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An infinite presentation for the mapping class group of a nonorientable surface

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We give an infinite presentation for the mapping class group of a nonorientable surface. The generating set consists of all Dehn twists and all crosscap pushing maps along simple loops.

57M05, 57M07, 57M20

1 Introduction

Let $\Sigma_{g,n}$ be a compact connected orientable surface of genus $g \geq 0$ with $n \geq 0$ boundary components. The *mapping class group* $\mathcal{M}(\Sigma_{g,n})$ of $\Sigma_{g,n}$ is the group of isotopy classes of orientation-preserving self-diffeomorphisms on $\Sigma_{g,n}$ fixing the boundary pointwise. A finite presentation for $\mathcal{M}(\Sigma_{g,n})$ was given by Hatcher and Thurston [6], Wajnryb [16], Harer [5], Gervais [4] and Labruère and Paris [9]. Gervais [3] obtained an infinite presentation for $\mathcal{M}(\Sigma_{g,n})$ by using Wajnryb's finite presentation for $\mathcal{M}(\Sigma_{g,n})$, and Luo [11] rewrote Gervais' presentation into a simpler infinite presentation; see Theorem 2.5.

Let $N_{g,n}$ be a compact connected nonorientable surface of genus $g \geq 1$ with $n \geq 0$ boundary components. The surface $N_g = N_{g,0}$ is a connected sum of g real projective planes. The mapping class group $\mathcal{M}(N_{g,n})$ of $N_{g,n}$ is the group of isotopy classes of self-diffeomorphisms on $N_{g,n}$ fixing the boundary pointwise. For $g \geq 2$ and $n \in \{0, 1\}$, a finite presentation for $\mathcal{M}(N_{g,n})$ was given by Lickorish [10], Birman and Chillingworth [1], Stukow [13] and Paris and Szepietowski [12]. Note that $\mathcal{M}(N_1)$ and $\mathcal{M}(N_{1,1})$ are trivial (see [2, Theorem 3.4]) and $\mathcal{M}(N_2)$ is finite (see [10, Lemma 5]). Stukow [14] rewrote the Paris–Szepietowski presentation into a finite presentation with Dehn twists and a "Y-homeomorphism" as generators; see Theorem 2.11.

In this paper, we give a simple infinite presentation for $\mathcal{M}(N_{g,n})$ (Theorem 3.1) when $g \geq 1$ and $n \in \{0, 1\}$. The generating set consists of all Dehn twists and all "crosscap pushing maps" along simple loops. We review the crosscap pushing map in Section 2. We prove Theorem 3.1 by applying Gervais' argument to Stukow's finite presentation.

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2 Preliminaries

2.1 Relations among Dehn twists and Gervais' presentation

Let S be either $N_{g,n}$ or $\Sigma_{g,n}$. We denote by $\mathcal{N}_S(A)$ a regular neighborhood of a subset A in S. For every simple closed curve c on S, we choose an orientation of c and fix it throughout this paper. However, for simple closed curves c_1 and c_2 on S and $f \in \mathcal{M}(S)$, by $f(c_1) = c_2$ we mean $f(c_1)$ is isotopic to c_2 or the inverse curve of c_2. If S is a nonorientable surface, we also fix an orientation of $\mathcal{N}_S(c)$ for each two-sided simple closed curve c. For a two-sided simple closed curve c on S, denote by t_c the right-handed Dehn twist along c on S. In particular, for a given explicit two-sided simple closed curve, an arrow on a side of the simple closed curve indicates the direction of the Dehn twist; see Figure 1.

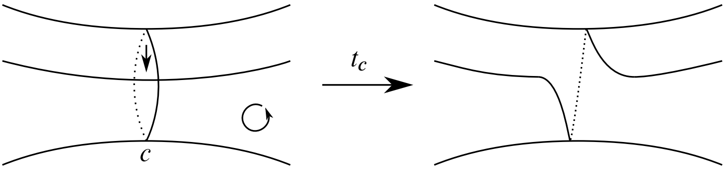

Figure 1: The right-handed Dehn twist t_c along a two-sided simple closed curve c on S

Recall the following relations on $\mathcal{M}(S)$ among Dehn twists along two-sided simple closed curves on S.

Lemma 2.1 *For a two-sided simple closed curve c on S which bounds a disk or a Möbius band in S, we have $t_c = 1$ on $\mathcal{M}(S)$.*

Lemma 2.2 (the braid relation (i)) *For a two-sided simple closed curve c on S and $f \in \mathcal{M}(S)$, we have*

$$t_{f(c)}^{\varepsilon_{f(c)}} = f t_c f^{-1},$$

where $\varepsilon_{f(c)} = 1$ if the restriction $f|_{\mathcal{N}_S(c)}\colon \mathcal{N}_S(c) \to \mathcal{N}_S(f(c))$ is orientation-preserving and $\varepsilon_{f(c)} = -1$ if the restriction is orientation-reversing.

When f in Lemma 2.2 is a Dehn twist t_d along a two-sided simple closed curve d and the geometric intersection number $|c \cap d|$ of c and d is m, we denote by T_m the braid relation.

Let $c_1, c_2, \ldots, c_k$ be two-sided simple closed curves on S. The sequence $c_1, c_2, \ldots, c_k$ of simple closed curves on S is a *k–chain on S* if $c_1, c_2, \ldots, c_k$ satisfy $|c_i \cap c_{i+1}| = 1$ for each $i = 1, 2, \ldots, k-1$ and $|c_i \cap c_j| = 0$ for $|j - i| > 1$.

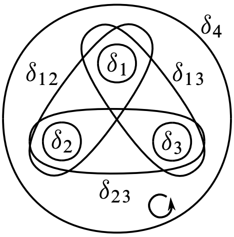

Figure 2: Simple closed curves $\delta_{12}, \delta_{23}, \delta_{13}, \delta_1, \delta_2, \delta_3$ and δ_4 on Σ

Lemma 2.3 (the k–chain relation) *Let $c_1, c_2, \dots, c_k$ be a k–chain on S, and let δ_1 and δ_2 (resp. δ) be distinct boundary components (resp. the boundary component) of $\mathcal{N}_S(c_1 \cup c_2 \cup \dots \cup c_k)$ when k is odd (resp. even). Then we have*

$$(t_{c_1}^{\varepsilon_{c_1}} t_{c_2}^{\varepsilon_{c_2}} \cdots t_{c_k}^{\varepsilon_{c_k}})^{k+1} = t_{\delta_1}^{\varepsilon_{\delta_1}} t_{\delta_2}^{\varepsilon_{\delta_2}} \quad \text{when } k \text{ is odd,}$$
$$(t_{c_1}^{\varepsilon_{c_1}} t_{c_2}^{\varepsilon_{c_2}} \cdots t_{c_k}^{\varepsilon_{c_k}})^{2k+2} = t_{\delta}^{\varepsilon_{\delta}} \quad \text{when } k \text{ is even,}$$

where $\varepsilon_{c_1}, \varepsilon_{c_2}, \dots, \varepsilon_{c_k}, \varepsilon_{\delta_1}, \varepsilon_{\delta_2}, \varepsilon_{\delta} \in \{1, -1\}$, and $t_{c_1}^{\varepsilon_{c_1}}, t_{c_2}^{\varepsilon_{c_2}}, \dots, t_{c_k}^{\varepsilon_{c_k}}, t_{\delta_1}^{\varepsilon_{\delta_1}}, t_{\delta_2}^{\varepsilon_{\delta_2}}$ and $t_{\delta}^{\varepsilon_{\delta}}$ are right-handed Dehn twists for some orientation of $\mathcal{N}_S(c_1 \cup c_2 \cup \dots \cup c_k)$.

Lemma 2.4 (the lantern relation) *Let Σ be a subsurface of S which is diffeomorphic to $\Sigma_{0,4}$, and let $\delta_{12}, \delta_{23}, \delta_{13}, \delta_1, \delta_2, \delta_3$ and δ_4 be simple closed curves on Σ as in Figure 2. Then we have*

$$t_{\delta_{12}}^{\varepsilon_{\delta_{12}}} t_{\delta_{23}}^{\varepsilon_{\delta_{23}}} t_{\delta_{13}}^{\varepsilon_{\delta_{13}}} = t_{\delta_1}^{\varepsilon_{\delta_1}} t_{\delta_2}^{\varepsilon_{\delta_2}} t_{\delta_3}^{\varepsilon_{\delta_3}} t_{\delta_4}^{\varepsilon_{\delta_4}},$$

where $\varepsilon_{\delta_{12}}, \varepsilon_{\delta_{23}}, \varepsilon_{\delta_{13}}, \varepsilon_{\delta_1}, \varepsilon_{\delta_2}, \varepsilon_{\delta_3}$ and ε_{δ_4} are 1 or -1, and $t_{\delta_{12}}^{\varepsilon_{\delta_{12}}}$, $t_{\delta_{23}}^{\varepsilon_{\delta_{23}}}$, $t_{\delta_{13}}^{\varepsilon_{\delta_{13}}}$, $t_{\delta_1}^{\varepsilon_{\delta_1}}$, $t_{\delta_2}^{\varepsilon_{\delta_2}}$, $t_{\delta_3}^{\varepsilon_{\delta_3}}$ and $t_{\delta_4}^{\varepsilon_{\delta_4}}$ are right-handed Dehn twists for some orientation of Σ.

Luo's presentation for $\mathcal{M}(\Sigma_{g,n})$, an improvement of Gervais', is as follows.

Theorem 2.5 [3; 11] *For $g \geq 0$ and $n \geq 0$, we have a presentation for $\mathcal{M}(\Sigma_{g,n})$ with generators $\{t_c \mid c :$ scc (simple closed curve) on $\Sigma_{g,n}\}$ and relations*

- (0′) *$t_c = 1$ when c bounds a disk in $\Sigma_{g,n}$,*
- (I′) *all the braid relations T_0 and T_1,*
- (II) *all the 2–chain relations,*
- (III) *all the lantern relations.*

2.2 Relations among the crosscap pushing maps and Dehn twists

Let μ be a one-sided simple closed curve on $N_{g,n}$ and α a simple closed curve on $N_{g,n}$ such that μ and α intersect transversely at one point. Recall that α is oriented. For these simple closed curves μ and α, we denote by $Y_{\mu,\alpha}$ a self-diffeomorphism on $N_{g,n}$ which is described as the result of pushing the Möbius band $\mathcal{N}_{N_{g,n}}(\mu)$ once along α. We call $Y_{\mu,\alpha}$ a *crosscap pushing map*. In particular, if α is two-sided, we call $Y_{\mu,\alpha}$ a *Y-homeomorphism* (or *crosscap slide*), where a *crosscap* means a Möbius band in the interior of a surface. The Y-homeomorphism was originally defined by Lickorish [10]. We have the following fundamental relation on $\mathcal{M}(N_{g,n})$, which we also call the *braid relation*.

Lemma 2.6 (the braid relation (ii)) *Let μ be a one-sided simple closed curve on $N_{g,n}$ and α a simple closed curve on $N_{g,n}$ such that μ and α intersect transversely at one point. For $f \in \mathcal{M}(N_{g,n})$, we have*

$$Y_{f(\mu),f(\alpha)}^{\varepsilon_{f(\alpha)}} = fY_{\mu,\alpha}f^{-1},$$

where $\varepsilon_{f(\alpha)} = 1$ if the fixed orientation of $f(\alpha)$ coincides with that induced by the orientation of α, and $\varepsilon_{f(\alpha)} = -1$ otherwise.

We describe crosscap pushing maps from a different point of view. Let $e\colon D' \hookrightarrow \operatorname{int} S$ be a smooth embedding of the unit disk $D' \subset \mathbb{C}$, let $D := e(D')$, and let S' be the surface obtained from $S - \operatorname{int} D$ by the identification of antipodal points of ∂D. We call the manipulation that gives S' from S the *blowup of S on D*. Note that the image $M \subset S'$ of $\mathcal{N}_{S-\operatorname{int} D}(\partial D) \subset S - \operatorname{int} D$ with respect to the blowup of S on D is a crosscap. Conversely, the *blowdown of S' on M* is the following manipulation that gives S from S'. We paste a disk on the boundary obtained by cutting S along the center line μ of M. The blowdown of S' on M is the inverse manipulation of the blowup of S on D.

Let μ be a one-sided simple closed curve on $N_{g,n}$ and let $\overline{S}$ be the surface which is obtained from $N_{g,n}$ by the blowdown of $N_{g,n}$ on $\mathcal{N}_{N_{g,n}}(\mu)$. Note that $\overline{S}$ is diffeomorphic to $N_{g-1,n}$ or $\Sigma_{h,n}$ for $g = 2h+1$. Denote by x_μ the center point of a disk D_μ that is pasted on the boundary obtained by cutting S along μ. Let $e\colon D' \hookrightarrow D_\mu \subset \overline{S}$ be a smooth embedding of the unit disk $D' \subset \mathbb{C}$ to $\overline{S}$ such that $D_\mu = e(D')$ and $e(0) = x_\mu$. Let $\mathcal{M}(\overline{S}, x_\mu)$ be the group of isotopy classes of self-diffeomorphisms on $\overline{S}$ fixing the boundary $\partial\overline{S}$ and the point x_μ, where isotopies also fix the boundary $\partial\overline{S}$ and x_μ. Then we have the *blowup homomorphism*

$$\varphi_\mu\colon \mathcal{M}(\overline{S}, x_\mu) \to \mathcal{M}(N_{g,n})$$

that is defined as follows. For $h \in \mathcal{M}(\overline{S}, x_\mu)$, we take a representative h' of h which satisfies either of the following conditions: (a) $h'|_{D_\mu}$ is the identity map on D_μ, or (b) $h'(x) = e(\overline{e^{-1}(x)})$ for $x \in D_\mu$, where $\overline{e^{-1}(x)}$ is the complex conjugation of $e^{-1}(x) \in \mathbb{C}$. Such h' is compatible with the blowup of $\overline{S}$ on D_μ; thus $\varphi_\mu(h) \in \mathcal{M}(N_{g,n})$ is induced and well defined; cf [15, Subsection 2.3].

The *point pushing map*

$$j_{x_\mu}\colon \pi_1(\overline{S}, x_\mu) \to \mathcal{M}(\overline{S}, x_\mu)$$

is a homomorphism that is defined as follows. For $\gamma \in \pi_1(\overline{S}, x_\mu)$, we describe $j_{x_\mu}(\gamma) \in \mathcal{M}(\overline{S}, x_\mu)$ as the result of pushing the point x_μ once along γ. The point pushing map comes from the Birman exact sequence. Note that for $\gamma_1, \gamma_2 \in \pi_1(\overline{S}, x_\mu)$, $\gamma_1\gamma_2$ means $\gamma_1\gamma_2(t) = \gamma_2(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $\gamma_1\gamma_2(t) = \gamma_1(2t-1)$ for $\frac{1}{2} \leq t \leq 1$.

Following Szepietowski [15], we define the composition of the homomorphisms:

$$\psi_{x_\mu} := \varphi_\mu \circ j_{x_\mu}\colon \pi_1(\overline{S}, x_\mu) \to \mathcal{M}(N_{g,n}).$$

For each closed curve α on $N_{g,n}$ which transversely intersects with μ at one point, we take a loop $\overline{\alpha}$ on $\overline{S}$ based at x_μ such that $\overline{\alpha}$ has no self-intersection points on D_μ and α is the image of $\overline{\alpha}$ with respect to the blowup of $\overline{S}$ on D_μ. If α is simple, we take $\overline{\alpha}$ as a simple loop. The next two lemmas follow from the description of the point pushing map; see [8, Lemmas 2.2, 2.3].

Lemma 2.7 *For a simple closed curve α on $N_{g,n}$ which transversely intersects with a one-sided simple closed curve μ on $N_{g,n}$ at one point, we have*

$$\psi_{x_\mu}(\overline{\alpha}) = Y_{\mu,\alpha}.$$

Lemma 2.8 *For a one-sided simple closed curve α on $N_{g,n}$ which transversely intersects with a one-sided simple closed curve μ on $N_{g,n}$ at one point, we take $\mathcal{N}_{\overline{S}}(\overline{\alpha})$ such that the interior of $\mathcal{N}_{\overline{S}}(\overline{\alpha})$ contains D_μ. Suppose that $\overline{\delta_1}$ and $\overline{\delta_2}$ are distinct boundary components of $\mathcal{N}_{\overline{S}}(\overline{\alpha})$, and δ_1 and δ_2 are two-sided simple closed curves on $N_{g,n}$ which are images of $\overline{\delta_1}$ and $\overline{\delta_2}$ with respect to the blowup of $\overline{S}$ on D_μ, respectively. Then we have*

$$Y_{\mu,\alpha} = t_{\delta_1}^{\varepsilon_{\delta_1}} t_{\delta_2}^{\varepsilon_{\delta_2}},$$

where ε_{δ_1} and ε_{δ_2} are 1 or -1, depending on the orientations of α, $\mathcal{N}_{N_{g,n}}(\delta_1)$ and $\mathcal{N}_{N_{g,n}}(\delta_2)$; see Figure 3.

By definition of the homomorphism ψ_{x_μ} and Lemma 2.7, we have the following lemma.

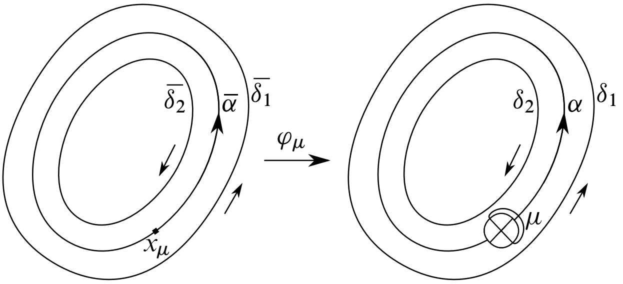

Figure 3: If the orientations of α, $\mathcal{N}_{N_{g,n}}(\delta_1)$ and $\mathcal{N}_{N_{g,n}}(\delta_2)$ are as above, then we have $Y_{\mu,\alpha} = t_{\delta_1} t_{\delta_2}^{-1}$. The $\otimes$–mark means that antipodal points of ∂D_μ are identified.

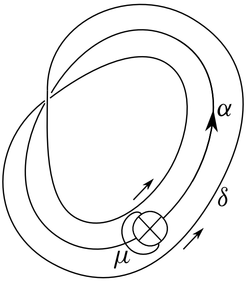

Figure 4: If the orientations of α and $\mathcal{N}_{N_{g,n}}(\delta)$ are as above, then $Y_{\mu,\alpha}^2 = t_{\delta_1}$.

Lemma 2.9 *Let α and β be simple closed curves on $N_{g,n}$ which transversely intersect with a one-sided simple closed curve μ on $N_{g,n}$ at one point each. Suppose the product $\overline{\alpha}\overline{\beta}$ of $\overline{\alpha}$ and $\overline{\beta}$ in $\pi_1(\overline{S}, x_\mu)$ is represented by a simple loop on $\overline{S}$, and $\alpha\beta$ is a simple closed curve on $N_{g,n}$ which is the image of the representative of $\overline{\alpha}\overline{\beta}$ with respect to the blowup of $\overline{S}$ on D_μ. Then we have*

$$Y_{\mu,\alpha\beta} = Y_{\mu,\alpha} Y_{\mu,\beta}.$$

Finally, we recall the following relation between a Dehn twist and a Y-homeomorphism.

Lemma 2.10 *Let α be a two-sided simple closed curve on $N_{g,n}$ which transversely intersects with a one-sided simple closed curve μ on $N_{g,n}$ at one point, and let δ be the boundary of $\mathcal{N}_{N_{g,n}}(\alpha \cup \mu)$. Then we have*

$$Y_{\mu,\alpha}^2 = t_\delta^\varepsilon,$$

where ε is 1 or -1, depending on the orientations of α and $\mathcal{N}_{N_{g,n}}(\delta)$; see Figure 4.

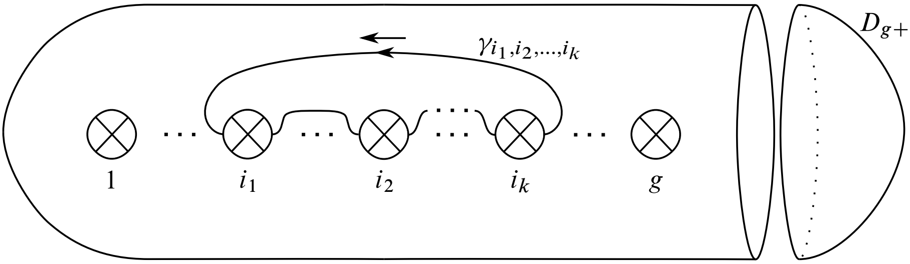

Figure 5: Simple closed curve $\gamma_{i_1,i_2,\ldots,i_k}$ on $N_{g,n}$

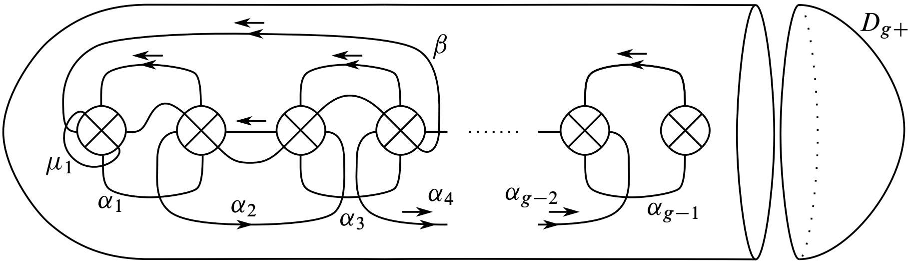

Figure 6: Simple closed curves $\alpha_1, \ldots, \alpha_{g-1}, \beta$ and μ_1 on $N_{g,n}$

Lemma 2.10 follows from relations in Lemmas 2.1, 2.8 and 2.9.

2.3 Stukow's finite presentation for $\mathcal{M}(N_{g,n})$

Let $e_i\colon D' \hookrightarrow \Sigma_0$ for $i = 1, 2, \ldots, g+1$ be smooth embeddings of the unit disk $D' \subset \mathbb{C}$ to a 2–sphere Σ_0 such that $D_i := e_i(D')$ and D_j are disjoint for distinct $1 \leq i, j \leq g+1$. Then we take a model of N_g (resp. $N_{g,1}$) as the surface obtained from Σ_0 (resp. $\Sigma_0 - \operatorname{int} D_{g+1}$) by the blowups on $D_1, \ldots, D_g$ and we describe the identification of ∂D_i by the $\otimes$–mark as in Figures 5 and 6. When $n \in \{0, 1\}$, for $1 \leq i_1 < i_2 < \cdots < i_k \leq g$, let $\gamma_{i_1,i_2,\ldots,i_k}$ be the simple closed curve on $N_{g,n}$ as in Figure 5. Then we define the simple closed curves $\alpha_i := \gamma_{i,i+1}$ for $i = 1, \ldots, g-1$, $\beta := \gamma_{1,2,3,4}$ and $\mu_1 := \gamma_1$ (see Figure 6), and the mapping classes $a_i := t_{\alpha_i}$ for $i = 1, \ldots, g-1$, $b := t_\beta$ and $y := Y_{\mu_1,\alpha_1}$. Then the following finite presentation for $\mathcal{M}(N_{g,n})$ is obtained by Lickorish [10] for $(g,n) = (2,0)$, Stukow [13] for $(g,n) = (2,1)$, Birman and Chillingworth [1] for $(g,n) = (3,0)$ and Theorem 3.1 and Proposition 3.3 in [14] for the other (g,n) such that $g \geq 3$ and $n \in \{0, 1\}$.

Theorem 2.11 [10; 1; 13; 14] *For $(g,n) = (2,0)$, $(2,1)$ and $(3,0)$, we have the following presentation for $\mathcal{M}(N_{g,n})$:*

$$\mathcal{M}(N_2)=\langle a_1,y \mid a_1^2=y^2=(a_1y)^2=1\rangle \cong \mathbb{Z}_2\oplus\mathbb{Z}_2,$$
$$\mathcal{M}(N_{2,1})=\langle a_1,y \mid ya_1y^{-1}=a_1^{-1}\rangle,$$
$$\mathcal{M}(N_3)=\langle a_1,a_2,y \mid a_1a_2a_1=a_2a_1a_2,\ y^2=(a_1y)^2=(a_2y)^2=(a_1a_2)^6=1\rangle.$$

If $g\geq 4$ *and* $n\in\{0,1\}$ *or* $(g,n)=(3,1)$, *then* $\mathcal{M}(N_{g,n})$ *admits a presentation with generators* $a_1,\dots,a_{g-1},y$, *and* b *for* $g\geq 4$. *Writing* $[x_1,x_2]=x_1x_2x_1^{-1}x_2^{-1}$, *the defining relations are*

(A1) $[a_i,a_j]=1$ *for* $g\geq 4$ *and* $|i-j|>1$,

(A2) $a_ia_{i+1}a_i=a_{i+1}a_ia_{i+1}$ *for* $i=1,\dots,g-2$,

(A3) $[a_i,b]=1$ *for* $g\geq 4$ *and* $i\neq 4$,

(A4) $a_4ba_4=ba_4b$ *for* $g\geq 5$,

(A5) $(a_2a_3a_4b)^{10}=(a_1a_2a_3a_4b)^6$ *for* $g\geq 5$,

(A6) $(a_2a_3a_4a_5a_6b)^{12}=(a_1a_2a_3a_4a_5a_6b)^9$ *for* $g\geq 7$,

(A7a) $[b_2,b]=1$ *for* $g=6$,

(A7b) $[a_{g-5},b_{(g-2)/2}]=1$ *for* $g\geq 8$ *even, where* $b_0=a_1$, $b_1=b$ *and* $b_{i+1}=(b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3}b_i)^5(b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3})^{-6}$ *for* $1\leq i\leq (g-4)/2$,

(B1) $y(a_2a_3a_1a_2ya_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1})=(a_2a_3a_1a_2ya_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1})y$ *for* $g\geq 4$,

(B2) $y(a_2a_1y^{-1}a_2^{-1}ya_1a_2)y=a_1(a_2a_1y^{-1}a_2^{-1}ya_1a_2)a_1$,

(B3) $[a_i,y]=1$ *for* $g\geq 4$ *and* $i=3,\dots,g-1$,

(B4) $a_2(ya_2y^{-1})=(ya_2y^{-1})a_2$,

(B5) $ya_1=a_1^{-1}y$,

(B6) $byby^{-1}=\big(a_1a_2a_3(y^{-1}a_2y)a_3^{-1}a_2^{-1}a_1^{-1}\big)\big(a_2^{-1}a_3^{-1}(ya_2y^{-1})a_3a_2\big)$ *for* $g\geq 4$,

(B7) $[(a_4a_5a_3a_4a_2a_3a_1a_2ya_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1}),b]=1$ *for* $g\geq 6$,

(B8) $\big((ya_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})b(a_4a_3a_2a_1y^{-1})\big)\big((a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})b^{-1}(a_4a_3a_2a_1)\big)$
$=\big((a_4^{-1}a_3^{-1}a_2^{-1})y(a_2a_3a_4)\big)\big(a_3^{-1}a_2^{-1}y^{-1}a_2a_3\big)\big(a_2^{-1}ya_2\big)y^{-1}$ *for* $g\geq 5$,

(C1) $(a_1a_2\cdots a_{g-1})^g=1$ *for* $g\geq 4$ *even and* $n=0$,

(C2) $[a_1,\rho]=1$ *for* $g\geq 4$ *and* $n=0$, *where* $\rho=(a_1a_2\cdots a_{g-1})^g$ *for* g *odd and* $\rho=(y^{-1}a_2a_3\cdots a_{g-1}ya_2a_3\cdots a_{g-1})^{(g-2)/2}y^{-1}a_2a_3\cdots a_{g-1}$ *for* g *even,*

(C3) $\rho^2=1$ *for* $g\geq 4$ *and* $n=0$,

(C4) $(y^{-1}a_2a_3\cdots a_{g-1}ya_2a_3\cdots a_{g-1})^{(g-1)/2}=1$ *for* $g\geq 4$ *odd and* $n=0$.

3 Presentation for $\mathcal{M}(N_{g,n})$

The main theorem in this paper is as follows:

Theorem 3.1 *For $g \geq 1$ and $n \in \{0,1\}$, we have a presentation for $\mathcal{M}(N_{g,n})$ with generating set*

$$X = \{t_c \mid c \text{ a two-sided scc on } N_{g,n}\} \\ \cup \{Y_{\mu,\alpha} \mid \mu \text{ a one-sided scc on } N_{g,n},\ \alpha \text{ a scc on } N_{g,n},\ |\mu \cap \alpha| = 1\}$$

and relations

(0) $t_c = 1$ *when c bounds a disk or a Möbius band in $N_{g,n}$,*

(I) *all the braid relations*

   (i) $f t_c f^{-1} = t_{f(c)}^{\varepsilon_{f(c)}}$ for $f \in X$,

   (ii) $f Y_{\mu,\alpha} f^{-1} = Y_{f(\mu),f(\alpha)}^{\varepsilon_{f(\alpha)}}$ *for $f \in X$,*

(II) *all the 2–chain relations,*

(III) *all the lantern relations,*

(IV) *all the relations in Lemma 2.9, ie $Y_{\mu,\alpha\beta} = Y_{\mu,\alpha} Y_{\mu,\beta}$,*

(V) *all the relations in Lemma 2.8, ie $Y_{\mu,\alpha} = t_{\delta_1}^{\varepsilon_{\delta_1}} t_{\delta_2}^{\varepsilon_{\delta_2}}$.*

In (I) and (IV) one can substitute the right-hand side of (V) for each generator $Y_{\mu,\alpha}$ with one-sided α. Then one can remove the generators $Y_{\mu,\alpha}$ with one-sided α and relations (V) from the presentation.

We denote by G the group with the presentation in Theorem 3.1. Let $\iota\colon \Sigma_{h,m} \hookrightarrow N_{g,n}$ be a smooth embedding and let G' be the group whose presentation has all Dehn twists along simple closed curves on $\Sigma_{h,m}$ as generators and relations (0′), (I′), (II) and (III) in Theorem 2.5. By that theorem, $\mathcal{M}(\Sigma_{h,m})$ is isomorphic to G', and we have the homomorphism $G' \to G$ defined by the correspondence of t_c to $t_{\iota(c)}^{\varepsilon_{\iota(c)}}$, where $\varepsilon_{\iota(c)} = 1$ if the restriction $\iota|_{\mathcal{N}_{\Sigma_{h,m}}(c)}\colon \mathcal{N}_{\Sigma_{h,m}}(c) \to \mathcal{N}_{N_{g,n}}(\iota(c))$ is orientation-preserving, and $\varepsilon_{\iota(c)} = -1$ if it is orientation-reversing. Then we remark the following.

Remark 3.2 The composition $\iota_*\colon \mathcal{M}(\Sigma_{h,m}) \to G$ of the isomorphism $\mathcal{M}(\Sigma_{h,m}) \to G'$ and the homomorphism $G' \to G$ is a homomorphism. In particular, if a product $t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_k}^{\varepsilon_k}$ of Dehn twists along simple closed curves $c_1, c_2, \ldots, c_k$ on a connected compact orientable subsurface of $N_{g,n}$ is equal to the identity map in the mapping class group of the subsurface, then $t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_k}^{\varepsilon_k}$ is equal to 1 in G. That means such a relation $t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_k}^{\varepsilon_k} = 1$ is obtained from relations (0)–(III).

Set $X^{\pm} := X \cup \{x^{-1} \mid x \in X\}$. By relations (I), we have the following lemma.

Lemma 3.3 *For* $f \in G$, *suppose that* $f = f_1 f_2 \cdots f_k$, *where* $f_1, f_2, \dots, f_k \in X^{\pm}$. *Then we have*

(i) $f t_c f^{-1} = t_{f(c)}^{\varepsilon_{f(c)}}$,

(ii) $f Y_{\mu,\alpha} f^{-1} = Y_{f(\mu),f(\alpha)}^{\varepsilon_{f(\alpha)}}$.

The next lemma follows from an argument of combinatorial group theory; for instance, see [7, Lemma 4.2.1, page 42].

Lemma 3.4 *For groups* Γ, Γ' *and* F, *a surjective homomorphism* $\pi\colon F \to \Gamma$ *and a homomorphism* $\nu\colon F \to \Gamma'$, *we define a map* $\nu'\colon \Gamma \to \Gamma'$ *by* $\nu'(x) := \nu(\widetilde{x})$ *for* $x \in \Gamma$, *where* $\widetilde{x} \in F$ *is a lift of* x *with respect to* π; *see the diagram below:*

$$\begin{tikzcd}
F \arrow[d, two heads, "\pi"'] \arrow[dr, "\nu"] & \\
\Gamma \arrow[r, dashed, "\nu'"'] & \Gamma'
\end{tikzcd}$$

If $\ker \pi \subset \ker \nu$, *then* ν' *is well-defined and a homomorphism.*

Proof of Theorem 3.1 $\mathcal{M}(N_1)$ and $\mathcal{M}(N_{1,1})$ are trivial; see [2]. Assume $g \geq 2$ and $n \in \{0, 1\}$. Then we obtain Theorem 3.1 if $\mathcal{M}(N_{g,n})$ is isomorphic to G. Let $\varphi\colon G \to \mathcal{M}(N_{g,n})$ be the surjective homomorphism defined by $\varphi(t_c) := t_c$ and $\varphi(Y_{\mu,\alpha}) := Y_{\mu,\alpha}$.

Set $X_0 := \{a_1, \dots, a_{g-1}, b, y\} \subset \mathcal{M}(N_{g,n})$ for $g \geq 4$ and $X_0 := \{a_1, \dots, a_{g-1}, y\} \subset \mathcal{M}(N_{g,n})$ for $g = 2, 3$. Let $F(X_0)$ be the free group which is freely generated by X_0 and let $\pi\colon F(X_0) \to \mathcal{M}(N_{g,n})$ be the natural projection (by Theorem 2.11). We define the homomorphism $\nu\colon F(X_0) \to G$ by $\nu(a_i) := a_i$ for $i = 1, \dots, g-1$, $\nu(b) := b$ and $\nu(y) := y$, and a map $\psi = \nu'\colon \mathcal{M}(N_{g,n}) \to G$ by

- $\psi(a_i^{\pm 1}) := a_i^{\pm 1}$ for $i = 1, \dots, g-1$,
- $\psi(b^{\pm 1}) := b^{\pm 1}$,
- $\psi(y^{\pm 1}) := y^{\pm 1}$, and
- $\psi(f) := \nu(\widetilde{f})$ for the other $f \in \mathcal{M}(N_{g,n})$,

where $\widetilde{f} \in F(X_0)$ is a lift of f with respect to π; see the diagram below:

$$\begin{tikzcd}
F(X_0) \arrow[d, two heads, "\pi"'] \arrow[dr, "\nu"] & \\
\mathcal{M}(N_{g,n}) \arrow[r, dashed, "\psi"'] & G
\end{tikzcd}$$

If ψ is a homomorphism, $\varphi \circ \psi = \mathrm{id}_{\mathcal{M}(N_{g,n})}$ by the definition of φ and ψ. Thus to prove that ψ is an isomorphism it suffices to show that ψ is a homomorphism and surjective.

3.1 Proof that ψ is a homomorphism

$\mathcal{M}(N_1)$ and $\mathcal{M}(N_{1,1})$ are trivial [2, Theorem 3.4]. For $(g,n) \in \{(2,0),(2,1),(3,0)\}$, relations of the presentation in Theorem 2.11 are clearly obtained from relations (0)–(V). Thus by Lemma 3.4, ψ is a homomorphism.

Assume $g \geq 4$ or $(g,n) = (3,1)$. By Lemma 3.4, if the relations of the presentation in Theorem 2.11 are obtained from relations (0)–(V), then ψ is a homomorphism.

The group generated by $a_1, \dots, a_{g-1}$ and b with relations (A1)–(A7b) as defining relations is isomorphic to $\mathcal{M}(\Sigma_{h,1})$ (resp. $\mathcal{M}(\Sigma_{h,2})$) for $g = 2h+1$ (resp. $g = 2h+2$) by Theorem 3.1 in [12], and relations (A1)–(A7b) are relations on the mapping class group of the orientable subsurface $\mathcal{N}_{N_{g,n}}(\alpha_1 \cup \dots \cup \alpha_{g-1})$ of $N_{g,n}$. Hence relations (A1)–(A7b) are obtained from relations (0)–(III) by Remark 3.2.

Stukow [14] gave geometric interpretations for relations (B1)–(B8) in Section 4 in [14]. By the interpretation, relations (B1)–(B5) and (B7) are obtained from relations (I) (use Lemma 3.3), relation (B6) is obtained from relations (0), (I), (III), (IV) and (V) (use Lemmas 2.10 and 3.3), and relation (B8) is obtained from relations (I), (IV) and (V) (use Lemma 3.3). Thus ψ is a homomorphism when $n = 1$.

We assume $n = 0$. By Remark 3.2, k–chain relations are obtained from relations (0)–(III) for each k. Relation (C1) is interpreted in G as follows:

$$(a_1 a_2 \cdots a_{g-1})^g = t_{\gamma_{1,2,\dots,g}} t_{\gamma_{1,2,\dots,g}}^{-1} = 1 \quad \text{by (0)–(III)}.$$

Thus relation (C1) is obtained from relations (0)–(III).

Relation (C2) is clearly obtained from relations (I) by Lemma 3.3.

When g is odd, by using the $(g-1)$–chain relation, relation (C3) is interpreted in G as follows:

$$\begin{aligned}
\rho^2 = (a_1 a_2 \cdots a_{g-1})^{2g} &= t_{\partial \mathcal{N}_{N_g}(\gamma_{1,2,\dots,g})}^{\varepsilon} && \text{by (0)–(III)} \\
&= 1 && \text{by (0),}
\end{aligned}$$

where ε is 1 or -1. Note that $\mathcal{N}_{N_g}(\gamma_{1,2,\dots,g})$ is a Möbius band in N_g. Thus relation (C3) is obtained from relations (0)–(III) when g is odd.

When g is even, we rewrite the left-hand side ρ^2 of relation (C3) by braid relations. Set $A := a_2 a_3 \cdots a_{g-1}$. Note that

$$Y_{\mu_1,\gamma_{1,2,3}} A^2 Y_{\mu_1,\gamma_{1,2,\dots,2i-1}} A^{-2} = Y_{\mu_1,\gamma_{1,2,\dots,2i+1}}$$

for $i = 2, \dots, (g-2)/2$ by relations (I) and (IV), and then we have

$$\begin{aligned}
\rho &= y^{-1} A (y A y^{-1} A)^{(g-2)/2} \\
&= y^{-1} A (y a_2 y^{-1} a_3 \cdots a_{g-1} A)^{(g-2)/2} && \text{by (I)} \\
&= y^{-1} A (\underline{y (a_2 y^{-1} a_2^{-1})} A^2)^{(g-2)/2} \\
&= y^{-1} A (Y_{\mu_1,\gamma_{1,2,3}} A^2)^{(g-2)/2} && \text{by (I), (IV)} \\
&= y^{-1} A Y_{\mu_1,\gamma_{1,2,3}} A^2 \cdots Y_{\mu_1,\gamma_{1,2,3}} A^2 Y_{\mu_1,\gamma_{1,2,3}} A^2 Y_{\mu_1,\gamma_{1,2,3}} A^2 \\
&= y^{-1} A Y_{\mu_1,\gamma_{1,2,3}} A^2 \cdots Y_{\mu_1,\gamma_{1,2,3}} A^2 \underline{Y_{\mu_1,\gamma_{1,2,3}} A^2 Y_{\mu_1,\gamma_{1,2,3}} A^{-2}} A^4 \\
&= y^{-1} A Y_{\mu_1,\gamma_{1,2,3}} A^2 \cdots Y_{\mu_1,\gamma_{1,2,3}} A^2 Y_{\mu,\gamma_{1,2,3,4,5}} A^4 && \text{by (I), (IV)} \\
&= y^{-1} A Y_{\mu_1,\gamma_{1,2,3}} A^2 \cdots \underline{Y_{\mu_1,\gamma_{1,2,3}} A^2 Y_{\mu,\gamma_{1,2,3,4,5}} A^{-2}} A^6 \\
&= y^{-1} A Y_{\mu_1,\gamma_{1,2,3}} A^2 \cdots Y_{\mu_1,\gamma_{1,2,3,4,5,6,7}} A^6 && \text{by (I), (IV)} \\
&\ \vdots \\
&= y^{-1} A Y_{\mu_1,\gamma_{1,2,\dots,g-1}} A^{g-2} && \text{by (I), (IV)} \\
&= \underline{y^{-1} \cdot A Y_{\mu_1,\gamma_{1,2,\dots,g-1}} A^{-1}} \cdot A^{g-1} \\
&= Y_{\mu_1,\gamma_{1,2,\dots,g}} A^{g-1} && \text{by (I), (IV).}
\end{aligned}$$

Since $Y_{\mu_1,\gamma_{1,2,\dots,g}}$ commutes with a_i for $i = 2, \dots, g-1$, and $\partial\mathcal{N}_{N_g}(\mu_1 \cup \gamma_{1,2,\dots,g}) = \partial\mathcal{N}_{N_g}(\alpha_2 \cup \cdots \cup \alpha_{g-1})$ (see Figure 7), we have

$$\begin{aligned}
\rho^2 &= Y_{\mu_1,\gamma_{1,2,\dots,g}} A^{g-1} Y_{\mu_1,\gamma_{1,2,\dots,g}} A^{g-1} \\
&= Y_{\mu_1,\gamma_{1,2,\dots,g}}^2 A^{2g-2} && \text{by (I)} \\
&= Y_{\mu_1,\gamma_{1,2,\dots,g}}^2 t_{\partial\mathcal{N}_{N_g}(\alpha_2 \cup \cdots \cup \alpha_{g-1})} && \text{by (0)–(III)} \\
&= t_{\partial\mathcal{N}_{N_g}(\alpha_2 \cup \cdots \cup \alpha_{g-1})}^{-1} t_{\partial\mathcal{N}_{N_g}(\alpha_2 \cup \cdots \cup \alpha_{g-1})} && \text{by Lemma 2.10} \\
&= 1.
\end{aligned}$$

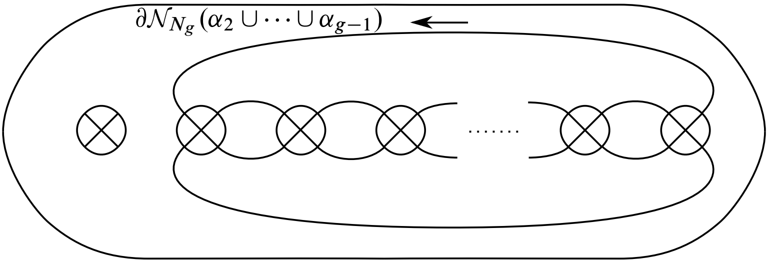

Figure 7: Simple closed curve $\partial\mathcal{N}_{N_g}(\alpha_2 \cup \cdots \cup \alpha_{g-1})$ on N_g

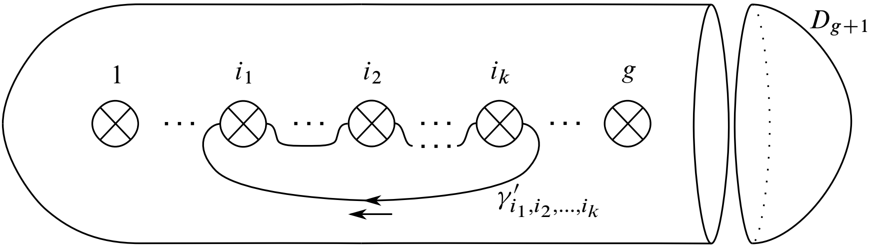

Figure 8: Simple closed curve $\gamma'_{i_1,i_2,\dots,i_k}$ on $N_{g,n}$

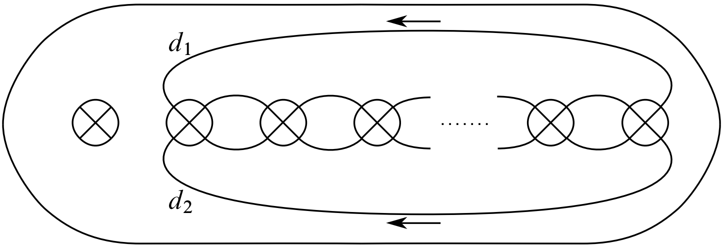

Figure 9: Simple closed curve d_1 and d_2 on $N_{g,n}$

Recall that the relations in Lemma 2.10 are obtained from relations (0), (IV) and (V). Thus relation (C3) is obtained from relations (0)–(V) when g is even.

Finally, we also rewrite the left-hand side $(y^{-1}a_2a_3\cdots a_{g-1}ya_2a_3\cdots a_{g-1})^{(g-1)/2}$ of relation (C4) by braid relations. Note that g is odd. For $1 \le i_1 < i_2 < \cdots < i_k \le g$, we denote by $\gamma'_{i_1,i_2,\dots,i_k}$ the simple closed curve on $N_{g,n}$ as in Figure 8. Note that

$$Y_{\mu_1,\gamma'_{1,2,3}} A^2 Y_{\mu_1,\gamma'_{1,2,\dots,2i-1}} A^{-2} = Y_{\mu_1,\gamma'_{1,2,\dots,2i+1}}$$

for $i = 2, \dots, (g-1)/2$ and $\partial\mathcal{N}_{N_g}(\mu_1 \cup \gamma_{1,2,\dots,g}) = d_1 \sqcup d_2$ as in Figure 9. By a similar argument as for relation (C3) when g is even, we have

$$\begin{aligned}
&(y^{-1}a_2a_3\cdots a_{g-1}ya_2a_3\cdots a_{g-1})^{(g-1)/2} \\
&\qquad = (y^{-1}AyA)^{(g-1)/2} \\
&\qquad = (\underline{y^{-1}(a_2ya_2^{-1})}A^2)^{(g-1)/2} && \text{by (I)} \\
&\qquad = (Y_{\mu_1,\gamma'_{1,2,3}}A^2)^{(g-1)/2} && \text{by (I), (IV)} \\
&\qquad = Y_{\mu_1,\gamma'_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma'_{1,2,3}}A^2Y_{\mu_1,\gamma'_{1,2,3}}A^2Y_{\mu_1,\gamma'_{1,2,3}}A^2 \\
&\qquad = Y_{\mu_1,\gamma'_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma'_{1,2,3}}A^2\underline{Y_{\mu_1,\gamma'_{1,2,3}}A^2Y_{\mu_1,\gamma'_{1,2,3}}A^{-2}}A^4 \\
&\qquad = Y_{\mu_1,\gamma'_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma'_{1,2,3}}A^2Y_{\mu_1,\gamma'_{1,2,3,4,5}}A^4 && \text{by (I), (IV)} \\
&\qquad = Y_{\mu_1,\gamma'_{1,2,3}}A^2\cdots \underline{Y_{\mu_1,\gamma'_{1,2,3}}A^2Y_{\mu_1,\gamma'_{1,2,3,4,5}}A^{-2}}A^6 \\
&\qquad = Y_{\mu_1,\gamma'_{1,2,3}}A^2\cdots Y_{\mu,\gamma'_{1,2,3,4,5,6,7}}A^6 && \text{by (I), (IV)} \\
&\qquad \vdots \\
&\qquad = Y_{\mu,\gamma_{1,2,\dots,g}}A^{g-1} && \text{by (I), (IV)} \\
&\qquad = Y_{\mu,\gamma_{1,2,\dots,g}}t_{d_1}t_{d_2} && \text{by (0)–(III)} \\
&\qquad = t_{d_1}^{-1}t_{d_2}^{-1}t_{d_1}t_{d_2} && \text{by (V)} \\
&\qquad = 1, && \text{by (I)}
\end{aligned}$$

where simple closed curves d_1 and d_2 are, as in Figure 9, boundary components of $\mathcal{N}_{N_g}(\alpha_2 \cup \cdots \cup \alpha_{g-1})$. Therefore, relation (C4) is obtained from relations (I), (II), (IV) and (V), and $\psi\colon \mathcal{M}(N_{g,n}) \to G$ is a homomorphism.

3.2 Surjectivity of ψ

To prove the surjectivity of ψ, we show that there exist lifts of the t_c and the $Y_{\mu,\alpha}$ with respect to ψ for the cases below:

(1) t_c: c is nonseparating and $N_{g,n} - c$ is nonorientable.

(2) t_c: c is nonseparating and $N_{g,n} - c$ is orientable.

(3) t_c: c is separating.

(4) $Y_{\mu,\alpha}$: α is two-sided and $N_{g,n} - \alpha$ is nonorientable.

(5) $Y_{\mu,\alpha}$: α is two-sided and $N_{g,n}-\alpha$ is orientable.

(6) $Y_{\mu,\alpha}$: α is one-sided.

Set $X_0^{\pm} := X_0 \cup \{x^{-1} \mid x \in X_0\}$, and for a simple closed curve c on $N_{g,n}$, we denote by $(N_{g,n})_c$ the surface obtained from $N_{g,n}$ by cutting $N_{g,n}$ along c.

Case 1 Since $(N_{g,n})_c$ is diffeomorphic to $N_{g-2,n+2}$ and $g \geq 3$, there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \dots, f_k \in X_0^{\pm}$ such that $f(\alpha_1) = c$. Note that $\psi(f_i) = f_i \in X^{\pm} \subset G$ for $i = 1, 2, \dots, k$. Thus we have

$$\begin{aligned}
\psi(f a_1 f^{-1}) &= \psi(f)\psi(a_1)\psi(f)^{-1} \\
&= f_1 f_2 \cdots f_k a_1 f_k^{-1} \cdots f_2^{-1} f_1^{-1} \\
&= t_{f(\alpha_1)}^{\varepsilon} && \text{by Lemma 3.3} \\
&= t_c^{\varepsilon},
\end{aligned}$$

where ε is 1 or -1. Thus $f a_1^{\varepsilon} f^{-1} \in \mathcal{M}(N_{g,n})$ is a lift of $t_c \in G$ with respect to ψ for some $\varepsilon \in \{-1, 1\}$.

Case 2 We remark that g is even in this case. When $g = 2$, such a simple closed curve c is unique and $c = \alpha_1$. Thus $a_1 \in \mathcal{M}(N_{g,n})$ is the lift of $t_c \in G$ with respect to ψ. When $g = 4$, since $(N_{g,n})_c$ is diffeomorphic to $\Sigma_{1,n+2}$, there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \dots, f_k \in X_0^{\pm}$ such that $f(\beta) = c$. By a similar argument as in case 1, $f b^{\varepsilon} f^{-1} \in \mathcal{M}(N_{g,n})$ is a lift of $t_c \in G$ with respect to ψ for some $\varepsilon \in \{-1, 1\}$.

Assume $g \geq 6$ even. Then there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \dots, f_k \in X_0^{\pm}$ such that $f(\gamma_{1,2,\dots,g}) = c$. Since $\alpha_1 \cup \alpha_3 \cup \gamma_{5,6,\dots,g} \cup \gamma_{1,2,\dots,g}$ bounds a subsurface of $N_{g,n}$ which is diffeomorphic to $\Sigma_{0,4}$ (see Figure 10), we have $b t_{\gamma_{3,4,\dots,g}} t_{\gamma_{1,2,5,\dots,g}} = t_{\gamma_{1,2,\dots,g}} a_1 a_3 t_{\gamma_{5,6,\dots,g}}$ by a lantern relation. Note that $b, t_{\gamma_{3,4,\dots,g}}, t_{\gamma_{1,2,5,\dots,g}}, a_1, a_3$, and $t_{\gamma_{5,6,\dots,g}}$ are Dehn twists of type (1), and $t_{\gamma_{3,4,\dots,g}}$, $t_{\gamma_{1,2,5,\dots,g}}$ and $t_{\gamma_{5,6,\dots,g}} \in G$ have lifts h_1, h_2 and h_3 in $\mathcal{M}(N_{g,n})$ with respect to ψ, respectively. Thus we have

$$\begin{aligned}
&\psi(f b h_1 h_2 a_1^{-1} a_3^{-1} h_3^{-1} f^{-1}) \\
&\quad = f_1 f_2 \cdots f_k b t_{\gamma_{3,4,\dots,g}} t_{\gamma_{1,2,5,\dots,g}} a_1^{-1} a_3^{-1} t_{\gamma_{5,6,\dots,g}}^{-1} f_k^{-1} \cdots f_2^{-1} f_1^{-1} \\
&\quad = f_1 f_2 \cdots f_k t_{\gamma_{1,2,\dots,g}} f_k^{-1} \cdots f_2^{-1} f_1^{-1} && \text{by (III)} \\
&\quad = t_c^{\varepsilon}, && \text{by Lemma 3.3}
\end{aligned}$$

where ε is 1 or -1. Thus $f(b h_1 h_2 a_1^{-1} a_3^{-1} h_3^{-1})^{\varepsilon} f^{-1} \in \mathcal{M}(N_{g,n})$ is a lift of $t_c \in G$ with respect to ψ for some $\{-1, 1\}$.

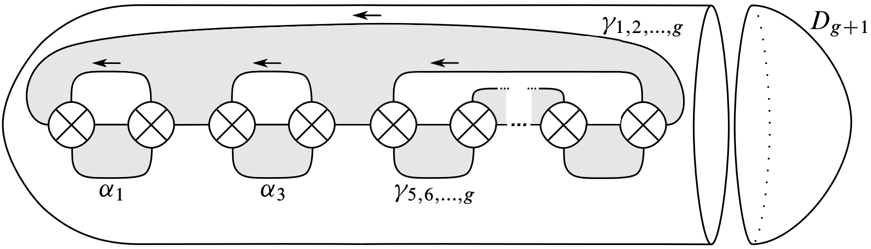

Figure 10: $\alpha_1 \cup \alpha_3 \cup \gamma_{5,6,\dots,g} \cup \gamma_{1,2,\dots,g}$ bound a subsurface of $N_{g,n}$ which is diffeomorphic to $\Sigma_{0,4}$

Case 4 Since $N_{g,n} - \operatorname{int} \mathcal{N}_{N_{g,n}}(\mu \cup \alpha)$ is diffeomorphic to $N_{g-2,n+1}$, and the two-sided simple closed curve on $N_{2,1}$ is unique, there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \dots, f_k \in X_0^{\pm}$ such that $f(\alpha_1) = \alpha$ and $f(\mu_1) = \mu$. Thus we have

$$\psi(fyf^{-1}) = f_1 f_2 \cdots f_k y f_k^{-1} \cdots f_2^{-1} f_1^{-1} = Y_{\mu,\alpha}^{\varepsilon} \quad \text{by Lemma 3.3},$$

where ε is 1 or -1. Thus $fy^{\varepsilon} f^{-1} \in \mathcal{M}(N_{g,n})$ is a lift of $Y_{\mu,\alpha} \in G$ with respect to ψ for some $\varepsilon \in \{-1, 1\}$.

Case 5 We remark that g is even in this case. Since $N_{g,n} - \operatorname{int} \mathcal{N}_{N_{g,n}}(\mu \cup \alpha)$ is diffeomorphic to $\Sigma_{(g-2)/2,n+1}$ and the two-sided simple closed curve on $N_{2,1}$ is unique, there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \dots, f_k \in X_0^{\pm}$ such that $f(\gamma_{1,2,\dots,g}) = \alpha$ and $f(\mu_1) = \mu$. Note that $Y_{\mu_1,\gamma_{1,2}}, Y_{\mu_1,\gamma_{1,3}}, \dots, Y_{\mu_1,\gamma_{1,g}}$ are Y-homeomorphisms of type (4), and $Y_{\mu_1,\gamma_{1,3}}, Y_{\mu_1,\gamma_{1,4}}, \dots, Y_{\mu_1,\gamma_{1,g}} \in G$ have lifts $h_3, h_4, \dots, h_g \in \mathcal{M}(N_{g,n})$ with respect to ψ, respectively. Thus we have

$$\begin{aligned} &\psi(f h_g \cdots h_4 h_3 y f^{-1}) \\ &\quad = f_1 f_2 \cdots f_k Y_{\mu_1,\gamma_{1,g}} \cdots Y_{\mu_1,\gamma_{1,4}} Y_{\mu_1,\gamma_{1,3}} y f_k^{-1} \cdots f_2^{-1} f_1^{-1} \\ &\quad = f_1 f_2 \cdots f_k Y_{\mu_1,\gamma_{1,2,\dots,g}} f_k^{-1} \cdots f_2^{-1} f_1^{-1} && \text{by (IV)} \\ &\quad = Y_{\mu,\alpha}^{\varepsilon}, && \text{by Lemma 3.3} \end{aligned}$$

where ε is 1 or -1. Thus $f(h_g \cdots h_4 h_3 y)^{\varepsilon} f^{-1} \in \mathcal{M}(N_{g,n})$ is a lift of $Y_{\mu,\alpha} \in G$ with respect to ψ for some $\varepsilon \in \{-1, 1\}$.

Case 3 Let Σ be the component of $(N_{g,n})_c$ which has one boundary component. Let us suppose that Σ is orientable; then there exists a k–chain $c_1, c_2, \dots, c_k$ on $N_{g,n}$ such that $\mathcal{N}_{N_{g,n}}(c_1 \cup c_2 \cup \cdots \cup c_k) = \Sigma$. By the chain relation, $(t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_k}^{\varepsilon_k})^{2k+2} = t_c$ for some $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{-1, 1\}$. Note that $t_{c_1}, t_{c_2}, \dots, t_{c_k}$ are Dehn twists of type (1) and $t_{c_1}, t_{c_2}, \dots, t_{c_k} \in G$ have lifts $h_1, h_2, \dots, h_k \in \mathcal{M}(N_{g,n})$ with respect to ψ, respectively. Thus we have

$$\psi((h_1^{\varepsilon_1} h_2^{\varepsilon_2} \cdots h_k^{\varepsilon_k})^{2k+2}) = (t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_k}^{\varepsilon_k})^{2k+2} = t_c \quad \text{by (0)–(III).}$$

Thus $(h_1^{\varepsilon_1} h_2^{\varepsilon_2} \cdots h_k^{\varepsilon_k})^{2k+2} \in \mathcal{M}(N_{g,n})$ is a lift of $t_c \in G$ with respect to ψ.

When Σ is nonorientable, we proceed by induction on the genus g' of Σ. For $g' = 1$, we have $t_c = 1$ by relation (0). When $g' = 2$ and $N_{g,n} - \Sigma$ is nonorientable, there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \dots, f_k \in X_0^{\pm}$ such that $f(\partial \mathcal{N}_{N_{g,n}}(\mu_1 \cup \alpha_1)) = c$. Hence $f y^2 f^{-1} = t_c^{\varepsilon}$ for some $\varepsilon \in \{-1, 1\}$. Then we have

$$\begin{aligned} \psi(f y^2 f^{-1}) &= f_1 f_2 \cdots f_k\, y^2 f_k^{-1} \cdots f_2^{-1} f_1^{-1} \\ &= f_1 f_2 \cdots f_k t^{\varepsilon'}_{\partial \mathcal{N}_{N_{g,n}}(\mu_1 \cup \alpha_1)} f_k^{-1} \cdots f_2^{-1} f_1^{-1} && \text{by Lemma 2.10} \\ &= t_c^{\varepsilon} && \text{by Lemma 3.3,} \end{aligned}$$

where ε' is 1 or -1. Thus $f y^{2\varepsilon} f^{-1} \in \mathcal{M}(N_{g,n})$ is a lift of $t_c \in G$ with respect to ψ. When $g' = 2$ and $N_{g,n} - \Sigma$ is orientable, g is even, and there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \dots, f_k \in X_0^{\pm}$ with $f(\partial \mathcal{N}_{N_{g,n}}(\mu_1 \cup \gamma_{1,2,\dots,g})) = c$. Hence $f Y^2_{\mu_1, \gamma_{1,2,\dots,g}} f^{-1} = t_c^{\varepsilon}$ for some $\varepsilon \in \{-1, 1\}$. Since $Y_{\mu_1, \gamma_{1,2,\dots,g}}$ is a Y-homeomorphism of type (5), there exists a lift $h \in \mathcal{M}(N_{g,n})$ of $Y_{\mu_1, \gamma_{1,2,\dots,g}} \in G$ with respect to ψ. By a similar argument as above, $f h^{2\varepsilon} f^{-1} \in \mathcal{M}(N_{g,n})$ is a lift of $t_c \in G$ with respect to ψ.

Suppose $g' \geq 3$. We take a diffeomorphism $f \colon \Sigma \to N_{g',1}$ and simple closed curves $c_1, c_2, \dots, c_6$ and $c' := \partial N_{g',1} = f(c)$ on $N_{g',1}$ as in Figure 11. Note that $c' \cup c_4 \cup c_5 \cup c_6$ bounds a subsurface of $N_{g',1}$ which is diffeomorphic to $\Sigma_{0,4}$, and we have

$$t^{\varepsilon_1}_{f^{-1}(c_1)}\, t^{\varepsilon_2}_{f^{-1}(c_2)}\, t^{\varepsilon_3}_{f^{-1}(c_3)}\, t^{\varepsilon_4}_{f^{-1}(c_4)} = t_c^{\varepsilon} \in G$$

for some $\varepsilon_1, \dots, \varepsilon_4, \varepsilon \in \{-1, 1\}$ by relations (0) and (III). Since each c_i for $i = 1, 2, \dots, 6$ bounds a subsurface of $N_{g,n}$ which is diffeomorphic to a nonorientable surface of genus $g_i < g'$ with one boundary component, and the complement of the subsurface is nonorientable, each $f^{-1}(c_i)$ $(i = 1, 2, \dots, 6)$ satisfies the inductive assumption. Hence $t_{f^{-1}(c_1)}, t_{f^{-1}(c_2)}, t_{f^{-1}(c_3)}, t_{f^{-1}(c_4)} \in G$ have lifts $h_1, h_2, h_3, h_4 \in \mathcal{M}(N_{g,n})$ with respect to ψ, respectively. Thus we have

$$\psi(h_1^{\varepsilon_1} h_2^{\varepsilon_2} h_3^{\varepsilon_3} h_4^{\varepsilon_4}) = t^{\varepsilon_1}_{f^{-1}(c_1)} t^{\varepsilon_2}_{f^{-1}(c_2)} t^{\varepsilon_3}_{f^{-1}(c_3)} t^{\varepsilon_4}_{f^{-1}(c_4)} = t_c^{\varepsilon} \quad \text{by (0), (III).}$$

Thus $h_1^{\varepsilon_1} h_2^{\varepsilon_2} h_3^{\varepsilon_3} h_4^{\varepsilon_4} \in \mathcal{M}(N_{g,n})$ is a lift of $t_c \in G$ with respect to ψ.

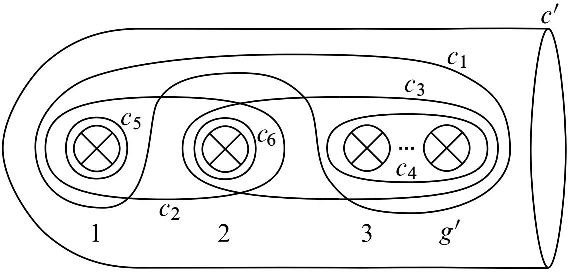

Figure 11: Simple closed curves $c_1, c_2, \dots, c_6$ and c' on $N_{g',1}$

Case 6 Let δ_1 and δ_2 be two-sided simple closed curves on $N_{g,n}$ such that $\delta_1 \sqcup \delta_2 = \partial \mathcal{N}_{N_{g,n}}(\mu \cap \alpha)$. By Lemma 2.8, we have $Y_{\mu,\alpha} = t_{\delta_1}^{\varepsilon_1} t_{\delta_2}^{\varepsilon_2}$ for some ε_1, $\varepsilon_2 \in \{-1, 1\}$ and, by the above arguments, $t_{c_1}, t_{c_2} \in G$ have lifts $h_1, h_2 \in \mathcal{M}(N_{g,n})$ with respect to ψ, respectively. Thus we have

$$\psi(h_1^{\varepsilon_1} h_2^{\varepsilon_2}) = t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} = Y_{\mu,\alpha} \quad \text{by (V)}.$$

Thus $h_1^{\varepsilon_1} h_2^{\varepsilon_2} \in \mathcal{M}(N_{g,n})$ is a lift of $Y_{\mu,\alpha} \in G$ with respect to ψ. Therefore, we have that $\psi\colon \mathcal{M}(N_{g,n}) \to G$ is surjective, completing the proof of Theorem 3.1. □

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