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Rational SO(2)–equivariant spectra

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We prove that the category of rational SO(2)–equivariant spectra has a simple algebraic model. Furthermore, all of our model categories and Quillen equivalences are monoidal, so we can use this classification to understand ring spectra and module spectra via the algebraic model.

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1 Introduction

Rational equivariant cohomology theories This paper is a contribution to the study of equivariant cohomology theories, and gives a rather complete analysis for one class of theories. To start with, G–equivariant cohomology theories are represented by G–spectra, so that the category of G–equivariant cohomology theories and stable natural transformations between them is equivalent to the homotopy category of G–spectra, and it is natural to study the homotopy theory of G–spectra. One cannot expect a complete analysis of either cohomology theories or spectra integrally, but if we rationalize, things are greatly simplified, whilst valuable geometric and group-theoretic structures remain. Henceforth we restrict attention to rational cohomology theories and rational spectra without further comment. The general conjecture states that there is a nice algebraic model for rational G–spectra and, more precisely, a graded abelian category $\mathcal{A}(G)$ and a Quillen equivalence

$$G\text{–spectra} \simeq d\mathcal{A}(G),$$

where $d\mathcal{A}(G)$ consists of differential objects of $\mathcal{A}(G)$. The category $\mathcal{A}(G)$ is of injective dimension equal to the rank of G and of a form that is easy to use in calculations. Of course one would like the Quillen equivalence to reflect as much structure as possible. The conjecture is known for quite a number of groups in some form, and we refer to Greenlees and Shipley [16] for a summary of what is known. In the present paper we are concerned with the specific case of the circle group, and with giving a zigzag of Quillen equivalences which are symmetric monoidal.

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The circle group We will entirely focus on the circle group, because it plays a critical role in understanding the case of all other infinite compact Lie groups. As an added benefit, it is significantly simpler than any other group, so we can focus on the critical features without being distracted by extraneous complication. We refer to the group as $\mathrm{SO}(2)$, because we have in mind as first applications its role as a subgroup of $O(2)$ (in Barnes [5]) and $\mathrm{SO}(3)$ (in Kędziorek [20]).

Our main result is as follows:

Main Theorem *The model category of rational $\mathrm{SO}(2)$–equivariant spectra is Quillen equivalent to the algebraic model $d\mathcal{A}(\mathrm{SO}(2))_{\mathrm{dual}}$. Furthermore, these Quillen equivalences are all symmetric monoidal, hence the homotopy category of rational $\mathrm{SO}(2)$–equivariant spectra and the homotopy category of the algebraic model $D(\mathcal{A}(\mathrm{SO}(2)))$ are equivalent as symmetric monoidal categories.*

The algebraic model is described in Section 2 below.

Rings and commutative rings Our main theorem establishes a zigzag of symmetric monoidal Quillen equivalences between the symmetric monoidal model category of rational $\mathrm{SO}(2)$–spectra and the symmetric monoidal model category $d\mathcal{A}(\mathrm{SO}(2))_{\mathrm{dual}}$. In particular we may use Schwede and Shipley [24, Theorem 3.12] to see that the model category of ring spectra is Quillen equivalent to the category of monoids in $\mathcal{A}(\mathrm{SO}(2))$. This means that a ring object R_a in $d\mathcal{A}(\mathrm{SO}(2))$ corresponds to a ring object R_{top} in $\mathrm{SO}(2)$–spectra in a homotopy-invariant fashion. Furthermore, the category of R_a–modules is Quillen equivalent to the category of R_{top}–modules.

However, it is essential to emphasize that if R_a is commutative, it does not follow that R_{top} will be a commutative $\mathrm{SO}(2)$–ring spectrum. The reason is that the correspondence between R_a and R_{top} is not simply applying the symmetric monoidal functors. Instead it involves derived functors and hence fibrant and cofibrant approximations. These approximations are only in the category of rings rather than in the category of commutative rings. This is inevitable, since for example the ring spectrum $R_{\mathrm{top}} = \widetilde{E}\mathcal{F}$ corresponds to a small and explicit commutative ring R_a, but it is well known — see McClure [22] — that $\widetilde{E}\mathcal{F}$ is not a commutative ring in orthogonal $\mathrm{SO}(2)$–spectra.

Greenlees [9] showed that if C is a generalized elliptic curve over a $\mathbb{Q}$–algebra, there is an associated $\mathrm{SO}(2)$–spectrum EC representing elliptic cohomology. Indeed the proof proceeds by writing down an object EC_a in $\mathcal{A}(\mathrm{SO}(2))$, and taking $EC = EC_{\mathrm{top}}$ to be the corresponding $\mathrm{SO}(2)$–spectrum. It is transparent from the construction that EC_a is a commutative ring in $\mathcal{A}(\mathrm{SO}(2))$, and it is a consequence of the present work that EC is a ring spectrum. As commented above, this does not prove that EC is a

commutative ring spectrum, though it is easily verified to be homotopy commutative and compatible with the homotopy ring structure used by Ando and Greenlees [1].

Contribution of this paper To place the contribution of this paper in the study of rational SO(2)–spectra, we need to give a little history. A description of the homotopy category of rational SO(2)–spectra was given by Greenlees [8]. This took the form of an equivalence of homotopy categories

$$\mathrm{Ho}(\mathrm{SO}(2)\text{–spectra}) \simeq D(\mathcal{A}(\mathrm{SO}(2)))$$

for the abelian category $\mathcal{A}(\mathrm{SO}(2))$ (described in Section 2 below) without giving a Quillen equivalence of model categories inducing it. Since $\mathcal{A}(\mathrm{SO}(2))$ is rather simple and of injective dimension 1, this gives a practical means for calculating the space $[X, Y]_*^{\mathrm{SO}(2)}$ of maps for arbitrary (rational) SO(2)–spectra X and Y up to extension. Since $\mathcal{A}(\mathrm{SO}(2))$ is (in a sense that will appear later) evenly graded, the extensions split, and so [8] gives a complete description of the category $\mathrm{Ho}(\mathrm{SO}(2)\text{–spectra})$. Unfortunately, [8] claimed that the above equivalence of homotopy categories is an equivalence of triangulated categories, but there is a gap in this argument. Patchkoria, who noticed this gap, gave in [23] (and more recent work) an illuminating systematic analysis of lifting equivalences of homotopy categories to ones that preserve triangulations and other structures. The purported argument for $\mathcal{A}(\mathrm{SO}(2))$ in [8] fits into Patchkoria's formalism, but does not satisfy the conditions necessary to apply Patchkoria's results. Shipley [26] showed that if the claimed triangulated equivalence of homotopy categories in [8] existed, it would lift to a Quillen equivalence of model categories. Work then began to give an algebraic model for the homotopy category of G–spectra for a torus G (eventually leading to Greenlees and Shipley [16]); it was soon apparent that the only way to approach this is to first prove a Quillen equivalence between G–spectra and $d\mathcal{A}(G)$ and then deduce the equivalence of homotopy categories as a consequence. This general project has taken some time, and has a complicated history of its own [13; 14; 15; 16], but the special case of the circle is much simpler than the general case, and easily explained. The underlying strategy applied in [16] is the same as that adopted here for the circle group, but there are some significant differences of implementation adopted from Barnes [2; 5] and Kędziorek [19; 20].

Meanwhile, work began on the group $O(2)$ (culminating in the model of Barnes [5]) and the group SO(3) (culminating in the model of Kędziorek [19]). Those models depended on the Quillen equivalence for SO(2); they originally built upon Greenlees and Shipley [16], but the technical context adopted here has advantages for them. The proof for the general torus is considerably more complicated than that for the circle, principally because SO(2) has only two connected subgroups (namely the trivial group and the whole group) rather than infinitely many for higher-dimensional tori.

Accordingly, it is much easier to see the essential structure of the argument in the case of the circle. It is therefore desirable to give a separate account for $\mathrm{SO}(2)$ to show the simplicity of the argument, and to provide the input to the work on $O(2)$ and $\mathrm{SO}(3)$.

Perhaps a more important reason for publishing a separate account for $\mathrm{SO}(2)$ is that at present we can prove more for the circle group than for a general torus. The category of G–spectra is a monoidal model category, and $\mathcal{A}(G)$ is a monoidal abelian category. One would like to have a monoidal equivalence between G–spectra and $d\mathcal{A}(G)$. Of course this requires more care and some more delicate analysis than a simple Quillen equivalence. As the first step, one needs a monoidal model structure on $d\mathcal{A}(G)$. The abelian category $\mathcal{A}(G)$ does not have enough projectives and the injective model structure on $d\mathcal{A}(G)$ used in earlier work is certainly not monoidal. On the other hand, for $G = \mathrm{SO}(2)$, constructing a monoidal model structure on $d\mathcal{A}(\mathrm{SO}(2))$ is the primary task of Barnes [4]. This result relies on some explicit constructions in $\mathcal{A}(\mathrm{SO}(2))$ from [8] that are not made explicit in Greenlees [10; 11] for higher tori. It is expected that a similar construction will work for other groups, but additional work will be necessary. Once a monoidal model structure is defined on $d\mathcal{A}(G)$, one would need to ensure that all Quillen pairs making up the equivalence are monoidal. At present, this is only accessible for $G = \mathrm{SO}(2)$.

The Hasse–Tate isotropy square The overarching strategy for building an algebraic model is to break the category of $\mathrm{SO}(2)$–spectra into parts, give an algebraic model of each part and then assemble an algebraic model for all spectra from the algebraic models of the parts.

To analyse an individual $\mathrm{SO}(2)$–spectrum it is natural to use isotropy separation, to assemble the spectrum from information at the family $\mathcal{F}$ of finite subgroups and the information at $\mathrm{SO}(2)$ itself. This can be implemented using the Tate square

$$\begin{array}{ccc} X & \longrightarrow & X \wedge \widetilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ F(E\mathcal{F}_+, X) & \longrightarrow & F(E\mathcal{F}_+, X) \wedge \widetilde{E}\mathcal{F} \end{array}$$

which expresses X as the homotopy pullback of its $\mathcal{F}$–completion, $F(E\mathcal{F}_+, X)$, and its localization away from $\mathcal{F}$, namely $X \wedge \widetilde{E}\mathcal{F}$, over the Tate object, $F(E\mathcal{F}_+, X) \wedge \widetilde{E}\mathcal{F}$. Thus X is the homotopy pullback of a punctured square diagram (ie a diagram of shape $\mathscr{P} = (\bullet \to \bullet \leftarrow \bullet)$). The basic idea is to do this at the level of model categories. We would like to assemble the category of all $\mathrm{SO}(2)$–spectra from the category of $\mathcal{F}$–complete objects and objects localized away from $\mathcal{F}$. The way we do it here is to take suitable model categories of $\mathcal{F}$–complete spectra, of spectra away from $\mathcal{F}$,

and Tate spectra, and then construct a model structure on the category $S^\bullet$–mod of $\mathscr{P}$–diagrams of such objects: a cellularization (K_{top}–cell–$S^\bullet$–mod) of this model category of $\mathscr{P}$–diagrams is then shown to be Quillen equivalent to the original category of SO(2)–spectra essentially using the fact that the Tate square is a homotopy pullback. The machinery of Greenlees and Shipley [15] was built for this purpose.

The alternative, adopted in Greenlees and Shipley [16], is to say that the category of SO(2)–spectra is equivalent to the category of $\mathbb{S}$–modules in SO(2)–spectra, where $\mathbb{S}$ is the sphere spectrum. We then consider the special case of the Tate square in which $X = \mathbb{S}$ and say that $\mathbb{S}$ is the pullback of a diagram of rings, so that the module category of $\mathbb{S}$ is Quillen equivalent to a cellularization of the model category of modules over the $\mathscr{P}$–diagram of rings.

After this, the general strategy in either case is to show the $\mathscr{P}$–diagram of model categories is equivalent to a simpler one that can be made algebraically explicit. In the present paper, several of the monoidal functors are taken from Barnes [2; 5] and Kędziorek [19; 20] and, since we work in a context where $\widetilde{E}\mathcal{F}$ is not a commutative ring, we adopt their methods for the formality argument in Section 4.1.

Summary of the zigzag of Quillen equivalences To illustrate the zigzag of Quillen equivalences in the Main Theorem we present a diagram of key steps:

$$
\begin{array}{lcl}
 & L_{\mathbb{S}_{\mathbb{Q}}}\mathbb{T}\mathrm{Sp} & (\text{in } \mathbb{T}\mathrm{Sp}) \\
(\text{Proposition 3.2.5}) & \big\downarrow S^\bullet\wedge- \quad \big\uparrow \mathrm{pb} & \\
 & K_{\mathrm{top}}\text{–cell–}S^\bullet\text{–mod} & (\text{in } \mathbb{T}\mathrm{Sp}) \\
(\text{Corollary 3.3.6}) & \big\uparrow \zeta_\# \quad \big\downarrow (-)^{\mathbb{T}} & \\
 & K_{\mathrm{top}}^{\mathbb{T}}\text{–cell–}S_{\mathrm{top}}^\bullet\text{–mod} & (\text{in Sp}) \\
(\text{Corollary 3.4.6}) & \big\uparrow \text{zigzag of Quillen equivalences} \big\downarrow & \\
 & K_t\text{–cell–}S_t^\bullet\text{–mod} & (\text{in Ch}(\mathbb{Q})) \\
(\text{Section 4.1}) & \big\uparrow \text{zigzag of Quillen equivalences} \big\downarrow & \\
 & K_a\text{–cell–}S_a^\bullet\text{–mod} & (\text{in Ch}(\mathbb{Q})) \\
(\text{Proposition 4.2.4}) & \big\uparrow \mu \quad \big\downarrow \Gamma & \\
 & d\mathcal{A}(\mathbb{T})_{\mathrm{dual}} & (\text{in Ch}(\mathbb{Q}))
\end{array}
$$

At the top we have our preferred model for rational $\mathbb{T}$–spectra (namely the left Bousfield localization $L_{\mathbb{S}_{\mathbb{Q}}}\mathbb{T}\,\mathrm{Sp}$ of the category of orthogonal $\mathbb{T}$–spectra at the rational sphere spectrum (Definition 3.2.1)) and at the bottom we have our algebraic model $d\mathcal{A}(\mathbb{T})_{\mathrm{dual}}$.

The first step, moving into categories of $\mathscr{P}$–diagrams, was suggested in the previous subsection, and the other steps will be described in the body of the paper. The reader may wish to refer to this diagram now, but the notation will be introduced as we proceed. In the diagram, left Quillen functors are placed on the left and $\mathbb{T} := \mathrm{SO}(2)$. References to specific results are given on the left, and on the right there is an indication of the ambient category. In the following the subscript "top" indicates that the corresponding object has a topological origin, whereas the subscript "t" indicates that the object is algebraic, but has been produced by applying the results of Shipley [27]. The subscript "a" indicates that the object is algebraic in nature and has an explicit description. The symbols $S^{\bullet}_{(-)}$ refer to particular $\mathscr{P}$–diagrams of model categories, and the various categories $S^{\bullet}_{(-)}$–mod are generalizations of the notion of a module over a diagram of rings; see Section 3.1. These model categories are cellularized (ie right Bousfield localized) at the sets of objects $K_{(-)}$, which at every level of the diagram are the derived images of the usual stable generators $\mathbb{T}/H_+$ of $\mathbb{T}$–spectra, where H varies through closed subgroups of $\mathbb{T}$.

Notation From now on we will write $\mathbb{T}$ for the group $\mathrm{SO}(2)$. We also stick to the convention of drawing the left adjoint above the right one in any adjoint pair. We use $\mathrm{Ch}(\mathbb{Q})$ for the category of chain complexes of rational vector spaces, Sp for the category of orthogonal spectra, $G\,\mathrm{Sp}$ for the category of orthogonal G–spectra and Sp^{Σ} for the category of symmetric spectra.

2 The algebraic model $d\mathcal{A}(\mathbb{T})$

In this section we recall the algebraic category $\mathcal{A}(\mathbb{T})$ as developed by the second author [8]. This category is naturally enriched in graded abelian groups. We use the notation $d\mathcal{A}(\mathbb{T})$ for the category of objects in $\mathcal{A}(\mathbb{T})$ with a differential and call it the algebraic model for rational $\mathbb{T}$–spectra. A nonmonoidal model structure for the category $d\mathcal{A}(\mathbb{T})$ is given in [8]. Work of the first author [4] builds upon this and constructs a monoidal model structure on $d\mathcal{A}(\mathbb{T})$.

We call $\mathcal{A}(\mathbb{T})$ the *abelian model* for rational $\mathbb{T}$–spectra and $d\mathcal{A}(\mathbb{T})$ the *algebraic model* for rational $\mathbb{T}$–spectra. The model structures we construct on $d\mathcal{A}(\mathbb{T})$ are such that $\mathrm{Ho}(d\mathcal{A}(\mathbb{T}))$ is equivalent to the derived category of the abelian model, $D(\mathcal{A}(\mathbb{T}))$, which is equivalent to the homotopy category of rational $\mathbb{T}$–spectra by [8].

2.1 The abelian model $\mathcal{A}(\mathbb{T})$

The abelian model for rational $\mathbb{T}$–spectra is established in [8]. We introduce this category, explain how to turn it into a differential graded category and then define the injective model structure.

Definition 2.1.1 Let $\mathcal{F}$ be the set of finite subgroups of $\mathbb{T}$. Let $\mathcal{O}_{\mathcal{F}}$ be the graded ring $\prod_{n\geqslant 1}\mathbb{Q}[c_n]$ with c_n of degree -2. Let e_n be the idempotent arising from projection onto factor n. In general, let ϕ be a subset of $\mathcal{F}$ and define e_ϕ to be the idempotent coming from projection onto the factors in ϕ. We let c be the unique element of $\mathcal{O}_{\mathcal{F}}$ such that $c_n = e_n c$ for all $n \geq 1$.

We use the notation $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} = \operatorname{colim}_{n\geqslant 1}\mathcal{O}_{\mathcal{F}}[c_1^{-1},\dots,c_n^{-1}]$. It is easy to see that $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is a ring. The notation arises since this ring can also be described in terms of inverting a certain set of Euler classes. As a vector space, $(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}})_{2k}$ is $\prod_{n\geqslant 1}\mathbb{Q}$ for $k\leqslant 0$ and is $\bigoplus_{n\geqslant 1}\mathbb{Q}$ for $n>0$.

For any $\mathcal{O}_{\mathcal{F}}$ module N, we define $\mathcal{E}^{-1}N$ to be $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes_{\mathcal{O}_{\mathcal{F}}} N$.

Definition 2.1.2 We define the *abelian model* $\mathcal{A} = \mathcal{A}(\mathbb{T})$ as follows. Its class of objects is the collection of triples (N, U, β) where N is an $\mathcal{O}_{\mathcal{F}}$–module, U is a graded rational vector space and

$$\beta\colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes U$$

is an $\mathcal{O}_{\mathcal{F}}$–module map such that $\mathcal{E}^{-1}\beta$ is an isomorphism.[1] We will often refer to β as the *structure map*.

A map (θ,ϕ) in $\mathcal{A}$ is a commutative square

$$\begin{array}{ccc} N & \xrightarrow{\beta} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes U \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\otimes\phi} \\ N' & \xrightarrow{\beta'} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes U' \end{array}$$

where θ is a map of $\mathcal{O}_{\mathcal{F}}$–modules and ϕ is a map of graded rational vector spaces.

The relation between this category and rational $\mathbb{T}$–equivariant spectra is given by the following pair of theorems from [8].

Theorem 2.1.3 *The homotopy category of rational $\mathbb{T}$–equivariant spectra is equivalent to the derived category of $\mathcal{A}$.*

[1] The tensor product in the target of β is over $\mathbb{Q}$, which we omit from the notation.

For a rational $\mathbb{T}$–equivariant spectrum X, let $\pi_*^{\mathcal{A}}(X)$ be the following object of $\mathcal{A}$, which is its counterpart in $\mathcal{A}$:

$$\pi_*^{\mathcal{A}}(X) = \big(\pi_*^{\mathbb{T}}(X \wedge DE\mathcal{F}_+) \to \pi_*^{\mathbb{T}}(X \wedge DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}) \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \pi_*(\Phi^{\mathbb{T}} X)\big).$$

For details of the spectra $DE\mathcal{F}_+$ and $\widetilde{E}\mathcal{F}$ see Definition 3.2.2. The spectrum $\Phi^{\mathbb{T}} X$ is the geometric $\mathbb{T}$–fixed points of X.

There is also an Adams short exact sequence which explains how to calculate maps in the homotopy category of rational $\mathbb{T}$–equivariant spectra:

Theorem 2.1.4 *Let X and Y be rational $\mathbb{T}$–equivariant spectra. Then the sequence below is exact:*

$$0 \to \mathrm{Ext}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(\Sigma X), \pi_*^{\mathcal{A}}(Y)) \to [X, Y]_*^{\mathbb{T}} \to \mathrm{Hom}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \to 0.$$

In [8] a model structure is given for the category of objects in $\mathcal{A}$ that have a differential. We define what it means to have a differential and then introduce the model structure. We will leave the proof that $\mathcal{A}$ has all small limits and colimits to the next subsection (see also [8]).

We can consider $\mathcal{O}_{\mathcal{F}}$ as an object of $\mathrm{Ch}(\mathbb{Q})$ with trivial differential and, as such, it is a commutative algebra in $\mathrm{Ch}(\mathbb{Q})$. An $\mathcal{O}_{\mathcal{F}}$–module in $\mathrm{Ch}(\mathbb{Q})$ is an $\mathcal{O}_{\mathcal{F}}$–module in graded vector spaces N along with maps $d_n\colon N_n \to N_{n-1}$. Note that these maps satisfy the relations

$$d_{n-1} \circ d_n = 0, \quad cd_n = d_{n-2}c.$$

Definition 2.1.5 We define the category $d\mathcal{A} = d\mathcal{A}(\mathbb{T})$ as follows. Its class of objects is the collection of triples (N, U, β) where N is a rational chain complex with an action of $\mathcal{O}_{\mathcal{F}}$, U is a rational chain complex and

$$\beta\colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$$

is a $\mathcal{O}_{\mathcal{F}}$–module map in $\mathrm{Ch}(\mathbb{Q})$ such that $\mathcal{E}^{-1}\beta$ is an isomorphism.

A map (θ, ϕ) in $d\mathcal{A}$ is then a commutative square as for $\mathcal{A}$ such that θ is a map in the category of $\mathcal{O}_{\mathcal{F}}$–modules in $\mathrm{Ch}(\mathbb{Q})$ and ϕ is a map of $\mathrm{Ch}(\mathbb{Q})$.

We call this category the *algebraic model for rational $\mathbb{T}$–spectra*.

Note that the category $d\mathcal{A}$ is not the same as $\mathrm{Ch}(\mathcal{A})$, since $\mathcal{A}$ is a graded category and in $d\mathcal{A}$ we do not introduce an additional grading; instead we take objects of $\mathcal{A}$ with a differential.

The following result is the subject of [8, Appendix B]:

Proposition 2.1.6 *The category $d\mathcal{A}$ has a model structure where the class of weak equivalences is exactly the class of quasi-isomorphisms. The class of cofibrations is the class of monomorphisms.* (This is called the *injective model structure*. We write $d\mathcal{A}_i$ for this model structure.)

As we shall see shortly, the category $\mathcal{A}$ has a monoidal product which induces a monoidal product on $d\mathcal{A}$. But the injective model structure does not make $d\mathcal{A}$ into a monoidal model category. This failure occurs because of c–torsion, just as the injective model structure on $\mathrm{Ch}(\mathbb{Z})$ is not monoidal due to torsion.

This is a serious defect, as we are unable to compare the monoidal product in $d\mathcal{A}$ to the smash product of $\mathbb{T}$–spectra. This defect is further complicated by the lack of projective objects of $\mathcal{A}$. There is however a cofibrantly generated monoidal model structure on $d\mathcal{A}$ which is Quillen equivalent to the injective model structure. It is constructed in [4] and we recall it in the next subsection.

2.2 The monoidal model structure

This subsection has three aims, namely to prove that $\mathcal{A}$ and $d\mathcal{A}$ have all small limits and colimits (see also [8]), define the monoidal product and recall the dualizable model structure on $d\mathcal{A}$ (see [4]), which is monoidal. To do so, we will need to relate $\mathcal{A}$ to a larger category $\widehat{\mathcal{A}}$, which we introduce next.

We let $\widehat{\mathcal{A}}$ be category of triples $(N, U, \alpha\colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ where N is an $\mathcal{O}_{\mathcal{F}}$–module, U is graded $\mathbb{Q}$–module and the map α is a map of $\mathcal{O}_{\mathcal{F}}$–modules. A map of such diagrams is a commutative diagram as below where θ is a map of $\mathcal{O}_{\mathcal{F}}$–modules, and ϕ is a map of graded $\mathbb{Q}$–modules:

$$\begin{array}{ccc} N & \longrightarrow & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle \mathrm{Id}\otimes\phi} \\ N' & \longrightarrow & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U' \end{array}$$

Thus $\widehat{\mathcal{A}}$ is $\mathcal{A}$ without the restriction that the structure map of an object should be an isomorphism after $\mathcal{E}$ is inverted. There is an adjunction

$$\iota\colon \mathcal{A} \rightleftarrows \widehat{\mathcal{A}} :\Gamma_h,$$

where ι is the inclusion. The functor ι is full and faithful. The explicit construction of the right adjoint Γ_h, which we call the *torsion functor*, is quite intricate and therefore we leave the details to [8, Section 20.2].

Our first use of the torsion functor Γ_h is to define limits in $\mathcal{A}$. It follows from [8, Section 20.2] that the adjunction (ι, Γ_h) passes to categories with differentials, as does the following definition:

Definition 2.2.1 Let I be some small category and let $\{N_i \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U_i\}$ be the objects of some I–shaped diagram in $\mathcal{A}$. The colimit over I is

$$\operatorname{colim}_i N_i \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (\operatorname{colim}_i U_i).$$

The limit is formed by applying the functor ι, taking limits in $\widehat{\mathcal{A}}$ and then applying Γ_h. In more detail, we construct the following pullback square:

$$\begin{array}{ccc} M & \longrightarrow & \lim(N_i) \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \lim(U_i) & \longrightarrow & \lim(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U_i) \end{array}$$

The limit of the I–shaped diagram $\{N_i \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U_i\}$ is $\Gamma_h f$.

Now we turn to the monoidal product of $\mathcal{A}$ and $d\mathcal{A}$.

Definition 2.2.2 For $\beta\colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$ and $\beta'\colon N' \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U'$ in $d\mathcal{A}$, their *tensor product* is

$$\beta \otimes \beta'\colon N \otimes_{\mathcal{O}_{\mathcal{F}}} N' \to (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U) \otimes_{\mathcal{O}_{\mathcal{F}}} (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U') \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (U \otimes_{\mathbb{Q}} U').$$

The unit of this monoidal product is the object $S^0 = (i\colon \mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q})$.

This monoidal product is related to the smash product of spectra, as we can see from the short exact sequence of [8],

$$0 \to \pi_*^{\mathcal{A}}(X) \otimes \pi_*^{\mathcal{A}}(Y) \to \pi_*^{\mathcal{A}}(X \wedge Y) \to \Sigma\operatorname{Tor}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \to 0.$$

This monoidal structure is closed, that is, there is an internal function object describing the $d\mathcal{A}$–object of maps between two objects. This functor is more complicated than the tensor product and requires use of the torsion functor Γ_h.

Definition 2.2.3 Consider two elements of $d\mathcal{A}$,

$$A = (\beta\colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U) \quad \text{and} \quad B = (\beta'\colon N' \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U').$$

The *function object* $F(A, B)$ is the map $\Gamma_h\delta$, where δ is defined by the pullback square:

$$\begin{array}{ccc} Q & \xrightarrow{\delta} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathrm{Hom}_{\mathbb{Q}}(U, U') \\ \downarrow & & \downarrow \\ & & \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U') \\ & & \downarrow \\ \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, N') & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U') \end{array}$$

The monoidal product and function object are related by a natural isomorphism by [8, Lemma 22.6.2]. Let A, B and C be objects of $d\mathcal{A}$; then

$$d\mathcal{A}(A \otimes B, C) \cong d\mathcal{A}(A, F(B, C)).$$

Definition 2.2.4 For $K \in \mathrm{Ch}(\mathbb{Q})$ we define $LK \in d\mathcal{A}$ as

$$LK = (i \otimes \mathrm{Id}_K \colon \mathcal{O}_{\mathcal{F}} \otimes K \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes K).$$

Note that $LK = S^0 \otimes K$, where $S^0 = (i \colon \mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q})$. For A and B in $d\mathcal{A}$, we define $\mathcal{A}(A, B)_*$ to be the graded set of maps of $\mathcal{A}$ (ignoring the differential). We then equip this graded $\mathbb{Q}$–module with the differential induced by the convention $df_n = d_B f_n + (-1)^{n+1} f_n d_A$. This construction gives a functor

$$R \colon d\mathcal{A} \to \mathrm{Ch}(\mathbb{Q}), \quad RA := \mathcal{A}(S^0, A)_*.$$

The functors L and R form an adjoint pair between $\mathrm{Ch}(\mathbb{Q})$ and $d\mathcal{A}$. Furthermore, they give $d\mathcal{A}$ the structure of a closed $\mathrm{Ch}(\mathbb{Q})$–module in the sense of [18, Section 4.1].

This module structure and the closed monoidal product interact to give $d\mathcal{A}$ a tensor product, a cotensor product and an enrichment over $\mathrm{Ch}(\mathbb{Q})$. Let $K \in \mathrm{Ch}(\mathbb{Q})$ and $A = (\beta \colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ in $d\mathcal{A}$. Their *tensor product* $A \otimes K$ is defined to be $A \otimes LK$. Thus $A \otimes K$ is given by

$$\beta \otimes \mathrm{Id}_K \colon N \otimes_{\mathbb{Q}} K \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (U \otimes_{\mathbb{Q}} K).$$

There is a *cotensor product* A^K defined to be $F(LK, A)$. The *enrichment* is given by $RF(A, B)$ for A and B in $d\mathcal{A}$. This enrichment, tensor and cotensor are related by the natural isomorphisms

$$d\mathcal{A}(A, B^K) \cong d\mathcal{A}(A \otimes K, B) = d\mathcal{A}(A \otimes LK, B) \cong \mathrm{Ch}(\mathbb{Q})(K, RF(A, B)).$$

Now we are ready to recall the monoidal model structure on $d\mathcal{A}$ from [4] and compare it to several other model categories, in particular the injective model structure on $d\mathcal{A}$

introduced in [8]. This monoidal model structure is defined in terms of the (strongly) dualizable objects of $d\mathcal{A}$.

Definition 2.2.5 An object $A \in \mathcal{A}$ is said to be (strongly) *dualizable* if for any $B \in \mathcal{A}$ the canonical map

$$F(A, S^0) \otimes B \to F(A, B)$$

is an isomorphism. The *functional dual* of an object B is the object $DB = F(B, S^0)$.

Let $\mathcal{P}$ be a set of representatives for the isomorphisms classes of dualizable objects in $\mathcal{A}$. Such a set exists by [4, Corollary 5.8]. The following theorem summarizes [4, Section 6]:

Theorem 2.2.6 *There is a cofibrantly generated model structure on $d\mathcal{A}$ with weak equivalences the class generated by the homology isomorphisms. The generating cofibrations have the form*

$$S^{n-1} \otimes P \to D^n \otimes P$$

for $P \in \mathcal{P}$ and $n \in \mathbb{Z}$, where S^n is the chain complex consisting of one copy of $\mathbb{Q}$ in degree n and 0 elsewhere and D^n consists of two copies of $\mathbb{Q}$ in degrees n and $n-1$ with the identity as the only nontrivial differential.

(We call this model structure the *dualizable model structure* and denote it by $d\mathcal{A}_{\mathrm{dual}}$. The dualizable model structure is proper, symmetric monoidal and satisfies the monoid axiom.)

Since all cofibrations in the dualizable model structure are in particular monomorphisms we get the following comparison with the injective model structure of [8], which we write as $d\mathcal{A}_i$ (see Proposition 2.1.6 for the description of the injective model structure).

Lemma 2.2.7 *The identity functor from $d\mathcal{A}_{\mathrm{dual}}$ to $d\mathcal{A}_i$ is the left adjoint of a Quillen equivalence,*

$$\mathrm{Id}\colon d\mathcal{A}_{\mathrm{dual}} \rightleftarrows d\mathcal{A}_i \colon \mathrm{Id}.$$

The object S^0 is clearly dualizable. Similarly, if V is a finite-dimensional vector space, then $S^0 \otimes V$ is dualizable. As a consequence, we have the following lemma:

Lemma 2.2.8 *There is a strong symmetric monoidal Quillen pair*

$$L\colon \mathrm{Ch}(\mathbb{Q}) \rightleftarrows d\mathcal{A}_{\mathrm{dual}} \colon R,$$

where $LV = S^0 \otimes V$ and $RA = \mathcal{A}(S^0, A)_*$ (see Definition 2.2.4). Thus, $d\mathcal{A}_{\mathrm{dual}}$ is a closed $\mathrm{Ch}(\mathbb{Q})$–model category.

3 Obtaining an algebraic category

The method of this section is the synthesis of three ideas. The first idea is to use the Hasse–Tate square from the introduction to separate the homotopical information of $\mathbb{T}$–equivariant spectra into pieces where we can remove equivariance without losing any information.

For $\mathbb{T}$–equivariant spectra, the relevant decomposition is to separate the homotopical information coming from finite subgroups from the homotopical information coming from the whole group. For this separation we will need a diagram of model categories rather than a diagram of commutative rings. We establish the categorical foundations in the next subsection and then perform the separation in Section 3.2.

The second is that the correct way to remove equivariance is to take fixed points. The primary example is that taking $\mathbb{T}$–fixed points gives a Quillen equivalence from $DE\mathbb{T}_+$–modules in rational $\mathbb{T}$–equivariant spectra to $DB\mathbb{T}_+$–modules in rational spectra. Here $DE\mathbb{T}_+$ is the Spanier–Whitehead dual of $E\mathbb{T}_+$ in $\mathbb{T}$–spectra and $DB\mathbb{T}_+$ is the Spanier–Whitehead dual of $B\mathbb{T}_+$ in the category of spectra. See Section 3.3.

With the separation complete and equivariance removed, we use the results of [27] to move to an algebraic setting in Section 3.4. That is, we obtain a Quillen equivalence between rational $\mathbb{T}$–spectra and some combined cellularization–localization of an algebraic category.

The next step is to simplify that algebraic category into the algebraic model $\mathcal{A}(\mathbb{T})$, by directly calculating the effects of these cellularizations and localizations. This is the essence of the third idea: to leave any examination of localizations or cellularizations until one is working with an algebraic category. This occurs in Section 4.1, where we simplify the category created by the results of [27] and remove a localization. Finally, in Section 4.2 we remove a cellularization to get to the algebraic model.

3.1 Diagrams of model categories

We will use several model categories that are built from diagrams of model categories. This idea has been studied in some detail in [15]. In this section we introduce the relevant structures and leave most of the proofs to the reference. We will only use one shape of diagram, the pullback diagram $\mathscr{P}$:

$$\bullet \to \bullet \leftarrow \bullet.$$

Pullbacks of model categories are also considered in detail in [7].

Definition 3.1.1 A *$\mathscr{P}$–diagram of model categories* $R^\bullet$ is a pair of Quillen pairs

$$L\colon \mathcal{A} \rightleftarrows \mathcal{B} : R, \quad F\colon \mathcal{C} \rightleftarrows \mathcal{B} : G,$$

with L and F the left adjoints. We will usually draw this as

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}.$$

A standard example comes from a $\mathscr{P}$–diagram of rings $R = (R_1 \xrightarrow{f} R_2 \xleftarrow{g} R_3)$. Using the adjoint pairs of extension and restriction of scalars we obtain a $\mathscr{P}$–diagram of model categories $R^\bullet$:

$$R_1\text{–mod} \underset{f^*}{\overset{R_2\otimes_{R_1}-}{\rightleftarrows}} R_2\text{–mod} \underset{g^*}{\overset{R_2\otimes_{R_3}-}{\rightleftarrows}} R_3\text{–mod}.$$

Definition 3.1.2 Given a $\mathscr{P}$–diagram of model categories $R^\bullet$ we can define a new category, $R^\bullet$–mod. The objects of this category are pairs of morphisms $\alpha\colon La \to b$ and $\gamma\colon Fc \to b$ in $\mathcal{B}$. We usually abbreviate a pair $(\alpha\colon La \to b,\ \gamma\colon Fc \to b)$ to a quintuple $(a, \alpha, b, \gamma, c)$. We find this notation suggestive but emphasize that objects of $R^\bullet$–mod are not usually modules over a diagram of rings.

A morphism in $R^\bullet$–mod from $(a, \alpha, b, \gamma, c)$ to $(a', \alpha', b', \gamma', c')$ is a triple of maps $x\colon a \to a'$ in $\mathcal{A}$, $y\colon b \to b'$ in $\mathcal{B}$ and $z\colon c \to c'$ in $\mathcal{C}$ such that we have a commuting diagram in $\mathcal{B}$:

$$\begin{array}{ccccc} La & \xrightarrow{\alpha} & b & \xleftarrow{\gamma} & Fc \\ \downarrow{\scriptstyle Lx} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle Fz} \\ La' & \xrightarrow{\alpha'} & b' & \xleftarrow{\gamma'} & Fc' \end{array}$$

Note that we could also have defined an object as a sequence $(a, \overline{\alpha}, b, \overline{\gamma}, c)$, where $\overline{\alpha}\colon a \to Rb$ is a map in $\mathcal{A}$ and $\overline{\gamma}\colon c \to Gb$ is a map in $\mathcal{C}$.

We say that a map (x, y, z) in $R^\bullet$–mod is an objectwise cofibration if x is a cofibration of $\mathcal{A}$, y is a cofibration of $\mathcal{B}$ and z is a cofibration of $\mathcal{C}$. We define objectwise weak equivalences similarly.

Lemma 3.1.3 [15, Proposition 3.3] *Consider a $\mathscr{P}$–diagram of model categories $R^\bullet$, with each category cellular and proper,*

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}.$$

The category $R^\bullet$–mod admits a cellular proper model structure with cofibrations and weak equivalences defined objectwise. (This is called the *diagram injective* model structure.)

Whilst there is also a diagram projective model structure, in this paper we only use the diagram injective model structure (and cellularizations thereof) on diagrams of model categories.

Now consider maps of $\mathscr{P}$–diagrams of model categories. Let $R^\bullet$ and $S^\bullet$ be two $\mathscr{P}$–diagrams, where $R^\bullet$ is as above and $S^\bullet$ is

$$\mathcal{A}' \underset{R'}{\overset{L'}{\rightleftarrows}} \mathcal{B}' \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{C}'.$$

Now we assume that we have Quillen adjunctions such that $P_2 L$ is naturally isomorphic to $L'P_1$ and $P_2 F$ is naturally isomorphic to $F'P_3$, given by

$$\begin{aligned} P_1\colon \mathcal{A} &\rightleftarrows \mathcal{A}' :Q_1, \\ P_2\colon \mathcal{B} &\rightleftarrows \mathcal{B}' :Q_2, \\ P_3\colon \mathcal{C} &\rightleftarrows \mathcal{C}' :Q_3. \end{aligned}$$

We then obtain a Quillen adjunction (P, Q) between $R^\bullet$–mod and $S^\bullet$–mod. For example, the left adjoint P takes the object $(a, \alpha, b, \gamma, c)$ to $(P_1 a, P_2\alpha, P_2 b, P_2\gamma, P_3 c)$. The commutativity assumptions ensure that this is an object of $S^\bullet$–mod. It is easy to see the following:

Lemma 3.1.4 *If the Quillen adjunctions (P_i, Q_i) are Quillen equivalences then the adjunction (P, Q) between $R^\bullet$–*mod *and $S^\bullet$–*mod *is a Quillen equivalence.*

Now we turn to monoidal considerations. There is an obvious monoidal product for $R^\bullet$–mod, provided that each of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ is monoidal and that the left adjoints L and F are strong monoidal,

$$(a, \alpha, b, \gamma, c) \wedge (a', \alpha', b', \gamma', c') := (a \wedge a', \alpha \wedge \alpha', b \wedge b', \gamma \wedge \gamma', c \wedge c').$$

Let $S_{\mathcal{A}}$ be the unit of $\mathcal{A}$, let $S_{\mathcal{B}}$ be the unit of $\mathcal{B}$ and let $S_{\mathcal{C}}$ be the unit of $\mathcal{C}$. Since L and F are monoidal, we have maps $\eta_{\mathcal{A}}\colon LS_{\mathcal{A}} \to S_{\mathcal{B}}$ and $\eta_{\mathcal{C}}\colon FS_{\mathcal{C}} \to S_{\mathcal{B}}$. The unit of the monoidal product on $R^\bullet$–mod is $(S_{\mathcal{A}}, \eta_{\mathcal{A}}, S_{\mathcal{B}}, \eta_{\mathcal{C}}, S_{\mathcal{C}})$.

It is worth noting that this category has an internal function object when $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are closed monoidal categories and thus itself is closed.

Lemma 3.1.5 *Consider a $\mathscr{P}$–diagram of model categories $R^\bullet$ such that each vertex is a cellular monoidal model category. Assume further that the two adjunctions of the diagram are strong monoidal Quillen pairs. Then $R^\bullet$–*mod *is also a monoidal model category. If each vertex also satisfies the monoid axiom, so does $R^\bullet$–*mod*.*

Proof Since the cofibrations and weak equivalences are defined objectwise, the pushout product and monoid axioms hold provided they do so in each model category in the diagram $R^\bullet$. □

We can also extend our monoidal considerations to maps of diagrams. Return to the setting of a map (P, Q) of $\mathscr{P}$–diagrams from $R^\bullet$ to $S^\bullet$ as described above. If we assume that each of the adjunctions (P_1, Q_1), (P_2, Q_2) and (P_3, Q_3) is a symmetric monoidal Quillen equivalence, then we see that (P, Q) is a symmetric monoidal Quillen equivalence.

With these formalities out of the way, we are ready to move from the model category of rational $\mathbb{T}$–spectra to modules over a $\mathscr{P}$–diagram of model categories.

3.2 Isotropy separation

In this subsection we separate the homotopical information of rational $\mathbb{T}$–spectra into three parts. The first part takes care of the homotopical information coming from the finite cyclic subgroups. The second part deals with the homotopical information coming from $\mathbb{T}$. The third part is a comparison term which enforces some compatibility conditions on the two other parts.

We achieve this separation by replacing the category of rational $\mathbb{T}$–spectra with a Quillen equivalent category $S^\bullet$–mod, for $S^\bullet$ a $\mathscr{P}$–diagram of model categories (see Definition 3.2.3).

Before we do that, let us first recall some basic definitions and properties for $\mathbb{T}$–spectra.

Definition 3.2.1 Let $\mathbb{T}\,\mathrm{Sp}$ be the category of $\mathbb{T}$–equivariant orthogonal spectra indexed on a complete $\mathbb{T}$–universe $\mathcal{U}$ considered with the stable model structure.

This model category is monoidal, proper and cellular [21]. The weak equivalences are those maps f such that $\pi_*^H(f)$ is an isomorphism for all closed subgroups H of $\mathbb{T}$.

Following [3, Section 5] and using [21, Theorem IV.6.3], we localize this model category at the rational sphere spectrum $\mathbb{S}_\mathbb{Q}$. That is, we leave the underlying category unchanged and alter the model structure. We call the weak equivalences of the localized model structure *rational equivalences*: a map f is a rational equivalence if $\pi_*^H(f)\otimes\mathbb{Q}$ is an isomorphism for all closed subgroups H of $\mathbb{T}$. We call this model structure the *rational model structure* and use the notation $L_{\mathbb{S}_\mathbb{Q}}\mathbb{T}\,\mathrm{Sp}$.

The localized model category is still proper, cellular, monoidal and stable.

Definition 3.2.2 Let $\mathcal{F}$ be the collection of finite cyclic subgroups of $\mathbb{T}$. There is a universal space for this family, called $E\mathcal{F}$, where, by definition, $E\mathcal{F}^H$ is nonequivariantly contractible for each finite cyclic subgroup H and $E\mathcal{F}^{\mathbb{T}} = \varnothing$. We define $\widetilde{E}\mathcal{F}$ via the cofibre sequence of $\mathbb{T}$–spaces

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E}\mathcal{F}.$$

We define $DE\mathcal{F}_+$ to be $F(E\mathcal{F}_+, N^{\#}\mathbb{S})$. Here $N^{\#}$ is the lax monoidal right adjoint described in [21, Theorem IV.3.9] from EKMM $\mathbb{T}$–equivariant $\mathbb{S}$–modules to $\mathbb{T}\mathrm{Sp}$.

Recall that $N^{\#}$ is the right adjoint of a Quillen equivalence when $\mathbb{T}\mathrm{Sp}$ is considered with the positive stable model structure (see [21, Chapter IV] for more details). The spectrum $DE\mathcal{F}_+$ is a commutative ring spectrum, which is fibrant in the positive stable model structure on $\mathbb{T}\mathrm{Sp}$.

We can use the above cofibre sequence to produce the Hasse–Tate homotopy pullback square of $\mathbb{T}$–equivariant spectra [12, Section 17]:

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & \widetilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ DE\mathcal{F}_+ & \longrightarrow & DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F} \end{array}$$

To see that it is a homotopy pullback square, note that the homotopy fibres of the top and bottom row are weakly equivalent (where the bottom row is the top one smashed with $DE\mathcal{F}_+$).

We have three model categories:

- $L_{\mathbb{S}_{\mathbb{Q}}}(DE\mathcal{F}_+\text{–mod})$, which captures the behaviour of the finite cyclic groups.
- $L_{\mathbb{S}_{\mathbb{Q}} \wedge \widetilde{E}\mathcal{F}}\mathbb{T}\mathrm{Sp}$, which captures the behaviour of $\mathbb{T}$.
- $L_{\mathbb{S}_{\mathbb{Q}} \wedge DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}}(DE\mathcal{F}_+\text{–mod})$, which captures the interaction of the first two.

Now we can give our diagram of model categories that separates the behaviour of the finite cyclic groups from the rest.

Definition 3.2.3 We define $S^{\bullet}$ to be the $\mathscr{P}$–diagram of model categories

$$L_{\mathbb{S}_{\mathbb{Q}}}(DE\mathcal{F}_+\text{–mod}) \underset{\text{Id}}{\overset{\text{Id}}{\rightleftarrows}} L_{\mathbb{S}_{\mathbb{Q}} \wedge DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}}(DE\mathcal{F}_+\text{–mod}) \underset{U}{\overset{DE\mathcal{F}_+ \wedge -}{\rightleftarrows}} L_{\mathbb{S}_{\mathbb{Q}} \wedge \widetilde{E}\mathcal{F}}\mathbb{T}\mathrm{Sp}.$$

Since all of the model categories in the diagram are cellular, proper, monoidal model categories, we have a cellular proper stable monoidal model category $S^{\bullet}$–mod that satisfies the monoid axiom.

Given an $X \in \mathbb{T}\mathrm{Sp}$, we have an $S^\bullet$–module

$$S^\bullet \wedge X := (DE\mathcal{F}_+ \wedge X, \mathrm{Id}, DE\mathcal{F}_+ \wedge X, \mathrm{Id}, X).$$

The functor $S^\bullet \wedge -$ has a right adjoint. Let $A = (a, \alpha, b, \gamma, c)$ be an $S^\bullet$–module. Then there are maps in $\mathbb{T}\mathrm{Sp}$, namely $a \to b$ and $c \to DE\mathcal{F}_+ \wedge c \to b$, where in the composite the first map is the unit of the adjunction $(DE\mathcal{F}_+ \wedge -, U)$ and the second map is γ. Thus we have a diagram in $\mathbb{T}\mathrm{Sp}$: $a \to b \leftarrow c$. We write $\mathrm{pb}\, A$ for the pullback of this diagram in $\mathbb{T}\mathrm{Sp}$. We assemble this construction into the following result, the proof of which is entirely routine:

Proposition 3.2.4 *There is a strong symmetric monoidal Quillen adjunction*

$$S^\bullet \wedge -\colon L_{S_{\mathbb{Q}}}(\mathbb{T}\mathrm{Sp}) \rightleftarrows S^\bullet\text{–mod} : \mathrm{pb}\,.$$

We want to turn this adjunction into a Quillen equivalence. To do so, we apply the cellularization principle of [13, Proposition 2.7]. The idea is to cellularize (right Bousfield localize; see also Section 5.1) the right-hand model category so that this adjunction induces a Quillen equivalence. In general, A–cell–$\mathcal{M}$ denotes the cellularization of the model category $\mathcal{M}$ with respect to a set of objects A in $\mathcal{M}$, which we call cells.

The generators for the homotopy category of $L_{\mathbb{S}_{\mathbb{Q}}}(\mathbb{T}\mathrm{Sp})$ are all suspensions and desuspensions of objects of the form $\mathbb{T}/H_+$ for H a subgroup of $\mathbb{T}$. For later purposes (see Section 4.2), we want a set of cells with simpler algebraic models. For every natural $n > 1$, let

$$\sigma_n = \mathbb{T}_+ \wedge_{C_n} e_{C_n}\mathbb{S},$$

where e_{C_n} is the idempotent in the Burnside ring for C_n (cyclic group of order n) corresponding to C_n. By [8, Lemma 2.1.5],

$$\mathbb{T}/C_{n+} = \bigvee_{C_m \subseteq C_n} \sigma_m;$$

hence we know that the set

$$K = \{\Sigma^k \mathbb{S} \mid k \in \mathbb{Z}\} \cup \{\Sigma^k \sigma_n \mid n > 1,\ k \in \mathbb{Z}\}$$

is a set of (cofibrant and homotopically small) generators for $L_{\mathbb{S}_{\mathbb{Q}}}(\mathbb{T}\mathrm{Sp})$.

Let K_{top} be the set of images of the objects from K under the functor $S^\bullet \wedge -$ (up to isomorphism). The elements of this set K_{top} will be called *basic cells*.

To apply the cellularization principle of [13] we need to know that these cells are homotopically small (this is also known as small or compact; see Definition 5.1.4).

First note that if X is homotopically small in $\mathbb{T}\mathrm{Sp}$ then it is so in $L_{\mathbb{S}_{\mathbb{Q}}}\mathbb{T}\mathrm{Sp}$ (since rationalization is a smashing localization).

Now consider the three elements of $S^\bullet$–mod

$$(*,*,DE\mathcal{F}_+\wedge X,*,*),\quad (*,*,DE\mathcal{F}_+\wedge X,\mathrm{Id},X),\quad (DE\mathcal{F}_+\wedge X,\mathrm{Id},DE\mathcal{F}_+\wedge X,*,*).$$

It is routine to check that these are cofibrant and homotopically small whenever X is cofibrant and homotopically small in $\mathbb{T}\mathrm{Sp}$. Finally, let X be cofibrant in $\mathbb{T}\mathrm{Sp}$. There is a homotopy pushout diagram, where the final term is $S^\bullet\wedge X$:

$$\begin{array}{ccc}(*,*,DE\mathcal{F}_+\wedge X,*,*) & \longrightarrow & (*,*,DE\mathcal{F}_+\wedge X,\mathrm{Id},X)\\ \downarrow & & \downarrow\\ (DE\mathcal{F}_+\wedge X,\mathrm{Id},DE\mathcal{F}_+\wedge X,*,*) & \longrightarrow & (DE\mathcal{F}_+\wedge X,\mathrm{Id},DE\mathcal{F}_+\wedge X,\mathrm{Id},X)\end{array}$$

Homotopically small objects are preserved by homotopy pushouts (consider the associated cofibre sequence). Hence $S^\bullet\wedge X$ is homotopically small in $S^\bullet$–mod whenever X is cofibrant and homotopically small. Since these two conditions hold for the generators of $\mathbb{T}\mathrm{Sp}$, we see that every element of K_{top} is homotopically small.

Note that the model category $L_{DE\mathcal{F}_+\wedge\widetilde{E}\mathcal{F}}DE\mathcal{F}_+$–mod is the same as the model category $L_{\Sigma^*f}DE\mathcal{F}_+$–mod, where $f\colon DE\mathcal{F}_+\to DE\mathcal{F}_+\wedge\widetilde{E}\mathcal{F}$ and Σ^*f is the set of all (integer) suspensions and desuspensions of f. This is a similar result to [6, Lemma 4.14], since $\widetilde{E}\mathcal{F}$–localization (in $\mathbb{T}\mathrm{Sp}$) is given by smashing with the map of $\mathbb{T}$–spaces $S^0\to\widetilde{E}\mathcal{F}$.

Proposition 3.2.5 *There is a Quillen equivalence*

$$S^\bullet\wedge -\colon L_{S_{\mathbb{Q}}}(\mathbb{T}\mathrm{Sp})\rightleftarrows K_{\mathrm{top}}\text{–cell–}S^\bullet\text{–mod :pb.}$$

Proof This follows from the cellularization principle, [13, Proposition 2.7]. It suffices to show that the derived unit is a weak equivalence on the set K of generators for the left-hand side, which are shifts of the objects σ_n for $n>1$ and $\mathbb{S}$. Each such object is cofibrant and homotopically small, as are the elements of K_{top}.

The derived left adjoint on cofibrant objects (such as the elements of K) is simply the left adjoint. The right derived functor on objects of the form $S^\bullet\wedge k$ for $k\in K$ is weakly equivalent to taking a homotopy pullback of the diagram

$$\begin{array}{ccc} & & \mathbb{S}_{\mathbb{Q}}\wedge\widetilde{E}\mathcal{F}\wedge k\\ & & \downarrow{\scriptstyle \mathrm{Id}\wedge\mathrm{Id}\wedge\lambda\wedge\mathrm{Id}}\\ \mathbb{S}_{\mathbb{Q}}\wedge DE\mathcal{F}_+\wedge k & \xrightarrow[\mathrm{Id}\wedge a\wedge\mathrm{Id}\wedge\mathrm{Id}]{} & \mathbb{S}_{\mathbb{Q}}\wedge\widetilde{E}\mathcal{F}\wedge DE\mathcal{F}_+\wedge k\end{array}$$

where the map $a\colon S^0 \to \widetilde{E}\mathcal{F}$ (of $\mathbb{T}$–spaces) is the map to the cofibre and λ is the unit map. Since homotopy pullbacks commute with smash products, the homotopy pullback of the above is weakly equivalent to the homotopy pullback of

$$\begin{array}{ccc} & & \widetilde{E}\mathcal{F} \\ & & \downarrow \\ DE\mathcal{F}_+ & \longrightarrow & DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F} \end{array}$$

(in the category $\mathbb{T}\mathrm{Sp}$) smashed with $\mathbb{S}_{\mathbb{Q}} \wedge k$. But the homotopy pullback of the diagram above is $\mathbb{S}$, as discussed after Definition 3.2.2. Hence the derived unit is a weak equivalence (in $L_{\mathbb{S}_{\mathbb{Q}}}(\mathbb{T}\mathrm{Sp})$) on the cells $k \in K$. □

We will show in Proposition 5.1.6 below that this Quillen equivalence is actually a symmetric monoidal Quillen equivalence.

Thus we have separated the homotopical information of $\mathbb{T}\mathrm{Sp}$ into a diagram of three model categories. The advantage of doing so is that we may now remove the equivariance from the model category whilst keeping the correct homotopy category.

3.3 Removing equivariance

Now we are going to remove equivariance using the inflation–fixed points adjunction $(\varepsilon, (-)^{\mathbb{T}})$.

Recall the functor $(-)^{\mathbb{T}}$ of [21, Section 3]. It takes a spectrum indexed on a complete $\mathbb{T}$–universe $\mathcal{U}$ to the $\mathbb{T}$–trivial universe $\mathcal{U}^{\mathbb{T}}$ and then applies the space-level fixed point functor levelwise. We begin by extending this functor to categories of modules over $\mathbb{T}$–equivariant ring spectra.

If A is a commutative ring object in $\mathbb{T}$–equivariant spectra then $A^{\mathbb{T}}$ is a commutative ring object in spectra. We want to compare A–modules in $\mathbb{T}$–equivariant spectra and $A^{\mathbb{T}}$–modules in spectra. Using [14, Section 4] there is a Quillen adjunction

$$A \wedge_{\varepsilon^* A^{\mathbb{T}}} \varepsilon^*(-)\colon A^{\mathbb{T}}\text{–mod} \rightleftarrows A\text{–mod} :(-)^{\mathbb{T}}$$

between right transferred model structures (fibrations and weak equivalences are defined in terms of the underlying categories). To simplify the notation, if $\zeta\colon \varepsilon^* A^{\mathbb{T}} \to A$ is the inclusion of fixed points, we write

$$\zeta_\# = A \wedge_{\varepsilon^* A^{\mathbb{T}}} \varepsilon^*(-)$$

for the left adjoint.

We consider several cases of this kind of adjunction and use them to build up an adjunction between $S^\bullet$–mod and a new diagram of model categories $S^\bullet_{\mathrm{top}}$–mod. We then show that this adjunction gives a Quillen equivalence, after cellularizing.

Proposition 3.3.1 *For $\zeta\colon \varepsilon^* DE\mathcal{F}^{\mathbb{T}}_+ \to DE\mathcal{F}_+$ the inclusion of fixed points, the adjunction*

$$\zeta_\#\colon L_{\mathbb{S}_\mathbb{Q}}(DE\mathcal{F}^{\mathbb{T}}_+\text{–mod}) \rightleftarrows L_{\mathbb{S}_\mathbb{Q}}(DE\mathcal{F}_+\text{–mod}) :(-)^{\mathbb{T}}$$

is a symmetric monoidal Quillen equivalence.

Proof We have a Quillen equivalence by [14, Corollaries 8.1 and 9.2]. The left adjoint is strong symmetric monoidal, so the result follows. □

We now left Bousfield localize the model categories in this adjunction. We localize the right-hand side at the set of maps $\Sigma^* f$, where $f\colon DE\mathcal{F}_+ \to DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}$. Let $(\Sigma^* f)^{\mathbb{T}}$ be the set of maps obtained by applying the derived right adjoint to the maps in $\Sigma^* f$. By [17, Theorem 3.3.20(1)(b)] we obtain the following result:

Proposition 3.3.2 *The adjunction*

$$\zeta_\#\colon L_{(\Sigma^* f)^{\mathbb{T}}} L_{\mathbb{S}_\mathbb{Q}}(DE\mathcal{F}^{\mathbb{T}}_+\text{–mod}) \rightleftarrows L_{\Sigma^* f} L_{\mathbb{S}_\mathbb{Q}}(DE\mathcal{F}_+\text{–mod}) :(-)^{\mathbb{T}}$$

is a symmetric monoidal Quillen equivalence.

Our final version is where we take A to be the sphere spectrum, so the left adjoint is just ε^*. By [21, Section V, Proposition 3.10] the adjunction

$$\varepsilon^*\colon \mathrm{Sp} \rightleftarrows \mathbb{T}\,\mathrm{Sp} :(-)^{\mathbb{T}}$$

is a symmetric monoidal Quillen adjunction. We localize it to obtain a Quillen equivalence:

Proposition 3.3.3 *The adjunction*

$$\varepsilon^*\colon L_{\mathbb{S}_\mathbb{Q}}(\mathrm{Sp}) \rightleftarrows L_{\mathbb{S}_\mathbb{Q}\wedge\widetilde{E}\mathcal{F}}(\mathbb{T}\,\mathrm{Sp}) :(-)^{\mathbb{T}}$$

is a symmetric monoidal Quillen equivalence.

Proof Since ε^* is strong monoidal and $\varepsilon^*(\mathbb{S}_\mathbb{Q}) = \mathbb{S}_\mathbb{Q}$, the above adjunction is a composite of two adjunctions, the second being identity adjunction between $L_{\mathbb{S}_\mathbb{Q}}(\mathbb{T}\,\mathrm{Sp})$ and further localization at $\widetilde{E}\mathcal{F}$, namely $L_{\mathbb{S}_\mathbb{Q}\wedge\widetilde{E}\mathcal{F}}(\mathbb{T}\,\mathrm{Sp})$.

To verify that this is a Quillen equivalence we will work with the derived unit and the derived counit on generators. The generator for the left-hand side is $\mathbb{S}$. The generators

for the right-hand side are $\mathbb{S} = \mathbb{T}/\mathbb{T}_+$ and $(\mathbb{T}/C_n)_+$ for $n \geqslant 1$. But $(\mathbb{T}/C_n)_+$ is weakly equivalent to a point in $L_{\mathbb{S}_\mathbb{Q} \wedge \widetilde{E}\mathcal{F}}(\mathbb{T}\,\mathrm{Sp})$ (that is, $(\mathbb{T}/C_n)_+ \wedge \widetilde{E}\mathcal{F} \simeq *$). So we only need to consider $\mathbb{S}$ for the right-hand side.

The derived functor of $(-)^\mathbb{T}$ acts as the geometric $\mathbb{T}$–fixed point functor, because, by definition, for any $H \leq G$, $\phi^H(X) = (X \wedge \widetilde{E}[\not\supseteq H])^H$. With this in mind, it is routine to check that the derived unit and counit are weak equivalences on the generators. It follows that this adjunction is a Quillen equivalence. □

We combine the previous three propositions to compare $S^\bullet$–mod and a new model category of $S^\bullet_{\mathrm{top}}$–mod, where $S^\bullet_{\mathrm{top}}$ is defined by:

Definition 3.3.4 We define $S^\bullet_{\mathrm{top}}$ to be the $\mathscr{P}$–diagram of model categories and adjoint Quillen pairs

$$L_{\mathbb{S}_\mathbb{Q}}(DE\mathcal{F}^\mathbb{T}_+\text{–mod}) \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} L_{\{(\Sigma^* f)^\mathbb{T}\}} L_{\mathbb{S}_\mathbb{Q}}(DE\mathcal{F}^\mathbb{T}_+\text{–mod}) \underset{U}{\overset{DE\mathcal{F}^\mathbb{T}_+ \wedge -}{\leftrightarrows}} L_{\mathbb{S}_\mathbb{Q}}(\mathbb{T}\,\mathrm{Sp}),$$

where U denotes the forgetful functor.

By construction, the functor $(-)^\mathbb{T}$ induces a functor between $S^\bullet$–mod and $S^\bullet_{\mathrm{top}}$–mod. Since each of the components is a symmetric monoidal Quillen equivalence, we obtain the following from Lemma 3.1.4:

Theorem 3.3.5 *The adjunction*

$$\zeta_\#\colon S^\bullet_{\mathrm{top}}\text{–mod} \rightleftarrows S^\bullet\text{–mod} :(-)^\mathbb{T}$$

is a symmetric monoidal Quillen equivalence.

We now extend this Quillen equivalence to a cellularized version. Define $K^\mathbb{T}_{\mathrm{top}}$ to be the set of cells given by applying the derived functor of $(-)^\mathbb{T}$ to K_{top}. By the cellularization principle of [13, Proposition 2.7], we see that the Quillen equivalence above is preserved by cellularization.

Corollary 3.3.6 *The adjunction below is a Quillen equivalence:*

$$\zeta_\#\colon K^\mathbb{T}_{\mathrm{top}}\text{–cell–}S^\bullet_{\mathrm{top}}\text{–mod} \rightleftarrows K_{\mathrm{top}}\text{–cell–}S^\bullet\text{–mod} :(-)^\mathbb{T}.$$

As in the previous section, the above Quillen equivalence is symmetric monoidal, but for clarity we postpone the proof of that fact to Section 5.2.

The model category $K^\mathbb{T}_{\mathrm{top}}$–cell–$S^\bullet_{\mathrm{top}}$–mod is constructed from model categories of nonequivariant spectra. Hence we have removed the equivariance. The reward for

doing so is in the next section, where we can replace our categories based on spectra with categories based on rational chain complexes. Such categories are our first approximation to the algebraic model.

3.4 Passing to algebra

We will replace the model category $K^{\mathbb{T}}_{\mathrm{top}}$–cell–$S^{\bullet}_{\mathrm{top}}$–mod by a Quillen equivalent $\mathrm{Ch}(\mathbb{Q})$–model category. The results of [27] and the general theory of diagrams of model categories allow us to do so. To apply the work of [27], we must work with $H\mathbb{Q}$–modules in symmetric spectra. So we give two Quillen equivalences: the first moves us from orthogonal spectra to symmetric spectra, the second from symmetric spectra to $H\mathbb{Q}$–modules.

In more detail, recall $\mathbb{U}$, the forgetful functor from orthogonal spectra (in based topological spaces) to symmetric spectra (in based simplicial sets) and call $\mathbb{P}$ its left adjoint. Define $\mathbb{U}S^{\bullet}_{\mathrm{top}}$ to be the $\mathscr{P}$–diagram of model categories

$$L_{\mathbb{S}_{\mathbb{Q}}}(\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_{+}\text{–mod}) \underset{\text{Id}}{\overset{\text{Id}}{\rightleftarrows}} L_{\{\mathbb{U}(\Sigma^* f)^{\mathbb{T}}\}}L_{\mathbb{S}_{\mathbb{Q}}}(\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_{+}\text{–mod}) \underset{U}{\overset{\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_{+}\wedge -}{\longleftrightarrow}} L_{\mathbb{S}_{\mathbb{Q}}}\mathrm{Sp}^{\Sigma}.$$

The functor $\mathbb{U}$ preserves all weak equivalences, so we do not need to apply fibrant replacement when constructing the set $\mathbb{U}(\Sigma^* f)^{\mathbb{T}}$ and the commutative ring spectrum $\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_{+}$.

Proposition 3.4.1 *The adjunction*

$$\mathbb{U}^{\bullet}\colon S^{\bullet}_{\mathrm{top}}\text{–mod} \leftrightarrows \mathbb{U}S^{\bullet}_{\mathrm{top}}\text{–mod} :\mathbb{P}^{\bullet}$$

is a strong symmetric monoidal Quillen equivalence.

Proof The adjunction $(\mathbb{P}, \mathbb{U})$ is a Quillen equivalence between $L_{\mathbb{S}_{\mathbb{Q}}}\,\mathrm{Sp}$ and $L_{S_{\mathbb{Q}}}\mathrm{Sp}^{\Sigma}$. Furthermore the left adjoint is strong symmetric monoidal, so the result follows by Lemma 3.1.4. □

The second step is to pass from symmetric spectra to $H\mathbb{Q}$–modules using the adjunction $(H\mathbb{Q}\wedge -, U)$. This is a Quillen equivalence between $L_{\mathbb{S}_{\mathbb{Q}}}\mathrm{Sp}^{\Sigma}$ and $H\mathbb{Q}$–mod, and the left adjoint is strong symmetric monoidal. Thus by the same argument as above we get the following:

Proposition 3.4.2 *The adjunction*

$$H\mathbb{Q}\wedge -^{\bullet}\colon \mathbb{U}S^{\bullet}_{\mathrm{top}}\text{–mod} \rightleftarrows H\mathbb{Q}\wedge\mathbb{U}S^{\bullet}_{\mathrm{top}}\text{–mod} :U^{\bullet}$$

is a strong symmetric monoidal Quillen equivalence, where $H\mathbb{Q}\wedge\mathbb{U}S^\bullet_{\mathrm{top}}$ denotes the following diagram of model categories:

$$H\mathbb{Q}\wedge\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_+\text{–mod}\underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}}L_{\{H\mathbb{Q}\wedge\mathbb{U}(\Sigma^*f)^{\mathbb{T}}\}}(H\mathbb{Q}\wedge\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_+\text{–mod})\underset{U}{\overset{\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_+\wedge-}{\longleftarrow}}H\mathbb{Q}\text{–mod}.$$

Here $H\mathbb{Q}\wedge\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_+$ denotes first the cofibrant replacement in the model category of commutative ring spectra and then application of $H\mathbb{Q}\wedge-$.

Now we are ready to use the results from [27] to move from topology to algebra on $\mathscr{P}$–diagrams. Let Θ be the derived functor described in [27, Section 2.2]. This functor Θ induces an equivalence between $H\mathbb{Q}$–modules and rational chain complexes.

Definition 3.4.3 By [27, Theorem 1.2] there is a commutative rational differential graded algebra $\widehat{S}_t$, which is naturally weakly equivalent to $\Theta(H\mathbb{Q}\wedge\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_+)$, such that the model category of $\widehat{S}_t$–modules (in $\mathrm{Ch}(\mathbb{Q})$) is Quillen equivalent to the model category of $H\mathbb{Q}\wedge\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_+$–modules (in spectra).

Remark 3.4.4 It is essential for the formality argument in Section 4.1 that the ring spectrum $H\mathbb{Q}\wedge\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_+$ is commutative. Without this, one is unable to replace the ring $\widehat{S}_t$ by the simpler ring $\mathcal{O}_{\mathcal{F}}$, nor can one understand the localising set A'' (defined in the next section) in terms of the inclusion $\mathcal{O}_{\mathcal{F}}\to\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$.

Let $S^\bullet_t$ be the $\mathscr{P}$–diagram of model categories below, where $\Theta(H\mathbb{Q}\wedge\mathbb{U}(\Sigma^*f)^{\mathbb{T}})$ denotes the image of the set of maps $H\mathbb{Q}\wedge\mathbb{U}(\Sigma^*f)^{\mathbb{T}}$ in the category of $\widehat{S}_t$–modules under the derived functor:

$$\widehat{S}_t\text{–mod}\underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}}L_{\Theta(H\mathbb{Q}\wedge\mathbb{U}(\Sigma^*f)^{\mathbb{T}})}(\widehat{S}_t\text{–mod})\underset{U}{\overset{\widehat{S}_t\otimes-}{\longleftarrow}}\mathrm{Ch}(\mathbb{Q}).$$

Proposition 3.4.5 *There is a zigzag of symmetric monoidal Quillen equivalences*

$$H\mathbb{Q}\wedge\mathbb{U}S^\bullet_{\mathrm{top}}\text{–mod}\simeq S^\bullet_t\text{–mod}.$$

Proof There is a zigzag of symmetric monoidal adjunctions between $H\mathbb{Q}$–modules and $\mathrm{Ch}(\mathbb{Q})$. By [27, Corollary 2.15], this zigzag consists of Quillen equivalences. We can extend this zigzag from $H\mathbb{Q}$–modules to $H\mathbb{Q}\wedge\mathbb{U}DE\mathcal{F}^{\mathbb{T}}_+$–modules in a natural way.

We can extend further to diagrams of model categories. Thus we obtain a zigzag of adjunctions between $H\mathbb{Q}\wedge\mathbb{U}S^\bullet_{\mathrm{top}}$–mod and $S^\bullet_t$–mod. At each stage, we have localized the middle category of the diagram at the derived image (ie image under the derived functor) of the set of maps $\{H\mathbb{Q}\wedge\mathbb{U}(\Sigma^*f)^{\mathbb{T}}\}$. We apply Lemma 3.1.4 to see that we have a symmetric monoidal Quillen equivalence, as claimed. □

Corollary 3.4.6 *Denote the derived images (ie images under the derived functor) of the cells* $K^{\mathbb{T}}_{\mathrm{top}}$ *in* $S^{\bullet}_t$*–*mod *by* K_t*. Then there is a zigzag of Quillen equivalences*

$$K^{\mathbb{T}}_{\mathrm{top}}\text{–cell–}S^{\bullet}_{\mathrm{top}}\text{–mod} \simeq K_t\text{–cell–}S^{\bullet}_t\text{–mod}.$$

Since cellularization is compatible with Quillen equivalences, all Quillen equivalences presented above are still Quillen equivalences after cellularizing at the derived images of the cells from the set $K^{\mathbb{T}}_{\mathrm{top}}$. By the discussion in Sections 5.1 and 5.2, the above zigzag consists of symmetric monoidal Quillen equivalences.

4 Simplifying the algebraic category

We have shown so far that the category of rational $\mathbb{T}$–spectra has an algebraic model of the form K_t–cell–$S^{\bullet}_t$–mod. However, since this category is not well understood, in this section we perform several steps to obtain a more concrete and easier algebraic model.

4.1 Removing the localization

In this section we have two tasks: replace the commutative dga $\widehat{S}_t$ of Definition 3.4.3 by something simpler and remove the localization of the middle model category, $L_{\Theta(H\mathbb{Q}\wedge\mathbb{U}(\Sigma^* f)^{\mathbb{T}})}(\widehat{S}_t\text{–mod})$.

The main idea is to use a formality argument, similar to the one in [16, Section 10]. However, the important difference lies in adapting the formality argument to one for *modules* over a commutative dga. This is enough to simplify the middle model category in $S^{\bullet}_t$.

The construction of Θ comes with an isomorphism between $H_*(\Theta X)$ and $\pi_*(X)$ for any $H\mathbb{Q}$–module X. It follows that the homology of $\widehat{S}_t$ is determined by the rational homotopy groups of $DE\mathcal{F}^{\mathbb{T}}_+$. We prove that the homology of $\widehat{S}_t \simeq \Theta(H\mathbb{Q}\wedge\mathbb{U}\,DE\mathcal{F}^{\mathbb{T}}_+)$ is so well-structured that $\widehat{S}_t$ is quasi-isomorphic to its homology. We then use this to understand the set of maps $A = \Theta(H\mathbb{Q}\wedge\mathbb{U}(\Sigma^* f)^{\mathbb{T}})$.

Recall that $\mathcal{O}_{\mathcal{F}}$ is the graded ring $\prod_{n\geqslant 1}\mathbb{Q}[c_n]$ with each c_n of degree -2, and $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is the colimit over n of $\mathcal{O}_{\mathcal{F}}[c_1^{-1},\dots,c_n^{-1}]$; see Section 2.1.

Lemma 4.1.1 *We have isomorphisms of graded rings*

$$\begin{aligned} H_*(\widehat{S}_t) &\cong H_*\big(\Theta(H\mathbb{Q}\wedge\mathbb{U}(DE\mathcal{F}^{\mathbb{T}}_+))\big) \cong \pi_*(H\mathbb{Q}\wedge\mathbb{U}(DE\mathcal{F}^{\mathbb{T}}_+)) \\ &\cong \pi_*(DE\mathcal{F}^{\mathbb{T}}_+)\otimes\mathbb{Q} \cong \pi^{\mathbb{T}}_*(DE\mathcal{F}_+)\otimes\mathbb{Q} \cong \mathcal{O}_{\mathcal{F}}, \end{aligned}$$

where the last isomorphism comes from [8].

Note that for the step $\pi_*(DE\mathcal{F}_+^{\mathbb{T}}) \otimes \mathbb{Q} \cong \pi_*^{\mathbb{T}}(DE\mathcal{F}_+) \otimes \mathbb{Q}$ we require $DE\mathcal{F}_+$ to be a (positive) fibrant spectrum.

We want to create a zigzag of quasi-isomorphisms between $\widehat{S}_t$ and $\mathcal{O}_{\mathcal{F}}$. For each $n \geqslant 1$ there is a cycle x_n inside $\widehat{S}_t$ which represents e_n (projection onto factor n) in homology. It follows that the homology of $\widehat{S}_t[(x_n)^{-1}]$ is equal to e_n applied to the homology of $\widehat{S}_t$. Note that for this argument to hold, we need to know that $\widehat{S}_t$ is a commutative dga, which requires that $DE\mathcal{F}_+$ be a commutative ring object in $\mathbb{T}$–spectra.

Define $\widetilde{S}_t = \prod_{n\geqslant 1} \widehat{S}_t[x_n^{-1}]$. There is a canonical map $\alpha\colon \widehat{S}_t \to \widetilde{S}_t$, which is a homology isomorphism. For each $n \geqslant 1$, pick a representative a_n in $\widehat{S}_t[x_n^{-1}]$ for the homology class of c_n. We thus have a map $\mathbb{Q}[c_n] \to \widehat{S}_t[x_n^{-1}]$ which sends c_n to a_n. Define $\beta\colon \mathcal{O}_{\mathcal{F}} \to \widetilde{S}_t$ as the product over n of these maps. We now have our zigzag of quasi-isomorphisms.

Let A' be the image of the set A under (derived) extension of scalars along α. Define a new $\mathscr{P}$–diagram of model categories, $\widetilde{S}_t^{\bullet}$, as

$$\widetilde{S}_t\text{–mod} \underset{\text{Id}}{\overset{\text{Id}}{\rightleftarrows}} L_{A'}(\widetilde{S}_t\text{–mod}) \underset{U}{\overset{\widetilde{S}_t\otimes -}{\leftrightarrows}} \text{Ch}(\mathbb{Q}).$$

Extension and restriction of scalars along $\alpha\colon \widehat{S}_t \to \widetilde{S}_t$ induce a symmetric monoidal Quillen equivalence between $\widehat{S}_t^{\bullet}$–mod and $\widetilde{S}_t^{\bullet}$–mod.

We repeat this construction once more using β. Let A'' be the image of the set A' under restriction of scalars along β. Define a new diagram of model categories, $\widetilde{S}_a^{\bullet}$, as

$$\mathcal{O}_{\mathcal{F}}\text{–mod} \underset{\text{Id}}{\overset{\text{Id}}{\rightleftarrows}} L_{A''}(\mathcal{O}_{\mathcal{F}}\text{–mod}) \underset{U}{\overset{\mathcal{O}_{\mathcal{F}}\otimes -}{\leftrightarrows}} \text{Ch}(\mathbb{Q}).$$

Extension and restriction of scalars along $\beta\colon \mathcal{O}_{\mathcal{F}} \to \widetilde{S}_t$ induce a symmetric monoidal Quillen equivalence between $\widetilde{S}_a^{\bullet}$–mod and $\widetilde{S}_t^{\bullet}$–mod.

We summarize these results in the following:

Proposition 4.1.2 *The adjoint pairs of extension and restriction of scalars along α and β induce symmetric monoidal Quillen equivalences*

$$S_t^{\bullet}\text{–mod} \simeq \widetilde{S}_t^{\bullet}\text{–mod} \simeq \widetilde{S}_a^{\bullet}\text{–mod}.$$

Let $K_{\tilde{t}}$ be the derived images of the cells K_t in $\widetilde{S}_t^{\bullet}$–mod and let $K_{\tilde{a}}$ be the derived images in $\widetilde{S}_a^{\bullet}$–mod. Then we have Quillen equivalences between K_t–cell–$S_t^{\bullet}$–mod, $K_{\tilde{t}}$–cell–$\widetilde{S}_t^{\bullet}$–mod and $K_{\tilde{a}}$–cell–$\widetilde{S}_a^{\bullet}$–mod.

Our next task is to understand the set of maps in A'' so that we can remove the localization in the middle model category in the diagram of model categories $\widetilde{S}_a^\bullet$. We show that there is a zigzag of homology isomorphisms between

$$\Theta(\mathbb{U}\,DE\mathcal{F}_+^{\mathbb{T}}) \to \Theta(\mathbb{U}(\widetilde{E}\mathcal{F}\wedge DE\mathcal{F}_+)^{\mathbb{T}}) \quad \text{and} \quad j\colon \mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}.$$

It will follow that we can replace the set A'' by the set of all shifts of j without changing the effect of the localization. That is, we will show that the model categories $L_{A''}(\mathcal{O}_{\mathcal{F}}\text{–mod})$ and $L_{\Sigma^* j}(\mathcal{O}_{\mathcal{F}}\text{–mod})$ are equal.

The zigzag of homology isomorphisms of $\mathcal{O}_{\mathcal{F}}$ modules that we will use is as follows. Factor $\Theta(\mathbb{U}\,DE\mathcal{F}_+^{\mathbb{T}}) \to \Theta(\mathbb{U}(\widetilde{E}\mathcal{F}\wedge DE\mathcal{F}_+)^{\mathbb{T}})$ into a cofibration followed by an acyclic fibration (with intermediate term R). Let C be the pushout of the top square below:

$$\begin{array}{ccccc}
\Theta(\mathbb{U}\,DE\mathcal{F}_+^{\mathbb{T}}) & \rightarrowtail & R & \xrightarrow{\simeq} & \Theta(\mathbb{U}(\widetilde{E}\mathcal{F}\wedge DE\mathcal{F}_+)^{\mathbb{T}}) \\
\wr\downarrow & & \downarrow\wr & & \\
\widetilde{S}_t & \xrightarrow{f} & C & & \\
\wr\uparrow & & \uparrow{\scriptstyle \mathrm{Id}} & & \\
\mathcal{O}_{\mathcal{F}} & \xrightarrow{a} & C & & \\
{\scriptstyle \mathrm{Id}}\downarrow & & \downarrow & & \\
\mathcal{O}_{\mathcal{F}} & \longrightarrow & \mathcal{E}^{-1}C & & \\
{\scriptstyle \mathrm{Id}}\uparrow & & \uparrow{\scriptstyle \mathcal{E}^{-1}a} & & \\
\mathcal{O}_{\mathcal{F}} & \xrightarrow{j} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} & &
\end{array}$$

Since $\mathcal{O}_{\mathcal{F}}$–mod is left proper it follows that $R \to C$ is a quasi-isomorphism. The functor defined by $M \mapsto \mathcal{E}^{-1}M$ on $\mathcal{O}_{\mathcal{F}}$–modules M is exact. It follows that $C \to \mathcal{E}^{-1}C$ is a homology isomorphism, since $\mathcal{E}^{-1}$ is already inverted on homology. The map f induces a homology isomorphism once $\mathcal{E}$ has been inverted, hence so does a. It follows that $\mathcal{E}^{-1}a$ is a homology isomorphism.

Thus we have shown that model categories $L_{A''}(\mathcal{O}_{\mathcal{F}}\text{–mod})$ and $L_{\Sigma^* j}(\mathcal{O}_{\mathcal{F}}\text{–mod})$ are equal. Now we are ready to remove the localization altogether.

Lemma 4.1.3 *The adjunction induced by the inclusion of rings $j\colon \mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ induces a symmetric monoidal Quillen equivalence*

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} -\colon L_{\Sigma^* j}(\mathcal{O}_{\mathcal{F}}\text{–mod}) \rightleftarrows \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\text{–mod} : j^*.$$

Proof The cofibrations are unchanged by localization. The weak equivalences of the model category $L_{\Sigma^* j}(\mathcal{O}_{\mathcal{F}}\text{–mod})$ are those maps f such that

$$H_*(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} f) = \mathcal{E}^{-1}H_*(f)$$

is an isomorphism. The left adjoint preserves (and detects) these new weak equivalences, so we have a symmetric monoidal Quillen adjunction as claimed. The object $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is a homotopically small generator for (the homotopy category of) $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$–mod. If we can show that the derived counit of this adjunction is a weak equivalence then it will follow that we have a Quillen equivalence. This follows since the counit map is an isomorphism on the generator,

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}. \qquad \square$$

We use the above result to remove the localization from the middle term in our diagram of model categories. We have a commuting diagram of model categories as below, where U denotes the forgetful functor:

$$\begin{array}{ccccc}
\mathcal{O}_{\mathcal{F}}\text{–mod} & \underset{\text{Id}}{\overset{\text{Id}}{\rightleftarrows}} & L_{\Sigma^* j}(\mathcal{O}_{\mathcal{F}}\text{–mod}) & \underset{U}{\overset{\mathcal{O}_{\mathcal{F}}\otimes -}{\rightleftarrows}} & \mathbb{Q}\text{–mod} \\
\Big\uparrow \text{Id} \;\; \Big\downarrow \text{Id} & & \Big\uparrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes_{\mathcal{O}_{\mathcal{F}}} - \;\; \Big\downarrow j^* & & \Big\uparrow \text{Id} \;\; \Big\downarrow \text{Id} \\
\mathcal{O}_{\mathcal{F}}\text{–mod} & \underset{j^*}{\overset{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes_{\mathcal{O}_{\mathcal{F}}} -}{\rightleftarrows}} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\text{–mod} & \underset{U}{\overset{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes -}{\rightleftarrows}} & \mathbb{Q}\text{–mod}
\end{array}$$

We denote the bottom row by $S_a^{\bullet}$, the left adjoint from top to bottom by $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} -$, the right adjoint by j^* and we summarize the above in the following:

Proposition 4.1.4 *The adjunction (described above)*

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} -\colon \widetilde{S}_a^{\bullet}\text{–mod} \rightleftarrows S_a^{\bullet}\text{–mod} : j^*$$

is a symmetric monoidal Quillen equivalence, and thus the adjunction

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} -\colon K_{\widetilde{a}}\text{–cell–}\widetilde{S}_a^{\bullet}\text{–mod} \rightleftarrows K_a\text{–cell–}S_a^{\bullet}\text{–mod} : j^*$$

is a Quillen equivalence, where K_a is the derived image of $K_{\widetilde{a}}$ under the left adjoint.

Again the adjunction at the level of cellularized categories is a symmetric monoidal Quillen equivalence, by discussion in Section 5.2.

4.2 Removing the cellularization

We now compare K_a–cell–$S_a^{\bullet}$–mod and the algebraic model $d\mathcal{A}_{\text{dual}}$ of Section 2. The point is to move from a category whose weak equivalences are quite complicated to define to a model category whose weak equivalences are the quasi-isomorphisms. The idea behind this step is similar to one presented in [16, Sections 12 and 13].

We first introduce an adjoint pair relating $S^\bullet_a$–mod and $d\widehat{\mathcal{A}}$. An object

$$\beta\colon M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$$

of $d\widehat{\mathcal{A}}$ gives an object of $S^\bullet_a$–mod defined by

$$(M,\ \mathcal{E}^{-1}\beta,\ \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V,\ \mathrm{Id},\ V).$$

This functor, which we call $\varkappa$, includes $d\widehat{\mathcal{A}}$ into $S^\bullet_a$–mod. It has a right adjoint Γ_v. Let $(a, \alpha, b, \gamma, c)$ be an object of $S^\bullet_a$–mod. Then we can draw the diagram of $\mathcal{O}_{\mathcal{F}}$–modules

$$a \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} a \to b \leftarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes c.$$

If we take the pullback P of this in the category of $\mathcal{O}_{\mathcal{F}}$–modules in $\mathrm{Ch}(\mathbb{Q})$ we obtain a map $\delta\colon P \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes c$. This map δ is an object of $d\widehat{\mathcal{A}}$. For more details see [11, Section 7]. We call this adjoint pair $(\varkappa, \Gamma_v)$ and we note that it is a strong symmetric monoidal adjunction.

We can compose this adjunction with the adjunction (ι, Γ_h) which relates $d\widehat{\mathcal{A}}$ to $d\mathcal{A}$ (see Section 2.2). We let $\mu = \varkappa \circ \iota$ and $\Gamma = \Gamma_h \circ \Gamma_v$.

Lemma 4.2.1 *The adjunction (μ, Γ) between the categories $d\mathcal{A}$ and $S^\bullet_a$–mod is symmetric monoidal.*

This adjunction is also studied in [4, Section 7], where it is called (inc, Γ) and $S^\bullet_a$ is called $R^\bullet_a$.

Recall that, up to a weak equivalence (and ignoring shifts), the cells K_{top} consist of objects of the form

$$S^\bullet \wedge k = (k \wedge DE\mathcal{F}_+ \to k \wedge DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F} \leftarrow k \wedge \widetilde{E}\mathcal{F}),$$

where $k \in K$, ie $k = \mathbb{S}$ or $k = \sigma_n$ for $n > 1$ (see Section 3.2).

Thus we have to calculate the cells in K_a, ie the derived images of cells from K (or equivalently from K_{top}) in $S^\bullet_a$–mod. Since all required Quillen equivalences are symmetric monoidal (which follows from Section 5), they preserve the unit (up to weak equivalence) and the unit is always cellular. So the derived image of $\mathbb{S} \in L_{\mathbb{S}_{\mathbb{Q}}}\mathbb{T}\,\mathrm{Sp}$ is the unit in $S^\bullet_a$–mod,

$$\mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \leftarrow \mathbb{Q}.$$

We will use the simplified notation S^0 for this object. As for the other cells, consider some $S^\bullet \wedge \sigma_n \in K_{\mathrm{top}}$. Let $k_n = (A \to B \leftarrow C)$ be its derived image in $S^\bullet_a$–mod. To recap this process, one takes homotopy $\mathbb{T}$–fixed points of $S^\bullet \wedge \sigma_n$ to get an object of $K^{\mathbb{T}}_{\mathrm{top}}$ and then one applies the derived functor Θ from [27], to get an object of K_t.

Finally, one applies a number of algebraic adjunctions from Section 4.1 to get the object k_n of K_a. All of these adjunctions are constructed by taking Quillen equivalences (which preserve the unit up to weak equivalence) on each of the component categories. It follows that we have isomorphisms

$$H_*(A) = [\mathcal{O}_{\mathcal{F}}, A]_*^{\mathcal{O}_{\mathcal{F}}\text{–mod}} \cong [DE\mathcal{F}_+, DE\mathcal{F}_+ \wedge \sigma_n]_*^{DE\mathcal{F}_+\text{–mod}} \cong [\mathbb{S}, DE\mathcal{F}_+ \wedge \sigma_n]_*^{\mathbb{T}}.$$

Similar isomorphisms also hold for the other two components so, by the calculations of [8, Example 5.8.1], we have

$$\begin{aligned} H_*(A) &= \pi_*^{\mathbb{T}}(DE\mathcal{F}_+ \wedge \sigma_n) = \mathbb{Q}_n\langle 1\rangle, \\ H_*(B) &= \pi_*^{\mathbb{T}}(DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F} \wedge \sigma_n) = 0, \\ H_*(C) &= \pi_*^{\mathbb{T}}(\widetilde{E}\mathcal{F} \wedge \sigma_n) = 0, \end{aligned}$$

where $\mathbb{Q}_n\langle 1\rangle$ is the torsion $\mathcal{O}_{\mathcal{F}}$–module consisting of a copy of $\mathbb{Q}$ in factor n and degree 1. It is immediate that there is a homology isomorphism

$$\widetilde{\sigma}_n = (\mathbb{Q}_n\langle 1\rangle \to 0 \leftarrow 0) \to (A \to B \leftarrow C) = k_n$$

given by simply picking a suitable representative cycle for $1 \in \mathbb{Q}_n\langle 1\rangle$. We therefore have the following description of the cells:

Lemma 4.2.2 *The set of cells K_a is given (up to weak equivalence) by all shifts of objects of the form $\widetilde{\sigma}_n$ for $n \geqslant 1$ and all shifts of $S^0 = (\mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \leftarrow \mathbb{Q})$.*

The above argument on the behaviour of the derived adjunction extends to the following useful result, which tells us that (after applying homology) our derived functors agree with the functor $\pi_*^{\mathcal{A}}$ of [8].

Theorem 4.2.3 *Let X be a rational $\mathbb{T}$–equivariant spectrum. Let ΥX be its derived image in $S_a^{\bullet}$–mod. Then $H_*(\Upsilon X) \cong \mu\pi_*^{\mathcal{A}}(X)$.*

The adjunction (μ, Γ) is shown to be a symmetric monoidal Quillen equivalence between $d\mathcal{A}$ with the dualizable model structure and a cellularization of $S_a^{\bullet}$–mod in [4, Theorem 7.6]. The cells for this cellularization are taken to be the "algebraic spheres". An algebraic sphere is an object of the form

$$S^{\nu} = (\mathcal{O}_{\mathcal{F}}(\nu) \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q} \leftarrow \mathbb{Q}),$$

where $\mathcal{O}_{\mathcal{F}}(\nu)$ is the subset of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ consisting of all those x such that $c^{\nu}x \in \mathcal{O}_{\mathcal{F}}$, for $\nu\colon \mathcal{F} \to \mathbb{Z}_{\geqslant 0}$ of finite support. We also allow negative spheres $S^{-\nu}$ and shifts of such objects. Essentially these are just "partial shifts" of the unit, where we have shifted

finitely many factors of $\mathcal{O}_{\mathcal{F}}$ by some varying amount. We let $\{S^\nu\}$ denote the set of such objects.

To show that (μ, Γ) is a Quillen equivalence between $d\mathcal{A}$ with the dualizable model structure and the cellularization of $S_a^\bullet$–mod at the set of cells K_a, we want to use [4, Theorem 7.6], which says that $d\mathcal{A}$ with the dualizable model structure is Quillen equivalent to the cellularization of $S_a^\bullet$–mod at the set of cells $\{S^\nu\}$. Hence, it is enough to show that these two cellularizations agree (that is, produce the same model structure). We will prove that the algebraic spheres can be built via cofibre sequences and coproducts in $S_a^\bullet$–mod from cells in K_a and vice versa. It will follow that the class of K_a–cellular objects equals the class of $\{S^\nu\}$–cellular objects. Hence we will see that the K_a–cellular equivalences and the $\{S^\nu\}$–cellular equivalences agree and that the model categories K_a–cell–$S_a^\bullet$–mod and $\{S^\nu\}$–cell–$S_a^\bullet$–mod are equal.

The unit S^0 (and all its suspensions) is in both sets: in K_a and in the set of "algebraic spheres". So consider the algebraic sphere S^{ν_1} for the function $\nu_1\colon \mathcal{F} \to \mathbb{Z}_{\geqslant 0}$ sending a trivial subgroup to 1 and all other subgroups to 0. There is a cofibre sequence (in $S_a^\bullet$–mod)

$$S^0 \to S^{\nu_1} \to \Sigma\sigma_1,$$

where Σ denotes the suspension. This shows that we can build S^{ν_1} from σ_1 and S^0 and that we can build σ_1 from algebraic spheres. We can also create the negative sphere $S^{-\nu_1}$ using the cofibre sequence

$$S^{-\nu_1} \to S^0 \to \Sigma^{-1}\sigma_1.$$

To build any algebraic sphere we apply the above argument repeatedly. Note that by the definition of an algebraic sphere we need only finitely many steps. Equally we can make all σ_i for $i \geqslant 1$ from the algebraic spheres.

By [4, Theorem 7.6] we have the following:

Proposition 4.2.4 *The pair (μ, Γ) induces a symmetric monoidal Quillen equivalence between the model categories $d\mathcal{A}_{\mathrm{dual}}$ and K_a–cell–$S_a^\bullet$–mod.*

This finishes the proof that $d\mathcal{A}_{\mathrm{dual}}$ provides an algebraic model for the category of rational $\mathbb{T}$–spectra. We leave the consideration that all our Quillen equivalences are in fact symmetric monoidal to the last section.

5 Symmetric monoidal equivalences

All of the adjunctions in the zigzag between $d\mathcal{A}_{\mathrm{dual}}$ and $\mathbb{T}\,\mathrm{Sp}$ have been compatible with the monoidal properties of the categories. By examining the cellularized model

structures more clearly we are able to show that each of these model categories is a proper, stable, cellular, monoidal model category that satisfies the monoid axiom. We are thus able to conclude that this zigzag of Quillen equivalences consists of *monoidal* Quillen equivalences. It follows that we also have Quillen equivalences of model categories of ring objects and modules over ring objects.

Our method is to prove a monoidal version of the cellularization principle [13, Proposition 2.7]; see Propositions 5.1.6 and 5.1.7.

5.1 Cellularization of stable model categories

A cellularization of a model category is a right Bousfield localization at a set of objects. Such a localization exists by [17, Theorem 5.1.1] whenever the model category is right proper and cellular. When we are in a stable context the results of [6] can be used.

Those results, which we shall introduce in the next subsection, allow us to understand the sets of generating cofibrations for our cellularized model categories and see that they are all symmetric monoidal and cellular.

In this subsection we recall the notion of cellularization (when $\mathcal{C}$ is stable) and some of basic definitions and results.

Definition 5.1.1 Let $\mathcal{C}$ be a stable model category and K a stable set of objects of $\mathcal{C}$, ie a class of K–cellular objects of $\mathcal{C}$ that is closed under desuspension.[2] We say that a map $f\colon A \to B$ of $\mathcal{C}$ is a *K–cellular equivalence* if the induced map

$$[k, f]^{\mathcal{C}}_*\colon [k, A]^{\mathcal{C}}_* \to [k, B]^{\mathcal{C}}_*$$

is an isomorphism of graded abelian groups for each $k \in K$. An object $Z \in \mathcal{C}$ is said to be *K–cellular* if

$$[Z, f]^{\mathcal{C}}_*\colon [Z, A]^{\mathcal{C}}_* \to [Z, B]^{\mathcal{C}}_*$$

is an isomorphism of graded abelian groups for any K–cellular equivalence f.

Definition 5.1.2 A *right Bousfield localization* or *cellularization* of $\mathcal{C}$ with respect to a set of objects K is a model structure K–cell–$\mathcal{C}$ on $\mathcal{C}$ such that

- the weak equivalences are Kcellular equivalences;
- the fibrations of K–cell–$\mathcal{C}$ are the fibrations of $\mathcal{C}$;
- the cofibrations of K–cell–$\mathcal{C}$ are defined via left lifting property.

[2] Note that the class is always closed under suspension.

By [17, Theorem 5.1.1], if $\mathcal{C}$ is a right proper, cellular model category and K a set of objects in $\mathcal{C}$, then the cellularization K–cell–$\mathcal{C}$ of $\mathcal{C}$ with respect to K exists and is a right proper model category. The cofibrant objects of K–cell–$\mathcal{C}$ are called *K–cofibrant* and are precisely the K–cellular and cofibrant objects of $\mathcal{C}$.

We recall some definitions and results from [6] and prove our monoidal version of the cellularization principle. We use $\widehat{c}_K$ for a cofibrant replacement functor in K–cell–$\mathcal{C}$.

Definition 5.1.3 Let K be a set of cofibrant objects in a monoidal model category $\mathcal{C}$. We say that K is *monoidal* if the following two conditions hold:

- Any object of the form $k \otimes k'$ for k, $k' \in K$ is K–cellular.
- For $\widehat{c}_K S_{\mathcal{C}}$ a K–cofibrant replacement of the unit $S_{\mathcal{C}}$ of $\mathcal{C}$ and any $k \in K$, the map $\widehat{c}_K S_{\mathcal{C}} \otimes k \to k$ is a K–cellular equivalence.

The cellularization of a right proper, cellular, stable model category at a stable set of cofibrant objects K is very well behaved (see [6, Theorem 5.9]), in particular it is proper, cellular and stable. Moreover, the second condition of the above definition holds automatically when the unit of $\mathcal{C}$ is K–cellular.

There is another important property we will often want the cells to satisfy, which makes right localization behave in an even more tractable manner; see [6, Section 9]. This property is variously called small, compact or finite. We choose to call it homotopically small to avoid those over-used terms.

Definition 5.1.4 We say that an object X of a stable model category $\mathcal{C}$ is *homotopically small* if, in the homotopy category, $\big[X, \coprod_i Y_i\big]^{\mathcal{C}}$ is canonically isomorphic to $\bigoplus_i [X, Y_i]^{\mathcal{C}}$; see [25, Definition 2.1.2].

Using [25, Lemma 2.2.1] it is routine to check that if K consists of homotopically small objects of $\mathcal{C}$ then K is a set of generators for K–cell–$\mathcal{C}$. Hence we know a set of generators for each of our cellularizations.

Notice that derived functors of both left and right Quillen equivalences preserve homotopically small objects. Now we may turn to monoidal considerations. The following theorem is [6, Theorem 7.2]:

Theorem 5.1.5 *Let $\mathcal{C}$ be a proper, monoidal, cellular, stable model category. Let K be a monoidal and stable set of cofibrant objects of $\mathcal{C}$. Then K–cell–$\mathcal{C}$ is a proper, monoidal, cellular, stable model category. Furthermore, if $\mathcal{C}$ satisfies the monoid axiom then so does K–cell–$\mathcal{C}$.*

The next two results are our upgraded version of the cellularization principle; see [13, Proposition 2.7]. They have slightly different assumptions according to whether the given cells are on the left or right of the adjunction. The first has the cells on the left and behaves as expected. The second starts with cells on the right of the adjunction and here we need to assume that the adjunction is a Quillen equivalence to start with. In both cases we have also assumed that a cofibrant replacement of the unit is in the set of cells (and hence is homotopically small). This simplifies the proofs but is not needed when the adjunction is already a Quillen equivalence.

For the following we let $\widehat{c}$ be the cofibrant replacement functor of $\mathcal{C}$, let $\widehat{c}_K$ be the cofibrant replacement functor of K–cell–$\mathcal{C}$ and let $\widehat{f}$ be the fibrant replacement functor of $\mathcal{D}$.

Proposition 5.1.6 *Consider a symmetric monoidal Quillen adjunction between a pair of proper, cellular, stable, monoidal model categories,*

$$L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R.$$

Let K be a stable and monoidal set of cofibrant objects of $\mathcal{C}$ which contains a cofibrant replacement of the unit. Assume that each element of K and LK is homotopically small and that the unit map $k \to R\widehat{f}Lk$ is a weak equivalence of $\mathcal{C}$ for each $k \in K$. Then LK is a stable monoidal set of cofibrant objects of $\mathcal{D}$ and the unit of $\mathcal{D}$ is in LK (up to weak equivalence). Moreover, we have an induced symmetric monoidal Quillen equivalence

$$L\colon K\text{–cell–}\mathcal{C} \rightleftarrows LK\text{–cell–}\mathcal{D} : R.$$

Proof We apply the cellularization principle [13, Proposition 2.7] to see that (L, R) is a Quillen equivalence on the cellularized categories.

We must show that LK satisfies both parts of the definition of a monoidal set. For the first part, let k and k' be objects of K. Then $Lk \wedge Lk'$ is weakly equivalent to $L(k \wedge k')$, which is LK–cofibrant and hence is LK–cellular. For the second part, the map $L(\widehat{c}S_{\mathcal{C}}) \to S_{\mathcal{D}}$ is a weak equivalence since (L, R) is a monoidal Quillen pair. Hence $S_{\mathcal{D}}$ is in LK (up to weak equivalence) and the second condition holds automatically.

Now we know that LK–cell–$\mathcal{D}$ is a cellular monoidal model category. We must show that (L, R) is a symmetric monoidal Quillen adjunction on the cellularized model categories. We know that the map $L(\widehat{c}S_{\mathcal{C}}) \to S_{\mathcal{D}}$ is a weak equivalence. The comonoidal map $L(X \wedge Y) \to LX \wedge LY$ is also a weak equivalence for any cofibrant X and Y. Hence the proof is complete. □

Proposition 5.1.7 *Consider a symmetric monoidal Quillen equivalence between a pair of proper, cellular, stable, monoidal model categories*

$$L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R.$$

Let H be a stable and monoidal set of cofibrant objects of $\mathcal{D}$ which contains a cofibrant replacement of the unit of $\mathcal{D}$. Assume that every element of H is homotopically small. Then $\hat{c}R\hat{f}H$ is a stable monoidal set of homotopically small cofibrant objects of $\mathcal{C}$ which contains the unit up to weak equivalence. Furthermore we have an induced symmetric monoidal Quillen equivalence

$$L\colon \hat{c}R\hat{f}H\text{–cell–}\mathcal{C} \rightleftarrows H\text{–cell–}\mathcal{D} : R.$$

Proof We apply the cellularization principle [13, Proposition 2.7] to see that (L, R) is a Quillen equivalence on the cellularized categories. We must prove that $K = \hat{c}R\hat{f}H$ is a monoidal set and that the unit of $\mathcal{C}$ is in K (up to weak equivalence).

It is simple to check that L takes K–cellular equivalences between cofibrant objects to H–cellular equivalences. Now consider the pair of maps, for k and k' elements of K,

$$L\hat{c}_K(k \wedge k') \xrightarrow{Lq} L(k \wedge k') \xrightarrow{\upsilon} Lk \wedge Lk'$$

The map υ is the comonoidal map of L and hence is a weak equivalence as (L, R) is monoidal. Since the codomain of υ is H–cellular, so is the domain of υ. The map Lq is L applied to a K–cellular equivalence between cofibrant objects, hence it is a H–cellular equivalence. We have shown that Lq is a H–cellular equivalence between H–cellular objects of $\mathcal{D}$ and thus must be a weak equivalence. Since (L, R) is a Quillen equivalence before cellularization, q must be a weak equivalence of $\mathcal{C}$. Thus $k \wedge k'$ must be K–cellular.

To complete the proof that K is monoidal it will suffice to prove that $S_{\mathcal{C}}$ is K–cellular. Thus we now show that the unit of $\mathcal{C}$ is in K up to weak equivalence. Since (L, R) is a symmetric monoidal Quillen pair, the composite map

$$L\hat{c}S_{\mathcal{C}} \to LS_{\mathcal{C}} \to S_{\mathcal{D}} \to \hat{f}S_{\mathcal{D}}$$

is a weak equivalence. Hence the adjoint $\hat{c}S_{\mathcal{C}} \to R\hat{f}S_{\mathcal{D}}$ is a weak equivalence. Thus we see that $\hat{c}S_{\mathcal{C}}$ is in K up to weak equivalence. We have now shown that the set K is monoidal and that K–cell–$\mathcal{C}$ is a symmetric monoidal model category.

The proof that this adjunction is symmetric monoidal on the cellularized model categories follows the same pattern as the previous case. □

5.2 Application to the classification

We start with the Quillen equivalence of Proposition 3.2.5,

$$S^\bullet\wedge -\colon L_{\mathbb{S}_\mathbb{Q}}(\mathbb{T}\,\mathrm{Sp}) \rightleftarrows K_{\mathrm{top}}\text{–cell–}S^\bullet\text{–mod} \;:\mathrm{pb}.$$

The set of cells K_{top} is given by $S^\bullet\wedge -$ applied to the set K of generators of $L_{\mathbb{S}_\mathbb{Q}}(\mathbb{T}\,\mathrm{Sp})$. We know that this set is stable and every element is homotopically small and cofibrant. By the proof of Proposition 5.1.6, it also follows that K_{top} is a monoidal set. Thus we may apply Proposition 5.1.6 to see that the adjunction $(S^\bullet\wedge -, \mathrm{pb})$ is symmetric monoidal.

We then have a large number of symmetric monoidal Quillen equivalences relating $S^\bullet$–mod and $S^\bullet_a$–mod. Our initial set of cells K_{top} is monoidal, stable, contains the unit and every element is homotopically small. Hence Propositions 5.1.6 and 5.1.7 tell us that K_{top}–cell–$S^\bullet$–mod and K_a–cell–$S^\bullet_a$–mod are Quillen equivalent via symmetric monoidal Quillen equivalences.

Theorem 5.2.1 *The model category of rational $\mathbb{T}$–spectra, $\mathbb{T}\,\mathrm{Sp}$, is Quillen equivalent to the algebraic model $d\mathcal{A}_{\mathrm{dual}}$. Furthermore, these Quillen equivalences are all symmetric monoidal. Hence the homotopy categories of $\mathbb{T}\,\mathrm{Sp}$ and $d\mathcal{A}_{\mathrm{dual}}$ are equivalent as symmetric monoidal categories.*

Proof This now follows by combining Proposition 3.2.5, Corollaries 3.3.6 and 3.4.6, Section 4.1 and Proposition 4.2.4 with Propositions 5.1.7 and 5.1.6. □

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