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A simple construction of taut submanifolds

DISHANT M PANCHOLI

We show that any integral second cohomology class of a closed manifold X^n, $n \geq 4$, admits, as a Poincaré dual, a submanifold N such that $X \setminus N$ has a handle decomposition with no handles of index bigger than $(n+1)/2$. In particular, if X is an almost complex manifold of dimension at least 6, the complement can be given a structure of a Stein manifold.

53D15; 57R17, 53D05

1 Introduction

Decomposition of manifolds into simple pieces has been a very useful way of studying geometric structures on manifolds. For example, using Giroux's [8] notion of *open book decomposition* supporting a contact structure, one can construct invariants of contact structures on a manifold (mainly in dimension 3). John Etnyre [4] extended these ideas to show that an almost contact orientable closed 5–manifold admits an open book decomposition, with pages topologically Stein, that supports a given almost contact structure. Here, by a topologically Stein manifold, we mean an almost complex manifold M^{2n} that admits a handle decomposition with no handle of index bigger than n. Yakov Eliashberg, during his visit to CMI, Chennai, India, proposed a possible decomposition of an almost symplectic manifold that is based on certain observations about Kähler and symplectic manifolds that we discuss below.

Let (M, ω) be a Kähler manifold of integral class. By the Kodaira embedding theorem, it admits an embedding in $\mathbb{C}P^n$, for some n, such that the pull back of the hyperplane class is $k[\omega]$, with $k > 0$ large. This implies that the intersection D of a generic hyperplane with the image of the embedding is Poincaré dual to $k[\omega]$. Since the complement of a hyperplane in $\mathbb{C}P^n$ is affine, the complement of D is a closed properly embedded submanifold of $\mathbb{C}^n$; in particular, it is Stein. For symplectic manifolds, using techniques developed by Donaldson [2], one can show that a suitably large multiple of the given integral symplectic class admits a Poincaré dual whose complement is topologically Stein; see, for example, Biran [1].

In this note, we first establish the following topological result:

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Theorem 1.1 *Let X be a smooth manifold of dimension $n \geq 4$ and $\alpha \in H^2(X, \mathbb{Z})$. There exists an oriented submanifold M^{n-2} of X that satisfies the following:*

(1) *$[M] \in H_{n-2}(X, \mathbb{Z})$ is Poincaré dual to α.*

(2) *$X \setminus M$ admits an exhausting Morse function with unique index-0 critical point and no critical points of index bigger than $(n+1)/2$.*

When the manifold X has dimension bigger than 5 and is simply connected, this theorem was established by Freedman [5; 6; 7] in the theory he developed during his thesis. The same result was also established by M Kato and Y Mastumoto [10]. A connected codimension-2 submanifold whose complement admits a handle decomposition with a unique zero-handle and no handles of index bigger than $(n+1)/2$ is known as a *taut* submanifold.

A closed almost complex manifold (M^{2n}, J) which admits a class $[\omega] \in H^2(M, \mathbb{R})$ satisfying the property that $[\omega]^n \neq 0$ is known as an almost symplectic manifold. An almost symplectic manifold is denoted by a triple $(M, J, [\omega])$. We combine Theorem 1.1 with the topological characterization of Stein manifolds due to Eliashberg [3] to partially answer the previously mentioned question when the dimension of the manifold is at least 6.

Corollary 1.2 *Let $(X, J, [\omega])$ be an almost symplectic closed manifold of dimension bigger than or equal to 6. If $[\omega] \in H^2(X, \mathbb{R})$ admits an integral lift, then $[\omega]$ admits an oriented submanifold M as its Poincaré dual, which satisfies the property that the complement of M in X admits a Stein structure. Furthermore, the Stein structure is such that the associated almost complex structure is homotopic to the given almost complex structure J on $X \setminus M$.*

C H Taubes in [13] showed that the connected sum of three copies of $\mathbb{C}P^2$ has no symplectic structure, but it is almost symplectic. We hope that the decomposition that we obtained above will be useful in discovering additional topological conditions that allow us to construct a symplectic structure on an almost symplectic manifold. For this, one needs to strengthen Theorem 1.1 and show that the Poincaré dual is also almost complex.

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2 Cell complexes admitting good complements

We will establish Theorem 1.1 in the case when the dimension of X is even, as the argument when the dimension of X is odd is identical. The main purpose of this section is to establish the following:

Lemma 2.1 *Let X be an orientable manifold of dimension $2n$ and M a codimension-2 orientable connected submanifold. Let $\mathcal{N}(M)$ be an open tubular neighborhood of M. Assume that $X \setminus \mathcal{N}(M)$ is given by attaching a $(2n-l)$–handle to a submanifold U of X, $0 < l < n$.*

Then there exists a cell complex C satisfying the following properties:

(1) *C is the union of M and an l–dimensional cell D_l embedded in X with $M \cap D_l = S^{l-1} = \partial D_l$ embedded in M with trivial normal bundle.*
(2) *The complement of an open regular neighborhood $\mathcal{R}(C)$ is (ambiently) isotopic to U.*

Remark 2.2 In order to understand the cell complex discussed in the previous lemma, it is best to visualize the case in which M has a trivial normal bundle in X. In this case, take any section of the normal bundle s of a fixed positive norm ϵ, for a small ϵ, with respect to a Riemannian metric on X. The space $Y = \{ts(x) \mid t \in [0,\epsilon], x \in M\}$ is a manifold of dimension $2n-1$, with M being one of its boundary components. Now, attach an l–handle to the manifold Y along the sphere S^{l-1} in M using the disk D_l. Notice that a regular neighborhood of this new manifold in X is precisely a regular neighborhood of the cell complex discussed in the previous lemma.

Proof of Lemma 2.1 By assumption, the $(l-1)$–dimensional belt sphere S_0 of H_{2n-l} is contained in $\partial\mathcal{N}(M)$. This implies that X is diffeomorphic to $\mathcal{N}(M) \cup H_l \cup U$, where H_l is attached to $\mathcal{N}(M)$ along S_0; see, for example, [11]. Identify $\mathcal{N}(M)$ with the normal bundle and write $\pi\colon \mathcal{N}(M) \to M$ for the natural projection. S_0 is then smoothly homotopic to π_{S_0} by projection along the fibers.

It follows from [9, Theorem 2.13 on page 55] that, after homotopy, $\pi(S_0)$ can be assumed to be embedded. By choosing a connection in $\pi\colon \mathcal{N}(M) \to M$, this homotopy can be lifted to a homotopy of S_0, yielding a new embedded sphere S_1 contained in the boundary of $\mathcal{N}(M)$. Applying the same result, again we can assume that S_1 and S_0 are homotopic through embeddings in $\mathcal{N}(M)$. This implies that the manifolds resulting from $\mathcal{N}(M)$ by performing surgeries along S_0 and S_1 are ambient isotopic, and so are their complements.

The cell D_l in the statement is the union of the belt disc of H_{2n-l} and the embedded cylinder given by the homotopy between S_1 and $\pi(S_1)$ using π. □

3 Proof of Theorem 1.1: from cell complexes to closed manifolds

We establish Theorem 1.1 by starting with an arbitrary manifold M which is Poincaré dual to α and then step by step we remove all the handles of index grater than n from the complement of M by modifying M suitably. Naturally, our argument is by an induction on the number of handles of index bigger than n of $X \setminus \mathcal{N}(M)$. The key inductive step is settled in the following subsection.

3.1 Inductive step

Lemma 3.1 *Let M be a $(2n-2)$–dimensional closed submanifold of X^{2n}. Let $\pi\colon \mathcal{N}(M) \to M$ be an arbitrary 2–disk bundle over it contained in X. Attach a $(k+1)$–handle H_{k+1} to $\mathcal{N}(M)$ with attaching sphere $\mathbb{S}^k \sim S \in \partial\mathcal{N}(M)$ in such a way that $\pi(S)$ is embedded in M with trivial normal bundle. There exists a codimension-2 closed submanifold of $\widetilde{M}$ contained in $\mathcal{N}(M) \cup H_{k+1}$ satisfying:*

(1) *$\widetilde{M}$ is cobordant to M inside $\mathcal{N}(M) \cup H_{k+1}$.*

(2) *Denote the tubular neighborhood of $\widetilde{M}$ by $\mathcal{N}(\widetilde{M})$. Then there exists a sphere $\mathbb{S}^{2n-3+k}$ contained in $\partial\mathcal{N}(\widetilde{M})$ with trivial normal bundle and a choice of framing of its normal bundle, such that $\mathcal{N}(\widetilde{M}) \cup H_{2n-2-k}$ and $\mathcal{N}(M) \cup H_{k+1}$ are diffeomorphic as manifolds with boundary.*

Proof We suggest that the reader take a look at the sequence of pictures given in Figure 1 before starting to read the proof, as the proof just describes the sequence.

Also, for the sake of simplicity, we can assume that the disk bundle $\mathcal{N}(M)$ is trivial. A proof in this case implies the proof for any $\mathcal{N}(M)$ as the process that we are about to describe takes place in a neighborhood of $\pi(S)$, where the restriction of the disk bundle $\mathcal{N}(M)$ is trivial.

The process consists of ambiently attaching the sphere $\pi(S)$ to produce a cobordism C, which is contained in $\mathcal{N}(M) \cup H_{k+1}$, whose upper boundary we denote by $\widetilde{M}$; see Figure 1 (second from the top). Since the 2–disk bundle is trivial, we have that a neighborhood of $\pi(S)$ is diffeomorphic to $\mathbb{S}^k \times D^{2n-2-k} \times D^1 \times D^1 \subset M \times D^1 \times D^1$.

In these coordinates, the attaching sphere S is $(\mathbb{S}^k \times \{0\}) \times \{1\} \times \{0\}$, which is contained in $\mathbb{S}^k \times D^{2n-2-k} \times D^1$. Note that the last D^1 is the second coordinate in the disk bundle. Also, notice that the first D^1 is the ambient direction that we fill.

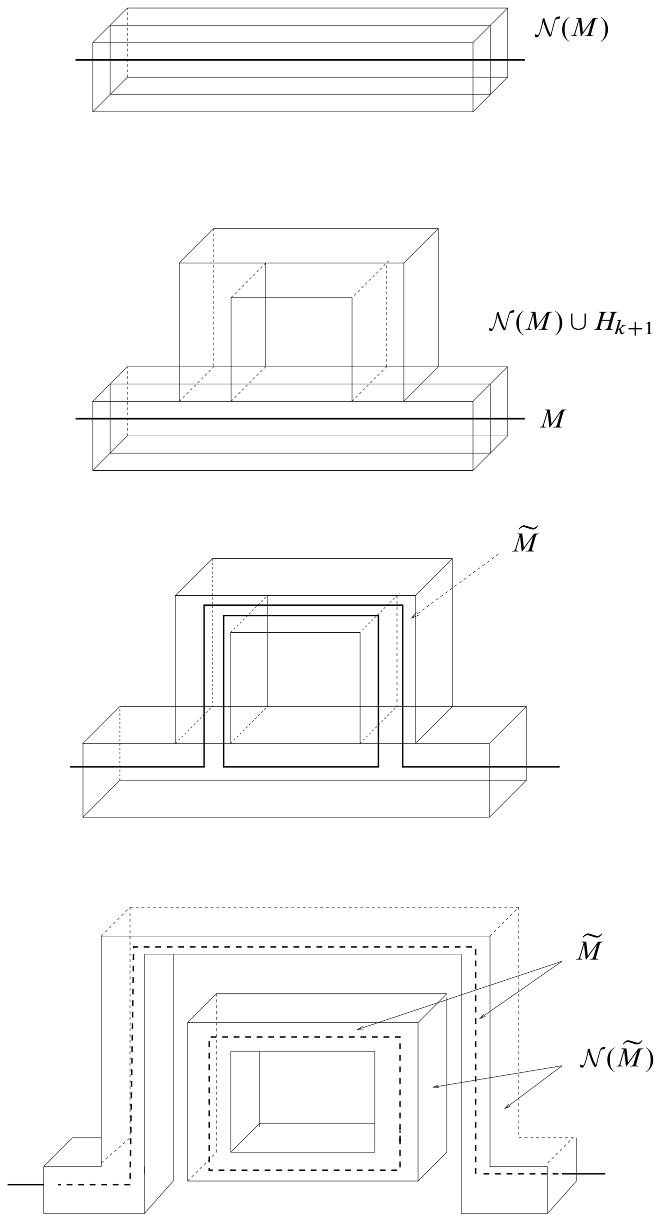

Figure 1: These figures depict the process discussed in the proof of Lemma 3.1 in the case when the dimension of X is 3 and $k=0$. The top figure shows the neighborhood $\mathcal{N}(M)$ near S. The second figure (from the top) shows $\mathcal{N}(M) \cup H_{k+1}$. The third figure depicts $\widetilde{M}$ inside $\mathcal{N}(M) \cup H_{k+1}$, while the bottom figure shows $\widetilde{M}$ together with $\mathcal{N}(\widetilde{M})$.

Now the handle attached to M to create the new submanifold $\widetilde{M}$ is read in these coordinates as

$$K = \left(D^{k+1} \times \mathbb{S}^{2n-3-k}\left(\tfrac{1}{2}\right)\right) \times \{0\} \subset (D^{k+1} \times D^{2n-2-k}) \times D^1.$$

The radius of the sphere is chosen to be $\frac{1}{2} < 1$ to place $\widetilde{M}$ in the interior of the handle.

A neighborhood $\mathcal{N}(M)$ of K contained in $\widetilde{M}$ is then

$$\left(D^{k+1} \times \mathbb{S}^{2n-3-k}\left(\tfrac{1}{2}\right) \times D^1\left(\tfrac{1}{4}\right)\right) \times D^1 \subset (D^{k+1} \times D^{2n-2-k}) \times D^1.$$

Observe that in order to get the whole original handle, we just need to attach an index-$(2n-2-k)$ handle to the boundary of it. In coordinates, the attaching sphere reads as

$$\left(\{0\} \times \mathbb{S}^{2n-3-k}\left(\tfrac{1}{2}\right)\right) \times D^1 \subset \left(D^{k+1} \times \mathbb{S}^{2n-3-k}\left(\tfrac{1}{2}\right) \times D^1\left(\tfrac{1}{4}\right)\right).$$

See Figure 1 (bottom) for a pictorial description of $\mathcal{N}(\widetilde{M})$. □

3.2 Proof of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1 First of all, observe that, since α is second cohomology class, it is possible to construct a connected Poincaré dual which is a manifold. The reason is that $H^2(X, \mathbb{Z})$ is in one to one correspondence with rank-1 complex vector bundles over X, and hence the zero set of a generic section of the bundle corresponding to the given class provides us with a manifold as its Poincaré dual. This Poincaré dual M can be assumed to be connected as the codimension of it in X is 2, allowing us to form an embedded band connected sum.

This implies that $X \setminus \mathcal{N}(M)$ admits a Morse function with only one index-zero critical point and no critical point of index bigger than $2n-1$. Equivalently, it admits a handle decomposition with unique zero handle and no handle of index $2n$; see, for example, [11]. Let j be the number of handles of index bigger than n. We know that we can assume (see, for example, [12]) that handles are added in ascending order of their indices and that two handles of same index are disjoint. Let H be a handle of highest index. Let $\widehat{H}$ be the corresponding dual handle attached to a neighborhood $\mathcal{N}(M)$. Say the index of H is $2n-(k+1) > n$. It follows from Lemma 3.1 that we can modify M to $\widetilde{M}$, which is also Poincaré dual to α, such that $\mathcal{N}(M) \cup \widehat{H}$ is obtained from $\mathcal{N}(\widetilde{M})$ by attaching a handle of index $2n-k-2$. Using the dual handle decomposition, we conclude that the complement of $\widetilde{M}$ is obtained from the complement of $\mathcal{N}(M) \cup \widehat{H}$ by attaching a handle of index $k+1 \leq n$.

Since the complement of $\mathcal{N}(M) \cup \widehat{H}$ has at most $j-1$ handles of index bigger than $k-1$, so is the case for the complement of $\widetilde{M}$. Therefore, by an induction on the number of handles of index bigger than n, there exists $\widetilde{M}$ Poincaré dual to α with its complement admitting no handles of index bigger than n. This implies that the complement of $\widetilde{M}$ admits a Morse function with no critical point of index bigger than n; see, for example, [11] . □

Proof of Corollary 1.2 Eliashberg [3] showed that an open almost complex manifold (V^{2n}, J) admits a Stein structure homotopic, as an almost complex structure, to a given J provided it admits an exhausting Morse function with unique index-0 critical point and no critical point of index bigger than n. So if $(X, J, [\omega])$ is a closed almost complex manifold, then application of Theorem 1.1 implies that there exists a submanifold M of X which is Poincaré dual to $[\omega]$ and which satisfies the hypothesis of Eliashberg's theorem. This completes the argument. □

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Chennai Mathematical Institute, H1 SIPCOT IT Park
Siruseri, Kelambakkam, Chennai 600130, India

dishant@cmi.ac.in

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