Infinite loop spaces and nilpotent K–theory

ALEJANDRO ADEM
JOSÉ MANUEL GÓMEZ
JOHN A LIND
ULRIKE TILLMANN

Using a construction derived from the descending central series of the free groups, we produce filtrations by infinite loop spaces of the classical infinite loop spaces $B\mathrm{SU}$, $B\mathrm{U}$, $B\mathrm{SO}$, $B\mathrm{O}$, $B\mathrm{Sp}$, $B\mathrm{GL}_\infty(R)^+$ and $Q_0(\mathbb{S}^0)$. We show that these infinite loop spaces are the zero spaces of nonunital E_∞–ring spectra. We introduce the notion of q–nilpotent K–theory of a CW–complex X for any $q \geq 2$, which extends the notion of commutative K–theory defined by Adem and Gómez, and show that it is represented by $\mathbb{Z} \times B(q, U)$, where $B(q, U)$ is the qth term of the aforementioned filtration of $B\mathrm{U}$.

For the proof we introduce an alternative way of associating an infinite loop space to a commutative $\mathbb{I}$–monoid and give criteria for when it can be identified with the plus construction on the associated limit space. Furthermore, we introduce the notion of a commutative $\mathbb{I}$–rig and show that they give rise to nonunital E_∞–ring spectra.

1 Introduction

Let G denote a locally compact, Hausdorff topological group such that $1_G \in G$ is a nondegenerate base point. It is well known that we can obtain a model for the classifying space BG as the geometric realization of the classical bar construction B_*G. Now fix an integer $q \geq 2$ and let Γ_n^q be the qth stage of the descending central series of the free group on n letters F_n, with the convention $\Gamma_n^1 = F_n$. Consider the set of homomorphisms $B_n(q, G) := \operatorname{Hom}(F_n / \Gamma_n^q, G)$. If $e_1, \ldots, e_n$ are generators of F_n, then evaluation on the classes corresponding to $e_1, \ldots, e_n$ provides a natural inclusion $B_n(q, G) \subset G^n$. Using this inclusion we can give $B_n(q, G)$ the subspace topology. Therefore $B_n(q, G)$ is precisely the space of ordered n–tuples in G generating a subgroup of G with nilpotence class less than q. For each fixed $q \geq 2$ the collection $\{B_n(q, G)\}_{n \geq 0}$ forms a simplicial space with face and degeneracy maps induced by those in the bar construction. The geometric realization of this simplicial space is

denoted by $B(q, G)$. These spaces were first introduced by Adem, Cohen and Torres Giese [1], where many of their basic properties were established. They give rise to a natural filtration of the classifying space

$$B(2, G) \subset B(3, G) \subset \cdots \subset B(q, G) \subset B(q + 1, G) \subset \cdots \subset BG.$$

For $q = 2$ we obtain $B_{\mathrm{com}} G := B(2, G)$, which is constructed by assembling the different spaces of ordered commuting n–tuples in the group G. Adem and Gómez [2] showed that for Lie groups this space plays the role of a classifying space for commutativity. More generally $B(q, G)$ is a classifying space for G–bundles of transitional nilpotency class less than q.

For the infinite unitary group $U = \mathrm{colim}_{n \to \infty} U(n)$, it is well known that BU is the infinite loop space underlying a nonunital E_∞–ring spectrum, namely the homotopy fiber of the Postnikov section $ku \to H\mathbb{Z}$. In other words, BU is a so-called nonunital E_∞–ring space. A basic question is whether the above gives rise to a filtration of BU by nonunital E_∞–ring spaces. The main purpose of this paper is to show that indeed this is the case, not only for U but also for other linear groups.

Theorem 1.1 *The spaces $B(q, \mathrm{SU})$, $B(q, U)$, $B(q, \mathrm{SO})$, $B(q, O)$ and $B(q, \mathrm{Sp})$ provide a filtration by nonunital E_∞–ring spaces of the classical infinite loop spaces $B\mathrm{SU}$, BU, $B\mathrm{SO}$, BO and $B\mathrm{Sp}$, respectively.*

The q–nilpotent K–theory of a space X is defined using isomorphism classes of bundles on X whose transition functions generate subgroups of nilpotence class less than q. We show that $K_{q\text{–nil}}(X) \cong [X, \mathbb{Z} \times B(q, U)]$, from which we obtain:

Corollary 1.2 *$K_{q\text{–nil}}(-)$ is the zeroth term of a generalized multiplicative cohomology theory.*

In particular we obtain a sequence of multiplicative cohomology theories

$$K_{\mathrm{com}}(X) = K_{2\text{–nil}}(X) \to K_{3\text{–nil}}(X) \to \cdots \to K_{q\text{–nil}}(X) \to \cdots \to K(X).$$

We also show that $B(q, U) \to BU$ splits as a map of infinite loop spaces, whence we see that topological K–theory is a direct summand in $K_{q\text{–nil}}$.

The infinite loop space structure on $B(q, G)$ for $G = U$, SU, SO, O, Sp is obtained by using the machinery of commutative $\mathbb{I}$–monoids first introduced by Bökstedt and developed by Schlichtkrull [19], Sagave and Schlichtkrull [18] and Lind [9]. Here $\mathbb{I}$ is the category of finite sets and injections. In addition to the usual construction, we associate an infinite loop space to a commutative $\mathbb{I}$–monoid by restricting the usual

homotopy colimit construction to the subcategory $\mathbb{P}$ of finite sets and isomorphisms. This allows us to identify the homotopy type of the homotopy colimit under certain conditions. Another addition to infinite loop space theory is the introduction of the notion of a commutative $\mathbb{I}$–rig, which we show to give rise to a bipermutative category and hence an E_∞–ring spectrum.

Our main examples above all arise from commutative $\mathbb{I}$–rigs where we can identify the infinite loop space as the plus construction of the associated limit space. A more complicated situation arises for $Q_0(\mathbb{S}^0) \simeq B\Sigma_\infty^+$ and $B\mathrm{GL}_\infty(R)^+$. Our methods give rise to natural sequences of E_∞–ring spaces but the terms are not easy to describe.

The outline of this article is as follows. In Section 2 we use the machinery of commutative $\mathbb{I}$–monoids to produce two associated infinite loop spaces, one of which is a nonunital E_∞–ring space when the $\mathbb{I}$–monoid is an $\mathbb{I}$–rig. In Section 3 we show that these are homotopy equivalent and identify them under suitable assumptions. Then in Section 4 we apply these results to prove Theorem 1.1 and show that the spaces $B(q, U)$ for $q \geq 2$ are infinite loop spaces and that BU splits off. Finally, in Section 5 we introduce the notion of q–nilpotent K–theory and show that it is represented by the infinite loop spaces $\mathbb{Z} \times B(q, U)$, answering the question raised for commutative K–theory in [2].

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2 Commutative $\mathbb{I}$–monoids and infinite loop spaces

The standard construction of the infinite loop space structure on BU from the permutative category of complex vector spaces and their isomorphisms does not restrict to give an infinite loop space structure on $B(q, U)$. Instead we are going to use certain constructions on commutative $\mathbb{I}$–monoids. More precisely, we will give two constructions of permutative categories from commutative $\mathbb{I}$–monoids. For the case of interest the permutative categories are actually bipermutative and hence give rise to E_∞–ring spectra. We start by setting up some notations and basic definitions following [19; 18; 9]. We will use [5] as a reference for bipermutative categories and the associated multiplicative infinite loop space machinery.

2.1   The category $\mathbb{I}$ and its subcategories $\mathbb{P}$ and $\mathbb{N}$

These three categories are skeletons of the category of finite sets and injections, the category of finite sets and isomorphisms, and the translation category associated to the monoid of natural numbers. We will use the following notation.

For every integer $n \geq 0$, let $\boldsymbol{n}$ denote the set $\{1, 2, \dots, n\}$. When $n = 0$ we use the convention $\boldsymbol{0} := \varnothing$. Let $\mathbb{I}$ denote the category whose objects are the elements of the form $\boldsymbol{n}$ for all integers $n \geq 0$ with morphisms given by all injective maps. Note that in particular $\boldsymbol{0}$ is an initial object in the category $\mathbb{I}$ and $\mathbb{I}$ is a symmetric monoidal category under the concatenation $\boldsymbol{m} \sqcup \boldsymbol{n} := \{1, 2, \dots, m + n\}$ with the symmetry morphism given by the (m, n)–shuffle map

$$\tau_{\boldsymbol{m}, \boldsymbol{n}} \colon \boldsymbol{m} \sqcup \boldsymbol{n} \to \boldsymbol{n} \sqcup \boldsymbol{m}.$$

It is also symmetric monoidal under the Cartesian product

$$\boldsymbol{m} \times \boldsymbol{n} := \{1 = (1, 1), 2 = (1, 2), \dots, n + 1 = (2, 1), \dots, mn = (m, n)\}$$

given by lexicographic ordering. By definition, $\boldsymbol{0} \times \boldsymbol{n} = \boldsymbol{0} = \boldsymbol{n} \times \boldsymbol{0}$. The associated symmetry morphism is given by a permutation

$$\tau_{\boldsymbol{m}\boldsymbol{n}}^{\times} \colon \boldsymbol{m} \times \boldsymbol{n} \to \boldsymbol{n} \times \boldsymbol{m}.$$

The latter monoidal product is distributive over the former. More precisely, left distributivity

$$\delta_{\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k}}^{l} \colon \boldsymbol{m} \times \boldsymbol{k} \sqcup \boldsymbol{n} \times \boldsymbol{k} \to (\boldsymbol{m} \sqcup \boldsymbol{n}) \times \boldsymbol{k}$$

is given by the identity and right distributivity is given by a permutation

$$\delta_{\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k}}^{r} \colon \boldsymbol{m} \times \boldsymbol{n} \sqcup \boldsymbol{m} \times \boldsymbol{k} \to \boldsymbol{m} \times (\boldsymbol{n} \sqcup \boldsymbol{k}).$$

These two structures make $\mathbb{I}$ into a bipermutative category, as in [5, Definition 3.6].

The category $\mathbb{I}$ has two natural subcategories. Let $\mathbb{P}$ be the totally disconnected subcategory containing all objects and all isomorphisms $\sigma \colon \boldsymbol{n} \to \boldsymbol{n}$ but no other morphisms, and let $\mathbb{N}$ denote the connected subcategory containing all objects, their identities and only the canonical inclusions $j \colon \boldsymbol{n} \to \boldsymbol{m}$. While $\mathbb{P}$ is a bipermutative subcategory, $\mathbb{N}$ does not inherit any monoidal structure from $\mathbb{I}$.

2.2   Definitions of commutative $\mathbb{I}$–monoids and $\mathbb{I}$–rigs

An $\mathbb{I}$–space is a functor $X \colon \mathbb{I} \to \mathrm{Top}$. Every morphism in $\mathbb{I}$ can be factored as a composition of a canonical inclusion $j \colon \boldsymbol{n} \hookrightarrow \boldsymbol{m}$ and a permutation $\sigma \colon \boldsymbol{m} \to \boldsymbol{m}$. Therefore an $\mathbb{I}$–space $X \colon \mathbb{I} \to \mathrm{Top}$ determines a sequence of spaces $X(\boldsymbol{n})$ together

with an induced action of the symmetric group Σ_n for $n \geq 0$, and structural maps $j_n \colon X(\mathbf{n}) \to X(\mathbf{n+1})$ that are equivariant in the sense that $j_n(\sigma \cdot x) = \sigma \cdot j_n(x)$ for every $\sigma \in \Sigma_n$ and $x \in X(\mathbf{n})$. On the right-hand side we see σ as element in Σ_{n+1} via the canonical inclusion $\Sigma_n \hookrightarrow \Sigma_{n+1}$. Vice versa, given such a sequence of Σ_n–spaces $X(\mathbf{n})$ and compatible structure maps j_n, they give rise to an $\mathbb{I}$–space if and only if for $m \geq n$ and any two elements $\sigma, \sigma' \in \Sigma_m$ which restrict to the same permutation of $\mathbf{n}$ we have $\sigma(x) = \sigma'(x)$ for all $x \in j(X(\mathbf{n}))$. We note that this condition is not satisfied by the sequence $X(\mathbf{n}) = \Sigma_n$ with the left or right multiplication action, but *is* satisfied by the sequence $X(\mathbf{n}) = \mathbf{n}$ with the natural permutation action since $\mathbf{n} \cong \mathbb{I}(\mathbf{1}, \mathbf{n})$.

We say that an $\mathbb{I}$–space is an $\mathbb{I}$–monoid if it comes equipped with a natural transformation

$$\mu_{\mathbf{m},\mathbf{n}} \colon X(\mathbf{m}) \times X(\mathbf{n}) \to X(\mathbf{m} \sqcup \mathbf{n})$$

of functors defined on $\mathbb{I} \times \mathbb{I}$ and a natural transformation

$$\eta_{\mathbf{n}} \colon * \to X(\mathbf{n})$$

from the constant $\mathbb{I}$–space $*(\mathbf{n}) = *$ to X satisfying associativity and unit axioms for $* \in X(\mathbf{0})$. We say that X is a commutative $\mathbb{I}$–monoid if μ is commutative, meaning that the diagram

$$
\begin{array}{ccc}
X(\mathbf{m}) \times X(\mathbf{n}) & \xrightarrow{\mu_{\mathbf{m},\mathbf{n}}} & X(\mathbf{m} \sqcup \mathbf{n}) \\
\tau \downarrow & & \downarrow \tau_{\mathbf{m},\mathbf{n}} \\
X(\mathbf{n}) \times X(\mathbf{m}) & \xrightarrow{\mu_{\mathbf{n},\mathbf{m}}} & X(\mathbf{n} \sqcup \mathbf{m})
\end{array}
$$

commutes, where $\tau(x, y) = (y, x)$.

An $\mathbb{I}$–rig is a commutative $\mathbb{I}$–monoid equipped with a natural transformation

$$\pi_{\mathbf{m},\mathbf{n}} \colon X(\mathbf{m}) \times X(\mathbf{n}) \to X(\mathbf{m} \times \mathbf{n})$$

of functors defined on $\mathbb{P} \times \mathbb{P}$ and an element $1 \in X(\mathbf{1})$ satisfying associativity and unit axioms, as well as left distributivity, ie that the diagram

$$
\begin{array}{ccc}
(X(\mathbf{m}) \times X(\mathbf{n})) \times X(\mathbf{k}) & \xrightarrow{\pi_{\mathbf{m} \sqcup \mathbf{n},\mathbf{k}} \circ (\mu_{\mathbf{m},\mathbf{n}} \times 1)} & X((\mathbf{m} \sqcup \mathbf{n}) \times \mathbf{k}) \\
{\scriptstyle (1 \times \tau \times 1) \circ (1 \times 1 \times \Delta)} \downarrow & & \uparrow {\scriptstyle \delta^l_{\mathbf{m},\mathbf{n},\mathbf{k}}} \\
X(\mathbf{m}) \times X(\mathbf{k}) \times X(\mathbf{n}) \times X(\mathbf{k}) & \xrightarrow{\mu_{\mathbf{m} \times \mathbf{k}, \mathbf{n} \times \mathbf{k}} \circ (\pi_{\mathbf{m},\mathbf{k}} \times \pi_{\mathbf{n},\mathbf{k}})} & X(\mathbf{m} \times \mathbf{k} \sqcup \mathbf{n} \times \mathbf{k})
\end{array}
$$

commutes, and right distributivity, which is given by an analogous commutative diagram. Here $\triangle$ is the diagonal map. We emphasize that π is only required to be natural on the subcategory $\mathbb{P} \times \mathbb{P}$ of $\mathbb{I} \times \mathbb{I}$.[1]

A commutative $\mathbb{I}$–rig is an $\mathbb{I}$–rig in which π is commutative in the sense that the diagram

$$
\begin{array}{ccc}
X(\boldsymbol{m}) \times X(\boldsymbol{n}) & \xrightarrow{\pi_{\boldsymbol{m},\boldsymbol{n}}} & X(\boldsymbol{m} \times \boldsymbol{n}) \\
{\scriptstyle \tau} \downarrow & & \downarrow {\scriptstyle \tau^{\times}_{\boldsymbol{m},\boldsymbol{n}}} \\
X(\boldsymbol{n}) \times X(\boldsymbol{m}) & \xrightarrow{\pi_{\boldsymbol{n},\boldsymbol{m}}} & X(\boldsymbol{n} \times \boldsymbol{m})
\end{array}
$$

commutes. A natural transformation T between two $\mathbb{I}$–spaces X and Y defines a map of commutative $\mathbb{I}$–monoids ($\mathbb{I}$–rigs) if it commutes with μ (and π) in the sense that $T \circ \mu_{\boldsymbol{m},\boldsymbol{n}} = \mu_{\boldsymbol{m},\boldsymbol{n}} \circ T \times T$ (and $T \circ \pi_{\boldsymbol{m},\boldsymbol{n}} = \pi_{\boldsymbol{m},\boldsymbol{n}} \circ T \times T$). We have thus defined a category of $\mathbb{I}$–spaces, a category of commutative $\mathbb{I}$–monoids and a category of $\mathbb{I}$–rigs.

2.3 Associated (bi)permutative translation categories

We will use the following notation for translation categories. If $Y \colon \mathcal{C} \to \mathrm{Top}$ is a functor from a category $\mathcal{C}$ to the category of topological spaces, we let $\mathcal{C} \ltimes Y$ denote the translation category on Y. The translation category, also known as the Grothendieck construction, is a topological category whose objects are pairs (c, x) consisting of an object c of $\mathcal{C}$ and a point $x \in Y(c)$. A morphism in $\mathcal{C} \ltimes Y$ from (c, x) to (c', x') is a morphism $\alpha \colon c \to c'$ in $\mathcal{C}$ satisfying the equation $Y(\alpha)(x) = x'$. For example, if $\mathcal{C} = G$ is a group, thought of as a one object category, then the translation category $G \ltimes Y$ is the action groupoid for the G–space Y and its classifying space is the homotopy orbit space $B(G \ltimes Y) = EG \times_G Y$. In general, the classifying space $B(\mathcal{C} \ltimes Y)$ is homeomorphic to the homotopy colimit $\mathrm{hocolim}_{\mathcal{C}} Y$ of Y over $\mathcal{C}$ defined using the bar construction.

Suppose now that X is a commutative $\mathbb{I}$–monoid. Then the translation category $\mathbb{I} \ltimes X$ is a permutative category, as we now explain. The monoidal structure $\oplus$ is defined on objects $(\boldsymbol{m}, x)$ and $(\boldsymbol{n}, y)$ by

$$
(\boldsymbol{m}, x) \oplus (\boldsymbol{n}, y) = (\boldsymbol{m} \sqcup \boldsymbol{n}, \mu_{\boldsymbol{m},\boldsymbol{n}}(x, y))
$$

---

[1] In fact, we do not know of any nontrivial examples where π may be extended to a natural transformation of functors defined on $\mathbb{I} \times \mathbb{I}$. The examples of $\mathbb{I}$–rigs that we discuss in Section 2.5 do not satisfy this additional naturality condition. Indeed, as we will see in the following sections, an $\mathbb{I}$–rig that does satisfy this condition and has each level $X(\boldsymbol{n})$ a connected space would give rise to a connected E_∞–ring space $\mathrm{hocolim}_{\mathbb{I}} X$. An E_∞–ring space whose multiplicative unit and additive unit lie in the same path component is contractible, so such examples would only give rise to trivial E_∞–ring spectra.

and on morphisms $\alpha\colon (\boldsymbol{m}, x) \to (\boldsymbol{m}', x')$ and $\beta\colon (\boldsymbol{n}, y) \to (\boldsymbol{n}', y')$ by letting

$$\alpha \oplus \beta\colon (\boldsymbol{m}, x) \oplus (\boldsymbol{n}, y) \to (\boldsymbol{m}', x') \oplus (\boldsymbol{n}', y')$$

be determined by the morphism

$$\alpha \sqcup \beta\colon \boldsymbol{m} \sqcup \boldsymbol{n} \to \boldsymbol{m}' \sqcup \boldsymbol{n}'$$

in the category $\mathbb{I}$. Notice that $X(\alpha \sqcup \beta)(\mu_{\boldsymbol{m},\boldsymbol{n}}(x, y)) = \mu_{\boldsymbol{m}',\boldsymbol{n}'}(x', y')$ by the naturality of μ, so that this is well-defined. The associativity and unit conditions for X imply that $\mathbb{I} \ltimes X$ is a strict monoidal category with strict unit object $(\boldsymbol{0}, *)$ determined by the unit η of the $\mathbb{I}$–monoid X. The commutativity of X implies that $\mathbb{I} \ltimes X$ is a permutative category, see for example [5, Definition 3.1]. Note that the permutative structure on $\mathbb{I} \ltimes X$ restricts to the subcategory $\mathbb{P} \ltimes X$.

Suppose now that X is a commutative $\mathbb{I}$–rig. Then by the same reasoning as above, there is another permutative category structure on $\mathbb{P} \ltimes X$ with product $\otimes$ induced by π and strict unit object $(\boldsymbol{1}, 1)$. The distributivity axioms for X translate to distributivity axioms for bipermutative categories [5, Definition 3.6].

Furthermore, a natural transformation T between two $\mathbb{I}$–spaces X and Y induces a functor $\mathbb{I} \ltimes X \to \mathbb{I} \ltimes Y$. If X and Y are commutative $\mathbb{I}$–monoids ($\mathbb{I}$–rigs) and T is a morphism of such then the induced functor of translation categories is a functor of (bi)permutative categories.

We have thus proved the following result:

Proposition 2.1 *The assignment $X \mapsto \mathbb{I} \ltimes X$ defines a functor from the category of commutative $\mathbb{I}$–monoids to the category of permutative categories, and the assignment $X \mapsto \mathbb{P} \ltimes X$ defines a functor from the category of commutative $\mathbb{I}$–monoids ($\mathbb{I}$–rigs) to the category of (bi)permutative categories.*

2.4 Construction of two infinite loop spaces

Let X be a commutative $\mathbb{I}$–monoid. As explained in [12], the classifying space of a permutative category is an E_∞–space structured by an action of the Barratt–Eccles operad. We have proved the next theorem.

Theorem 2.2 *Suppose that $X\colon \mathbb{I} \to \mathrm{Top}$ is a commutative $\mathbb{I}$–monoid. Then the homotopy colimit*

$$\mathrm{hocolim}_{\mathbb{I}} X = B(\mathbb{I} \ltimes X)$$

is an E_∞–space.

Without further assumptions on X, this E_∞–space need not be grouplike (ie the monoid $\pi_0(\mathrm{hocolim}_{\mathbb{I}} X)$ need not be a group). However, we can always form the group completion $\Omega B(\mathrm{hocolim}_{\mathbb{I}} X)$ to get the associated infinite loop space. Note that an algebra over the Barratt–Eccles operad has an underlying monoid structure that is always strictly associative (and homotopy commutative) so that the usual functorial construction of the classifying space for monoids built using the bar construction can be applied. We will always use this model for B in defining the group completion functor $\Omega B(-)$. The consistency results in [12] guarantee that the group completion $\Omega B(\mathrm{hocolim}_{\mathbb{I}} X)$ defines an infinite loop space weakly equivalent to that obtained using any other delooping machine.

Schlichtkrull [19] defined a different infinite loop space associated to X, using the language of Γ–spaces. Schlichtkrull's construction is the same as May's construction [14] of a Γ–space applied to the permutative category $\mathbb{I} \ltimes X$. By the uniqueness result of [14], the infinite loop space $\Omega B(\mathrm{hocolim}_{\mathbb{I}} X)$ is equivalent to that defined by Schlichtkrull.

We now give a different construction of an infinite loop space associated to X. To start note the decomposition of categories

$$\mathbb{P} \ltimes X = \bigsqcup_{n \geq 0} \Sigma_n \ltimes X(\boldsymbol{n}),$$

where Σ_n is seen as a category with one object. Thus $\mathbb{P} \ltimes X$ is a topological category with classifying space

$$M := \mathrm{hocolim}_{\mathbb{P}} X = B(\mathbb{P} \ltimes X) \simeq \bigsqcup_{n \geq 0} E\Sigma_n \times_{\Sigma_n} X(\boldsymbol{n}).$$

As $\mathbb{P} \ltimes X$ is a permutative category, $M = B(\mathbb{P} \ltimes X)$ is an E_∞–space and thus its group completion, ΩBM, is an infinite loop space. The reduction maps $X(\boldsymbol{n}) \to *$ define a map of permutative categories $\mathbb{P} \ltimes X \to \mathbb{P} \ltimes *$ and hence a map of infinite loop spaces

$$\rho^X \colon \Omega B(\mathrm{hocolim}_{\mathbb{P}} X) \to \Omega B(\mathrm{hocolim}_{\mathbb{P}} *).$$

In particular, the homotopy fiber hofib ρ^X is naturally an infinite loop space.

When X is a commutative $\mathbb{I}$–rig, we process the associated bipermutative category $\mathbb{P} \ltimes X$ using the machinery of Elmendorf and Mandell. To a bipermutative category $\mathcal{C}$, they functorially associate a commutative symmetric ring spectrum [5, Corollary 3.9 and Theorem 9.3.8]. By [5, Theorem 4.6] and the original work of Segal [22], its underlying infinite loop space is weak homotopy equivalent to $\Omega BB\mathcal{C}$. By a theorem due to Schwede [21] and later refined by Mandell and May [10, Section 1], the

homotopy category of commutative symmetric ring spectra is equivalent to that of E_∞–ring spectra. We write $K\mathcal{C}$ for the E_∞–ring spectrum associated to $\mathcal{C}$ under this equivalence of homotopy categories. The underlying infinite loop space of an E_∞–ring spectrum is an E_∞–ring space, as defined in [13, Chapter VI], so we may functorially associate to each bipermutative category an E_∞–ring space $\Omega^\infty K\mathcal{C}$. Moreover, by [9, Theorem 1.2], the space $\Omega^\infty K\mathcal{C}$ is weak homotopy equivalent to the group completion $\Omega BB\mathcal{C}$.

We now apply this machinery to the morphism $\mathbb{P} \ltimes X \to \mathbb{P} \ltimes *$ of bipermutative categories. We obtain a map of E_∞–ring spectra

$$K(\mathbb{P} \ltimes X) \to K(\mathbb{P} \ltimes *)$$

which is equivalent to ρ^X after applying Ω^∞. The homotopy fiber of a map of E_∞–ring spectra is a nonunital E_∞–ring spectrum. By a nonunital E_∞–ring space, we mean the underlying infinite loop space of a nonunital E_∞–ring spectrum. Since Ω^∞ preserves homotopy fiber sequences, this means that the homotopy fiber of a map of E_∞–ring spaces is a nonunital E_∞–ring space. We have proved the next theorem.

Theorem 2.3 *For any commutative $\mathbb{I}$–monoid X the homotopy fiber hofib ρ^X of*

$$\rho^X \colon \Omega B(\operatorname{hocolim}_\mathbb{P} X) \to \Omega B(\operatorname{hocolim}_\mathbb{P} *).$$

is an infinite loop space. If furthermore X is a commutative $\mathbb{I}$–rig, then hofib ρ^X is a nonunital E_∞–ring space.

2.5 The main example

For any group G, conjugation by G or action by any other automorphism of G induces a well-defined action on $B_n(q, G) = \operatorname{Hom}(F_n/\Gamma_n^q, G)$ by postcomposition. The action is also compatible with the simplicial face and degeneracy maps in the bar construction and hence induces an action on $B(q, G)$.

For every $q \geq 2$ we define an $\mathbb{I}$–space $B(q, U(-))$ by setting $\boldsymbol{n} \mapsto B(q, U(n))$ with morphisms induced by the natural inclusions and the action of Σ_n on $B(q, U(n))$ given by conjugation through permutation matrices. Being induced by the natural action of Σ_n on $\boldsymbol{n}$, it can be checked that this compatible sequence defines indeed an $\mathbb{I}$–space.

We give $B(q, U(-))$ the structure of an $\mathbb{I}$–monoid by defining the unit map $\eta_{\boldsymbol{n}} \colon * \to B(q, U(n))$ to be the inclusion of the base-point and defining the monoid structure map

$$\mu_{\boldsymbol{n},\boldsymbol{m}} \colon B(q, U(n)) \times B(q, U(m)) \to B(q, U(n+m))$$

to be induced by the block sum of matrices. To see that $\mu_{n,m}$ is well-defined note that block sum defines a group homomorphism $U(n) \times U(m) \to U(n+m)$. When taking elements of the symmetric groups to permutation matrices, the disjoint union of sets corresponds to block sum of matrices. Thus μ defines a natural transformation of functors defined on $\mathbb{I} \times \mathbb{I}$. One checks compatibility with τ and hence $B(q, U(-))$ is a commutative $\mathbb{I}$–monoid.

Next we note that tensor product of matrices induces a well-defined map

$$\pi_{n,m} \colon B(q, U(n)) \times B(q, U(m)) \to B(q, U(nm)).$$

To see this note that tensor product commutes with matrix multiplication and hence induces a homomorphism $U(n) \times U(m) \to U(nm)$. The map is equivariant for the symmetric group actions because the permutation matrix associated to the product of two permutations is the same as the tensor product of the corresponding permutation matrices. Hence π is a natural transformation of functors defined on the category $\mathbb{P} \times \mathbb{P}$. Note, however, that π is not natural for proper injections. The map π is compatible with τ and the distributivity of block sum and tensor product of matrices induces distributivity maps for μ and π. We have shown:

Theorem 2.4 $B(q, U(-))$ *is a commutative* $\mathbb{I}$–*rig.*

As a consequence, we may apply Theorems 2.2 and 2.3 to get a pair of infinite loop spaces, the latter of which carries a nonunital E_∞–ring structure. In the next section, we will show that these two infinite loop spaces are equivalent.

3 Identifying and comparing the infinite loop spaces

Let X be a commutative $\mathbb{I}$–monoid. We will first identify hofib ρ^X under certain assumptions and then show it is homotopy equivalent as an infinite loop space to $\mathrm{hocolim}_\mathbb{I} X$.

Consider the space

$$X_\infty := \mathrm{hocolim}_{n \in \mathbb{N}} X(n).$$

Note that $X_\infty \simeq \mathrm{colim}_{n \in \mathbb{N}} X(n)$ if the structural maps $j_n \colon X(n) \to X(n+1)$ are cofibrations. In our applications this will always be the case. Let X_∞^+ denote Quillen's plus construction applied with respect to the maximal perfect subgroup of $\pi_1(X_\infty)$ (which we take to be understood to be done in each component separately, if X_∞ is not connected). Also recall that a space Z is abelian if $\pi_1(Z)$ is abelian and acts trivially on homotopy groups $\pi_*(Z)$. It is well known that H–spaces are abelian.

Theorem 3.1 *Let $X\colon \mathbb{I} \to \mathrm{Top}$ be a commutative $\mathbb{I}$–monoid. Assume that*

- *the action of Σ_∞ on $H_*(X_\infty)$ is trivial;*
- *the inclusions induce natural isomorphisms $\pi_0(X(\boldsymbol{n})) \simeq \pi_0(X_\infty)$ of finitely generated abelian groups with multiplication compatible with the Pontrjagin product and in the center of the homology Pontrjagin ring;*
- *the commutator subgroup of $\pi_1(X_\infty)$ is perfect (for each component) and X_∞^+ is abelian.*

Then hofib $\rho^X \simeq X_\infty^+$ and, in particular, X_∞^+ is an infinite loop space.

Proof Let $M = \mathrm{hocolim}_{\mathbb{P}} X = B(\mathbb{P} \ltimes X)$ and m be the point corresponding to the base point in $X(\boldsymbol{1})$ (in the identity component of $\pi_0(X(\boldsymbol{1}))$). Then

$$\mathrm{Tel}(M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} \cdots) \simeq \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty).$$

As $\mathbb{P} \ltimes X$ is a symmetric monoidal category, its classifying space M is a homotopy commutative topological monoid. The hypotheses imply that $\pi_0(M)$ is in the center of $H_*(M)$. Hence $H_*(M)[\pi_0(M)^{-1}]$ can be constructed by right fractions, so that we may apply the group completion theorem [15; 17]. Therefore there is a map

$$f\colon \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty) \to \Omega BM$$

which induces an isomorphism on homology with all systems of local coefficients on ΩBM. Furthermore, the fundamental group (of each component) of $E\Sigma_\infty \times_{\Sigma_\infty} X_\infty$ has a perfect commutator subgroup by [17], and f extends to a homology equivalence between abelian spaces

$$f^+\colon \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+ \to \Omega BM,$$

which is thus a homotopy equivalence. This shows, in particular, that the space $\mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+$ is an infinite loop space as ΩBM is the group completion of an E_∞–space.

Consider now the fibration sequence

(1) $$X_\infty \to E\Sigma_\infty \times_{\Sigma_\infty} X_\infty \xrightarrow{p} B\Sigma_\infty$$

and the associated map of plus constructions

$$p^+\colon \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+ \to \mathbb{Z} \times B\Sigma_\infty^+.$$

Since f^+ is a homotopy equivalence and $\Omega B(\mathrm{hocolim}_{\mathbb{P}} *) \simeq \mathbb{Z} \times B\Sigma_\infty^+$, we can identify the homotopy fiber of p^+ with hofib ρ^X. By assumption the action of Σ_∞ on X_∞ is homologically trivial. We are also assuming that X_∞^+ is abelian and in particular

nilpotent. Under these conditions the fiber sequence (1) remains a fiber sequence after passing to plus constructions; see [4, Theorem 1.1]. Thus we have a homotopy fibration

$$X_\infty^+ \to \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+ \to \mathbb{Z} \times B\Sigma_\infty^+.$$

This shows that the homotopy fiber of p^+ is X_∞^+ and so $X_\infty^+ \simeq \mathrm{hofib}\, \rho^X$. $\square$

Remark 3.2 For any commutative $\mathbb{I}$–monoid X, the multiplication on $M_X := \bigsqcup_{n \geq 0} X(n)$ is commutative up to the action of the shuffle maps $\tau_{m,n}$. These are induced by the action of the symmetric group. So, assuming that these actions are trivial in homology, it follows that the Pontrjagin product is commutative on the level of homology. In particular $\pi_0(M_X)$ is in the center of the Pontrjagin ring $H_*(M_X)$. Thus by the group completion theorem [15], the map

$$\mathbb{Z} \times X_\infty \to \Omega B(M_X)$$

is a homology isomorphism. In recent work, Gritschacher [7] has shown that without any further assumption, the commutator subgroup of $\pi_1(X_\infty)$ is always perfect and that X_∞^+ is always an abelian space. In other words, the assumptions in Theorem 3.1 on $\pi_1(X_\infty)$ and X_∞^+ are actually consequences.[2]

In contrast, the condition that the symmetric groups act homologically trivially is necessary. To see this consider the commutative $\mathbb{I}$–space X with $X(n) := Z^n$ for some pointed connected space Z. Then, by the parametrized version of the Barratt–Priddy–Quillen theorem (see for example [12; 22]),

$$\Omega B(\mathrm{hocolim}_\mathbb{P} X) \simeq Q(Z_+)$$

and thus $\mathrm{hofib}\, \rho^X \simeq \mathrm{hofib}\, p^+ \simeq Q(Z)$ while $X_\infty \simeq \mathrm{hocolim}_n Z^n$. Here $Q = \Omega^\infty \Sigma^\infty$ and Z_+ denotes the space Z with an additional base point.

We now turn to the question of comparing the infinite loop spaces $\mathrm{hofib}\, \rho^X$ and $\mathrm{hocolim}_\mathbb{I} X$. Suppose that X is a commutative $\mathbb{I}$–monoid. Consider the following commutative diagram of strict functors between permutative categories:

$$
\begin{array}{ccc}
\mathbb{P} \ltimes X & \xrightarrow{\alpha_X} & \mathbb{I} \ltimes X \\
\rho^X \downarrow & & \downarrow \rho_1^X \\
\mathbb{P} \ltimes * & \xrightarrow{\alpha_*} & \mathbb{I} \ltimes *
\end{array}
$$

The horizontal maps are induced by the inclusion $\mathbb{P} \to \mathbb{I}$. In the above diagram $*$ is the terminal commutative $\mathbb{I}$–monoid and the vertical maps ρ^X and ρ_1^X are induced by

---

[2]As we do not know whether M_X is homotopy commutative, the results of [17] cannot be applied directly to conclude that the induced map $\mathbb{Z} \times X_\infty^+ \to \Omega B(M_X)$ is a homotopy equivalence.

the projection maps to a point. Passing to the level of classifying spaces and applying group completion we obtain a commutative diagram of infinite loop spaces:

(2)
$$
\begin{array}{ccc}
\Omega B(\operatorname{hocolim}_{\mathbb{P}} X) & \xrightarrow{\ \alpha_X\ } & \Omega B(\operatorname{hocolim}_{\mathbb{I}} X) \\
\rho^X \downarrow & & \downarrow \rho_1^X \\
\Omega B(\operatorname{hocolim}_{\mathbb{P}} *) & \xrightarrow{\ \alpha_*\ } & \Omega B(\operatorname{hocolim}_{\mathbb{I}} *) \simeq *
\end{array}
$$

Note that the empty set is an initial object for $\mathbb{I}$ and hence $\operatorname{hocolim}_{\mathbb{I}} * = B\mathbb{I} \simeq *$.

The above diagram induces an infinite loop map between the homotopy fibers of the maps ρ^X and ρ_1^X. By definition the homotopy fiber on the left is the space $\operatorname{hofib} \rho^X$. Also, since $\operatorname{hocolim}_{\mathbb{I}} *$ is contractible, the homotopy fiber on the right can be identified with $\Omega B(\operatorname{hocolim}_{\mathbb{I}} X)$. This shows that we have a map of infinite loop spaces

$$
\operatorname{hofib} \rho^X \xrightarrow{\ g\ } \Omega B(\operatorname{hocolim}_{\mathbb{I}} X).
$$

Note that ρ^X has a canonical splitting of permutative categories induced by the unit $* \to X$ of the $\mathbb{I}$–monoid X. Thus it follows from the following theorem that g is a homotopy equivalence whenever the stated conditions on X are satisfied.

Theorem 3.3 *Let X be a commutative $\mathbb{I}$–monoid such that all maps $j\colon X(\boldsymbol{n}) \to X(\boldsymbol{m})$ induced by injections $j\colon \boldsymbol{n} \to \boldsymbol{m}$ are monomorphisms. Furthermore, assume that, for all $x \in X(\boldsymbol{n})$ and $y \in X(\boldsymbol{m})$, the sum $\mu_{\boldsymbol{n},\boldsymbol{m}}(x, y)$ is in the image of a map induced by a nonidentity order preserving injection if and only if x or y is. Then*

$$
\alpha_X \times \rho^X : \Omega B(\operatorname{hocolim}_{\mathbb{P}} X) \to \Omega B(\operatorname{hocolim}_{\mathbb{I}} X) \times \Omega B(\operatorname{hocolim}_{\mathbb{P}} *)
$$

is a weak homotopy equivalence of infinite loop spaces which is natural for commutative $\mathbb{I}$–monoids.

Notice that, when X is a commutative $\mathbb{I}$–rig, we may use the theorem to transfer the nonunital E_∞–ring space structure on $\operatorname{hofib} \rho^X$ along g to obtain a nonunital E_∞–ring space structure on the group completion of $\operatorname{hocolim}_{\mathbb{I}} X$.

A version of the theorem was proved by Fiedorowicz and Ogle [6] in the setting of simplicial sets. This was revisited in Gritschacher [7, Section 4]. For convenience of the reader we sketch a streamlined argument following [7].

Proof Given $x \in X(\boldsymbol{n})$ we can write it as $x = j_x(\bar{x})$, where $\bar{x} \in X(\bar{\boldsymbol{n}})$, $j_x\colon \bar{\boldsymbol{n}} \to \boldsymbol{n}$ is an order-preserving injection and $\bar{\boldsymbol{n}}$ is minimal. We call x reduced if $x = \bar{x}$. Note that $\bar{x}$ and j_x are uniquely determined. Denote by $\bar{X}(\boldsymbol{n})$ the set of reduced elements in $X(\boldsymbol{n})$. The assignment $\boldsymbol{n} \mapsto \bar{X}(\boldsymbol{n})$ defines a $\mathbb{P}$–diagram. By the assumption on μ the commutative $\mathbb{I}$–monoid structure of X induces the structure of a permutative category on $\mathbb{P} \ltimes \bar{X}$.

Assume now that X is discrete. Then the assignment $(\boldsymbol{n}, x) \mapsto (\overline{\boldsymbol{n}}, \overline{x})$ on objects extends to define a functor

$$R_X \colon \mathbb{I} \ltimes X \to \mathbb{P} \ltimes \overline{X}.$$

It has a right inverse given by the inclusion $\iota_X \colon \mathbb{P} \ltimes \overline{X} \to \mathbb{I} \ltimes X$. Furthermore, the maps j_x define a natural transformation from $\iota_X \circ R_X$ to the identity on $\mathbb{I} \ltimes X$. Hence, R_X defines a homotopy deformation retract on classifying spaces. We also note that by our assumption on μ, the functor R_X is a strict symmetric monoidal functor.

The inclusions $\mathbb{P} \ltimes \overline{X} \to \mathbb{P} \ltimes X$ and $\mathbb{P} \to \mathbb{P} \ltimes X$ combine via the monoidal product functor to a functor

$$T_X \colon (\mathbb{P} \ltimes \overline{X}) \times \mathbb{P} \to \mathbb{P} \ltimes X$$

that maps the object $((\overline{\boldsymbol{n}}, \overline{x}), \boldsymbol{n})$ to $(\overline{\boldsymbol{n}} + \boldsymbol{n}, j(\overline{x}))$, where j is the canonical inclusion $\overline{\boldsymbol{n}} \hookrightarrow \overline{\boldsymbol{n}} + \boldsymbol{n}$. We claim this is a homotopy equivalence on classifying spaces. Indeed, an analysis of the effect of permutations on reduced points shows that the functor is bijective on automorphism groups of objects. As both source and target categories are groupoids and every isomorphism class of the target category has a representative in the image, this is an equivalence of categories. We note that T_X is not a strict monoidal functor (only up to conjugation by a block permutation). However, the left inverse functor $(\boldsymbol{n}, x) \mapsto ((\overline{\boldsymbol{n}}, \overline{x}), \boldsymbol{n} - \overline{\boldsymbol{n}})$ does commute strictly with the monoidal structure. Hence, this defines a homotopy equivalence of monoids on classifying spaces, and induces a homotopy equivalence of group completions. Compare [6, Lemma 1.7].

Consider now the map of permutative categories

$$\alpha_X \times \rho^X \colon \mathbb{P} \ltimes X \to (\mathbb{I} \ltimes X) \times \mathbb{P}$$

and take the group completion of their classifying spaces

$$(3) \qquad \alpha_X \times \rho^X \colon \Omega B(B(\mathbb{P} \ltimes X)) \to \Omega B(B(\mathbb{I} \ltimes X)) \times \Omega B(B\mathbb{P}).$$

We claim that this is a weak homotopy equivalence which is natural in commutative $\mathbb{I}$–monoids. To see this precompose with the map of group completed classifying spaces induced by T_X and postcompose with the map induced by $R_X \times \mathrm{Id}$. The resulting composite is homotopic to the endofunctor of $(\mathbb{P} \ltimes \overline{X}) \times \mathbb{P}$ given by

$$((\overline{\boldsymbol{n}}, \overline{x}), \boldsymbol{m}) \mapsto ((\overline{\boldsymbol{n}}, \overline{x}), \overline{\boldsymbol{n}} + \boldsymbol{m}).$$

This map is the identity on the first component and an equivalence on the second component because we are working with group-complete monoids.

Using the naturality of the weak homotopy equivalence in (3) and applying it to boundary and face maps allows us to extend it to $\mathbb{I}$–diagrams in simplicial sets. More

precisely, for any commutative $\mathbb{I}$–monoid X in simplicial sets that satisfies levelwise the condition on μ, we have a map of simplicial permutative categories which is a weak homotopy equivalence on applying $\Omega B(B(-))$ to each simplicial level, and hence a weak homotopy equivalence on total spaces:

$$\alpha_X \times \rho^X : |\boldsymbol{n} \mapsto \Omega B(B(\mathbb{P} \ltimes X(\boldsymbol{n})))| \simeq |\boldsymbol{n} \mapsto \Omega B(B(\mathbb{I} \ltimes X(\boldsymbol{n}))) \times \Omega B(B\mathbb{P})|.$$

As Ω commutes with Cartesian product, and as $|\boldsymbol{n} \mapsto \Omega Z(\boldsymbol{n})| \simeq \Omega|\boldsymbol{n} \mapsto Z(\boldsymbol{n})|$ whenever each $Z(\boldsymbol{n})$ is connected (see [11, Theorem 12.3]), we also have

$$\alpha_X \times \rho^X : \Omega|\boldsymbol{n} \mapsto B(B(\mathbb{P} \ltimes X(\boldsymbol{n})))| \simeq \Omega|\boldsymbol{n} \mapsto B(B(\mathbb{I} \ltimes X(\boldsymbol{n}))) \times B(B\mathbb{P})|.$$

Furthermore, as realizations of multisimplicial sets can be taken in any order, we deduce that

$$\alpha_X \times \rho^X : \Omega B\big(B(\mathbb{P} \ltimes |\boldsymbol{n} \mapsto X(\boldsymbol{n})|)\big) \simeq \Omega B\big(B(\mathbb{I} \ltimes |\boldsymbol{n} \mapsto X(\boldsymbol{n})|)\big) \times \Omega B(B\mathbb{P}).$$

Compare [6, Lemma 1.8]. Finally, by replacing every space by its singular simplicial set, any $\mathbb{I}$–diagram X in topological spaces gives rise to an $\mathbb{I}$–diagram in simplicial sets, taking commutative $\mathbb{I}$–monoids to simplicial ones. Note that the conditions on μ are pointwise conditions and are automatically satisfied by the singular p–simplices for each p. As a space is weakly homotopy equivalent to the realization of its singular simplicial set, the theorem follows. $\qquad\square$

Example 3.4 Consider the commutative $\mathbb{I}$–space X with $X(\boldsymbol{n}) := Z^{\boldsymbol{n}}$, where Z is a well-pointed connected space. Note that in this case Σ_n does not act trivially on $H_*(Z^{\boldsymbol{n}})$ and hence Theorem 3.1 does not apply. As before, by the parametrized version of the Barratt–Priddy–Quillen theorem,

$$\Omega B(\mathrm{hocolim}_{\mathbb{P}} X) \simeq Q(Z_+) \simeq Q(\mathbb{S}^0) \times Q(Z)$$

and hence hofib $\rho^X \simeq Q(Z)$. Thus, by Theorem 3.3 we also have $\mathrm{hocolim}_{\mathbb{I}} X \simeq Q(Z)$, which is in agreement with a result of Schlichtkrull [20].

4 Constructing filtrations by infinite loop spaces

In this section we use the results obtained in the previous sections to produce filtrations of classical infinite loop spaces by sequences of infinite loop spaces arising from the descending central series of the free groups.

Theorem 4.1 *The spaces $B(q, U)$, $B(q, \mathrm{SU})$, $B(q, \mathrm{SO})$ $B(q, O)$ and $B(q, \mathrm{Sp})$ provide a filtration by nonunital E_∞–ring spaces of the classical nonunital E_∞–ring spaces BU, $B\mathrm{SU}$, $B\mathrm{SO}$, BO and $B\mathrm{Sp}$, respectively.*

Proof Consider first the case of BU. Recall that the spaces $B(q, U)$ provide a filtration of the space BU

$$B(2, U) \subset B(3, U) \subset \cdots \subset B(q, U) \subset B(q+1, U) \subset \cdots \subset BU.$$

We will show that this filtration is a filtration by nonunital E_∞–ring spaces. For this notice that by the main example in Section 2, each $\boldsymbol{n} \mapsto B(q, U(n))$ for $q \geq 2$ is a commutative $\mathbb{I}$–rig. In what follows we are going to show that the conditions of Theorem 3.1 are satisfied, and hence $B(q, U) \simeq \mathrm{hofib}\, \rho^{B(q, U(-))}$ is a nonunital E_∞–ring space by Theorem 2.3.

The conjugation action of Σ_n on $B(q, U(n))$ is homologically trivial because this action factors through the conjugation action of $U(n)$. The conjugation action by any element in $U(n)$ is trivial, up to homotopy, since the action of the identity matrix is trivial and $U(n)$ is path-connected. This implies in particular that the action of Σ_∞ on $B(q, U)$ is homologically trivial.

Note that $B(q, U(n))$ and hence $B(q, U)$ is path connected. Next, we argue that the space $B(q, U)$ is an H–space under direct sum multiplication. To be more precise, consider the injection $\mathbb{N} \sqcup \mathbb{N} \to \mathbb{N}$ defined by $(1, 2, 3, 4, \dots) \cup (1', 2', 3', 4', \dots) \mapsto (1, 2, 1', 2', 3, 4, 3', 4', \dots)$. It defines a map of vector spaces $\mathbb{C}^\infty \times \mathbb{C}^\infty \to \mathbb{C}^\infty$ and hence a continuous homomorphisms $U \times U \to U$. The image of $U(n)$ in U under right or left multiplication by the identity matrix I differs from the image under the standard inclusion by conjugation of an even permutation. As such a permutation is in the path-component of the identity matrix, we see that the multiplication is unital up to homotopy.

H–spaces have abelian fundamental group and hence Theorem 3.1 applies. We conclude that $B(q, U) \simeq \mathrm{hofib}\, \rho^{B(q, U(-))}$ for every $q \geq 2$ and is a nonunital E_∞–ring space by Theorem 2.3. The very same arguments can be used to prove analogous statements for the commutative $\mathbb{I}$–rig $\boldsymbol{n} \mapsto B(q, SU(n))$, and $\boldsymbol{n} \mapsto B(q, \mathrm{Sp}(n))$ for any $q \geq 2$.

In case of the commutative $\mathbb{I}$–rig $\boldsymbol{n} \mapsto B(q, SO(n))$ we note that Σ_n is not a subgroup of $SO(n)$. Nevertheless, the alternating group A_n is contained in $SO(n)$ and by the same argument as above acts therefore trivially on the homology of $B(q, SO(n))$. Furthermore, when n is odd, any odd permutation is represented by a matrix with determinant equal to -1. Hence it can be path-connected to the diagonal matrix $-I$ with constant entry -1. As $-I$ is in the center of $O(n)$ it acts trivially by conjugation on $B(q, SO(n))$ and hence also on its homology. But then so does any odd permutation. This proves that when n is odd the action of Σ_n on $B(q, SO(n))$ is homologically trivial. This in turn implies that the action of Σ_∞ on $B(q, SO)$ is

homologically trivial. We also have that $B(q, \mathrm{SO})$ is an H–space and hence abelian. Thus $B(q, \mathrm{SO}) \simeq \mathrm{hofib}\, \rho^{B(q,\mathrm{SO}(-))}$ for every $q \geq 2$ and it is a nonunital E_∞–ring space by Theorem 2.3. This line of argument can also be used to prove the analogous statement for the commutative $\mathbb{I}$–rig $\boldsymbol{n} \mapsto B(q, O(n))$. $\qquad\square$

As remarked in [1, Theorem 6.3], the natural map $\Omega B(q, G) \to \Omega BG$ admits a splitting up to homotopy. It is given by a factorization of the usual homotopy equivalence $G \to \Omega BG$. Indeed we have that $\Sigma G = F_1 B(q, G) = F_1 BG$, where F_1 denotes the first layer in the usual filtration of the geometric realization of these simplicial spaces. Hence, the adjoint of $\Sigma G \to BG$ factors through $\Omega B(q, G)$. Note that this splitting does not in general admit a delooping; see [1, Section 6] for a counterexample. Nevertheless, we have the following theorem. Here $E(q, G)$ denotes the pull-back of the universal G–bundle EG over BG. It is homotopy equivalent to the homotopy fiber of the inclusion $B(q, G) \to BG$.

Theorem 4.2 *For all $q \geq 2$, and $G = U$, SU, SO, O and Sp, there is a homotopy split fibration of infinite loop spaces*

$$E(q, G) \to B(q, G) \to BG.$$

In particular there is a splitting of spaces

$$B(q, G) \simeq BG \times E(q, G).$$

Both are natural in the entry q, meaning that both are compatible with the filtration maps.

In order to prove the theorem, we will need to know the fundamental group of $B(q, G)$ for the groups in question. We have the following general result:

Lemma 4.3 *Let G be a topological group with a CW–structure. Assume $\pi_0(G)$ is abelian and that the natural homomorphism $G \to \pi_0(G)$ splits. Then, for all $q \geq 2$,*

$$\pi_1(B(q, G)) = \pi_0(G).$$

Proof Consider $\Sigma G = F_1 B(q, G) = F_1 BG$. As the 1–skeleton of the realization of a (good) simplicial space is contained in the first filtration [11, Proposition 11.4], any map from S^1 to $B(q, G)$ will factor through ΣG. Hence the map $\Sigma G \to B(q, G)$ is surjective on fundamental groups.

The fundamental group of a suspension ΣX for any space X has fundamental group the free group over the set $\pi_0(X) - \{1\}$; hence we have

$$\pi_1(\Sigma G) = F\big(g \mid g \in \pi_0(G) - \{1\}\big).$$

The inclusion $\Sigma G \to BG$ induces the surjective map of fundamental groups $\pi_1(\Sigma G) \to \pi_0(G)$ which sends a generator g to the *element* $g \in \pi_0(G)$ and, more generally, the word $g_1 \bullet \cdots \bullet g_k$ to the product of the elements $g_1 \cdots g_k$. To see this geometrically, consider $\pi_0(G)$ as a subgroup of G, and note that the 2–simplex (g, h) defines a homotopy from the 2–letter word $g \bullet h$ to the product element gh.

We now note that, as $\pi_0(G)$ is abelian, the 2–simplex (g, h) is contained in $B_2(q, G)$ for $q \geq 2$. Hence all the above relations are already satisfied in $\pi_1(B(q, G))$. As the factorization $\pi_1(\Sigma G) \to \pi_1(B(q, G)) \to \pi_1(BG)$ is surjective, the result follows. □

Proof of Theorem 4.2 As $EG_\infty \simeq *$, for every $q \geq 2$ we have a homotopy fibration sequence $E(q, G_\infty) \to B(q, G_\infty) \to BG_\infty$. As the map on the right is a map of infinite loop spaces, the homotopy fiber $E(q, G_\infty)$ is an infinite loop space. It remains to show that it splits.

Let G_n denote one of the groups $U(n)$, $SU(n)$, $SO(n)$, $O(n)$ or $Sp(n)$, so that $G_\infty = \mathrm{colim}_n G_n$ denotes the group U, SU, SO, O or Sp, respectively. For each fixed $q \geq 2$, the assignment $\boldsymbol{n} \mapsto \Omega B(q, G_n)$ defines a commutative $\mathbb{I}$–rig with μ given by block sum and π given by tensor product of matrices. In the same way the assignment $\boldsymbol{n} \mapsto \Omega BG_n$ also defines a commutative $\mathbb{I}$–rig and the inclusion map $\Omega B(q, G_n) \to \Omega BG_n$ defines a morphism of commutative $\mathbb{I}$–rigs.

We claim that the commutative $\mathbb{I}$–rigs G_-, $\Omega B(q, G_-)$ and ΩBG_- satisfy the hypotheses of Theorem 3.1. Indeed, except in the case $G = O$, the group $G_n \simeq \Omega BG_n$ is path-connected for every $n \geq 0$ and, as $\pi_0(\Omega B(q, G_n)) \cong \pi_1(B(q, G_n))$ is trivial by Lemma 4.3, $\Omega B(q, G_n)$ is also path-connected. When $G = O$,

$$\pi_0(\Omega B(q, O(n))) = \pi_1 B(q, O(n)) = \mathbb{Z}/2\mathbb{Z}$$

for each $n \geq 1$ by Lemma 4.3. The multiplication in $\pi_0 \Omega B(q, O(n))$ is compatible with direct sum and stabilization. This checks the second condition in Theorem 3.1.

Except in the cases $G = SO$ or $G = O$, the action of Σ_n is homologically trivial as conjugation by any element in the path component of the identity is trivial, up to homotopy, and G_n is path-connected. This implies that Σ_∞ acts homologically trivially on G_∞, $\Omega B(q, G_\infty)$ and ΩBG_∞. The same conclusion can be obtained for $G = SO$ or $G = O$ using a similar argument as in the proof of Theorem 4.1. Hence the first condition from Theorem 3.1 holds.

To verify the third condition, observe that the commutator group of $\pi_1(\Omega B(q, G_n)) \cong \pi_2(B(q, G_n))$ is trivial, as this group is abelian in all cases. Finally, $\Omega B(q, G_\infty)$ is an abelian space since it is a loop space and hence in particular an H–space.

By Theorem 3.1 we thus have maps of E_∞–spaces

$$G_\infty \to \Omega B(q, G_\infty) \to \Omega B G_\infty$$

whose composition is a homotopy equivalence. Taking classifying spaces is compatible with E_∞–space structures and hence the above splitting deloops to give the splitting of the theorem. □

We have concentrated so far on compact groups such as $O(n)$ and $U(n)$, although the methods clearly extend to other linear groups. Using some results by Pettet and Souto [16] and Bergeron [3] we can prove the following theorem:

Theorem 4.4 *Suppose that G is the group of complex or real points in a reductive linear algebraic group (defined over $\mathbb{R}$ in the real case). Let $K \subset G$ be a maximal compact subgroup. Then the inclusion map $i \colon B(q, K) \to B(q, G)$ is a homotopy equivalence for every $q \geq 2$.*

Proof By [3, Theorem I] it follows that the inclusion map $i_n \colon B_n(q, K) \to B_n(q, G)$ is a homotopy equivalence for all $q \geq 2$ and all $n \geq 0$. Thus the inclusion map induces a simplicial map $i_* \colon B_*(q, K) \to B_*(q, G)$ that is a levelwise homotopy equivalence. Since G is assumed to be the group of complex or real points in a reductive linear algebraic group (defined over $\mathbb{R}$ in the real case), we can identify G with a Zariski closed subgroup of $\mathrm{SL}_N(\mathbb{C})$ for some $N \geq 0$. Also, for every $n \geq 0$ we can see the space $B_n(q, G)$ as an algebraic variety since it is defined in terms of iterated commutators of elements in G and such equations can be defined in terms of polynomial functions. Moreover, the subspace $S_n^1(q, G) \subset B_n(q, G)$ consisting of all n–tuples in $B_n(q, G)$ for which at least one of the coordinates is equal to 1_G is an algebraic subvariety of $B_n(q, G)$. By the semialgebraic triangulation theorem (see [8, Section 1]) it follows that $B_n(q, G)$ has the structure of a CW–complex in such a way that $S_n^1(q, G)$ is a subcomplex. In particular, it follows that the pair $(B_n(q, G), S_n^1(q, G))$ is a strong NDR pair. This proves that $B_*(q, G)$ is a proper simplicial space. The same is true for $B_*(q, K)$. Using the gluing lemma — for example see [12, Theorem A.4] — we obtain the result of the theorem. □

Our tools can also be used to obtain a similar filtration for the infinite loop space defining algebraic K–theory for any discrete ring R. Indeed, suppose that R is a discrete ring with unit and let $q \geq 2$. Consider the commutative $\mathbb{I}$–rig $B(q, \mathrm{GL}_-(R))$ defined by $\mathbf{n} \mapsto B(q, \mathrm{GL}_n(R))$. As before the morphisms are induced by the natural inclusions and the conjugation action of Σ_n on $B(q, \mathrm{GL}_n(R))$. The multiplication map

$$\mu_{\mathbf{n},\mathbf{m}} \colon B(q, \mathrm{GL}_n(R)) \times B(q, \mathrm{GL}_m(R)) \to B(q, \mathrm{GL}_{n+m}(R))$$

is also given by the block sum and π by tensor product of matrices. Note that Theorem 3.3 applies to give

$$\operatorname{hocolim}_{\mathbb{I}} B(q, \mathrm{GL}_-(R)) \simeq \operatorname{hofib} \rho^{B(q, \mathrm{GL}_-(R))}.$$

By Theorem 2.3, this space has the structure of a nonunital E_∞–ring space. This way we obtain a filtration of nonunital E_∞–ring spaces:

$$\operatorname{hocolim}_{\mathbb{I}} B(2, \mathrm{GL}_-(R)) \subset \cdots \subset \operatorname{hocolim}_{\mathbb{I}} B(q, \mathrm{GL}_-(R)) \subset \cdots \subset \operatorname{hocolim}_{\mathbb{I}} B\mathrm{GL}_-(R).$$

As is well known, the conjugation action of Σ_n on $B\mathrm{GL}_n(R)$ is homologically trivial. It follows from Theorems 3.1 and 3.3 that we have an equivalence

$$B\mathrm{GL}_\infty(R)^+ \simeq \operatorname{hofib} \rho^{B\mathrm{GL}_-(R)} \simeq \operatorname{hocolim}_{\mathbb{I}} B\mathrm{GL}_-(R).$$

Thus the above gives a filtration of nonunital E_∞–ring spaces with final space weakly homotopy equivalent to the algebraic K–theory of R. However, unlike the case of $B\mathrm{GL}_n(R)$, we do not know whether the conjugation action of Σ_n on $B(q, \mathrm{GL}_n(R))$ is homologically trivial, and we expect that the natural map

$$B(q, \mathrm{GL}_\infty(R)) \to \operatorname{hocolim}_{\mathbb{I}} B(q, \mathrm{GL}_-(R))$$

is not a homology isomorphism.

In a similar way we can obtain a filtration of $Q(\mathbb{S}^0)$. For this note that the conjugation action of Σ_n on $B\Sigma_n$ is homologically trivial. Therefore, by the Barratt–Priddy–Quillen theorem, the level zero component of $Q(\mathbb{S}^0)$ is equivalent to the homotopy colimit over $\mathbb{I}$ of the classifying spaces of the symmetric groups:

$$Q_0(\mathbb{S}^0) \simeq (B\Sigma_\infty)^+ \simeq \operatorname{hofib} \rho^{B\Sigma_-} \simeq \operatorname{hocolim}_{\mathbb{I}} B\Sigma_-.$$

Consider the commutative $\mathbb{I}$–rig $B(q, \Sigma_-)$ defined by $\boldsymbol{n} \mapsto B(q, \Sigma_n)$. The structural maps are given by conjugation of Σ_n and inclusions in an analogous way as above. Then by Theorem 2.2 we have a filtration of nonunital E_∞–ring spaces

$$\operatorname{hocolim}_{\mathbb{I}} B(2, \Sigma_-) \subset \cdots \subset \operatorname{hocolim}_{\mathbb{I}} B(q, \Sigma_-) \subset \cdots \subset \operatorname{hocolim}_{\mathbb{I}} B\Sigma_- \simeq Q_0(\mathbb{S}^0).$$

As in the case of $B(q, \mathrm{GL}_n(R))$, the conjugation action of Σ_n on $B(q, \Sigma_n)$ may fail to be homologically trivial (for example this is the case for the conjugation action of Σ_3 on $B(2, \Sigma_3)$; see [1]). The conditions of Theorem 3.3 are satisfied but the homotopy types of the spaces $\operatorname{hocolim}_{\mathbb{I}} B(q, \Sigma_-) \simeq \operatorname{hofib} \rho^{B(q, \Sigma_-)}$ remain to be determined.

Corollary 4.5 *The spaces*

$$\text{hocolim}_{\mathbb{I}}\, B(q, \text{GL}_-(R)) \simeq \text{hofib}\, \rho^{B(q,\text{GL}_-(R))},$$

$$\text{hocolim}_{\mathbb{I}}\, B(q, \Sigma_-) \simeq \text{hofib}\, \rho^{B(q,\Sigma_-)}$$

provide filtrations of nonunital E_∞–ring spaces with final target the classical nonunital E_∞–ring spaces $B\text{GL}_\infty(R)^+$ and $Q_0(\mathbb{S}^0)$.

5 Transitional nilpotence, bundles and K–theory

In this section we extend the notions of transitionally commutative bundles and commutative K–theory as defined in [2] to more general q–nilpotent notions for $q \geq 2$, reflecting the filtration induced by the descending central series of the free groups. We will show that these geometrically defined theories are represented by the infinite loop spaces $\mathbb{Z} \times B(q, U)$.

Definition 5.1 For a CW–complex X a principal G–bundle $\pi\colon E \to X$ is said to have *transitional nilpotency class* at most q if there exists an open cover $\{U_i\}_{i \in I}$ of X such that the bundle $\pi\colon E \to X$ is trivial over each U_i and for every $x \in X$ the group generated by the collection $\{\rho_{i,j}(x)\}_{i,j}$ is a group of nilpotency class at most q. Here $\rho_{i,j}\colon U_i \cap U_j \to G$ denotes the transition functions, and i and j run through all indices in I for which $x \in U_i \cap U_j$. The minimum of all such numbers q is said to be transitional nilpotency class of $\pi\colon E \to X$.

The principal G–bundle $p_q\colon E(q, G) \to B(q, G)$ is universal for all principal G–bundles with transitional nilpotency class less than q.

Theorem 5.2 *Assume that G is an algebraic subgroup of $\text{GL}_N(\mathbb{C})$ for some $N \geq 0$, X is a finite CW–complex and that $\pi\colon E \to X$ is a principal G–bundle over X. Then, for any $q \geq 2$, the classifying map $f\colon X \to BG$ of π factors through $B(q, G)$ (up to homotopy) if and only if π has transitional nilpotency class less than q.*

Proof The case $q = 2$ was treated in [2, Theorem 2.2] and in fact this theorem is true for any Lie group in this case. The proof goes through verbatim also for $q > 2$ using the fact that when G is an algebraic subgroup of $\text{GL}_N(\mathbb{C})$, then the simplicial space $B_*(q, G)$ is proper, as was pointed out in the proof of Theorem 4.4. □

As $[\Sigma X, BG] = [X, \Omega BG]$ and the canonical map $\Omega B(q, G) \to \Omega BG$ always admits a splitting up to homotopy, any principal G–bundle on a suspension ΣX has transitional nilpotency class less than q for all q. However, the nilpotency structure is not unique in general, not even up to isomorphism in the sense of the following definition:

Definition 5.3 Let $\pi_0\colon E_0 \to X$ and $\pi_1\colon E_1 \to X$ be two principal G–bundles with transitional nilpotency class less than q. We say that these bundles are q–transitionally isomorphic if there exists a principal G–bundle $p\colon E \to X \times [0,1]$ with transitional nilpotency class less than q such that $\pi_0 = p_{|p^{-1}(X\times\{0\})}$ and $\pi_1 = p_{|p^{-1}(X\times\{1\})}$.

A complex vector bundle $\pi\colon E \to X$ is said to have transitional nilpotency class less than q if the corresponding frame bundle, under a fixed Hermitian metric on E, has transitional nilpotency class less than q. Theorem 4.2 can then be interpreted to say that any vector bundle is stably of transitional nilpotency class less than q for all $q \geq 2$, and there is a functorial choice of such a structure. The set $\mathrm{Vect}_{q\text{–nil}}(X)$ of q–transitionally isomorphism classes of complex vector bundles over X with transitional nilpotency class less than q is a monoid under the direct sum of vector bundles. The q–nilpotent K–theory of X is defined as the associated Grothendieck group.

Definition 5.4 $K_{q\text{–nil}}(X) := \mathrm{Gr}(\mathrm{Vect}_{q\text{–nil}}(X))$.

Tensor products induce a natural multiplication on $K_{q\text{–nil}}(X)$ just as in classical K–theory.

Theorem 5.5 *For any finite CW–complex X there is a natural isomorphism of rings*

$$K_{q\text{–nil}}(X) \cong [X, \mathbb{Z} \times B(q, U)].$$

Hence, it is the zeroth term of a multiplicative generalized cohomology theory.

Proof Let X be a finite CW–complex. By working one path-connected component at a time, we may assume without loss of generality that X is path-connected. By Theorem 5.2,

$$\mathrm{Vect}_{q\text{–nil}}(X) = \left[X, \bigsqcup_{n\geq 0} B(q, U(n))\right]$$

as abelian monoids, where the addition is induced by direct sum of matrices on the right hand side. Any injection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ induces a linear injection $\mathbb{C}^\infty \times \mathbb{C}^\infty \to \mathbb{C}^\infty$, which in turn induces an H–space product on $\mathbb{Z} \times B(q, U)$. The natural inclusions $B(q, U(n)) \to B(q, U)$ define a map

$$\left[X, \bigsqcup_{n\geq 0} B(q, U(n))\right] \to [X, \mathbb{Z} \times B(q, U)].$$

As the symmetric groups act by homotopy equivalences on $B(q, U)$, we see that the above map is compatible with the product structure on both sets, ie it is a map of

monoids. By the universal property of the Grothendieck construction, this map factors through a unique map of abelian groups

$$K_{q\text{–nil}}(X) \to [X, \mathbb{Z} \times B(q, U)].$$

As X is compact, any map $X \to B(q, U)$ factors through some $B(q, U(n))$ for some large enough n. Hence the above map is surjective.

To prove that it is injective, suppose that the image of $[A] - [B] \in K_{q\text{–nil}}(X)$ in $[X, \mathbb{Z} \times B(q, U)]$ is zero. Let us write $f_B \colon X \to B(q, U)$ for the image of a map representing B in the colimit $B(q, U) = \mathrm{colim}_{n \in \mathbb{N}} B(q, U(n))$. Since $B(q, U)$ is a grouplike H–space, the induced product on $\mathrm{Map}(X, B(q, U))$ is also a grouplike H–space structure. Let $f_{B'} \colon X \to B(q, U)$ be a homotopy inverse for f_B under this product. Since X is compact, we may factor $f_{B'}$ through a finite stage of the colimit and find a corresponding bundle B' over X with transitional nilpotency class less than q which is classified by the map $f_{B'}$. It follows that $B \oplus B'$ is stably q–transitionally isomorphic to the trivial bundle ϵ_k of rank $k = \dim B + \dim B'$. By our assumption, we see that the image of $[A \oplus B'] - [\epsilon_k]$ in $[X, \mathbb{Z} \times B(q, U)]$ is also zero. This means that $A \oplus B'$ is stably q–transitionally isomorphic to a trivial bundle, say $A \oplus B' \oplus \epsilon_t \cong \epsilon_{k+t}$. We then have the relation

$$[A] - [B] = [A \oplus B' \oplus \epsilon_t] - [\epsilon_{k+t}] = 0$$

in $K_{q\text{–nil}}(X)$, which completes the proof. $\qquad\square$

This answers the question raised in [2] for $q = 2$. Moreover, we have a sequence of cohomology theories and maps between them,

$$K_{\mathrm{com}}(X) = K_{2\text{–nil}}(X) \to K_{3\text{–nil}}(X) \to \cdots \to K_{q\text{–nil}}(X) \to \cdots \to K(X).$$

By Theorem 4.2, topological K–theory splits off q–nilpotent K–theory for all $q \geq 2$. These theories are not well understood and would seem to warrant further attention. For example in [2] it was shown that $K_{\mathrm{com}}(\mathbb{S}^i) \cong K(\mathbb{S}^i)$ for $0 \leq i \leq 3$, but that $K_{\mathrm{com}}(\mathbb{S}^4) \neq K(\mathbb{S}^4)$.

We leave it to the reader to formulate q–nilpotent versions of real and hermitian K–theory.

References

[1]   **A Adem**, **F R Cohen**, **E Torres Giese**, *Commuting elements, simplicial spaces and filtrations of classifying spaces*, Math. Proc. Cambridge Philos. Soc. 152 (2012) 91–114 MR

[2] **A Adem**, **J M Gómez**, *A classifying space for commutativity in Lie groups*, Algebr. Geom. Topol. 15 (2015) 493–535 MR

[3] **M Bergeron**, *The topology of nilpotent representations in reductive groups and their maximal compact subgroups*, Geom. Topol. 19 (2015) 1383–1407 MR

[4] **A J Berrick**, *The plus-construction and fibrations*, Quart. J. Math. Oxford Ser. 33 (1982) 149–157 MR

[5] **A D Elmendorf**, **M A Mandell**, *Rings, modules, and algebras in infinite loop space theory*, Adv. Math. 205 (2006) 163–228 MR

[6] **Z Fiedorowicz**, **C Ogle**, *Algebraic K–theory and configuration spaces*, Topology 29 (1990) 409–418 MR

[7] **S Gritschacher**, *Modules over operads and iterated loop spaces*, Transfer thesis, University of Oxford (2014)

[8] **H Hironaka**, *Triangulations of algebraic sets*, from "Algebraic geometry" (R Hartshorne, editor), Proc. Sympos. Pure Math. 29, Amer. Math. Soc., Providence, RI (1975) 165–185 MR

[9] **J A Lind**, *Diagram spaces, diagram spectra and spectra of units*, Algebr. Geom. Topol. 13 (2013) 1857–1935 MR

[10] **M A Mandell**, **J P May**, *Equivariant orthogonal spectra and S–modules*, Mem. Amer. Math. Soc. 755, Amer. Math. Soc., Providence, RI (2002) MR

[11] **J P May**, *The geometry of iterated loop spaces*, Lectures Notes in Mathematics 271, Springer, Berlin (1972) MR

[12] **J P May**, *E_∞ spaces, group completions, and permutative categories*, from "New developments in topology" (G Segal, editor), London Math. Soc. Lecture Note Ser. 11, Cambridge Univ. Press (1974) 61–93 MR

[13] **J P May**, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Mathematics 577, Springer, Berlin (1977) MR

[14] **J P May**, *The spectra associated to permutative categories*, Topology 17 (1978) 225–228 MR

[15] **D McDuff**, **G Segal**, *Homology fibrations and the "group-completion" theorem*, Invent. Math. 31 (1975/76) 279–284 MR

[16] **A Pettet**, **J Souto**, *Commuting tuples in reductive groups and their maximal compact subgroups*, Geom. Topol. 17 (2013) 2513–2593 MR

[17] **O Randal-Williams**, *"Group-completion", local coefficient systems and perfection*, Q. J. Math. 64 (2013) 795–803 MR

[18] **S Sagave**, **C Schlichtkrull**, *Diagram spaces and symmetric spectra*, Adv. Math. 231 (2012) 2116–2193 MR

[19] **C Schlichtkrull**, *Units of ring spectra and their traces in algebraic K–theory*, Geom. Topol. 8 (2004) 645–673 MR

[20] **C Schlichtkrull**, *The homotopy infinite symmetric product represents stable homotopy*, Algebr. Geom. Topol. 7 (2007) 1963–1977 MR

[21] **S Schwede**, *S–modules and symmetric spectra*, Math. Ann. 319 (2001) 517–532 MR

[22] **G Segal**, *Categories and cohomology theories*, Topology 13 (1974) 293–312 MR

Department of Mathematics, University of British Columbia
1984 Mathematics Road, Room 121, Vancouver BC V6T 1Z2, Canada

Departmento de Matemáticas, Universidad Nacional de Colombia
Medellín, AA 3840, Colombia

Department of Mathematics, Reed College
3203 SE Woodstock Blvd., Portland, OR 97202, United States

Mathematical Institute, Oxford University
Oxford, OX2 6GG, United Kingdom

adem@math.ubc.ca, jmgomez0@unal.edu.co, john.alexander.lind@gmail.com,
tillmann@maths.ox.ac.uk