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# On a question of Etnyre and Van Horn-Morris

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The purpose of this note is to answer Question 6.12 of Etnyre and Van Horn-Morris [*Monoids in the mapping class group*, Geom. Topol. Monographs 19 (2015) 319–365], asking when the set of mapping classes whose fractional Dehn twist coefficient is greater than a given constant forms a monoid.

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## 1 Introduction

Let $S$ be a compact oriented surface with nonempty boundary. Let $\mathrm{Mod}(S)$ denote the mapping class group of $S$, the group of isotopy classes of homeomorphisms of $S$ that fix the boundary $\partial S$ pointwise. Let $c(-,C)\colon \mathrm{Mod}(S)\to\mathbb{Q}$ denote the *fractional Dehn twist coefficient* (FDTC) of $\phi\in\mathrm{Mod}(S)$ with respect to the connected component $C$ of $\partial S$. The FDTC plays a fundamental role in the study of (contact) 3–manifolds. See Honda, Kazez and Matić [4] and Ito and Kawamuro [7] for the definition and basic properties of the FDTC which are used in this paper. For $r\in\mathbb{R}$ we define the following sets (see Etnyre and Van Horn-Morris [2, page 344]):

$$\mathrm{FDTC}_{r,C}(S) := \{\phi\in\mathrm{Mod}(S)\mid c(\phi,C)\geq r\}\cup\{\mathrm{id}_S\},$$
$$\mathrm{FDTC}_{r}(S) := \{\phi\in\mathrm{Mod}(S)\mid c(\phi,C)\geq r \text{ for all } C\subset\partial S\}\cup\{\mathrm{id}_S\}.$$

Etnyre and Van Horn-Morris ask [2, Question 6.12]: *For which $r\in\mathbb{R}$ does the set $\mathrm{FDTC}_r(S)$ form a monoid?* The following theorem answers this question:

**Theorem 1.1** *Let $S$ be a surface that is not a pair of pants and has negative Euler characteristic. Let $C$ be a boundary component of $S$. The set $\mathrm{FDTC}_{r,C}(S)$ — and hence $\mathrm{FDTC}_r(S)$ — is a monoid if and only if $r>0$.*

**Remark 1.2** In [2, page 344] it is shown that $\mathrm{FDTC}_r(S)$ is a monoid for $r>1$.

**Remark 1.3** If $S$ is a pair of pants then $\mathrm{FDTC}_{r,C}(S)$ is a monoid if and only if $r\geq 0$.

Theorem 1.1 states that $\mathrm{FDTC}_0(S)$ is not a monoid. But $\mathrm{FDTC}_0(S)$ contains the monoid $\mathrm{Veer}^+(S)$ of *right-veering mapping classes* (see [4] for the definition of right-veering mapping classes).

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**Corollary 1.4** *We have*

$$\bigcup_{r>0} \mathrm{FDTC}_r(S) \subsetneq \mathrm{Veer}^+(S) \subsetneq \mathrm{FDTC}_0(S).$$

Corollary 1.4 shows that the statement $\mathrm{Veer}^+(S) = \mathrm{FDTC}_0(S)$ in [2, page 345] does not hold.

As discussed in [2], given a surface $S$, the set of mapping classes in $\mathrm{Mod}(S)$ compatible with the contact 3–manifolds with a certain property, such as tight and fillable, often forms a monoid. Conversely, a contact 3–manifold has a certain property when the monodromy lies in a submonoid of $\mathrm{Mod}(S)$ which is not directly related to 3–dimensional topology such as $\mathrm{Veer}^+(S)$.

The monoid $\mathrm{Veer}^+(S)$ contains the tight monoid $\mathrm{Tight}(S)$, as shown in [4]. Corollary 1.4 shows a submonoid structure of $\mathrm{Veer}^+(S)$. It is announced in Wand [8] that $\bigcup_{r>1} \mathrm{FDTC}_r(S) \subset \mathrm{Tight}(S)$; see also [6] for the planar surface case. In [5] we show that $\mathrm{FDTC}_1(S) \not\subset \mathrm{Tight}(S)$. Classification and detection of tight contact structures are central problems in contact topology, and the monoids $\mathrm{FDTC}_r(S)$ are expected to play important roles.

## 2 Basic study of quasimorphisms

As shown in [7, Corollary 4.17], the FDTC map $c(-, C)\colon \mathrm{Mod}(S) \to \mathbb{Q}$ is not a homomorphism but a homogeneous quasimorphism if the surface $S$ has negative Euler characteristic. In order to prove Theorem 1.1 we first study general homogeneous quasimorphisms and obtain a monoid criterion (Theorem 2.2).

Let $G$ be a group. A map $q\colon G \to \mathbb{R}$ is called a *homogeneous quasimorphism* if

$$D(q) := \sup_{g,h\in G} |q(gh) - q(g) - q(h)| < \infty,$$

$$q(g^n) = nq(g) \quad \text{for all } g \in G \text{ and } n \in \mathbb{Z}.$$

The value $D(q)$ is called the *defect* of $q$. A typical example of homogeneous quasimorphism is the *translation number* $\tau\colon \widetilde{\mathrm{Homeo}}^+(S^1) \to \mathbb{R}$ defined by

$$\tau(g) = \lim_{n\to\infty} \frac{g^n(0)}{n} = \lim_{n\to\infty} \frac{g^n(x) - x}{n}.$$

Here $\widetilde{\mathrm{Homeo}}^+(S^1)$ is the group of orientation-preserving homeomorphisms of $\mathbb{R}$ that are lifts of orientation-preserving homeomorphisms of $S^1$. The limit $\tau(g)$ does not depend on the choice of $x \in \mathbb{R}$. The following is an important property of $\tau$ we will use:

($*$) If $0 < \tau(g)$ then $x < g(x)$ for all $x \in \mathbb{R}$.

Given a quasimorphism $q\colon G \to \mathbb{R}$ and $r \in \mathbb{R}$ let

$$G_r = G_r^q := \{g \in G \mid g = \mathrm{id}_G \text{ or } q(g) \geq r\}.$$

It is easy to see that:

**Proposition 2.1** *The set $G_r$ forms a monoid if $r \geq D(q)$.*

Remark 1.2 is an immediate consequence of Proposition 2.1.

The following theorem gives another a monoid criterion for $G_r$:

**Theorem 2.2** *Let $q\colon G \to \mathbb{R}$ be a homogeneous quasimorphism which is a pull-back of the translation number quasimorphism $\tau$; namely, there is a homomorphism $f\colon G \to \widetilde{\mathrm{Homeo}}^+(S^1)$ such that $q = \tau \circ f$. Then $\max\{q(g), q(h)\} \leq q(gh)$ if $q(g), q(h) > 0$. Consequently, for $r, s > 0$ and $t = \max\{r, s, r+s-D(q)\}$ we have*

$$G_r \cdot G_s := \{gh \mid g \in G_r,\ h \in G_s\} \subset G_t.$$

*In particular, $G_r$ forms a monoid for $r > 0$.*

**Proof** Assume to the contrary that there exist $g, h \in G$ such that $0 < q(h), q(g)$ but $q(gh) < \max\{q(g), q(h)\}$. We treat the case $q(h) \leq q(g)$. A similar argument applies for the case $q(g) < q(h)$.

Since $q(gh) < q(g)$ there exists an integer $n > 0$ such that

$$q(g^n) - q((gh)^n) = n(q(g) - q(gh)) > D(q). \tag{1}$$

By the definition of the defect we have

$$|q(g^{-n}(gh)^n) + q(g^n) - q((gh)^n)| \leq D(q). \tag{2}$$

By (1) and (2) we get

$$q(g^{-n}(gh)^n) \leq -q(g^n) + q((gh)^n) + D(q) < -D(q) + D(q) = 0.$$

Letting $\mathtt{G} = f(g)$ and $\mathtt{H} = f(h)$, by the property $(*)$ we have $(\mathtt{G}^{-n}(\mathtt{GH})^n)(0) < 0$.

On the other hand, since $0 < q(h) = \tau(\mathtt{H})$ by the property $(*)$ we have $\mathtt{H}(x) > x$ for all $x \in \mathbb{R}$. Thus, $\mathtt{G}(\mathtt{H}(x)) > \mathtt{G}(x)$. By induction on $n$, we have $(\mathtt{GH})^n(x) > \mathtt{G}^n(x)$. Setting $x = 0$ we get $(\mathtt{G}^{-n}(\mathtt{GH})^n)(0) > (\mathtt{G}^{-n}\mathtt{G}^n)(0) = 0$, which is a contradiction. □

# 3 Proof of Theorem 1.1

**Proof of Theorem 1.1** According to [7, Theorem 4.16], if $\chi(S) < 0$ then the FDTC has $c(\phi, C) = (\tau \circ \Theta_C)(\phi)$ for some homomorphism $\Theta_C\colon \mathrm{Mod}(S) \to \widetilde{\mathrm{Homeo}}^+(S^1)$.

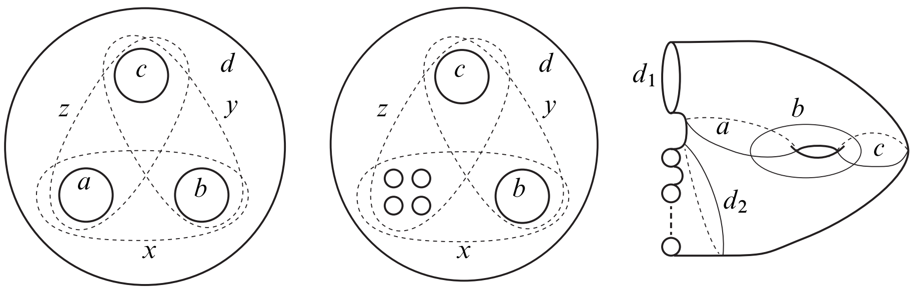


Figure 1

This fact along with Theorem 2.2 shows that $\mathrm{FDTC}_{r,C}(S)$ is a monoid if $\chi(S) < 0$ and $r > 0$.

Since $\mathrm{FDTC}_r(S)$ is the intersection of $\mathrm{FDTC}_{r,C}(S)$ for all the boundary components of $S$ the set $\mathrm{FDTC}_r(S)$ is also a monoid if $\chi(S) < 0$ and $r > 0$.

Next we show that $\mathrm{FDTC}_{r,C}(S)$ is not a monoid for $r \leq 0$. For any nonseparating simple closed curve $\gamma$ and any boundary component $C'$ of $S$ we have $c(T_\gamma^{\pm 1}, C') = 0$. Therefore, for every boundary component $C$ we have

$$T_\gamma^{\pm 1} \in \mathrm{FDTC}_{0,C}(S) \subset \mathrm{FDTC}_{r,C}(S). \tag{3}$$

**Case 1** Recall that for any surface $S$ of genus $g \geq 2$ the group $\mathrm{Mod}(S)$ is generated by Dehn twists about nonseparating simple closed curves (see [3, page 114]). If $\mathrm{FDTC}_{r,C}(S)$ were a monoid then this fact and (3) would imply that $\mathrm{FDTC}_{0,C}(S) = \mathrm{FDTC}_{r,C}(S) = \mathrm{Mod}(S)$, which is clearly absurd. Thus $\mathrm{FDTC}_{r,C}(S)$ is not a monoid if $g \geq 2$ and $r \leq 0$.

**Case 2** If $g = 0$ and $|\partial S| = 4$, let $a$, $b$, $c$, $d$ be the boundary components and $x$, $y$, $z$ be the simple closed curves as shown in Figure 1 (left). Let $r \leq 0$ and $C \in \{a, b, c, d\}$. Since $x$, $y$, $z$ are nonseparating,

$$T_x^{\pm 1},\ T_y^{\pm 1},\ T_z^{\pm 1} \in \mathrm{FDTC}_{0,C}(S) \subset \mathrm{FDTC}_{r,C}(S).$$

By the *lantern relation*, for any positive integer $n$ with $-n < r$ we have

$$c((T_x T_y T_z)^{-n}, C) = c(T_a^{-n} T_b^{-n} T_c^{-n} T_d^{-n}, C) = -n;$$

thus, $(T_x T_y T_z)^{-n} \notin \mathrm{FDTC}_{r,C}(S)$. This shows that $\mathrm{FDTC}_{r,C}(S)$ is not a monoid for all $r \leq 0$ and $C \in \{a, b, c, d\}$.

**Case 3** If $g = 0$ and $n = |\partial S| > 4$, add $n - 3$ additional boundary components $a_1, \dots, a_{n-3}$ in the place of $a$, as shown in Figure 1 (center). By a similar argument using the lantern relation, we can show that $\mathrm{FDTC}_{r,C}(S)$ is not a monoid for all $r \leq 0$

and any $C = b, c, d$. By the symmetry of the surface we can further show that $\mathrm{FDTC}_{r,C}(S)$ is not a monoid for all $r \le 0$ and $C = a_1, \dots, a_{n-3}$.

**Case 4** If $g = 1$ and $|\partial S| = 1$, the group $\mathrm{Mod}(S)$ is generated by Dehn twists about nonseparating simple closed curves. Thus this case is subsumed into Case 1.

**Case 5** If $g = 1$ and $|\partial S| \ge 2$, applying the 3–chain relation [3, Proposition 4.12] to the simple closed curves in Figure 1 (right) we get

$$c((T_a T_b T_c)^{-4n}, d_1) = c((T_{d_1})^{-n}(T_{d_2})^{-n}, d_1) = -n.$$

By the same argument as in Case 2 we can show that $\mathrm{FDTC}_{r,d_1}(S)$ is not a monoid for all $r \le 0$.

Parallel arguments show that $\mathrm{FDTC}_r(S)$ does not form a monoid for $r \le 0$. □

**Proof of Corollary 1.4** Let $\gamma \subset S$ be a nonseparating simple closed curve. By (3) we observe that

$$T_\gamma \in \mathrm{Veer}^+(S) \setminus \Big( \bigcup_{r>0} \mathrm{FDTC}_r(S) \Big) \quad \text{and} \quad T_\gamma^{-1} \in \mathrm{FDTC}_0(S) \setminus \mathrm{Veer}^+(S). \qquad \square$$

**Corollary 3.1** *If $\chi(S) < 0$ then for $r, s > 0$ and $x = \max\{r, s, r+s-1\}$ we have:*

(1) $\mathrm{FDTC}_r(S) \cdot \mathrm{FDTC}_s(S) \subset \mathrm{FDTC}_x(S)$.

(2) $\mathrm{FDTC}_r(S) \cdot \mathrm{Tight}(S) \subset \mathrm{FDTC}_r(S) \cdot \mathrm{Veer}^+(S) \subset \mathrm{FDTC}_r(S)$.

**Proof** (1) follows from Theorem 2.2 and the fact that the defect of the FDTC is 1.

The first inclusion of (2) follows from $\mathrm{Tight}(S) \subset \mathrm{Veer}^+(S)$ [4]. To see the second inclusion of (2), we note that a right-veering $\phi \in \mathrm{Mod}(S)$ has the property $(*')$, which is similar to $(*)$, where $<$ is replaced with $\le$ [4; 7]:

$(*')$ With $\Phi := \Theta_C(\phi) \in \widetilde{\mathrm{Homeo}}^+(S^1)$, if $\phi \in \mathrm{Veer}^+(S)$ then $x \le \Phi(x)$ for all $x \in \mathbb{R}$.

The same argument as in the proof of Theorem 2.2 gives the second inclusion. □

**Remark 3.2** Although $\mathrm{Veer}^+(S) \subset \mathrm{FDTC}_0(S)$, it is not true that

$$\mathrm{FDTC}_r(S) \cdot \mathrm{FDTC}_0(S) \subset \mathrm{FDTC}_r(S).$$

Let $A$ and $B$ be simple closed curves on a torus $S$ with one hole which form a basis of $H_1(S)$. We have $c(T_A^{\pm 1}, \partial S) = c(T_B^{\pm 1}, \partial S) = 0$ and $c(T_A T_B, \partial S) = \frac{1}{6}$. On the other hand, $c((T_A T_B) \cdot T_B^{-1}, \partial S) = 0 \not> \frac{1}{6}$.

We do not know, at the time of this writing, the contact and symplectic properties that are related to the monoid $\mathrm{FDTC}_r(S)$ for $0 < r \le 1$. Moreover, in general, given a quasimorphism $q\colon \mathrm{Mod}(S) \to \mathbb{R}$ and $r \in \mathbb{R}$, as the mapping class group admits a huge number of quasimorphisms [1], it would be interesting to know when the subset $\mathrm{Mod}(S)^q_r$ forms a monoid and how $\mathrm{Mod}(S)^q_r$ is related to the topology and geometry of the corresponding (contact) 3–manifolds.

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