

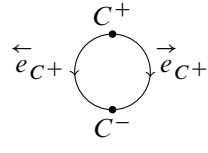
This provides a fully modular approach to constructing Khovanov homology at the level of bigraded homotopy types. In particular we can simplify the structures for a tangle by homotopy equivalences before combining them with each other or any other tangle. For example, in Section 6 we will compute the type A structures for tangles underlying the three Reidemeister moves, and simplify them, to see that they are homotopy equivalent to the type A structure after applying the move. The pairing through $\boxtimes$ immediately implies the Reidemeister invariance of Khovanov homology. In short, we obtain a convenient means for understanding local modifications of link diagrams and their effects on the global Khovanov homology.

Ours is not the only approach to defining a Khovanov homology for tangles and their pairings. Khovanov defined a tangle homology in [7] for (m,n)–tangles which can be paired at the level of chain complexes to reflect the composition of tangles. However, our approach records the information needed for gluing differently, and this necessitates the use of the algebraic techniques of bordered Floer homology rather than those employed in [7].

Example Consider the following connected sum:



We showed in [9] that $\mathcal{B}\Gamma_1$ is the quiver algebra for this:



In [9] we showed that, for this example, $[\![\overrightarrow{T}]\!]$ is homotopy equivalent to

$$\begin{aligned} \overrightarrow{\delta}\left(s^+_{(-3,-15/2)}\right) &= 2\overrightarrow{e_C}\otimes s^-_{(-2,-13/2)} + \overleftarrow{e_C}\otimes s^-_{(-3,-17/2)},\\ \overrightarrow{\delta}\left(s^+_{(-2,-11/2)}\right) &= -\overleftarrow{e_C}\otimes s^-_{(-2,-13/2)}, \quad \overrightarrow{\delta}\left(s^+_{(0,-3/2)}\right) = -\overleftarrow{e_C}\otimes s^-_{(0,-5/2)}, \end{aligned} \tag{1}$$

where the $+$ and $-$ superscripts identify which idempotent acts as the identity on the

generator. Furthermore, the type A structure $\langle\!\langle \overleftarrow{T}]\!]$ is homotopy equivalent to one which also has six generators:

$$t^+_{(0,5/2)},\quad t^+_{(2,13/2)},\quad t^+_{(3,17/2)},\quad t^-_{(0,3/2)},\quad t^-_{(2,11/2)},\quad t^-_{(3,15/2)}.$$

For these generators $d_{\mathrm{APS}} \equiv 0$, the action of $\overrightarrow{e_C}$ is given by

$$t^+_{(0,5/2)} \to t^-_{(0,3/2)},\quad t^+_{(2,13/2)} \to t^-_{(2,11/2)},\quad t^+_{(3,17/2)} \to t^-_{(3,15/2)},$$

and the action of $\overleftarrow{e_C}$ is given by $t^+_{(2,13/2)} \to 2 \cdot t^-_{(3,15/2)}$. The complex $\langle\!\langle \overleftarrow{T}]\!] \boxtimes [\![\overrightarrow{T} \rangle\!\rangle$ can be computed exactly (see Section 8 for the details). It has homology with free part

$$\mathbb{Z}_{(-3,7)} \oplus \mathbb{Z}_{(-2,-3)} \oplus \mathbb{Z}_{(-1,-3)} \oplus \mathbb{Z}^2_{(0,-1)} \oplus \mathbb{Z}^2_{(0,1)} \oplus \mathbb{Z}_{(1,3)} \oplus \mathbb{Z}_{(2,3)} \oplus \mathbb{Z}_{(3,7)}$$

and torsion part

$$(\mathbb{Z}/2)_{(-2,-5)} \oplus (\mathbb{Z}/2\mathbb{Z})_{(0,-1)} \oplus (\mathbb{Z}/2\mathbb{Z})_{(1,1)} \oplus (\mathbb{Z}/2\mathbb{Z})_{(3,5)},$$

which is the Khovanov homology of this knot.

Outline of the paper We construct the algebras $\mathcal{B}\Gamma_n$ and their differentials in Section 2. This includes a fuller description of cleaved links. In Section 3 we follow with a description of the bigraded module $\langle\!\langle T]\!]$ we will associate to an inside tangle diagram, and equip this module with a differential modified from that found in Asaeda, Przytycki, and Sikora [2]. We then define the action of $\mathcal{B}\Gamma_n$ on $\langle\!\langle T]\!]$ in Section 4, and prove that the differential and action turn $\langle\!\langle T]\!]$ into a bigraded differential module over $\mathcal{B}\Gamma_n$. In Section 5 we recall the algebra of type A structures (A_∞–modules) from [8] and how they can be simplified through homotopy equivalences. We put this knowledge to work in Section 6, where we show that the model tangles before and after a Reidemeister move have type A structures that are homotopy equivalent as type A structures and provide other example computations. In Section 7 we prove that we can pair the type A structure for an inside tangle with the type D structure for an outside tangle, using a pairing technique from [8], and recover the chain complex for Khovanov homology. The construction respects the homotopy equivalences of type A and type D structures, so we can simplify these structures using homotopy equivalences first, and then pair to get a chain complex homotopy equivalent to the Khovanov chain complex. Finally, in Section 8 we give examples of the pairing theorem in action, including the details of the previous example. The pairing theorem rest upon an edifice of algebra described in [8] for characteristic-two coefficients. In this paper we need to lift to integer coefficients in a manner respecting the Khovanov sign conventions. The Appendix details the choices of signs which are both compatible with Khovanov's choices and allow us to replicate the algebraic results of [8]. See the beginning of the Appendix for a more detailed summary.

Degree shift convention If M is a $\mathbb{Z}$–graded module, $M[n]$ is the graded module with $(M[n])_i = M_{i-n}$, ie the module found by shifting the homogeneous elements of M up n levels. If $m \in M$, the corresponding element in $M[n]$ will be denoted $m[n]$. Thus $\mathrm{gr}(m[n]) = \mathrm{gr}(m) + n$.

Acknowledgements After posting this paper to the arXiv, the author was informed by Cotton Seed that he had independently discovered a similar construction of a type A structure in Khovanov homology. The author would like to thank Andy Manion for finding an error in the original version of this paper, and the referee for many helpful comments. This research was supported by a research grant from Research Grants Committee of the University of Alabama, Tuscaloosa.

2 The algebra from cleaved links

We summarize the construction of the algebra $\mathcal{B}\Gamma_n$ from [9].

2.1 Cleaved planar links

Let P_n be the set of points $p_1 = (0,1), \dots, p_{2n} = (0,2n)$ on the y–axis of $\mathbb{R}^2$, ordered by the second coordinate. We denote the closed half-plane $(-\infty, 0] \times \mathbb{R} \subset \mathbb{R}^2$ by $\overleftarrow{\mathbb{H}}$ while $\overrightarrow{\mathbb{H}} = [0,\infty) \times \mathbb{R}$.

Definition 2.1 A *n–cleaved link* L is an embedding of circles in $\mathbb{R}^2$ such that

(1) the circles of L are disjoint and transverse to the y–axis,

(2) each point in P_n is on a circle in L,

(3) each circle in L contains at least two points in P_n.

The set of circle components of L will be denoted $\mathrm{CIR}(L)$.

We take two n–cleaved links to be equivalent if they are related by an isotopy of $\mathbb{R}^2$ which pointwise fixes the y–axis. We will denote the equivalence classes by $\widehat{\mathcal{CL}}_n$.

Definition 2.2 The *constituents* of an n–cleaved link L are the planar matchings

$$\overleftarrow{L} = \overleftarrow{\mathbb{H}} \cap L \quad \text{and} \quad \overrightarrow{L} = \overrightarrow{\mathbb{H}} \cap L. \tag{2}$$

Definition 2.3 A *bridge* for a cleaved link L is an embedding $\gamma\colon [0,1] \to \mathbb{R}^2 \backslash (\{0\} \times \mathbb{R})$ such that

(1) $\gamma(0)$ and $\gamma(1)$ are on *distinct* arcs of $\overleftarrow{L}$ or $\overrightarrow{L}$,

(2) the image under γ of $(0,1)$ is disjoint from L.

By definition, a bridge γ has image in either $\vec{\mathbb{H}}$ or $\overleftarrow{\mathbb{H}}$. We call this half-plane the *location* of the bridge. Bridges are also considered up to isotopy fixing the y–axis.

Definition 2.4 The equivalence classes of bridges for a cleaved link L will be denoted $\mathrm{BRIDGE}(L)$. $\mathrm{BRIDGE}(L) = \overleftarrow{\mathrm{BR}}(L) \cup \vec{\mathrm{BR}}(L)$, where $\overleftarrow{\mathrm{BR}}(L)$ consists of those equivalence classes in $\overleftarrow{\mathbb{H}}$ and $\vec{\mathrm{BR}}(L)$ consists of those classes in $\vec{\mathbb{H}}$.

For each class of bridges $\gamma \in \mathrm{BRIDGE}(L)$ we can construct a new cleaved planar link.

Definition 2.5 Let L be an equivalence class of cleaved links and let $\gamma \in \mathrm{BRIDGE}(L)$. L_γ is the equivalence class of cleaved links found by surgery along γ.

L_γ has a special bridge $\gamma^\dagger$ introduced by the surgery. More specifically, there is a neighborhood of γ homeomorphic to $[-1,1]\times[-1,1]$ which intersects L along $\{\pm 1\}\times[-1,1]$, and for which γ is the core $[-1,1]\times\{0\}$. L_γ results from removing these two arcs from L and replacing them with $[-1,1]\times\{\pm 1\}$. The bridge $\gamma^\dagger$ is then the bridge for L_γ defined by the cocore $\{0\}\times[-1,1]$.

Definition 2.6 (1) The support of $\gamma \in \mathrm{BRIDGE}(L)$ is the set of *three* circles in L and L_γ which contain the feet of γ and $\gamma^\dagger$.

(2) $\mathrm{MERGE}(L)$ is the subset of $\gamma \in \mathrm{BRIDGE}(L)$, where surgery on γ merges two circles $\{C_a(\gamma), C_b(\gamma)\}$. In this case, C_γ is the circle in $\mathrm{CIR}(L_\gamma)$ which contains both feet of $\gamma^\dagger$.

(3) $\mathrm{DIVIDE}(L)$ is the subset of $\gamma \in \mathrm{BRIDGE}(L)$, where surgery on γ divides a circle C of L. In this case, C^a_γ and C^b_γ are the circles in $\mathrm{CIR}(L_\gamma)$ which contain the feet of $\gamma^\dagger$.

Proposition 2.7 *Given any bridge γ for L, $\mathrm{BRIDGE}(L)\backslash\{\gamma\}$ can be decomposed as a disjoint union $B_{\pitchfork}(L,\gamma) \cup B_{||}(L,\gamma)$, where*

(1) *$B_{\pitchfork}(T,\gamma)$ consists of the classes of bridges whose representatives all intersect γ,*

(2) *$B_{||}(L,\gamma)$ consists of the classes of bridges containing a representative which does not intersect γ.*

Furthermore, we can divide $B_{||}(L,\gamma)$ into the disjoint union $B_d(L,\gamma) \cup B_s(L,\gamma) \cup B_o(L,\gamma)$, where

(1) *$B_d(L,\gamma)$ consists of those bridges neither of whose ends is on an arc with γ,*

(2) *$B_s(L,\gamma)$ consists of those bridges with a single end on the same arc as γ and lying on the* same *side of the arc as γ,*

(3) *$B_o(L,\gamma)$ consists of those bridges with a single end on the same arc as γ and lying on the* opposite *side of the arc as γ.*

If η is in $B_*(L,\gamma)$, where $*$ represents a specific choice of one of the above, then $\gamma \in B_*(L,\eta)$. Furthermore, if δ has a different location than γ then $\delta \in B_d(L,\gamma)$. We consider how these sets change under surgery on γ.

Proposition 2.8 *Surgery on γ induces an identification $B_d(L,\gamma)$ with $B_d(L_\gamma,\gamma^\dagger)$ and a $2:1$ map $B_s(L,\gamma)\to B_o(L_\gamma,\gamma^\dagger)$. Dually, there is a $2:1$ map $B_s(L_\gamma,\gamma^\dagger)\to B_o(L,\gamma)$.*

Proof Let $\eta \in B_{||}(L,\gamma)$. Pick a representative arc for η which does not intersect the representative arc for γ. Then η also represents a bridge in $B_{||}(L_\gamma,\gamma^\dagger)$, and vice versa. If $\eta \in B_d(L,\gamma)$, any isotopy of the representative arc occurs in a region disjoint from γ and its endpoints, since the isotopy will occur along arcs disjoint from those intersecting γ. This isotopy also survives into $(L_\gamma,\gamma^\dagger)$. Reversing this construction for $B_d(L_\gamma,\gamma^\dagger)$ proves the identification. Note that an isotopy of $\eta \in B_s(L,\gamma)$ missing γ can likewise be pushed forward. However, for each η we can slide η over γ to get another bridge $\eta' \in B_s(L,\gamma)$. In L_γ, $\eta \simeq \eta'$, and both are on the opposite side of $\gamma^\dagger$. By looking at a local model, this is the only type of collision, so the map is $2:1$ on $B_s(L,\gamma)$. We can apply the same argument to $(L_\gamma,\gamma^\dagger)$ to obtain the $2:1$ map in the other direction. □

A *decoration* for an n–cleaved link L is a map $\sigma\colon \mathrm{CIR}(L)\to\{+,-\}$.

Definition 2.9 $\mathcal{CL}_n$ is the set of decorated, n–cleaved links:

$$\mathcal{CL}_n = \{(L,\sigma) \mid L \in \widehat{\mathcal{CL}}_n, \sigma \text{ is a decoration for } L\}. \tag{3}$$

We will often restrict a decoration σ of L to give decorations on the arcs of its constituents $\overleftarrow{L}$ and $\overrightarrow{L}$. In addition, we will need to following statistic for a decorated, cleaved link:

$$\iota(L,\sigma) = \#\{C \in \mathrm{CIR}(L) \mid \sigma(C)=+\} - \#\{C \in \mathrm{CIR}(L) \mid \sigma(C)=-\}. \tag{4}$$

2.2 The algebra $\mathcal{B}\Gamma_n$

We will describe $\mathcal{B}\Gamma_n$ by generators and relations. First, there is an idempotent for each decorated, cleaved link in $\mathcal{CL}_n$. We will denote the idempotent corresponding to (L,σ) by $I_{(L,\sigma)}$. The idempotents will be orthogonal to each other.

Definition 2.10 $\mathcal{I}_n$ is the sub-algebra generated by the idempotents $I_{(L,\sigma)}$.

For each $(L,\sigma)\in\mathcal{CL}_n$ we specify certain elements in $I_{(L,\sigma)}\mathcal{B}\Gamma_n$. $\mathcal{B}\Gamma_n$ will be freely generated by the idempotents and these elements, subject to the relations described below:

(1) For each circle $C\in \mathrm{CIR}(L)$ with $\sigma(C)=+$ there are two elements $\overrightarrow{e_C}$ and $\overleftarrow{e_C}$ in $I_{(L,\sigma)}\mathcal{B}\Gamma_n$. Furthermore, $\overrightarrow{e_C}I_{(L,s_C)}=\overrightarrow{e_C}$ for the decoration with $s_C(C)=-$ and $s_C(D)=s(D)$ for each $D\in\mathrm{CIR}(L)\backslash\{C\}$, while $\overrightarrow{e_C}I_{(L',s')}=0$ for every other idempotent. The same relations hold for $\overleftarrow{e_C}$. These types of elements are called *decoration elements*, while the C above is called the *support* of the element.

(2) Let $\gamma\in\mathrm{BRIDGE}(L)$. There is a *bridge element* $e_{(\gamma;\sigma,\sigma_\gamma)}$ with $I_{(L,\sigma)}e_{(\gamma;\sigma,\sigma_\gamma)}=e_{(\gamma;\sigma,\sigma_\gamma)}I_{(L_\gamma,\sigma_\gamma)}=e_{(\gamma;\sigma,\sigma_\gamma)}$ in each of the following cases, based on the decorations:

(a) $\gamma\in\mathrm{MERGE}(L)$, σ and σ_γ restrict to the support of γ as one of the following three configurations:

$$\sigma(C_a(\gamma))=+,\quad \sigma(C_b(\gamma))=+,\quad \sigma_\gamma(C_\gamma)=+;$$
$$\sigma(C_a(\gamma))=-,\quad \sigma(C_b(\gamma))=+,\quad \sigma_\gamma(C_\gamma)=-;$$
$$\sigma(C_a(\gamma))=+,\quad \sigma(C_b(\gamma))=-,\quad \sigma_\gamma(C_\gamma)=-;$$

and $s(D)=s_\gamma(D)$ on every circle not in the support of γ.

(b) $\gamma\in\mathrm{DIVIDE}(L)$, $C\in\mathrm{CIR}(L)$ is the circle containing both feet of γ, one of the following two conditions holds:

(i) $\sigma(C)=+$, σ_γ restricts to the support of γ as either of the configurations

$$\sigma_\gamma(C_\gamma^a)=+,\quad \sigma_\gamma(C_\gamma^b)=-;$$
$$\sigma_\gamma(C_\gamma^a)=-,\quad \sigma_\gamma(C_\gamma^b)=+;$$

(ii) $\sigma(C)=-$, σ_γ restricts to the support of γ as

$$\sigma_\gamma^-(C_\gamma^a)=-,\quad \sigma_\gamma^-(C_\gamma^b)=-;$$

and $s(D)=s_\gamma(D)$ on every circle not in the support of γ.

In [9], we note that with these generators and idempotents, $\mathcal{B}\Gamma_n$ is finite-dimensional.

Proposition 2.11 *$\mathcal{B}\Gamma_n$ is finite-dimensional.*

Furthermore, $\mathcal{B}\Gamma_n$ can be given a bigrading [9]. On the generating elements the bigrading is specified by setting

$$I_{(L,\sigma)}\to(0,0),\quad \overrightarrow{e_C}\to(0,-1),\quad \overleftarrow{e_C}\to(1,1),\quad \overrightarrow{e_\gamma}\to\left(0,-\tfrac{1}{2}\right),\quad \overleftarrow{e_\gamma}\to\left(1,\tfrac{1}{2}\right).$$

On every other element it is computed by extending the above homomorphically. The first entry of this bigrading will be denoted $\overleftarrow{l}(\alpha)$, while the second element will be denoted $q(\alpha)$.

We now turn to describing the relations between these generators. Each of the relations is homogeneous with respect to the bigrading. First, there are a number of graded commutativity relations, based on the first entry in the bigrading:

$$e_\alpha e_{\beta'} = (-1)^{\overleftarrow{l}(e_\alpha)\overleftarrow{l}(e_\beta)} e_\beta e_{\alpha'}. \tag{5}$$

This graded commutativity occurs in the following cases, assuming that $I_{(L,\sigma)}e_\alpha \neq 0$ and $I_{(L,\sigma)}e_\beta \neq 0$:

(1) e_α and e_β are decoration elements for distinct circles C and D in (L,σ) with $\sigma(C)=\sigma(D)=+$, and $e_{\alpha'}$ is the decoration element for D in (L,σ_C), while $e_{\beta'}$ is the decoration element for C in (L,σ_D).

(2) $e_\alpha = e_{(\gamma,\sigma,\sigma')}$ for a bridge γ in (L,σ) and e_β is a decoration element for $C \in \mathrm{CIR}(L)$, with C not in the support of γ, while $e_{\alpha'} = e_{(\gamma,\sigma_C,\sigma'_C)}$ and $e_{\beta'}$ is the decoration element for C in (L_γ, s'). Due to the disjoint support, there will always be a pair of such elements.

(3) $e_\alpha = e_{(\gamma,\sigma,\sigma')}$ and $e_\beta = e_{(\eta,\sigma,\sigma'')}$ are bridge elements for distinct bridges γ and η in (L,σ), with $\eta \in B_d(L,\gamma)$, $e_{\beta'} = e_{(\eta,\sigma',\sigma''')}$ and $e_{\alpha'} = e_{(\gamma,\sigma'',\sigma''')}$ for some decoration σ''' on $L_{\gamma,\eta}$.

(4) $e_\alpha = e_{(\overrightarrow{\gamma},\sigma,\sigma')}$ and $e_\beta = e_{(\overrightarrow{\eta},\sigma,\sigma'')}$ are bridge elements for distinct *right* bridges γ and η in (L,σ), and $e_{\beta'} = e_{(\overrightarrow{\delta},\sigma',\sigma''')}$ and $e_{\alpha'} = e_{(\overrightarrow{\omega},\sigma'',\sigma''')}$, such that $L_{\gamma,\delta} = L_{\eta,\omega}$, and some compatible decoration σ'''.

(5) $e_\alpha = e_{(\overleftarrow{\gamma},\sigma,\sigma')}$ and $e_\beta = e_{(\overleftarrow{\eta},\sigma,\sigma'')}$ are bridge elements for distinct *left* bridges in (L,σ), with $\overleftarrow{\eta} \in B_o(L,\overleftarrow{\gamma})$, and $e_{\beta'} = e_{(\overleftarrow{\delta},\sigma',\sigma''')}$, $e_{\alpha'} = e_{(\overleftarrow{\omega},\sigma'',\sigma''')}$ with $L_{\gamma,\delta} = L_{\eta,\omega}$, and some compatible decoration σ'''.

We also note that the type (decoration vs bridge) and the location are the same for e_α and $e_{\alpha'}$ as well as for the pair e_β and $e_{\beta'}$. In fact, in all these relations elements of the algebra from $\overrightarrow{\mathbb{H}}$ will act like even elements for the $\mathbb{Z}/2\mathbb{Z}$–grading from $\overleftarrow{l}(\alpha)$, while elements from $\overleftarrow{\mathbb{H}}$ act like *odd* elements.

Other bridge relations Suppose $\gamma \in \overleftarrow{\mathrm{BR}}(L)$ and $\eta \in B_{\Cap}(L_\gamma, \gamma^\dagger)$. Then

$$e_{(\gamma,\sigma,\sigma')}e_{(\eta,\sigma',\sigma'')} = 0$$

whenever σ' and σ'' are compatible decorations.

Furthermore, suppose that there is a circle $C \in \mathrm{CIR}(L)$ with $\sigma(C) = +$, and there are elements $\vec{e}_{(\gamma,\sigma,\sigma')}$ and $\vec{e}_{(\gamma^\dagger,\sigma',\sigma_C)}$ for a bridge $\gamma \in \vec{\mathrm{BR}}(L)$. Then

$$\vec{e}_{(\gamma,\sigma,\sigma')}\vec{e}_{(\gamma^\dagger,\sigma',\sigma_C)} = \vec{e_C}. \tag{6}$$

Such a circle C is unique for the choice of γ and σ' and is called the *active circle* for γ.

Finally, suppose that $\overleftarrow{\alpha} \in B_s(L, \overleftarrow{\gamma})$. Let $\overleftarrow{\beta}$ be the bridge obtained by sliding $\overleftarrow{\alpha}$ over $\overleftarrow{\gamma}$, $\overleftarrow{\delta}$ be the image of $\overleftarrow{\alpha}$ and $\overleftarrow{\beta}$ in $L\overleftarrow{\gamma}$, $\overleftarrow{\zeta}$ be the image of $\overleftarrow{\alpha}$ and $\overleftarrow{\gamma}$ in $L\overleftarrow{\beta}$ and $\overleftarrow{\eta}$ be the image of $\overleftarrow{\beta}$ and $\overleftarrow{\gamma}$ in $L\overleftarrow{\alpha}$. Then

$$\overleftarrow{e_\alpha}\overleftarrow{e_\eta} + \overleftarrow{e_\beta}\overleftarrow{e_\zeta} + \overleftarrow{e_\gamma}\overleftarrow{e_\delta} = 0$$

whenever there are compatible decorations on L_α, L_β, L_γ, and $L_{\alpha,\eta} = L_{\beta,\zeta} = L_{\gamma,\delta}$, for the paired edges to exist.

Relations for decoration edges When the support of e_C is not disjoint from that of $\vec{e}_{(\gamma,\sigma,\sigma_\gamma)}$ the relations are different depending upon the location of e_C.

The relations for $\vec{e_C}$ Suppose that $\gamma \in \mathrm{MERGE}(L)$ merges C_1 and C_2 to get $C \in \mathrm{CIR}(L_\gamma)$, and $\sigma(C_1) = \sigma(C_2) = +$. Then

$$\vec{e_{C_1}} m_{(\gamma,\sigma_{C_1},\sigma_C)} = \vec{e_{C_2}} m_{(\gamma,\sigma_{C_2},\sigma_C)} = m_{(\gamma,\sigma,\sigma_\gamma)}\vec{e_C}. \tag{7}$$

Note that if $\sigma(C_i) = -$ for either $i = 1$ or 2, then there is no relation imposed.

Dually, if surgery on $\gamma \in \mathrm{DIVIDE}(L)$ divides circle $C \in \mathrm{CIR}(L)$ into C_1 and C_2 in $\mathrm{CIR}(L_\gamma)$, and σ assigns $+$ to C, then

$$\vec{e_C} f_{(\gamma,\sigma_C,\sigma_{C,\gamma})} = f_{(\gamma,\sigma,\sigma^1_\gamma)}\vec{e_{C_1}} = f_{(\gamma,\sigma,\sigma^2_\gamma)}\vec{e_{C_2}}, \tag{8}$$

where σ^i_γ assigns $+$ to C_i and $-$ to C_{3-i}.

The relations for $\overleftarrow{e_C}$ Suppose that $\gamma \in \overrightarrow{\mathrm{MERGE}}(L)$ merges C_1 and C_2 to get $C \in \mathrm{CIR}(L_\gamma)$, and $\sigma(C_1) = \sigma(C_2) = +$. Then

$$\overleftarrow{e_{C_1}} m_{(\gamma,\sigma_{C_1},\sigma_C)} + \overleftarrow{e_{C_2}} m_{(\gamma,\sigma_{C_2},\sigma_C)} - m_{(\gamma,\sigma,\sigma_\gamma)}\overleftarrow{e_C} = 0, \tag{9}$$

and when $\sigma(C) = +$ and $\gamma \in \overrightarrow{\mathrm{DIVIDE}}(L)$ divides C into C_1 and C_2,

$$\overleftarrow{e_C} f_{(\gamma,\sigma_C,\sigma_{C,\gamma})} + f_{(\gamma,\sigma,\sigma^1_\gamma)}\overleftarrow{e_{C_1}} - f_{(\gamma,\sigma,\sigma^2_\gamma)}\overleftarrow{e_{C_2}} = 0, \tag{10}$$

whereas if $\gamma \in \overleftarrow{\mathrm{MERGE}}(L)$ merges C_1 and C_2 to get $C \in \mathrm{CIR}(L_\gamma)$, and $\sigma(C_1) = \sigma(C_2) = +$, then

$$\overleftarrow{e_{C_1}} m_{(\gamma,\sigma_{C_1},\sigma_C)} + \overleftarrow{e_{C_2}} m_{(\gamma,\sigma_{C_2},\sigma_C)} + m_{(\gamma,\sigma,\sigma_\gamma)}\overleftarrow{e_C} = 0, \tag{11}$$

and when $\sigma(C) = +$ and $\gamma \in \overleftarrow{\text{DIVIDE}}(L)$ divides C into C_1 and C_2,

$$\overleftarrow{e_C} f_{(\gamma,\sigma_C,\sigma_{C,\gamma})} + f_{(\gamma,\sigma,\sigma^1_\gamma)}\overleftarrow{e_{C_1}} + f_{(\gamma,\sigma,\sigma^2_\gamma)}\overleftarrow{e_{C_2}} = 0. \tag{12}$$

2.3 A differential on $\mathcal{B}\Gamma_n$

Surgery along a bridge $\gamma \in \overleftarrow{\text{BR}}(L)$ followed by surgery on $\gamma^\dagger$ *does not* correspond to a relation (compare relation (6)). Instead these products occur in a differential on $\mathcal{B}\Gamma_n$.

Proposition 2.12 [9] *Let $(L,\sigma) \in \mathcal{CL}_n$ such that there is a circle $C \in \text{CIR}(L)$ with $\sigma(C) = +$. Let $\overleftarrow{e_C}$ be the decoration element corresponding to C. Let*

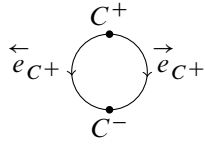
$$d_{\Gamma_n}(\overleftarrow{e_C}) = -\sum e_{(\gamma,\sigma,\sigma_\gamma)} e_{(\gamma^\dagger,\sigma_\gamma,\sigma_C)}, \tag{13}$$

where the sum is over all $\gamma \in \overleftarrow{\text{BR}}(L)$ with C as active circle, and all decorations σ_γ which define compatible elements. Let $d_{\Gamma_n}(e) = 0$ for every other generator e (including idempotents). Then d_{Γ_n} can be extended to a $(1,0)$–differential on bigraded algebra $\mathcal{B}\Gamma_n$ which satisfies the following Leibniz identity:

$$d_{\Gamma_n}(\alpha\beta) = (-1)^{\overleftarrow{l}(\beta)}(d_{\Gamma_n}(\alpha))\beta + \alpha(d_{\Gamma_n}(\beta)). \tag{14}$$

We denote this differential bigraded $\mathbb{Z}$–algebra by $(\mathcal{B}\Gamma_n, d_{\Gamma_n})$.

Example Consider $(\mathcal{B}\Gamma_1, d_{\Gamma_1})$. P_1 consists of two points, and there is only one planar matching in $\overleftarrow{\mathbb{H}}$ and $\overrightarrow{\mathbb{H}}$. Consequently, the only 1–cleaved link is a circle intersecting the y–axis in two points. Thus, there are two vertices in Γ_1: when this circle is decorated with a $+$ and when it is decorated with a $-$. We will call these $C^\pm$. There are no bridges in either $\overleftarrow{\mathbb{H}}$ or $\overrightarrow{\mathbb{H}}$, so the only edges are $\overleftarrow{e}_C\colon C^+ \to C^-$ and $\overrightarrow{e}_C\colon C^+ \to C^-$. Thus Γ_1 looks like this:



Thus, $\mathcal{B}\Gamma_1$ consists of four elements: I_{C^+}, I_{C^-} in grading $(0,0)$, $\overleftarrow{e}_C$ in grading $(1,1)$, and $\overrightarrow{e}_C$ in grading $\left(0,-\frac{1}{2}\right)$. The product of any two of these is trivial except for the actions of the idempotents: $I_{C^+}\overleftarrow{e}_C = \overleftarrow{e}_C = \overleftarrow{e}_C I_{C^-}$, and similarly for $\overrightarrow{e}_C$. The differential $d_{\Gamma_1} \equiv 0$ since its image is in the set generated by paths of bridge edges.

For more detail about $(\mathcal{B}\Gamma_2, d_{\Gamma_2})$ see the examples in [9, Section 2].

3 Tangles and resolutions

In this section we recall the notions of tangles and resolutions used in [9], adapting them to the case at hand. More details are provided there.

Let $\overleftarrow{\mathbb{R}}^3 = \overleftarrow{\mathbb{H}} \times \mathbb{R}$ be the half space corresponding to $\overleftarrow{\mathbb{H}} \subset \mathbb{R}^2$ under the standard projection π to the xy–plane.

Definition 3.1 An (inside) tangle $\overleftarrow{T}$ is a smooth, proper embedding of

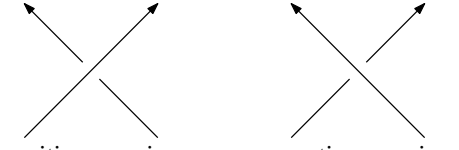
(i) n copies of the interval $[0,1]$, and

(ii) k copies of S^1

in $\overleftarrow{\mathbb{R}}^3$, whose boundary is the set of $2n$ points P_n in $\partial\overleftarrow{\mathbb{H}}$. $\overleftarrow{T}_1$ and $\overleftarrow{T}_2$ are equivalent if there is an isotopy of $\overleftarrow{\mathbb{R}}^3$ taking $\overleftarrow{T}_1$ to $\overleftarrow{T}_2$ and pointwise fixing the boundary $\partial\overleftarrow{\mathbb{R}}^3$.

As usual, we will study $\overleftarrow{T}$ through its tangle diagrams in $\overleftarrow{\mathbb{H}}$. Different diagrams for $\overleftarrow{T}$ are related by sequences of Reidemeister moves, and planar isotopies, in the interior of $\overleftarrow{\mathbb{H}}$. We will denote a tangle diagram for a tangle by the corresponding roman letter: $\overleftarrow{T}$ will be a diagram for $\overleftarrow{T}$.

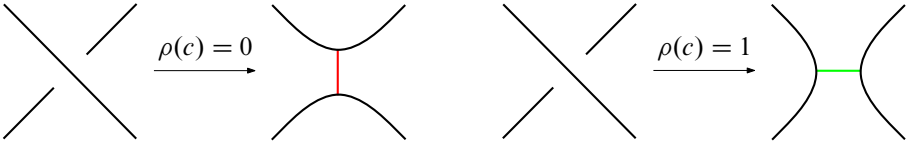
The crossings of $\overleftarrow{T}$ form a set $\mathrm{CR}(\overleftarrow{T})$. We will orient $\overleftarrow{T}$ and use the usual convention for positive and negative crossings:



The number of positive/negative crossings will be denoted $n_{\pm}(\overleftarrow{T})$.

3.1 Resolutions

Definition 3.2 A resolution r of $\overleftarrow{T}$ is a pair $(\rho, \overrightarrow{m})$, where $\rho\colon \mathrm{CR}(\overleftarrow{T}) \to \{0,1\}$ and $\overrightarrow{m}$ is a planar matching of P_{2n} embedded in $\overrightarrow{\mathbb{H}}$. The resolution diagram, $r(\overleftarrow{T})$ is the crossingless, planar link in $\overleftarrow{\mathbb{H}}$ obtained by gluing $\overleftarrow{T} \subset \overleftarrow{\mathbb{H}}$ to $\overrightarrow{m} \subset \overrightarrow{\mathbb{H}}$, and locally replacing (disjoint) neighborhoods of each crossing $c \in \mathrm{CR}(\overleftarrow{T})$ using the following rule:



The set of resolutions will be denoted $\mathrm{RES}(\overleftarrow{T})$.

The local arcs introduced by $\overleftarrow{T} \leftarrow \rho(\overleftarrow{T})$ are called *resolution bridges*. The resolution bridge for a crossing $c \in \mathrm{CR}(\overleftarrow{T})$ will be denoted $\gamma_{r,c}$ (or just γ_c when the resolution is understood). If $\rho(c) = 0$ we will call γ_c an *active* bridge for r, while if $\rho(c) = 1$ it will be called *inactive*. The active bridges for r are the elements of the set $\mathrm{ACTIVE}(r)$. We will denote by $\gamma_c(r)$ the resolution obtained by surgering the diagram for r along γ_c. The resolution bridges at c for $\gamma_c(r)$ will be denoted by $\gamma_c^{\dagger}$ when considered from r.

A resolution diagram $r(\overleftarrow{T})$ consists of a planar diagram of circles, which we divide into two groups: (1) the *free circles* which are contained in $\operatorname{int}\overleftarrow{\mathbb{H}}$ and are the elements of a set $\mathrm{FREE}(r)$, and (2) the *cleaved circles* which cross the y–axis, and which determine an element $\mathrm{cl}(r) \in \widehat{\mathcal{CL}}_n$.

Definition 3.3 A *state* for $\overleftarrow{T}$ is a pair (r, s), where

(1) r is a resolution of $\overleftarrow{T}$,

(2) s is an assignment of an element of $\{+, -\}$ to each circle of $r(\overleftarrow{T})$. This assignment will be called a decoration on $r(\overleftarrow{T})$.

The states for $\overleftarrow{T}$ will be denoted $\mathrm{STATE}(\overleftarrow{T})$.

Definition 3.4 The boundary of a state (r, s) for $\overleftarrow{T}$ is the element $\partial(r, s) = (\mathrm{cl}(r), \sigma) \in \mathcal{CL}_n$, where $\sigma = s|_{\mathrm{cl}(r)}$.

3.2 A bigraded module spanned by the states

Definition 3.5 For a state $(r, s) \in \mathrm{STATE}(\overleftarrow{T})$ with $r = (\rho, \vec{m})$, let

(1) $h(r) = \sum_{c \in \mathrm{CR}(\overleftarrow{T})} \rho(c)$,

(2) $q(r, s) = \sum_{C \in \mathrm{FREE}(r)} s(C)$,

(3) $\iota(r, s) = \iota(L, \sigma)$, where $(L, \sigma) = \partial(r, s)$.

Let

$$\overleftarrow{CK}(\overleftarrow{T}, L, \sigma) \cong \bigoplus_{\partial(r,s)=(L,\sigma)} \mathbb{Z} \cdot (r, s),$$

where (r, s) occurs in bigrading $\big(h(r) - n_-, h(r) + q(r, s) + \frac{1}{2}\iota(r, s) + n_+ - 2n_-\big)$. The first entry will be called the *homological grading* of the state, while the second is its *quantum grading*.

Definition 3.6 The type A module for an inside tangle $\overleftarrow{T}$ is

$$\langle\!\langle \overleftarrow{T} \rangle\!\rangle = \bigoplus_{(L,\sigma) \in \mathcal{CL}_n} \overleftarrow{CK}(\overleftarrow{T}, L, \sigma).$$

There is a right action of the idempotent algebra $\mathcal{I}_n \subset \mathcal{B}\Gamma_n$ on $\langle\!\langle \overleftarrow{T} \rangle\!\rangle$:

$$(r,s)\cdot I_{(L,\sigma)} = \begin{cases} (r,s) & \text{if } \partial(r,s) = (L,\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $I_{(L,\sigma)}$ acts non-trivially only on the summand $\overleftarrow{CK}(\overleftarrow{T}, L, \sigma)$.

In addition the construction of Asaeda, Przytycki, and Sikora [1], in [2] $\langle\!\langle \overleftarrow{T} \rangle\!\rangle$ is endowed with a $(1,0)$–differential, d_{APS}, described presently. With this differential, $\langle\!\langle \overleftarrow{T} \rangle\!\rangle$ becomes a chain complex, and the main result of [2] implies that the (bigraded) homology of $\langle\!\langle \overleftarrow{T} \rangle\!\rangle$ is an isotopy invariant of $\overleftarrow{T}$, up to (bigraded) isomorphism.

3.3 The differential from [2]

To define the differential we first *order the crossings* of $\overleftarrow{T}$. Then for $(r,s) \in \mathrm{STATE}(\overleftarrow{T})$, we define

$$d_{\mathrm{APS}}(r,s) = \sum_{\gamma \in \mathrm{ACTIVE}(r)} (-1)^{I(\rho,\gamma)} D_{\gamma,\rho}(r,s),$$

where $r = (\rho, \vec{m})$, $I(\rho,\gamma) = \sum_{c_\gamma < c'} \rho(c')$ is the number of ρ–inactive crossings which occur after the crossing c corresponding to γ, and $D_{\gamma,\rho}$ is a map defined at each active arc. This map is prescribed by the following recipe:

(1) *Khovanov case (i)* Suppose surgery on γ merges the *free* circles C_1 and C_2 in ρ to get a *free circle* C in $\gamma(\rho)$.
   (a) If $s(C_1) = s(C_2) = +$, then $D_{\gamma,\rho}(r,s) = (\gamma(r), s')$, where $s'(C) = +$ and $s'(D) = s(D)$ for every circle $D \neq C_1, C_2, C$, free or cleaved.
   (b) If either $s(C_1) = -$ and $s(C_2) = +$ or $s(C_1) = -$ and $s(C_2) = +$, then $D_{\gamma,\rho}(r,s) = (\gamma(r), s'')$, where $s''(C) = -$ and $s''(D) = s(D)$ for every circle $D \neq C_1, C_2, C$, free or cleaved.
   (c) If $s(C_1) = s(C_2) = -$, then $D_{\gamma,\rho}(r,s) = 0$.

(2) *Khovanov case (ii)* Suppose γ has both feet on the same *free circle* C in r, so that surgering C along γ produces two new *free circles* C_1 and C_2 in $\gamma(r)$.
   (a) If $s(C) = +$, then $D_{\gamma,\rho}(r,s) = (\gamma(r), s_{+-}) + (\gamma(r), s_{-+})$, where we define $s_{+-}(C_1) = +$, $s_{-}(C_2) = -$ and $s_{+-}(D) = s(D)$ for every other circle in $\mathrm{CIR}(r)$. We similarly define s_{-+}, with the roles of C_1 and C_2 reversed.
   (b) If $s(C) = -$, then $D_{\gamma,\rho}(r,s) = (\gamma(r), s_{--})$, where $s_{--}(C_1) = s_{--}(C_2) = -$ and $s_{--}(D) = s(D)$ for every other circle in $\mathrm{CIR}(r)$.

(3) Suppose γ has both feet on the same arc A in $\overleftarrow{H} \cap r$. Then $\gamma(r)$ will have a new *free* circle component C. Then $D_{\gamma,\rho}(r,s) = (\gamma(r), s')$, where $s'(C) = -$ and $s'(D) = s(D)$ for every other circle in $\mathrm{CIR}(\gamma(r))$.

(4) If γ has one foot on a cleaved circle C and the other foot on a *free* circle D, then surgery on γ will merge D into C, leaving the other circles unchanged. If $s(D) = +$, then $D_{\gamma,\rho}(r,s) = (\gamma(r), s')$ with $s'(C') = s(C')$ for every other circle in $\mathrm{CIR}(r)$, including C, while if $s(D) = -$ then $D_{\gamma,\rho}(r,s) = 0$.

(5) In every case not covered on this list, $D_{\gamma,\rho}(r,s) = 0$.

3.4 Some classes of active bridges

For a state (r,s), we can use s to group the bridges in $\mathrm{ACTIVE}(r)$ into (overlapping) classes:

(1) $\mathrm{INTERIOR}(r,s)$ is the subset $\mathrm{ACTIVE}(r)$ consisting of all γ for which $D_{\gamma,\rho}(r,s)$ is nonzero. That is,
   (a) both feet of γ are on elements of $\mathrm{FREE}(r)$, or
   (b) one foot of γ is on $C \in \mathrm{cl}(r)$ and the other foot is on $C' \in \mathrm{FREE}(r)$ with $s(C') = +$, or
   (c) both feet are on the same arc of $C \cap \overleftarrow{\mathbb{H}}$ for some $C \in \mathrm{cl}(r)$.

(2) $\mathrm{DEC}(r,s)$ is the subset $\mathrm{ACTIVE}(r)$ consisting of those γ where
   (a) both feet are on the same arc of $C \cap \overleftarrow{\mathbb{H}}$ for some $C \in \mathrm{cl}(r)$ with $s(C) = +$, or
   (b) one foot of γ is on $C \in \mathrm{cl}(r)$ with $s(C) = +$ and the other foot is on $C' \in \mathrm{FREE}(r)$ with $s(C') = -$.

(3) $\overleftarrow{\mathrm{BR}}(r)$ is the subset $\mathrm{ACTIVE}(r)$ consisting of those γ such that either
   (a) γ has one foot on $C_1 \in \mathrm{cl}(r)$ and the other on a distinct circle $C_2 \in \mathrm{cl}(r)$, or
   (b) γ has both feet on some $C \in \mathrm{cl}(r)$, but they are on different arcs of $C \cap \overleftarrow{\mathbb{H}}$.

If $r = (\rho, \vec{m})$ we will let $\mathrm{BRIDGE}(r) = \overleftarrow{\mathrm{BR}}(r) \cup \overrightarrow{\mathrm{BR}}(\vec{m})$ and $\overrightarrow{\mathrm{BR}}(r) = \overrightarrow{\mathrm{BR}}(\vec{m})$. There is a natural map $\mathrm{BRIDGE}(r) \to \mathrm{BRIDGE}(\mathrm{cl}(r))$.

4 The type A structure for an inside tangle

Given a diagram $\overleftarrow{T}$ for an inside tangle $\overleftarrow{\mathcal{T}}$, we describe a type A structure on $\langle\!\langle \overleftarrow{T} \rangle\!\rangle$ over $\mathcal{B}\Gamma_n$. This structure is specified by two *bigrading preserving* maps

$$m_1 \colon \langle\!\langle \overleftarrow{T} \rangle\!\rangle \to \langle\!\langle \overleftarrow{T} \rangle\!\rangle[(-1,0)], \tag{15}$$

$$m_2 \colon \langle\!\langle \overleftarrow{T} \rangle\!\rangle \otimes_{\mathcal{I}} \mathcal{B}\Gamma_n \to \langle\!\langle \overleftarrow{T} \rangle\!\rangle. \tag{16}$$

Let $\xi = (r,s)$ be a generator of $\langle\!\langle \overleftarrow{T} \rangle\!\rangle$ with $\partial\xi = (L,\sigma)$, and let $e \in \mathcal{B}\Gamma_n$ be a generator.

For m_1 we let $m_1(\xi) = d_{\mathrm{APS}}(\xi)$, the differential on $\langle\!\langle \overleftarrow{T} \rangle\!\rangle$. d_{APS} maps (r,s) in bigrading (h,q) to an element in $(h+1,q)$. This is bigrading preserving into $\langle\!\langle \overleftarrow{T} \rangle\!\rangle[(-1,0)]$.

To define the action m_2 we start by describing the action of the generators of $\mathcal{B}\Gamma_n$. Let $\xi = (r, s)$ be a generator of $[\![\overleftarrow{T}]\!]$. Then $m_2(\xi \otimes_{\mathcal{I}} e)$ is computed as follows:

(1) When e is an idempotent, $m_2(\xi \otimes I_{(L,\sigma)}) = \xi \cdot I_{(L,\sigma)}$: the idempotent action defined above.

(2) When $e = \overrightarrow{e_C}$ for some $C \in \partial\xi$ with $\sigma(C) = +$, $m_2(\xi \otimes_{\mathcal{I}} \overrightarrow{e_C}) = (r, s_C)$, where $s_C(C) = -$ but equals s on all other circles in $\mathrm{CIR}(r)$.

(3) When $e = \overleftarrow{e_C}$ for some $C \in \partial\xi$ with $\sigma(C) = +$,

$$m_2(\xi \otimes_{\mathcal{I}} \overleftarrow{e_C}) = \sum_{\gamma \in \mathrm{DEC}((r,s),C)} (-1)^{I(r,\mathrm{CR}(\gamma))}(r_\gamma, s_\gamma),$$

where $\mathrm{DEC}((r, s), C)$ are those active arcs which can change the decoration on C, r_γ is the result of surgery on γ, and s_γ is the new decoration with $s'(C) = -$ (and $s'(D) = +$ if a new circle is created).

(4) When $e = e_{\eta,\sigma,\sigma'}$ for some $\eta \in \overleftarrow{\mathrm{BR}}(L)$,

$$m_2(\xi \otimes_{\mathcal{I}} e_{\eta,\sigma,\sigma'}) = \sum_{\substack{\gamma \in \mathrm{ACTIVE}(r) \\ \mathrm{cl}(\gamma) = \eta}} (-1)^{I(r,\mathrm{CR}(\gamma))}(r_\gamma, s_\gamma),$$

where r_γ is surgery along γ and s_γ is the decoration on r_γ which equals s on $\mathrm{FREE}(r)$ and σ' on $\mathrm{cl}(r)$.

(5) When $e = e_{\eta,\sigma,\sigma'}$ for some $\eta \in \overrightarrow{\mathrm{BR}}(L)$ and $r = (\rho, \overrightarrow{m})$, let $r' = (\rho, \overrightarrow{m_\eta})$ and s' equal σ' on the cleaved circles and s on $\mathrm{FREE}(r')$, $m_2(\xi \otimes_{\mathcal{I}} e_{\eta,\sigma,\sigma'}) = (r', s')$.

(6) In all other cases $m_2(\xi \otimes_{\mathcal{I}} e) = 0$. Note that $(r, s) \otimes_{\mathcal{I}} e_1 = 0$ unless $\partial(r, s)$ is the source of e_1 since otherwise $I_{\partial(r,s)} \cdot e_1 = 0$.

Proposition 4.1 *The map m_2 constructed above is bigrading preserving.*

Proof (1) This is immediate since the bigrading of $I_{(L,\sigma)}$ is $(0, 0)$.

(2) When $e = \overrightarrow{e_C}$, if ξ is in bigrading (h, q), then $\xi \otimes_{\mathcal{I}} \overrightarrow{e_C}$ is in bigrading $(h, q) + (0, -1) = (h, q-1)$, whereas $m_2(\xi \otimes_{\mathcal{I}} \overrightarrow{e_C})$ is in bigrading $(h, q-1)$ since we changed a $\left(0, +\frac{1}{2}\right)$ cleaved circle to a $\left(0, -\frac{1}{2}\right)$ cleaved circle.

(3) When $e = \overleftarrow{e_C}$, the bigrading of $\xi \otimes_{\mathcal{I}} \overleftarrow{e_C}$ is $(h, q) + (1, 1)$. For $m_2(\xi \otimes_{\mathcal{I}} \overleftarrow{e_C})$ we consider the bigrading in two cases. If we merge a $-$ free circle, then $q = \bar{q} + \frac{1}{2} - 1$ and the bigrading of (r_γ, s_γ) is $\left(h, \bar{q} - \frac{1}{2}\right) + (1, 1) = (h + 1, q + 1)$, where the additional $(1, 1)$ comes from the grading shifts associated with 1–resolving a crossing. If we divide a $+$ cleaved circle then the bigrading change is from $\left(h, \bar{q} + \frac{1}{2}\right) + (1, 1) = \left(h+1, q + \frac{3}{2}\right)$ to $\left(h, \bar{q} + 1 - \frac{1}{2}\right) + (1, 1) = \left(h + 1, q + \frac{3}{2}\right)$. In either case, there is a $(0, 0)$ change in bigrading.

(4) When $e = e_{\eta,\sigma,\sigma'}$ for some $\eta \in \overleftarrow{\mathrm{BR}}(L)$, if η merges two plus circles then $\xi \otimes_{\mathcal{I}} e_{\eta,\sigma,\sigma'}$ has bigrading $\left(h, \bar{q} + \frac{1}{2} + \frac{1}{2}\right) + \left(1, \frac{1}{2}\right) = \left(h+1, \bar{q} + \frac{3}{2}\right)$, while $m_2(\xi \otimes_{\mathcal{I}} e_{\eta,\sigma,\sigma'})$ has $\left(h, \bar{q} + \frac{1}{2}\right) + (1,1)$, since we change the resolution at a crossing. If η merges a $+$ and a $-$ we have $\left(h, \bar{q} + \frac{1}{2} - \frac{1}{2}\right) + \left(1, \frac{1}{2}\right) = \left(h+1, \bar{q} + \frac{1}{2}\right)$ before and $\left(h, \bar{q} - \frac{1}{2}\right) + (1,1)$ after. If η divides a $+$ circle then we start with $\left(h, \bar{q} + \frac{1}{2}\right) + \left(1, \frac{1}{2}\right)$ and end with $\left(h, \bar{q} + \frac{1}{2} - \frac{1}{2}\right) + (1,1)$, while if η divides a $-$ circle we start with $\left(h, \bar{q} - \frac{1}{2}\right) + \left(1, \frac{1}{2}\right)$ and end with $\left(h, \bar{q} - \frac{1}{2} - \frac{1}{2}\right) + (1,1)$. Each of these is a $(0,0)$ change.

(5) When $e = e_{\eta,\sigma,\sigma'}$ for some $\eta \in \overrightarrow{\mathrm{BR}}(L)$ and $r = (\rho, \vec{m})$, if surgery on η merges two $+$ cleaved circles, then the bigrading of $\xi \otimes_{\mathcal{I}} e_{\eta,\sigma,\sigma'}$ is $\left(h, \bar{q} + \frac{1}{2} + \frac{1}{2}\right) + \left(0, -\frac{1}{2}\right)$, while that of $m_2(\xi \otimes_{\mathcal{I}} e_{\eta,\sigma,\sigma'})$ is $\left(h, \bar{q} + \frac{1}{2}\right)$ (as there is no crossing change). Likewise for a $+$ and $-$ circle: $\left(h, \bar{q} + \frac{1}{2} - \frac{1}{2}\right) + \left(0, -\frac{1}{2}\right) \to \left(h, \bar{q} - \frac{1}{2}\right)$, while for a divide of a $+$ circle: $\left(h, \bar{q} + \frac{1}{2}\right) + \left(0, -\frac{1}{2}\right) \to \left(h, \bar{q} + \frac{1}{2} - \frac{1}{2}\right)$, and a divide of a $-$ circle: $\left(h, \bar{q} - \frac{1}{2}\right) + \left(0, -\frac{1}{2}\right) \to \left(h, \bar{q} - \frac{1}{2} - \frac{1}{2}\right)$. In all cases there is a $(0,0)$ change in bigrading.

(6) This is true by definition true since the image of m_2 is 0. □

This specifies m_2 on the generators of $\mathcal{B}\Gamma_n$. To define m_2 on all elements, we impose the following relation: if $p_1, p_2 \in \mathcal{B}\Gamma_n$, we define

$$m_2(\xi \otimes p_1 p_2) = m_2(m_2(\xi \otimes p_1) \otimes p_2).$$

Proposition 4.2 *If two products of the generators p_1 and p_2 define equal elements in $\mathcal{B}\Gamma_n$, then $m_2(\xi \otimes p_1) = m_2(\xi \otimes p_2)$ for every $\xi \in \langle\!\langle \overleftarrow{T} \rangle\!\rangle$.*

Thus the rules above fully specify m_2: $\langle\!\langle \overleftarrow{T} \rangle\!\rangle \otimes_{\mathcal{I}} \mathcal{B}\Gamma_n \to \langle\!\langle \overleftarrow{T} \rangle\!\rangle$.

Proof It suffices to show that $m_2(\xi \otimes \rho) = 0$ whenever ρ is a relation defining $\mathcal{B}\Gamma_n$. We start with the relations from disjoint supports. Suppose C and D are distinct cleaved circles with $\sigma(C) = \sigma(D) = +$. Then

$$\begin{aligned}(17)\quad m_2(\xi \otimes (\vec{e_C}\vec{e_D} - \vec{e_D}\vec{e_C})) &= m_2(m_2(\xi \otimes \vec{e_C}) \otimes \vec{e_D}) - m_2(m_2(\xi \otimes \vec{e_D}) \otimes \vec{e_C}) \\ &= (r, s_{C,D}) - (r, s_{C,D}) = 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}(18)\quad & m_2(\xi \otimes (\overleftarrow{e_C}\vec{e_D} - \vec{e_D}\overleftarrow{e_C})) \\ &= m_2(m_2(\xi \otimes \overleftarrow{e_C}) \otimes \vec{e_D}) - m_2(m_2(\xi \otimes \vec{e_D}) \otimes \overleftarrow{e_C}) \\ &= \sum_{\gamma \in \mathrm{DEC}((r,s),C)} (-1)^{I(r,\mathrm{CR}(\gamma))}(r_\gamma, s_{\gamma,D}) - \sum_{\gamma \in \mathrm{DEC}((r,s),C)} (-1)^{I(r,\mathrm{CR}(\gamma))}(r_\gamma, s_{D,\gamma}) \\ &= 0.\end{aligned}$$

To compute $m_2(\xi \otimes (\overleftarrow{e_C}\overleftarrow{e_D} + \overleftarrow{e_D}\overleftarrow{e_C}))$ note that each $m_2(m_2(\xi \otimes \overleftarrow{e_C}) \otimes \overleftarrow{e_D})$ and $m_2(m_2(\xi \otimes \overleftarrow{e_D}) \otimes \overleftarrow{e_C})$ are sums over pairs of edges $\gamma \in \mathrm{DEC}(r,s,C)$ and $\gamma' \in \mathrm{DEC}(r,s,D)$. In one case we sum over (γ,γ') pairs and in the other (γ',γ) pairs. In each case we obtain $(r_{\gamma,\gamma'}, s_{\gamma,\gamma'})$ with $s_{\gamma,\gamma'}$ uniquely determined by the requirement that C and D are decorated with $-$. Thus we need only look at the signs: for (γ,γ') we have $(-1)^{I(r,\mathrm{CR}(\gamma))+I(r_\gamma,\mathrm{CR}(\gamma'))}$ which is $-(-1)^{I(r,\mathrm{CR}(\gamma'))+I(r_{\gamma'},\mathrm{CR}(\gamma))}$. Consequently, they cancel in the sum.

Now suppose that C_1 and C_2 are cleaved circles in r with $s(C_1) = s(C_2) = +$. Let β be an active arc which merges C_1 and C_2 to get C and maps to $\gamma \in \overleftarrow{\mathrm{BR}}(L)$. We can partition the active arcs α which contribute to $\mathrm{DEC}(r_\beta, s_\beta, C)$ into the three sets: $\mathrm{DEC}(r,s,C_1)$, $\mathrm{DEC}(r,s,C_2)$, and α which also map to γ. To obtain $m_2(\xi \otimes m_\gamma \overleftarrow{e_C})$ we sum over all such β and α arcs: $\sum_{(\beta,\alpha)}(-1)^{I(\beta)+I(r_\beta,\alpha)}(r_{\beta,\alpha}, s_{\beta\alpha})$. For α isotopic as a bridge to β the term for (α,β) occurs in this sum, with sign $(-1)^{I(\alpha)+I(r_\alpha,\beta)}$. This cancels the term from (β,α). Thus

$$
\begin{aligned}
(19)\qquad m_2(\xi \otimes m_\gamma \overleftarrow{e_C}) &= \sum_{\beta,\alpha \in \mathrm{DEC}(r,s,C_1)} (-1)^{I(\beta)+I(r_\beta,\alpha)}(r_{\beta,\alpha}, s_{\beta\alpha}) \\
&\quad + \sum_{\beta,\alpha\ \mathrm{DEC}(r,s,C_2)} (-1)^{I(\beta)+I(r_\beta,\alpha)}(r_{\beta,\alpha}, s_{\beta\alpha}) \\
&= -\sum_{\alpha \in \mathrm{DEC}(r,s,C_1),\beta} (-1)^{I(\alpha)+I(r_\alpha,\beta)}(r_{\alpha,\beta}, s_{\alpha\beta}) \\
&\quad - \sum_{\alpha \in \mathrm{DEC}(r,s,C_2),\beta} (-1)^{I(\alpha)+I(r_\alpha,\beta)}(r_{\alpha,\beta}, s_{\alpha\beta}) \\
&= -m_2(\xi \otimes \overleftarrow{e_{C_1}} m_\gamma) - m_2(\xi \otimes \overleftarrow{e_{C_2}} m_\gamma),
\end{aligned}
$$

which verifies that m_2 is compatible with relation (11). Exactly the same argument applies to for $\gamma \in \overrightarrow{\mathrm{BR}}(L)$, although we no longer need to sum over the representatives of γ since there is only one such bridge. More significantly, all the terms occur with sign $(-1)^{I(\alpha)}$ since surgery on γ does not affect the signs. The conclusion becomes

$$m_2(\xi \otimes m_\gamma \overleftarrow{e_C}) = m_2(\xi \otimes \overleftarrow{e_{C_1}} m_\gamma) + m_2(\xi \otimes \overleftarrow{e_{C_2}} m_\gamma),$$

which is compatible with relation (9). The case where surgery on γ is divisive follows from the same line of reasoning.

For $\overrightarrow{e_C}$ and γ merging C_1 and C_2 the situation is easier. First, suppose $\gamma \in \overrightarrow{\mathrm{BR}}(L)$. Then

$$m_2(\xi \otimes m_\gamma \overrightarrow{e_C}) = (r, s_{\gamma,C}),$$

while

$$m_2(\xi \otimes \overrightarrow{e_{C_1}} m_\gamma) = m_2((r, s_{C_1}) \otimes m_\gamma) = (r, s_{C,\gamma}).$$

As these are equal, and as C_2 plays a symmetric role, m_2 is compatible with this type of relation. Again, the case for dividing is similar. For $\overrightarrow{e_C}$ and $\gamma \in \overleftarrow{\mathrm{BR}}(L)$, we have

$$m_2(\xi \otimes m_\gamma \overrightarrow{e_C}) = \sum_\alpha (-1)^{I(\alpha)}(r_\alpha, s_{\alpha,C}),$$

where the sum is over active arcs which map to γ. On the other hand, while

$$m_2(\xi \otimes \overrightarrow{e_{C_1}} m_\gamma) = m_2((r, s_{C_1}) \otimes m_\gamma) = \sum_\alpha (-1)^{I(\alpha)}(r_\alpha, s_{C,\alpha}).$$

Thus m_2 is compatible with $m_\gamma \overrightarrow{e_C} = \overrightarrow{e_{C_1}} m_\gamma$ for all $\gamma \in \mathrm{BRIDGE}(L)$.

Suppose that γ and γ' are in $\mathrm{BRIDGE}(L)$ and that there is a commuting square:

$$\begin{array}{ccc} (L,\sigma) & \xrightarrow{e_{(\gamma,\sigma,\sigma_{01})}} & (L_\gamma, \sigma_{01}) \\ {\scriptstyle e_{(\gamma',\sigma,\sigma_{10})}}\downarrow & & \downarrow{\scriptstyle e_{(\gamma',\sigma_{01},\sigma'')}} \\ (L_{\gamma'}, \sigma_{10}) & \xrightarrow{e_{(\gamma,\sigma_{01},\sigma'')}} & (L_{\gamma,\gamma'}, \sigma'') \end{array}$$

Then we have the following cases:

(1) Both γ and γ' are in $\overrightarrow{\mathrm{BR}}(L)$ we need to see $m_2(\xi \otimes (e_\gamma e_{\gamma'} - e_{\gamma'} e_\gamma)) = 0$. However, both terms resulting from expanding m will equal $(r_{\gamma\gamma'}, s'')$, where $r_{\gamma,\gamma'}$ is identical to r in $\overleftarrow{\mathbb{H}}$ but equals $L_{\gamma\gamma'}$ in $\overrightarrow{\mathbb{H}}$, and s'' is s on $\mathrm{FREE}(r)$ but σ'' on $\mathrm{cl}(r)$. Since both terms are identical, the difference will be zero and m_2 is compatible with this case.

(2) If γ in $\overleftarrow{\mathrm{BR}}(L)$ but $\gamma' \in \overrightarrow{\mathrm{BR}}(L)$, then we need $m_2(\xi \otimes (e_\gamma e_{\gamma'} - e_{\gamma'} e_\gamma)) = 0$. The action of e_γ followed by $e_{\gamma'}$ (or vice versa) will give $\sum_\alpha (-1)^{I(r,\alpha)}(r_{\alpha,\gamma'}, s'')$, where the sum is over all active arcs for r which have image γ in $\mathrm{cl}(r)$. Since surgery on γ' does not affect the sign, we see that the two terms will cancel, and m_2 is compatible with this case.

(3) Suppose both γ and γ' are in $\overleftarrow{\mathrm{BR}}(L)$, then $m_2(\xi \otimes e_\gamma e_{\gamma'})$ is the sum over pairs of active arcs (α, α') for r which map to γ and γ' when considered as bridges. Each pair also contributes to $m_2(\xi \otimes e_{\gamma'} e_\gamma)$ but in the reversed order (α', α). The decorations of the result are determined by σ'', so we need only check the signs of each term. The sign for (α, α') is $(-1)^{I(r,\mathrm{CR}(\alpha))+I(r_\alpha,\mathrm{CR}(\alpha'))}$ while that for (α', α) is $(-1)^{I(r,\mathrm{CR}(\alpha'))+I(r_{\alpha'},\mathrm{CR}(\alpha))}$. Due to the ordering of the crossings, one of these will be $+1$ and the other -1. Consequently, $m_2(\xi \otimes e_\gamma e_{\gamma'}) = -m_2(\xi \otimes e_{\gamma'} e_\gamma)$ which is compatible with the relation for $\overleftarrow{\mathrm{BR}}(L)$.

Note that a similar argument to that in (1) works for all pairs of right bridge edges that form commutative squares, so it will be omitted. For left bridge elements, there are two further cases to consider:

(a) Suppose $\delta \in \overleftarrow{B}_o(L,\gamma)$ and $\delta_1, \delta_2 \in B_s(L_\gamma, \gamma^\dagger)$ map to δ under surgery along $\gamma^\dagger$. Likewise, suppose γ_1 and γ_2 map to γ under surgery along $\delta^\dagger$. If we orient the mutual arc for δ and γ, then δ_1 corresponds to the version of δ before γ along the mutual arc, and δ_2 corresponds to that after γ. Likewise for $\gamma_i, i = 1,2$ and δ. There can then be (anti-)commutative squares

$$\begin{array}{ccc} (L,\sigma) & \xrightarrow{e_{(\gamma,\sigma,\sigma_{01})}} & (L_\gamma,\sigma_{01}) \\ {\scriptstyle e_{(\delta,\sigma,\sigma_{10})}}\downarrow & & \downarrow{\scriptstyle e_{(\delta_1,\sigma_{01},\sigma'')}} \\ (L_\delta,\sigma_{10}) & \xrightarrow{e_{(\gamma_2,\sigma_{01},\sigma'')}} & (L_{\gamma,\delta_1},\sigma'') \end{array} \qquad \begin{array}{ccc} (L,\sigma) & \xrightarrow{e_{(\gamma,\sigma,\sigma_{01})}} & (L_\gamma,\sigma_{01}) \\ {\scriptstyle e_{(\delta,\sigma,\sigma_{10})}}\downarrow & & \downarrow{\scriptstyle e_{(\delta_2,\sigma_{01},\sigma'')}} \\ (L_\delta,\sigma_{10}) & \xrightarrow{e_{(\gamma_1,\sigma_{01},\sigma'')}} & (L_{\gamma,\delta_2},\sigma'') \end{array}$$

where L_{γ,δ_2} and L_{γ,δ_1} have different left matchings. Two resolution arcs a_1 and a_2 for ξ, one corresponding to γ and one corresponding to δ, will be counted in the action of either γ followed by δ_1, or γ followed by δ_2 (but not both). Reversing the order means contributing to δ followed by γ_2, or δ followed by γ_1. As these contribute with the usual sign conventions, the contributions of the pair will cancel in either the action of $e_\gamma e_{\delta_1} + e_\delta e_{\gamma_2}$ or $e_\gamma e_{\delta_2} + e_\delta e_{\gamma_1}$, which verifies that the action respects the anti-commutativity in this case.

(b) Now suppose that $\overleftarrow{\alpha}$ is in $B_s(L,\overleftarrow{\gamma})$ and that $\overleftarrow{\beta}$ is the bridge obtained by sliding $\overleftarrow{\alpha}$ over $\overleftarrow{\gamma}$. Let $\overleftarrow{\delta}$ be the image of $\overleftarrow{\alpha}$ and $\overleftarrow{\beta}$ in $L_{\overleftarrow{\gamma}}$, $\overleftarrow{\zeta}$ be the image of $\overleftarrow{\alpha}$ and $\overleftarrow{\gamma}$ in $L_{\overleftarrow{\beta}}$ and $\overleftarrow{\eta}$ be the image of $\overleftarrow{\beta}$ and $\overleftarrow{\gamma}$ in $L_{\overleftarrow{\alpha}}$. The action of $\overleftarrow{e_\alpha}\overleftarrow{e_\eta}$ is the sum over active resolution arcs a_1 and a_2 for ξ with a_1 representing $\overleftarrow{\alpha}$ and a_2 representing $\overleftarrow{\eta}$ in L_α. Thus, a_2 represents one of β or γ in L. Reversing the order thus gives a contribution to either $\overleftarrow{e_\beta}\overleftarrow{e_\zeta}$ or $\overleftarrow{e_\gamma}\overleftarrow{e_\delta}$. However, pairs representing β and γ also contribute to $\overleftarrow{e_\beta}\overleftarrow{e_\zeta}$, while their reverse contributes to $\overleftarrow{e_\gamma}\overleftarrow{e_\delta}$. Thus, if we consider all ordered pairs (a_1,a_2) of resolution arcs which represent pairs of α, β, or γ, in either order, we will count both (a_1,a_2) and (a_2,a_1) for each such pair, and they will contribute with opposite signs (due to the Khovanov sign conventions) in the action of $\overleftarrow{e_\alpha}\overleftarrow{e_\eta} + \overleftarrow{e_\beta}\overleftarrow{e_\zeta} + \overleftarrow{e_\gamma}\overleftarrow{e_\delta}$ on ξ. Consequently, all the terms in this action will cancel, verifying that it acts as 0.

For the additional bridge relations, suppose $\gamma \in \overrightarrow{\mathrm{BR}}(L)$. Then there is a relation $e_{\gamma,\sigma,\sigma'} e_{\gamma^\dagger\sigma',\sigma_C} = \overrightarrow{e_C}$. In this case

$$\begin{aligned} (20)\quad m_2(\xi \otimes (e_{\gamma,\sigma,\sigma'} e_{\gamma^\dagger\sigma',\sigma_C} - \overrightarrow{e_C})) &= ((\rho, \overrightarrow{m}_{\gamma\gamma^\dagger}), s_C) - (r, s_C) \\ &= ((\rho,\overrightarrow{m}), s_C) - (r, s_C) = (r, s_C) - (r, s_C) = 0, \end{aligned}$$

where $\vec{m}_{\gamma\gamma^\dagger} = \vec{m}$ follows from the result that surgery on a bridge γ for L, followed by surgery on $\gamma^\dagger$, recovers L.

Now suppose that $\gamma \in \overleftarrow{\mathrm{BR}}(L)$ and $\eta \in \overleftarrow{\mathrm{BR}}(L_\gamma)$ intersects $\gamma^\dagger$ non-trivially. We need to see that $m_2(\xi \otimes e_\gamma e_\eta) = 0$ since $e_\gamma e_\eta = 0$. However, in $m_2(m_2(\xi \otimes e_\gamma) \otimes e_\eta)$ the action of e_η will result in a sum over active arcs in r_γ which map to η in $\mathrm{cl}(r)$. Each active arc comes from a crossing, and thus must already be present in r for it to be present in r_γ. This excludes there being any active arc for η in r_γ. Consequently the sum is 0 and we have verified that m_2 is compatible with this relation.

Since $m_2(\xi \otimes R) = 0$ whenever R is a relation for $\mathcal{B}\Gamma_n$, we have shown that m_2 is well defined. □

Proposition 4.3 *For $\xi = (r, s)$ a generator of $\langle\!\langle \overleftarrow{T} \rangle\!\rangle$ and $\rho_1, \rho_2 \in \mathcal{B}\Gamma_n$, the maps m_1 and m_2 above satisfy the following properties:*

$$0 = m_1(m_1(\xi)), \tag{21}$$

$$0 = (-1)^{\overleftarrow{l}(\rho_1)} m_2(m_1(\xi) \otimes \rho_1) + m_2(\xi \otimes d_\Gamma(\rho_1)) - m_1(m_2(\xi \otimes \rho_1)), \tag{22}$$

$$0 = m_2(m_2(\xi \otimes \rho_1) \otimes \rho_2) - m_2(\xi \otimes \rho_1\rho_2). \tag{23}$$

Note These are the relations for $\langle\!\langle \overleftarrow{T} \rangle\!\rangle$ to be an A_∞–module over the differential graded algebra $\mathcal{B}\Gamma_n$, as in [8], with $m_n = 0$ for $n \geq 3$.

Proof That $m_1(m_1(\xi)) = 0$ is a byproduct of $m_1 = d$ being a differential (see also the proof that $\vec{\delta}$ is a D structure for an outside tangle). Furthermore, that m_2 defines a right action follows from how we bootstrapped m_2 from the definition on generators. Thus, we need only verify that d and m_2 are compatible with μ_Γ through the equation

$$d(m_2(\xi \otimes \rho_1)) = (-1)^{\overleftarrow{l}(\rho_1)} m_2(d(\xi) \otimes \rho_1) + m_2(\xi \otimes d_\Gamma(\rho_1)).$$

It suffices to prove this for ρ_1 of length 0 or 1 since we can bootstrap the relation for longer words using

$$\begin{aligned} d(\xi\cdot(\alpha\beta)) &= d((\xi\cdot\alpha)\cdot\beta) \\ &= (-1)^{\overleftarrow{l}(\beta)} d(\xi\cdot\alpha)\cdot\beta + (\xi\cdot\alpha)\cdot d_\Gamma(\beta) \\ &= (-1)^{\overleftarrow{l}(\beta)+\overleftarrow{l}(\alpha)} (d(\xi)\cdot\alpha)\cdot\beta + (-1)^{\overleftarrow{l}(\beta)} (\xi\cdot d_\Gamma(\alpha))\cdot\beta + \xi\cdot(\alpha d_\Gamma(\beta)) \\ &= (-1)^{\overleftarrow{l}(\beta)+\overleftarrow{l}(\alpha)} \big[d(\xi)\cdot(\alpha\beta)\big] + \xi\cdot\big[(-1)^{\overleftarrow{l}(\beta)} d_\Gamma(\alpha)\beta + \alpha d_\Gamma(\beta)\big] \\ &= (-1)^{\overleftarrow{l}(\beta)+\overleftarrow{l}(\alpha)} \big[d(\xi)\cdot(\alpha\beta) + \xi\cdot d_\Gamma(\alpha\beta)\big]. \end{aligned} \tag{24}$$

For length 0, we have $\rho_1 = I_{(L,\sigma)}$ for some idempotent. If $\partial\xi \neq (L, \sigma)$ then both sides are zero since $m_2(\xi \otimes I_{(L,\sigma)}) = m_2(0) = 0$, $d(\xi)$ has the same boundary as ξ

so $d(\xi)\otimes I_{(L,\sigma)}=0$, and $d_\Gamma(I_{(L,\sigma)})=0$ for every idempotent. On the other hand, if $\partial\xi=(L,\sigma)$ the last term still vanishes, and

$$d\big(m_2(\xi\otimes I_{(L,\sigma)})\big)=d(\xi)=m_2\big(d(\xi)\otimes I_{(L,\sigma)}\big).$$

For length-one words, we need to check when $\rho_1=\overrightarrow{e_C},\overleftarrow{e_C}$, or e_γ for $\gamma\in\mathrm{BRIDGE}(L)$, where $\partial\xi=(L,\sigma)$. We know $d_\Gamma(\overrightarrow{e_C})=0$ and $\overleftarrow{l}\,(\overrightarrow{e_C})=0$, so for $\overrightarrow{e_C}$ we need only verify that $d(m_2(\xi\otimes\overrightarrow{e_C}))=m_2(d(\xi)\otimes\overrightarrow{e_C})$. If ξ has $\sigma(C)=-$ then both are 0, whereas if $\sigma(C)=+$ then both equal $\Sigma_\alpha(-1)^{I(\alpha)}(r_\alpha,s_{\alpha,C})$, where the sum is over all active, non-bridging arcs α and $s_{\alpha,C}$ is any decoration compatible with d, r_α, and assigning C a $-$. For e_γ with $\gamma\in\overrightarrow{\mathrm{BR}}(L)$, the only difference is that the sum is over terms $(r_{\alpha,\gamma},s_{\alpha,\gamma})$.

For e_γ with $\gamma\in\overleftarrow{\mathrm{BR}}(L)$, d_Γ still vanishes but $\overleftarrow{l}=1$. We then have

$$d(m_2(\xi\otimes e_\gamma))=\sum_{\alpha,\beta}(-1)^{I(r,\alpha)+I(r_\alpha,\beta)}(r_{\alpha,\beta},s_{\alpha,\beta}), \tag{25}$$

where the sum is over all active arcs α which map to γ and all active arcs β which contribute to d (as well as all compatible decorations on $r_{\alpha,\beta}$. On the other hand,

$$m_2(d(\xi)\otimes e_\gamma)=\sum_{\beta,\alpha}(-1)^{I(r,\beta)+I(r_\beta,\alpha)}(r_{\beta,\alpha},s_{\beta,\alpha})$$

due to the ordering of the crossings the signs will be different for each (α,β) term, so

$$d(m_2(\xi\otimes e_\gamma))=-m_2(d(\xi)\otimes e_\gamma)=(-1)^{\overleftarrow{l}\,(e_\gamma)}m_2(d(\xi)\otimes e_\gamma).$$

We are left with verifying the formula for $\overleftarrow{e_C}$. We start with

$$d(m_2(\xi\otimes\overleftarrow{e_C}))=\sum_{\alpha,\beta}(-1)^{I(r,\alpha)+I(r_\alpha,\beta)}(r_{\alpha,\beta},s_{\alpha,\beta}),$$

where the sum is over all active arcs $\alpha\in\mathrm{DEC}(r,s,C)$ and β contributing to d on r_α. Furthermore,

$$m_2(d(\xi)\otimes\overleftarrow{e_C})=\sum_{\beta',\alpha'}(-1)^{I(r,\beta')+I(r_{\beta'},\alpha')}(r_{\beta',\alpha'},s_{\beta',\alpha'}),$$

where the sum is over all β' contributing to d on r and over all α' contributing to $\mathrm{DEC}(r_{\beta'},s_{\beta'},C)$.

Some ordered pairs (α,β) contribute a term to $d(m_2(\xi\otimes\overleftarrow{e_C}))$ while the pair (β,α) contributes to $m_2(d(\xi)\otimes\overleftarrow{e_C})$. The corresponding terms in the sum occur with coefficients that are equal but have opposite signs, and thus cancel. We let R be the sum over the pairs (α,β) in $d(m_2(\xi\otimes\overleftarrow{e_C}))$, and define $\Psi_1=d(m_2(\xi\otimes\overleftarrow{e_C}))-R$.

Then $m_2(d(\xi)\otimes \overleftarrow{e_C}) = -R + \Psi_2$ for some Ψ_2. We can characterize the terms which contribute to Ψ_1 and Ψ_2. These terms correspond to pairs (α,β) coming from bridges $\gamma, \gamma^\dagger$ with C as their active circle. A sign check, however, shows that $\Psi_1+\Psi_2 = m_2(\xi\otimes\sum_\gamma e_\gamma e_{\gamma^\dagger})$, where the sum is over all γ with active circle C. As this sum is just the action of $-d_\Gamma(\overleftarrow{e_C})$ we obtain the relation

$$\begin{aligned}(26)\qquad d(m_2(\xi\otimes\overleftarrow{e_C})) &= R+\Psi_1\\ &= -(-R+\Psi_2)+\Psi_2+\Psi_1\\ &= -m_2(d(\xi)\otimes\overleftarrow{e_C})+(\Psi_1+\Psi_2)\\ &= -m_2(d(\xi)\otimes\overleftarrow{e_C})-m_2(\xi\otimes -d_\Gamma(\overleftarrow{e_C}))\\ &= (-1)^{\overleftarrow{l}(\overleftarrow{e_C})}\big(m_2(d(\xi)\otimes\overleftarrow{e_C})\big)+m_2(\xi\otimes d_\Gamma(\overleftarrow{e_C}))\end{aligned}$$

as required.

We have now verified that the action of the length one words is compatible with the (right) Leibniz relation, and thus using the bootstrap, that d is a (right) differential on the module $\langle\!\langle \overleftarrow{T}]\!]$. □

5 Background on type A structures and their simplification

5.1 Algebra

In this section we redefine A_∞–algebras and module to be consistent with our sign conventions. We begin with some notation for handling signs and gradings

Definition 5.1 Let $W = W_0\oplus W_1$ be a $\mathbb{Z}/2\mathbb{Z}$–graded module. $|\mathbb{I}_W|\colon W\to W$ is the signed identity defined by linearly extending

$$|\mathbb{I}_W|(w) = (-1)^{\mathrm{gr}(w)}w$$

for homogeneous $w\in A$.

Note By $|\mathbb{I}|^j$ we mean the composition of $|\mathbb{I}|$ with itself j times. Furthermore, by $|\mathbb{I}|^{j\otimes n}$ we will mean the function $|\mathbb{I}|^j\otimes\cdots\otimes|\mathbb{I}|^j$, where there are n factors. For an element α, $|\mathbb{I}|^j(\alpha)$ will be shortened to $|\alpha|^j$. Thus, on a homogeneous element α, $|\alpha|^j = (-1)^{j\,\mathrm{gr}(\alpha)}\alpha$, and $\big||\alpha|^j\big|^k = |\alpha|^{j+k}$.

Definition 5.2 An A_∞–algebra A over a ring R is a graded module A equipped with maps $\mu_n\colon A^{\otimes n}\to A[n-2]$ for each $n\in\mathbb{N}$ which satisfy the relation

$$0 = \sum_{\substack{i+j=n+1\\ k\in\{1,\dots,n-j+1\}}} (-1)^{j(i+1)+(k+1)(j+1)}\mu_i\big(\mathbb{I}^{\otimes(k-1)}\otimes\mu_j\otimes|\mathbb{I}|^{j\otimes(n-k-j+1)}\big).$$

Definition 5.3 A right module over a $\mathbb{Z}/2\mathbb{Z}$–graded differential R–algebra (A,μ_1,μ_2) is an R–module M together with maps $m_1\colon M\to M[-1]$ and $m_2\colon M\otimes_R A\to M$ such that

$$0 = m_1\circ m_1, \tag{27}$$

$$0 = m_2(m_1\otimes|\mathbb{I}|)+m_2(\mathbb{I}\otimes\mu_1)-m_1(m_2), \tag{28}$$

$$0 = m_2(m_2\otimes\mathbb{I})-m_2(\mathbb{I}\otimes\mu_2). \tag{29}$$

A right module as above is a special case of the A_∞–modules found in [8] (when using the sign conventions in this paper).

Definition 5.4 [8] A right A_∞–module M over an A_∞–algebra A is a set of maps $\{m_i\}_{i\in\mathbb{N}}$ with $m_i\colon M\otimes A^{\otimes(i-1)}\to M[i-2]$, and satisfying the following relations for each $n\geq 1$:

$$\begin{aligned}0 = {}& \sum_{i+j=n+1}(-1)^{j(i+1)}m_i\big(m_j\otimes|\mathbb{I}|^{j\otimes(i-1)}\big)\\ &+\sum_{i+j=n+1,k>0}(-1)^{k(j+1)+j(i+1)}m_i\big(\mathbb{I}^{\otimes k}\otimes\mu_j\otimes|\mathbb{I}|^{j\otimes(i-k-1)}\big).\end{aligned} \tag{30}$$

M is said to be *strictly unital* if for any $\xi\in M$, $m_2(\xi\otimes 1_A)=\xi$, but for $n>1$, $m_n(\xi\otimes a_1\otimes a_2\otimes\cdots\otimes a_{n-1})=0$ if any $a_i=1_A$.

Our right modules correspond to $m_i=0$ for $i\geq 2$. Nevertheless, we will think of these as objects in the category of right A_∞–modules. The morphisms in this category are the following.

Definition 5.5 [8] An A_∞–morphism Ψ of right A–modules M and M' is a set of maps $\psi_i\colon M\otimes A^{\otimes(i-1)}\to M'[i-1]$ for $i\in\mathbb{N}$, satisfying

$$\begin{aligned}&\sum_{i+j=n+1}(-1)^{(i+1)(j+1)}m_i'\big(\psi_j\otimes|\mathbb{I}|^{(j+1)\otimes(i-1)}\big)\\ &\quad=\sum_{i+j=n+1}(-1)^{j(i+1)}\psi_i\big(m_j\otimes|\mathbb{I}|^{j\otimes(i-1)}\big)\\ &\qquad+\sum_{i+j=n+1,k>0}(-1)^{j(i+1)+k(j+1)}\psi_i\big(\mathbb{I}^{\otimes k}\otimes\mu_j\otimes|\mathbb{I}|^{j\otimes(i-k-1)}\big).\end{aligned} \tag{31}$$

Ψ is *strictly unital* if $\psi_i(\xi\otimes a_1\otimes\cdots\otimes a_{i-1})=0$ when $a_j=1_A$ for some j and $i>1$. The *identity* morphism I_M is the collection of maps $i_1(\xi)=\xi$, $i_j=0$ for $j>1$.

Definition 5.6 [8] Let Ψ be an A_∞–morphism from M to M', and let Φ be an A_∞–morphism from M' to M''. The composition $\Phi * \Psi$ is the morphism whose component maps for $n \geq 1$ are

$$(\Phi * \Psi)_n = \sum_{i+j=n+1} (-1)^{(i+1)(j+1)} \phi_i\big(\psi_j \otimes |\mathbb{I}|^{(j+1)\otimes(i-1)}\big).$$

Definition 5.7 [8] Let Ψ, Φ be A_∞–morphisms from M to M'. Ψ and Φ are homotopic if there is a set of maps $\{h_i\}$ with $h_i\colon M \otimes A^{\otimes(i-1)} \to M'[i]$ such that

$$\begin{aligned}(32)\quad \psi_i - \phi_i = &\sum_{i+j=n+1} (-1)^{(i+1)j} m'_i\big(h_j \otimes |\mathbb{I}|^{j\otimes(i-1)}\big) \\ &+ \sum_{i+j=n+1} (-1)^{(i+1)j} h_i\big(m_j \otimes |\mathbb{I}|^{j\otimes(i-1)}\big) \\ &+ \sum_{i+j=n+1,k>0} (-1)^{k(j+1)+j(i+1)} h_i\big(\mathbb{I}^{\otimes k} \otimes \mu_j \otimes |\mathbb{I}|^{j\otimes(i-k-1)}\big)\end{aligned}$$

and for $i > 1$, $h_i(\xi \otimes a_1 \otimes \cdots \otimes a_{i-1}) = 0$ when $a_j = 1_A$ for some j.

The sign convention used in the previous definitions is that of Keller [5] with the Koszul sign rule

$$(f \otimes g)(x \otimes y) = (-1)^{|f||y|}(f(x) \otimes g(y)).$$

Thus, as can be checked directly, the composition of morphisms is a morphism for this sign convention, and homotopy of morphisms is an equivalence relation (or see the Appendix). With these definitions, we are equipped to consider right A_∞–modules up to homotopy equivalence. The following is our version of a standard result in the study of A_∞–modules.

Proposition 5.8 *Let $(M, \{m_i\})$ be a strictly unital, right A_∞–module over $(A, \{\mu_i\})$, and let $(\overline{M}, \overline{m}_1)$ be a chain complex. Suppose there exist chain maps $\iota\colon (\overline{M}, \overline{m}_1) \to (M, m_1)$ and $\pi\colon (M, m_1) \to (\overline{M}, \overline{m}_1)$, and a map $H\colon M \to M[1]$ satisfying the following relations:*

$$\pi \circ \iota = \mathbb{I}_{\overline{M}}, \tag{33}$$

$$\iota \circ \pi - \mathbb{I}_M = m_1 \circ H + H \circ m_1, \tag{34}$$

$$H \circ \iota = 0, \tag{35}$$

$$\pi \circ H = 0, \tag{36}$$

$$H^2 = 0. \tag{37}$$

Then there are maps $\overline{m}_i\colon \overline{M}\otimes A^{\otimes(i-1)}\to\overline{M}$ for $i\geq 2$ such that $\{\overline{m}_i\}_{i=1}^{\infty}$ defines a strictly unital right A_∞–module structure on $\overline{M}$. This structure is homotopy equivalent to $(M,\{m_i\})$ through strictly unital morphisms which extend π and ι.

The proof supplies an explicit formula for computing $\overline{m}_i$ and the morphisms in the homotopy equivalence. First, we introduce some notation to simplify the formulas.

Definition 5.9 For positive integers $i_1,\dots,i_k$, let

$$N(i_1,\dots,i_k)=\sum_j(i_j-1)\quad\text{and}\quad\alpha(i_1,\dots,i_k)=\sum_{1\leq r<s\leq k}(i_r-1)(i_s-1).$$

Definition 5.10 Let $i_j\geq 2$ be integers for $j=1,\dots,k$. By $[i_1,\dots,i_k]$ we will mean the composition

$$(m_{i_1})\big(H\otimes|\mathbb{I}|^{\otimes(i_1-1)}\big)\big(m_{i_2}\otimes|\mathbb{I}|^{i_2\otimes(i_1-1)}\big)\cdots\big(H\otimes|\mathbb{I}|^{\otimes(I-i_k)}\big)\big(m_{i_k}\otimes|\mathbb{I}|^{i_k\otimes(I-i_k)}\big),$$

where we alternate between applying m_{i_j} to the first i_j entries in the tensor product and applying H to the first factor in the tensor product.

Using this notation, we can define the action, morphisms, and homotopy. First, for $n\geq 2$ define a map $M\otimes A^{\otimes(n-1)}\to M[n-2]$ by

$$\Sigma_n=\sum_{\substack{N(i_1,i_2,\dots,i_k)=n-1\\ i_j\geq 2}}(-1)^{\alpha(i_1,\dots,i_k)}[i_1,\dots,i_k].$$

We use Σ_n to define $\overline{m}_n$ for $n\geq 1$:

$$\overline{m}_n:=\pi\circ\Sigma_n\circ\big(\iota\otimes\mathbb{I}^{\otimes(n-1)}\big).$$

For $n=1$ we use the boundary map $\overline{m}_1$. Then $\{\overline{m}_i\}_{i=1}^{\infty}$ equips $\overline{M}$ with the structure of a right A_∞–module.

The morphisms which induce the homotopy equivalence are similarly defined. For $n=1$ we will use $\pi_1=\pi$ and $\omega_1=\iota$, while for $n>1$ we use

$$\pi_n:=(-1)^n\big(\pi\circ\Sigma_n\circ\big(H\otimes|\mathbb{I}|^{\otimes(n-1)}\big)\big),$$
$$\omega_n:=H\circ\Sigma_n\circ\big(\iota\otimes\mathbb{I}^{\otimes(n-1)}\big).$$

The additional H means that these are maps $\pi_n\colon M\otimes\mathbb{I}^{\otimes(n-1)}\to\overline{M}[n-1]$ and $\omega_n\colon\overline{M}\otimes\mathbb{I}^{\otimes(n-1)}\to M[n-1]$. As defined, these morphisms satisfy the relations $\Pi*\Omega=I_{\overline{M}}$ and $\Omega*\Pi\simeq_\Lambda I_M$, where $\lambda_1=H$,

$$\lambda_n:=(-1)^n\big(H\circ\Sigma_n\circ\big(H\otimes|\mathbb{I}|^{\otimes(n-1)}\big)\big),$$

and all homotopy equivalences occur in the category of (right) A_∞–modules.

We note that even when $m_i \equiv 0$ for $i > 2$, a homotopy equivalence as described in Proposition 5.8 can have higher order action terms. Indeed, the new module structure is given by

$$\overline{m}_n = (-1)^{\epsilon}\pi[2,2,\dots,2](\iota \otimes \mathbb{I}^n),$$

where there are exactly $n-1$ 2s inside the square brackets and $\epsilon = 0$ if $n \equiv 1,2$ modulo 4, and $\epsilon = 1$ if $n \equiv 3,4$ modulo 4. Thus, in all cases

$$\overline{m}_2 = \pi \circ m_2 \circ (\iota \otimes \mathbb{I}),$$

which comes from appropriately adjusting m_2. The effect of π, however, is substantial when doing calculations. With this observation and Proposition 5.8, we can, by directly analyzing the diagrams before and after a Reidemeister move, see that the A_∞–module structure is preserved up to homotopy equivalence. This affords us the difficult part of the next theorem.

Theorem 5.11 *Let $\overleftarrow{T}$ be an inside tangle with boundary P_{2n}.*

(1) *Let $\overleftarrow{T}$ be a diagram for $\overleftarrow{T}$ in $\overleftarrow{\mathbb{H}}$. If $\mathfrak{o}_1$ and $\mathfrak{o}_2$ are two orderings of $\mathrm{CR}(\overleftarrow{T})$, then $\langle\!\langle T, \mathfrak{o}_1 \rangle\!\rangle$ and $\langle\!\langle T, \mathfrak{o}_2 \rangle\!\rangle$ are isomorphic type A structures.*

(2) *If $\overleftarrow{T}_1$ and $\overleftarrow{T}_2$ are two diagrams for $\overleftarrow{T}$, then $\langle\!\langle \overleftarrow{T}_1 \rangle\!\rangle$ and $\langle\!\langle \overleftarrow{T}_2 \rangle\!\rangle$ are homotopy equivalent type A structure.*

Corollary 5.12 *The homotopy type of the type A structure $\langle\!\langle \overleftarrow{T} \rangle\!\rangle$, for any diagram T of an inside tangle $\overleftarrow{T}$, is a tangle invariant.*

We will not prove these theorems here, as the proofs are modifications of those for the type D structure for an outside tangle found in [9]. In addition, there are easier ways to prove these results once we have generalized the gluing theory in Section 7. Instead we content ourselves with computing some examples using Proposition 5.8 which will illustrate the argument.

How we will use this Suppose we have a chain complex $\{C_i \mid i \in \mathbb{Z}\}$ with explicit generators for each *free* chain group C_i. If the generators of C_i are $\{x_1,\dots,x_n\}$ and those for C_{i-1} are $\{y_1,\dots,y_m\}$ we can find a homotopy H as in Proposition 5.8 by searching through the images $\partial x_i = \sum a_i^j y_j$ to find one where $a_j^i = u$ is a unit, for some j. We will reorder the generators so that this occurs for $i = j = 1$. We can then construct a new chain complex where the other chain groups and boundary maps are taken to be the same, but C_i' is spanned by $x_2',\dots,x_n'$ and C_{i-1}' is spanned by $y_2',\dots,y_m'$. We let $\pi(x_i) = x_i'$ for $i > 1$ and $\pi(x_1) = 0$, and likewise for the y_j. Otherwise π is the identity. The new boundary $\partial_i'\colon C_{i'} \to C_{i-1}'$ is given by

$\partial' x_i' = (\pi \circ \partial)(x_i - a_i^1 u^{-1} x_1)$. If we let $\iota(x_i') = x_i - a_i^1 u^{-1} x_1$ and $H(y_1) = -u^{-1} x_1$ ($H(\beta) = 0$ otherwise), then we are in the situation envisioned in Proposition 5.8. The map π is the quotient map found by quotienting out the subcomplex generated by $\{x_1, \partial x_1\}$. The formulas involving π are a specific presentation of this quotient complex for the specific basis. ∂' is computed by calculating $\pi \circ \partial$ in this presentation.

We now compute $\overline{m}_2(x_i' \otimes e)$. Since $\overline{m}_2 = \pi \circ m_2 \circ (\iota \otimes \mathbb{I})$ we first compute

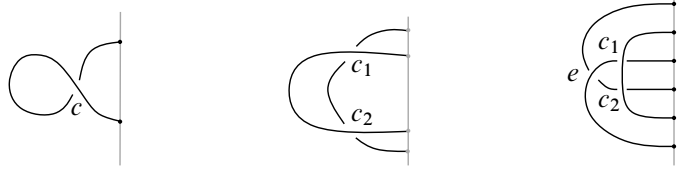
$$m_2((x_i - a_i^1 u^{-1} x_1) \otimes e) = m_2(x_i \otimes e) - a_i^1 u^{-1} m_2(x_1 \otimes e),$$

then compute π. In particular, suppose $\langle \partial x_j, y_1 \rangle = a_j^1 = 0$ but $m_2(x_j \otimes e) = a y_1 + Y$. Then $\overline{m}_2(x_j' \otimes e) = \pi(a y_1 + Y) = a(u^{-1} \sum_{j>1} a_1^j y_j) + Y$. This is the same process as for adjusting the boundary maps above.

However, suppose $\langle \partial x_j, y_1 \rangle = a_j^1 \neq 0$, but $m_2(x_j \otimes e_1) = a y_1 + Y$ and $m_2(x_1 \otimes e_2) = W$. Then $\overline{m}_3(x_j' \otimes e_1 \otimes e_2) = \pi(m_3(x_j \otimes e_1 \otimes e_2)) - \pi(m_2(H \circ m_2(x_j \otimes e_1) \otimes e_2))$. We concentrate upon $\pi(m_2(H \circ m_2(x_j \otimes e_1) \otimes e_2)) = \pi(m_2(H(a y_1 + Y) \otimes e_2)) = \pi(m_2(-u^{-1} a x_1 \otimes e_2)) = -u^{-1} a \cdot W$. We can thus pick up a higher order action.

6 Examples of the type A structure

Example I (Reidemeister tangles) The three tangles below appear in the local description of the Reidemeister moves; we will analyze each in turn:

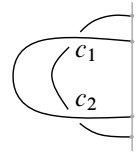


(a) **First move** For the RI move we have a tangle diagram R_I, over P_2 with one crossing. There are two resolutions, corresponding to $\rho = 0$ and $\rho = 1$, and the unique matching on P_2. Since the crossing is right handed, the 0 resolution has a single free circle. Writing the decoration on the cleaved circle first, we can think of the decorated resolutions as z_{++}, z_{+-}, z_{-+}, and z_{--} which occur in bigrading $\left(0, \frac{5}{2}\right)$, $\left(0, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}\right)$ and $\left(0, -\frac{1}{2}\right)$. For the 1–resolution there is only the cleaved circle, so we get two state t_+ in grading $\left(1, \frac{5}{2}\right)$ and t_- in $\left(1, \frac{3}{2}\right)$. We can compute d_{APS} as $d_{\mathrm{APS}}(z_{++}) = t_+$ and $d_{\mathrm{APS}}(z_{-+}) = t_-$. $\mathcal{B}\Gamma_1$ has two non-idempotent elements $\overleftarrow{e_C}$ and $\overrightarrow{e_C}$. The actions of these elements are $z_{+*}\overrightarrow{e_C} = z_{-*}$ and $t_+\overrightarrow{e_C} = t_-$, since $\overrightarrow{e}_C$ only changes the sign on the cleaved circle. On the other hand, $z_{+-}\overleftarrow{e_C} = t_-$ is the only non-trivial action for $\overleftarrow{e}_C$. Since t_+ is not in the image of the action, or d_{APS} except for z_{++} we can cancel

both to result in the homotopy equivalent structure with generators z_{-+}, z_{+-}, z_{--} and t_-.

We now cancel t_- through the image of z_{-+}. To compute the new action on z_{+-} and z_{--} we consider ι, which in this case is just inclusion, followed by $m_2(z_{*-} \otimes e)$ followed by projection. For z_{+-} there is the non-trivial action $m_2(z_{+-} \otimes \overrightarrow{e_C}) = z_{--}$, which projects to an action as well. However, while $m_2(z_{+-} \otimes \overleftarrow{e_C}) = t_-$, the projection will kill this image. Furthermore, all the higher actions vanish since m_2 acts trivially on z_{--}, and any computation of $\overline{m}_n$ for $n > 2$ starts with $H \circ m_2(z_{+-} \otimes \overleftarrow{e_C}) = z_{-+}$, but the action on z_{-+} is trivial for all non-idempotents. The idempotent will fix z_{-+}, but this will be killed under π, or H, and the computation cannot proceed. Thus, $\langle\!\langle R_I \rangle\!\rangle$ is isomorphic to $\alpha_+ = z_{+-}$ in grading $\left(0, \frac{1}{2}\right)$ and $\alpha_- = z_{--}$ in $\left(0, -\frac{1}{2}\right)$ with $d_{\mathrm{APS}} \equiv 0$ and the only non-trivial action term being $\alpha_+ \cdot \overrightarrow{e_C} = \alpha_-$. This is isomorphic to $\langle\!\langle U_2 \rangle\!\rangle$, where U_2 is the planar matching on P_2 found from untwisting the crossing.

(b) **Second move** For the RII move we analyze the tangle below, R_{II} over P_4, with two opposite crossings:



Thus $n_+ = 1$ and $n_- = 1$ for every choice of orientation. We label the crossings from top to bottom. Now consider the states corresponding to the 01–resolution. There is a free circle in this resolution, and we can divide the states into S^+_{01} and S^-_{01} based on the decoration of the circle (we do this regardless of the matching $\overrightarrow{m}$ used to construct the state). d_{APS} maps S^+_{01} isomorphically to S_{11} and S_{00} isomorphically to S^-_{01}. The action $m_2(\xi \otimes e)$ for ξ in S^+_{01} has image in S^+_{01} since it will not change the decoration on the $+$ free circle, and merging the $+$ free circle does not change the boundary of the state. Consequently if we cancel along the isomorphism from S^+_{01} to S_{11} the image of H is in S^+_{01} and the image of ι on $\nu \in S_{10}$ is a sum $\nu + \nu'$, where $\nu' \in S^+_{01}$. Thus $\pi \circ m_2 \circ (\iota \otimes \mathbb{I})$ will have image equal to the part of $m_2(\nu \otimes e)$ in S_{10}, since π will kill S^+_{01}. The only element e for which the image may not be in S_{10} is $\overleftarrow{e_\gamma}$ for the unique class of bridges γ in the boundary of any element in S_{10}. Its action would have image in S_{11}, but does not contribute to $\underline{m}_n$ for $n > 1$ since $H\colon S_{11} \to S^+_{01}$, and thus any additional actions stay in S^+_{01}, which will be killed by π.

The effect of the cancellation, therefore, is to reduce our module to $S_{10} \oplus S_{00} \oplus S^-_{01}$ with action defined by restricting the image of m_2 to the remaining summands. Now a similar argument applies to the isomorphism found by the image $d_{\mathrm{APS}}|S_{00}$ in S^-_{01}. Now, however, no element from S_{10} can have a term in its action or boundary within S_{01}, so

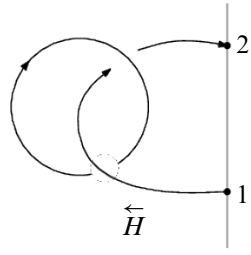
these will be unchanged. After the cancellation we obtain all the states in S_{10} having trivialized the $\overleftarrow{\gamma}$ action, but otherwise left the action unchanged. This is the same type A structure as for the matching of the top point in P_4 with the bottom, and the second with the third. Thus it is isomorphic to the structure obtained after removing the crossings with the RII move. Being in S_{10} means the states have no free circle, and just receive grading based on the cleaved circles. Furthermore, they are shifted by $(1,1)+(-1,1-2\cdot 1)=(0,0)$ when we account for the resolution and the crossings. Thus, as a bigraded type A structure the RII tangle is homotopy equivalent to the planar matching obtained from the RII move.

(c) **Third move (sketch)** Let R_b be the diagram before the move and R_a be the diagram after:

In each, if we 0–resolve the second crossing from the top we obtain a diagram with an RII move. As usual the states with a 1–resolution here give rise to the same type A structure. It is enough then to see what happens in the 0–resolved sub-module. As with the RII move we can use d_{APS} to leave S_{100} with its action intact, including the action of $e^{\leftarrow}_{\gamma_1}$ which has image in S_{110}. However, d_{APS} now maps S^+_{001} to $S_{101}\oplus S_{011}$, isomorphically to each factor, given by minus the Khovanov maps. We let ξ be the state in S_{101} then ξ' is the corresponding state, found by planar isotopy, in S_{011}. The effect of π is to identify ξ with $-\xi'$. Now, let ν be a state in S_{100} and let $\nu'=H\circ d_{\mathrm{APS}}(\nu)$ in S^+_{001}. Then $\iota(\nu)=\nu+\nu'$. The action $m_2((\nu+\nu')\otimes e^{\leftarrow}_{\gamma_2})=m_2(\nu\otimes e^{\leftarrow}_{\gamma_2})$ since the action of $e^{\leftarrow}_{\gamma_2}$ on S^+_{001} is trivial (γ_2 is used in the calculation of d_{APS} for these states). If $m_2(\nu\otimes e^{\leftarrow}_{\gamma_2})$ is non-zero in S_{101}, then the effect of π is to identify it with $-m_2(\nu\otimes e^{\leftarrow}_{\gamma_2})'$.

If we repeat this argument with R_a, with the same crossing ordering, we get S_{001} being the planar matching diagram and S^+_{100} being used in the cancellation process. For ν in S_{001} the effect of $e^{\leftarrow}_{\gamma_2}$ is the same as before, but as it takes image in S_{011} it occurs with a minus sign. On the other hand, the image of $e^{\leftarrow}_{\gamma_1}$ will be in S_{101}, occurring with a minus sign, due to the crossing ordering, and thus will be identified with $-(-\eta)$, where η is the image $m_2(\nu\otimes e^{\leftarrow}_{\gamma_1})$ from R_b in the previous paragraph. As such the actions of the bridges will be the same, and the APS-complexes will be the same. It is straightforward to see that the higher actions all vanish.

Example II (Hopf tangle) For the Hopf tangle over P_2



we will enumerate the crossings from the top of the page to the bottom, and write states with the decoration of the cleaved circle first. For the moment we will ignore orientations. There are four states $s^{00}_{\pm,\pm}$ in homological grading 0 and quantum gradings $\pm\frac{1}{2}\pm 1$. There are two states $s^{10}_{\pm}$ for the 10–resolution, and two states $s^{01}_{\pm}$ for the 01. These occur in the bigradings $\left(1, 1\pm\frac{1}{2}\right)$. Finally, there are four states $s^{11}_{\pm\pm}$ with bigrading $\left(2, 2\pm 1\pm\frac{1}{2}\right)$. For these states:

(1) d_{APS} is computed as

$$s^{00}_{++}\to s^{10}_{+}+s^{01}_{+},\quad s^{00}_{-+}\to s^{10}_{-}+s^{01}_{-},\quad s^{10}_{+}\to s^{11}_{+-},$$
$$s^{10}_{-}\to s^{10}_{--},\qquad s^{01}_{+}\to -s^{11}_{+-},\qquad s^{01}_{-}\to -s^{11}_{--}.$$

(2) The action of $\overrightarrow{e_C}$ simply takes $s^{*}_{+*}\to s^{*}_{-*}$, where $*$ matches anything in those spots.

(3) The action of $\overleftarrow{e_C}$ is given by

$$s^{00}_{+-}\to s^{10}_{-}+s^{01}_{-},\quad s^{10}_{+}\to s^{11}_{-+},\quad s^{01}_{+}\to -s^{11}_{-+}.$$

If we cancel s^{00}_{++} with s^{10}_{+} we will have no effect except to remove these generators, as s^{10}_{+} does not occur in the image of d_{APS} or the action for any other state. Once we have done that, we can cancel s^{01}_{+} with $-s^{11}_{+-}$ with no other effect, since s^{11}_{+-} only occurs in the image of a previously canceled state. s^{11}_{--} will then appear only in the image $d_{\mathrm{APS}}(s^{01}_{-})$ and $d_{\mathrm{APS}}(s^{10}_{-})$ (since s^{11}_{+-} has been canceled, otherwise we would also need to include it in the image of $\overrightarrow{e_C}$). As $d_{\mathrm{APS}}(s^{01}_{-})=-s^{11}_{--}$, we can cancel these without affecting the rest of the maps. Finally we can cancel s^{00}_{-+} with s^{10}_{-}. Now, s^{10}_{-} occurs as $s^{00}_{+-}\overleftarrow{e_C}$, but there are no other terms to consider, so the effect of the cancellation (through the projection π) is to cancel this portion of the action of $\overleftarrow{e}_C$.

Following these steps results in s^{00}_{+-} and s^{00}_{--} in bigrading $\left(0,-\frac{1}{2}\right)$ and $\left(0,-\frac{3}{2}\right)$, and s^{11}_{++} in bigrading $\left(2,\frac{7}{2}\right)$ and s^{-+} in bigrading $\left(2,\frac{5}{2}\right)$. The residual action is that of $\overrightarrow{e_C}$, which takes s^{00}_{+-} to s^{00}_{--} and s^{11}_{++} to s^{11}_{-+}.

The Hopf tangle will either have two positive or two negative crossings, depending upon the orientation of the components. If there are two positive crossings we will shift the bigrading up $(0,2)$. Otherwise, for negative crossings, we add $(-2,-4)$ to each bigrading.

Consequently, for the *positive Hopf tangles* we will have

$$\mathbb{F}_{(0,3/2)} \xrightarrow{\overrightarrow{e_C}} \mathbb{F}_{(0,1/2)} \quad \text{and} \quad \mathbb{F}_{(2,11/2)} \xrightarrow{\overrightarrow{e_C}} \mathbb{F}_{(2,9/2)}.$$

For *negative* Hopf tangles we will have

$$\mathbb{F}_{(-2,-9/2)} \xrightarrow{\overrightarrow{e_C}} \mathbb{F}_{(-2,-11/2)} \quad \text{and} \quad \mathbb{F}_{(0,-1/2)} \xrightarrow{\overrightarrow{e_C}} \mathbb{F}_{(0,-3/2)}.$$

7 Gluing inside and outside tangles

Let $\overleftarrow{\mathcal{T}_1}$ be an inside tangle for P_{2n} and $\overrightarrow{\mathcal{T}_2}$ be an outside tangle. We let $\mathcal{T} = \overleftarrow{\mathcal{T}_1} \# \overrightarrow{\mathcal{T}_2}$ be the link in $\mathbb{R}^3$ obtained by gluing $\overleftarrow{\mathbb{R}^3}$ to $\overrightarrow{\mathbb{R}^3}$ and thereby gluing $\overleftarrow{\mathcal{T}_1}$ to $\overrightarrow{\mathcal{T}_2}$ along P_{2n}. Likewise, if $\overleftarrow{T_1}$ is a diagram for $\overleftarrow{\mathcal{T}_1}$ in $\overleftarrow{\mathbb{H}}$ and $\overrightarrow{T_2}$ is a diagram for $\overrightarrow{\mathcal{T}_2}$ in $\overrightarrow{\mathbb{H}}$, we can glue these diagrams along P_{2n} to obtain a diagram T for $\mathcal{T}$.

In [9], we showed how to associate a bigraded type D structure to $\overrightarrow{T_2}$ whose homotopy type is an isotopy invariant of $\overrightarrow{\mathcal{T}_2}$. In particular, we constructed a bigrading preserving map

$$\overrightarrow{\delta}_T \colon [\![\overrightarrow{T}]\!\rangle \to \mathcal{B}\Gamma_n \otimes_{\mathcal{I}} [\![\overrightarrow{T}]\!\rangle[(-1,0)]$$

which satisfies the type D structure equation

$$(\mu_{\mathcal{B}\Gamma_n} \otimes \mathbb{I})(\mathbb{I} \otimes \overrightarrow{\delta}_T)\overrightarrow{\delta}_T + (d_{\Gamma_n} \otimes |\mathbb{I}|)\overrightarrow{\delta}_T = 0,$$

where $\mu_{\mathcal{B}\Gamma_n} \colon \mathcal{B}\Gamma_n \otimes \mathcal{B}\Gamma_n \to \mathcal{B}\Gamma_n$ is the multiplication map on $\mathcal{B}\Gamma_n$.

Definition 7.1 By $\langle\!\langle T_1]\!] \boxtimes [\![T_2]\!\rangle\!\rangle$ we mean the bigraded module

$$\langle\!\langle T_1]\!] \otimes_{\mathcal{I}_n} [\![T_2]\!\rangle\!\rangle$$

equipped with the map

$$\partial^{\boxtimes}(x \otimes y) = d_{\mathrm{APS}}(x) \otimes |y| + (m_{2,T_1} \otimes \mathbb{I})(x \otimes \overrightarrow{\delta}_{T_2}(y)).$$

Proposition 7.2 *$\partial^{\boxtimes}$ is a $(1,0)$–differential map on $\langle\!\langle T_1]\!] \boxtimes [\![T_2]\!\rangle\!\rangle$.*

Proof First, we rewrite $\partial^{\boxtimes}$ as an operator:

$$\partial^{\boxtimes} = d_{\mathrm{APS}} \otimes |\mathbb{I}| + (m_{2,T_1} \otimes \mathbb{I})(\mathbb{I} \otimes \vec{\delta}_{T_2}).$$

We note that $\mathbb{I} \otimes \vec{\delta}_{T_2}$ is a $(1,0)$–map $\langle\!\langle T_1]\!] \otimes_{\mathcal{I}} [\![T_2 \rangle\!\rangle \to \langle\!\langle T_1]\!] \otimes_{\mathcal{I}} \mathcal{B}\Gamma_n \otimes_{\mathcal{I}} [\![T_2 \rangle\!\rangle$, while $m_{2,T_1} \otimes \mathbb{I}$ preserves the bigrading as a map $\langle\!\langle T_1]\!] \otimes_{\mathcal{I}} \mathcal{B}\Gamma_n \otimes_{\mathcal{I}} [\![T_2 \rangle\!\rangle \to \langle\!\langle T_1]\!] \otimes_{\mathcal{I}} [\![T_2 \rangle\!\rangle$. In addition, d_{APS} is a $(1,0)$–map. Hence $\partial^{\boxtimes}$ is a $(1,0)$–map. We need to verify that $\partial^{\boxtimes}$ is a differential. This result follows from Section A.9 in the Appendix, and that when $m_{i,T_1} = 0$ for $i > 2$, $m_{1,T_1} = d_{\mathrm{APS}}$, implies that $\partial^{\boxtimes}$ above coincides with the definition in the Appendix. □

By $\langle\!\langle T \rangle\!\rangle$ we will mean the usual bigraded Khovanov complex over $\mathbb{Z}$, equipped with its invariant bigrading.

Proposition 7.3 $\langle\!\langle T \rangle\!\rangle \cong (\langle\!\langle T_1]\!] \boxtimes [\![T_2 \rangle\!\rangle, \partial^{\boxtimes})$.

Important comment We have not required that the orientations on T_1 and T_2 match along P_n. If they do, $\langle\!\langle T \rangle\!\rangle$ is exactly the Khovanov complex from [6], as described in [3]. However, the statement still holds even if the orientations do not match. The Khovanov complex in the latter case is for a link with a finite number of orientation changes, constructed in the same manner as before. Now, however, it has an invariant bigrading only as long as isotopies do not take a strand across a point where the orientation changes. In the latter case there is a bigrading shift of $\pm(1,3)$ due to the conversion of a negative crossing to a positive crossing, or vice versa.

Proof We start by identifying the generators of $\langle\!\langle T_1]\!] \otimes_{\mathcal{I}} [\![T_2 \rangle\!\rangle$ with the generators of $\langle\!\langle T \rangle\!\rangle$. For $(r_1, s_1) \otimes_{\mathcal{I}} (r_2, s_2) \neq 0$ we need that $I_{\partial(r_1,s_1)} \cdot (r_2, s_2) \neq 0$ since $(r_1, s_1) \cdot I_{\partial(r_1,s_1)} = (r_1, s_1)$. However, only $I_{\partial(r_2,s_2)} \cdot (r_2, s_2) \neq 0$, so $\partial(r_1, s_1) = \partial(s_2, r_2) = (L, \sigma)$. If $r_1 = (\rho_1, \vec{m}_1)$ and $r_2 = (\overleftarrow{m}_2, \rho_2)$, we use $\vec{m}_1 = \overleftarrow{L}$ to identify $\vec{m}_1$ with the *arcs* in $\rho_2(T_2)$, and likewise we can identify $\overleftarrow{m}_2 = \vec{L}$ with the arcs in $\rho_1(T_1)$. Since s_1 and s_2 restrict to σ, we can decorate $\rho_1(T_1) \# \rho_2(T_2)$ by $s_1 \# s_2$ to get a resolution diagram for T where every circle is unambiguously decorated with $\pm$.

Furthermore, we can reverse the construction. If ρ is a resolution of T, we let ρ_1 be ρ restricted to those crossings in $\overleftarrow{\mathbb{H}} \cap T = T_1$ and ρ_2 be ρ restricted to $\vec{\mathbb{H}} \cap T = T_2$. Furthermore, the arcs in $\rho_2(T_2)$ form an (outside) planar matching $\vec{m}_1$, and we define $r_1 = (\rho_1, \vec{m}_1)$. Likewise the arcs of $\rho_1(T_1)$ define an (inside) planar matching $\overleftarrow{m}_2$ and we let $r_2 = (\overleftarrow{m}_2, \rho_2)$. A generator of $\langle\!\langle T \rangle\!\rangle$ is a pair (ρ, s), where s is a decoration of $\mathrm{CIR}(\rho(T))$. By restriction s defines decorations, s_1, s_2 on $r_1(T_1)$ and $r_2(T_2)$ with $\partial(r_1, s_1) = \partial(r_2, s_2)$. It is straightforward to see that $(r_1, s_1) \otimes_{\mathcal{I}} (r_2, s_2) = (\rho, s)$, so this is the inverse of the previous map.

Furthermore, the bigrading of $(r_1,s_1)\otimes_{\mathcal{I}}(r_2,s_2)$ coming from the tensor product is identical to that of (ρ,s) from the construction of $\langle\!\langle T\rangle\!\rangle$. The bigrading of $(r_1,s_1)\otimes_{\mathcal{I}}(r_2,s_2)$ is the sum

$$\begin{aligned}h(r_1)-n_-(T_1),h(r_1)+q(r_1,s_1)+\tfrac{1}{2}\iota(\partial(r_1,s_1))+n_+(T_1)-2n_-(T_1)\\ +h(r_2)-n_-(T_2),h(r_2)+q(r_2,s_2)+\tfrac{1}{2}\iota(\partial(r_2,s_2))+n_+(T_2)-2n_-(T_2).\end{aligned}$$

However $h(r_1)+h(r_2)$ is the number of 1–resolutions in $\rho_1(T_1)$ added to the number in $\rho_2(T_2)$, which equals the total number in $\rho(T)$. Likewise, since they are counts over crossings, $n_+(T_1)+n_+(T_2)=n_+(T)$ and $n_-(T_1)+n_-(T_2)=n_-(T)$. Finally, $\iota(\partial(r_1,s_1))+\iota(\partial(r_2,s_2))=2\iota(L,\sigma)$ so the second entry in the bigrading equals the sum of the decorations on the free circles in $r_1(T_1)$ plus the sum of the decorations on the circles in (L,σ) plus the sum of the decorations on the free circles in $r_2(T_2)$. In $\rho(T)$ this is just the quantum grading for the usual Khovanov generator. Thus the bigrading of $(r_1,s_1)\otimes_{\mathcal{I}}(r_2,s_2)$ is $(h(\rho)-n_-(T),h(\rho)+q(\rho,s)+n_+(T)-2n_-(T))$ which is the bigrading of (r,s) in $\langle\!\langle T\rangle\!\rangle$. The tensor product identifies $\langle\!\langle T\rangle\!\rangle$ with $\langle\!\langle T_1\rrbracket\otimes_{\mathcal{I}}\llbracket T_2\rangle\!\rangle$ as bigraded modules over $\mathbb{Z}$.

To see that $\partial^{\boxtimes}$ is the $(1,0)$–Khovanov differential ∂_{KH} under this isomorphism, we must first specify the order of the crossings to be used in calculating the signs in ∂_{KH}. The chain isomorphism type of $\langle\!\langle T\rangle\!\rangle$ is unaffected by this choice of ordering [6]. If $\mathfrak{o}_i$ is the ordering of the crossings in T_i and, then $\mathfrak{o}_1||\mathfrak{o}_2$ is an ordering of the crossings for T, which we now fix. In short, all the crossings of the inside tangle T_1 come before all the crossings of T_2, and in the same order as in T_1.

We compute $\partial^{\boxtimes}$ in stages. First $(d_{\mathrm{APS}}\otimes|\mathbb{I}|)[(r_1,s_1)\otimes_{\mathcal{I}}(r_2,s_2)]$ is a sum over the crossings of T_1. For each crossing c we get either 0 or $(-1)^m(-1)^{h(r_2)}(r',s')\otimes_{\mathcal{I}}(r_2,s_2)$, where m is the number of 1–resolutions in (r_1,s_1) following c, $h(r_2)$ is the total number of 1–resolutions in r_2, and (r',s') is as specified previously, which has $\partial(r',s')=\partial(r_1,s_1)$. Consequently, $m+h(r_2)$ is the number of 1–resolutions of T following c in our fixed order, and $(r',s')\otimes_{\mathcal{I}}(r_2,s_2)$ is a generator of $\langle\!\langle T\rangle\!\rangle$. Following the definition of d_{APS} this is precisely a term in $\partial_{\mathrm{KH}}(r,s)$. In fact, the sum of these is precisely the terms in $\partial_{\mathrm{KH}}(r,s)$ which have the same decorated cleaved link, and occur from a crossing change in $T\cap\overleftarrow{\mathbb{H}}$. Those terms in $\partial_{\mathrm{KH}}(r,s)$ which have the same decorated cleaved link, and occur from a crossing change in $\overrightarrow{\mathbb{H}}\cap T$ correspond to terms in $(m_{2,T_1}\otimes\mathbb{I})(\mathbb{I}\otimes\overrightarrow{\delta}_{T_2})$ applied to $(r_1,s_1)\otimes_{\mathcal{I}}(r_2,s_2)$. In the definition of $\overrightarrow{\delta}_{T_2}(r_2,s_2)$ there is a term $I_{\partial(r_2,s_2)}\otimes d_{\mathrm{APS}}(r_2,s_2)$. Since $\partial(r_2,s_2)=\partial(r_1,s_1)$ we conclude that $(m_{2,T_1}\otimes\mathbb{I})((r_1,s_1)\otimes I_{\partial(r_2,s_2)}\otimes d_{\mathrm{APS}}(r_2,s_2))=(r_1,s_1\otimes_{\mathcal{I}}d_{\mathrm{APS}}(r_2,s_2)$. Note that the signs are also correct for ∂_{KH} since the sign of a term in $I_{\partial(r_2,s_2)}\otimes d_{\mathrm{APS}}(r_2,s_2)$ is $(-1)^m$, where m is the number of 1–resolutions in r_2 following the crossing that yields the term. As this crossing follows all those of T_1, the same sign is used in ∂_{KH}.

This leaves the terms of $\partial_{\mathrm{KH}}[(r_1,s_1)\otimes_{\mathcal{I}}(r_2,s_2)]$ which change the decorated, cleaved link. We divide them into two groups, based on whether the crossing change giving the term occurs in $\overleftarrow{\mathbb{H}}$ or $\overrightarrow{\mathbb{H}}$. We start with those in $\overrightarrow{\mathbb{H}}$. Each such crossing gives an active arc which is either in $\overrightarrow{\mathrm{BR}}(r_2)$ or $\mathrm{DEC}(r_2,s_2)$. In the first case, $\overrightarrow{\delta}_{T_2}(r_2,s_2)$ will have a term, or two terms, $(-1)^m(\overrightarrow{e}_\gamma\otimes(r',s'))$ which corresponds to the crossing change. As before, m is the number of 1–resolved crossings following the crossing. This is the same sign as in ∂_{KH}, and the decorations on the decorated, cleaved link also follow the pattern for Khovanov homology.

In $(m_{2,T_1}\otimes\mathbb{I})(\mathbb{I}\otimes\overrightarrow{\delta}_{T_2})$ we get the term $(-1)^m(m_{2,T_1}\otimes\mathbb{I})[(r_1,s_1)\otimes\overrightarrow{e}_\gamma\otimes(r',s')]$. From the definition of m_{2,T_1} the action of $\overrightarrow{e}_\gamma$ on $\langle\!\langle T_1\rangle\!\rangle$ is just to change the decorated, cleaved link to have the same boundary as (r',s') (which occurs purely in $\overrightarrow{m}_1$). Consequently, we obtain the tensor product of compatible pairs, and we replicate the term in ∂_{KH}. The case of an arc in $\mathrm{DEC}(r_2,s_2)$ is similar, except only the decoration on one circle changes, and not the underlying cleaved link. This is the effect of $\overrightarrow{e_C}$, for that circle, on $\langle\!\langle T_1\rangle\!\rangle$.

This leaves the terms of ∂_{KH} which come from crossing changes in $\overleftarrow{\mathbb{H}}$ that change the decorated, cleaved link. Let c be such a crossing, and γ be the active arc. Suppose γ has image in $\overleftarrow{\mathrm{BR}}(\partial(r_1,s_1))$, which we will denote by γ'. There is then a term in $\overrightarrow{\delta}_{T_2}(r_2,s_2)$ of the form $(-1)^{h(r_2)}(\overleftarrow{e_{\gamma'}}\otimes(r_2',s_2'))$, where (r_2',s_2') is the result of γ' surgery on $r(T_2)\cap\overleftarrow{\mathbb{H}}$ which reflects the decoration changes necessary for the Khovanov differential. In $(-1)^{h(r_2)}(m_{2,T_1}\otimes I)((r_1,s_1)\otimes\overleftarrow{e_{\gamma'}}\otimes(r_2',s_2'))$ we get a sum over all the terms in ∂_{KH} which correspond to γ' and the decoration changes for $\overleftarrow{e_{\gamma'}}$, but with sign $(-1)^{h(r_2)}(-1)^m$, where m is the number of 1–resolved crossings following that for γ (not γ') in ordering on the crossings of T_1. However, $h(r_2)+m$ is the number of 1–resolved crossings following that for γ in the ordering on T. Thus, the sign is the same as that for ∂_{KH}. If $\gamma\in\mathrm{DEC}(r_1,s_1)$ then the argument is the same except that the term in ∂_{KH} comes from the action of $(-1)^{h(r_2)}(\overleftarrow{e_C}\otimes(r_2,s_{2,C}))$, where C is the cleaved circle whose decoration changes. Note that a crossing change can occur in multiple terms, but that with the decoration changes included, each crossing and decoration change occurs in precisely one way above. Thus we recover all the terms of $\partial_{\mathrm{KH}}(r,s)$ with the correct signs from the ordering of crossings. □

The advantage of using $\langle\!\langle T_1\rangle\!\rangle\boxtimes[\![T_2]\!]$ arises from the ability to separately simplify $\langle\!\langle T_1\rangle\!\rangle$ and $[\![T_2]\!]$ without changing the homotopy type of $\langle\!\langle T_1\rangle\!\rangle\boxtimes[\![T_2]\!]$. We show this in the Appendix through a series of propositions which replicate, for our sign conventions, results from [8]. In particular, Proposition A.43 and the corollary to A.45 imply the following result.

Proposition 7.4 *Suppose (N,δ) is homotopy equivalent, as a type D structure over $\mathcal{B}\Gamma_n$, to $[\![T_2]\!]$, and $(M,\{m_i\})$ is homotopy equivalent to $\langle\!\langle T_1\rangle\!\rangle$, as a type A structure. Then $(M,\{m_i\})\boxtimes(N,\delta)\simeq\langle\!\langle T_1\rangle\!\rangle\boxtimes[\![T_2]\!]\simeq\langle\!\langle T_1\#T_2\rangle\!\rangle$.*

In the preceding proposition, we assume that the homotopy equivalences preserve the quantum grading.

We have seen in Section 5 how to affect such a homotopy equivalence by simplifying the chain complex $(\langle\!\langle T_1\rangle\!\rangle, d_{\mathrm{APS}})$. A similar result holds for the type D structure on $\langle\!\langle T_2\rangle\!\rangle$: simplifications of the chain complex with differential d_{APS} results in a homotopy equivalent type D structure on the simplified complex. Over a field, $\mathbb{F}$, such simplifications show that $(\langle\!\langle T_1\rangle\!\rangle\otimes\mathbb{F}, d_{\mathrm{APS}})\simeq H_{*,\mathbb{F}}(\langle\!\langle T_1\rangle\!\rangle)$, where the homology is taken with respect to d_1 and similarly for $([\![T_2]\!]\otimes\mathbb{F}, d_{\mathrm{APS}})$. These homologies are determined by the tangle homology of Asaeda, Przytycki, and Sikora. The following corollary is a consequence.

Corollary 7.5 *There is a type A structure on $H_{*,\mathbb{F}}(\langle\!\langle T_1\rangle\!\rangle)$ and a type D structure on $H_{*,\mathbb{F}}([\![T_2]\!])$ for which*

$$\langle\!\langle T\rangle\!\rangle\simeq H_{*,\mathbb{F}}(\langle\!\langle T_1\rangle\!\rangle)\boxtimes H_{*,\mathbb{F}}([\![T_2]\!]).$$

For example, this result applies to the rational coefficient theory and the theory over $\mathbb{Z}/2\mathbb{Z}$.

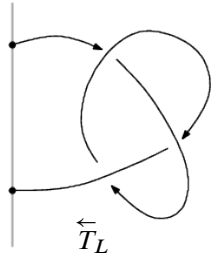
8 Examples of pairing type A and type D structures

Example I (Reidemeister invariance of Khovanov homology) Suppose that L and L' are two link diagrams for an oriented link in S^3. Furthermore, suppose they differ by a Reidemeister move. If $D^2\subset\mathbb{R}^2$ is the local region in which the move occurs, we can use ∂D^2 and the orientation on $\mathbb{R}^2$ to think of $R=L\cap D^2$ as the inside tangle, and $\vec{L}=L\cap(\mathbb{R}^2\setminus D)$ as the outside tangle. Then $\langle\!\langle L\rangle\!\rangle\cong\langle\!\langle R\rangle\!\rangle\boxtimes[\![\vec{L}]\!]$. If we let $R'=L'\cap D^2$ then $\langle\!\langle L'\rangle\!\rangle\cong\langle\!\langle R'\rangle\!\rangle\boxtimes[\![\vec{L}]\!]$. In Section 6 we computed the type A structure for three of the tangles involved in the Reidemeister moves. In each, we saw that the type A structure was homotopy equivalent to the structure obtained for the tangle after applying the Reidemeister move. Due to the results in Section A.9, this implies that

$$\langle\!\langle L\rangle\!\rangle\cong\langle\!\langle R\rangle\!\rangle\boxtimes[\![\vec{L}]\!]\simeq\langle\!\langle R'\rangle\!\rangle\boxtimes[\![\vec{L}]\!]\cong\langle\!\langle L'\rangle\!\rangle.$$

This gives a new perspective on the locality arguments for invariance in various forms of Khovanov homology.

Example II In [9] we computed the type D structure $[\![\vec{T}_L]\!]$ for the following tangle $\vec{T}_L$, based on the left handed trefoil T_L:



This structure is the map $\vec{\delta}$, where

$$\begin{aligned}
\vec{\delta}\left(s^+_{(-3,-15/2)}\right) &= 2\vec{e_C} \otimes s^-_{(-2,-13/2)} + \overleftarrow{e_C} \otimes s^-_{(-3,-17/2)},\\
\vec{\delta}\left(s^+_{(-2,-11/2)}\right) &= -\overleftarrow{e_C} \otimes s^-_{(-2,-13/2)},\\
\vec{\delta}\left(s^+_{(0,-3/2)}\right) &= -\overleftarrow{e_C} \otimes s^-_{(0,-5/2)},
\end{aligned} \tag{38}$$

where the superscript indicates the decoration on the cleaved circle C in the corresponding resolutions, and the subscript is the bigrading. We use the pairing theorem to compute several connected sums.

(i) With the unknot We can think of the unknot U as a cleaved circle on P_2, where $\overleftarrow{U} = U \cap \overleftarrow{\mathbb{H}}$ and $\vec{U} = U \cap \vec{\mathbb{H}}$ are the unique planar matchings. Then the left handed trefoil is the connected sum $U \# T_L$ which we can think of as gluing $\overleftarrow{U}$ with the tangle above. The type A structure $\langle\!\langle \overleftarrow{U}]\!]$ is isomorphic to $\mathbb{Z} f_{(0,1/2)} \oplus \mathbb{Z} f_{(0,-1/2)}$, which is the idempotent decomposition for I_{C+} and I_{C-} (see the example in Section 2). The action of $\overleftarrow{e_C}$ is trivial since there are no crossings in the standard diagram. On the other hand $\vec{e_C}$ takes $f_{(0,1/2)}$ to $f_{(0,-1/2)}$. Since $d_{\mathrm{APS}} \equiv 0$ for $\overleftarrow{U}$, we need only compute $(m_2 \otimes \mathbb{I})(\mathbb{I} \otimes \vec{\delta})$. Using the idempotents we see that there are six generators. Furthermore, the only terms in $(m_2 \otimes \mathbb{I})(\mathbb{I} \otimes \vec{\delta})$ come from $\vec{e_C}$. This gives the following chain complex for $\langle\!\langle \overleftarrow{U}]\!] \boxtimes [\![\vec{T}_L]\!]$:

$$\begin{aligned}
f_{(0,1/2)} \otimes s^+_{(-3,-15/2)} &\xrightarrow{\cdot 2} f_{(0,-1/2)} \otimes s^-_{(-2,-13/2)},\\
f_{(0,1/2)} \otimes s^+_{(-2,-11/2)} &\to 0,\\
f_{(0,1/2)} \otimes s^+_{(0,-3/2)} &\to 0,\\
f_{(0,-1/2)} \otimes s^-_{(0,-5/2)} &\to 0,\\
f_{(0,-1/2)} \otimes s^-_{(-3,-17/2)} &\to 0,
\end{aligned}$$

whose homology consists of a $\mathbb{Z}$–summand in bigradings $(-2,-5)$, $(0,-1)$, $(0,-3)$, $(-3,-9)$ and a $\mathbb{Z}/2\mathbb{Z}$ summand in bigrading $(-2,-7)$. This is the Khovanov homology of the left handed trefoil.

(ii) With the positive Hopf tangle in Section 6 To compute the Khovanov homology of the connected sum of the left handed trefoil with the Hopf link with $+1$ linking number, recall that the Hopf tangle with positive crossings has $\langle\!\langle H]\!]$ with the following action:

$$r_{(0,3/2)} \xrightarrow{\vec{e_C}} r_{(0,1/2)} \quad \text{and} \quad r_{(2,11/2)} \xrightarrow{\vec{e_C}} r_{(2,9/2)},$$

where the first entry in each corresponds to the $+$ decoration. As a consequence, $\langle\!\langle H]\!] \otimes_{\mathcal{I}} [\![T_L]\!\rangle$ has the following generators:

$$\begin{aligned}
&r_{(0,3/2)} \otimes s^+_{(-3,-15/2)}, \quad r_{(2,11/2)} \otimes s^+_{(-3,-15/2)}, \quad r_{(0,3/2)} \otimes s^+_{(-2,-11/2)},\\
&r_{(2,11/2)} \otimes s^+_{(-2,-11/2)}, \quad r_{(0,3/2)} \otimes s^+_{(0,-3/2)}, \quad r_{(2,11/2)} \otimes s^+_{(0,-3/2)},\\
&r_{(0,1/2)} \otimes s^-_{(-2,-13/2)}, \quad r_{(2,9/2)} \otimes s^-_{(-2,-13/2)}, \quad r_{(0,1/2)} \otimes s^-_{(-3,-17/2)},\\
&r_{(2,9/2)} \otimes s^-_{(-3,-17/2)}, \quad r_{(0,1/2)} \otimes s^-_{(0,-5/2)}, \quad r_{(2,9/2)} \otimes s^-_{(0,-5/2)}.
\end{aligned}$$

Since $d_{\mathrm{APS}} \equiv 0$ after the simplification, we need only compute $(m_2 \otimes \mathbb{I})(\mathbb{I} \otimes \vec{\delta})$ on these twelve generators. Since the action m_2 is trivial except for on $\vec{e_C}$, we can ignore all the terms in $\vec{\delta}$ except for those with $\vec{e_C}$. This leaves the following as the only non-trivial maps in $\partial^{\boxtimes}$:

$$\begin{aligned}
r_{(0,3/2)} \otimes s^+_{(-3,-15/2)} &\to 2\big(r_{(0,1/2)} \otimes s^-_{(-2,-13/2)}\big),\\
r_{(2,11/2)} \otimes s^+_{(-3,-15/2)} &\to 2\big(r_{(2,9/2)} \otimes s^-_{(-2,-13/2)}\big).
\end{aligned}$$

Taking the homology of this new complex gives two $\mathbb{Z}/2\mathbb{Z}$ summands in bigradings $(-2,-6)$ and $(0,-2)$, and eight $\mathbb{Z}$ summands for the remaining generators in the corresponding bigrading:

$$\begin{aligned}
&r_{(0,3/2)} \otimes s^+_{(-2,-11/2)} \to (-2,-4), \quad r_{(2,11/2)} \otimes s^+_{(-2,-11/2)} \to (0,0),\\
&r_{(0,3/2)} \otimes s^+_{(0\ ,\ -3/2)} \to (0,0), \quad r_{(2,11/2)} \otimes s^+_{(0\ ,\ -3/2)} \to (2,4),\\
&r_{(0,1/2)} \otimes s^-_{(-3,-17/2)} \to (-3,-8), \quad r_{(2\ ,\ 9/2)} \otimes s^-_{(-3,-17/2)} \to (-1,-4),\\
&r_{(0,1/2)} \otimes s^-_{(0\ ,\ -5/2)} \to (0,-2), \quad r_{(2\ ,\ 9/2)} \otimes s^-_{(0,-5/2)} \to (2,2),
\end{aligned}$$

which is isomorphic to the Khovanov homology of the connected sum of the positive Hopf link with the left handed trefoil.

(iii) With a right-handed trefoil We give some details on the computation in the introduction. We consider the tangle $\overleftarrow{T}_R$ found by removing an arc from the right-handed trefoil. We can compute $\langle\!\langle \overleftarrow{T}_R]\!]$ directly. The result is a bigraded module spanned

by $t^+_{(0,5/2)}$, $t^+_{(2,13/2)}$, $t^+_{(3,17/2)}$, $t^-_{(0,3/2)}$, $t^-_{(2,11/2)}$, $t^-_{(3,15/2)}$, where the superscript identifies the corresponding idempotent. The action of $\vec{e}_C$ is given by $t^+_{(0,5/2)} \to t^-_{(0,3/2)}$, $t^+_{(2,13/2)} \to t^-_{(2,11/2)}$, $t^+_{(3,17/2)} \to t^-_{(3,15/2)}$. The action of $\overleftarrow{e}_C$ is $t^+_{(2,13/2)} \to 2t^-_{(3,15/2)}$. Consequently, the module $\langle\!\langle \overleftarrow{T_R}\rangle\!\rangle \otimes_{\mathcal{I}} [\![\vec{T}_L]\!]\rangle$ has eighteen generators. Those, along with their images under $\partial^{\boxtimes}$ are shown in the following list:

$$
\begin{aligned}
t^+_{(0,5/2)} \otimes s^+_{(-3,-15/2)} &\to \quad 2t^-_{(0,3/2)} \otimes s^-_{(-2,-13/2)},\\
t^+_{(0,5/2)} \otimes s^+_{(-2,-11/2)} &\to 0,\\
t^+_{(0,5/2)} \otimes s^+_{(\ 0,\ -3/2)} &\to 0,\\
t^+_{(2,13/2)} \otimes s^+_{(-3,-15/2)} &\to \quad 2t^-_{(2,11/2)} \otimes s^-_{(-2,-13/2)} + 2t^-_{(3,15/2)} \otimes s^-_{(-3,-17/2)},\\
t^+_{(2,13/2)} \otimes s^+_{(-2,-11/2)} &\to -2t^-_{(3,15/2)} \otimes s^-_{(-2,-13/2)},\\
t^+_{(2,13/2)} \otimes s^+_{(\ 0,\ -3/2)} &\to -2t^-_{(3,15/2)} \otimes s^-_{(0,-5/2)},\\
t^+_{(3,17/2)} \otimes s^+_{(-3,-15/2)} &\to \quad 2t^-_{(3,15/2)} \otimes s^-_{(-2,-13/2)},\\
t^+_{(3,17/2)} \otimes s^+_{(-2,-11/2)} &\to 0, \quad t^+_{(3,17/2)} \otimes s^+_{(\ 0,\ -3/2)} \to 0,\\
t^-_{(0,3/2)} \otimes s^-_{(-3,-17/2)} &\to 0, \quad t^-_{(0,3/2)} \otimes s^-_{(-2,-13/2)} \to 0, \quad t^-_{(0,3/2)} \otimes s^-_{(0,-5/2)} \to 0,\\
t^-_{(2,11/2)} \otimes s^-_{(-3,-17/2)} &\to 0, \quad t^-_{(2,11/2)} \otimes s^-_{(-2,-13/2)} \to 0, \quad t^-_{(2,11/2)} \otimes s^-_{(0,-5/2)} \to 0,\\
t^-_{(3,15/2)} \otimes s^-_{(-3,-17/2)} &\to 0, \quad t^-_{(3,15/2)} \otimes s^-_{(-2,-13/2)} \to 0, \quad t^-_{(3,15/2)} \otimes s^-_{(0,-5/2)} \to 0.
\end{aligned}
$$

From this we see immediately that there are $\mathbb{Z}$ summands for each of

- $t^+_{(0,5/2)} \otimes s^+_{(-2,-11/2)}$ in bigrading $(-2,-3)$,
- $t^+_{(0,5/2)} \otimes s^+_{(0,-3/2)}$ in $(0,1)$,
- $t^+_{(3,17/2)} \otimes s^+_{(-2,-11/2)}$ in $(1,3)$,
- $t^+_{(3,17/2)} \otimes s^+_{(0,-3/2)}$ in $(3,7)$,
- $t^-_{(0,3/2)} \otimes s^-_{(-3,-17/2)}$ in $(-3,-7)$,
- $t^-_{(0,3/2)} \otimes s^-_{(0,-5/2)}$ in $(0,-1)$,
- $t^-_{(2,11/2)} \otimes s^-_{(-3,-17/2)}$ in $(-1,-3)$, and
- $t^-_{(2,11/2)} \otimes s^-_{(0,-5/2)}$ in $(2,3)$.

The remaining generators occur in the non-zero rows for $\partial^{\boxtimes}$. We will have a $\mathbb{Z}/2\mathbb{Z}-$summand for $t^-_{(3,15/2)} \otimes s^-_{(0,-5/2)}$ in $(3,5)$ and for $t^-_{(0,3/2)} \otimes s^-_{(-2,-13/2)}$ in $(-2,-5)$.

The equality

$$\partial^{\boxtimes}\big(t^{+}_{(2,13/2)}\otimes s^{+}_{(-3,-15/2)}\big)=2t^{-}_{(2,11/2)}\otimes s^{-}_{(-2,-13/2)}+2t^{-}_{(3,15/2)}\otimes s^{-}_{(-3,-17/2)}$$

gives a $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ in $(0,-1)$. In addition, that

$$\partial^{\boxtimes}\big(t^{+}_{(2,13/2)}\otimes s^{+}_{(-2,-11/2)}\big)=-2t^{-}_{(3,15/2)}\otimes s^{-}_{(-2,-13/2)}\quad\text{and}$$
$$\partial^{\boxtimes}\big(t^{+}_{(3,17/2)}\otimes s^{+}_{(-3,-15/2)}\big)=2t^{-}_{(3,15/2)}\otimes s^{-}_{(-2,-13/2)}$$

means that $t^{+}_{(2,13/2)}\otimes s^{+}_{(-2,-11/2)}+t^{+}_{(3,17/2)}\otimes s^{+}_{(-3,-15/2)}$ generates a $\mathbb{Z}$ summand in homology in bigrading $(0,1)$, while $t^{-}_{(3,15/2)}\otimes s^{-}_{(-2,-13/2)}$ generates a $\mathbb{Z}/2\mathbb{Z}$ summand in $(1,1)$.

Consequently, the Khovanov homology of this connected sum has free part

$$\mathbb{Z}_{(-3,7)}\oplus\mathbb{Z}_{(-2,-3)}\oplus\mathbb{Z}_{(-1,-3)}\oplus\mathbb{Z}^2_{(0,-1)}\oplus\mathbb{Z}^2_{(0,1)}\oplus\mathbb{Z}_{(1,3)}\oplus\mathbb{Z}_{(2,3)}\oplus\mathbb{Z}_{(3,7)}$$

while the torsion part is

$$(\mathbb{Z}/2)_{(-2,-5)}\oplus(\mathbb{Z}/2)_{(0,-1)}\oplus(\mathbb{Z}/2)_{(1,1)}\oplus(\mathbb{Z}/2)_{(3,5)}.$$

This agrees with the Khovanov homology of the knot as computed by Bar-Natan and Green's JavaKh program [4]. Note that we have correctly computed the torsion terms. Furthermore, we provide a modular approach: we can simplify $[\![\vec{T}]\!]$ using homotopy equivalences before knowing with which type D structure it will pair.

Appendix: Graded modules, type A structures, conventions

Summary of the appendix In this appendix, we record the sign conventions used for type A (A_∞) algebras and modules, review type D structures, and prove the algebraic results about the pairing of these structures. This gives a characteristic-zero account of the algebra in bordered Floer homology [8], where it is stated only in characteristic two. We start by reviewing basics about graded modules and tensor algebras. We then define a category which will allow us to show that the sign conventions we need for this paper will also work with the algebra from [8]. The approach is already present in [8], although not with signs and not as explicitly as we will use it. An object of this category, equipped with a certain differential captures the structural equations for type A structures with the correct sign conventions for this paper. We then describe type D structures in the same language, and show that this description captures the sign conventions use in [9]. We also show that type D structures with these conventions still form an A_∞–category. Finally, we define the pairing of our

type A and type D structures, and prove that the pairing is appropriately functorial for homotopy equivalences in the two factors.

Let M be a $\mathbb{Z}$–graded module over a ring R and let M_i be the module of elements in grading $i \in \mathbb{Z}$. For a homogeneous element $m \in M$, $|m|$ will denote the grading of m: if $m \in M_i$ then $|m| = i$.

A module map $f\colon M \to M'$ has *order* r if the composition $M_i \hookrightarrow M \xrightarrow{f} M'$ has image in M_{i+r} for each $i \in \mathbb{Z}$.

Degree shift convention If M is a $\mathbb{Z}$–graded module, $M[n]$ is the graded module with $(M[n])_i = M_{i-n}$, ie the module found by shifting the homogeneous elements of M up n levels. If $m \in M$, the corresponding element in $M[n]$ will be denoted $m[n]$. Thus $|m[n]| = |m| + n$.

An order r map $f\colon M \to M'$ induces order 0 maps $M \to M'[-r]$ and $M[r] \to M'$, along with maps of different orders $M[n] \to M[s]$. These will also be denoted by f, except where confusion could arise.

The identity on M will be denoted $\mathbb{I}_M$. We will also have need of a graded version of the identity.

Definition A.1 $|\mathbb{I}_M|\colon M \to M$ is the 0–order map defined by setting

$$|\mathbb{I}_M|(m) = (-1)^{|m|}m$$

for homogeneous $m \in M$ and linearly extending to M. $|\mathbb{I}_M|^j$ is the j–fold composition of $|\mathbb{I}_M|$.

If $m \in M_i$ then $|\mathbb{I}_M|^j(m) = (-1)^{ij}m$. Consequently, $|\mathbb{I}_M|^j \circ |\mathbb{I}_M|^k = |\mathbb{I}_M|^{j+k}$ while $(|\mathbb{I}_M|^j)^k = |\mathbb{I}_M|^{jk}$.

In addition, shifting changes the sign:

$$|\mathbb{I}_{M[n]}| = (-1)^n|\mathbb{I}_M| \quad \text{and} \quad |\mathbb{I}_{M[n]}|^j = (-1)^{jn}|\mathbb{I}_M|.$$

A.1 Tensor algebras

We fix a $\mathbb{Z}$–graded R–module A. As usual,

$$\mathcal{T}^*(A) = \bigoplus_{i=0}^{\infty} A^{\otimes n},$$

where $A^{\otimes 0} = R$ and for $n > 0$, $A^{\otimes n} = A \otimes_R A \otimes_R \cdots \otimes_R A$ using exactly n factors. $A^{\otimes n}$ is graded using the standard rule $|a_1 \otimes \cdots \otimes a_n| = \sum |a_i|$.

Furthermore, $\mathcal{T}^*(A)$ has a filtration

$$R \subset \mathcal{T}^1(A) \subset \cdots \subset \mathcal{T}^k(A) \subset \cdots,$$

where $\mathcal{T}^k(A) = \bigoplus_{i=0}^{k} A^{\otimes n}$.

By $\mathbb{I}_A^{\otimes n}$ we will mean the identity on $A^{\otimes n}$ thought of as the map $\mathbb{I}_A \otimes \mathbb{I}_A \otimes \cdots \otimes \mathbb{I}_A$. In general, we will only use the subscript when we need to distinguish A; by default, $\mathbb{I}^{\otimes n}$ will be the identity on $A^{\otimes n}$. Furthermore, by $|\mathbb{I}|^{j\otimes n}$ we will mean the map $|\mathbb{I}|^j \otimes \cdots \otimes |\mathbb{I}|^j$ on $A^{\otimes n}$.

Definition A.2 For any $\mathbb{Z}$–graded module, $\mathcal{T}_A^*(M)$ is the $\mathbb{Z}$–graded R–module $M \otimes_R \mathcal{T}^*(A)$ filtered by the submodules $M \otimes \mathcal{T}^k(A)$ for $k = 0, 1, 2, \ldots$.

Definition A.3 Let $\mathfrak{T}_A$ be the category whose objects are the R–modules $\mathcal{T}_A^*(M)$ for each $\mathbb{Z}$–graded module M, and whose morphisms, $\mathcal{T}_A(M, M')$, are filtered R–module maps $\Phi\colon \mathcal{T}_A^*(M) \to \mathcal{T}_A^*(M')$.

Definition A.4 Let $\Phi \in \mathcal{T}_A(M, M')$. For $i, j \in \mathbb{N}$, the ij^{th} *component* of Φ is the map

$$\Phi_{ij}\colon M \otimes A^{\otimes(i-1)} \hookrightarrow \mathcal{T}_A^*(M) \xrightarrow{\Phi^*} \mathcal{T}^*(M') \to M' \otimes A^{\otimes(j-1)}.$$

Since Φ is filtered, $\Phi_{ij} = 0$ unless $j \le i$.

A.2 The INF–subcategory of $\mathcal{T}_A^*$

Proposition A.5 *Let $\mathcal{C}_A(M, M') \subset \mathcal{T}_A(M, M')$ be those module maps $\Phi\colon \mathcal{T}_A^*(M) \to \mathcal{T}_A^*(M')$ such that Φ has order r for some $r \in \mathbb{Z}$, and*

$$\Phi_{nm} = \Phi_{n-m+1,1} \otimes |\mathbb{I}|^{(n+m+r)\otimes(m-1)} \tag{39}$$

for every $n, m \in \mathbb{N}$ with $1 \le m \le n$. Then $\mathcal{C}_A(M, M')$ are the sets of morphisms for a subcategory $\mathcal{C}_A$ of $\mathcal{T}_A$ whose objects are all the objects of $\mathcal{T}_A$.

Proof First, we verify that $\mathbb{I}_{\mathcal{T}_A^*(M)} \in \mathcal{C}_A(M, M)$. I_{nm} is non-zero only if $n = m$. When $n = m$ the right side of (39) equals $\mathbb{I}_{11} \otimes |\mathbb{I}|^{(n+n+0)\otimes(n-1)}$. However, $I_{11} = \mathbb{I}_M$, and $|\mathbb{I}|^{(n+n+0)\otimes(n-1)} = \mathbb{I}^{\otimes(l-1)}$ since an even entry in the first factor in the exponent of $|\mathbb{I}|$ will not change the sign. Thus, $I_{nn} = \mathbb{I}_M \otimes \mathbb{I}^{\otimes(n-1)}$, which is the identity on $M \otimes A^{\otimes(n-1)}$. On the other hand, if $n > m$, then $I_{nm} \equiv 0$ and $I_{n-m+1,1} \equiv 0$ as well. Thus, $\mathbb{I}_{\mathcal{T}_A^*(M)} \in \mathcal{C}_A(M, M)$ for every M.

We now need to verify that composition of morphisms with the property in (39) will still have this property. Suppose $\Phi \in \mathcal{C}_A(M, M')$ has order r and $\Psi \in \mathcal{C}_A(M', M'')$

has order s, and components of each satisfy (39). Then the order $(r+s)$ morphism $\Psi\circ\Phi$ has components

$$(40)\quad \begin{aligned}(\Psi\circ\Phi)_{nm}&=\sum_{m\le k\le n}\Psi_{km}\circ\Phi_{nk}\\&=\sum_{m\le k\le n}\big(\Psi_{k-m+1,1}\otimes|\mathbb{I}|^{(k+m+s)\otimes(m-1)}\big)\circ\big(\Phi_{n-k+1,1}\otimes|\mathbb{I}|^{(n+k+r)\otimes(k-1)}\big)\\&=\sum_{m\le k\le n}\Psi_{k-m+1,1}\big(\Phi_{n-k+1,1}\otimes|\mathbb{I}|^{(n+k+r)\otimes(k-m)}\big)\otimes|\mathbb{I}|^{(n+m+r+s)\otimes(m-1)}.\end{aligned}$$

If we let $i=k-m+1$ and $j=n-k+1$, then $n+k\equiv j+1$ modulo 2, as k changes from m to n, i changes from 1 to $n-m+1$, so we can rewrite the previous result as

$$(41)\quad (\Psi\circ\Phi)_{nm}=\Bigg(\sum_{i+j=n-m+2}\Psi_{i,1}\big(\Phi_{j,1}\otimes|\mathbb{I}|^{(j+r+1)\otimes(i-1)}\big)\Bigg)\otimes|\mathbb{I}|^{(n+m+(r+s))\otimes(m-1)}.$$

On the other hand,

$$(42)\quad \begin{aligned}(\Psi\circ\Phi)_{n-m+1,1}&=\sum_{1\le i\le n-m+1}\Psi_{i,1}\circ\Phi_{n-m+1,i}\\&=\sum_{1\le i\le n-m+1}\Psi_{i,1}\big(\Phi_{n-m-i+2,1}\otimes|\mathbb{I}|^{(n+m+1+i+r)\otimes(i-1)}\big)\\&=\sum_{i+j=n-m+2}\Psi_{i,1}\big(\Phi_{j,1}\otimes|\mathbb{I}|^{(j+r+1)\otimes(i-1)}\big)\end{aligned}$$

when we let $j=n-m+2-i$. Thus $\Psi\circ\Phi$ satisfies (39). □

Definition A.6 For $\Phi\in\mathcal{C}(M,M')$ of order r, the *core* of Φ is the set of order r module maps $\Phi^*=\{\phi_k\mid n\in\mathbb{N}\}$, where $\phi_k=\Phi_{k1}\colon M\otimes A^{k-1}\to M'$. Given a set of order r module maps $R=\{\rho_k\mid n\in\mathbb{N}\}$ with $\rho_k\colon M\otimes A^{k-1}\to M'$, the *extension* of R is the map $\bar{R}\in\mathcal{C}(M,M')$ with components

$$(43)\qquad \bar{R}_{nm}=\rho_{n-m+1}\otimes|\mathbb{I}|^{(n+m+r)\otimes(m-1)}.$$

The argument in Proposition A.5 shows that these are inverses: for $\Phi\in\mathcal{C}(M,M')$, $\bar{\Phi}^*=\Phi$, while for $R=\{\rho_k\mid n\in\mathbb{N}\}$, $(\bar{R})^*$ equals R. Consequently, we can describe $\mathcal{C}_A$ completely in terms of a composition on the cores Φ^* which directly reflects the usual module map composition for filtered maps on $\mathcal{T}_A^*(M)$. This allows us to pull the operations of $\mathcal{C}_A$ back to the category of R–modules.

Definition A.7 $\mathcal{C}_A^*$ is the category whose objects are $\mathbb{Z}$–graded R–modules, and whose morphisms $\Phi^*\colon M \nrightarrow M'$ are sets $\Phi^* = \{\phi_i \mid i \in \mathbb{N}\}$ of R–module maps $\phi_i\colon M \otimes A^{\otimes(i-1)} \to M'$ such that every ϕ_j has order r for some $r \in \mathbb{Z}$. The identity $\mathbb{I}_M^*\colon M \nrightarrow M$ is the set of 0–order module maps with $(\mathbb{I}_M^*)_1 = \mathbb{I}_M$ and $(\mathbb{I}_{M_i}^* = 0$ for $i > 1$. The composition of an order r morphism $\Phi^*\colon M \nrightarrow M'$ with an order s morphism $\Psi^*\colon M' \nrightarrow M''$ is the set of order $r+s$ module maps given by

$$(\Psi^* * \Phi^*)_k = \sum_{i+j=k+1} \psi_i\big(\phi_j \otimes |\mathbb{I}|^{(j+r+1)\otimes(i-1)}\big)$$

for $k = 1, 2, \ldots$.

Proposition A.8 *There is a functor $\mathcal{F}\colon \mathcal{C}_A^* \to \mathcal{C}_A$ which takes $M \to \mathcal{T}_A^*(M)$ and $\Phi^*\colon M \nrightarrow M'$ to its extension $\Phi\colon \mathcal{T}_A^*(M) \to \mathcal{T}_A^*(M')$.*

Proof This result is implied directly by Proposition A.5. □

We will generally work in $\mathcal{C}_A$, and then pull back our results to $\mathcal{C}_A^*$.

A.3 INF–structures

Since A is a $\mathbb{Z}$–graded R–module, we may take $M = A$ above. Then $\mathcal{T}_A^*(A) = A \otimes \mathcal{T}^*(A) \cong \bigoplus_{n=1}^{\infty} A^{\otimes n}$. Let $P\colon \mathcal{T}_A(A) \to \mathcal{T}_A^*(A)$ be an order r map in $\mathcal{T}_A(A, A)$. Then we can form a new map $P + (\mathbb{I}_A \otimes P)$ in $\mathcal{T}_A(A, A)$. This is evidently still filtered.

Definition A.9 An INF–algebra structure on A is an order 1 map $D \in \mathcal{T}(A, A)$ such that

(1) $D \circ D = 0$, and

(2) $D + (\mathbb{I} \otimes D)$ is in $\mathcal{C}(A, A)$.

For an INF–algebra structure D, we will let $\mu = D + \mathbb{I} \otimes D$. Then the core of μ is a collection of maps $\mu^* = \{\mu_i\colon A \otimes \mathcal{A}^{\otimes(i-1)} \to A\}$ in $\mathcal{C}_A^*(A, A)$. When we have a prescribed μ in mind, we will write D_μ for the corresponding structure.

Definition A.10 A right INF–module M over (A, D_μ) is a $\mathbb{Z}$–graded module and an order 1 morphism $D_M \in \mathcal{T}(M, M)$ such that

(1) $D_M \circ D_M = 0$, and

(2) $D_M + \mathbb{I} \otimes D_\mu$ is in $\mathcal{C}(M, M)$.

Notice that A with the map D_μ is a right module over (A, D_μ). A right INF–module over (A, D_μ) is a chain complex with an additional requirement placed on its boundary map. We can similarly adapt the notion of chain map and chain homotopy to this context.

Definition A.11 An INF–module map between right INF–modules (M, D_M) and $(M', D_{M'})$ (over (A, D_μ)) is an order 0 morphism $\Psi \in \mathcal{C}(M, M')$ such that $\Psi \circ D_M = D_{M'} \circ \Psi$.

Definition A.12 An INF–homotopy between INF–module maps Φ and Ψ, each mapping (M, D_M) to $(M', D_{M'})$, is an order -1 map $H \in \mathcal{C}(M, M')$ such that $\Phi - \Psi = H \circ D_M + D_{M'} \circ H$.

Since chain complexes form a category, and the maps are drawn from the morphisms of $\mathcal{C}(M, M')$, we obtain a category of right INF–modules. Furthermore, we can quotient by chain homotopies to obtain a notion of chain homotopy equivalence.

A.4 INF–structures in terms of the core category

Let (A, D_μ) be an INF–algebra. The following identity is an immediate consequence of the definition

$$D_\mu = \mu - (\mathbb{I} \otimes D_\mu).$$

From this identity we obtain

$$D_\mu = \mu - \mathbb{I} \otimes \mu + \mathbb{I} \otimes \mathbb{I} \otimes \mu + \cdots = \sum_{l=0}^{\infty} (-1)^l (\mathbb{I}^{\otimes l} \otimes \mu).$$

Note that the sum is actually finite on any summand $A^{\otimes n}$.

If we wish to write out the relations for INF–algebras, modules, morphisms, etc, in terms of their cores, we encounter the difficulty that D_μ is not itself in $\mathcal{C}_A(A, A)$. Furthermore, composing with it is not likely to be in $\mathcal{C}_A(A, A)$, either. However, a graded commutator with $I \otimes D_\mu$ will be an extension. Before we prove this, we must understand commutators with $|\mathbb{I}|^j$.

Proposition A.13 *If $R \in \mathcal{C}_A(A, A)$ has order r, then*

$$|\mathbb{I}_A|^j \circ R_{k,1} = (-1)^{rj} R_{k,1} \circ |\mathbb{I}|^{j \otimes k}.$$

Proposition A.14 *If $\Phi \in \mathcal{C}_A(M, M')$ has order r, then $\Phi(\mathbb{I} \otimes D) - (-1)^r(\mathbb{I} \otimes D)\Phi$ is in $\mathcal{C}_A(M, M')$ and has core $\{(\Phi(\mathbb{I} \otimes D))_{n,1} \mid n \in \mathbb{N}\}$.*

Proof We compute the components of $(\mathbb{I} \otimes D)\Phi$:

$$
\begin{aligned}
&((\mathbb{I} \otimes D)\Phi)_{lm} \\
&\quad = \sum_{m \le k \le l} (\mathbb{I} \otimes D)_{km} \circ \Phi_{lk} \\
&\quad = \sum_{m \le k \le l} \left(\sum_{s=0}^{\infty} (-1)^s \mathbb{I} \otimes \mathbb{I}^s \otimes \mu_{k-s-1,m-s-1} \right) \circ \left(\Phi_{l-k+1,1} \otimes |\mathbb{I}|^{(l+k+r)\otimes(k-1)} \right) \\
&\quad = \sum_{m \le k \le l} \Bigg[\left(\sum_{s=0}^{\infty} (-1)^s \mathbb{I} \otimes \mathbb{I}^s \otimes \mu_{k-m+1,1} \otimes |\mathbb{I}|^{(k+m+1)\otimes(m-s-2)} \right) \\
&\qquad\qquad \left(\Phi_{l-k+1,1} \otimes |\mathbb{I}|^{(l+k+r)\otimes(k-1)} \right) \Bigg] \\
&\quad = \sum_{m \le k \le l} \Bigg[\sum_{s=0}^{\infty} (-1)^{s+l+k+r} \left(\Phi_{l-k+1,1} \otimes |\mathbb{I}|^{(l+k+r)\otimes(m-1)} \right) \\
&\qquad\qquad \left(\mathbb{I} \otimes \mathbb{I}^{l-k+s} \otimes \mu_{k-m+1,1} \otimes |\mathbb{I}|^{(k+m+1)\otimes(m-s-2)} \right) \Bigg] \\
&\quad = \sum_{m \le k \le l} \Bigg[\sum_{s'=l-k}^{\infty} (-1)^{s'+r} \left(\Phi_{l-k+1,1} \otimes |\mathbb{I}|^{(l+k+r)\otimes(m-1)} \right) \\
&\qquad\qquad \left(\mathbb{I} \otimes \mathbb{I}^{s'} \otimes \mu_{k-m+1,1} \otimes |\mathbb{I}|^{(k+m+1)\otimes(m+l-s'-k-2)} \right) \Bigg] \\
&\quad = (-1)^r \sum_{m \le k \le l} \left(\Phi_{l-k+1,1} \otimes |\mathbb{I}|^{(l+k+r)\otimes(m-1)} \right) \\
&\qquad\qquad \left(\sum_{s'=l-k}^{\infty} (-1)^{s'} \mathbb{I} \otimes \mathbb{I}^{s'} \otimes \mu_{k-m+1,1} \otimes |\mathbb{I}|^{(k+m+1)\otimes(m+l-s'-k-2)} \right) \\
&\quad = -(-1)^r \sum_{m \le k \le l} \left(\Phi_{l-k+1,1} \otimes |\mathbb{I}|^{(l+k+r)\otimes(m-1)} \right) \\
&\qquad\qquad \left(\sum_{s'=0}^{l-k-1} (-1)^{s'} \mathbb{I} \otimes \mathbb{I}^{s'} \otimes \mu_{k-m+1,1} \otimes |\mathbb{I}|^{(k+m+1)\otimes(m+l-s'-k-2)} \right) \\
&\qquad\qquad + (-1)^r \sum_{m \le k \le l} \Phi_{l+m-k,m}(\mathbb{I} \otimes D)_{l,l+m-k} \\
&\quad = -(-1)^r \Bigg[\sum_{m \le k \le l} \Phi_{l-k+1,1} \left(\sum_{s'=0}^{l-k-1} (-1)^{s'} (\mathbb{I} \otimes \mathbb{I}^{s'} \otimes \mu_{k-m+1,1} \otimes |\mathbb{I}|^{(k+m+1)\otimes(l-s'-k-1)}) \right) \\
&\qquad\qquad \otimes |\mathbb{I}|^{(l+m+r+1)\otimes(m-1)} \Bigg] + (-1)^r (\Phi(\mathbb{I} \otimes D))_{l,m}
\end{aligned}
$$

However, $\Phi_{l-k+1,1}$ will consume the M factor as well as the first $l-k$ factors of A. Since s' only has range up to $l-k-1$,the μ term must feed into an argument

of $\Phi_{l-k+1,1}$. By the identity (43), $(-1)^r(\mathbb{I}\otimes D)\Phi-\Phi(\mathbb{I}\otimes D)$ is the extension of $\{\rho_n\}$, where

$$\begin{aligned}\rho_n &= -\sum_{1\le k\le n}\Phi_{n-k+1,1}\Big(\sum_{s=0}^{n-k-1}(-1)^s\big(\mathbb{I}\otimes\mathbb{I}^s\otimes\mu_{k,1}\otimes|\mathbb{I}|^{k\otimes(n-s-k-1)}\big)\Big)\\ &= \sum_{1\le k\le n}\Phi_{n-k+1,1}\Big(\sum_{s=0}^{n-k-1}(-1)^s\big(\mathbb{I}\otimes\mathbb{I}^s\otimes\mu_{n-s-1,n-s-k}\big)\Big).\end{aligned}$$

Thus $\rho_n=(-\Phi(\mathbb{I}\otimes D))_{n,1}$. □

To write out the requirement that (A,D_μ) be an INF–algebra in terms of the μ_i, we first note that $0=D_\mu\circ D_\mu=\mu\circ\mu-(\mathbb{I}\otimes D_\mu)\mu-\mu(\mathbb{I}\otimes D_\mu)$. Using the preceding proposition, $0=\mu\circ\mu-\overline{\{(\mu(\mathbb{I}\otimes D_\mu))_{n1}\}}$. However, $\mu\circ\mu=\overline{\mu^*\ast\mu^*}$, so for each $n\in\mathbb{N}$, $(\mu*\mu)_n-(\mu(\mathbb{I}\otimes D_\mu))_{n,1}=0$. Unpacking the definitions above produces the following:

$$\sum_{\substack{i+j=n+1\\1\le l\le i}}\mu_i\big(\mu_j\otimes|\mathbb{I}|^{j\otimes(i-l)}\big)-\sum_{\substack{i+j=n+1\\0\le l\le i-2}}(-1)^l\mu_i\big(\mathbb{I}\otimes\mathbb{I}^{\otimes l}\otimes\mu_j\otimes|\mathbb{I}|^{j\otimes(i-l-2)}\big)=0,$$

which is equivalent to

$$\sum_{\substack{i+j=n+1\\1\le l\le i}}(-1)^{l+1}\mu_i\big(\mathbb{I}^{\otimes(l-1)}\otimes\mu_j\otimes|\mathbb{I}|^{j\otimes(i-l)}\big)=0.$$

Definition A.15 An A_∞–algebra structure on a $\mathbb{Z}$–graded R–module A is an INF–algebra structure D_μ on $A[-1]$.

If $\mu^*=\{\mu_i\}$, then $\mu_i\colon (A[-1])^{\otimes i}\to A[-1]$ being order 1 means $\mu_i\colon((A[-1])^{\otimes i})_k\to(A[-1])_{k+1}$ or $(A^{\otimes i})_{k+i}\to A_{k+2}$. If we let $k'=k+i$ then $\mu_i\colon A^{\otimes i}_{k'}\to A_{k'-i+2}=A[i-2]_{k'}$. Thus, μ_n is a grading preserving map $A^{\otimes n}\to A[n-2]$. Alternatively, in terms of A with its original grading, each μ_n needs to be an order $2-n$ map $A^{\otimes n}\to A$.

The preceding relation for an INF–structure on $A[-1]$ is

$$\sum_{\substack{i+j=n+1\\1\le l\le i}}(-1)^{l+1}\mu_i\big(\mathbb{I}^{\otimes(l-1)}\otimes\mu_j\otimes|\mathbb{I}|^{j\otimes(i-l)}_{A[-1]}\big)=0.$$

Since $|\mathbb{I}|_{A[-1]}=-|\mathbb{I}|_A$ in terms of the grading on A we obtain

$$\sum_{\substack{i+j=n+1\\1\le l\le i}}(-1)^{j(i-l)+l+1}\mu_i\big(\mathbb{I}^{\otimes(l-1)}\otimes\mu_j\otimes|\mathbb{I}|^{j\otimes(i-l)}_{A}\big)=0.$$

However, $j(i-l)+l+1 \equiv ji+lj+l+1 \equiv j(i+1)+(l+1)(j+1)$ so

$$\sum_{\substack{i+j=n+1\\ 1\le l\le i}} (-1)^{j(i+1)+(j+1)(l+1)} \mu_i\big(\mathbb{I}^{\otimes(l-1)} \otimes \mu_j \otimes |\mathbb{I}|_A^{j\otimes(i-l)}\big) = 0.$$

We therefore see that our definition of an A_∞–algebra is equivalent to the standard one.

Definition A.16 An A_∞–algebra A over a ring R is a $\mathbb{Z}$–graded R–module A equipped with maps $\mu_n\colon A^{\otimes n} \to A[n-2]$ for each $n \in \mathbb{N}$, which satisfy the relation

$$0 = \sum_{\substack{i+j=n+1\\ l\in\{1,\dots,i\}}} (-1)^{j(i+1)+(j+1)(l+1)} \mu_i\big(\mathbb{I}^{\otimes(l-1)} \otimes \mu_j \otimes |\mathbb{I}|^{j\otimes(i-l)}\big).$$

Definition A.17 A (right) A_∞–module structure on a $\mathbb{Z}$–graded R–module M, over a A_∞–algebra (A,μ), is a (right) INF–module structure $D_{M[-1]}$ on $M[-1]$ over $(A[-1], D_\mu)$.

Following the argument above, suppose the core of $D + (\mathbb{I}\otimes D)$ is a set of order 1 maps $m_i\colon M[-1]\otimes(A[-1])^{\otimes(i-1)} \to M[-1]$ with extension $\overline{m}$. Then $D\circ D \equiv 0$ is equivalent to $\overline{m}\circ\overline{m} - (\mathbb{I}_{A[-1]}\otimes D_\mu)\overline{m} - \overline{m}(\mathbb{I}_{A[-1]}\otimes D_\mu) \equiv 0$. Pushing this identity back to one involving the maps $m_i\colon M\otimes A^{\otimes(i-1)} \to M[i-2]$ yields the following.

Definition A.18 [8] A right A_∞–module M over an A_∞–algebra A is a set of maps $\{m_i\}_{i\in\mathbb{N}}$ with $m_i\colon M\otimes A^{\otimes(i-1)} \to M[i-2]$, and satisfying the following relations for each $n \ge 1$:

$$\begin{aligned}(44)\quad 0 = &\sum_{i+j=n+1} (-1)^{j(i+1)} m_i\big(m_j \otimes |\mathbb{I}|^{j\otimes(i-1)}\big)\\ &+ \sum_{i+j=n+1,k>0} (-1)^{k(j+1)+j(i+1)} m_i\big(\mathbb{I}^{\otimes k}\otimes\mu_j\otimes|\mathbb{I}|^{j\otimes(i-k-1)}\big).\end{aligned}$$

M is said to be strictly unital if for any $\xi\in M$, $m_2(\xi\otimes\mathbb{I}_A) = \xi$, but for $n>1$, $m_n(\xi\otimes a_1\otimes a_2\otimes\cdots\otimes a_{n-1}) = 0$ if any $a_i = \mathbb{I}_A$.

The definition for an INF–morphism unpacks similarly.

Definition A.19 An A_∞–morphism Ψ from M to M' over (A,μ) is an INF–morphism from $(M[-1], D_{M[-1]})$ to $(M'[-1], D_{M'[-1]})$ over $(A[-1], D_\mu)$.

The same argument as above allows us to write this requirement in terms of the core maps for Ψ, conceived of as order 0 module maps $\psi_i\colon M[-1]\otimes(A[-1])^{\otimes(i-1)} \to M'[-1]$. The requirement that $\Psi\circ D_{M[-1]} = D_{M'[-1]}\circ\Psi$ becomes $\Psi\circ\overline{m} - \overline{m}'\circ\Psi = \Psi\circ(\mathbb{I}\otimes D_\mu) - (\mathbb{I}\otimes D_\mu)\circ\Psi$. By our proposition, $\Psi\circ(\mathbb{I}\otimes D_\mu) - (\mathbb{I}\otimes D_\mu)\circ\Psi$ is the

extension of $\{(\Psi(\mathbb{I}\otimes D_\mu))_{n1}\}$, since Ψ is order 0. Writing out his relation in terms of the cores, and adjusting $|\mathbb{I}|_{A[-1]} = -|\mathbb{I}|_A$ as above yields the standard definition.

Definition A.20 [8] An A_∞–morphism Ψ of right A–modules M and M' is a set of maps $\psi_i\colon M\otimes A^{\otimes(i-1)}\to M'[i-1]$ for $i\in\mathbb{N}$, satisfying

$$\begin{aligned}(45)\quad &\sum_{i+j=n+1}(-1)^{(i+1)(j+1)}m'_i\big(\psi_j\otimes|\mathbb{I}|^{(j+1)\otimes(i-1)}\big)\\ &\quad=\sum_{i+j=n+1}(-1)^{j(i+1)}\psi_i\big(m_j\otimes|\mathbb{I}|^{j\otimes(i-1)}\big)\\ &\qquad\quad+\sum_{i+j=n+1,k>0}(-1)^{j(i+1)+k(j+1)}\psi_i\big(\mathbb{I}^{\otimes k}\otimes\mu_j\otimes|\mathbb{I}|^{j\otimes(i-k-1)}\big).\end{aligned}$$

Ψ is *strictly unital* if $\psi_i(\xi\otimes a_1\otimes\cdots\otimes a_{i-1})=0$ when $a_j=\mathbb{I}_A$ for some j and $i>1$. The *identity* morphism I_M is the collection of maps $i_1(\xi)=\xi$, $i_j=0$ for $j>1$.

Likewise, if we have two morphisms of A_∞–modules $\Phi\colon (M'[-1], D_{M'[-1]})\to(M''[-1], D_{M''[-1]})$ and $\Psi\colon (M[-1], D_{M[-1]})\to(M'[-1], D_{M'[-1]})$ over (A,μ), when we take their composition $\Phi\circ\Psi$, we can write it in terms of the cores of Φ and Ψ, and then adjust the signed identities to be on A. This process gives the following.

Definition A.21 [8] Let Ψ be an A_∞–morphism from M to M', and let Φ be an A_∞–morphism from M' to M''. The composition $\Phi*\Psi$ is the morphism whose component maps for $n\geq 1$ are

$$(\Phi*\Psi)^1_n=\sum_{i+j=n+1}(-1)^{(i+1)(j+1)}\phi_i\big(\psi_j\otimes|\mathbb{I}|^{(j+1)\otimes(i-1)}\big).$$

This is almost the same composition defined in $\mathcal{C}^*_{A[-1]}$, but in transferring to A, we use $|\mathbb{I}|^{(j+1)\otimes(i-1)}_{A[-1]}=(-1)^{(j+1)(i-1)}|\mathbb{I}|^{(j+1)\otimes(i-1)}_A$. This accounts for the additional sign.

Definition A.22 Two A_∞–morphisms Ψ,Φ from M to M' over (A,μ) are homotopic if they are homotopic as INF–morphisms from $\big(M[-1], D_{M[-1]}\big)$ to $\big(M'[-1], D_{M'[-1]}\big)$ over $(A[-1], D_\mu)$.

If H is a homotopy, it is order 1. Writing out the conditions in terms of its core, using the commutator proposition, and adjusting the signs in using $|\mathbb{I}|_A$ instead of $|\mathbb{I}|_{A[-1]}$ produces an equivalent definition.

Definition A.23 [8] Let Ψ, Φ be A_∞–morphisms from M to M'. Ψ and Φ are homotopic if there is a set of maps $\{h_i\}$ with $h_i\colon M \otimes A^{\otimes(i-1)} \to M'[i]$ such that

$$\begin{aligned}(46)\quad \psi_i - \phi_i = &\sum_{i+j=n+1} (-1)^{(i+1)j} m_i'\big(h_j \otimes |\mathbb{I}|^{j\otimes(i-1)}\big)\\ &+ \sum_{i+j=n+1} (-1)^{(i+1)j} h_i\big(m_j \otimes |\mathbb{I}|^{j\otimes(i-1)}\big)\\ &+ \sum_{i+j=n+1,k>0} (-1)^{k(j+1)+j(i+1)} h_i\big(\mathbb{I}^{\otimes k} \otimes \mu_j \otimes |\mathbb{I}|^{j\otimes(i-k-1)}\big),\end{aligned}$$

and for $i > 1$, $h_i(\xi \otimes a_1 \otimes \cdots \otimes a_{i-1}) = 0$ when $a_j = \mathbb{I}_A$ for some j.

In short, all the notions of an A_∞–object O come from the same notion for an INF–object applied to $O[-1]$, and then adjusting the signs on $|\mathbb{I}|_{O[-1]}$ to get a formula without grading shifts.

A.5 Incorporating a factor on the right

Let A and N be $\mathbb{Z}$–graded R–modules (as above). We can lift maps $\mathcal{T}^*(A) \to \mathcal{T}^*(A)$ to maps which take account of N.

Definition A.24 Let $\Psi\colon \mathcal{T}^*(A) \to \mathcal{T}^*(A)$ be an order r module map. Ψ_N is the map $\mathcal{T}^*(A) \otimes N \to \mathcal{T}^*(A) \otimes N$ with component maps $A^{\otimes n} \otimes N \to A^{\otimes m} \otimes N$ given by

$$(\Psi_N)_{n,m} = \Psi_{n,m} \otimes |\mathbb{I}|_N^{n-m+r}.$$

We can also extend maps with domain N.

Definition A.25 Let $\phi\colon N \to \mathcal{T}^*(A) \otimes N'$ be a degree r map with projections $\phi_i\colon N \to A^{\otimes i} \otimes N'$. The extension of ϕ is the degree r map $\bar{\phi}\colon \mathcal{T}^*(A) \otimes N \to \mathcal{T}^*(A) \otimes N'$ with component $\bar{\phi}_{nm}\colon A^{\otimes n} \otimes N \to A^{\otimes m} \otimes N'$ given by

$$\bar{\phi}_{nm} = (-1)^{nr}(\mathbb{I}_A^n \otimes \phi_{m-n})$$

for $n \le m$ and 0 otherwise.

Proposition A.26 *$\bar{\phi}$ is the extension of ϕ if and only if $\bar{\phi} = \phi \oplus (-1)^r(\mathbb{I}_A \otimes \bar{\phi})$ under the isomorphism $\mathcal{T}^*(A) \otimes N \cong N \oplus A \otimes \mathcal{T}^*(A) \otimes N$.*

Proof For $m \ge n > 0$ we have

$$(\mathbb{I}_A \otimes \bar{\phi})_{nm} = (\mathbb{I}_A \otimes \bar{\phi}_{n-1,m-1}) = (-1)^{(n-1)r}(\mathbb{I}_A \otimes \mathbb{I}_A^{n-1} \otimes \phi_{m-n}) = (-1)^r \bar{\phi}_{nm}.$$

If $n = 0$ then $(\mathbb{I}_A \otimes \bar{\phi})_{nm} = 0$ but $\bar{\phi}_{0m} = \phi_m$. □

Example (1) We think of $\psi\colon N \to A \otimes N$ as a map $N \to \mathcal{T}^*(A) \otimes N$ by setting $\psi_i = 0$ except for $\psi_1 = \psi$. In this case, $\overline{\psi}$ only has non-zero entries $\overline{\psi}_{n,n+1} = (-1)^{nr}(\mathbb{I}_A^{\otimes n} \otimes \psi)$.

(2) $\mathbb{I}_N\colon N \to N$ can be considered as a degree 0 map $\iota\colon N \to \mathcal{T}^*(A) \otimes N$ by setting $\iota_i = 0$ except for $\iota_0 = \mathbb{I}_N$. In this case, $\overline{\iota}_{nn} = \mathbb{I}_A^n \otimes \mathbb{I}_N$ while $\overline{\iota}_{nm} = 0$ for $n \neq m$. Thus $\overline{\iota} = \mathbb{I}_{\mathcal{T}^*(A)\otimes N}$.

We now fix an INF–structure D_μ on A. Let μ be the corresponding map on $\mathcal{T}^*(A)$ with core maps $\mu^* = \{\mu_i\}$ and extension $\mu_N\colon \mathcal{T}^*(A) \otimes N \to \mathcal{T}^*(A) \otimes N$. The core of μ_N is the map $\mu_N^*\colon \mathcal{T}^*(A) \otimes N \to A \otimes N$ found by extending each μ_i to $A^{\otimes n} \otimes N \to A \otimes N$:

$$\bigoplus_{n=1}^{\infty} (\mu_n \otimes |\mathbb{I}_N|^n).$$

A similar set of identities obtain for these maps, and the extension of D_μ.

Proposition A.27 *Let $D_{\mu,N}\colon \mathcal{T}^*(A) \otimes N \to \mathcal{T}^*(A) \otimes N$ be the extension of D_μ. Then $D_{\mu,N} + \mathbb{I} \otimes D_{\mu,N} = \mu_N$, and $D_{\mu,N} \circ D_{\mu,N} = 0$.*

Proof We know that $(D_\mu)_{n,m} = \mu_{n,m} - (\mathbb{I} \otimes D_\mu)_{n,m}$, while on the other hand, $(\mathbb{I} \otimes D_\mu)_{n,m} = \mathbb{I} \otimes (D_\mu)_{n-1,m-1}$. Thus we have the equation $(D_\mu)_{n,m} \otimes |\mathbb{I}|_N^{n-m+1} = \mu_{n,m} \otimes |\mathbb{I}|_N^{n-m+1} - \mathbb{I} \otimes (D_\mu)_{n-1,m-1} \otimes |\mathbb{I}|_N^{n-m+1}$. Consequently, $(D_{\mu,N})_{n,m} = \mu_{n,m} \otimes |\mathbb{I}|_N^{n-m+1} - \mathbb{I} \otimes (D_{\mu,N})_{n-1,m-1}$. Thus $D_{\mu,N} + \mathbb{I} \otimes D_{\mu,N} = \mu_N$. □

As a consequence of the proposition,

$$D_{\mu,N} = \sum_{l=0}^{\infty} (-1)^l (\mathbb{I}^{\otimes l} \otimes \mu_N).$$

A.6 Type *D* structures

Definition A.28 A type D structure on N over (A, D_μ) is an order 0 map $\Delta\colon N \to \mathcal{T}^*(A) \otimes N$ such that

(1) $\Delta_0 = \mathbb{I}_N$,

(2) $(\mathbb{I}_A^{\otimes n} \otimes \Delta_m)\Delta_n = \Delta_{m+n}$, and

(3) $D_{\mu,N} \circ \Delta = 0$.

Definition A.29 A type D structure Δ on N (over (A, D_μ)) is *bounded* if there is an $N \in \mathbb{N}$ such that $\Delta_n \equiv 0$ whenever $n \geq N$.

From now on all type D structures in this paper will be bounded, unless otherwise stated.

If we let $\delta = \Delta_1$ then $(\mathbb{I}_A^{\otimes n} \otimes \Delta_m)\Delta_n = \Delta_{m+n}$ implies that

$$\Delta_0 = \mathbb{I}_N, \quad \Delta_1 = \delta, \quad \Delta_n = (\mathbb{I}^{\otimes(n-1)} \otimes \delta)\Delta_{n-1}. \tag{47}$$

We will denote type D structures by this core map: (N, δ), where $\delta\colon N \to A \otimes N$.

Note that we may also extend $\delta\colon N \to A \otimes N$ as the map $\overline{\delta}\colon \mathcal{T}^*(A) \otimes N \to \mathcal{T}^*(A) \otimes N$ with $\overline{\delta}_{n,n+1} = \mathbb{I}_A^{\otimes n} \otimes \delta$, and that we can similarly extend Δ.

Proposition A.30 *Let Δ be the map for δ, then Δ satisfies the following identities:*

(1) $(\overline{\Delta}\Delta)_n = (n+1)\Delta_n$,

(2) $\Delta = \mathbb{I}_N \oplus (\mathbb{I}_A \otimes \Delta)\delta$, *and*

(3) $\Delta = \mathbb{I}_N \oplus \overline{\delta}\Delta$.

Proof Item (1) follows by noting that $(\mathbb{I}^k \otimes \Delta_l)$ is $\overline{\Delta}_{k,k+l}$ and $(\mathbb{I}^k \otimes \Delta_l)\Delta_k = \Delta_{l+k}$. In the composition, l and k are independent, so we obtain Δ_n in each of the $(n+1)$ ways we can write $n+1 = l+k$ with $l, k \geq 0$. For item (2), note that

$$(\mathbb{I}_A \otimes \Delta)_n\delta = (\mathbb{I}_A \otimes \Delta_{n-1})\delta = (\mathbb{I}^{\otimes(n-1)} \otimes \delta)(\mathbb{I} \otimes \Delta_{n-2})\delta = (\mathbb{I}^{\otimes(n-1)} \otimes \delta)(\mathbb{I}_A \otimes \Delta)_{n-1}\delta.$$

Thus the components of $(\mathbb{I}_A \otimes \Delta)\delta$ follow the same definition as Δ. Furthermore, $(\mathbb{I}_A \otimes \Delta)_1\delta = (\mathbb{I}_A \otimes \mathbb{I}_N)\delta = \delta$, but $(\mathbb{I}_A \otimes \Delta)_0 = 0$ since there must be at least one A–factor. Item (2) follows after adjusting the 0^{th} level to compensate. For item (3), we compute $(\overline{\delta}\Delta)_{0m} = \overline{\delta}_{m-1,m} \circ \Delta_{m-1}$ for $m \geq 1$. Thus, this component equals $(\mathbb{I}^{\otimes(m-1)} \otimes \delta)\Delta_{m-1} = \Delta_m$ for $m \geq 1$. However, $\Delta_{00} = \mathbb{I}_N$. □

The definition above uses the map $D_{\mu,N}$, but we can use the identities to replace this condition with one depending solely on the core map μ_N^*.

Proposition A.31 *(N, δ) being a type D structure for (A, D_μ) is equivalent to either $\mu_N\Delta = 0$ or*

$$\mu_N^*\Delta = \sum_{n=1}^{\infty} \big(\mu_n \otimes |\mathbb{I}_N|^n\big)\Delta_n \equiv 0.$$

Proof First, note that $D_{\mu,N}\Delta = \mu_N\Delta - (\mathbb{I}_A \otimes D_{\mu,N})\Delta$. If we replace the second Δ on the right with $\Delta = \mathbb{I}_N + (\mathbb{I}_A \otimes \Delta)\delta$ we will get $D_{\mu,N}\Delta = \mu_N\Delta - (\mathbb{I}_A \otimes D_{\mu,N})(\mathbb{I}_A \otimes \Delta)\delta$

since $(\mathbb{I}_A \otimes \Delta)$ is zero on $N \subset \mathcal{T}^*(A) \otimes N$. However, $(\mathbb{I}_A \otimes D_{\mu,N})(\mathbb{I}_A \otimes \Delta) = (\mathbb{I}_A \otimes D_{\mu,N}\Delta))$. So

$$\mu_N \Delta = D_{\mu,N}\Delta + (\mathbb{I}_A \otimes D_{\mu,N}\Delta)\delta.$$

Thus, when $D_{\mu,N}\Delta = 0$, then $\mu_N \Delta = 0$. On the other hand, if $\mu_N \Delta = 0$ then $D_{\mu,N}\Delta = -(\mathbb{I}_A \otimes D_{\mu,N}\Delta)\delta$. Iterating this yields $D_{\mu,N}\Delta = (\mathbb{I}_A \otimes (\mathbb{I}_A \otimes D_{\mu,N}\Delta)\delta)\delta = (\mathbb{I}_A^{\otimes 2} \otimes D_{\mu,N}\Delta)(\mathbb{I}_A \otimes \delta)\delta = (\mathbb{I}_A^{\otimes 2} \otimes D_{\mu,N}\Delta)\Delta_2$. By induction, we can show that $D_{\mu,N}\Delta = (-1)^n(\mathbb{I}_A^{\otimes n} \otimes D_{\mu,N}\Delta)\Delta_n$. Since δ is assumed to be bounded, $\Delta_n \equiv 0$ for n large enough. Thus $D_{\mu,N}\Delta = 0$ when $\mu\Delta = 0$.

To complete the argument, we show that the identity in the proposition is equivalent to $\mu_N \Delta = 0$ or, more simply, $\mu_N^* \Delta = 0$. It follows from the definition of μ that

$$(\mu\Delta)_{0n} = \sum_{j-i=n-1} \big(\mu_i \otimes |\mathbb{I}_A|^{i\otimes(j-i)} \otimes |\mathbb{I}_N|^i\big)\Delta_j$$

for $n \geq 1$. But $\Delta_j = \overline{\delta}^{j-i}\Delta_i$ for $j \geq 1$. Since δ is order 0, we then have

$$(\mu\Delta)_{0n} = \overline{\delta}^{n-1} \sum_{j-i=n-1} \big(\mu_i \otimes |\mathbb{I}_N|^i\big)\Delta_i$$

since the later application of $\overline{\delta}$ produces factors on the *right* of the tensor products of A. Rewriting the sum to be in terms of $i = j - n + 1$, and noting that any $j \geq n$ is possible, we get

$$(\mu\Delta)_{0n} = \overline{\delta}^{n-1} \sum_{i=0}^{\infty} \big(\mu_i \otimes |\mathbb{I}_N|^i\big)\Delta_i,$$

from which the statement in the proposition follows directly. □

We now consider maps between type D structures and their compositions.

Definition A.32 Let (N, δ) and (N', δ') be bounded type D structures for (A, D_μ). An order r type D map $\psi\colon (N, \delta) \hookrightarrow (N', \delta')$ is an order r map of graded modules $\psi\colon N \to A \otimes N'$.

Definition A.33 Let $\psi_i\colon (N_i, \delta_i) \hookrightarrow (N_{i+1}, \delta_{i+1})$, $i = 1, \ldots, n$ be order r_i type D maps. Define $M_n(\psi_n, \ldots, \psi_1)$ to be the order $1 + \sum r_i$ map given by

$$M_n(\psi_n, \ldots, \psi_1) = \mu^*_{N_{n+1}} \overline{\Delta}_{n+1} \overline{\psi}_n \overline{\Delta}_n \cdots \overline{\Delta}_2 \overline{\psi}_1 \Delta_1.$$

Here is the basic proposition relating the M_n compositions to the INF–algebra (A, μ).

Proposition A.34 *Let Δ_i, $i=1,\dots,n+1$ be type D structures for (A,D_μ) and let ψ_i be a degree r_i map from (N_i,Δ_i) to (N_{i+1},Δ_{i+1}). Then*

$$\begin{aligned}(48)\quad & D_{\mu,N_{n+1}}\overline{\Delta}_{n+1}\overline{\psi}_n\overline{\Delta}_n\cdots\overline{\Delta}_2\overline{\psi}_1\Delta_1\\ &=\sum_{\substack{1\le i\le n\\ 0\le l\le n-i}}(-1)^{\sum_{p=n-L+2}^{n}\deg\psi_p}\overline{\Delta}_{n+1}\overline{\psi}_n\overline{\Delta}_n\cdots\overline{\Delta}_{n-l+1}\overline{M}_i\overline{\Delta}_{n-l-i+1}\cdots\overline{\Delta}_2\overline{\psi}_1\Delta_1,\end{aligned}$$

where $\overline{M}_i=\overline{M_i(\psi_{n-l},\dots,\psi_{n-l-i+1})}$.

Proof We consider the image of $\xi\in N_1$ under the map found by alternating the ψ_i and the Δ_i:

$$\overline{\Delta}_{n+1}\overline{\psi}_n\overline{\Delta}_n\cdots\overline{\Delta}_2\overline{\psi}_1\Delta_1.$$

The image of ξ is then a sum of terms of the form

$$(49)\quad \epsilon_{k_1,k_2,\dots,k_{n+1}}a_1^1\otimes\cdots\otimes a_{k_1}^1\otimes\gamma_1\otimes a_1^2\otimes\cdots a_{k_n}^n\otimes\gamma_n\otimes a_1^{n+1}\otimes\cdots a_{k_{n+1}}^{n+1}\otimes\xi',$$

where $a_j^i\in A$, each $\gamma_i\in A$ marks the factor coming from a ψ_i, and ξ' is some element of N_{n+1}. The sign in front equals

$$\epsilon_{k_1,k_2,\dots,k_n}=(-1)^{k_1r_1+(k_1+1+k_2)r_2+\cdots+(k_1+1+\cdots+1+k_n)r_n}.$$

These signs come from the signs in $\overline{\psi}_i$ for each of $i=1,\dots,n$: $(k_1+1+\cdots+1+k_i)r_i$, comes from the number of factors preceding ψ_i, including the $i-1$ factors arising from ψ_j with $j<i$, times the degree of ψ_i.

To this we will apply the map

$$D_{\mu,N_{n+1}}=\sum_{\substack{1\le i\le M\\ 0\le l\le M-i}}(-1)^l\big(\mathbb{I}^{\otimes l}\otimes\mu_i\otimes|\mathbb{I}|^{i\otimes(M-l-i)}\otimes|\mathbb{I}_{N_{n+1}}|^i\big)=0$$

with $M=k_1+1+\cdots+1+k_n+1+k_{n+1}$. To simplify the computation let L be the number of γ_k–factors which are after the closing parenthesis for μ_i and I be the number of such factors inside μ_i. Finally, let $\chi_{s_1,\dots,s_n}=s_1+1+s_2+\cdots+1+s_n$. We will fix the value of I for a minute. After applying $D_{\mu,N_{n+1}}$ we obtain terms of the form

$$\begin{aligned}(50)\quad & \epsilon_{k_1,k_2,\dots,k_n}\cdot(-1)^{\chi_{k_1,\dots,k_{n-L-I},s}}a_1^1\otimes\cdots\otimes a_s^{n-L-I+1}\\ &\otimes\mu_i\big(a_{s+1}^{n-L-I+1}\otimes\cdots\otimes\gamma_{n-L-I+1}\otimes\cdots\otimes\gamma_{n-L}\otimes a_1^{n-L+1}\otimes\cdots\otimes a_{s'}^{n-L+1}\big)\\ &\otimes|a_{s'}^{n-L+1}|^i\otimes\cdots\otimes|\gamma_n|^i\otimes|a_1^{n+1}|^i\otimes\cdots|a_{k_{n+1}}^{n+1}|^i\otimes|\xi'|^i,\end{aligned}$$

where the additional sign comes from $(-1)^l$ in the definition of $D_{\mu,N}$.

We now do some sign accounting. First, $\epsilon_{k_1,k_2,\dots,k_n} = \prod_{t=1}^{n+1}(-1)^{\chi_{k_1,\dots,k_t}\cdot r_t}$. Consequently, we can use this sign to replace each γ_u with $(-1)^{\chi_{k_1,\dots,k_u} r_u}\gamma_u$. Note that this is the sign which would be used in an application of $\overline{\psi}_u$ in the product above. For $\gamma_{n-L},\dots,\gamma_{n-L-I+1}$, however, we rewrite $\chi_{k_1,\dots,k_u} r_u$ as $\chi_{k_1,\dots,k_{n-L-I},s} r_u + p_u r_u$. Then p_u is the number of factors inside μ_i which precede γ_u. We can then bring the χ–sign from the front into the factors, and rewrite the portion which uses μ_i as

$$(-1)^{(1+\sum_{s=1}^{I} r_{n-L-I+s})\chi_{k_1,\dots,k_{n-L-I},s}}$$
$$\mu_i\Big(a_{s+1}^{n-L-I+1}\otimes\cdots\otimes(-1)^{p_{n-L-I+1}r_{n-L-I+1}}\gamma_{n-L-I+1}\otimes\cdots$$
$$\otimes(-1)^{p_{n-L}r_{n-L}}\gamma_{n-L}\otimes a_1^{n-L+1}\otimes\cdots\otimes a_{s'}^{n-L+1}\Big).$$

The sign in front is the same as the sign introduced in extending to get $\overline{M}_I(\psi_{n-L},\dots,\psi_{n-L-I+1})$, a degree $1+\sum_{s=1}^{I} r_{n-L-I+s}$ map, after skipping $\chi_{k_1,\dots,k_{n-L-I},s}$ preceding A–factors. Each p_u is the number of factors preceding the application of ψ_u in $M_I(\psi_{n-L-I+1},\dots,\psi_{n-L})$ before extending. There is another sign which is also added when we change to $\overline{M}_I$: in $\overline{M}_I$ we use μ_N^* not μ^*. The action on N–factor introduces another sign: that in $|\xi'|^i$ versus ξ', where ξ' is the term in the N–factor coming right after the application of μ_i. This sign is $(-1)^{i\deg(\tilde{\xi})}$.

Last we consider the terms on the third line. We note that the sign introduced is -1 raised to $i(\sum\deg a_j^l + \sum\deg\gamma_t + \deg(\xi'))$ from the signed identity terms, times (-1) raised to the sum of $\chi_{k_1,\dots,k_p} r_p$ for $n\geq p\geq n-L+1$. In the INF–relation for ψ_j we apply $\overline{\psi}_r$ after we have used μ_i to contract i factors to 1 factor. Thus the exponent we need differs from $\chi_{k_1,\dots,k_p} r_p$ by $(i-1)r_p$. This occurs for each of the $n-(n-L+1)+1 = L$ factors after μ_i. Thus, the sign is different by $(-1)^{\sum_{p=n-L+2}^{n}(i-1)r_p}$. In addition, $\sum\deg a_j^l + \sum\deg\gamma_t + \deg(\xi')$ is $\deg(\tilde{\xi}) + \sum_{p=n-L+2}^{n} r_p$, where $\tilde{\xi}$ is the result in the N–factor immediately after applying $\overline{M}_i$. This introduces another $i\sum_{p=n-L+2}^{n} r_p$ in the exponent. As a consequence, the sign remaining after combining is $(-1)^{i\deg(\tilde{\xi})+\sum_{p=n-L+2}^{n} r_p}$. Combining with the sign above, we are left with $(-1)^{\sum_{p=n-L+2}^{n} r_p}$.

Thus $(-1)^l\big(\mathbb{I}^{\otimes l}\otimes\mu_j\otimes|\mathbb{I}|^{j\otimes(i-l)}\big)$ applied to each term in

$$(\overline{\Delta}_{n+1}\overline{\psi}_n\overline{\Delta}_n\cdots\overline{\Delta}_2\overline{\psi}_1\Delta_1)(\xi)$$

is the same as a term in

$$(-1)^{\sum_{p=n-L+2}^{n} r_p}(\overline{\Delta}_{n+1}\overline{\psi}_n\overline{\Delta}_n\cdots\overline{\Delta}_{n-L+2}\overline{M}_J\overline{\Delta}_{n-L-J+2}\cdots\overline{\Delta}_2\overline{\psi}_1\Delta_1)(\xi),$$

where $\overline{M}_J = \overline{M}_J(\psi_{n-L+1},\dots,\psi_{n-L-J+2})$. If we add over all the terms we obtain the desired identity.

Note that we are interpreting $I=0$ as the case where μ_i is applied solely to A–factors which come from $\overline{\Delta}$s. In the final summation these will all cancel since Δ_i is a type D structure. □

We note that for type D morphisms, $\sum_{p=n-L+2}^{n} r_p = L-1$ and the signs will mimic the INF–relations used above.

Proposition A.35 *The compositions $M_n, n\in\mathbb{N}$, satisfy the following* INF*–relations:*

$$\sum_{\substack{i+j=n+1\\ 1\le l\le i}} (-1)^{\sum_{p=n-L+2}^{n} \deg \psi_p} M_i(\psi_n,\dots,\psi_{n-l+2}, M_j, \psi_{n-l-j+1},\dots,\psi_1) = 0,$$

where $M_j = M_j(\psi_{n-l+1},\dots,\psi_{n-l-j+2})$.

Proof We compose $\mu^*_{N_{n+1}}$ to $D_{\mu,N_{n+1}}\overline{\Delta}_{n+1}\overline{\psi}_n\overline{\Delta}_n\cdots\overline{\Delta}_2\overline{\psi}_1\Delta_1$. Since (A,μ) is an INF–algebra, $\mu^*_{N_{n+1}} D_{\mu,N_{n+1}} = 0$. On the other hand, using Proposition A.34, we see that this implies

$$\mu^*_{N_{n+1}}\Bigg(\sum_{\substack{1\le i\le n\\ 0\le l\le n-i}} (-1)^{\sum_{p=n-L+2}^{n}\deg\psi_p}\overline{\Delta}_{n+1}\overline{\psi}_n\overline{\Delta}_n\cdots\overline{\Delta}_{n-l+1}\overline{M}_i\overline{\Delta}_{n-l-i+1}\cdots\overline{\Delta}_2\overline{\psi}_1\Delta_1\Bigg) = 0,$$

where $\overline{M}_i = \overline{M_i(\psi_{n-l},\dots,\psi_{n-l-i+1})}$.

Moving $\mu^*_{N_{n+1}}$ inside the summation, and then using the definition of M_n, we obtain the INF–relations we desire. □

We now concentrate on M_1 and M_2. Note that $M_2(\psi_2,\psi_1)$ has degree $1+r_1+r_2$. If we limit ψ_i to have degree -1, then $M_2(\psi_2,\psi_1)$ will also have degree -1. Thus M_2 defines a product on the degree -1 maps. Indeed, the INF–relation on -1 maps has the simpler form

$$\sum_{\substack{i+j=n+1\\ 1\le l\le i}} (-1)^{l-1} M_i(\psi_n,\dots,\psi_{n-l+2}, M_j(\psi_{n-l+1},\dots,\psi_{n-l-j+2}), \psi_{n-l-j+1},\dots,\psi_1) = 0.$$

From the INF–relations we see that M_1 is a boundary map. We will call a map ψ with $M_1(\psi)=0$ a *closed* map.

Definition A.36 A type D morphism $\psi\colon (N,\delta)\circ\!\!\to(N',\delta')$ is a closed order -1 module map $\psi\colon N\to A\otimes N'$.

Proposition A.37 *A degree -1 map $\psi\colon N \to A \otimes N'$ is closed if and only if*

$$D_{\mu,N'} \circ \overline{\Delta}' \circ \overline{\psi} \circ \Delta \equiv 0.$$

Proof By Proposition A.34, we know

$$D_{\mu,N'} \circ \overline{\Delta}' \circ \overline{\psi} \circ \Delta = \overline{\Delta}' \overline{M_1(\psi)} \Delta.$$

If $M_1(\psi) = 0$, then $D_{\mu,N'} \circ \overline{\Delta}' \circ \overline{\psi} \circ \Delta = 0$ since $\overline{M_1(\psi)} = 0$. On the other hand, if the left hand side is 0, we get that $\overline{\Delta}' \overline{M_1(\psi)} \Delta = 0$. This map has image in $\mathcal{T}^*(A) \otimes N'$. If we look at the image in $A \otimes N'$, we see that it equals $\overline{\Delta}'_{11} \overline{M_1(\psi)}_{01} \Delta_{00} = M_1(\psi)$. Thus, when $D_{\mu,N'} \circ \overline{\Delta}' \circ \overline{\psi} \circ \Delta = 0$ we have $M_1(\psi) = 0$. □

We will now restrict ourselves to degree -1 maps of type D structures. The INF$-$identity for $n = 3$ reduces to

$$M_1(M_2(\psi_2, \psi_1)) + M_2(M_1(\psi_2), \psi_1) - M_2(\psi_2, M_1(\psi_1)) = 0.$$

We see from this identity that M_2 will take closed maps to closed maps, thereby defining a product on the kernel of M_1. Furthermore, M_1 is a (signed, right) differential for the composition $-M_2$. We formalize this as follows.

Proposition A.38 *If $\psi\colon (N, \delta) \circ\!\!\to (N', \delta')$ and $\phi\colon (N', \delta') \circ\!\!\to (N'', \delta'')$ are two type D morphisms, then $M_2(\phi, \psi)\colon (N, \delta) \circ\!\!\to (N'', \delta'')$ is a type D morphism. The composition $\phi * \psi$ is the type D morphism $-M_2(\phi, \psi)$.*

We require that A be (strictly) unital with identity $1_A \in A_{-1}$[1]. The identity is a two-sided identity for μ_2 (which will map $A_{-1} \otimes A_{-1} \to A_{-2+1}$), but its presence as any argument in the application of another core map μ_i will mean the image is 0.

Proposition A.39 *Let $\mathbb{I}_{(N,\delta)}\colon (N, \delta) \circ\!\!\to (N, \delta)$ be the map $N \to A \otimes N$ defined by $x \to 1_A \otimes x$. Then $\mathbb{I}_{(N,\delta)}$ is a type D morphism with $M_2(\mathbb{I}_{(N',\delta')}, \psi) = \psi$ and $M_2\big(\phi, \mathbb{I}_{(N,\delta)}\big) = \phi$. Furthermore, the presence of $I_{(N_i,\delta_i)}$ as an argument in M_n, $n \geq 3$ results in 0.*

Proof First, $I_{(N,\delta)}$ has degree -1 since 1_A is in A_{-1}. Second, we show that $I_{(N,\delta)}$ is a morphism, ie it is closed for M_1:

$$\mu_N^* \overline{\Delta} \overline{I}_{(N,\delta)} \Delta = 0.$$

Note that the image in $A^{\otimes n} \otimes N$ will be non-zero only for $n \geq 1$. If it is non-zero, then its image will be linear combinations of terms with a 1_A in some factor of $A^{\otimes n}$,

[1] Recall that we will let A be $A'[-1]$ for some A', thus $(A'[-1])_{-1} = A'_{-1+1} = A'_0$.

due to the presence of $I_{(N,\delta)}$. This factor will be fed into a core map μ_i in μ_N^*. When $i=1$ or $i>2$, the image will then be zero. The only potentially non-zero terms of the composition applied to ξ are those with μ_2. If $\delta(\xi)=\sum c_i\otimes x_i$ we have

$$\begin{aligned}(51)\quad &(\mu_2\otimes I_N)(-\mathbb{I}_A\otimes I_{(N,\delta)})\delta(\xi)+(\mu_2\otimes I_N)(\mathbb{I}_A\otimes\delta)I_{(N,\delta)}(\xi)\\ &=(\mu_2\otimes I_N)(-\mathbb{I}_A\otimes I_{(N,\delta)})\Big(\sum c_i\otimes x_i\Big)+(\mu_2\otimes I_N)(\mathbb{I}_A\otimes\delta)(1_A\otimes\xi)\\ &=(\mu_2\otimes I_N)\Big(-\sum c_i\otimes 1_A\otimes x_i\Big)+(\mu_2\otimes I_N)\Big(1_A\otimes\Big(\sum c_i\otimes x_i\Big)\Big)\\ &=(\mu_2\otimes I_N)\Big(\sum(1_A\otimes c_i-c_i\otimes 1_A)\otimes x_i\Big)\\ &=0.\end{aligned}$$

To verify that $I_{(N,\delta)}$ composes as the identity on both sides, we compute

$$\mu_{N'}^*\circ\overline{\Delta}'\circ\overline{\mathbb{I}_{(N,\delta)}}\circ\overline{\Delta}'\circ\overline{\psi}\circ\Delta.$$

As above, the strict unitality of the maps μ_i mean that the only terms in this composition which are non-zero will be those which feed two factors of A into $\mu_{N'}$. These must come from $\overline{\mathbb{I}_{(N,\delta)}}$ and $\overline{\psi}$. Thus the entire composition collapses to a sum of terms $(\mu_2\otimes\mathbb{I}_N)(-c_i\otimes 1_A\otimes x_i')=-\mu_2(c_i,1_A)\otimes x_i'=-c_i\otimes x_i'$, where $\psi(\xi)=\sum c_i\otimes x_i'$. So $\psi * I_{(N,\delta)}=-M_2(I_{(N,\delta)},\psi)=\psi$. A similar argument shows that $\mathbb{I}_{(N,\delta)}$ acts as an identity on the left (recalling the order reversal in the product).

To see that $\mathbb{I}_{(N,\delta)}$ in an argument of M_n for $n>2$ we note that since there are n morphisms the μ_i map applied will have $i\geq n>2$. Furthermore, at least one argument in that μ_i will come from $\mathbb{I}_{(N,\delta)}$ and thus be 1_A. Since the μ_i form a strictly unital INF–algebra, this means that the result must be 0. □

However, the product $-M_2$ is not associative. Instead, again from the INF–relations, M_2 satisfies the generalized associativity relation

$$\begin{aligned}(52)\quad &M_2(M_2(\psi_3,\psi_2),\psi_1)-M_2(\psi_3,M_2(\psi_2,\psi_1))\\ &\quad=-M_1(M_3(\psi_3,\psi_2,\psi_1))-M_3(M_1(\psi_3),\psi_2,\psi_1)\\ &\qquad+M_3(\psi_3,M_1(\psi_2),\psi_1)-M_3(\psi_3,\psi_2,M_1(\psi_1)).\end{aligned}$$

We can simplify this relation by quotienting by the image of M_1. To this end we declare equivalent any two -1–morphisms ψ and ϕ if there is a degree -2 map $H\colon N\to A\otimes N'$ with

$$\psi-\phi=M_1(H)=\mu_{N'}^*\overline{\Delta}'\overline{H}\Delta.$$

We call such morphisms *homotopic*, following the terminology in [8]. However, equivalent maps represent the same homology class under M_1.

That M_1 is a differential for M_2 implies that M_2 defines a composition on the equivalence classes under homotopy. The generalized associativity relation implies that the composition M_2 is associative once restricted to equivalence classes. As usual, once we have the INF–structure above, we obtain an INF–structure on the homology: the set of closed morphisms after modding out by homotopy. From the arguments above, and the INF–relations we obtain the following.

Proposition A.40 *Let $\mathcal{D}$ be the collection of type D structures (N,δ) over (A, D_μ). Let $\mathrm{MOR}((N,\delta),(N',\delta'))$ be the homotopy equivalence classes of the set of closed degree -1 type D maps. Then $\mathcal{D}$, with these morphism sets, forms a category if we take*

(1) *the composition*

$$\mathrm{MOR}((N,\delta),(N',\delta'))\otimes_R \mathrm{MOR}((N',\delta'),(N'',\delta''))\to \mathrm{MOR}((N,\delta),(N'',\delta''))$$

to be induced from $(\phi,\psi)\to -M_2(\psi,\phi)$, and

(2) *the identity morphism at (N,δ) to be $\mathbb{I}_{(N,\delta)}$.*

A.7 For A a DGA

We are interested in the following case: A' is such that $A=A'[-1]$ has an INF–structure with $\mu_i=0$ for $i\geq 3$. This makes A' into a differential graded algebra. In this case $M_i\equiv 0$ for $i\geq 3$ since these require the use of μ_n for $n\geq i$ due to the number of $A'[-1]$–factors involved. Examining the INF–relation we see that M_2 defines an associative composition on type D morphisms, before quotienting by homotopy. Furthermore, $\mathbb{I}_{(N,\delta)}$ is still the identity map. Thus, in this case, type D structures with type D morphisms form a category *before quotienting by the homotopy relation*. Modding out by homotopy is then quotienting this category by an ideal.

We can write out the conditions for being a type D structure, a type D morphism, and composition of type D structure in this setting. We note that these only require grading shifts and not changes in sign. First, a type D structure is an order 0 map $\delta\colon N\to A'[-1]\otimes N\cong (A'\otimes N)[-1]$ satisfying

$$\mu_N^*\Delta=\sum_{n=1}^{\infty}(\mu_n\otimes|\mathbb{I}_N|^n)\Delta_n\equiv 0.$$

Since $\mu_n=0$ for $n>2$ we can simplify this to

$$(\mu_2\otimes\mathbb{I}_N)\Delta_2+(\mu_1\otimes|\mathbb{I}_N|)\Delta_1=0.$$

Since $\Delta_2 = (\mathbb{I}_{A'[-1]} \otimes \delta)\delta$ and $\Delta_1 = \delta$ we obtain the relation

$$(\mu_2 \otimes \mathbb{I}_N)(\mathbb{I}_{A'} \otimes \delta)\delta + (\mu_1 \otimes |\mathbb{I}_N|)\delta = 0.$$

By a similar argument, we see that a morphism of type D structure will be an order -1 map $\psi\colon N \to (A'[-1] \otimes N')$. Thus, ψ maps N_k to $(A' \otimes N)[-1]_{k-1} \cong (A' \otimes N)_k$. Thus we can take a type D morphism to be an order 0 map $N \to A' \otimes N'$ which satisfies $\mu^*_{N'}\overline{\Delta}'\overline{\psi}\Delta = 0$. This simplifies to

$$(\mu_2 \otimes \mathbb{I}_N)(\overline{\Delta}'\overline{\psi}\Delta)_{02} + (\mu_1 \otimes |\mathbb{I}_N|)(\overline{\Delta}'\overline{\psi}\Delta)_{01} = 0.$$

$\overline{\psi}$ will increase the number of A'–factors by one, so $(\overline{\Delta}'\overline{\psi}\Delta)_{01} = (\mathbb{I}_{A'} \otimes \mathbb{I}_{N'})\psi\mathbb{I}_N$ since we must use Δ'_0 and Δ_0 or else have too many factors. On the other hand, in the first term we may use either Δ_1 or Δ'_1, but not both. Then

$$(\overline{\Delta}'\overline{\psi}\Delta)_{02} = (\mathbb{I}_{A'[-1]} \otimes \delta')\psi - (\mathbb{I}_{A'[-1]} \otimes \psi)\delta.$$

Under our isomorphisms, this becomes

$$(\mu_2 \otimes \mathbb{I}_N)(\mathbb{I}_{A'[-1]} \otimes \delta')\psi - (\mu_2 \otimes \mathbb{I}_N)(\mathbb{I}_{A'[-1]} \otimes \psi)\delta + (\mu_1 \otimes |\mathbb{I}_N|)\psi = 0.$$

The composition of two morphisms $\psi\colon (N,\delta) \hookrightarrow (N',\delta')$ and $\phi\colon (N',\delta') \hookrightarrow (N'',\delta'')$ can be computed from $-M_2(\phi,\psi) = -\mu^*_{N''}\overline{\Delta}''\overline{\phi}\overline{\Delta}'\overline{\psi}\Delta$. Both $\overline{\psi}$ and $\overline{\phi}$ introduce $A'[-1]$–factors. Hence, the contributions of $\overline{\Delta}''$, $\overline{\Delta}'$ and Δ must either be the identity on the respective modules, or introduce $A'[-1]$–factors which force $\mu^*_{N''}$ to evaluate to 0, as $\mu_i = 0$ for $i > 2$. Thus,

$$-M_2(\phi,\psi) = -(\mu_2 \otimes \mathbb{I}_{N''})(-\mathbb{I}_{A'} \otimes \phi)\psi = (\mu_2 \otimes \mathbb{I}_{N''})(\mathbb{I}_{A'} \otimes \phi)\psi.$$

Furthermore, a homotopy $H\colon N \to (A'[-1] \otimes N')$ is a degree -2 map, and thus can be thought of as a map $N_k \to (A' \otimes N'[-1])_{k-2} \cong (A' \otimes N'[+1])_k$. It is thus an order 0 map $N \to (A' \otimes N')[+1]$. Furthermore, as above, we can compute $M_1(H) = (\mu_2 \otimes \mathbb{I}_N)(\overline{\Delta}'\overline{H}\Delta)_{02} + (\mu_1 \otimes |\mathbb{I}_N|)(\overline{\Delta}'\overline{H}\Delta)_{01}$ which simplifies using

$$(\overline{\Delta}'\overline{H}\Delta)_{02} = (\mathbb{I}_{A'[-1]} \otimes \delta')H + (\mathbb{I}_{A'[-1]} \otimes H)\delta,$$

since H has even order. Thus if ψ and ϕ are homotopic type D morphisms, with homotopy H, then

$$\psi - \phi = (\mu_2 \otimes \mathbb{I}_N)(\mathbb{I}_{A'[-1]} \otimes \delta')H + (\mu_2 \otimes \mathbb{I}_N)(\mathbb{I}_{A'[-1]} \otimes H)\delta + (\mu_1 \otimes |\mathbb{I}_N|)H,$$

or, after applying the shift isomorphisms

$$\psi - \phi = (\mu_2 \otimes \mathbb{I}_N)(\mathbb{I}_{A'} \otimes \delta')H + (\mu_2 \otimes \mathbb{I}_N)(\mathbb{I}_{A'} \otimes H)\delta + (\mu_1 \otimes |\mathbb{I}_N|)H.$$

A.8 Pairing

Since $D_{M,N} \circ D_{M,N} = 0$ and $D_{M,N} = m_N - \mathbb{I}_M \otimes D_{\mu,N}$ we see that $m_N m_N = m_N(\mathbb{I}_M \otimes D_{\mu,N}) + (\mathbb{I}_M \otimes D_{\mu,N})m_N$. $(\mathbb{I}_M \otimes D_{\mu,N})m_N$ has image in $\bigoplus_{n>0} M \otimes A^n \otimes N$ since there is always an A–factor remaining in the codomain of $D_{\mu,N}$. Thus, after restricting to have domain and codomain $M \otimes_R N$,

$$m_N^* \circ m_N = m_N^*(\mathbb{I}_M \otimes D_{\mu,N}).$$

This works for any right INF–module M.

Now suppose that we have type D structures (N_i, Δ_i) for $i = 1, \dots, n+1$. Let $\psi_i \colon (N_i, \delta_i) \hookrightarrow (N_{i+1}, \delta_{i+1})$, $i = 1, \dots, n$ be order r_i type D maps. We will now apply both sides of $m^*_{N_{n+1}} \circ m_{N_{n+1}} = m^*_{N_{n+1}}(\mathbb{I}_M \otimes D_{\mu,N_{n+1}})$ to

$$\xi = \mathbb{I}_M \otimes (\overline{\Delta}_{n+1}\overline{\psi}_n\overline{\Delta}_n \cdots \overline{\Delta}_2\overline{\psi}_1\Delta_1).$$

We let

$$\begin{aligned}&\Omega_n(\psi_n, \dots, \psi_1)\\ &\quad = m^*_{N_{n+1}}(\mathbb{I}_M \otimes \overline{\Delta}_{n+1})(\mathbb{I}_M \otimes \overline{\psi}_n)(\mathbb{I}_M \otimes \overline{\Delta}_n)\cdots(\mathbb{I}_M \otimes \overline{\Delta}_2)(\mathbb{I}_M \otimes \overline{\psi}_1)(\mathbb{I}_M \otimes \Delta_1)\end{aligned}$$

for $n \geq 1$, and $\Omega_0 = m^*_{N_1}\Delta_1$.

By Proposition A.34,

$$\begin{aligned}(53)\quad m_N^*(\mathbb{I}_M \otimes D_{\mu,N})(\xi) = \\ \sum_{\substack{i+j=n+1\\ 1\leq l\leq i}} (-1)^{\sum_{p=n-l+2}^{n} \deg \psi_p}\Omega_i(\psi_n, \dots, \psi_{n-l+2}, M_j, \psi_{n-l-j+1}, \dots, \psi_1),\end{aligned}$$

where $M_j = M_j(\psi_{n-l+1}, \dots, \psi_{n-l-j+2})$.

Similar to the proof of Proposition A.34 we can analyze $(m_N^* \circ m_N)(\xi)$. There are two differences between this argument and that in the proof of Proposition A.34. The first occurs in the signs: there we removed j factors of A by applying μ_j and replaced it with a new factor (the image) which resulted in a difference of sign of $(j-1)r_k$ for each ψ_k occurring after the application of μ_j. Here, however, applying m_j removes $j-1$ factors and merges them into the M factor out front. This also results in a change of $(j-1)r_k$. The second difference is that $m_N^*\Delta = \Omega_0 = \partial^{\boxtimes}$ and not zero as before. Furthermore, when we apply m_j followed by m_i we obtain a composition of Ω_i and Ω_j since each has image in $M \otimes N$. Putting these observations together with the proof of Proposition A.34 we get

$$(m_N^* \circ m_N)(\xi) = \sum_{i+j=n} (-1)^{\sum_{p=j+1}^{n} \deg \psi_p}\Omega_i(\psi_n, \dots, \psi_{j+1})\Omega_j(\psi_j, \dots, \psi_1),$$

where both i and j on the left side can equal 0. Consequently,

$$\sum_{i+j=n} (-1)^{\sum_{p=j+1}^{n} \deg \psi_p} \Omega_i(\psi_n, \dots, \psi_{j+1}) \Omega_j(\psi_j, \dots, \psi_1) \tag{54}$$
$$= \sum_{\substack{i+j=n+1 \\ 1 \le l \le i}} (-1)^{\sum_{p=n-l+2}^{n} \deg \psi_p} \Omega_i(\psi_n, \dots, \psi_{n-l+2}, M_j, \psi_{n-l-j+1}, \dots, \psi_1),$$

where $M_j = M_j(\psi_{n-l+1}, \dots, \psi_{n-l-j+2})$.

A.9 Pairing a left INF–module and a type D structure

Let (A, D_μ) be an INF–algebra with $\mu^* = \{\mu_i\}$, and let M be a right INF–module over (A, D_μ) with differential D_M. We let $m = D_M + \mathbb{I}_M \otimes D_\mu$, and $m^* = \{m_i\}$ be the corresponding core maps $m_n\colon M \otimes A^{\otimes(n-1)} \to M$. In addition, we let (N, δ) be a type D structure over (A, D_μ).

Definition A.41 Define $M \boxtimes N$ to be the graded module $M \otimes_R N$, and $\partial^{\boxtimes}\colon M \boxtimes N \to (M \boxtimes N)[-1]$ to be the map

$$\partial^{\boxtimes} = m_N^*(\mathbb{I}_M \otimes \Delta) = \sum_{k=0}^{\infty} (m_{k+1} \otimes |\mathbb{I}_N|^{k+1}) \circ (\mathbb{I}_M \otimes \Delta_k).$$

Theorem A.42 [8] *$(M \boxtimes N, \partial^{\boxtimes})$ is a chain complex.*

Proof We note that $\partial^{\boxtimes} = \Omega_0$ for $N_1 = N$. Taking the relation for $i = j = 0$ we obtain $\Omega_0\Omega_0 = 0$, since the right hand side contains no terms. This shows that $\partial^{\boxtimes}$ is a boundary map. □

Proposition A.43 *For each type D structure (N, δ) over (A, D_μ) there is a functor $\mathcal{F}_{(N,\delta)}$ from the category of right* INF*–modules over (A, D_μ) to the category of chain complexes. $\mathcal{F}_{(N,\delta)}$ is defined by*

$$\mathcal{F}_{(N,\delta)}(M, D_M) = (M \boxtimes N, \partial^{\boxtimes}), \quad \mathcal{F}_{(N,\delta)}(\Phi) = \Phi_N^*(\mathbb{I}_M \otimes \Delta), \tag{55}$$

where $\Phi \in \mathcal{C}(M, M')$ is a morphism of right INF*–modules over (A, D_μ). We will denote $\mathcal{F}_{(N,\delta)}\Phi$ by $\Phi \boxtimes \mathbb{I}_N$. Furthermore, if Φ and Ψ are homotopic then $\Phi \boxtimes \mathbb{I}_N$ is chain homotopic to $\Psi \boxtimes \mathbb{I}_N$. Thus, $\mathcal{F}_{(N,\delta)}$ induces a functor from the homotopy category of right* INF*–modules to the homotopy category of chain complexes.*

Before proving this proposition, we introduce a useful lemma.

Lemma A.44 *Let $\Phi \in \mathcal{C}(M, M')$ have order r. Then*

$$(\mathbb{I}_{M'} \otimes \Delta) \circ (\Phi_N^*) \circ (\mathbb{I}_M \otimes \Delta) = \Phi_N \circ (\mathbb{I}_M \otimes \Delta).$$

Proof The image of $(\mathbb{I}_{M'} \otimes \Delta) \circ (\Phi_N^*) \circ (\mathbb{I}_M \otimes \Delta)$ in $M \otimes A^{\otimes n} \otimes N$ has the form

$$\sum_l (\mathbb{I}_{M'} \otimes \Delta_n)(\Phi_{l+1}^* \otimes |\mathbb{I}_N|^{l+r})(\mathbb{I}_M \otimes \Delta_l).$$

Since $(\mathbb{I}_M \otimes \Delta_n)$ does not change the M factor, and $(\Phi_{l+1}^* \otimes |\mathbb{I}_N|^{l+r})$ only affects the M factor and the l available A–factors, we can rewrite

$$(\mathbb{I}_{M'} \otimes \Delta_n)(\Phi_{l+1}^* \otimes |\mathbb{I}_N|^{l+r}) = (\Phi_{l+1}^* \otimes \mathbb{I}_A^{\otimes n} \otimes \mathbb{I}_N)(\mathbb{I}_M \otimes \mathbb{I}_A^{\otimes l} \otimes \Delta_n)(\mathbb{I}_M \otimes \mathbb{I}_A^{\otimes l} \otimes |\mathbb{I}_N|^{l+r}).$$

Furthermore, as Δ_n preserves grading, so

$$\Delta_n(|\mathbb{I}_N|^{l+r}) = (|\mathbb{I}_A|^{(l+r)\otimes n} \otimes |\mathbb{I}_N|^{l+r})\Delta_n.$$

Therefore,

$$(\mathbb{I}_{M'} \otimes \Delta_n)(\Phi_{l+1}^* \otimes |\mathbb{I}_N|^{l+r}) = (\Phi_{l+1}^* \otimes \mathbb{I}_A^{\otimes n} \otimes \mathbb{I}_N)(\mathbb{I}_M \otimes \mathbb{I}_A^{\otimes l} \otimes |\mathbb{I}_A|^{(l+r)\otimes n} \otimes |\mathbb{I}_N|^{l+1})(\mathbb{I}_M \otimes \mathbb{I}_A^{\otimes l} \otimes \Delta_n).$$

However,

$$(\Phi_{l+1}^* \otimes \mathbb{I}_A^{\otimes n} \otimes \mathbb{I}_N)(\mathbb{I}_M \otimes \mathbb{I}_A^{\otimes l} \otimes |\mathbb{I}_A|^{(l+r)\otimes n} \otimes |\mathbb{I}_N|^{l+r}) = \Phi_{n+l+1,n+1} \otimes |\mathbb{I}_N|^{l+1},$$

which is an entry in Φ_N. If we precompose with $\mathbb{I}_M \otimes \Delta_l$ we can then replace $(\mathbb{I}_M \otimes \mathbb{I}_A^{\otimes l} \otimes \Delta_n)(\mathbb{I}_M \otimes \Delta_l)$ with $\mathbb{I}_M \otimes \Delta_{l+n}$. As a consequence, the image in $M \otimes A^{\otimes n} \otimes N$ is the sum $\sum_l (\Phi_{n+l+1,n+1} \otimes |\mathbb{I}_N|^{l+r})(\mathbb{I}_M \otimes \Delta_{l+n})$, which is also the entry in $\Phi_N(\mathbb{I}_M \times \Delta)$ which maps $M \otimes N$ to $M \otimes A^{\otimes n} \otimes N$. □

Proof of Proposition A.43 Let $\Phi \in \mathcal{C}(M, M')$ be an order 0 morphism from M to M' with core $\Phi^* = \{\phi_k\}$. We define

$$\phi \boxtimes \mathbb{I}_N = \Phi_N^*(\mathbb{I}_M \otimes \Delta).$$

To see that this is a chain map, we compute $(\Phi \boxtimes \mathbb{I}_N)\partial^{\boxtimes} = \Phi_N^*(\mathbb{I}_M \otimes \Delta)(m_N^*)(\mathbb{I}_M \otimes \Delta)$. Using the previous lemma we can simplify this to $\Phi_N^*(m_N)(\mathbb{I}_M \otimes \Delta)$. Since Φ is a morphism of right INF–modules, we have $\Phi_N D_{M,N} = D_{M',N}\Phi_N$, or

$$\Phi_N(m_N - (\mathbb{I}_M \otimes D_{\mu,N})) = (m'_N - (\mathbb{I}_M \otimes D_{\mu,N}))\Phi_N.$$

If we look at those terms with image in $M \otimes N$ we obtain $\Phi_N^* m_N - \Phi_N^*(\mathbb{I}_M \otimes D_{\mu,N}) = (m'_N)^* \Phi_N$. Thus

$$(\Phi \boxtimes \mathbb{I}_N)\partial^{\boxtimes} = \big(\Phi_N^*(\mathbb{I}_M \otimes D_{\mu,N}) + (m'_N)^*\Phi_N\big)(\mathbb{I}_M \otimes \Delta) = (m'_N)^*\Phi_N(\mathbb{I}_M \otimes \Delta).$$

On the other hand, $\partial^{\boxtimes}(\Phi \boxtimes \mathbb{I}_N) = (m_N^*)(\mathbb{I}_M \otimes \Delta)(\Phi_N^*)(\mathbb{I}_M \otimes \Delta)$, which reduces to $(m_N^*)\Phi_N(\mathbb{I}_M \otimes \Delta)$, using the lemma above. Thus, $(\Phi \boxtimes \mathbb{I}_N)$ is a chain map.

Let $\Psi\colon M \to M'$ and $\Phi\colon M' \to M''$ be morphisms of right INF–modules. Then $(\Phi * \Psi)\boxtimes \mathbb{I}_N$ is the map $(\Phi*\Psi)^*_N(\mathbb{I}_M \otimes \Delta) = \Phi^*_N \Psi_N(\mathbb{I}_M \otimes \Delta)$. Since $\Psi_N(\mathbb{I}_M \otimes \Delta) = (\mathbb{I}_M \otimes \Delta)\Psi^*_N(\mathbb{I}_M \otimes \Delta)$, we see that $(\Phi * \Psi) \boxtimes \mathbb{I}_N = \Phi^*_N(\mathbb{I}_M \otimes \Delta)\Psi^*_N(\mathbb{I}_M \otimes \Delta) = (\Phi \boxtimes \mathbb{I}_N)(\Psi \boxtimes \mathbb{I}_N)$ as required. Furthermore, $\mathbb{I}^\infty_M \boxtimes \mathbb{I}_N = (\mathbb{I}^\infty_{M,N})^*(\mathbb{I}_M \otimes \Delta)$. Since $(\mathbb{I}^\infty_{M,N})^*_k$ will be non-zero only for $k = 1$, we see that the only non-zero term is $(\mathbb{I}^\infty_M)^*_1 \otimes |\mathbb{I}_N|^0)(\mathbb{I}_M \otimes \Delta_0) = (\mathbb{I}_M \otimes \mathbb{I}_N)(\mathbb{I}_M \otimes \mathbb{I}_N) = \mathbb{I}_{M \otimes_R N}$. Our map preserves the identity morphisms.

Finally, we verify that the functor preserves homotopy relations. Suppose $\Phi_N - \Psi_N = D_{M',N} H_N + H_N D_{M,N}$ for some homotopy map: an order -1 map in $\mathcal{C}(M, M')$. Since $D_{M',N} H_N + H_N D_{M,N} = (m'_N + (\mathbb{I}_{M'} \otimes D_{\mu,N}) H_N + H_N (m_N + (\mathbb{I}_M \otimes D_{\mu,N})$, the only terms with image in $M \otimes N$ will be those without a $D_{\mu,N}$ term. Thus, $\Phi^*_N - \Psi^*_N = (m')^*_N H_N + H^*_N m_N$.

Now, let $\mathbb{H} = H^*_N(\mathbb{I}_M \otimes \Delta)$. Then, using Lemma A.44 above,

$$
\begin{aligned}
(56)\quad \partial^{\boxtimes}\mathbb{H} + \mathbb{H}\partial^{\boxtimes} &\\
&= (m')^*_N)(\mathbb{I}_M \otimes \Delta) H^*_N(\mathbb{I}_M \otimes \Delta) + H^*_N(\mathbb{I}_M \otimes \Delta)(m^*_N)(\mathbb{I}_M \otimes \Delta)\\
&= ((m')^*_N) H_N(\mathbb{I}_M \otimes \Delta) + H^*_N m_N(\mathbb{I}_M \otimes \Delta)\\
&= (\Phi^*_N - \Psi^*_N)(\mathbb{I}_M \otimes \Delta)\\
&= (\Phi \boxtimes \mathbb{I}_N) - (\Psi \boxtimes \mathbb{I}_N). \qquad \square
\end{aligned}
$$

Consequently, homotopy equivalent INF–modules will result in chain equivalences of the chain complexes.

Proposition A.45 *For each right INF–module (M, D_M) over (A, D_μ) there is a functor $\mathcal{G}_{(M,D_M)}$ from the category $\mathcal{D}$ of homotopy classes of type D structures over (A, D_μ) to the homotopy category of chain complexes. $\mathcal{G}_{(M,D_M)}$ is defined by*

$$
\begin{aligned}
(57)\qquad \mathcal{G}_{(M,D_M)}(N, \delta) &= (M \boxtimes N, \partial^{\boxtimes}),\\
\mathcal{G}_{(M,D_M)}(\psi) &= [m^*_N(\mathbb{I}_M \otimes \overline{\Delta}')(\mathbb{I}_M \otimes \overline{\psi})(\mathbb{I}_M \otimes \Delta)],
\end{aligned}
$$

where ψ represents a homotopy class of morphisms of type D structure over (A, D_μ), and the image $\mathcal{G}_{(M,D_M)}(\psi)$ is the homotopy class of the chain map inside the brackets. We will denote $\mathcal{G}_{(M,D_M)}(\psi)$ by $\mathbb{I}_M \boxtimes \psi$.

Corollary A.46 *When (A, D_μ) has $\mu_i = 0$ for $i \geq 2$ then the functor $\mathcal{G}_{(M,D_M)}$ can be extended to a functor on the category whose objects are type D structures over (A, D_μ) and whose morphisms are all the type D morphisms between two type D structures. Homotopic morphisms will be taken by $\mathcal{G}_{(M,D_M)}$ to chain homotopy equivalent chain maps.*

Proof Let $\psi\colon N \to A \otimes N'$ be an order -1 morphism of type D structures. We define $\mathbb{I}_M \boxtimes \psi\colon M \boxtimes N \to M \boxtimes N'$ to be $\Omega_1(\psi)$, or

$$\mathbb{I}_M \boxtimes \psi = m_N^*(\mathbb{I}_M \otimes \overline{\Delta}')(\mathbb{I}_M \otimes \overline{\psi})(\mathbb{I}_M \otimes \Delta).$$

By taking $n = 1$ in the pairing relation we obtain

$$(-1)^1 \Omega_1(\psi)\Omega_0 + (-1)^0 \Omega_0 \Omega_1(\psi) = \Omega_1(M_1(\psi)),$$

so if ψ is a type D morphism, we obtain $(\mathbb{I}_M \boxtimes \psi)\partial^{\boxtimes} = \partial^{\boxtimes}(\mathbb{I}_M \boxtimes \psi)$. Thus, $(\mathbb{I}_M \boxtimes \psi)$ is a chain map.

Suppose that H is homotopy of type D morphisms ψ and $\phi\colon \psi - \phi = M_1(H)$. If we apply the same identity to H we obtain

$$(-1)^2 \Omega_1(H)\Omega_0 + (-1)^0 \Omega_0 \Omega_1(H) = \Omega_1(M_1(H)),$$

or

$$\Omega_1(H)\partial^{\boxtimes} + \partial^{\boxtimes}\Omega_1(H) = \Omega_1(\psi - \phi).$$

If we let

$$\mathbb{I}_M \boxtimes H = \Omega_1(H) = m_N^*(\mathbb{I}_M \otimes \overline{\Delta}')(\mathbb{I}_M \otimes \overline{H})(\mathbb{I}_M \otimes \Delta),$$

then

$$(\mathbb{I}_N \boxtimes \psi) - (\mathbb{I}_N \boxtimes \phi) = (\mathbb{I}_M \boxtimes H)\partial^{\boxtimes} + \partial^{\boxtimes}(\mathbb{I}_M \boxtimes H),$$

so homotopic type D morphisms will be taken to chain homotopic chain maps. Thus the functor takes morphisms in the homotopy category of type D morphisms to morphisms in the homotopy category of chain complexes.

We have seen that the map $\psi \mapsto \mathbb{I}_M \boxtimes \psi$ takes homotopy classes to homotopy classes. We now verify that it maps the identity correctly, and preserves compositions. The image of $\mathbb{I}_{(N,\delta)}$ is the map $m_N^*(\mathbb{I}_M \otimes \overline{\Delta}')(\mathbb{I}_M \otimes \overline{\mathbb{I}_{(N,\delta)}})(\mathbb{I}_M \otimes \Delta)$. This introduces a 1_A into each term. Since M is strictly unital, the only term remaining will be that employing m_2. We are thus able to add only one A–factor, so $m_{N,2}^*(\mathbb{I}_M \otimes \mathbb{I}_{(N,\delta)}) = m_{N,2}^*(\mathbb{I}_M \otimes 1_A \otimes \mathbb{I}_N)$ or $\mathbb{I}_M \otimes \mathbb{I}_N$.

To verify that the functor preserves compositions *in the homotopy categories*, recall that the composition of two type D morphisms ψ and ϕ is given by

$$M_2(\phi, \psi) = \mu_{N''}^* \overline{\Delta}'' \overline{\phi} \overline{\Delta}' \overline{\psi} \Delta.$$

As a consequence, we may use the pairing relations, when ϕ and ψ are type D morphisms and $n = 2$, to get

$$\begin{aligned}(58)\quad &(-1)^0 \partial^{\boxtimes}\Omega_2(\phi, \psi) + (-1)^1 \Omega_1(\phi)\Omega_1(\psi) + (-1)^2 \Omega_2(\phi, \psi)\partial^{\boxtimes} \\ &\quad = (-1)^0 \Omega_1(M_2(\phi, \psi)) + (-1)^0 \Omega_2(M_1(\phi), \psi) + (-1)^1 \Omega_2(\phi, M_1(\psi)).\end{aligned}$$

Since $M_1(\phi) = M_1(\psi) = 0$, this identity becomes

$$\partial^{\boxtimes}\Omega_2(\phi,\psi) + \Omega_2(\phi,\psi)\partial^{\boxtimes} = \Omega_1(\phi)\Omega_1(\psi) + \Omega_1(M_2(\phi,\psi)).$$

Thus the map $\Omega_1(\phi)\Omega_1(\psi)$ is chain homotopic to $\Omega_1(-M_2(\phi,\psi))$. Consequently, after modding out by homotopies,

$$(\mathbb{I}\boxtimes\phi)(\mathbb{I}\boxtimes\psi) \simeq (\mathbb{I}\boxtimes(\phi * \psi)),$$

and we have verified that the map preserves compositions and thus is a functor. □

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