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On the cohomology equivalences between bundle-type quasitoric manifolds over a cube

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The aim of this article is to establish the notion of bundle-type quasitoric manifolds and provide two classification results on them: (i) $(\mathbb{C}P^2 \# \mathbb{C}P^2)$–bundle type quasitoric manifolds are weakly equivariantly homeomorphic if their cohomology rings are isomorphic, and (ii) quasitoric manifolds over I^3 are homeomorphic if their cohomology rings are isomorphic. In the latter case, there are only four quasitoric manifolds up to weakly equivariant homeomorphism which are not bundle-type.

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1 Introduction

A quasitoric manifold M over a simple polytope P, which was introduced by Davis and Januszkiewicz [8], is a $2n$–dimensional smooth manifold with a locally standard $T^n = (S^1)^n$–action for which the orbit space is identified with P. Quasitoric manifolds are defined as a topological counterpart of toric varieties. Actually, as the toric varieties are in one-to-one correspondence with the fans, the quasitoric manifolds over P are in one-to-one correspondence with a kind of combinatorial objects, called characteristic maps on P. Moreover, any smooth projective toric variety turns out to be a quasitoric manifold if we regard T^n as acting on it through the inclusion to $(\mathbb{C}^\times)^n$.

On the classification of quasitoric manifolds, Masuda posed the following *cohomological rigidity problem for quasitoric manifolds* in [11], where he affirmatively solved the equivariant version of it.

Problem 1.1 *Are two quasitoric manifolds homeomorphic if their cohomology rings are isomorphic as graded rings?*

Since then toric topologists have studied the topological classification of quasitoric manifolds from the viewpoint of cohomological rigidity, and now we have some classification results which give partial affirmative answers for this problem. First, the cohomological rigidity of quasitoric manifolds over the simplex Δ^n, for $n = 1, 2, \ldots$, is shown in Davis and Januszkiewicz [8]. Second, the cohomological rigidity of quasitoric

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manifolds over the convex m–gon, for $m = 4, 5, \dots$, is an immediate corollary of the classification theorem of Orlik and Raymond [13]. Third, over the product of two simplices, the cohomological rigidity is proved by Choi, Park and Suh [6]. Finally, over the dual cyclic polytope $C^n(m)^*$, where $n \geq 4$ or $m - n = 3$, it is shown by the author [9]. In addition, there are some results on the cohomological rigidity of Bott manifolds, a special subclass of quasitoric manifolds over cubes, by Choi, Masuda and Suh [5], Choi [2] and Choi, Masuda and Murai [3]. On the other hand, we have found no counterexample to this problem.

In this article we mainly consider the cohomological rigidity of "bundle-type" quasitoric manifolds over the cube I^n, of which we give the precise definition later. Bundle-type quasitoric manifolds form a large subclass of quasitoric manifolds. For instance, up to weakly equivariant homeomorphism, the equivariant connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ is the only quasitoric manifold over I^2 which is not bundle-type (Proposition 3.1 and Remark 3.2), and there are only four quasitoric manifolds over I^3 which are not bundle-type (Lemma 4.10). Note that there are infinitely many quasitoric manifolds over I^n, where $n \geq 2$, up to weakly equivariant homeomorphism.

The goal of this article is to show the following two theorems. Here a $(\mathbb{C}P^2 \# \mathbb{C}P^2)$–bundle type quasitoric manifold means an iterated $(\mathbb{C}P^2 \# \mathbb{C}P^2)$–bundle over a point equipped with a good torus action, of which the precise definition is given in Section 2.2.

Theorem 1.2 *Suppose there is a graded ring isomorphism $\varphi\colon H^*(M';\mathbb{Z}) \to H^*(M;\mathbb{Z})$ between the cohomology rings of two $(\mathbb{C}P^2 \# \mathbb{C}P^2)$–bundle type quasitoric manifolds M and M'. Then there exists a weakly equivariant homeomorphism $f\colon M \to M'$ which induces φ in cohomology.*

Theorem 1.3 *Suppose there is a graded ring isomorphism $\varphi\colon H^*(M';\mathbb{Z}) \to H^*(M;\mathbb{Z})$ between the cohomology rings of two quasitoric manifolds M and M' over I^3. Then there exists a homeomorphism $f\colon M \to M'$ which induces φ in cohomology.*

This article is organized as follows. In Section 2, we review the basics of quasitoric manifolds and give the precise definitions of the terms bundle-type quasitoric manifold and so on. In Section 3, we prove the key lemma of this article (Lemma 3.7) and prove Theorem 1.2. In Section 4, we classify the quasitoric manifolds over I^3 up to weakly equivariant homeomorphism. Finally, we give the proof of Theorem 1.3 in Section 5.

2 Preliminaries

2.1 Basics of quasitoric manifolds

First, let us begin with the definition of a quasitoric manifold. The reader can find a more detailed explanation in eg Buchstaber and Panov [1] and [9]. Here we always assume that $\mathbb{C}^n$ is equipped with the standard T^n–action, ie the action defined by $tz := (t_1z_1, \dots, t_nz_n)$, where $t = (t_1, \dots, t_n) \in T^n$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

For two T^n–spaces X and Y, a map $f\colon X \to Y$ is called *weakly equivariant* if there exists a $\psi \in \mathrm{Aut}(T^n)$ such that $f(tx) = \psi(t)f(x)$ for any $t \in T^n$ and $x \in X$, where $\mathrm{Aut}(T^n)$ denotes the group of continuous automorphisms of T^n. We say a smooth T^n–action on a $2n$–dimensional differentiable manifold M is *locally standard* if for each $z \in M$ there exists a triad (U, V, φ) consisting of a T^n–invariant open neighborhood U of z, a T^n–invariant open subset V of $\mathbb{C}^n$, and a weakly equivariant diffeomorphism $\varphi\colon U \to V$.

The orbit space of a locally standard T^n–action is naturally regarded as a *manifold with corners*, by which we mean a Hausdorff space locally homeomorphic to an open subset of $(\mathbb{R}_{\geq 0})^n$ with the transition functions preserving the depth. Here $\operatorname{depth} x$ of $x \in (\mathbb{R}_{\geq 0})^n$ is defined as the number of zero components of x. By definition, for a manifold with corners X, we can define the *depth* of $x \in X$ by $\operatorname{depth} x := \operatorname{depth} \varphi(x)$, where φ is an arbitrary local chart around x. Then a map f between two manifolds with corners is said to *preserve the corners* if $\operatorname{depth} \circ f = \operatorname{depth}$.

An n–dimensional convex polytope is called *simple* if it has exactly n facets at each vertex. We regard a simple polytope as a manifold with corners in the natural way, and define a quasitoric manifold as follows.

Definition 2.1 A *quasitoric manifold over a simple polytope P* is a pair (M, π) consisting of a $2n$–dimensional smooth manifold M equipped with a locally standard T^n–action and a continuous surjection $\pi\colon M \to P$ which descends to a homeomorphism from M/T^n to P preserving the corners. We omit the projection π unless it is misleading.

Next we recall the two ways to construct a quasitoric manifold. In this section P always denotes an n–dimensional simple polytope with exactly m facets and $\mathcal{F}(P)$ denotes the face poset of P. In addition, we define $\mathfrak{T}^n$ as the set of subtori of T^n.

Definition 2.2 A *characteristic map on P* is a map $\ell\colon \mathcal{F}(P) \to \mathfrak{T}^n$ such that

(i) $\dim \ell(F) = n - \dim F$ for each face F,

(ii) $\ell(F) \subseteq \ell(F')$ if $F' \subseteq F$, and

(iii) if a face F is the intersection of k distinct facets $F_1, \dots, F_k$, then the inclusions $\ell(F_i) \to \ell(F)$, $i = 1, \dots, k$, induce an isomorphism $\ell(F_1) \times \cdots \times \ell(F_k) \to \ell(F)$.

Remark 2.3 For each face F of P, we denote the relative interior of F by relint F. Given a quasitoric manifold M over P, we obtain a characteristic map ℓ_M on P by

$$\ell_M(F) := (T^n)_z,$$

where z is an arbitrary point of $\pi^{-1}(\text{relint } F)$ and $(T^n)_z$ denotes the isotropy subgroup at z. Actually we can easily check the conditions of Definition 2.2 since the T^n–action is locally standard.

Construction 2.4 For each point $q \in P$, we denote the minimal face containing q by $G(q)$. Then we obtain a quasitoric manifold $(M(\ell), \pi)$ over P by setting

$$M(\ell) := (T^n \times P)/\sim_\ell,$$

where $(t_1, q_1) \sim_\ell (t_2, q_2)$ if and only if $q_1 = q_2$ and $t_1 t_2^{-1} \in \ell(G(q_1))$, and $\pi\colon M(\ell) \to P$ denotes the map induced by $\mathrm{pr}_2\colon T^n \times P \to P$. Obviously the T^n–action on $T^n \times P$ by multiplication on the first component descends to a T^n–action on $M(\ell)$.

We can define a differentiable structure on $M(\ell)$ as follows. We regard P as a subset of $\mathbb{R}^n$ and denote the hyperplane $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = 0\}$ by H_i, for $i = 1, \dots, n$. For a vertex v of P, we denote by U_v the open subset of P obtained by deleting all faces not containing v from P, and take n facets $F_1, \dots, F_n$ of P such that $v = F_1 \cap \dots \cap F_n$. Additionally, we take an affine transformation $\overline{\varphi}_v$ of $\mathbb{R}^n$ which maps U_v onto an open subset of $(\mathbb{R}_{\geq 0})^n$ and F_i into H_i. If we take an automorphism ψ_v of T^n which maps $\ell(F_i)$ into the i^{th} coordinate subtorus for each $i = 1, \dots, n$, then the map $\psi_v \times \overline{\varphi}_v\colon T^n \times U_v \to T^n \times (\mathbb{R}_{\geq 0})^n$ descends to a weakly equivariant homeomorphism φ_v from $\pi^{-1}(U_v)$ to some T^n–invariant open subset of $\mathbb{C}^n$. We can check that the atlas $\{(\pi^{-1}(U_v), \varphi_v)\}$ gives a differentiable structure on $M(\ell)$. Clearly the T^n–action on $M(\ell)$ is locally standard and the orbit space is identified with P, ie $M(\ell)$ is a quasitoric manifold over P. Moreover, by definition, we have $\ell = \ell_{M(\ell)}$.

In this article, we define an isomorphism of quasitoric manifolds as follows: for two quasitoric manifolds (M, π) and (M', π') over P, a map $f\colon M \to M'$ is called an *isomorphism of quasitoric manifolds* if it is a T^n–equivariant homeomorphism such that $\pi' \circ f = \pi$.

By using the blow-up method of Davis [7], we see that for any quasitoric manifold M over P there exists a T^n–equivariant surjection $T^n \times P \to M$ which descends to an isomorphism $M(\ell_M) \to M$ of quasitoric manifolds. Thus we obtain the following.

Proposition 2.5 *The correspondence $\ell \mapsto M(\ell)$ gives a bijection from the set of characteristic maps on P to the set of isomorphism classes of quasitoric manifolds over P, and the inverse is given by $M \mapsto \ell_M$.*

The second way to construct a quasitoric manifold uses a characteristic matrix and a moment-angle manifold. Below the term *facet labeling of* P means a bijection from $\{1,\dots,m\}$ to the set of the facets of P.

Definition 2.6 An $(n\times m)$–matrix $\lambda=(\lambda_1,\dots,\lambda_m)$ of integers is called a *characteristic matrix on* P *with respect to the facet labeling* $F_1,\dots,F_m$ if it satisfies the following *nonsingularity condition*: if $F_{i_1},\dots,F_{i_n}$ meet at a vertex, then $\det(\lambda_{i_1},\dots,\lambda_{i_n})=\pm 1$.

Hereafter, unless mentioned otherwise, we fix a facet labeling $F_1,\dots,F_m$ of P.

Remark 2.7 Given a characteristic matrix λ on P, we can define a characteristic map ℓ_λ by

$$\ell_\lambda(F_{i_1}\cap\cdots\cap F_{i_k}):=\operatorname{im}(\lambda_{i_1},\dots,\lambda_{i_k}),$$

where we identify S^1 with $\mathbb{R}/\mathbb{Z}$ and regard $(\lambda_{i_1},\dots,\lambda_{i_k})$ as a homomorphism from T^k to T^n. Obviously, any characteristic map is obtained from some characteristic matrix in this way.

Construction 2.8 Let K_P be the simplicial complex on $[m]:=\{1,\dots,m\}$ defined by $K_P:=\{J_F\mid F\in\mathcal{F}(P)\}$, where $J_F:=\{i\in[m]\mid F\subseteq F_i\}$. We regard D^2 as the unit disc of $\mathbb{C}$ and define

$$(D^2,S^1)^J:=\bigl\{(z_1,\dots,z_m)\in(D^2)^m\bigm| |z|_i=1 \text{ if } i\notin J\bigr\}$$

for each $J\subseteq[m]$. Then the *moment-angle manifold* $\mathcal{Z}_P$ is defined as the union

$$\bigcup_{J\in K_P}(D^2,S^1)^J\subseteq(D^2)^m,$$

which is equipped with the T^m–action defined by

$$(t_1,\dots,t_m)\cdot(z_1,\dots,z_m)=(t_1z_1,\dots,t_mz_m).$$

We can define an embedding $\varepsilon\colon P\to\mathcal{Z}_P$ as follows. Denote the barycentric subdivision of K_P by K_P'. If we take $b_F\in\operatorname{relint}F$ for each face F, then the correspondence $J_F\mapsto b_F$ gives a triangulation of P by K_P'. Then we define $\varepsilon\colon|K_P'|\to\mathcal{Z}_P$ so that $\varepsilon(J_F)=(c_1(F),\dots,c_m(F))$ for the vertices and it restricts to an affine map on each simplex, where $c_i(F)=0$ if $F\subseteq F_i$ and $c_i(F)=1$ otherwise. Note that ε descends to a homeomorphism from $P\cong|K_P'|$ to $\mathcal{Z}_P/T^m$. If we define $G(q)$ as in Construction 2.4 and $\ell_P\colon\mathcal{F}(P)\to\mathfrak{T}^m$ by

$$\ell_P(F):=\{(t_1,\dots,t_m)\in T^m\mid t_i=1 \text{ if } F\not\subseteq F_i\},$$

then the correspondence $(t,q) \mapsto t \cdot \varepsilon(q)$ will give an equivariant homeomorphism from $(T^m \times P)/{\sim}$ to $\mathcal{Z}_P$, where $(t_1,q_1) \sim (t_2,q_2)$ if and only if $q_1 = q_2$ and $t_1 t_2^{-1} \in \ell_P(G(q_1))$. Moreover, as in Construction 2.4, we can define a differentiable structure on $(T^m \times P)/{\sim} \cong \mathcal{Z}_P$, and then the T^m–action on $\mathcal{Z}_P$ is smooth.

Let λ be a characteristic matrix on P. If we regard λ as a homomorphism from T^m to T^n, then we can check that $T_\lambda := \ker \lambda$ acts on $\mathcal{Z}_P$ freely. Thus we obtain a manifold $M(\lambda) := \mathcal{Z}_P / T_\lambda$ with a smooth action of $T^m / T_\lambda \cong T^n$, where the isomorphism is induced by λ. We can easily check that this T^n–action on $M(\lambda)$ is locally standard. Actually, the map $\lambda \times \mathrm{id}_P \colon T^m \times P \to T^n \times P$ descends to an equivariant diffeomorphism from $M(\lambda)$ to $M(\ell_\lambda)$. We define $\pi \colon M(\lambda) \to P$ as the composite of the quotient map $M(\lambda) \to M(\lambda)/T^n = \mathcal{Z}_P/T^m$ and ε^{-1}, and then $(M(\lambda), \pi)$ is a quasitoric manifold over P.

Clearly, we have the following proposition.

Proposition 2.9 *For a characteristic matrix λ on P, the two quasitoric manifolds $M(\lambda)$ and $M(\ell_\lambda)$ are smoothly isomorphic.*

Definition 2.10 For a quasitoric manifold M over P, a *characteristic matrix of M* means a characteristic matrix λ on P such that $M(\lambda)$ is isomorphic to M. In other words, λ is called a characteristic matrix of M if $\ell_\lambda = \ell_M$.

Next we consider the weakly equivariant homeomorphisms between quasitoric manifolds. We denote by $[m]_\pm$ the set of $2m$ integers $\pm 1, \dots, \pm m$ and regard $\mathbb{Z}/2$ as acting on $[m]_\pm$ by multiplication with -1. Additionally, we define a map $\mathrm{sgn} \colon [m]_\pm \to \mathbb{Z}/2$ so that $x = \mathrm{sgn}(x) \cdot |x|$, where we identify $\mathbb{Z}/2$ with the multiplicative group $\{\pm 1\}$.

Definition 2.11 We define R_m as the group of $(\mathbb{Z}/2)$–equivariant permutations of $[m]_\pm$ and $p \colon R_m \to \mathfrak{S}_m$ as the canonical surjection to the symmetric group. In addition, we define $\iota \colon R_m \to \mathrm{GL}_m(\mathbb{Z})$ so that $e_i \cdot \iota(\rho) = \mathrm{sgn}_i(\rho) \cdot e_{\sigma(i)}$ for $i = 1, \dots, m$, where $\{e_1, \dots, e_m\}$ is the standard basis of $\mathbb{Z}^m$, $\sigma := p(\rho)$ and $\mathrm{sgn}_i(\rho) := \mathrm{sgn}(\rho(i))$.

Remark 2.12 The map $\iota \colon R_m \to \mathrm{GL}_m(\mathbb{Z})$ defined above is an antihomomorphism. Actually, if we take $\rho_i \in R_m$ and put $\sigma_i := p(\rho_i)$ for $i = 1, 2$, we can check that $\iota(\rho_1 \circ \rho_2) = \iota(\rho_2) \cdot \iota(\rho_1)$ as follows. For a fixed $j \in \{1, \dots, m\}$, if we put $k := \sigma_2(j)$, then $\mathrm{sgn}_j(\rho_1 \circ \rho_2) = \mathrm{sgn}_j(\rho_2) \cdot \mathrm{sgn}_k(\rho_1)$. Therefore

$$\begin{aligned} e_j \cdot \iota(\rho_1 \circ \rho_2) &= \mathrm{sgn}_j(\rho_1 \circ \rho_2) \cdot e_{\sigma_1 \circ \sigma_2(j)} = \mathrm{sgn}_j(\rho_2) \cdot (\mathrm{sgn}_k(\rho_1) \cdot e_{\sigma_1(k)}) \\ &= \mathrm{sgn}_j(\rho_2) \cdot (e_k \cdot \iota(\rho_1)) = (\mathrm{sgn}_j(\rho_2) \cdot e_{\sigma_2(j)}) \cdot \iota(\rho_1) = e_j \cdot \iota(\rho_2) \cdot \iota(\rho_1). \end{aligned}$$

Definition 2.13 For a simple polytope P, we denote by $\mathrm{Aut}(P)$ the group of combinatorial self-equivalences of P and regard it as a subgroup of the symmetric group $\mathfrak{S}_m$ by using the facet labeling. Then we denote by $R(P)$ the subgroup $p^{-1}(\mathrm{Aut}(P))$ of R_m. Moreover, we define Λ_P as the set of characteristic matrices on P and a left action of $\mathrm{GL}_n(\mathbb{Z}) \times R(P)$ on Λ_P by $(\psi, \rho) \cdot \lambda := \psi \cdot \lambda \cdot \iota(\rho)$.

Definition 2.14 Let P be a simple polytope, λ and λ' be characteristic matrices on P, and $f\colon M(\lambda) \to M(\lambda')$ be a weakly equivariant homeomorphism. We denote by $\bar{f}$ the corner-preserving self-homeomorphism of P induced by f. Then a pair $(\psi, \rho) \in \mathrm{GL}_n(\mathbb{Z}) \times R(P)$ is called the *representation of* f if the following hold:

(i) $f(tx) = \psi(t) f(x)$ for any $t \in T^n$ and $x \in M(\lambda)$, where we identify $\mathrm{GL}_n(\mathbb{Z})$ with $\mathrm{Aut}(T^n)$ through the left action on $\mathbb{R}^n/\mathbb{Z}^n = T^n$.

(ii) If we denote by σ_f the combinatorial self-equivalence of P induced by $\bar{f}$, then $\sigma_f = p(\rho)$.

(iii) $\lambda' = (\psi, \rho) \cdot \lambda$.

It is easy to see that for any weakly equivariant homeomorphism $f\colon M(\lambda) \to M(\lambda')$ there exists a unique representation of f. Conversely, we have the following proposition.

Proposition 2.15 *For any pair $(\psi, \rho) \in \mathrm{GL}_n(\mathbb{Z}) \times R(P)$ and a characteristic matrix λ on P, there exists a weakly equivariant homeomorphism $f\colon M(\lambda) \to M(\lambda')$ of which the representation is (ψ, ρ). Here λ' denotes the characteristic matrix $(\psi, \rho) \cdot \lambda$ on P.*

Proof First, by using the triangulation of P given in Construction 2.8, we can construct a corner-preserving self-homeomorphism $\bar{f}$ of P which induces $p(\rho)$. Since $\lambda' = \psi \cdot \lambda \cdot \iota(\rho)$, we see $\psi(\ell(F)) \subseteq \ell'(\sigma(F))$ for each face F of P, where $\sigma := p(\rho)$, $\ell := \ell_\lambda$ and $\ell' := \ell_{\lambda'}$. This implies that $(\psi(t_1), \bar{f}(q_1)) \sim_{\ell'} (\psi(t_2), \bar{f}(q_2))$ if $(t_1, q_1) \sim_\ell (t_2, q_2)$, where $\sim_\ell$ and $\sim_{\ell'}$ are defined in the same way as Construction 2.4. Thus we see that $\psi \times \bar{f}\colon T^n \times P \to T^n \times P$ descends to a weakly equivariant homeomorphism $f\colon M(\ell) \to M(\ell')$, of which the representation is obviously (ψ, ρ). $\square$

Corollary 2.16 *If we denote by $\mathcal{M}_P^{\mathrm{weh}}$ the set of weakly equivariant homeomorphism classes of quasitoric manifolds over P, then the correspondence $\lambda \mapsto M(\lambda)$ gives a bijection from $\Lambda_P/(\mathrm{GL}_n(\mathbb{Z}) \times R(P))$ to $\mathcal{M}_P^{\mathrm{weh}}$.*

Then we consider the cohomology ring of a quasitoric manifold $M = M(\lambda)$ over P. The following computation is due to [8].

Let us define the *Davis–Januszkiewicz space* DJ_P as the union

$$\bigcup_{J \in K_P} BT^J \subseteq BT^m = (\mathbb{C}P^\infty)^m,$$

where $BT^J := \{(y_1, \dots, y_m) \in BT^m \mid y = * \text{ if } i \notin J\}$ and $*$ denotes the basepoint of $\mathbb{C}P^\infty$. K_P is the simplicial complex defined in Construction 2.8. Denote the Borel constructions of M and $\mathcal{Z}_P$ by $\mathcal{B}_{T^n}(M)$ and $\mathcal{B}_{T^m}(\mathcal{Z}_P)$, respectively, ie $\mathcal{B}_{T^n}(M)$ (resp. $\mathcal{B}_{T^m}(\mathcal{Z}_P)$) denotes the quotient of $ET^n \times M$ (resp. $ET^m \times \mathcal{Z}_P$) by the action of T^n (resp. T^m) defined by $t \cdot (x, y) := (xt, t^{-1}y)$. Then we have a homotopy commutative diagram

$$\begin{tikzcd}
\mathcal{Z}_P \arrow[r] \arrow[d] & M \arrow[d] \\
\mathcal{B}_{T^m}(\mathcal{Z}_P) \arrow[r] \arrow[d] & \mathcal{B}_{T^n}(M) \arrow[d] \\
BT^m \arrow[r, "B\lambda"] & BT^n
\end{tikzcd}$$

where the columns are fiber bundles, the middle horizontal map is a homotopy equivalence, and the bottom one is the map induced by $\lambda\colon T^m \to T^n$. By using homotopy colimit, we can construct a homotopy equivalence from DJ_P to $\mathcal{B}_{T^m}(\mathcal{Z}_P)$ such that the diagram

$$\begin{tikzcd}
 & \mathcal{B}_{T^m}(\mathcal{Z}_P) \arrow[d] \\
\mathrm{DJ}_P \arrow[ur] \arrow[r] & BT^m
\end{tikzcd}$$

commutes up to homotopy, where the horizontal arrow is the inclusion.

Thus we obtain the following theorem.

Theorem 2.17 (Davis and Januszkiewicz) *Let P be an n–dimensional simple polytope with m facets and λ be a characteristic matrix on P. Then $M(\lambda)$ is the homotopy fiber of the map $B\lambda \circ \mathrm{incl}\colon \mathrm{DJ}_P \to BT^n$, where* incl *denotes the inclusion into BT^m.*

Since it is also shown by Davis and Januszkiewicz (in the proof of [8, Theorem 3.1]) that any quasitoric manifold has a CW structure without odd dimensional cells, we immediately obtain the following corollary.

Corollary 2.18 (Davis and Januszkiewicz) *Let P be an n–dimensional simple polytope with m facets, $\lambda = (\lambda_{i,j})$ be a characteristic matrix on P, and put $M := M(\lambda)$.*

Then the integral cohomology ring of M is given by

$$H^*(M;\mathbb{Z}) = \mathbb{Z}[v_1,\dots,v_m]/(\mathcal{I}_P + \mathcal{J}_\lambda).$$

Here $v_i := j^* t_i \in H^2(M;\mathbb{Z})$ for $i = 1,\dots,m$, where $j\colon M \to \mathrm{DJ}_P$ is the inclusion of fiber and the t_i form the canonical basis of $H^2(\mathrm{DJ}_P;\mathbb{Z})$, and $\mathcal{I}_P$ and $\mathcal{J}_\lambda$ are the ideals

$$\begin{aligned} \mathcal{I}_P &= (v_{i_1}\cdots v_{i_k} \mid F_{i_1}\cap\cdots\cap F_{i_k} = \varnothing), \\ \mathcal{J}_\lambda &= (\lambda_{i,1}v_1 + \cdots + \lambda_{i,m}v_m \mid i = 1,\dots,n). \end{aligned}$$

Lemma 2.19 *For each $i = 1,\dots,m$, the generator $v_i \in H^2(M;\mathbb{Z})$ of Corollary 2.18 is equal to the Poincaré dual of the submanifold $M_i := \pi^{-1}(F_i)$.*

We make some preparations before the proof of this lemma. For the sake of simplicity, we make the following conventions:

- Unless otherwise mentioned, a space means a Hausdorff space and a map means a continuous map. An action is also assumed to be continuous.
- A structure group $\mathfrak{F}$ of a fiber bundle with fiber F is always assumed to act on F effectively. Moreover, $\mathfrak{F}$ is assumed to have the following property: for a space X and a possibly noncontinuous map $f\colon X \to \mathfrak{F}$, the map f is continuous if the map $X \times F \to F$ defined by $(x,y) \mapsto f(x)\cdot y$ is continuous.

Definition 2.20 Let G and $\mathfrak{F}$ be two topological groups and regard $\mathfrak{F}$ as acting on a space F. A *G–equivariant fiber bundle with fiber F and structure group $\mathfrak{F}$* is a map $p\colon E \to B$ between G–spaces satisfying the following conditions:

(i) p is a fiber bundle with fiber F and structure group $\mathfrak{F}$.

(ii) p is G–equivariant.

(iii) For each $g \in G$ and $x \in B$, if we take local trivializations $\phi\colon U \times F \to p^{-1}(U)$ and $\phi'\colon U' \times F \to p^{-1}(U')$ around x and gx, respectively, then there exists an $f \in \mathfrak{F}$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(x) \arrow[r, "g"] & p^{-1}(gx) \\ \{x\}\times F \arrow[u, "\phi|_{\{x\}\times F}"] \arrow[r, "g\times f"] & \{gx\}\times F \arrow[u, "\phi'|_{\{gx\}\times F}"'] \end{tikzcd}$$

If G is a Lie group, then a G–equivariant fiber bundle is called smooth if it is smooth as a fiber bundle, and the G–actions on the total space and the base space are smooth. Here we say a fiber bundle is smooth if the fiber, the total space, and the base space are differentiable manifolds and the local trivializations can be chosen to be diffeomorphisms.

Lemma 2.21 *Let G be a topological group, H be a closed normal subgroup of G, $p\colon E \to P$ be a G–equivariant fiber bundle with fiber F and structure group $\mathfrak{F}$, and put $\overline{E} := E/H$ and $B := P/H$. If the quotient map $q\colon P \to B$ is a principal H–bundle, then $\overline{p}\colon \overline{E} \to B$ induced by p can be equipped with a G/H–equivariant fiber bundle structure with fiber F and structure group $\mathfrak{F}$ so that the quotient map $\widetilde{q}\colon E \to \overline{E}$ is a bundle map covering q.*

Proof To summarize the setting of the lemma, we have the following commutative diagram:

$$\begin{tikzcd} F \arrow[r] & E \arrow[r, "p"] \arrow[d, "\widetilde{q}"'] & P \arrow[d, "q"] \\ & \overline{E} \arrow[r, "\overline{p}"] & B \end{tikzcd}$$

Let $\mathcal{A}$ be the set consisting of triads (U, s, β), where U is an open subset of B, s is a section of $q\colon q^{-1}(U) \to U$, and $\beta\colon V \times F \to p^{-1}(V)$ is a local trivialization of p such that $s(U) \subseteq V$. If we define $\phi_\alpha\colon U \times F \to \overline{p}^{-1}(U)$ by $\phi_\alpha(x, y) := \widetilde{q} \circ \beta(s(x), y)$ for each $\alpha = (U, s, \beta) \in \mathcal{A}$, then it is clearly bijective. Note that, since $\overline{E}$ is a quotient by a group action, $\widetilde{q}$ is an open map and restricts to a quotient map $\widetilde{q}^{-1}(W) \to W$ for any open subset W of $\overline{E}$. Since $\beta \circ (s \times \mathrm{id}_F)$ is a topological embedding and $\widetilde{q} \circ \beta \circ (s \times \mathrm{id}_F) \circ \phi_\alpha^{-1} = \mathrm{id}_{\overline{p}^{-1}(U)}$ is continuous, ϕ_α^{-1} is also continuous. Thus we see that $\overline{p}$ is a fiber bundle with fiber F.

Next, let us show that the transition functions associated with the local trivializations $\{(U, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ take values in $\mathfrak{F}$. Let $\alpha = (U, s, \beta)$ and $\alpha' = (U', s', \beta')$ be elements of $\mathcal{A}$ and assume $U \cap U' \neq \varnothing$. Due to the second convention made before Definition 2.20, we only have to show that for each $x \in U \cap U'$ there exists an $f \in \mathfrak{F}$ such that $\phi_\alpha(x, y) = \phi_{\alpha'}(x, f \cdot y)$ for any $y \in F$. Fix $x \in U \cap U'$ and take $h \in H$ such that $s'(x) = h \cdot s(x)$. Since p is a G–equivariant fiber bundle, there exists an $f \in \mathfrak{F}$ such that $h \cdot \beta(s(x), y) = \beta'(s'(x), f \cdot y)$ for any $y \in F$. Then, since $\widetilde{q}(h \cdot \beta(s(x), y))$ is equal to $\widetilde{q}(\beta(s(x), y))$, we have $\phi_\alpha(x, y) = \phi_{\alpha'}(x, f \cdot y)$ for any $y \in F$. Thus we see that $\overline{p}$ is a fiber bundle with structure group $\mathfrak{F}$.

Finally, we show that the condition (iii) of Definition 2.20 holds for $\overline{p}$. Fix $g \in G$ and $x \in B$ and let $\alpha = (U, s, \beta)$ and $\alpha' = (U', s', \beta')$ be elements of $\mathcal{A}$ so that $x \in U$

and $gx \in U'$. We can take $h \in H$ such that $s'(gx) = h \cdot (g \cdot s(x))$. If we put $g' := hg$, then there exists an $f \in \mathfrak{F}$ such that $g' \cdot \beta(s(x), y) = \beta'(s'(gx), f \cdot y)$ for any $y \in F$, since p is a G–equivariant fiber bundle. Then, since G acts on $\bar{E}$ via G/H, we have $g \cdot \phi_\alpha(x, y) = \phi_{\alpha'}(gx, f \cdot y)$. Thus the proof is completed. □

Proof of Lemma 2.19 Fix $i \in \{1, \ldots, m\}$ and let X be the inverse image of M_i under the quotient map from $\mathcal{Z}_P$ to $M = M(\lambda)$. Then

$$X = \big\{(z_1, \ldots, z_m) \in \mathcal{Z}_P \mid z_i = 0\big\}$$

(see Construction 2.8). We define a normal bundle $\nu(X)$ of X in $\mathcal{Z}_P$ by

$$\nu(X) := \big\{(z_1, \ldots, z_m) \in \mathcal{Z}_P \bigm| |z_i| < 1\big\},$$

with the projection $\nu(X) \to X$ given by $(z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_{i-1}, 0, z_{i+1}, \ldots, z_m)$. Then $\mathrm{pr}_i \colon \mathcal{Z}_P \to D^2$ restricts to a bundle map $\nu(X) \to \operatorname{int} D^2$ covering $X \to \{0\}$. By Lemma 2.21, since $\nu(X)$ is a T^m–equivariant vector bundle and $\mathcal{Z}_P \to M(\lambda)$ is a principal T_λ–bundle, $\nu(M_i) := \nu(X)/T_\lambda$ gives a T^n–equivariant normal bundle of M_i in M. Moreover, by using Lemma 2.21 again, we see that $\mathcal{B}_{T^n}(\nu(M_i)) \to \mathcal{B}_{T^n}(M_i)$ and $\mathcal{B}_{T^m}(\nu(X)) \to \mathcal{B}_{T^m}(X)$ also have vector bundle structures. Thus we have the following diagram, where each square is a bundle map:

$$\begin{tikzcd}
\nu(M_i) \arrow[r] \arrow[d] & \mathcal{B}_{T^n}(\nu(M_i)) \arrow[d] & \mathcal{B}_{T^m}(\nu(X)) \arrow[l] \arrow[r] \arrow[d] & \mathcal{B}_{T^1}(\operatorname{int} D^2) \arrow[d] \\
M_i \arrow[r] & \mathcal{B}_{T^n}(M_i) & \mathcal{B}_{T^m}(X) \arrow[l] \arrow[r] & BT^1
\end{tikzcd}$$

Then, let us put $(A, B)^c := (A, A \setminus B)$ and $\mathcal{B}_{T^k}(A, B) := (\mathcal{B}_{T^k} A, \mathcal{B}_{T^k} B)$ for a pair (A, B) of T^k–spaces, and consider the following commutative diagram:

$$\begin{tikzcd}
H^2(\mathcal{B}_{T^1}(D^2,\{0\})^c) \arrow[r, "\mathrm{pr}_i^*"] \arrow[d, "r_1"'] & H^2(\mathcal{B}_{T^m}(\mathcal{Z}_P,X)^c) \arrow[d] & H^2(\mathcal{B}_{T^n}(M,M_i)^c) \arrow[l, "\cong"'] \arrow[r] \arrow[d] & H^2((M,M_i)^c) \arrow[d, "r_2"] \\
H^2(\mathcal{B}_{T^1}(D^2)) \arrow[r, "\mathrm{pr}_i"] & H^2(\mathcal{B}_{T^m}(\mathcal{Z}_P)) & H^2(\mathcal{B}_{T^n}(M)) \arrow[l, "\cong"'] \arrow[r] & H^2(M)
\end{tikzcd}$$

Here $H^*(\cdot)$ denotes the integral cohomology and each vertical arrow denotes the restriction. Let us denote the Thom class of $\mathcal{B}_{T^1}(\operatorname{int} D^2)$ by τ and regard it as an element of $H^2(\mathcal{B}_{T^1}(D^2, \{0\})^c)$ through the excision isomorphism. Moreover, we denote the composite of the upper (resp. lower) horizontal arrows by γ_1 (resp. γ_2). Due to the above diagram of bundle maps, γ_1 maps τ to the Thom class of $\nu(M_i)$, and therefore $r_2 \circ \gamma_1(\tau)$ is the Poincaré dual of M_i. Moreover, since $r_1(\tau)$ is the canonical generator of $H^2(\mathcal{B}_{T^1}(D^2)) \cong H^2(BT^1)$, we have $\gamma_2 \circ r_1(\tau) = v_i$. Thus the proof is completed. □

Note that, with the notation of Definition 2.14, a weakly equivariant homeomorphism f maps $\pi^{-1}(F_i)$ to $\pi'^{-1}(F_{\sigma_f(i)})$ for each $i = 1, \dots, m$, where π (resp. π') denotes the projection from $M(\lambda)$ (resp. $M(\lambda')$) to P. By taking into account the orientations of the normal bundles, we have the following.

Corollary 2.22 *Let λ, λ' be two characteristic matrices on P and $f\colon M(\lambda) \to M(\lambda')$ be a weakly equivariant homeomorphism represented by $(\psi, \rho) \in \mathrm{GL}_n(\mathbb{Z}) \times R(P)$. Then, taking generators $v_1, \dots, v_m \in H^*(M(\lambda); \mathbb{Z})$ and $v'_1, \dots, v'_m \in H^*(M(\lambda'); \mathbb{Z})$ as in Corollary 2.18, we have*

$$f^*(v'_1, \dots, v'_m) = (v_1, \dots, v_m) \cdot \iota(\rho)^{-1}.$$

To close this subsection, we introduce two theorems which we will use for the classification of quasitoric manifolds over I^3.

Theorem 2.23 [8, Corollary 6.8] *With the notation in Corollary 2.18, we have the following formulae for the total Stiefel–Whitney class and the total Pontrjagin class:*

$$w(M) = \prod_{i=1}^{m} (1 + v_i) \quad \text{and} \quad p(M) = \prod_{i=1}^{m} (1 - v_i^2).$$

Theorem 2.24 (Jupp's classification of certain 6–manifolds [10]) *Let M and N be closed, one-connected, smooth 6–manifolds with torsion-free cohomology. If a graded ring isomorphism $\alpha\colon H^*(N; \mathbb{Z}) \to H^*(M; \mathbb{Z})$ preserves the second Stiefel–Whitney classes and the first Pontrjagin classes, then there exists a homeomorphism $f\colon M \to N$ which induces α in cohomology.*

2.2 Bundle-type quasitoric manifolds

Given a quasitoric manifold M, we denote by $\mathfrak{D}(M)$ the group of smooth automorphisms of M equipped with the compact-open topology (recall that an isomorphism between quasitoric manifolds (M, π) and (M', π') means an equivariant homeomorphism $f\colon M \to M'$ satisfying $\pi' \circ f = \pi$). The following proposition is immediate from the definition of a smooth equivariant fiber bundle (Definition 2.20).

Proposition 2.25 *Let M_i be a quasitoric manifold acted on by T_i, for $i = 1, 2$, and suppose that $p\colon M \to M_2$ is a smooth T_2–equivariant fiber bundle with fiber M_1 and structure group $\mathfrak{D}(M_1)$. Then there exists a unique T_1–action on M such that $t_1 \cdot \phi(x, y) = \phi(x, t_1 y)$ for any local trivialization $\phi\colon U \times M_1 \to p^{-1}(U)$ of p and $t_1 \in T_1$. Moreover, this action of T_1 on M is smooth and commutes with the action of T_2.*

Definition 2.26 Let M_i be a quasitoric manifold over P_i acted on by T_i, for $i = 1, 2$. Then a *quasitoric M_1–bundle over M_2* is a smooth T_2–equivariant fiber bundle $p\colon M \to M_2$ with fiber M_1, structure group $\mathfrak{D}(M_1)$, and total space equipped with the action of $T := T_1 \times T_2$ defined by $(t_1, t_2) \cdot x := t_1(t_2 x)$, where the T_1–action is the one defined in Proposition 2.25.

We prove later that the quasitoric bundle M is a quasitoric manifold over $P_1 \times P_2$.

Definition 2.27 Let $\mathcal{M}$ be a class of quasitoric manifolds and consider a sequence

$$B_l \xrightarrow{p_{l-1}} B_{l-1} \xrightarrow{p_{l-2}} \cdots \xrightarrow{p_1} B_1 \xrightarrow{p_0} B_0,$$

where B_0 is a point. Then B_l is called an *l–stage $\mathcal{M}$–bundle type quasitoric manifold* if p_i, for $i = 0, \dots, l-1$, is a quasitoric M_i–bundle for some $M_i \in \mathcal{M}$.

Now the term $(\mathbb{C}P^2 \# \mathbb{C}P^2)$–bundle type quasitoric manifold in Theorem 1.2 is defined as follows: let us define $\mathcal{M}(\mathbb{C}P^2 \# \mathbb{C}P^2)$ as the class of quasitoric manifolds which are homeomorphic to $\mathbb{C}P^2 \# \mathbb{C}P^2$, and use the term *$(\mathbb{C}P^2 \# \mathbb{C}P^2)$–bundle type quasitoric manifold* instead of $\mathcal{M}(\mathbb{C}P^2 \# \mathbb{C}P^2)$–bundle type quasitoric manifold.

Suppose that, for $i = 1, 2$, the facets of P_i are labeled by $F_{i,1}, \dots, F_{i,m_i}$ and λ_i is a characteristic matrix of M_i with respect to this facet labeling. If we give a facet labeling of $P_1 \times P_2$ by $F_1, \dots, F_{m_1+m_2}$, where

$$F_j := \begin{cases} F_{1,j} \times P_2 & \text{for } 1 \le j \le m_1, \\ P_1 \times F_{2,j-m_1} & \text{for } m_1 + 1 \le j \le m_1 + m_2, \end{cases}$$

then we have the following.

Proposition 2.28 *Let (M_i, π_i) be a quasitoric manifold over P_i acted on by T_i, for $i = 1, 2$, and $p\colon M \to M_2$ be a quasitoric M_1–bundle over M_2. Then M is a quasitoric manifold over $P_1 \times P_2$, which has a characteristic matrix in the form*

$$\begin{pmatrix} \lambda_1 & * \\ 0 & \lambda_2 \end{pmatrix}.$$

Conversely, if a quasitoric manifold M over $P_1 \times P_2$ has a characteristic matrix in the above form, then λ_i is a characteristic matrix on P_i, for $i = 1, 2$, and M is isomorphic to the total space of a quasitoric $M(\lambda_1)$–bundle over $M(\lambda_2)$.

We use the following lemma to prove this proposition.

Lemma 2.29 *For $i = 1, 2$, let (M_i, π_i) be a quasitoric manifold over P_i acted on by T_i, and let $p\colon M \to M_2$ be a quasitoric M_1–bundle over M_2. Moreover, take $x \in M_2$ and put $M_x := p^{-1}(x)$ and $T' := \mathrm{pr}_2^{-1}((T_2)_x)$, where $(T_2)_x$ denotes the isotropy subgroup at $x \in M_2$. Then the action of T on M restricts to a T'–action on the fiber M_x, and there exists a homomorphism $\rho\colon T' \to T_1$ such that $t \cdot y = \rho(t) \cdot y$ for any $t \in T'$ and $y \in M_x$. In particular, for each $z \in M_x$, there is a split exact sequence*

$$0 \longrightarrow (T_1)_z \longrightarrow T_z \longrightarrow (T_2)_x \longrightarrow 0.$$

Proof Take a local trivialization $\phi\colon U \times M_1 \to p^{-1}(U)$ of p around x and define $\varphi\colon M_1 \to M_x$ by $\varphi(y) := \phi(x, y)$. Since p is a T_2–equivariant fiber bundle with structure group $\mathfrak{D}(M_1)$, there is a map $\gamma\colon T' \to \mathfrak{D}(M_1)$ such that $t \cdot \varphi(y) = \varphi(\gamma(t)(y))$ for any $t \in T'$ and $y \in M_1$, which is clearly a homomorphism. Moreover, since T_1 acts on ${\pi_1}^{-1}(\mathrm{int}\, P_1)$ freely and each $\gamma(t)$, where $t \in T'$, descends to id_{P_1}, there is a unique $s_{q,y,t} \in T_1$ for each triple $q \in \mathrm{int}\, P_1$, $y \in {\pi_1}^{-1}(q)$ and $t \in T'$ such that $\gamma(t)(y) = s_{q,y,t} \cdot y$. Since $\gamma(t)$ is T_1–equivariant, $s_{q,y,t}$ does not depend on y and therefore we can put $s(q, t) := s_{q,y,t}$. Moreover, for each q, the correspondence $t \mapsto s(q, t)$ gives a homomorphism from T' to T_1. Thus we see that the correspondence $q \mapsto s(q, \cdot)$ gives a map from $\mathrm{int}\, P_1$ to $\mathrm{Hom}(T', T_1)$, the set of continuous homomorphisms equipped with the compact-open topology. Since $\mathrm{Hom}(T', T_1)$ is discrete and $\mathrm{int}\, P_1$ is connected, this map is constant. If we define ρ as the value of this map, then $\gamma(t)(y) = \rho(t) \cdot y$ for any $t \in T'$ and $y \in {\pi_1}^{-1}(\mathrm{int}\, P_1)$. This identity holds for any $t \in T'$ and $y \in M_1$ since ${\pi_1}^{-1}(\mathrm{int}\, P_1)$ is dense in M_1. Thus we obtain the former part of the lemma.

For each $z \in M_x$, the correspondence $t \mapsto (\rho(t)^{-1}, t)$ gives a section of $\mathrm{pr}_2\colon T_z \to (T_2)_x$, and $(T_1)_z$ clearly coincides with the kernel of $\mathrm{pr}_2\colon T_z \to T_2$. Thus we obtain the latter part of the lemma. □

Proof of Proposition 2.28 First, we show that the T–action on M is locally standard. Recall that any quasitoric manifold has a CW structure without odd dimensional cells, and therefore it is simply connected and its odd-degree cohomology vanishes. By using the long exact sequence of homotopy groups and the Serre spectral sequence associated with p, we see that M is also simply connected and has vanishing odd-degree cohomology. Then the local standardness follows immediately from the following theorem of Masuda: a torus manifold with vanishing odd-degree cohomology is locally standard [12, Theorem 4.1]. Here a torus manifold means an even-dimensional closed connected orientable smooth manifold equipped with an effective smooth action of the half-dimensional torus which has at least one fixed point.

Next, we prove that M/T is homeomorphic to $P_1 \times P_2$ as a manifold with corners. It is clear that p descends to a T_2–equivariant fiber bundle $\overline{p}\colon M/T_1 \to M_2$ with fiber P_1 and structure group $\{\mathrm{id}_{P_1}\}$ by the definition of $\mathfrak{D}(M_1)$. Thus we see that there is a T_2–equivariant homeomorphism $f\colon M/T_1 \to P_1 \times M_2$, where T_2 acts on $P_1 \times M_2$ by the action on the second component, such that (i) for any local trivialization $\overline{\phi}\colon U \times P_1 \to \overline{p}^{-1}(U)$ of $\overline{p}$ and any $x \in U$ the map $P_1 \to P_1$ defined by $q \mapsto \mathrm{pr}_1 \circ f \circ \overline{\phi}(x,q)$ is the identity, and (ii) $\overline{p} = \mathrm{pr}_2 \circ f$. Then f clearly descends to a homeomorphism $\overline{f}\colon M/T \to P_1 \times P_2$. We can prove that $\overline{f}$ preserves corners as follows. If we take $x \in M$ and denote by $\overline{x} \in M/T$ the equivalence class containing x, then $\operatorname{depth} \overline{x} = \dim T_x$ by definition. On the other hand, $\operatorname{depth} \overline{f}(\overline{x})$ is equal to $\operatorname{depth} \overline{x}_1 + \operatorname{depth} \overline{x}_2$, where $\overline{f}(\overline{x}) = (\overline{x}_1, \overline{x}_2) \in P_1 \times P_2$. If we take a local trivialization $\phi\colon U \times M_1 \to p^{-1}(U)$ around $p(x)$ and put $(x_2, x_1) := \phi^{-1}(x)$, then $\operatorname{depth} \overline{x}_i = \dim(T_i)_{x_i}$, for $i = 1, 2$, since $\overline{x}_i = \pi_i(x_i)$. Then, by Lemma 2.29, we have $\operatorname{depth} \overline{x} = \operatorname{depth} \overline{f}(\overline{x})$.

Next, we consider the characteristic matrix λ of M. We denote by π the projection $M \to M/T \cong P_1 \times P_2$. Take $x \in M$ and a local trivialization $\phi\colon U \times M_1 \to p^{-1}(U)$ of p around $p(x)$, and put $S_j := \ell_M(F_j)$ for $j = 1, \dots, m_1 + m_2$, where ℓ_M denotes the characteristic map associated with M. If $x \in \pi^{-1}(\operatorname{relint} F_j)$ for some $j \in \{1, \dots, m_1\}$, then $\mathrm{pr}_2(S_j) \subseteq T_2$ fixes $p(x) \in \pi_2^{-1}(\operatorname{int} P_2)$ and therefore $S_j \subseteq T_1$. Since φ is T_1–equivariant on each fiber, we see $S_j = \ell_{M_1}(F_{1,j})$ for $j = 1, \dots, m_1$. On the other hand, if $x \in \pi^{-1}(\operatorname{relint} F_j)$ for some $j \in \{m_1 + 1, \dots, m_1 + m_2\}$, then $(T_1)_x = \{0\}$ and $\mathrm{pr}_2(S_j)$ fixes $p(x) \in \pi_2^{-1}(\operatorname{relint} F_{2,j-m_1})$. Therefore we have $\mathrm{pr}_2(S_j) = \ell_{M_2}(F_{2,j-m_1})$. Thus we obtain the former part of the proposition.

Finally, we prove the latter part. It is clear that λ_i, for $i = 1, 2$, is a characteristic matrix on P_i. We can assume that $M = M(\lambda) := \mathcal{Z}_{P_1 \times P_2}/T_\lambda$ (see Construction 2.8) since they are isomorphic. Put $m := m_1 + m_2$ and identify T^{m_1} with $T^{m_1} \times \{0\} \subseteq T^m$. Then $T_{\lambda_1} \subseteq T_\lambda$ and $\overline{T}_\lambda := T_\lambda / T_{\lambda_1}$ is isomorphic to T_{λ_2} through the projection to T^{m_2}. If we regard T^m as acting on $\mathcal{Z}_{P_2}$ through the projection to T^{m_2}, then, since $\mathcal{Z}_{P_1 \times P_2} = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}$ and $M(\lambda) = (M(\lambda_1) \times \mathcal{Z}_{P_2})/\overline{T}_\lambda$, we have the following commutative diagram:

$$\begin{tikzcd}
M(\lambda_1) \arrow[r] & M(\lambda_1) \times \mathcal{Z}_{P_2} \arrow[r, "\mathrm{pr}_2"] \arrow[d] & \mathcal{Z}_{P_2} \arrow[d] \\
 & M(\lambda) \arrow[r] & M(\lambda_2)
\end{tikzcd}$$

Here the upper row is a T^m/T_{λ_1}–equivariant fiber bundle with structure group $\mathfrak{D}(M_1)$ and the right vertical arrow is a principal $\overline{T}_\lambda$–bundle. Thus the proof is completed by Lemma 2.21. $\square$

Let P_i, for $i = 1, \dots, l$, be an n_i–dimensional simple polytope with a facet labeling $F_{i,1}, \dots, F_{i,m_i}$, and put $n := \sum n_i$, $m := \sum m_i$ and $P := P_1 \times \cdots \times P_l$. Given $(\psi_i, \rho_i) \in \mathrm{GL}_{n_i}(\mathbb{Z}) \times R(P_i)$, for $i = 1, \dots, l$, we define $(\psi_1, \rho_1) \times \cdots \times (\psi_l, \rho_l) \in \mathrm{GL}_n(\mathbb{Z}) \times R(P)$ as follows: define $\psi \in \mathrm{GL}_n(\mathbb{Z})$ and $\rho \in R(P)$ so that

$$\psi = \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \psi_l \end{pmatrix} \quad \text{and} \quad \iota(\rho) = \begin{pmatrix} \iota(\rho_1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \iota(\rho_l) \end{pmatrix},$$

and put $(\psi_1, \rho_1) \times \cdots \times (\psi_l, \rho_l) := (\psi, \rho)$. Here we label the facets of P by $F_1, \dots, F_m$, where

$$\begin{aligned} &F_{m_1+\cdots+m_{i-1}+j} \\ &\quad := P_1 \times \cdots \times P_{i-1} \times F_{i,j} \times P_{i+1} \times \cdots \times P_l \quad \text{for } i = 1, \dots, l \text{ and } j = 1, \dots, m_i. \end{aligned}$$

Lemma 2.30 *Let M_i be a quasitoric manifold over an n_i–dimensional simple polytope P_i, for $i = 1, \dots, l$, and consider a sequence*

$$B_l \xrightarrow{p_{l-1}} B_{l-1} \xrightarrow{p_{l-2}} \cdots \xrightarrow{p_2} B_2 \xrightarrow{p_1} M_1,$$

where each p_i is a quasitoric M_{i+1}–bundle. For each i, take a characteristic matrix λ_i of M_i and $(\psi_i, \rho_i) \in \mathrm{GL}_{n_i}(\mathbb{Z}) \times R(P_i)$, and put $\lambda'_i := (\psi_i, \rho_i) \cdot \lambda_i$. Then there exists a sequence

$$B'_l \xrightarrow{p'_{l-1}} B'_{l-1} \xrightarrow{p'_{l-2}} \cdots \xrightarrow{p'_2} B'_2 \xrightarrow{p'_1} M(\lambda'_1),$$

where each p'_i is a quasitoric $M(\lambda'_{i+1})$–bundle, and a weakly equivariant homeomorphism from B_l to B'_l represented by $(\psi_l, \rho_l) \times \cdots \times (\psi_1, \rho_1)$.

Proof Put $(\psi'_i, \rho'_i) := (\psi_i, \rho_i) \times (\psi_{i-1}, \rho_{i-1}) \times \cdots \times (\psi_1, \rho_1)$ for $i = 1, \dots, l$. By an iterated use of Proposition 2.28, we can take, for each i, a characteristic matrix μ_i of B_i in the form

$$\begin{pmatrix} \lambda_i & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_1 \end{pmatrix}.$$

Then, if we put $\mu'_i := (\psi'_i, \rho'_i) \cdot \mu_i$ for $i = 1, \dots, l$, we see that each $M(\mu'_{i+1})$ is a quasitoric $M(\lambda'_{i+1})$–bundle over $M(\mu'_i)$ by Proposition 2.28. The proof is completed by setting $B'_i := M(\mu'_i)$. □

2.3 Quasitoric manifolds over I^n

Now we restrict ourselves to the case $P = I^n$. Hereafter, we always use the facet labeling $F_1, \dots, F_{2n}$ of I^n defined by

$$F_i := \{(x_1, \dots, x_n) \in I^n \mid x_i = 0\} \quad \text{and} \quad F_{n+i} := \{(x_1, \dots, x_n) \in I^n \mid x_i = 1\}$$

for $i = 1, \dots, n$. Note that this facet labeling is different from the one used in Section 2.2. We easily see that $\mathrm{Aut}(I^n)$ is generated by $\rho_{i,j} := (i\ j)(i+n\ j+n)$ and $\rho_k := (k\ k+n)$, where $i, j, k = 1, \dots, n$ and we regard $\mathrm{Aut}(I^n)$ as a subgroup of the symmetric group $\mathfrak{S}_{2n}$ by using the facet labeling, as in Definition 2.13.

Definition 2.31 Let ξ be a square matrix of order n. We call ξ a *characteristic square on* I^n if each diagonal component of ξ is equal to 1 and $(E_n\ \xi)$ is a characteristic matrix on I^n. We denote by Ξ_n the set of characteristic squares on I^n. For the convenience of notation, we identify a characteristic square ξ with the characteristic matrix $(E_n\ \xi)$; for example, we write $M(\xi)$ instead of $M((E_n\ \xi))$.

Remark 2.32 Due to Proposition 2.15, any quasitoric manifold over I^n is weakly equivariantly homeomorphic to $M(\xi)$ for some characteristic square ξ.

Definition 2.33 For a characteristic square $\xi = (\xi_{i,j})$ on I^n, we define a graded ring $H^*(\xi)$, canonically isomorphic to $H^*(M(\xi); \mathbb{Z})$, as follows. Let $\mathbb{Z}[X_1, \dots, X_n]$ be the polynomial ring whose generators have degree 2, and $\mathcal{I}_\xi$ be the ideal generated by $u_i(\xi)X_i$, where $i = 1, \dots, n$ and $u_i(\xi) := \sum_{j=1}^n \xi_{i,j} X_j$. Then $H^*(\xi)$ is defined by

$$H^*(\xi) := \mathbb{Z}[X_1, \dots, X_n]/\mathcal{I}_\xi.$$

Next, we consider the bundle-type quasitoric manifolds over I^n.

Definition 2.34 Let ξ be a characteristic square on I^n and $n_1, \dots, n_l$ be positive integers summing up to n. Then ξ is called $(\xi_1, \dots, \xi_l)$*–type* if it is in the form

$$\begin{pmatrix} \xi_1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \xi_l \end{pmatrix},$$

where each ξ_i is a characteristic square on I^{n_i}.

Lemma 2.35 *Let ξ_i be a characteristic square on I^{n_i}, for $i=1,\dots,l$, and consider a sequence*

$$B_l \xrightarrow{p_{l-1}} B_{l-1} \xrightarrow{p_{l-2}} \cdots \xrightarrow{p_2} B_2 \xrightarrow{p_1} B_1,$$

where $B_1 = M(\xi_1)$. If each p_i is a quasitoric $M(\xi_{i+1})$–bundle, then B_l is weakly equivariantly homeomorphic to $M(\xi)$ for some $(\xi_l,\dots,\xi_1)$–type characteristic square ξ.

Proof Denote $\sum_{i=1}^{l} n_i$ by n. By an iterated use of Proposition 2.28, we see that B_l has a characteristic matrix λ in the form

$$\begin{pmatrix} E_{n_l} & * & \cdots & * & \xi_l & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * & \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & E_{n_1} & 0 & \cdots & 0 & \xi_1 \end{pmatrix}.$$

We denote the left $n\times n$ part of λ by A. Then, since A^{-1} is also in the form

$$\begin{pmatrix} E_{n_l} & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & E_{n_1} \end{pmatrix},$$

$A^{-1}\lambda = (E_n\ \xi)$ for some $(\xi_l,\dots,\xi_1)$–type characteristic square ξ. Thus the proof is completed by Proposition 2.15. □

3 $(\mathbb{C}P^2 \# \mathbb{C}P^2)$–bundle type quasitoric manifolds

In this section we give the proof of Theorem 1.2. We use the facet labeling $F_1,\dots,F_{2n}$ of I^n defined in Section 2.3. Let us begin with the following proposition.

Proposition 3.1 *Any quasitoric manifold over I^2 is weakly equivariantly homeomorphic to $M(\chi)$, where χ denotes a characteristic square in the following form:*

(1) $$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Proof Let λ be a characteristic matrix of a quasitoric manifold M over I^2. Since λ satisfies the nonsingularity condition, there is a pair $(\psi,\rho)\in \mathrm{GL}_2(\mathbb{Z})\times(\mathbb{Z}/2)^4$ such that

$$\psi\cdot\lambda\cdot\rho = \begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & b & 1 \end{pmatrix},$$

where a and b are integers satisfying $ab = 1\pm 1$, ie $(a,b)=(0,b)$, $(a,b)=(a,0)$ $(a,b)=\pm(1,2)$ or $(a,b)=\pm(2,1)$. Moreover, by multiplying the first row, the first column and the third column by -1 if necessary, we can assume that $b \geq 0$. If we put $\lambda' := \psi \cdot \lambda \cdot \rho$ and $\sigma := (1\ 2)(3\ 4) \in \mathrm{Aut}(I^2)$, then we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \lambda' \cdot \iota(\sigma) = \begin{pmatrix} 1 & 0 & 1 & b \\ 0 & 1 & a & 1 \end{pmatrix}.$$

Thus the proof is completed by Proposition 2.15. □

Throughout this section, we denote by κ^2 the latter characteristic square in (1).

Remark 3.2 We easily see that $M(\kappa^2)$ is homeomorphic to $\mathbb{C}P^2 \# \mathbb{C}P^2$. For instance, since $(E_2\, \kappa^2)$ is decomposed into a connected sum [9, Section 3.2] and $\mathbb{C}P^2$ is the only quasitoric manifold over Δ^2 [8, Example 1.18], $M(\kappa^2)$ is homeomorphic to $\mathbb{C}P^2 \# \mathbb{C}P^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. As $H^*(\kappa^2)$ (Definition 2.33) is not isomorphic to $H^*(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}; \mathbb{Z})$, we have $M(\kappa^2) \cong \mathbb{C}P^2 \# \mathbb{C}P^2$. Note that, by [4, Proposition 6.2] of Choi, Masuda and Suh, the other quasitoric manifolds $M(\chi)$ of Proposition 3.1 are the Hirzebruch surfaces. In particular, they are not homeomorphic to $\mathbb{C}P^2 \# \mathbb{C}P^2$.

Next, we consider the graded ring automorphisms of $H^*(\kappa^2)$. If we put $x := X_2$ and $y := u_2(\kappa^2) = X_1 + X_2$, using the notation of Definition 2.33, then

$$H^*(\kappa^2) = \mathbb{Z}[x, y]/(x^2 - y^2, xy).$$

Let us denote by $\mathrm{Aut}(H^*(\kappa^2))$ the group of graded ring automorphisms of $H^*(\kappa^2)$ and regard it as a subgroup of $\mathrm{GL}_2(\mathbb{Z})$ by identifying an automorphism φ with the matrix A defined by

$$\varphi \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Lemma 3.3
$$\mathrm{Aut}(H^*(\kappa^2)) = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \right\}.$$

Proof Let φ be an automorphism of $H^*(\kappa^2)$ identified with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that a and b (resp. c and d) are coprime. Since $\varphi(x)\varphi(y) = (ac + bd)y^2 = 0$ in $H^*(\kappa^2)$, we have $ac = -bd$. In particular, if $a \neq 0$, we see that a divides d and vice versa, implying $a = \pm d$. We put $\epsilon := d/a = \pm 1$, and then obtain $c = -\epsilon b$. Moreover, since φ is an automorphism, $\det A = \epsilon(a^2 + b^2) = \pm 1$. Thus we have $a = \epsilon d = \pm 1$ and $b = c = 0$. Similarly, if we assume $b \neq 0$, then we have $|b| = |c| = 1$ and $a = d = 0$. Thus the proof is completed. □

Recall (Definition 2.11) that we put $[m]_\pm := \{\pm 1, \dots, \pm m\}$ and defined R_m as the group of $(\mathbb{Z}/2)$–equivariant permutations of $[m]_\pm$. Let us describe $\rho \in R_m$ by $\rho = (\rho(1), \dots, \rho(m))$. Then, if we put $\tau_1 := (-1, -2, -3, -4)$, $\tau_2 := (3, -2, 1, 4)$ and $\tau_3 := (-1, 4, 3, 2)$, they belong to $R(I^2)$ and there are $\psi_i \in \mathrm{GL}_2(\mathbb{Z})$, for $i = 1, 2, 3$, such that $(\psi_i, \tau_i) \cdot (E_2\, \kappa^2) = (E_2\, \kappa^2)$. By Proposition 2.15, there are weakly equivariant self-homeomorphisms f_i, for $i = 1, 2, 3$, of $M(\kappa^2)$ represented by $(\psi_i^{-1}, \tau_i^{-1})$, and by Corollary 2.22, we have

$$f_1^* := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f_2^* := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad f_3^* := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

where we canonically identify $H^*(M(\kappa^2); \mathbb{Z})$ with $H^*(\kappa^2)$ (note that, in the notation of Corollary 2.18, $x = v_4$ and $y = -v_2$). Since these matrices generate $\mathrm{Aut}(H^*(\kappa^2))$ by Lemma 3.3, we have the following.

Lemma 3.4 *Any graded ring automorphism of $H^*(M(\kappa^2); \mathbb{Z})$ is induced by a weakly equivariant self-homeomorphism of $M(\kappa^2)$.*

Then we consider the isomorphisms between the cohomology rings of $(\mathbb{C}P^2 \# \mathbb{C}P^2)$–bundle type quasitoric manifolds.

Definition 3.5 We denote by $\mathcal{K}_n$ the set of $(\kappa^2, \dots, \kappa^2)$–type characteristic squares on I^{2n}. For $\xi = (\xi_{i,j}) \in \mathcal{K}_n$ and integers h and k such that $0 < h \le k \le n$, we define $\xi_{[h,k]} := (\xi_{i,j})_{2h-1 \le i,j \le 2k}$, which belongs to $\mathcal{K}_{k-h+1}$. We identify $H^*(\xi_{[h,n]})$ with the subring of $H^*(\xi)$ generated by $X_{2h-1}, \dots, X_{2n}$ and $H^*(\xi_{[h,k]})$ with the quotient ring $H^*(\xi_{[h,n]})/(X_{2k+1}, \dots, X_{2n})$, where we use the notation of Definition 2.33.

Lemma 3.6 *Any n–stage $(\mathbb{C}P^2 \# \mathbb{C}P^2)$–bundle type quasitoric manifold is weakly equivariantly homeomorphic to $M(\xi)$ for some $\xi \in \mathcal{K}_n$.*

Proof By Proposition 3.1 and Remark 3.2, if a quasitoric manifold M over I^2 is homeomorphic to $\mathbb{C}P^2 \# \mathbb{C}P^2$, then M is weakly equivariantly homeomorphic to $M(\kappa^2)$. Therefore, by Lemma 2.30, any $(\mathbb{C}P^2 \# \mathbb{C}P^2)$–bundle type quasitoric manifold is weakly equivariantly homeomorphic to a $\{M(\kappa^2)\}$–bundle type quasitoric manifold. Then the proof is completed by Lemma 2.35. □

Thus we see that we only have to consider $M(\xi)$, for $\xi \in \mathcal{K}_n$, to prove Theorem 1.2.

Next, let ξ' be a characteristic square on I^n, where $n > 2$, which is (κ^2, ξ'_0)–type for some characteristic square ξ'_0 on I^{n-2}, and denote the first and second rows of ξ' by $(1, 2, s_3, \dots, s_n)$ and $(1, 1, t_3, \dots, t_n)$, respectively. We continue to use the notation of Definition 2.33.

Lemma 3.7 *Let $\varphi\colon \mathbb{Z}[X_1, X_2] \to \mathbb{Z}[X_1, \dots, X_n]$ be a graded ring monomorphism which maps $\mathcal{I}_{\kappa^2}$ into $\mathcal{I}_{\xi'}$. Also put $\varphi(X_1) = \sum_{i=1}^n a_i X_i$ and $\varphi(X_2) = \sum_{i=1}^n b_i X_i$ and assume that for any prime p the* mod p *reductions of $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are linearly independent. Then one of the following holds:*

(i) $a_i = b_i = s_i = t_i = 0$ *for* $i = 3, 4, \dots, n$;

(ii) $a_1 = a_2 = b_1 = b_2 = 0$.

Proof We prove the lemma by showing that (i) holds under the assumption that $(a_1, a_2, b_1, b_2) \neq (0, 0, 0, 0)$. Since $\varphi(X_1(X_1 + 2X_2))$ and $\varphi(X_2(X_1 + X_2))$ belong to $\mathcal{I}_{\xi'}$, we have

$$\begin{aligned}
&\varphi(X_1(X_1+2X_2)) \\
&\qquad = \Big(\sum_{i=1}^n a_i X_i\Big)\Big\{\sum_{i=1}^n (a_i + 2b_i) X_i\Big\} \\
&\qquad \equiv \alpha_1 X_1\Big(X_1 + 2X_2 + \sum_{j=3}^n s_j X_j\Big) + \beta_1 X_2\Big(X_1 + X_2 + \sum_{j=3}^n t_j X_j\Big) \mod W,
\end{aligned}$$

$$\begin{aligned}
&\varphi(X_2(X_1+X_2)) \\
&\qquad = \Big(\sum_{i=1}^n b_i X_i\Big)\Big\{\sum_{i=1}^n (a_i + b_i) X_i\Big\} \\
&\qquad \equiv \alpha_2 X_1\Big(X_1 + 2X_2 + \sum_{j=3}^n s_j X_j\Big) + \beta_2 X_2\Big(X_1 + X_2 + \sum_{j=3}^n t_j X_j\Big) \mod W,
\end{aligned}$$

for some integers α_i and β_i, for $i = 1, 2$, where W denotes the submodule spanned by $\{X_p X_q \mid p, q \geq 3\}$. Since the coefficients of X_1^2 in $\varphi(X_1(X_1 + 2X_2))$ and in $\varphi(X_2(X_1 + X_2))$ are $a_1(a_1 + 2b_1)$ and $b_1(a_1 + b_1)$, respectively, we obtain $\alpha_1 = a_1(a_1 + 2b_1)$ and $\alpha_2 = b_1(a_1 + b_1)$. Similarly, we see $\beta_1 = a_2(a_2 + 2b_2)$ and $\beta_2 = b_2(a_2 + b_2)$. Thus we obtain the following equations:

$$\begin{aligned}
a_1(a_2 + 2b_2) + a_2(a_1 + 2b_1) &= 2a_1(a_1 + 2b_1) + a_2(a_2 + 2b_2), \\
a_1(a_i + 2b_i) + a_i(a_1 + 2b_1) &= a_1(a_1 + 2b_1)s_i && \text{for } i \geq 3, \\
a_2(a_i + 2b_i) + a_i(a_2 + 2b_2) &= a_2(a_2 + 2b_2)t_i && \text{for } i \geq 3, \\
b_1(a_2 + b_2) + b_2(a_1 + b_1) &= 2b_1(a_1 + b_1) + b_2(a_2 + b_2), \\
b_1(a_i + b_i) + b_i(a_1 + b_1) &= b_1(a_1 + b_1)s_i && \text{for } i \geq 3, \\
b_2(a_i + b_i) + b_i(a_2 + b_2) &= b_2(a_2 + b_2)t_i && \text{for } i \geq 3.
\end{aligned}$$

For convenience, we rewrite these equations as follows:

(2) $$(a_1 - a_2)(a_2 + 2b_2) = (2a_1 - a_2)(a_1 + 2b_1),$$

(3) $$a_1(a_i + 2b_i) = (s_i a_1 - a_i)(a_1 + 2b_1) \quad \text{for } i \geq 3,$$

(4) $$a_2(a_i + 2b_i) = (t_i a_2 - a_i)(a_2 + 2b_2) \quad \text{for } i \geq 3,$$

(5) $$(b_1 - b_2)(a_2 + b_2) = (2b_1 - b_2)(a_1 + b_1),$$

(6) $$b_1(a_i + b_i) = (s_i b_1 - b_i)(a_1 + b_1) \quad \text{for } i \geq 3,$$

(7) $$b_2(a_i + b_i) = (t_i b_2 - b_i)(a_2 + b_2) \quad \text{for } i \geq 3.$$

First, we assume that $a_1 - a_2$, $2a_1 - a_2$, $b_1 - b_2$ and $2b_1 - b_2$ are all nonzero. Let $k > 0$ be the greatest common divisor of $a_1 - a_2$ and $2a_1 - a_2$, and $l > 0$ be that of $b_1 - b_2$ and $2b_1 - b_2$. Suppose that r divides $a_1 + 2b_1$ and $a_2 + 2b_2$. If we assume that r does not divide k, then there is a prime number r' which divides $r/(k,r)$ but does not divide $k/(k,r)$, where (k,r) means the greatest common divisor. Then, by (3) and (4), r' divides $a_i + 2b_i$ for $i = 1, 2, \ldots, n$, but this contradicts the assumption of linear independence. Thus we see that any common divisor of $a_1 + 2b_1$ and $a_2 + 2b_2$ divides k. In particular, $a_1 + 2b_1$, $a_2 + 2b_2 \neq 0$. Similarly, we shall show that any common divisor of $a_1 + b_1$ and $a_2 + b_2$ divides l.

Let $p > 0$ be the greatest common divisor of $a_1 + 2b_1$ and $a_2 + 2b_2$, and $q > 0$ be that of $a_1 + b_1$ and $a_2 + b_2$. Since $(a_1 - a_2)/k$ and $(2a_1 - a_2)/k$ (resp. $(a_1 + 2b_1)/p$ and $(a_2 + 2b_2)/p$) are prime to each other, we obtain

$$\frac{(a_1 - a_2)/k}{(a_1 + 2b_1)/p} = \frac{(2a_1 - a_2)/k}{(a_2 + 2b_2)/p} = \pm 1$$

from (2). This can be written as

(8) $$\frac{a_1 - a_2}{a_1 + 2b_1} = \frac{2a_1 - a_2}{a_2 + 2b_2} = \pm \frac{k}{p} \in \mathbb{Z}.$$

Similarly, we obtain

(9) $$\frac{b_1 - b_2}{a_1 + b_1} = \frac{2b_1 - b_2}{a_2 + b_2} = \pm \frac{l}{q} \in \mathbb{Z}.$$

Define

$$k' := \frac{a_1 - a_2}{a_1 + 2b_1} \in \mathbb{Z} \quad \text{and} \quad l' := \frac{b_1 - b_2}{a_1 + b_1} \in \mathbb{Z}.$$

Then, from (8) and (9), we have the following equations:

(10) $$\begin{cases} (1 - k')a_1 - a_2 - 2k'b_1 = 0, \\ 2a_1 + (-1 - k')a_2 - 2k'b_2 = 0, \\ -l'a_1 + (1 - l')b_1 - b_2 = 0, \\ -l'a_2 + 2b_1 + (-1 - l')b_2 = 0. \end{cases}$$

Since we assume $a_i, b_i \neq 0$ for $i = 1, 2$, the determinant of the matrix

$$A := \begin{pmatrix} 1-k' & -1 & -2k' & 0 \\ 2 & -1-k' & 0 & -2k' \\ -l' & 0 & 1-l' & -1 \\ 0 & -l' & 2 & -1-l' \end{pmatrix}$$

equals 0. Therefore, since $\det A = (k'+l')^2 + (k'l'+1)^2$, we obtain $(k', l') = (1, -1)$ or $(k', l') = (-1, 1)$.

If $(k', l') = (1, -1)$, we have

$$a_1 = b_2 - 2b_1 \quad \text{and} \quad a_2 = -2b_1$$

from (10). If $b_1 = 0$, we easily obtain $a_j = b_j = s_j = t_j = 0$ for $j \geq 2$ from (3), (4), (6) and (7). On the other hand, if we assume $b_2 = 0$, we similarly obtain that $a_j = b_j = s_j = t_j = 0$ for $j \geq 2$ (but this contradicts the assumption of linear independence since $(a_1, \ldots, a_n) \equiv 0 \bmod 2$).

Then we can assume that $b_1, b_2 \neq 0$. Putting $b_i' := b_i / l$ for $i = 1, 2$, we obtain the following equations from (3), (4), (6) and (7):

$$(b_2' - 2b_1')(a_i + 2b_i) = \{s_i(b_2 - 2b_1) - a_i\} b_2' \quad \text{for } i \geq 3, \tag{11}$$

$$b_1'(a_i + 2b_i) = (-2t_i b_1 - a_i)(b_1' - b_2') \quad \text{for } i \geq 3, \tag{12}$$

$$b_1'(a_i + b_i) = (s_i b_1 - b_i)(b_2' - b_1') \quad \text{for } i \geq 3, \tag{13}$$

$$b_2'(a_i + b_i) = (t_i b_2 - b_i)(-2b_1' + b_2') \quad \text{for } i \geq 3. \tag{14}$$

If b_2 is odd, $b_2' - 2b_1'$ and b_2' (resp. b_1' and $b_1' - b_2'$) are prime to each other, and hence we obtain the following:

$$\frac{a_i + 2b_i}{b_2'} = s_i l - \frac{a_i}{b_2' - 2b_1'} \in \mathbb{Z} \quad \text{for } i \geq 3, \tag{15}$$

$$\frac{a_i + 2b_i}{b_1' - b_2'} = -2t_i l - \frac{a_i}{b_1'} \in \mathbb{Z} \quad \text{for } i \geq 3, \tag{16}$$

$$\frac{a_i + b_i}{b_2' - b_1'} = s_i l - \frac{b_i}{b_1'} \in \mathbb{Z} \quad \text{for } i \geq 3, \tag{17}$$

$$\frac{a_i + b_i}{b_2' - 2b_1'} = t_i l - \frac{b_i}{b_2'} \in \mathbb{Z} \quad \text{for } i \geq 3. \tag{18}$$

In particular, b_1' divides a_i and b_i for $i = 3, 4, \ldots, n$. Putting $a_i' := a_i / b_1'$ and

$b_i' := b_i/b_1'$ for $i \geq 3$, from (15) and (17) (resp. (16) and (18)), we obtain

$$\text{(19)}\quad \{(a_i'+2b_i')(b_2'-2b_1')+a_i'b_2'\}(b_2'-b_1')b_1' = \{(a_i'+b_i')b_1'+b_i'(b_2'-b_1')\}(b_2'-2b_1')b_2',$$

$$\text{(20)}\quad \{(a_i'+2b_i')b_1'+a_i'(b_1'-b_2')\}(b_2'-2b_1')b_2' = -2\{(a_i'+b_i')_2'+b_i'(b_2'-2b_1')\}b_1'(b_1'-b_2'),$$

for $i = 3, \dots, n$. Since b_1' is prime to each of $b_2'-2b_1'$, b_2' and $b_2'-b_1'$, we see from (19) that b_1' divides b_i' for $i = 3, \dots, n$ and from (20) that b_1' divides a_i' for $i = 3, \dots, n$. Repeating this procedure, we see that any power of b_1' divides a_i and b_i for $i = 3, \dots, n$. By similar arguments, we can show that any powers of b_2', $2b_1'-b_2'$ and $b_1'-b_2'$ divide a_i and b_i for $i = 3, \dots, n$. Since b_1', b_2', $2b_1'-b_2'$ and $b_1'-b_2'$ cannot be ± 1 simultaneously, $a_i = 0$ and $b_i = 0$ for $i = 3, \dots, n$. Then we obtain $s_i = 0$ and $t_i = 0$ for $i = 3, \dots, n$ from (15) and (16).

Otherwise, if b_2' is even (and hence b_1' is odd), put $b_2'' := b_2'/2$. Then we have

$$\begin{aligned}
(b_2''-b_1')(a_i+2b_i) &= \{s_i(b_2-2b_1)-a_i\}b_2'' && \text{for } i \geq 3,\\
b_1'(a_i+2b_i) &= (-2t_ib_1-a_i)(b_1'-2b_2'') && \text{for } i \geq 3,\\
b_1'(a_i+b_i) &= (s_ib_1-b_i)(2b_2''-b_1') && \text{for } i \geq 3,\\
b_2''(a_i+b_i) &= (t_ib_2-b_i)(b_2''-b_1') && \text{for } i \geq 3.
\end{aligned}$$

By an argument similar to that above, we again obtain that a_i, b_i, s_i and t_i are all 0 for $i = 3, \dots, n$. We obtain (i) in the same way if $(k', l') = (-1, 1)$.

Finally, we consider the case where at least one of a_1-a_2, $2a_1-a_2$, b_1-b_2 and $2b_1-b_2$ equals zero. Note that X_2^2 and X_pX_q, where $p = 1, 2$ and $q = 3, \dots, n$, form a basis of $H^4(\xi')/W$. Then, by considering the equation $\varphi(X_2(X_1+X_2)) = 0$ in $H^4(\xi')/W$, we see that $(a_1, a_2) = (0, 0)$ implies $(b_1, b_2) = (0, 0)$ and vice versa. Since we assume $(a_1, a_2, b_1, b_2) \neq (0, 0, 0, 0)$, as mentioned at the beginning of this proof, we have $(a_1, a_2) \neq (0, 0)$ and $(b_1, b_2) \neq (0, 0)$.

If $a_1-a_2 = 0$, then we have $0 = a_1(a_1+2b_1)$ by (2), which implies $a_1+2b_1 = 0$ since $(a_1, a_2) \neq (0, 0)$. Similarly, we obtain $a_i+2b_i = 0$ for $i \geq 3$ by (3), but this contradicts the assumption of linear independence since $(a_1, \dots, a_n) \equiv 0 \bmod 2$.

If $2a_1-a_2 = 0$, then we have $a_2+2b_2 = 0$ by (2), which implies $b_2 = -a_1$. By (4) and (3), we have $a_i+2b_i = 0$ for $i \geq 3$ and then $s_ia_1-a_i = 0$ for $i \geq 3$. Moreover, $a_i+t_ib_2 = 0$ by (7). We have $b_1(a_1+b_1) = 0$ by (5). If $b_1 = 0$, then (6) implies $b_i = 0$ for $i \geq 3$, and therefore we obtain $a_i = s_i = t_i$ for $i \geq 3$, since both a_1 and b_2 are nonzero. If $a_1+b_1 = 0$, then (6) implies $a_i+b_i = 0$ for $i \geq 3$, and therefore $a_i = b_i = s_i = t_i = 0$ for $i \geq 3$. We can show $a_i = b_i = s_i = t_i = 0$ for $i \geq 3$ similarly in the other two cases. □

For $\rho \in \mathfrak{S}_m$, the symmetric group, and a positive integer k, we define $\rho[k] \in \mathfrak{S}_{km}$ so that $\rho[k](ik - j) = \rho(i) \cdot k - j$ for $i = 1, \dots, m$ and $j = 0, \dots, k-1$.

Lemma 3.8 *Let $\xi' \in \mathcal{K}_n$ and let $s_{i,j}$ be its $(i, j)^{th}$ entry. Suppose that there exist two integers p and q which satisfy the following:*

(a) $0 < p < q \leq n$;

(b) $s_{i,j} = 0$ *if* $i = 2p-1, 2p$ *and* $j = 2p+1, \dots, 2q$.

Then there exist $\eta' \in \mathcal{K}_n$ and a weakly equivariant homeomorphism $f \colon M(\eta') \to M(\xi')$ such that $f^*(X_i) = X_{\sigma[2](i)}$ for $i = 1, \dots, 2n$, where σ denotes the cyclic permutation $(q\ q-1\ \dots\ p)$.

Proof Recall that $\rho_{i,j} := (i\ j)(i+n\ j+n) \in \mathrm{Aut}(I^n)$. If we put $\omega_k := \rho_{k,k+1}[2]$ and $\omega_{p,q} := \omega_{q-1} \circ \cdots \circ \omega_p$, then $\omega_{p,q} \in \mathrm{Aut}(I^{2n})$. Also, we put $\psi := \iota(\sigma[2])^{-1} \in \mathrm{GL}_{2n}(\mathbb{Z})$ and $\sigma_k := (k\ k+1) \in \mathfrak{S}_n$. Since ι is an antihomomorphism,

$$(\psi, \omega_{p,q}) \cdot (E_{2n}\, \xi') = \iota(\sigma_{q-1}[2])^{-1} \cdots \iota(\sigma_p[2])^{-1} \cdot (E_{2n}\, \xi') \cdot \iota(\omega_p) \cdots \iota(\omega_{q-1}).$$

We can easily check that $\iota(\sigma_p[2])^{-1} \cdot (E_{2n}\, \xi') \cdot \iota(\omega_p) = (E_{2n}\, \xi'')$ where $\xi'' = (s'_{i,j}) \in \mathcal{K}_n$ and the following is satisfied: $s'_{i,j} = 0$ if $i = 2p+1, 2p+2$ and $j = 2p+3, \dots, 2q$. By induction, we see that $(\psi, \omega_{p,q}) \cdot (E_{2n}\, \xi') = (E_{2n}\, \eta')$ for some $\eta' \in \mathcal{K}_n$. Then the proof is completed by Proposition 2.15 and Corollary 2.22. □

Lemma 3.9 *Let $\xi, \xi' \in \mathcal{K}_n$ and $\varphi \colon H^*(\xi) \to H^*(\xi')$ be a graded ring isomorphism. Then there exist $\eta' \in \mathcal{K}_n$ and a weakly equivariant homeomorphism $f \colon M(\eta') \to M(\xi')$ such that $f^* \circ \varphi$ preserves the ideal $(X_{2i-1}, \dots, X_{2n})$ for each $i = 1, \dots, n$.*

Proof First, by Lemma 3.7 and Lemma 3.8, there are $\eta'' \in \mathcal{K}_n$ and a weakly equivariant homeomorphism $f' \colon M(\eta'') \to M(\xi')$ such that $f'^* \circ \varphi$ preserves the ideal (X_{2n-1}, X_{2n}). Therefore, without loss of generality, we can assume that φ preserves (X_{2n-1}, X_{2n}).

We prove the lemma by induction on n. The lemma is trivial if $n = 1$. Suppose that the lemma holds for $n-1$. If φ preserves the ideal (X_{2n-1}, X_{2n}), then it descends to a graded ring isomorphism $\overline{\varphi} \colon H^*(\xi_{[1,n-1]}) \to H^*(\xi'_{[1,n-1]})$; see Definition 3.5. By the induction hypothesis, there are $\eta'_0 \in \mathcal{K}_{n-1}$ and a weakly equivariant homeomorphism $f_0 \colon M(\eta'_0) \to M(\xi'_{[1,n-1]})$ such that $f_0^* \circ \overline{\varphi}$ preserves the ideal $(X_{2i-1},\dots,X_{2n-2})$ for each $i=1,\dots,n-1$. Let (ψ_0,ρ_0) be the representation of f_0, and put $(\psi,\rho):=(\psi_0,\rho_0)\times(E_2,e)\in \mathrm{GL}_{2n}(\mathbb{Z})\times R(I^{2n})$, where e denotes the identity element of $R(I^2)$. The product $(\psi_0,\rho_0)\times(E_2,e)$ is defined before Lemma 2.30, but we should note that now we use a different facet labeling. If we define a characteristic square η' on I^{2n} so that $(\psi,\rho)\cdot(E_{2n}\,\eta')=(E_{2n}\,\xi')$, then there exists a weakly equivariant homeomorphism $f\colon M(\eta')\to M(\xi')$ represented by (ψ,ρ). We can easily check that $\eta'\in\mathcal{K}_n$ and $f^*\circ\varphi$ preserves the ideal $(X_{2i-1},\dots,X_{2n})$ for each $i=1,\dots,n$. □

Corollary 3.10 *Let $\xi,\xi'\in\mathcal{K}_n$ and $\varphi\colon H^*(\xi)\to H^*(\xi')$ be a graded ring isomorphism. Then there exist $\eta'\in\mathcal{K}_n$ and a weakly equivariant homeomorphism $f\colon M(\eta')\to M(\xi')$ such that*

$$f^*\circ\varphi\begin{pmatrix}X_1\\ \vdots\\ X_{2n}\end{pmatrix}=\begin{pmatrix}E_2 & * & \cdots & *\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & *\\ 0 & \cdots & 0 & E_2\end{pmatrix}\begin{pmatrix}X_1\\ \vdots\\ X_{2n}\end{pmatrix}.$$

Proof By Lemma 3.9, there exist $\eta''\in\mathcal{K}_n$ and a weakly equivariant homeomorphism $f_1\colon M(\eta'')\to M(\xi')$ such that

$$f_1^*\circ\varphi\begin{pmatrix}X_1\\ \vdots\\ X_{2n}\end{pmatrix}=\begin{pmatrix}\alpha_1 & * & \cdots & *\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & *\\ 0 & \cdots & 0 & \alpha_n\end{pmatrix}\begin{pmatrix}X_1\\ \vdots\\ X_{2n}\end{pmatrix},$$

where each of the α_i, for $i=1,\dots,n$, gives an automorphism of $H^*(\kappa^2)$ since $\xi_{[i,i]}=\xi'_{[i,i]}=\kappa^2$. Furthermore, by Lemma 3.4, there is a weakly equivariant self-homeomorphism $h_i\colon M(\kappa^2)\to M(\kappa^2)$ such that $h_i^*=\alpha_i$. For $i=1,\dots,n$, take the representation (ψ_i,ρ_i) of h_i, and put $(\psi,\rho):=(\psi_1,\rho_1)\times\cdots\times(\psi_n,\rho_n)$. If we define a characteristic square η' on I^{2n} so that $(E_{2n}\,\eta')=(\psi,\rho)\cdot(E_{2n}\,\eta'')$ and take a weakly equivariant homeomorphism $f_2\colon M(\eta')\to M(\eta'')$ represented by (ψ^{-1},ρ^{-1}), then $\eta'\in\mathcal{K}_n$ and

$$f_2^*\begin{pmatrix}X_1\\ \vdots\\ X_{2n}\end{pmatrix}=\begin{pmatrix}\alpha_1^{-1} & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \alpha_n^{-1}\end{pmatrix}\begin{pmatrix}X_1\\ \vdots\\ X_{2n}\end{pmatrix}.$$

If we put $f:=f_1\circ f_2$, then it satisfies the condition of the lemma. □

Lemma 3.11 *Let ξ_0 be a characteristic square on I^{n-2}, where $n\geq 3$, let ξ and ξ' be two (κ^2,ξ_0)–type characteristic squares on I^n, and let $\varphi\colon H^*(\xi)\to H^*(\xi')$ be a* *graded ring isomorphism such that*

$$\varphi\begin{pmatrix}X_1\\ \vdots\\ X_n\end{pmatrix}=\begin{pmatrix}E_2 & A\\ 0 & E_{n-2}\end{pmatrix}\begin{pmatrix}X_1\\ \vdots\\ X_n\end{pmatrix},$$

where A denotes some $(2\times(n-2))$–matrix of integers. Then we have $A=0$ and $\xi=\xi'$.

Proof We denote the first rows of A, ξ and ξ' by $(a_3,\dots,a_n)$, $(1,2,s_3,\dots,s_n)$ and $(1,2,s_3',\dots,s_n')$, respectively. Similarly, we denote their second rows by $(b_3,\dots,b_n)$, $(1,1,t_3,\dots,t_n)$ and $(1,1,t_3',\dots,t_n')$. Then, in $H^*(\xi')$, we have

$$\begin{aligned}\varphi(X_1(X_1&+2X_2+s_3X_3+\cdots+s_nX_n))\\ &=(X_1+a_3X_3+\cdots+a_nX_n)\big\{X_1+2X_2+(a_3+2b_3+s_3)X_3\\ &\qquad\qquad+\cdots+(a_n+2b_n+s_n)X_n\big\}\\ &=X_1\{(2a_3+2b_3+s_3-s_3')X_3+\cdots+(2a_n+2b_n+s_n-s_n')X_n\}\\ &\qquad+2X_2\{a_3X_3+\cdots+a_nX_n\}+(\text{a polynomial in } X_3,\dots,X_n)\\ &=0,\end{aligned}$$

$$\begin{aligned}\varphi(X_2(X_1&+X_2+t_3X_3+\cdots+t_nX_n))\\ &=(X_2+b_3X_3+\cdots+b_nX_n)\big\{X_1+X_2+(a_3+b_3+t_3)X_3\\ &\qquad\qquad+\cdots+(a_n+b_n+t_n)X_n\big\}\\ &=X_2\{(a_3+2b_3+t_3-t_3')X_3+\cdots+(a_n+2b_n+t_n-t_n')X_n\}\\ &\qquad+X_1\{b_3X_3+\cdots+b_nX_n\}+(\text{a polynomial in } X_3,\dots,X_n)\\ &=0.\end{aligned}$$

If we define W as the submodule of $H^4(\xi')$ generated by X_pX_q, where $p,q\geq 3$, then X_2^2 and X_iX_j, where $i=1,2$ and $j=3,\dots,n$, form a basis of $H^4(\xi')/W$. Therefore we obtain $a_i=b_i=s_i-s_i'=t_i-t_i'=0$, ie $A=0$ and $\xi=\xi'$. □

Corollary 3.12 *Let $\xi,\xi'\in\mathcal{K}_n$ and $\varphi\colon H^*(\xi)\to H^*(\xi')$ be a graded ring isomorphism such that*

$$\varphi\begin{pmatrix}X_1\\ \vdots\\ X_{2n}\end{pmatrix}=\begin{pmatrix}E_2 & A_{1,2} & \cdots & A_{1,n}\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & A_{n-1,n}\\ 0 & \cdots & 0 & E_2\end{pmatrix}\begin{pmatrix}X_1\\ \vdots\\ X_{2n}\end{pmatrix}.$$

Then $A_{i,j}=0$ for $1\leq i<j\leq n$, and $\xi=\xi'$. In particular, φ is induced by $\mathrm{id}_{M(\xi)}$.

Proof We prove the corollary by induction on n. If $n=2$, the corollary is immediate from Lemma 3.11. Suppose the corollary holds for $n-1$. Since φ restricts to a graded ring isomorphism from $H^*(\xi_{[2,n]})$ to $H^*(\xi'_{[2,n]})$, by the induction hypothesis, we obtain $A_{i,j}=0$ for $2\le i<j\le n$ and $\xi_{[2,n]}=\xi'_{[2,n]}$. Then we have $A_{1,j}=0$ for $1<j\le n$, and $\xi=\xi'$ by Lemma 3.11. □

Then the following theorem is immediate from Corollary 3.10 and Corollary 3.12.

Theorem 3.13 *Let $\xi,\xi'\in\mathcal{K}_n$ and $\varphi\colon H^*(\xi)\to H^*(\xi')$ be a graded ring isomorphism. Then there exists a weakly equivariant homeomorphism $f\colon M(\xi')\to M(\xi)$ such that $\varphi=f^*$.*

By Lemma 3.6 and Theorem 3.13, we obtain Theorem 1.2.

4 Computation of $\mathcal{M}^{\mathrm{weh}}_{I^3}$

Toward the proof of Theorem 1.3, in this section, we list all the quasitoric manifolds over I^3 up to weakly equivariant homeomorphism. We denote by $\mathcal{M}^{\mathrm{weh}}_{I^3}$ the set of weakly equivariant homeomorphism classes of quasitoric manifolds over I^3, as in Corollary 2.16.

Notation 4.1 To compute $\mathcal{M}^{\mathrm{weh}}_{I^3}$, we use the following notation. Recall that we denote by Ξ_3 the set of characteristic squares on I^3.

- We denote by $\phi\colon \Xi_3\to\mathcal{M}^{\mathrm{weh}}_{I^3}$ the surjection given by $\xi\mapsto M(\xi)$.
- For $V_1,V_2,V_3\subseteq\mathbb{Z}^2$, we define
$$\Xi(V_1,V_2,V_3):=\left\{\begin{pmatrix}1&x_1&x_2\\y_1&1&x_3\\y_2&y_3&1\end{pmatrix}\in\Xi_3\;\middle|\;\begin{pmatrix}x_i\\y_i\end{pmatrix}\in V_i\text{ for }i=1,2,3\right\}.$$
- We put $P_+:=\left\{\begin{pmatrix}k\\0\end{pmatrix}\;\middle|\;k\in\mathbb{Z}\right\}$, $\quad P_-:=\left\{\begin{pmatrix}0\\k\end{pmatrix}\;\middle|\;k\in\mathbb{Z}\right\}$,
$$N_+:=\left\{\pm\begin{pmatrix}2\\1\end{pmatrix}\right\},\qquad N_-:=\left\{\pm\begin{pmatrix}1\\2\end{pmatrix}\right\},$$
$$C_0:=P_+\cup P_-,\qquad C_2:=N_+\cup N_-.$$
- We put $C_{\epsilon_1,\epsilon_2,\epsilon_3}:=\Xi(C_{\epsilon_1},C_{\epsilon_2},C_{\epsilon_3})$ for $(\epsilon_1,\epsilon_2,\epsilon_3)\in\{0,2\}^3$.

- We define $\sigma_i, \tau_i \in \mathrm{Aut}(I^3)$, for $i = 1, 2, 3$, by
$$\sigma_1 := (1\ 2)(4\ 5), \quad \sigma_2 := (1\ 3)(4\ 6), \quad \sigma_3 := (2\ 3)(5\ 6), \quad \tau_i := (i\ \ i+3)$$
and $\delta_i \in \mathrm{GL}_3(\mathbb{Z})$, for $i = 1, 2, 3$, by
$$\delta_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \delta_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \delta_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$
Additionally, for $i = 1, 2, 3$, we take $\mu_i \in (\mathbb{Z}/2)^6$ such that the i^{th} and $(i+3)^{\text{th}}$ components are -1 and the other components are 1 and take $\nu_i \in \mathrm{GL}_3(\mathbb{Z})$ which acts on $\mathbb{Z}^3$ by multiplication by -1 on the i^{th} component.

Remark 4.2 Since $(\delta_i, \sigma_i) \cdot (E_3\, \xi)$ and $(\nu_i, \mu_i) \cdot (E_3\, \xi)$ (see Definition 2.13) are in the form $(E_3\, \xi')$ for each $\xi \in \Xi_3$ and $i = 1, 2, 3$, we can regard (δ_i, σ_i) and (ν_i, μ_i) as acting on Ξ_3.

Lemma 4.3 *The restriction of ϕ to $C_{0,0,0} \cup C_{0,0,2} \cup C_{0,2,2} \cup C_{2,2,2}$ is surjective.*

Proof Let ξ be a characteristic square on I^3 and write
$$\xi = \begin{pmatrix} 1 & x_1 & x_2 \\ y_1 & 1 & x_3 \\ y_2 & y_3 & 1 \end{pmatrix}.$$
From the nonsingularity condition, $1 - x_i y_i = \pm 1$ for $i = 1, 2, 3$. This implies that each ${}^t(x_i, y_i)$ belongs to C_0 or C_2. Therefore we obtain
$$\Xi_3 = \bigcup_{\epsilon_1,\epsilon_2,\epsilon_3 \in \{0,2\}} C_{\epsilon_1,\epsilon_2,\epsilon_3}.$$
Since $(\delta_1, \sigma_1) \cdot C_{\epsilon_1,\epsilon_2,\epsilon_3} = C_{\epsilon_1,\epsilon_3,\epsilon_2}$, we have $\phi(C_{\epsilon_1,\epsilon_2,\epsilon_3}) = \phi(C_{\epsilon_1,\epsilon_3,\epsilon_2})$. Similarly, we have $\phi(C_{\epsilon_1,\epsilon_2,\epsilon_3}) = \phi(C_{\epsilon_3,\epsilon_2,\epsilon_1})$ and $\phi(C_{\epsilon_1,\epsilon_2,\epsilon_3}) = \phi(C_{\epsilon_2,\epsilon_1,\epsilon_3})$. Hence we see that
$$\mathcal{M}^{\mathrm{weh}}_{I^3} = \bigcup_{\epsilon_1,\epsilon_2,\epsilon_3 \in \{0,2\}} \phi(C_{\epsilon_1,\epsilon_2,\epsilon_3}) = \phi(C_{0,0,0}) \cup \phi(C_{0,0,2}) \cup \phi(C_{0,2,2}) \cup \phi(C_{2,2,2}).$$
Thus we obtain the lemma. □

Let us put $P_{s_1,s_2,s_3} := \Xi(P_{s_1}, P_{s_2}, P_{s_3})$, where $s_i \in \{+,-\}$ for $i = 1, 2, 3$. Then we have
$$C_{0,0,0} = \bigcup_{s_1,s_2,s_3 \in \{+,-\}} P_{s_1,s_2,s_3}.$$

Moreover, (δ_i,σ_i) for $i=1,2,3$ act as follows:

$$\begin{tikzcd}
P_{+,+,+} \arrow[r, leftrightarrow, "(\delta_1{,}\sigma_1)"] \arrow[d, leftrightarrow, "(\delta_3{,}\sigma_3)"'] \arrow[dr, leftrightarrow, "(\delta_2{,}\sigma_2)"] & P_{-,+,+} \arrow[r, leftrightarrow, "(\delta_2{,}\sigma_2)"] \arrow[dr, leftrightarrow, "(\delta_3{,}\sigma_3)"] & P_{-,-,+} & P_{+,-,+} \arrow[d, leftrightarrow, "(\delta_2{,}\sigma_2)"] \\
P_{+,+,-} & P_{-,-,-} & P_{+,-,-} & P_{-,+,-}
\end{tikzcd}$$

Thus we obtain the following lemma.

Lemma 4.4 $$\phi(C_{0,0,0})=\phi(P_{+,+,+}\cup P_{+,-,+}).$$

Suppose that

$$\xi=\begin{pmatrix}1 & x_1 & 0\\ 0 & 1 & x_3\\ x_2 & 0 & 1\end{pmatrix}\in P_{+,-,+}.$$

Then, from the nonsingularity condition, we have $x_1x_2x_3=-1\pm 1$.

- If $x_1x_2x_3=0$, then $\xi\in P_{-,-,+}\cup P_{+,+,+}\cup P_{+,-,-}$.
- If $x_1x_2x_3=-2$, then there is a $\psi\in \mathrm{GL}_3(\mathbb{Z})$ such that $(\psi,\tau_1)\cdot(E_3\,\xi)=(E_3\,\xi')$, where

$$\xi'=\begin{pmatrix}1 & x_1 & 0\\ 0 & 1 & x_3\\ -x_2 & -x_1x_2 & 1\end{pmatrix}\in C_{0,0,2}.$$

Thus we obtain the following lemma.

Lemma 4.5 *Put $A_1:=P_{+,+,+}$. Then we have $\phi(C_{0,0,0})\subseteq\phi(A_1)\cup\phi(C_{0,0,2})$.*

For $C_{0,0,2}$, we prove the following lemma first. Put

$$C'_{0,0,2}:=\left\{\begin{pmatrix}1 & x_1 & x_2\\ y_1 & 1 & 2\\ y_2 & 1 & 1\end{pmatrix}\in C_{0,0,2}\right\}.$$

Lemma 4.6 $$\phi(C'_{0,0,2})=\phi(C_{0,0,2}).$$

Proof Since (δ_3,σ_3) gives a bijection between $\Xi(C_0,C_0,N_+)$ and $\Xi(C_0,C_0,N_-)$, we have $\phi(C_{0,0,2})=\phi(\Xi(C_0,C_0,N_+))$. By using (ν_3,μ_3), we see that we also have $\phi(C'_{0,0,2})=\phi(\Xi(C_0,C_0,N_+))$. □

Suppose that

$$\xi = \begin{pmatrix} 1 & x_1 & 0 \\ 0 & 1 & 2 \\ y_2 & 1 & 1 \end{pmatrix} \in C'_{0,0,2}.$$

From the nonsingularity condition, we have $2x_1y_2 = 1 \pm 1$.

- If $x_1y_2 = 0$, then $\xi \in \Xi(P_+, P_+, N_+) \cup \Xi(P_-, P_-, N_+)$.
- If $x_1y_2 = 1$, then $x_1 = \pm 1$ and
$$\xi = \begin{pmatrix} 1 & x_1 & 0 \\ 0 & 1 & 2 \\ x_1 & 1 & 1 \end{pmatrix}.$$

Similarly, if we assume

$$\xi = \begin{pmatrix} 1 & 0 & x_2 \\ y_1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \in C'_{0,0,2},$$

then we see that $\xi \in \Xi(P_+, P_+, N_+) \cup \Xi(P_-, P_-, N_+)$ or

$$\xi = \begin{pmatrix} 1 & 0 & 2a \\ a & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \xi = \begin{pmatrix} 1 & 0 & b \\ 2b & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix},$$

where a and b are ± 1. By using the action of (ν_1, μ_1), we obtain the following.

Lemma 4.7 *Put $A_2 := C'_{0,0,2} \cap \Xi(P_+, P_+, N_+)$, $A_3 := C'_{0,0,2} \cap \Xi(P_-, P_-, N_+)$,*

$$\chi_1 := \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad \chi_2 := \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad \textit{and} \quad \chi_3 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then we have $\phi(C_{0,0,2}) = \phi(A_2 \cup A_3 \cup \{\chi_1, \chi_2, \chi_3\})$.

For $C_{0,2,2}$, we have the following.

Lemma 4.8 *Put*

$$\chi_4 := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \quad \chi_5 := \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \quad \chi_6 := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}, \quad \chi_7 := \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix},$$

$$\chi_8 := \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad \chi_9 := \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \quad \textit{and} \quad \chi_{10} := \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then we have $\phi(C_{0,2,2}) = \phi(\{\chi_4, \dots, \chi_{10}\})$.

Proof Take $\xi \in C_{0,2,2}$. By using (δ_1, σ_1), (ν_2, μ_2) and (ν_3, μ_3), we can assume that ξ is in one of the following forms:

$$\xi = \begin{pmatrix} 1 & x & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \quad \xi = \begin{pmatrix} 1 & x & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}, \quad \xi = \begin{pmatrix} 1 & x & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \quad \text{or} \quad \xi = \begin{pmatrix} 1 & x & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

If ξ is in the first form, then we have $x = 1$ or 2 by the nonsingularity condition. Similarly, $x = 1$ if ξ is in the second form, $x = 2$ or 4 in the third form, and $x = 1$ or 2 in the fourth form. Thus we obtain the lemma. $\square$

Similarly, we have the following lemma for $C_{2,2,2}$.

Lemma 4.9 *Put*

$$\chi_{11} := \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then we have $\phi(C_{2,2,2}) = \{\phi(\chi_{11})\}$.

Proof Take

$$\xi = \begin{pmatrix} 1 & x_1 & x_2 \\ y_1 & 1 & x_3 \\ y_2 & y_3 & 1 \end{pmatrix} \in C_{2,2,2}.$$

By using (δ_1, σ_1), (ν_2, μ_2) and (ν_3, μ_3), we can assume $x_1 = 2$, $y_1 = 1$ and $x_2, y_2 > 0$. From the nonsingularity condition, we have that $2y_2x_3 + x_2y_3 = 5 \pm 1$. Then it is straightforward to obtain

$$\xi = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad \xi = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \quad \text{or} \quad \xi = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

Moreover, if we set the above to be ξ_1, ξ_2 and ξ_3, then we have $(\delta_3, \sigma_3) \cdot \xi_1 = \xi_2$ and $(\delta_2, \sigma_2) \cdot \xi_2 = \xi_3$. $\square$

Taking τ_i for $i = 1, 2, 3$ into account, we have the following diagram:

$$\begin{tikzcd}
\chi_1 \arrow[r, "\tau_1"] \arrow[d, "\tau_3"'] \arrow[dr, "\tau_2"] & \gamma_1 \arrow[r, "\tau_2"] \arrow[dr, "\tau_3"] & \gamma_2 \arrow[r, "\sigma_1"] & \chi_3 & \chi_4 \arrow[d, "\tau_1"] & \chi_6 \arrow[d, "\tau_3"] \\
\gamma_3 \arrow[d, "\sigma_3"'] & \gamma_4 \arrow[d, "\tau_3"'] \arrow[dr, "\sigma_1"] & \gamma_5 \arrow[r, "\sigma_3\circ\sigma_2"] & \chi_2 & \gamma_6 \arrow[d, "\sigma_3\circ\sigma_1"] & \gamma_7 \arrow[d, "\sigma_1"] \\
\chi_7 & \chi_{11} & \chi_9 & & \chi_3 & \chi_8
\end{tikzcd}$$

Here the arrow $\xi_1 \xrightarrow{\rho} \xi_2$ means that there exist $\psi \in \mathrm{GL}_3(\mathbb{Z})$ and $\mu \in (\mathbb{Z}/2)^6$ such that $(\psi, \mu \circ \rho) \cdot (E_3\,\xi_1) = (E_3\,\xi_2)$, and γ_i for $i = 1, \dots, 7$ denote the following characteristic squares:

$$\gamma_1 := \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \gamma_2 := \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \gamma_3 := \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad \gamma_4 := \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix},$$
$$\gamma_5 := \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \gamma_6 := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_7 := \begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

Summarizing Lemma 4.3, Lemma 4.5, Lemma 4.7, Lemma 4.8, Lemma 4.9 and the above diagram, we obtain the following. Note that χ_3 appears twice in the diagram and χ_5 and χ_{10} do not appear.

Lemma 4.10 $$\mathcal{M}^{\mathrm{weh}}_{I^3} = \phi(A_1 \cup A_2 \cup A_3 \cup \{\chi_1, \chi_5, \chi_6, \chi_{10}\}).$$

Definition 4.11 For $s, t \in \mathbb{Z}$, we define $\xi_{s,t}$ and $\xi^{s,t}$ by

$$\xi_{s,t} := \begin{pmatrix} 1 & s & t \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \in A_2 \qquad \text{and} \qquad \xi^{s,t} := \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 2 \\ t & 1 & 1 \end{pmatrix} \in A_3.$$

Note that $A_2 = \{\xi_{s,t}\}_{s,t \in \mathbb{Z}}$ and $A_3 = \{\xi^{s,t}\}_{s,t \in \mathbb{Z}}$.

5 Strong cohomological rigidity of $\mathcal{M}_{I^3}$

In this section, for $\xi \in \Xi_3$, we denote the generators of $H^*(\xi)$ by X, Y and Z instead of X_1, X_2 and X_3 (see Definition 2.33). Also, we define $H^*(\xi; \mathbb{Z}/2) := H^*(\xi)/2$, $w_2(\xi) := \sum_{i=1}^{6} u_i(\xi) \in H^2(\xi; \mathbb{Z}/2)$ and $p_1(\xi) := -\sum_{i=1}^{6} u_i(\xi)^2 \in H^4(\xi)$ and identify $w_2(\xi)$ and $p_1(\xi)$ with $w_2(M(\xi))$ and $p_1(M(\xi))$, respectively, through the canonical isomorphism between $H^*(\xi)$ and $H^*(M(\xi); \mathbb{Z})$ (see Theorem 2.23).

Definition 5.1 Let $\mathcal{M}_{I^3}$ be the set of homeomorphism classes of quasitoric manifolds over I^3 and ϕ_1 be the canonical surjection from $\mathcal{M}^{\mathrm{weh}}_{I^3}$ to $\mathcal{M}_{I^3}$. We define subsets $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ of $\mathcal{M}_{I^3}$ by

$$\mathcal{M}_1 := \phi_1 \circ \phi(A_1), \quad \mathcal{M}_2 := \phi_1 \circ \phi(A_2 \setminus \{\xi_{0,0}\}) \quad \text{and} \quad \mathcal{M}_3 := \phi_1 \circ \phi(A_3).$$

Additionally, we define $\mathcal{M}^{\mathrm{ceq}}_{I^3}$ as the quotient $\mathcal{M}_{I^3}/{\sim}$, where $M \sim M'$ if and only if $H^*(M; \mathbb{Z}) \cong H^*(M'; \mathbb{Z})$ as graded rings, and we denote the quotient map by $\phi_2\colon \mathcal{M}_{I^3} \to \mathcal{M}^{\mathrm{ceq}}_{I^3}$.

Definition 5.2 A class $\mathcal{C}$ of topological spaces is called *strongly cohomologically rigid* if for any graded ring isomorphism φ between the cohomology rings of $X, Y \in \mathcal{C}$ there exists a homeomorphism f between them such that $\varphi = f^*$.

Remark 5.3 By [4, Proposition 6.2], we see that $\mathcal{M}_1$ corresponds with the class of 3–stage Bott manifolds. Then we obtain the strong cohomological rigidity of $\mathcal{M}_1$ by [2, Theorem 3.1] which shows the strong cohomological rigidity of 3–stage Bott manifolds.

Lemma 5.4 *$M(\chi_5)$, $M(\chi_6)$, $M(\chi_{10}) \in \mathcal{M}_2$. Thus $\mathcal{M}_{I^3} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \{\chi_1\}$.*

Proof Define graded ring automorphisms α_5, α_6 and α_{10} of $\mathbb{Z}[X, Y, Z]$ so that

$$\alpha_i \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = A_i \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

where A_i for $i = 5, 6, 10$ denote the following matrices:

$$A_5 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_6 := \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_{10} := \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The α_i descend to isomorphisms $\alpha_5\colon H^*(\xi_{-1,-2}) \to H^*(\chi_5)$, $\alpha_6\colon H^*(\xi_{1,1}) \to H^*(\chi_6)$ and $\alpha_{10}\colon H^*(\xi_{-2,-2}) \to H^*(\chi_{10})$ and they preserve the second Stiefel–Whitney classes and the first Pontrjagin classes. Thus we obtain the lemma by Theorem 2.24. □

Lemma 5.5 *Let $\mathbb{Z}[Y, Z]$ be the polynomial ring generated by Y and Z of degree 2, and R be the quotient ring $\mathbb{Z}[Y, Z]/(Y(Y + 2Z), Z(Y + Z))$. Then R has no nonzero element of degree 2 whose square is equal to 0.*

Proof Let $W = sY + tZ$ be an element whose square is 0. Then

$$0 = W^2 = (sY + tZ)^2 = (-2s^2 + 2st - t^2)YZ = -\{s^2 + (s - t)^2\}YZ,$$

so we have $s = s - t = 0$, ie $W = 0$. □

Remark 5.6 For any $\xi \in \Xi(\mathbb{Z}^2, \mathbb{Z}^2, C_2)$, since $H^*(\xi)/(X)$ is isomorphic to R, the set $\{W \in H^2(\xi) \mid W^2 = 0\}$ is equal to $\mathbb{Z}X$ or $\{0\}$.

This remark immediately yields the following lemma.

Lemma 5.7 *Let M be a quasitoric manifold over I^3. Then there exists a nonzero W in $H^2(M; \mathbb{Z})$ such that $W^2 = 0$ if and only if $M \in \mathcal{M}_1 \cup \mathcal{M}_3$. In particular, $\phi_2(\mathcal{M}_1 \cup \mathcal{M}_3) \cap \phi_2(\mathcal{M}_2 \cup \{\chi_1\}) = \varnothing$.*

Lemma 5.8 $$\phi_2(\mathcal{M}_1)\cap\phi_2(\mathcal{M}_3)=\varnothing.$$

Proof Let $\xi_1\in A_1$ and $\xi_3\in A_3$ and suppose that there exists an isomorphism $\alpha\colon H^*(\xi_1)\to H^*(\xi_3)$. Since α preserves the elements whose squares are zero, α descends to an isomorphism $\bar\alpha\colon H^*(\xi_1)/(Z)\to H^*(\xi_3)/(X)$. However, $H^*(\xi_1)/(Z)$ has nonzero degree-2 elements whose squares are zero, but $H^*(\xi_3)/(X)\cong R$ does not. This is a contradiction. □

Lemma 5.9 *Let ξ_0 be a characteristic square on I^{n-1}, where $n\geq 3$, let ξ be a $(1,\xi_0)$–type characteristic square on I^n, and let $\varphi\colon \mathbb{Z}[X,Y,Z]\to\mathbb{Z}[X_1,\dots,X_n]$ be a graded ring monomorphism which maps X, Y and Z to $\sum_{i=1}^n a_iX_i$, $\sum_{i=1}^n b_iX_i$ and $\sum_{i=1}^n c_iX_i$, respectively. Moreover, we assume the following:*

(a) *φ maps $\mathcal{I}_{\chi_1}$ into $\mathcal{I}_\xi$.*

(b) *The* mod p *reductions of $(a_1,\dots,a_n)$, $(b_1,\dots,b_n)$ and $(c_1,\dots,c_n)$ are linearly independent for each prime p.*

Then we have $a_1=b_1=c_1=0$. In particular, there exists no graded ring isomorphism from $H^*(\chi_1)$ to $H^*(\xi_{s,t})$ for any integers s and t.

Proof Denote the first row of ξ by $(1,s_2,s_3,\dots,s_n)$. Since ξ is $(1,\xi_0)$–type and

$$\begin{aligned}
\varphi(X(X+2Z)) &= \Big(\sum_{i=1}^n a_iX_i\Big)\Big\{\sum_{i=1}^n (a_i+2c_i)X_i\Big\}\\
&= X_1\Big\{a_1(a_1+2c_1)X_1+\sum_{i=2}^n\{a_1(a_i+2c_i)+a_i(a_1+2c_1)\}X_i\Big\}\\
&\qquad\qquad\qquad+(\text{a polynomial in } X_2,\dots,X_n)\\
&= X_1\sum_{i=2}^n\{a_1(a_i+2c_i)+(a_i-s_ia_1)(a_1+2c_1)\}X_i\\
&\qquad\qquad\qquad+(\text{a polynomial in } X_2,\dots,X_n)\\
&=0\quad\text{in } H^*(\xi),
\end{aligned}$$

we obtain $a_1(a_i+2c_i)=(s_ia_1-a_i)(a_1+2c_1)$ for $i=2,\dots,n$. Note that, by the assumption of linear independence, $a_1+2c_1=0$ if $a_1=0$ and vice versa. If a_1 and a_1+2c_1 are nonzero, denoting by k the greatest common divisor of a_1 and a_1+2c_1, we see that a_1/k and $(a_1+2c_1)/k$ divide $s_ia_1-a_i$ and a_i+2c_i, respectively, for $i=2,\dots,n$. By assumption (b), we obtain $a_1/k=\pm1$ and $(a_1+2c_1)/k=\pm1$, namely, $a_1+c_1=0$ or $c_1=0$. This holds also in the case $a_1=a_1+2c_1=0$.

Similarly, for $i=2,\dots,n$ we have $b_1(a_i+b_i+2c_i)=(s_ib_1-b_i)(a_1+b_1+2c_1)$ and $c_1(b_i+c_i)=(s_ic_1-c_i)(b_1+c_1)$ from $\varphi(Y(X+Y+2Z))=0$ and $\varphi(Z(Y+Z))=0$, respectively, and then obtain that $a_1+2c_1=0$ or $a_1+2b_1+2c_1=0$ and that $b_1=0$ or $b_1+2c_1=0$ in the same way as above. We solve these equations to see that $a_1=b_1=c_1=0$. □

Remark 5.10 By Remark 5.3, Lemma 5.4, Lemma 5.7, Lemma 5.8 and Lemma 5.9, to show the strong cohomological rigidity of $\mathcal{M}_{I^3}$, we only have to show that of $\mathcal{M}_2$, $\mathcal{M}_3$ and $\{M(\chi_1)\}$.

Lemma 5.11 *Let ξ_0 be a characteristic square on I^{n-1}, where $n\geq 2$, let ξ be a $(1,\xi_0)$–type characteristic square on I^n, and let $\varphi\colon \mathbb{Z}[X_1,X_2]\to\mathbb{Z}[X_1,\dots,X_n]$ be a graded ring monomorphism which maps X_1 and X_2 to $\sum_{i=1}^n a_iX_i$ and $\sum_{i=1}^n b_iX_i$, respectively. Moreover, we assume the following:*

(a) *φ maps $\mathcal{I}_{\kappa^2}$ into $\mathcal{I}_\xi$.*

(b) *For any prime p, the* mod p *reductions of $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$ are linearly independent.*

Then we have $a_1=b_1=0$.

Proof Denote the first row of ξ by $(1,s_2,s_3,\dots,s_n)$. Since ξ is $(1,\xi_0)$–type and

$$\begin{aligned}
\varphi(X_1(X_1+2X_2)) &= \Big(\sum_{i=1}^n a_iX_i\Big)\Big\{\sum_{i=1}^n (a_i+2b_i)X_i\Big\}\\
&= X_1\Big\{a_1(a_1+2b_1)X_1+\sum_{i=2}^n\{a_1(a_i+2b_i)+a_i(a_1+2b_1)\}X_i\Big\}\\
&\qquad\qquad\qquad\qquad +(\text{a polynomial in } X_2,\dots,X_n)\\
&= X_1\sum_{i=2}^n\{a_1(a_i+2b_i)+(a_i-s_ia_1)(a_1+2b_1)\}X_i\\
&\qquad\qquad\qquad\qquad +(\text{a polynomial in } X_2,\dots,X_n)\\
&=0 \quad \text{in } H^*(\xi),
\end{aligned}$$

we obtain $a_1(a_i+2b_i)=(s_ia_1-a_i)(a_1+2b_1)$ for $i=2,\dots,n$. In the same way as the proof of Lemma 5.9, we obtain $a_1+b_1=0$ or $b_1=0$, which implies that the coefficient of X_1 in $\varphi(X_1+X_2)$ or $\varphi(X_2)$ is zero. Then we easily see that $\varphi(X_2(X_1+X_2))\neq 0$ in $H^*(\xi)$ unless both b_1 and a_1+b_1 are zero. □

Lemma 5.12 *$\mathcal{M}_2$ is strongly cohomologically rigid.*

Proof Let $\varphi\colon H^*(\xi_{s,t}) \to H^*(\xi_{x,y})$ be a graded ring isomorphism. By Lemma 5.11,

$$\varphi\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \left(\begin{array}{c|cc} a & b & c \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{θ}} \\ 0 & & \end{array}\right)\begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

where $a = \pm 1$ and θ is an automorphism of $H^*(\kappa^2)$. By Lemma 3.4, θ can be realized as a weakly equivariant self-homeomorphism of $M(\kappa^2)$, and therefore we can construct a weakly equivariant homeomorphism f from $M(\xi_{x,y})$ to some $M(\xi_{x',y'})$ such that

$$f^*\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \left(\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{θ}} \\ 0 & & \end{array}\right)\begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

in a way similar to the proof of Corollary 3.10. Thus we see that we can assume $a = 1$ and $\theta = E_2$. Since φ maps $\mathcal{I}_{\xi_{s,t}}$ into $\mathcal{I}_{\xi_{x,y}}$,

$$\begin{aligned} \varphi(X(X+sY+tZ)) &= (X+bY+cZ)\{X+(s+b)Y+(t+c)Z\} \\ &= X\{(s-x+2b)Y+(t-y+2c)Z\} \\ &\qquad + \{-2b(s+b)-c(t+c)+b(t+c)+c(s+b)\}YZ \\ &= 0 \quad \text{in } H^*(\xi_{x,y}). \end{aligned}$$

Thus we obtain

$$b = \tfrac{1}{2}(x-s), \quad c = \tfrac{1}{2}(y-t) \quad \text{and} \quad (s-t)^2 + s^2 = (x-y)^2 + x^2.$$

In particular, $s \equiv x$ and $t \equiv y$ modulo 2. Then we have

$$\varphi(w_2(\xi_{s,t})) = \varphi((s+1)Y+tZ)) = (s+1)Y+tZ = w_2(\xi_{x,y}) \quad \text{in } H^*(\xi_{x,y};\mathbb{Z}/2).$$

Similarly, since $\varphi(2X+sY+tZ)-(2X+xY+yZ) = 0$, we have

$$\begin{aligned} &p_1(\xi_{x,y}) - \varphi(p_1(\xi_{s,t})) \\ &\quad = \varphi(X)^2 + \varphi(X+sY+tZ)^2 - X^2 - (X+xY+yZ)^2 \\ &\quad = \varphi(2X+sY+tZ)^2 - (2X+xY+yZ)^2 \\ &\quad = \{\varphi(2X+sY+tZ)+(2X+xY+yZ)\}\{\varphi(2X+sY+tZ)-(2X+xY+yZ)\} \\ &\quad = 0 \quad \text{in } H^*(\xi_{x,y}). \end{aligned}$$

Thus we obtain the lemma by Theorem 2.24. □

Lemma 5.13 *Any graded ring isomorphism between the cohomology rings of two members of $\phi(A_3)$ is induced by a weakly equivariant homeomorphism. In particular, $\mathcal{M}_3$ is strongly cohomologically rigid.*

Proof Note $(\psi_3\psi_2, \sigma_3 \circ \sigma_2)\cdot\xi^{s,t}$ is a $(\kappa^2, 1)$–type characteristic square. Let ξ and ξ' be two $(\kappa^2, 1)$–type characteristic squares and $\varphi\colon H^*(\xi) \to H^*(\xi')$ be a graded ring isomorphism. Since φ preserves the elements of degree 2 whose squares are zero, we have

$$\varphi\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{θ}} & a \\ & & b \\ \hline 0 & 0 & c \end{array}\right)\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

As in the proof of the previous lemma, we can assume $c = 1$ and $\theta = E_2$. Then we have $\xi = \xi'$ and $\varphi = (\mathrm{id}_{M(\xi)})^*$ by Lemma 3.11. □

Lemma 5.14 *Let φ be a graded ring automorphism of $H^*(\chi_1)$. Then $\varphi = \pm\,\mathrm{id}$.*

Proof Take $A \in \mathrm{GL}_3(\mathbb{Z})$ so that

$$\varphi\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = A\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

and denote the i^{th} row of A by (a_i, b_i, c_i) for $i = 1, 2, 3$. Then the entries of A satisfy

$$(a_1 - b_1)(b_1 + 2b_3) = -b_1(a_1 + 2a_3), \tag{21}$$

$$(c_1 - 2b_1)(b_1 + 2b_3) = (c_1 - b_1)(c_1 + 2c_3), \tag{22}$$

$$(c_1 - 2a_1)(a_1 + 2a_3) = -a_1(c_1 + 2c_3), \tag{23}$$

$$(a_2 - b_2)(b_1 + b_2 + 2b_3) = -b_2(a_1 + a_2 + 2a_3), \tag{24}$$

$$(c_2 - 2b_2)(b_1 + b_2 + 2b_3) = (c_2 - b_2)(c_1 + c_2 + 2c_3), \tag{25}$$

$$(c_2 - 2a_2)(a_1 + a_2 + 2a_3) = -a_2(c_1 + c_2 + 2c_3), \tag{26}$$

$$(a_3 - b_3)(b_2 + b_3) = -b_3(a_2 + a_3), \tag{27}$$

$$(c_3 - 2b_3)(b_2 + b_3) = (c_3 - b_3)(c_2 + c_3), \tag{28}$$

$$(c_3 - 2a_3)(a_2 + a_3) = -a_3(c_2 + c_3). \tag{29}$$

By solving these equations modulo 2, we obtain

$$A \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b_3 & 1 \end{pmatrix} \mod 2.$$

Since a_1 is odd and b_1 is even, we have

$$\tfrac{1}{2}b_1 + b_3 \equiv -\tfrac{1}{2}b_1 \mod 2$$

from (21), which implies $b_3 \equiv 0 \mod 2$.

Moreover, we obtain that c_2-b_2, c_3-b_3 and c_3-2b_3 are equal to ± 1 as follows. Note that c_2-b_2, c_3-b_3 and c_3-2b_3 are odd. Let p be an odd prime, and consider equations (21)–(29) and $\det A=\pm 1$ modulo p. Then, by a direct calculation, one can show that there exists no solution with $c_2-b_2\equiv 0$, $c_3-b_3\equiv 0$ or $c_3-2b_3\equiv 0$ modulo p. This implies that no prime divides them, ie they are all equal to ± 1. Then we can solve (21)–(29) straightforwardly and obtain the lemma. □

The following theorem, which is a paraphrase of Theorem 1.3, is immediate from Remark 5.10, Lemma 5.12, Lemma 5.13 and Lemma 5.14.

Theorem 5.15 *$\mathcal{M}_{I^3}^{\mathrm{homeo}}$ is strongly cohomologically rigid.*

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