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Two-complete stable motivic stems over finite fields

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Let ℓ be a prime and $q = p^{\nu}$, where p is a prime different from ℓ. We show that the ℓ–completion of the n^{th} stable homotopy group of spheres is a summand of the ℓ–completion of the $(n,0)$ motivic stable homotopy group of spheres over the finite field with q elements, $\mathbb{F}_q$. With this, and assisted by computer calculations, we are able to explicitly compute the two-complete stable motivic stems $\pi_{n,0}(\mathbb{F}_q)^{\wedge}_2$ for $0 \le n \le 18$ for all finite fields and $\pi_{19,0}(\mathbb{F}_q)^{\wedge}_2$ and $\pi_{20,0}(\mathbb{F}_q)^{\wedge}_2$ when $q \equiv 1 \bmod 4$ assuming Morel's connectivity theorem for $\mathbb{F}_q$ holds.

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1 Introduction

The homotopy groups of spheres belong to the most important and puzzling invariants in topology. See Kochman [28] and the more recent works of Isaksen [26] and Wang and Xu [51] for amazing computer-assisted ways of computing these invariants based on the Adams spectral sequence. The Adams spectral sequence of topology is a well-studied method to calculate the stable homotopy groups of spheres; see Adams [2] and Ravenel [40]. With two-primary coefficients, the second page of the Adams spectral sequence has a description in terms of Ext groups over the mod 2 Steenrod algebra

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}^*}(\mathbb{F}_2, \mathbb{F}_2)$$

and converges to the two-complete stable homotopy groups of spheres $(\pi_n^s)^{\wedge}_2$. Extensive computer calculations of these Ext groups have been carried out by Bruner in [8] and [10]. However, even if one knew completely the answer for the Ext groups in the Adams spectral sequence, one is still not finished with computing the stable homotopy groups of spheres. One needs to know in addition the differentials and all the group extensions hidden in the associated graded groups of the filtration. Only partial results have been obtained in spite of an enormous effort.

Given any field k the stable motivic homotopy category $\mathcal{SH}_k$ over k has the structure of a triangulated category and encodes both topological information and arithmetic

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information about k. An application of this framework is the proof of Milnor's conjecture on Galois cohomology given by Voevodsky [48]. Just as for the stable homotopy category $\mathcal{SH}$, it is an interesting and deep problem to compute the stable motivic homotopy groups of spheres $\pi_{s,w}(k)$ over k, that is, $\mathcal{SH}_k(\Sigma^{s,w}\mathbb{1},\mathbb{1})$, where $\mathbb{1}$ denotes the motivic sphere spectrum over k. When k has finite mod 2 cohomological dimension and $s \geq w \geq 0$, the motivic Adams spectral sequence (MASS) converges to the two-completion of the stable motivic stems:

$$E_2^{f,(s,w)} = \mathrm{Ext}_{\mathcal{A}^{**}}^{f,(s+f,w)}(H^{**},H^{**}) \Longrightarrow (\pi_{s,w}\mathbb{1})^{\wedge}_2.$$

This is a trigraded spectral sequence, where $\mathcal{A}^{**}$ is the bigraded mod 2 motivic Steenrod algebra (see the work of Hoyois, Kelly and Østvær [22] and Voevodsky [48]), and H^{**} is the bigraded mod 2 motivic cohomology ring of k. A construction of the motivic Adams spectral sequence is given in Section 5. The calculational challenges are (1) to identify the motivic Ext groups, (2) to determine the differentials and (3) to reconstruct the abutment from the filtration quotients.

Based on the MASS, Dugger and Isaksen [12] have carried out calculations of the two-complete stable motivic homotopy groups of spheres up to the 34 stem over the complex numbers. Isaksen [25; 26] has extended this work largely up to the 70 stem. We are led to wonder: how do the stable motivic homotopy groups vary for different base fields? Morel [34] has given a complete description of the 0–line $\pi_{n,n}(k)$ in terms of Milnor–Witt K–theory. The 1–line $\pi_{n+1,n}(k)$ is determined by Hermitian and Milnor K–theory groups by the work of Röndigs, Spitzweck and Østvær [41], which generalizes the partial results obtained by Ormsby and Østvær in [39]. Ormsby has investigated the case of related invariants over p–adic fields in [37] and the rationals in Ormsby and Østvær [38], and Dugger and Isaksen [13] have analyzed the case over the real numbers. It is now possible to perform similar calculations over fields of positive characteristic, thanks to work on the motivic Steenrod algebra in positive characteristic by Hoyois, Kelly and Østvær [22]. In this paper we use computer-assisted motivic Ext group calculations in tandem with theoretical arguments to determine stable motivic stems $\pi_{n,0}$ in weight zero over finite fields.

We now state our main results. For a prime ℓ and an abelian group G, we write the ℓ–completion of G by $G^{\wedge}_{\ell}$.

Theorem 1.1 *Let $\bar{F}$ be an algebraically closed field of positive characteristic p. For all $s \geq w \geq 0$ or $s < w$, there are isomorphisms $\pi_{s,w}(\bar{F})[p^{-1}] \cong \pi_{s,w}(\mathbb{C})[p^{-1}]$.*

Proof By Proposition 5.14, when $s > w \geq 0$, the groups $\pi_{s,w}(\bar{F})$ and $\pi_{s,w}(\mathbb{C})$ are torsion. The isomorphism $\pi_{s,w}(\bar{F})[p^{-1}] \cong \pi_{s,w}(\mathbb{C})[p^{-1}]$ follows when $s > w \geq 0$

from Theorem 1.3 by summing up the ℓ–primary parts. When $s = w \geq 0$ the result follows by Morel's identification of the 0–line in [34]. If $s < w$ then Morel's connectivity theorem implies that both groups are trivial by Corollary 2.14. □

Let π_n^s denote the n^{th} topological stable stem. Over the complex numbers, Levine [29, Corollary 2] showed there is an isomorphism $\pi_n^s \cong \pi_{n,0}(\mathbb{C})$. We obtain a similar result over any algebraically closed field of positive characteristic p after inverting p.

Corollary 1.2 *Let $\overline{F}$ be an algebraically closed field of positive characteristic p. For all $n \geq 0$ the homomorphism $\mathbb{L}c\colon (\pi_n^s)[p^{-1}] \to \pi_{n,0}(\overline{F})[p^{-1}]$ is an isomorphism.*

We do not expect Levine's theorem to hold over a field which is not algebraically closed. Write $\mathbb{F}_q$ for the finite field with $q = p^\nu$ elements where p is a prime and $\widetilde{\mathbb{F}}_q$ for the union of the field extensions $\mathbb{F}_{q^i}$ over $\mathbb{F}_q$ with i odd. In this paper, we will see how the groups $\pi_{n,0}(\mathbb{F}_q)$ differ from π_n^s using motivic Adams spectral sequence calculations. Corollary 1.2 allows us to identify differentials in the mod 2 motivic Adams spectral sequence over a finite field and identify the two-complete groups $\pi_{n,0}(\mathbb{F}_q)^\wedge_2$ in a range. The analogous calculations with the mod ℓ motivic Adams spectral sequence for ℓ an odd prime are given by Wilson in [52]. The groups take the following form.

Theorem 1.3 *If Morel's connectivity theorem holds for the finite field $\mathbb{F}_q$, then for all $0 \leq n \leq 18$ there is an isomorphism*

$$\pi_{n,0}(\mathbb{F}_q)[p^{-1}] \cong (\pi_n^s \oplus \pi_{n+1}^s)[p^{-1}].$$

In particular, the group $\pi_{4,0}(\mathbb{F}_q)[p^{-1}]$ is trivial.

Proof Propositions 7.15 and 7.18 calculate the two-completion of $\pi_{n,0}(\mathbb{F}_q)$ for n satisfying $0 \leq n \leq 18$. For primes $\ell \neq 2$, the calculations are similar and given by Wilson in [52, Sections 6 and 7]. The ℓ–completions of $\pi_{n,0}(\mathbb{F}_q)$ are shown to agree with the ℓ–primary part of $\pi_{n,0}(\mathbb{F}_q)$ for $n > 0$ in Proposition 5.14. When $n = 0$, the result follows by Morel's identification of $\pi_{0,0}(\mathbb{F}_q)$ with the Grothendieck–Witt ring of $\mathbb{F}_q$, since $\mathrm{GW}(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, as shown by Scharlau in [42, Chapter 2, Section 3.3]. □

Remark 1.4 The above theorem depends on Morel's connectivity theorem to prove that the motivic Adams spectral sequence converges to the homotopy groups of the ℓ–completion of the sphere spectrum. The published proof of the theorem by Morel in [34] holds for infinite fields. A private message from Panin gives a new proof of Morel's connectivity theorem which is valid for finite fields. We therefore state our results under the assumption that Morel's connectivity theorem holds for finite fields.

However, our argument for Theorem 1.3 goes through with the field $\mathbb{F}_q$ replaced by $\widetilde{\mathbb{F}}_q$, where Morel's connectivity theorem holds by Proposition 7.22. The uneasy reader may replace $\mathbb{F}_q$ with $\widetilde{\mathbb{F}}_q$ throughout.

In the case of a finite field $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$, we use the ρ–Bockstein spectral sequence to identify the additive structure of the E_2 page of the MASS. Some hidden products in the ρ–Bockstein spectral sequence were identified with the help of computer calculations by Fu and Wilson, which can be found in [16].

It is interesting to note that the pattern $\pi_{n,0}(\mathbb{F}_q)^\wedge_2 \cong (\pi^s_n \oplus \pi^s_{n+1})^\wedge_2$ obtained in Theorem 1.3 does not hold in general. We show that if $q \equiv 1 \bmod 4$, then

$$\pi_{19,0}(\mathbb{F}_q)^\wedge_2 \cong (\mathbb{Z}/8 \oplus \mathbb{Z}/2) \oplus \mathbb{Z}/4 \quad \text{and} \quad \pi_{20,0}(\mathbb{F}_q)^\wedge_2 \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2.$$

We shall leave open for further investigations the question of whether or not an isomorphism $\pi_{n,0}(\mathbb{F}_q)^\wedge_2 \cong (\pi^s_n \oplus \pi^s_{n+1})^\wedge_2$ holds when $q \equiv 3 \bmod 4$ and $n = 19, 20$.

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2 The stable motivic homotopy category

We first sketch a construction of the stable motivic homotopy category that will be convenient for our purposes, and in the process, set our notation. Treatments of stable motivic homotopy theory can be found in Voevodsky [47], Jardine [27], Hu [23], Dundas, Röndigs and Østvær [15], Morel [32], Ayoub [4] and the Nordfjordeid lectures [14].

2.1 The unstable motivic homotopy category

A base scheme S is a Noetherian separated scheme of finite Krull dimension. We write Sm/S for the category of smooth schemes of finite type over S. A space over S is a simplicial presheaf on Sm/S. The collection of spaces over S forms the category $\mathrm{Spc}(S)$, where morphisms are natural transformations of functors. We write $\mathrm{Spc}_*(S)$ for the category of pointed spaces.

The first model category structure we endow $\mathrm{Spc}(S)$ with is the projective model structure; see, for example, Blander [5, Theorem 1.4], Dundas, Röndigs and Østvær [15, Theorem 2.7], Hirschhorn [19, Theorem 11.6.1].

Definition 2.1 A map $f\colon X \to Y$ in $\mathrm{Spc}(S)$ is a (global) weak equivalence if for any $U \in \mathrm{Sm}/S$ the map $f(U)\colon X(U) \to Y(U)$ of simplicial sets is a weak equivalence. The projective fibrations are those maps $f\colon X \to Y$ for which $f(U)\colon X(U) \to Y(U)$ is a Kan fibration for any $U \in \mathrm{Sm}/S$. The projective cofibrations are those maps in $\mathrm{Spc}(S)$ which satisfy the left lifting property for trivial projective fibrations. The projective model structure on $\mathrm{Spc}(S)$ consists of the global weak equivalences, the projective fibrations and the projective cofibrations.

The category $\mathrm{Spc}(S)$ equipped with the projective model structure is cellular, proper and simplicial; see Blander [5, Theorem 1.4]. Furthermore, $\mathrm{Spc}(S)$ has the structure of a simplicial monoidal model category, with product $\times$ and internal hom $\underline{\mathrm{Hom}}$.

The constant presheaf functor $c\colon \underline{\mathrm{sSet}} \to \mathrm{Spc}(S)$ associates to a simplicial set A the presheaf cA defined by $cA(U) = A$ for any $U \in \mathrm{Sm}/S$. The functor c is a left Quillen functor when $\mathrm{Spc}(S)$ is equipped with the projective model structure. Its right adjoint $\mathrm{Ev}_S\colon \mathrm{Spc}(S) \to \underline{\mathrm{sSet}}$ satisfies $\mathrm{Ev}_S(X) = X(S)$. One can show that representable presheaves and constant presheaves in $\mathrm{Spc}(S)$ are cofibrant in the projective model structure.

For a smooth scheme X over S, we write h_X for the representable presheaf of simplicial sets. We will occasionally abuse notation and write X for h_X. Although the representable presheaf functor embeds Sm/S into $\mathrm{Spc}(S)$, colimits which exist in Sm/S are not necessarily preserved in $\mathrm{Spc}(S)$. That is, if $X = \operatorname{colim} X_i$ in Sm/S, it need not be true that $h_X = \operatorname{colim} h_{X_i}$, for example, $\operatorname{colim}(h_{\mathbb{A}^1} \leftarrow h_{\mathbb{G}_m} \to h_{\mathbb{A}^1}) \neq h_{\mathbb{P}^1}$, as one can check by applying the Picard group functor. To fix this, one introduces the Nisnevich topology on Sm/S.

Morel and Voevodsky proved in [35, Section 3, Proposition 1.4] that the Nisnevich topology is generated by covers coming from the elementary distinguished squares. Recall that an elementary distinguished square is a pull-back square in Sm/S

$$\begin{array}{ccc} V' & \longrightarrow & X' \\ \downarrow & & \downarrow f \\ V & \xrightarrow{\ j\ } & X \end{array}$$

for which f is an étale map, j is an open embedding and $f^{-1}(X-V) \to X-V$ is an isomorphism, where these subschemes are given the reduced structure. Hence a presheaf of sets F on Sm/S is a Nisnevich sheaf if and only if for any elementary distinguished square the resulting square after applying F is a pull-back square.

Definition 2.2 For a pointed space $\mathcal{X}$ and $n \geq 0$, the n^{th} simplicial homotopy sheaf $\pi_n\mathcal{X}$ of $\mathcal{X}$ is the Nisnevich sheafification of the presheaf $U \mapsto \pi_n(\mathcal{X}(U))$.

Write W_{Nis} for the class of maps $f\colon \mathcal{X} \to \mathcal{Y}$ for which $f_*\colon \pi_n\mathcal{X} \to \pi_n\mathcal{Y}$ is an isomorphism of Nisnevich sheaves for all $n \geq 0$. The Nisnevich local model structure on $\mathrm{Spc}_*(S)$ is the left Bousfield localization of the projective model structure with respect to W_{Nis}.

Definition 2.3 Let $W_{\mathbb{A}^1}$ be the class of maps $\pi_X\colon (X \times \mathbb{A}^1)_+ \to X_+$ for $X \in \mathrm{Sm}/S$. The motivic model structure on $\mathrm{Spc}_*(S)$ is the left Bousfield localization of the projective model structure with respect to $W_{\mathrm{Nis}} \cup W_{\mathbb{A}^1}$. We write $\mathrm{Spc}_*^{\mathbb{A}^1}(S)$ for the category of pointed spaces equipped with the motivic model structure. The homotopy category of $\mathrm{Spc}_*^{\mathbb{A}^1}(S)$ is the pointed motivic homotopy category $\mathcal{H}_*^{\mathbb{A}^1}(S)$.

For pointed spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $[\mathcal{X}, \mathcal{Y}]$ for the set of maps $\mathcal{H}_*^{\mathbb{A}^1}(S)(\mathcal{X}, \mathcal{Y})$. The n^{th} motivic homotopy sheaf of a pointed space $\mathcal{X}$ over S is the sheaf $\pi_n\mathcal{X}$ associated to the presheaf $U \mapsto [S^n \wedge U_+, \mathcal{X}]$.

There are two circles in the category of pointed spaces: the constant simplicial presheaf S^1 pointed at its 0–simplex and the representable presheaf $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ pointed at 1. These determine a bigraded family of spheres $S^{i,j} = (S^1)^{\wedge i-j} \wedge \mathbb{G}_m^{\wedge j}$.

Definition 2.4 For a pointed space X over S and natural numbers i and j with $i \geq j$, write $\pi_{i,j}X$ for the set of maps $[S^{i,j}, X]$.

The category of pointed spaces $\mathrm{Spc}_*(S)$ equipped with the induced motivic model category structure has many good properties which make it amenable to Bousfield localization. In particular, $\mathrm{Spc}_*(S)$ is closed symmetric monoidal, pointed simplicial, left proper and cellular.

2.2 The stable Nisnevich local model structure

With the unstable motivic model category in hand, we now construct the stable motivic model category using the general framework laid out by Hovey in [20].

Let T be a cofibrant replacement of $\mathbb{A}^1/\mathbb{A}^1 - \{0\}$. Morel and Voevodsky have shown that T is weakly equivalent to $S^{2,1}$ in $\mathrm{Spc}_*^{\mathbb{A}^1}(S)$ [35, Section 3, Proposition 2.15]. The functor $T \wedge -$ on $\mathrm{Spc}_*^{\mathbb{A}^1}(S)$ is a left Quillen functor, and we may invert it by creating a category of T–spectra.

Definition 2.5 A T–spectrum X is a sequence of spaces $X_n \in \mathrm{Spc}_*^{\mathbb{A}^1}(S)$ equipped with structure maps $\sigma_n\colon T \wedge X_n \to X_{n+1}$. A map of T–spectra $f\colon X \to Y$ is a collection of maps $f_n\colon X_n \to Y_n$ which are compatible with the structure maps. We write $\mathrm{Spt}_T(S)$ for the category of T–spectra of spaces.

To start, the level model category structure on $\mathrm{Spt}_T(S)$ is defined by declaring a map $f\colon X \to Y$ to be a weak equivalence (respectively fibration) if every map $f_n\colon X_n \to Y_n$ is a weak equivalence (respectively fibration) in the motivic model structure on $\mathrm{Spc}_*(S)$. The cofibrations for the level model structure are determined by the left lifting property for trivial level fibrations.

Definition 2.6 Let X be a T–spectrum. For integers i and j, the (i,j) stable homotopy sheaf of X, written as $\pi_{i,j}X$, is the Nisnevich sheafification of the presheaf $U \mapsto \mathrm{colim}_n\, \pi_{i+2n,j+n} X_n(U)$. A map $f\colon X \to Y$ is a stable weak equivalence if for all integers i and j the induced maps $f_*\colon \pi_{i,j}X \to \pi_{i,j}Y$ are isomorphisms.

Definition 2.7 The stable model structure on $\mathrm{Spt}_T(S)$ is the model category where the weak equivalences are the stable weak equivalences and the cofibrations are the cofibrations in the level model structure. The fibrations are those maps with the right lifting property with respect to trivial cofibrations. We write $\mathcal{SH}_S$ for the homotopy category of $\mathrm{Spt}_T(S)$ equipped with the stable model structure.

The stable model structure on $\mathrm{Spt}_T(S)$ can be realized as a left Bousfield localization of the levelwise model structure, as defined by Hovey [20, Definition 3.3].

Just as for the category Spt_{S^1} of simplicial S^1–spectra, there is not a symmetric monoidal category structure on $\mathrm{Spt}_T(S)$ which lifts the smash product $\wedge$ in $\mathcal{SH}_S$. One remedy is to use a category of symmetric T–spectra $\mathrm{Spt}_T^{\Sigma}(S)$. The construction of this category is given by Hovey in [20, Definition 8.7] and Jardine in [27]. It is proven in [20, Theorem 9.1] that there is a zig-zag of Quillen equivalences from $\mathrm{Spt}_T^{\Sigma}(S)$ to $\mathrm{Spt}_T(S)$, hence $\mathcal{SH}_S$ is equivalent to the homotopy category of $\mathrm{Spt}_T^{\Sigma}(S)$ as well. Since Quillen equivalences induce equivalences of homotopy categories, the category $\mathcal{SH}_S$ is a symmetric monoidal triangulated category with shift functor $[1] = S^{1,0} \wedge -$.

Definition 2.8 If E is a T–spectrum over S, write $\pi_{i,j}E$ for $\mathcal{SH}_S(\Sigma^{i,j}\mathbb{1}, E)$. In the case where $E = \mathbb{1}$ and $S = \mathrm{Spec}(R)$ for a ring R, we simply write $\pi_{i,j}(R)$ for $\mathcal{SH}_S(\Sigma^{i,j}\mathbb{1}, \mathbb{1})$.

In addition to the category of T–spectra, we will find it convenient to work with the category of $(\mathbb{G}_m, S^1)$–bispectra; see Jardine [27] or the Nordfjordeid lectures [14].

Definition 2.9 Consider the simplicial circle S^1 as a space over S given by the constant presheaf. An S^1–spectrum over S is a sequence of spaces $X_n \in \mathrm{Spc}_*(S)$

equipped with structure maps $\sigma_n\colon S^1 \wedge X_n \to X_{n+1}$. A map of S^1–spectra over S is a sequence of maps $f_n\colon X_n \to Y_n$ that are compatible with the structure maps. The collection of S^1–spectra over S with compatible maps between them forms a category $\mathrm{Spt}_{S^1}(S)$.

First equip $\mathrm{Spt}_{S^1}(S)$ with the level model structure with respect to the Nisnevich local model structure on $\mathrm{Spc}_*(S)$. The n^{th} stable homotopy sheaf of an S^1–spectrum E over S is the Nisnevich sheaf $\pi_n E = \operatorname{colim} \pi_{n+j} E_j$. A map $f\colon E \to F$ of S^1–spectra over S is a simplicial stable weak equivalence if for all $n \in \mathbb{Z}$ the induced map $f_*\colon \pi_n E \to \pi_n F$ is an isomorphism of sheaves. The stable Nisnevich local model category structure on $\mathrm{Spt}_{S^1}(S)$ is obtained by localizing at the class of simplicial stable equivalences, as in Definition 2.7.

The motivic stable model category structure on $\mathrm{Spt}_{S^1}(S)$ is obtained from the simplicial stable model category structure by left Bousfield localization at the class of maps $W_{\mathbb{A}^1} = \{\Sigma^\infty X_+ \wedge \mathbb{A}^1 \to \Sigma^\infty X_+ \mid X \in \mathrm{Sm}/S\}$. Write $\mathrm{Spt}_{S^1}^{\mathbb{A}^1}(S)$ for the motivic stable model category $L_{W_{\mathbb{A}^1}} \mathrm{Spt}_{S^1}(S)$ and write $\mathcal{SH}_{S^1}^{\mathbb{A}^1}(S)$ for its homotopy category. The n^{th} motivic stable homotopy sheaf of an S^1–spectrum E is the Nisnevich sheaf $\pi_n^{\mathbb{A}^1} E$ associated to the presheaf $U \mapsto \mathcal{SH}_{S^1}^{\mathbb{A}^1}(S^n \wedge \Sigma^\infty U_+, E)$.

Definition 2.10 In the projective model structure on $\mathrm{Spc}_*(S)$, the space $\mathbb{G}_m$ pointed at 1 is not cofibrant. We abuse notation and write $\mathbb{G}_m$ for a cofibrant replacement of $\mathbb{G}_m$. A $(\mathbb{G}_m, S^1)$–bispectrum over S is a $\mathbb{G}_m$–spectrum of S^1–spectra. We write $\mathrm{Spt}_{\mathbb{G}_m, S^1}(S)$ for the category of $(\mathbb{G}_m, S^1)$–bispectra over S. Viewing $\mathrm{Spt}_{\mathbb{G}_m, S^1}(S)$ as the category of $\mathbb{G}_m$–spectra of S^1–spectra, we first equip $\mathrm{Spt}_{\mathbb{G}_m, S^1}(S)$ with the level model category structure with respect to the motivic stable model category structure on $\mathrm{Spt}_{S^1}(S)$. The motivic stable model category structure on $\mathrm{Spt}_{\mathbb{G}_m, S^1}(S)$ is the left Bousfield localization at the class of stable equivalences.

There are left Quillen functors

$$\Sigma^\infty_{S^1}\colon \mathrm{Spc}_*(S) \to \mathrm{Spt}_{S^1}(S) \quad \text{and} \quad \Sigma^\infty_{\mathbb{G}_m}\colon \mathrm{Spt}_{S^1}(S) \to \mathrm{Spt}_{\mathbb{G}_m, S^1}(S).$$

Additionally, the category $\mathrm{Spt}_{\mathbb{G}_m, S^1}(S)$ equipped with the motivic stable model structure is Quillen equivalent to the stable model category structure on $\mathrm{Spt}_T(S)$; see the Nordfjordeid lectures [14, page 216].

Definition 2.11 To any spectrum of simplicial sets $E \in \mathrm{Spt}_{S^1}$ we may associate the constant S^1–spectrum cE over S with value E. That is, cE is the sequence of spaces cE_n with the evident bonding maps. For a simplicial spectrum E, we also write cE for the $(\mathbb{G}_m, S^1)$–bispectrum $\Sigma^\infty_{\mathbb{G}_m} cE$. This defines a left Quillen functor $c\colon \mathrm{Spt}_{S^1} \to \mathrm{Spt}_{\mathbb{G}_m, S^1}(B)$ with right adjoint given by evaluation at S. Compare with Levine [29, Lemma 6.5].

2.3 Base change of stable model categories

Definition 2.12 Let $f\colon R \to S$ be a map of base schemes. Pull-back along f determines a functor $f^{-1}\colon \mathrm{Sm}/S \to \mathrm{Sm}/R$ which induces Quillen adjunctions

$$(f^*, f_*)\colon \mathrm{Spc}_*^{\mathbb{A}^1}(S) \to \mathrm{Spc}_*^{\mathbb{A}^1}(R) \quad \text{and} \quad (f^*, f_*)\colon \mathrm{Spt}_T(S) \to \mathrm{Spt}_T(R).$$

We now discuss some of the properties of base change. A more thorough treatment is given by Morel in [33, Section 5]. The map f_* sends a space $\mathcal{X}$ over R to the space $\mathcal{X} \circ f^{-1}$ over S. The adjoint f^* is given by the formula $(f^*\mathcal{Y})(U) = \mathrm{colim}_{U \to f^{-1}V}\, \mathcal{Y}(V)$. For a smooth scheme X over S, a standard calculation shows $f^*X = f^{-1}X$. Additionally, if cA is a constant simplicial presheaf on Sm/S, it follows that $f^*(cA) = cA$.

The Quillen adjunction (f^*, f_*) extends to both the model category of T–spectra and $(\mathbb{G}_m, S^1)$–bispectra by applying the maps f^*, and respectively f_*, termwise to a given spectrum. In the case of f^* for T–spectra, for instance, the bonding maps of f^*E are given by $T \wedge f^*E_n \cong f^*(T \wedge E_n) \to f^*(E_{n+1})$ as $f^*T = T$. The same reasoning shows that the adjunction (f^*, f_*) extends to $(\mathbb{G}_m, S^1)$–bispectra.

Write Q (respectively R) for the cofibrant (respectively fibrant) replacement functor in $\mathrm{Spt}_T(S)$. The derived functors $\mathbb{L}f^*$ and $\mathbb{R}f_*$ are given by the formulas $\mathbb{L}f^* = f^*Q$ and $\mathbb{R}f_* = f_*R$.

Let $f\colon C \to B$ be a smooth map. The functor $f_\#\colon \mathrm{Sm}/C \to \mathrm{Sm}/B$ sends $\alpha\colon X \to C$ to $f \circ \alpha\colon X \to B$ and, by restricting a presheaf on Sm/B to a presheaf on Sm/C, induces a functor $f_\#\colon \mathrm{Spc}_*^{\mathbb{A}^1}(B) \to \mathrm{Spc}_*^{\mathbb{A}^1}(C)$. The functor f^* is canonically equivalent to $f_\#$ on the level of spaces and spectra.

2.4 The connectivity theorem

Morel establishes the connectivity of the sphere spectrum over fields F by studying the effect of Bousfield localization at $W_{\mathbb{A}^1}$ of the stable Nisnevich local model category structure on $\mathrm{Spt}_{S^1}(F)$ (see Definition 2.9).

An S^1–spectrum E over S is said to be simplicially k–connected if for any $n \leq k$, the simplicial stable homotopy sheaves $\pi_n E$ are trivial. An S^1–spectrum E is k–connected if for all $n \leq k$ the motivic stable homotopy sheaves $\pi_n^{\mathbb{A}^1} E$ are trivial.

Theorem 2.13 (Morel's connectivity theorem) *If E is a simplicially k–connected S^1–spectrum over an infinite field F, then E is also k–connected.*

Morel's connectivity theorem has been proven when F is an infinite field in [34], but the argument there does not hold for finite fields. Private correspondence with Panin gives a new argument to prove Morel's connectivity theorem for finite fields as well.

The connectivity theorem along with the work of Morel in [32, Section 5] yield the following. This also follows from Voevodsky [47, Theorem 4.14].

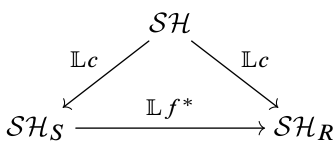
Corollary 2.14 *Over a field F where Morel's connectivity theorem holds, the sphere spectrum $\mathbb{1}$ is (-1)–connected. In particular, for all $s-w<0$ the groups $\pi_{s,w}(F)$ are trivial.*

3 Comparison to the stable homotopy category

The following result of Levine is crucial for our calculations [29, Theorem 1].

Theorem 3.1 *If $S=\operatorname{Spec}(\mathbb{C})$, the functor $\mathbb{L}c\colon \mathcal{SH}\to\mathcal{SH}_S$ is fully faithful.*

Proposition 3.2 *Let $f\colon R\to S$ be a map of base schemes. The following diagram of stable homotopy categories commutes:*



Proof The result follows by establishing $f^*\circ c=c$ on the level of model categories. For a constant space $cA\in\operatorname{Spc}(S)$, we have $f^*cA=cA$ by the calculation

$$(f^*cA)(U)=\operatorname*{colim}_{U\to f^{-1}V} cA(V)=A,$$

given the formula for f^* in Section 2.3. As the base change map is extended to T–spectra by applying f^* termwise, the claim follows. □

Proposition 3.3 *Let S be a base scheme equipped with a map $\operatorname{Spec}(\mathbb{C})\to S$. Then $\mathbb{L}c\colon\mathcal{SH}\to\mathcal{SH}_S$ is faithful.*

Proof For symmetric spectra X and Y, the map $\mathbb{L}c\colon\mathcal{SH}(X,Y)\to\mathcal{SH}(\mathbb{C})(cX,cY)$ factors through $\mathcal{SH}_S(cX,cY)$ by Proposition 3.2. Theorem 3.1 implies that the map $\mathbb{L}c\colon\mathcal{SH}(X,Y)\to\mathcal{SH}_S(cX,cY)$ must be injective. □

Corollary 3.4 *Write $W(\overline{\mathbb{F}}_p)$ for the ring of Witt vectors of $\overline{\mathbb{F}}_p$ and K for the fraction field of $W(\overline{\mathbb{F}}_p)$ (see Serre [43, Chapter II, Section 6] for a definition). Because we have maps $W(\overline{\mathbb{F}}_p)\to K\to\mathbb{C}$, the map $\mathbb{L}c\colon\pi_n^s\to\pi_{n,0}(W(\overline{\mathbb{F}}_p))$ is an injection.*

4 Motivic cohomology

Spitzweck has constructed a spectrum $H\mathbb{Z}$ in $\mathrm{Spt}_T^{\Sigma}(S)$ which represents motivic cohomology $H^{a,b}(X;\mathbb{Z})$ defined using Bloch's cycle complex when S is the Zariski spectrum of a Dedekind domain [45]. Spitzweck establishes enough nice properties of $H\mathbb{Z}$ so that we may construct the motivic Adams spectral sequence over general base schemes and establish comparisons between the motivic Adams spectral sequence over a Hensel local ring in which ℓ is invertible and its residue field.

4.1 Integral motivic cohomology

Definition 4.1 Over the base scheme $\mathrm{Spec}(\mathbb{Z})$, the spectrum $H\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$ is defined by Spitzweck in [45, Definition 4.27]. For a general base scheme S, we define $H\mathbb{Z}_S$ to be $f^*H\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$ where $f\colon S\to\mathrm{Spec}(\mathbb{Z})$ is the unique map.

Let $S=\mathrm{Spec}(D)$ for D a Dedekind domain. For $X\in\mathrm{Sm}/S$, Spitzweck shows there is a canonical isomorphism $\mathcal{SH}_S(\Sigma^\infty X_+,\Sigma^{a,b}H\mathbb{Z})\cong H^{a,b}(X;\mathbb{Z})$, where $H^{a,b}(-;\mathbb{Z})$ denotes Levine's motivic cohomology defined using Bloch's cycle complex [45, Corollary 7.19]. The isomorphism is functorial with respect to maps in Sm/S. Additionally, if $i\colon\{s\}\to S$ is the inclusion of a closed point with residue field $k(s)$, there is a commutative diagram for $X\in\mathrm{Sm}/S$:

$$\begin{array}{ccc} \mathcal{SH}_S(\Sigma^\infty X_+,\Sigma^{a,b}H\mathbb{Z}) & \xrightarrow{\cong} & H^{a,b}(X;\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathcal{SH}(k(s))(\mathbb{L}i^*\Sigma^\infty X_+,\Sigma^{a,b}H\mathbb{Z}) & \xrightarrow{\cong} & H^{a,b}(\mathbb{L}i^*X;\mathbb{Z}) \end{array}$$

If the residue field $k(s)$ has positive characteristic, there is a canonical isomorphism of ring spectra $\mathbb{L}i^*H\mathbb{Z}_S\cong H\mathbb{Z}_{k(s)}$ by Spitzweck [45, Theorem 9.16]. For a smooth map of base schemes $f\colon R\to S$, there is an isomorphism $\mathbb{L}f^*H\mathbb{Z}_S\cong H\mathbb{Z}_R$, because when f is smooth we have $\mathbb{L}f^*=f^*$; see Morel [33, page 44]. It is then straightforward to see that $f^*H\mathbb{Z}_S\cong H\mathbb{Z}_R$.

4.2 Motivic cohomology with coefficients $\mathbb{Z}/\ell$

For a prime ℓ, write $H\mathbb{Z}/\ell$ for the cofiber of the map $H\mathbb{Z}\xrightarrow{\ell\cdot}H\mathbb{Z}$ in $\mathcal{SH}_S$. The spectrum $H\mathbb{Z}/\ell$ represents motivic cohomology with $\mathbb{Z}/\ell$ coefficients. For a smooth scheme X over S, we write $H^{**}(X;\mathbb{Z}/\ell)$ for the motivic cohomology of X with $\mathbb{Z}/\ell$ coefficients. When S is the Zariski spectrum of a ring R, we write $H^{**}(R;\mathbb{Z}/\ell)$ for $H^{**}(\mathrm{Spec}(R);\mathbb{Z}/\ell)$. We will frequently omit Spec from our notation when the meaning is clear in other cases as well.

The now resolved Beilinson–Lichtenbaum conjecture allows us to calculate the mod 2 motivic cohomology of a finite field $\mathbb{F}_q$ of odd characteristic. In particular, there is an isomorphism $H^{**}(\mathbb{F}_q;\mathbb{Z}/2) \cong K_*^M(\mathbb{F}_q)/2[\tau]$ where τ has bidegree $(0,1)$ and elements of $K_n^M(\mathbb{F}_q)/2$ have bidegree (n,n). The group $K_1^M(\mathbb{F}_q)/2 \cong \mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$ is isomorphic to $\mathbb{Z}/2$. We write u for the nontrivial element of $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$ and ρ for the class of -1. It is well known that -1 is a square in $\mathbb{F}_q$ if and only if $q \equiv 1 \bmod 4$. Hence $H^{**}(\mathbb{F}_q;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau,u]/(u^2)$ and $u=\rho$ if and only if $q \equiv 3 \bmod 4$.

The mod 2 Bockstein homomorphism β is the motivic cohomology operation given by the connecting homomorphism in the long exact sequence of cohomology associated to the short exact sequence of coefficient groups

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

The Bockstein is a cohomology operation of bidegree $(1,0)$. On the mod 2 motivic cohomology of a finite field $\mathbb{F}_q$, the Bockstein is determined by $\beta(\tau)=\rho$ and $\beta(u)=0$ as it is a derivation. We remark that the Bockstein is trivial on the mod 2 motivic cohomology of a finite field $\mathbb{F}_q$ if and only if $q \equiv 1 \bmod 4$.

Proposition 4.2 *Let D be a Hensel local ring in which ℓ is invertible. Write F for the residue field of D and write $\pi\colon D \to F$ for the quotient map. Then the map $\pi^*\colon H^{**}(D;\mathbb{Z}/\ell) \to H^{**}(F;\mathbb{Z}/\ell)$ is an isomorphism of $\mathbb{Z}/\ell$–algebras. Furthermore, the action of the Bockstein is the same in either case.*

Proof The rigidity theorem for motivic cohomology in Geisser [17, Theorem 1.2(3)] gives the isomorphism. The map $\mathbb{L}\pi^*$ gives comparison maps for the long exact sequences which define the Bockstein over D and F. The rigidity theorem shows the long exact sequences are isomorphic, so the action of the Bockstein is the same in either case. □

4.3 Mod 2 motivic cohomology operations and cooperations

The mod 2 motivic Steenrod algebra over a base scheme S, which we write as $\mathcal{A}^{**}(S)$, is the algebra of bistable mod 2 motivic cohomology operations. A bistable cohomology operation is a family of operations $\theta_{**}\colon H^{**}(-;\mathbb{Z}/2) \to H^{*+a,*+b}(-;\mathbb{Z}/2)$ which are compatible with the suspension isomorphism for both the simplicial circle S^1 and the Tate circle $\mathbb{G}_m$.

When S is the Zariski spectrum of a characteristic 0 field, Voevodsky identified the structure of this algebra in [49; 50]. Voevodsky's calculation was extended to hold where the base is the Zariski spectrum of a field of positive characteristic $p \neq 2$ by Hoyois, Kelly and Østvær in [22]. In particular, the algebra $\mathcal{A}^{**}(S)$ is generated over $\mathbb{F}_2$ by the Steenrod squaring operations Sq^i of bidegree $(i, \lfloor i/2 \rfloor)$ and the operations given by cup products $x \cup -$ where $x \in H^{**}(S;\mathbb{Z}/2)$. The Steenrod squaring operations satisfy motivic Adem relations, which are given by Voevodsky in [49, Section 10] (a minor modification is needed in the case $a+b \equiv 1 \bmod 2$).

We record the structure of the mod 2 dual Steenrod algebra $\mathcal{A}_{**}(\mathbb{F}_q)$ for a finite field $\mathbb{F}_q$ of characteristic different from 2 in the following proposition.

Proposition 4.3 *Let $\mathbb{F}_q$ be a finite field of odd characteristic. The mod 2 dual Steenrod algebra is an associative commutative algebra of the form*

$$\mathcal{A}_{**}(\mathbb{F}_q) \cong H_{**}(\mathbb{F}_q)[\tau_i, \xi_j \mid i \geq 0, j \geq 1]/(\tau_i^2 - \tau\xi_{i+1} - \rho\tau_{i+1} - \rho\tau_0\xi_{i+1}),$$

where τ_i has bidegree $(2^{i+1}-1, 2^i-1)$ and ξ_i has bidegree $(2^{i+1}-2, 2^i-1)$. Note that if $q \equiv 1 \bmod 4$, the relation for τ_i^2 simplifies to $\tau_i^2 = \tau\xi_{i+1}$ as $\rho = 0$.

The structure maps for the Hopf algebroid $(H_{**}(\mathbb{F}_q), \mathcal{A}_{**}(\mathbb{F}_q))$, which we write simply as $(H_{**}, \mathcal{A}_{**})$, are as follows:

(a) *The left unit $\eta_L\colon H_{**} \to \mathcal{A}_{**}$ is given by $\eta_L(x) = x$.*

(b) *The right unit $\eta_R\colon H_{**} \to \mathcal{A}_{**}$ is determined as a map of $\mathbb{Z}/2$–algebras by $\eta_R(\rho) = \rho$ and $\eta_R(\tau) = \tau + \rho\tau_0$. In the case where ρ is trivial, that is, $q \equiv 1 \bmod 4$, the right and left unit agree: $\eta_R = \eta_L$.*

(c) *The augmentation $\epsilon\colon \mathcal{A}_{**} \to H_{**}$ kills τ_i and ξ_i, and for $x \in H_{**}$, it follows that $\epsilon(x) = x$.*

(d) *The coproduct $\Delta\colon \mathcal{A}_{**} \to \mathcal{A}_{**} \otimes_{H_{**}} \mathcal{A}_{**}$ is a map of graded $\mathbb{Z}/2$–algebras determined by*

$$\Delta(x) = x \otimes 1 \quad \text{for } x \in H_{**},$$

$$\Delta(\tau_i) = \tau_i \otimes 1 + 1 \otimes \tau_i + \sum_{j=0}^{i-1} \xi_{i-j}^{2^j} \otimes \tau_j,$$

$$\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i + \sum_{j=1}^{i-1} \xi_{i-j}^{2^j} \otimes \xi_j.$$

(e) *The antipode c is a map of $\mathbb{Z}/2$–algebras determined by*

$$c(\rho) = \rho, \qquad c(\tau) = \tau + \rho\tau_0,$$

$$c(\tau_i) = \tau_i + \sum_{j=0}^{i-1} \xi_{i-j}^{2^j} c(\tau_j), \qquad c(\xi_i) = \xi_i + \sum_{j=1}^{i-1} \xi_{i-j}^{2^j} c(\xi_j).$$

Proof The calculation can be found in the work of Hoyois, Kelly and Østvær [22] and Voevodsky [49]. □

We now investigate the structure of the Hopf algebroid of mod 2 cohomology cooperations over a Dedekind domain.

Definition 4.4 Let D be a Dedekind domain, and let C denote the set of sequences $(\epsilon_0, r_1, \epsilon_1, r_2, \dots)$ with $\epsilon_i \in \{0,1\}$, each r_i nonnegative, and only finitely many nonzero terms. The elements $\tau_i \in \mathcal{A}_{2^{i+1}-1,2^i-1}(D)$ and $\xi_i \in \mathcal{A}_{2^{i+1}-2,2^i-1}(D)$ are constructed by Spitzweck in [45, Corollary 11.23]. For any sequence $I = (\epsilon_0, r_1, \epsilon_1, r_2, \dots)$ in C, write $\omega(I)$ for the element $\tau_0^{\epsilon_0}\xi_1^{r_1}\cdots$ and $(p(I), q(I))$ for the bidegree of the operation $\omega(I)$.

Spitzweck calculates in [45, Theorem 11.24] that the dual Steenrod algebra is generated by the elements τ_i and ξ_j but does not identify the relations for τ_i^2. We record Spitzweck's calculation in the following proposition.

Proposition 4.5 *Let D be a Dedekind domain. As an $H\mathbb{Z}/2$ module, there is a weak equivalence $\bigvee_{I\in B}\Sigma^{p(I),q(I)}H\mathbb{Z}/2 \to H\mathbb{Z}/2 \wedge H\mathbb{Z}/2$. The map is given by $\omega(I)$ on the factor $\Sigma^{p(I),q(I)}H\mathbb{Z}/2$.*

To obtain the relations for τ_i^2, we find an analog of the result of Voevodsky [49, Theorem 6.10] when D is a Hensel local ring.

Proposition 4.6 *Let D be a Hensel local ring in which 2 is invertible and let F denote the residue field of D. Then the following isomorphism holds:*

$$H^{**}(B\mu_2, \mathbb{Z}/2) \cong H^{**}(D, \mathbb{Z}/2)[\![u, v]\!]/(u^2 = \tau v + \rho u).$$

Here v is the class $v_2 \in H^{2,1}(B\mu_2)$ defined by Spitzweck in [45, page 81] and $u \in H^{1,1}(B\mu_2; \mathbb{Z}/2)$ is the unique class satisfying $\widetilde{\beta}(u) = v$, where $\widetilde{\beta}$ is the integral Bockstein determined by the coefficient sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$.

Proof The motivic classifying space $B\mu_2$ over D (respectively F) fits into a triangle $B\mu_{2+} \to (\mathcal{O}(-2)_{\mathbb{P}^\infty})_+ \to \mathrm{Th}(\mathcal{O}(-2))$ by [49, (6.2)] and [45, (25)]. From this triangle, we obtain a long exact sequence in mod 2 motivic cohomology [49, (6.3)] and [45, (26)]. The comparison map $\mathbb{L}\pi^*\colon \mathcal{SH}_D \to \mathcal{SH}_F$ induces a homomorphism of these long exact sequences. The rigidity Proposition 4.2 and the 5-lemma then show that the comparison maps are all isomorphisms. As the desired relation holds in the motivic cohomology of $B\mu_2$ over F and the choices of u and v are compatible with base change, the result follows. □

With this result, the relations $\tau_i^2 = \tau\xi_{i+1} + \rho\tau_{i+1} + \rho\tau_0\xi_{i+1}$ in $\mathcal{A}_{**}(D)$ follow when D is a Hensel local ring in which 2 is invertible by the argument given by Voevodsky in [49, Theorem 12.6]. Furthermore, the calculation of Spitzweck in [45, Corollary 11.23] shows that the coproduct Δ is the same as in Proposition 4.3(d). The action of the Steenrod squaring operations $H^{**}(D)$ and $H^{**}(F)$ agree by the naturality of these cohomology operations, since these cohomology groups are isomorphic. This shows that the right unit η_R and the antipode c are given by the formulas in Proposition 4.3(b,e).

Remark 4.7 Let D be a Dedekind domain in which 2 is invertible and consider the map $f\colon \mathbb{Z}[\frac{1}{2}] \to D$. A key observation of Spitzweck in the proof of Theorem 11.24 in [45] is that the map $\mathbb{L}f^*\colon \mathcal{A}_{**}(\mathbb{Z}[\frac{1}{2}]) \to \mathcal{A}_{**}(D)$ satisfies $\mathbb{L}f^*\tau_i = \tau_i$ and $\mathbb{L}f^*\xi_i = \xi_i$ for all i. It follows that for a map $j\colon D \to \widetilde{D}$ of Dedekind domains in which 2 is invertible, $\mathbb{L}j^*\tau_i = \tau_i$ and $\mathbb{L}j^*\xi_i = \xi_i$ for all i.

Proposition 4.8 *Let D be a Hensel local ring in which 2 is invertible and let F denote the residue field of D. Then the comparison map $\pi^*\colon \mathcal{A}_{**}(D) \to \mathcal{A}_{**}(F)$ is an isomorphism of Hopf algebroids.*

Proof Remark 4.7 shows that the map $\pi^*\colon \mathcal{A}_{**}(D) \to \mathcal{A}_{**}(F)$ is an isomorphism of left $H_{**}(F)$ modules. The compatibility of the isomorphism with the coproduct, right unit and antipode was established above. □

The following definition is taken from Dugger and Isaksen [12, Definition 2.11].

Definition 4.9 A set of bigraded objects $X = \{x_{(a,b)}\}$ is said to be motivically finite if for any bigrading (a,b) there are only finitely many objects $y_{(a',b')} \in X$ for which $a \geq a'$ and $2b - a \geq 2b' - a'$. We say a bigraded algebra or module is motivically finite if it has a generating set which is motivically finite.

To motivate the preceding definition, observe that if $H^{**}(X)$ is a motivically finite $H^{**}(F)$ module, then $H^{**}(X)$ is a finite dimensional $\mathbb{F}_\ell$ vector space in each bidegree.

For a Hensel local ring D, the isomorphism $\mathcal{A}_{**}(D) \cong \mathcal{A}_{**}(F)$ of motivically finite algebras gives an isomorphism of their duals $\mathcal{A}^{**}(D) \cong \mathcal{A}^{**}(F)$. See Hoyois, Kelly and Østvær [22, Section 5.2] and Spitzweck [45, Remark 11.25] for the proof that the dual of the Hopf algebroid of cooperations is the Steenrod algebra.

The analogous results of this section hold for mod ℓ motivic cohomology over a base field or a Hensel local ring in which ℓ is invertible for odd primes ℓ. Precise statements can be found in Wilson [52].

5 Motivic Adams spectral sequence

The motivic Adams spectral sequence over a base scheme S may be defined using the appropriate notion of an Adams resolution; see Adams [2], Switzer [46] or Ravenel [40] for treatments in the topological case. We recount the definition for completeness and establish some basic properties of the motivic Adams spectral sequence under base change. We follow Dugger and Isaksen [12, Section 3] for the definition of the motivic Adams spectral sequence. See also the work of Hu, Kriz and Ormsby [24, Section 6].

Let p and ℓ be distinct primes and let $q = p^{\nu}$ for some integer $\nu \geq 1$. We will be interested in the specific case of the motivic Adams spectral sequence over a field and over a Hensel discrete valuation ring with residue field of characteristic p. We write H for the spectrum $H\mathbb{Z}/\ell$ over the base scheme S and $H^{**}(S)$ for the motivic cohomology of S with $\mathbb{Z}/\ell$ coefficients. The spectrum H is a ring spectrum and is cellular in the sense of Dugger and Isaksen [11] by work of Spitzweck [45, Corollary 11.4].

5.1 Construction of the mod ℓ MASS

Definition 5.1 Consider a spectrum X over the base scheme S and let $\overline{H}$ denote the spectrum in the cofibration sequence $\overline{H} \to \mathbb{1} \to H \to \Sigma\overline{H}$. The standard H–Adams resolution of X is the tower of cofibration sequences $X_{f+1} \to X_f \to W_f$ given by $X_f = \overline{H}^{\wedge f} \wedge X$ and $W_f = H \wedge X_f$:

$$\begin{array}{ccccccccc}
X_0 = X & & \xleftarrow{\;i_1\;} & & \overline{H}\wedge X & & \xleftarrow{\;i_2\;} & & \overline{H}\wedge\overline{H}\wedge X \longleftarrow \cdots \\
& {\scriptstyle j_0}\searrow & & \nearrow{\scriptstyle \partial_0}\,\bullet & & {\scriptstyle j_1}\searrow & & \nearrow{\scriptstyle \partial_1}\,\bullet & \\
& & H\wedge X & & & & H\wedge\overline{H}\wedge X & &
\end{array}$$

Compare this with [2, Section 15].

Definition 5.2 Let X be a T–spectrum over S and let $\{X_f, W_f\}$ be the standard H–Adams resolution of X. The motivic Adams spectral sequence for X with respect to H is the spectral sequence determined by the following exact couple:

$$\begin{array}{ccc}
\oplus\pi_{**}X_f & \xrightarrow{\;i_*\;} & \oplus\pi_{**}X_f \\
{\scriptstyle \partial_*}\nwarrow & & \swarrow{\scriptstyle j_*} \\
& \oplus\pi_{**}W_f &
\end{array}$$

The E_1 term of the motivic Adams spectral sequence is $E_1^{f,(s,w)} = \pi_{s,w}W_f$. The index f is called the Adams filtration, s is the stem and w is the motivic weight. The Adams filtration of $\pi_{**}X$ is given by $F_i\pi_{**}X = \mathrm{im}(\pi_{**}X_i \to \pi_{**}X)$.

Proposition 5.3 *Let $\mathfrak{S}$ denote the category of spectral sequences in the category of abelian groups. The associated spectral sequence to the standard H–Adams resolution defines a functor $\mathfrak{M}\colon \mathcal{SH}_S \to \mathfrak{S}$. Furthermore, the motivic Adams spectral sequence is natural with respect to base change.*

Proof The construction of the standard H–Adams resolution is functorial because $\mathcal{SH}_S$ is symmetric monoidal. Given $X \to X'$ we get induced maps of standard H–Adams resolutions $\{X_f, W_f\} \to \{X'_f, W'_f\}$. As $\pi_{**}(-)$ is a triangulated functor, we get an induced map of the associated exact couples and hence of spectral sequences $\mathfrak{M}(X) \to \mathfrak{M}(X')$.

Let $f\colon R \to S$ be a map of base schemes. The claim is that there is a natural transformation between $\mathfrak{M}\colon \mathcal{SH}_S \to \mathfrak{S}$ and $\mathfrak{M} \circ \mathbb{L} f^*\colon \mathcal{SH}_S \to \mathcal{SH}_R \to \mathfrak{S}$. Let $X \in \mathcal{SH}_S$ and let $\{X_f, W_f\}$ be the standard H_S–Adams resolution of X in $\mathcal{SH}_S$. We may as well assume X is cofibrant, in which case $QX = X$ where Q is the cofibrant replacement functor. Let $\{X'_f, W'_f\}$ denote the standard H_R–Adams resolution of $\mathbb{L} f^* X = f^* X$. Observe that we have $\{f^* X_f, f^* W_f\} = \{X'_f, W'_f\}$, since $f^*\mathbb{1} = \mathbb{1}$, $f^* H_S = H_R$ and $\mathbb{L} f^*$ is a monoidal functor. We therefore have a map $\{\mathbb{L} f^* X_f, \mathbb{L} f^* W_f\} \to \{X'_f, W'_f\}$. Applying $\mathbb{L} f^*\colon \mathcal{SH}_S(\Sigma^{s,w}\mathbb{1}, -) \to \mathcal{SH}_R(\Sigma^{s,w}\mathbb{1}, \mathbb{L} f^* -)$ to $\{X_f, W_f\}$ gives a map of exact couples and therefore a map $\Phi_X\colon \mathfrak{M}_S(X) \to \mathfrak{M}_R(\mathbb{L} f^* X)$. It is straightforward to verify that Φ determines a natural transformation. □

Corollary 5.4 *For a map of base schemes $f\colon R \to S$, there is a map of motivic Adams spectral sequences $\Phi\colon \mathfrak{M}_S(\mathbb{1}) \to \mathfrak{M}_R(\mathbb{1})$. The map Φ is furthermore compatible with the induced map $\pi_{**}(S) \to \pi_{**}(R)$.*

Definition 5.5 A particularly well-behaved family of spectra in $\mathcal{SH}_S$ are the cellular spectra in the sense of Dugger and Isaksen [11, Definition 2.10]. A spectrum $E \in \mathcal{SH}_S$ is cellular if it can be constructed out of the spheres $\Sigma^\infty S^{a,b}$ for any integers a and b by homotopy colimits. A cellular spectrum is of finite type if for some k it has a cell decomposition with no cells $S^{a,b}$ for $a - b < k$ and at most finitely many cells $S^{a,b}$ for any a and b; see Hu, Kriz and Ormsby [24, Section 2].

In the following proposition, Ext is taken in the category of $\mathcal{A}_{**}$–comodules. The homological algebra of comodules is investigated thoroughly in Adams [2], Switzer [46] and Ravenel [40].

Proposition 5.6 *Suppose X is a cellular spectrum over the base scheme S. The motivic Adams spectral sequence for X has E_2 page given by*

$$E_2^{f,(s,w)} \cong \mathrm{Ext}^{f,(s+f,w)}_{\mathcal{A}_{**}(S)}(H_{**}S, H_{**}X),$$

with differentials $d_r\colon E_r^{f,(s,w)} \to E_r^{f+r,(s-1,w)}$ for $r \geq 2$.

Proof Spitzweck proves that H is a cellular spectrum in [45, Corollary 11.4]. The argument given for [12, Proposition 6.10] by Dugger and Isaksen then goes through. The cellularity of X and H is sufficient to ensure that the Künneth theorem holds, which is needed in the argument. □

Corollary 5.7 *If X and X' are cellular spectra over S and $X \to X'$ induces an isomorphism $H_{**}X \to H_{**}X'$, then the induced map $\mathfrak{M}(X) \to \mathfrak{M}(X')$ is an isomorphism of spectral sequences from the E_2 page onwards.*

Corollary 5.8 *Let $f\colon R \to S$ be a map of base schemes and consider a cellular spectrum X over S. Suppose $f^*\colon H_{**}(S) \to H_{**}(R)$, $f^*\colon \mathcal{A}_{**}(S) \to \mathcal{A}_{**}(R)$ and $f^*\colon H_{**}X \to H_{**}(\mathbb{L}f^*X)$ are all isomorphisms. Then $\mathfrak{M}_S(X) \to \mathfrak{M}_R(\mathbb{L}f^*X)$ is an isomorphism of spectral sequences from the E_2 page onwards.*

Corollary 5.9 *Let D be a Hensel local ring in which ℓ is invertible and write F for the residue field of D. Then the comparison map $\mathfrak{M}(D) \to \mathfrak{M}(F)$ is an isomorphism at the E_2 page.*

Proof Propositions 4.2 and 4.8 and Corollary 5.8 give the result when $X = \mathbb{1}$. □

5.2 Convergence of the motivic Adams spectral sequence

To simplify the notation, write $\mathrm{Ext}(R)$ for $\mathrm{Ext}_{\mathcal{A}^{**}(R)}(H^{**}(R), H^{**}(R))$ when working over the base scheme $S = \mathrm{Spec}(R)$. For any abelian group G and any prime ℓ, we write $G_{(\ell)}$ for the ℓ–primary part of G and $G^{\wedge}_{\ell} = \varprojlim G/\ell^{\nu}$ for the ℓ–completion of G. If $\{X_f, W_f\}$ is the standard H–Adams resolution of a spectrum X, the H–nilpotent completion of X is the spectrum $X^{\wedge}_{H} = \mathrm{holim}_f\, X/X_f$ defined by Bousfield in [6, Section 5]. The H–nilpotent completion has a tower given by $C_i = \mathrm{holim}_f(X_i/X_f)$.

Proposition 5.10 *Let S be the Zariski spectrum of a field F with characteristic $p \neq \ell$ and let X be a cellular spectrum X over S of finite type (Definition 5.5). If either $\ell > 2$ and F has finite mod ℓ cohomological dimension, or $\ell = 2$ and $F[\sqrt{-1}]$ has finite mod 2 cohomological dimension, the motivic Adams spectral sequence converges to the homotopy groups of the H–nilpotent completion of X:*

$$E_2^{f,(s,w)} \Rightarrow \pi_{s,w}(X^{\wedge}_{H}).$$

Furthermore, there is a weak equivalence $X^{\wedge}_{H} \cong X^{\wedge}_{\ell}$.

Proof The argument given by Hu, Kriz and Ormsby in [24], which requires Morel's connectivity theorem for F, carries over to the positive characteristic case from the work of Hoyois, Kelly and Østvær [22]. See Ormsby and Østvær [39, Section 3.1] for the analogous argument for the motivic Adams–Novikov spectral sequence. □

We say a line $s = mf + b$ in the (f,s)–plane is a vanishing line for a bigraded group $G^{f,s}$ if $G^{f,s}$ is zero whenever $0 < s < mf + b$.

Proposition 5.11 *If $\bar{F}$ is an algebraically closed field of characteristic $p \neq \ell$, then a vanishing line for $\mathrm{Ext}^{**}(\bar{F}) \cong \mathrm{Ext}^{**}(W(\bar{F}))$ at the prime ℓ is $s = (2\ell-3)f$. If $\mathbb{F}_q$ is a finite field of characteristic $p \neq \ell$, then a vanishing line for $\mathrm{Ext}^{**}(\mathbb{F}_q) \cong \mathrm{Ext}^{**}(W(\mathbb{F}_q))$ at the prime ℓ is $s = (2\ell-3)f - 1$.*

Proof A vanishing line exists for $\mathrm{Ext}(\bar{F}) \cong \mathrm{Ext}(W(\bar{F}))$ when $\bar{F}$ is an algebraically closed fields by comparison with $\mathbb{C}$ and the topological case by work of Dugger and Isaksen [12]. The vanishing line $s = f(2\ell-3)$ from topology by Adams [1] is therefore a vanishing line for $\mathrm{Ext}(\bar{F}) \cong \mathrm{Ext}(W(\bar{F}))$.

For a finite field $\mathbb{F}_q$, the line $s = f(2\ell-3)-1$ is a vanishing line for $\mathrm{Ext}(\mathbb{F}_q) \cong \mathrm{Ext}(W(\mathbb{F}_q))$ by the identification of the E_2 page of the motivic Adams spectral sequence. When $\ell = 2$ this is given in Proposition 7.1 when $q \equiv 1 \bmod 4$ and the calculation of the ρ–BSS when $q \equiv 3 \bmod 4$. For odd ℓ, see Wilson [52]. □

We now discuss the convergence of the motivic Adams spectral sequence over the ring of Witt vectors associated to a finite field or an algebraically closed field. Consult Serre [43, Chapter II, Section 6] for a construction of the ring of Witt vectors associated to a field of positive characteristic.

Proposition 5.12 *Let $W(F)$ be the ring of Witt vectors of a field F that is either a finite field or an algebraically closed field of characteristic p and let ℓ be a prime different from p. The motivic Adams spectral sequence for $\mathbb{1}$ over $W(F)$ converges to $\pi_{**}(\mathbb{1}^\wedge_H)$ filtered by the Adams filtration, where $\mathbb{1}^\wedge_H$ is the H–nilpotent completion of $\mathbb{1}$.*

Proof The convergence $\mathfrak{M}_{W(F)}(\mathbb{1}) \Rightarrow \pi_{**}(\mathbb{1}^\wedge_H)$ follows by the argument given by Dugger and Isaksen [12, Corollary 6.15], given the vanishing line in the motivic Adams spectral sequence from Proposition 5.11. □

Proposition 5.13 *Let R and S be base schemes for which the motivic Adams spectral sequence for $\mathbb{1}$ converges to $\pi_{**}(\mathbb{1}^\wedge_H)$; see Propositions 5.10 and 5.12 for examples. A map of base schemes $f\colon R \to S$ yields a comparison map $\mathfrak{M}_S(\mathbb{1}^\wedge_H) \to \mathfrak{M}_R(\mathbb{1}^\wedge_H)$ which is compatible with the induced map*

$$\pi_{**}(\mathbb{1}^\wedge_H(S)) \to \pi_{**}(\mathbb{L} f^* \mathbb{1}^\wedge_H(S)) \to \pi_{**}(\mathbb{1}^\wedge_H(R)).$$

Proof Let $\{X_f(S), W_f(S)\}$ denote the standard H–Adams resolution of $\mathbb{1}$ over S. We now construct a map $\pi_{**}(\mathbb{1}^\wedge_H(S)) \to \pi_{**}(\mathbb{1}^\wedge_H(R))$. Recall from Proposition 5.3 that $f^*X_f(S) = X_f(R)$. Since $\mathbb{L}f^*$ is a triangulated functor, there are maps $\mathbb{L}f^*(\mathbb{1}/X_f(S)) \to \mathbb{1}/X_f(R)$ and so a map $\mathbb{L}f^*\mathbb{1}^\wedge_H(S) \to \mathbb{1}^\wedge_H(R)$ by the universal property for $\mathbb{1}^\wedge_H(R) = \operatorname{holim} \mathbb{1}/X_f(R)$. Write $C_i(S)$ for the tower of $\mathbb{1}^\wedge_H(S)$ over S defined above (and in Bousfield [6, Section 5]). Similar considerations give a map of towers $\mathbb{L}f^*C_i(S) \to C_i(R)$. Hence $\mathfrak{M}_S(\mathbb{1}^\wedge_H) \to \mathfrak{M}_R(\mathbb{1}^\wedge_H)$ is compatible with the induced map $\pi_{**}(\mathbb{1}^\wedge_H(S)) \to \pi_{**}(\mathbb{1}^\wedge_H(R))$. □

Proposition 5.14 *Let F be a field of characteristic p with finite mod ℓ cohomological dimension for all primes $\ell \neq p$ and suppose $H^{s,w}(F;\mathbb{Z}/\ell)$ is a finite dimensional vector space over $\mathbb{F}_\ell$ for all s and w. Furthermore, assume that the mod ℓ motivic Adams spectral sequence for $\mathbb{1}$ over F has a vanishing line, such as when F is a finite field or an algebraically closed field. Then the ℓ–primary part of $\pi_{s,w}(F)$ is finite whenever $s > w \geq 0$.*

Proof Ananyevsky, Levine and Panin show in [3] that the groups $\pi_{s,w}(F)$ are torsion for $s > w \geq 0$. It follows that the group $\pi_{s,w}(F)$ is the sum of its ℓ–primary subgroups $\pi_{s,w}(F)_{(\ell)}$. We set out to show that $\pi_{s,w}(F)_{(\ell)}$ is finite when $\ell \neq p$.

The motivic Adams spectral sequence converges to $\pi_{**}(\mathbb{1}^\wedge_\ell)$ by Proposition 5.10 (this requires Morel's connectivity theorem). The vanishing line in the motivic Adams spectral sequence shows that the Adams filtration of $\pi_{s,w}(\mathbb{1}^\wedge_\ell)$ has finite length, and as each group $E_2^{f,(s,w)}$ is a finite dimensional $\mathbb{F}_\ell$ vector space, we conclude the groups $\pi_{s,w}(\mathbb{1}^\wedge_\ell)$ are finite. From the long exact sequence of homotopy groups associated to the triangle $\mathbb{1}^\wedge_\ell \to \prod \mathbb{1}/\ell^\nu \to \prod \mathbb{1}/\ell^\nu$ defining $\mathbb{1}^\wedge_\ell$, we extract the short exact sequence of finite groups

(5-1) $$0 \to \varprojlim{}^1 \pi_{s+1,w}(\mathbb{1}/\ell^\nu) \to \pi_{s,w}(\mathbb{1}^\wedge_\ell) \to \varprojlim \pi_{s,w}(\mathbb{1}/\ell^\nu) \to 0.$$

Similarly, from the triangles $\mathbb{1} \xrightarrow{\ell^\nu\cdot} \mathbb{1} \to \mathbb{1}/\ell^\nu$ we extract the short exact sequences

$$0 \to \pi_{s,w}(\mathbb{1})/\ell^\nu \to \pi_{s,w}(\mathbb{1}/\ell^\nu) \to {}_{\ell^\nu}\pi_{s-1,w}(\mathbb{1}) \to 0,$$

which form a short exact sequence of towers. The maps in the tower $\{\pi_{s,w}(\mathbb{1})/\ell^\nu\}$ are given by the reduction maps $\pi_{s,w}(\mathbb{1})/\ell^\nu \to \pi_{s,w}(\mathbb{1})/\ell^{\nu-1}$. Since the tower $\{\pi_{s,w}(\mathbb{1})/\ell^\nu\}$ satisfies the Mittag–Leffler condition, we have $\varprojlim{}^1 \pi_{s,w}(\mathbb{1})/\ell^\nu = 0$. The associated long exact sequence for the inverse limit gives the exact sequence

(5-2) $$0 \to \pi_{s,w}(\mathbb{1})^\wedge_\ell \to \varprojlim \pi_{s,w}(\mathbb{1}/\ell^\nu) \to \varprojlim {}_{\ell^\nu}\pi_{s-1,w}(\mathbb{1}) \to 0.$$

The group $\varprojlim {}_{\ell^\nu}\pi_{s-1,w}(\mathbb{1})$ is the ℓ–adic Tate module of $\pi_{s-1,w}(\mathbb{1})$, which is torsion-free. As $\varprojlim \pi_{s,w}(\mathbb{1}/\ell^\nu)$ is finite by (5-1), the map $\varprojlim \pi_{s,w}(\mathbb{1}/\ell^\nu) \to \varprojlim {}_{\ell^\nu}\pi_{s-1,w}(\mathbb{1})$

is trivial. But since the sequence (5-2) is exact, the group $\varprojlim_{\ell^\nu} \pi_{s-1,w}(\mathbb{1})$ is trivial, $\pi_{s,w}(\mathbb{1})^\wedge_\ell \cong \varprojlim \pi_{s,w}(\mathbb{1}/\ell^\nu)$ and $\pi_{s,w}(\mathbb{1})^\wedge_\ell$ is finite.

Write $K(i)$ for the kernel of the canonical map $\pi_{s,w}(\mathbb{1})^\wedge_\ell \to \pi_{s,w}(\mathbb{1})/\ell^i$. The tower $\cdots \subseteq K(i) \subseteq K(i-1) \subseteq \cdots \subseteq K(1)$ consists of finite groups and so it must stabilize. Hence the tower

$$\cdots \to \pi_{s,w}(\mathbb{1})/\ell^\nu \to \pi_{s,w}(\mathbb{1})/\ell^{\nu-1} \to \cdots \to \pi_{s,w}(\mathbb{1})/\ell$$

must also stabilize. There is then some N for which $\ell^N\pi_{s,w}(\mathbb{1}) = \ell^\nu\pi_{s,w}(\mathbb{1})$ for all $\nu \geq N$, and so $\ell^N\pi_{s,w}(\mathbb{1})$ is ℓ–divisible. From the short exact sequence of towers $\ell^\nu\pi_{s,w}(\mathbb{1}) \to \pi_{s,w}(\mathbb{1}) \to \pi_{s,w}(\mathbb{1})/\ell^\nu$, taking the inverse limit yields the exact sequence

$$0 \to \ell^N\pi_{s,w}(\mathbb{1}) \to \pi_{s,w}(\mathbb{1}) \to \pi_{s,w}(\mathbb{1})^\wedge_\ell \to 0.$$

Since $\pi_{s,w}(\mathbb{1})^\wedge_\ell$ is finite, it is ℓ–primary and there is a short exact sequence

$$0 \to \ell^N\pi_{s,w}(\mathbb{1})_{(\ell)} \to \pi_{s,w}(\mathbb{1})_{(\ell)} \to \pi_{s,w}(\mathbb{1})^\wedge_\ell \to 0.$$

The group $\ell^N\pi_{s,w}(\mathbb{1})_{(\ell)}$ must be zero. Suppose for a contradiction that it is nonzero. Then $\ell^N\pi_{s,w}(\mathbb{1})_{(\ell)}$ must contain $\mathbb{Z}/\ell^\infty$ as a summand, which shows the ℓ–adic Tate module of $\pi_{s,w}(\mathbb{1})$ is nonzero, a contradiction. □

We now identify the groups $\pi_{s,s}(\mathbb{1}^\wedge_\ell)$ for $s \geq 0$.

Proposition 5.15 *Let F be a finite field or an algebraically closed field of characteristic $p \neq \ell$. When $s = w \geq 0$ or $s < w$, the motivic Adams spectral sequence of $\mathbb{1}$ over F converges to the ℓ–completion of $\pi_{s,w}(F)$.*

Proof If $s < w$, the convergence follows from Morel's connectivity theorem. When $s = w \geq 0$, Proposition 5.10 implies that at bidegree (s, w) the motivic Adams spectral sequence converges to the group $\pi_{s,w}(\mathbb{1}^\wedge_\ell)$. Since $\pi_{s-1,s}(\mathbb{1}) = 0$ by Morel's connectivity theorem, the short exact sequence (see, for example, Hu, Kriz and Ormsby [24, (2)])

$$0 \to \mathrm{Ext}(\mathbb{Z}/\ell^\infty, \pi_{s,s}(\mathbb{1})) \to \pi_{s,s}(\mathbb{1}^\wedge_\ell) \to \mathrm{Hom}(\mathbb{Z}/\ell^\infty, \pi_{s-1,s}(\mathbb{1})) \to 0$$

gives an isomorphism $\mathrm{Ext}(\mathbb{Z}/\ell^\infty, \pi_{s,s}(\mathbb{1})) \cong \pi_{s,s}(\mathbb{1}^\wedge_\ell)$. In [34, Corollary 1.25], Morel has calculated $\pi_{0,0}(F) \cong \mathrm{GW}(F)$ and $\pi_{s,s}(F) \cong W(F)$ for $s > 0$ where $W(F)$ is the Witt group of the field F. For the fields under consideration, $\mathrm{GW}(F)$ and $W(F)$ are finitely generated abelian groups. But for any finitely generated abelian group A, there is an isomorphism $\mathrm{Ext}(\mathbb{Z}/\ell^\infty, A) \cong A^\wedge_\ell$, given in Bousfield and Kan [7, Chapter VI, Section 2.1], which concludes the proof. □

6 Stable stems over an algebraically closed field

Let $\bar{F}$ be an algebraically closed field of positive characteristic p. Denote the ring of Witt vectors of $\bar{F}$ by $W = W(\bar{F})$, the field of fractions of W by $K = K(\bar{F})$, and the algebraic closure of K by $\bar{K} = \bar{K}(\bar{F})$. Note that K is a field of characteristic 0. The previous sections have set us up with enough machinery to compare the motivic Adams spectral sequences at a prime $\ell \neq p$ over the associated base schemes $\mathrm{Spec}(\bar{F})$, $\mathrm{Spec}(W)$ and $\mathrm{Spec}(\bar{K})$. We will often write the ring instead of the Zariski spectrum of the ring in our notation. For any Dedekind domain R, we write $\mathrm{Ext}(R)$ for the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{**}(R)}(H^{**}(R), H^{**}(R))$.

Proposition 6.1 *Let $\bar{F}$ be an algebraically closed field of positive characteristic p, and let ℓ be a prime different from p. The E_2 page of the mod ℓ motivic Adams spectral sequence for $\mathbb{1}$ over W, the ring of Witt vectors of $\bar{F}$, is given by*

$$E_2^{f,(s,w)}(W) \cong \mathrm{Ext}^{f,(s+f,w)}(W) \cong \mathrm{Ext}^{f,(s+f,w)}(\bar{F}).$$

Proof Since W is a Hensel local ring with residue field $\bar{F}$, Corollary 5.9 applies. □

Proposition 6.2 *Let $\bar{F}$ be an algebraically closed field of characteristic p. The homomorphism $f\colon W \to \bar{K}$ induces isomorphisms of graded rings*

$$f^*\colon H_{**}(W) \to H_{**}(\bar{K}) \quad \text{and} \quad f^*\colon \mathcal{A}_{**}(W) \to \mathcal{A}_{**}(\bar{K}).$$

Proof Since $H^{**}(S) \cong H_{-*,-*}(S)$, it suffices to establish isomorphisms for motivic cohomology. Because $H^{**}(W) \cong H^{**}(\bar{\mathbb{F}}_p)$, we have $H^{**}(W) \cong \mathbb{F}_\ell[\tau]$ where $\tau \in H^{0,1}(W) \cong \mu_\ell(W)$. We also have that $H^{**}(\bar{K}) \cong \mathbb{F}_\ell[\tau]$. To identify the ring map $f^*\colon H^{**}(W) \to H^{**}(R)$ it suffices to identify the value of $f^*(\tau)$. The homomorphism $f^*\colon H^{0,1}(W) \to H^{0,1}(\bar{K})$ may be identified with $\mu_\ell(W) \to \mu_\ell(\bar{K})$, which is an isomorphism. Hence $f^*\colon H^{**}(W) \to H^{**}(\bar{K})$ is an isomorphism. The argument given for Proposition 4.8 establishes that $f^*\colon \mathcal{A}_{**}(W) \to \mathcal{A}_{**}(\bar{K})$ is an isomorphism. □

Corollary 6.3 *Let $\bar{F}$ be an algebraically closed field of characteristic p. The homomorphisms $W \to \bar{K}$ and $W \to \bar{F}$ induce isomorphisms of motivic Adams spectral sequences for $\mathbb{1}$ from the E_2 page onwards. In particular,* $\mathrm{Ext}(\bar{F}) \cong \mathrm{Ext}(W) \cong \mathrm{Ext}(\bar{K})$.

Lemma 6.4 *Let $f\colon \bar{k} \to \bar{K}$ be an extension of algebraically closed fields of characteristic 0. For all s and $w \geq 0$, base change induces an isomorphism $\pi_{s,w}(\bar{k}) \to \pi_{s,w}(\bar{K})$.*

Proof Let ℓ be prime. The maps $f^*\colon H_{**}(\bar{k}) \to H_{**}(\bar{K})$ and $f^*\colon \mathcal{A}_{**}(\bar{k}) \to \mathcal{A}_{**}(\bar{K})$ are isomorphisms, hence the induced map of cobar complexes $f^*\colon \mathcal{C}^*(\bar{k}) \to \mathcal{C}^*(\bar{K})$ is an isomorphism. It follows that the map $\mathfrak{M}_{\bar{k}}(\mathbb{1}) \to \mathfrak{M}_{\bar{K}}(\mathbb{1})$ is an isomorphism from the E_2 page onwards. The homomorphism $\mathbb{L} f^*\colon \pi_{**}(\mathbb{1}^\wedge_H(\bar{k})) \to \pi_{**}(\mathbb{1}^\wedge_H(\bar{K}))$ is therefore an isomorphism since it is compatible with the map of spectral sequences. Propositions 5.14 and 5.15 identify $\pi_{s,w}(\mathbb{1}^\wedge_H)$ with $\pi_{s,w}(\mathbb{1})^\wedge_\ell$ for all $s \geq w \geq 0$ over both $\bar{k}$ and $\bar{K}$. By the work of Ananyevsky, Levine and Panin [3], the groups $\pi_{s,w}(\bar{k})$ and $\pi_{s,w}(\bar{K})$ are torsion for $s > w \geq 0$ and so they are the sum of their ℓ–primary parts. This establishes the result for $s > w \geq 0$. When $s = w \geq 0$, the result follows by Proposition 5.15 and Morel's identification of the groups $\pi_{n,n}(F)$. If $s < w$, the connectivity theorem applies and gives the isomorphism. □

Corollary 6.5 *Let $\bar{K}$ be an algebraically closed field of characteristic 0. For any $n \geq 0$, the map $\mathbb{L}c\colon \pi^s_n \to \pi_{n,0}(\bar{K})$ is an isomorphism.*

Proof The statement is true when $\bar{K} = \mathbb{C}$ by Levine's theorem. The previous proposition extends the result to an arbitrary algebraically closed field of characteristic 0. □

Theorem 6.6 *Let $\bar{F}$ be an algebraically closed field of characteristic p and let ℓ be a prime different from p. Then there is an isomorphism $\pi_{s,w}(\bar{F})^\wedge_\ell \cong \pi_{s,w}(\mathbb{C})^\wedge_\ell$ for all $s \geq w \geq 0$.*

Proof Consider the homomorphisms $\bar{F} \leftarrow W \to \bar{K}$. The induced maps on the motivic Adams spectral sequence are compatible with the maps of homotopy groups

$$\pi_{**}(\mathbb{1}^\wedge_H(\bar{F})) \leftarrow \pi_{**}(\mathbb{1}^\wedge_H(W)) \to \pi_{**}(\mathbb{1}^\wedge_H(\bar{K})).$$

By Corollary 6.3, the maps $\mathfrak{M}_{\bar{F}}(\mathbb{1}) \leftarrow \mathfrak{M}_W(\mathbb{1}) \to \mathfrak{M}_{\bar{K}}(\mathbb{1})$ are isomorphisms at the E_2 page, and so there are isomorphisms

$$\pi_{**}(\mathbb{1}^\wedge_H(\bar{F})) \cong \pi_{**}(\mathbb{1}^\wedge_H(W)) \cong \pi_{**}(\mathbb{1}^\wedge_H(\bar{K})).$$

For $s \geq w \geq 0$, Propositions 5.14 and 5.15 give isomorphisms

$$\pi_{s,w}(\mathbb{1}^\wedge_H(\bar{F})) \cong \pi_{s,w}(\bar{F})^\wedge_\ell \quad \text{and} \quad \pi_{s,w}(\mathbb{1}^\wedge_H(\bar{K})) \cong \pi_{s,w}(\bar{K})^\wedge_\ell.$$

The result now follows from Lemma 6.4. □

Corollary 6.7 *Let $\bar{F}$ be an algebraically closed field of characteristic p and let ℓ be a prime different from p. The homomorphism $\mathbb{L}c\colon (\pi^s_n)^\wedge_\ell \to \pi_{n,0}(\bar{F})^\wedge_\ell$ is an isomorphism for all $n \geq 0$.*

Proof The previous theorem yields the following diagram for all $n \geq 0$:

$$\begin{array}{ccccc} & & (\pi_n^s)^\wedge_\ell & & \\ & \swarrow^{\mathbb{L}c} & \downarrow{\scriptstyle \mathbb{L}c} & \searrow^{\mathbb{L}c}_{\cong} & \\ \pi_{n,0}(\bar{F})^\wedge_\ell & \xleftarrow{\cong} & \pi_{n,0}(\mathbb{1}^\wedge_H(W)) & \xrightarrow{\cong} & \pi_{n,0}(\bar{K})^\wedge_\ell \end{array}$$

The map $\mathbb{L}c\colon (\pi_n^s)^\wedge_\ell \to \pi_{n,0}(\bar{K})^\wedge_\ell$ is an isomorphism by Corollary 6.5, and so all of the maps in the above diagram are isomorphisms. □

Corollary 6.8 *For a finite field $\mathbb{F}_q$ with characteristic $p \neq \ell$, the group $(\pi_n^s)^\wedge_\ell$ is a summand of $\pi_{n,0}(\mathbb{F}_q)^\wedge_\ell$ for $n \geq 0$.*

Proof The map $\mathbb{L}c\colon \pi_n^s \to \pi_{n,0}(\bar{\mathbb{F}}_p)$ factors through $\pi_{n,0}(\mathbb{F}_q)$. Passing to the ℓ–completion, Corollary 6.7 implies the composition $(\pi_n^s)^\wedge_\ell \to \pi_{n,0}(\mathbb{F}_q)^\wedge_\ell \to \pi_{n,0}(\bar{\mathbb{F}}_p)^\wedge_\ell$ is an isomorphism. Hence the result. □

7 The motivic Adams spectral sequence for finite fields

We now analyze the two-complete stable stems $\hat{\pi}_{**}(\mathbb{F}_q) = \pi_{**}(\mathbb{F}_q)^\wedge_2$ when q is odd. The results of the previous section allow us to identify the n^{th} topological two-complete stable stem $\hat{\pi}_n^s = (\pi_n^s)^\wedge_2$ as a summand of $\hat{\pi}_{n,0}(\mathbb{F}_q)$. With this, we are able to analyze the MASS for $\mathbb{F}_q$ in a range. We remind the reader that these results assume Morel's connectivity theorem hold for $\mathbb{F}_q$, or the results hold without qualification for the fields $\widetilde{\mathbb{F}}_q$. For the remainder of this section, write H for the mod 2 motivic cohomology spectrum.

7.1 The E_2 page of MASS over $\mathbb{F}_q$ when $q \equiv 1 \bmod 4$

We will make frequent use of the calculation $H^{**}(\mathbb{F}_q;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau,u]/(u^2)$ which was given in Section 4. Recall τ and u are in bidegree $(0,1)$ and $(1,1)$, respectively.

Proposition 7.1 *The E_2 page of the mod 2 motivic Adams spectral sequence for the sphere spectrum over $\mathbb{F}_q$ with $q \equiv 1 \bmod 4$ is the trigraded algebra*

$$E_2 \cong \operatorname{Ext}(\mathbb{F}_q) \cong \mathbb{F}_2[\tau,u]/(u^2) \otimes_{\mathbb{F}_2[\tau]} \operatorname{Ext}(\bar{\mathbb{F}}_p).$$

We abuse notation and write τ and u for their duals. Hence in the above, τ and u are of bidegree $(0,-1)$ and $(-1,-1)$, respectively.

Proof Consult Dugger and Isaksen [12, Proposition 3.5] for a similar argument. Recall from Proposition 4.3 that we have $\mathcal{A}^{**}(\mathbb{F}_q) \cong \mathcal{A}^{**}(\overline{\mathbb{F}}_p) \otimes_{\mathbb{F}_2[\tau]} \mathbb{F}_2[\tau,u]/(u^2)$ and $H^{**}(\mathbb{F}_q) \cong H^{**}(\overline{\mathbb{F}}_p) \otimes \mathbb{F}_2[\tau,u]/(u^2)$. Since $\mathbb{F}_2[\tau,u]/(u^2)$ is flat as a module over $\mathbb{F}_2[\tau]$, a free resolution $H^{**}(\overline{\mathbb{F}}_p) \leftarrow P^\bullet$ by $\mathcal{A}^{**}(\overline{\mathbb{F}}_p)$ modules determines a free resolution $H^{**}(\mathbb{F}_q) \leftarrow P^\bullet \otimes \mathbb{F}_2[\tau,u]/(u^2)$. It is necessary here that $\mathrm{Sq}^1(\tau) = 0$ for $P^\bullet \otimes \mathbb{F}_2[\tau,u]/(u^2)$ to be a resolution of $\mathcal{A}^{**}(\mathbb{F}_q)$ modules. The canonical map

$$\mathrm{Hom}_{\mathcal{A}^{**}(\overline{\mathbb{F}}_p)}\big(-, H^{**}(\overline{\mathbb{F}}_p)\big) \otimes \mathbb{F}_2[\tau,u]/(u^2) \to \mathrm{Hom}_{\mathcal{A}^{**}(\mathbb{F}_q)}\big(- \otimes \mathbb{F}_2[\tau,u]/(u^2), H^{**}(\mathbb{F}_q)\big)$$

is a natural isomorphism, since a generating set for a module M over $\mathcal{A}^{**}(\overline{\mathbb{F}}_p)$ is also a generating set for $M \otimes \mathbb{F}_2[\tau,u]/(u^2)$ over $\mathcal{A}^{**}(\mathbb{F}_q)$ by Proposition 4.3. We conclude that $\mathrm{Ext}(\overline{\mathbb{F}}_p) \otimes \mathbb{F}_2[\tau,u]/(u^2) \cong \mathrm{Ext}(\mathbb{F}_q)$. □

By the previous proposition, the irreducible elements of $\mathrm{Ext}(\mathbb{C})$ are also irreducible elements of $\mathrm{Ext}(\mathbb{F}_q)$ when $q \equiv 1 \bmod 4$. The only additional irreducible element in $\mathrm{Ext}(\mathbb{F}_q)$ is the class u. The irreducible elements of $\mathrm{Ext}(\mathbb{F}_q)$ up to stem $s = 21$ can be found in Table 1. These were obtained by consulting Isaksen [26, Table 8] and independently verified by computer calculation by Fu and Wilson [16].

element	filtration (f,s,w)	element	filtration (f,s,w)	element	filtration (f,s,w)
u	$(0,-1,-1)$	c_0	$(3,8,5)$	e_0	$(4,17,10)$
τ	$(0,0,-1)$	Ph_1	$(5,9,5)$	P^2h_1	$(9,17,9)$
h_0	$(1,0,0)$	Ph_2	$(5,11,6)$	f_0	$(4,18,10)$
h_1	$(1,1,1)$	d_0	$(4,14,8)$	P^2h_2	$(9,19,10)$
h_2	$(1,3,2)$	h_4	$(1,15,8)$	c_1	$(3,19,11)$
h_3	$(1,7,4)$	Pc_0	$(7,16,9)$	$[\tau g]$	$(4,20,11)$

Table 1: The irreducible elements of $\mathrm{Ext}(\mathbb{F}_q)$ with $q \equiv 1 \bmod 4$ in stem $s \le 21$

We now investigate the motivic May spectral sequence over the finite field $\mathbb{F}_q$ when $q \equiv 1 \bmod 4$. We will find it useful for calculating Massey products in the MASS.

Definition 7.2 Write J for the cokernel of the map $\eta_L\colon H_{**} \to \mathcal{A}_{**}$ in the category of bigraded $\mathbb{F}_2$ vector spaces and consider the increasing filtration of $\mathcal{A}_{**}$ given by

$$F_n\mathcal{A}_{**} = \ker(\mathcal{A}_{**} \xrightarrow{\Delta^n} \mathcal{A}_{**}^{\otimes n+1} \to J^{\otimes n+1}).$$

This filtration on $\mathcal{A}_{**}$ induces a filtration on the cobar complex $(\mathcal{C}, d)$ defined by Ravenel in [40, Definition A1.2.11]. The filtration of the cobar complex is compatible and leads to a spectral sequence [40, Theorem A1.3.9] called the motivic May spectral sequence.

Following the work of Dugger and Isaksen [12, Section 5], we are able to identify the structure of the motivic May spectral sequence over a finite field $\mathbb{F}_q$ when $q \equiv 1 \mod 4$.

Proposition 7.3 *The associated graded Hopf algebroid $E^0\mathcal{A}_{**}$ to the filtration $F^*\mathcal{A}_{**}$ of the motivic dual Steenrod algebra over a finite field $\mathbb{F}_q$ when $q \equiv 1 \mod 4$ is the exterior algebra over $H_{**}(\mathbb{F}_q) \cong \mathbb{F}_2[\tau, u]/(u^2)$*

$$E^0\mathcal{A}_{**} \cong E_{H_{**}(\mathbb{F}_q)}(\tau_i, \xi_j^{2^k} \mid i \geq 0, j \geq 1, k \geq 0).$$

If each generator ζ_i of $E^0\mathcal{A}_*^{\text{top}}$ is assigned the weight of τ_{i-1} for $i \geq 1$ and $\zeta_i^{2^j}$ is assigned the weight of $\xi_i^{2^{j-1}}$ for $j \geq 1$, there is an isomorphism of trigraded algebras

$$E^0\mathcal{A}_{**} \cong \mathbb{F}_2[\tau, u]/(u^2) \otimes_{\mathbb{F}_2} E^0\mathcal{A}_*,$$

where $\mathcal{A}_*$ denotes the topological dual Steenrod algebra, which was studied by Milnor in [31].

Proof Since $u \in F^0\mathcal{A}_{**}(\mathbb{F}_q)$ and $\mathcal{A}_{**}(\mathbb{F}_q) \cong \mathbb{F}_2[\tau, u]/(u^2) \otimes_{\mathbb{F}_2[\tau]} \mathcal{A}_{**}(\mathbb{C})$, there are isomorphisms $F^n\mathcal{A}_{**}(\mathbb{F}_q) \cong F^n\mathcal{A}_{**}(\mathbb{C}) \otimes_{\mathbb{F}_2[\tau]} \mathbb{F}_2[\tau, u]/(u^2)$. Over $\mathbb{C}$, there is an isomorphism

$$E^0\mathcal{A}_{**}(\mathbb{C}) \cong \mathbb{F}_2[\tau] \otimes_{\mathbb{F}_2} E^0\mathcal{A}_*,$$

which follows by dualizing the result of Dugger and Isaksen in [12, Proposition 5.2(a)]. The result now follows as $\mathbb{F}_2[\tau] \to \mathbb{F}_2[\tau, u]/(u^2)$ is flat. □

Proposition 7.4 *The E_2 page of the motivic May spectral sequence over a finite field $\mathbb{F}_q$ with $q \equiv 1 \mod 4$ is given by*

$$\begin{aligned} E_2^{m,f,s,w} &= \operatorname{Ext}_{E^0\mathcal{A}_{**}(\mathbb{F}_q)}^{f,(s+f,w,m)}(H_{**}(\mathbb{F}_q), H_{**}(\mathbb{F}_q)) \\ &\cong \mathbb{F}_2[\tau, u]/(u^2) \otimes_{\mathbb{F}_2[\tau]} \operatorname{Ext}_{E^0\mathcal{A}_{**}(\mathbb{C})}^{f,(s+f,w,m)}(H_{**}(\mathbb{C}), H_{**}(\mathbb{C})), \end{aligned}$$

where f is the Adams filtration (or homological degree), s is the stem, w is the motivic weight and m is the May filtration. The differential d_r changes grading as $d_r\colon E_r^{m,f,s,w} \to E_r^{m+r-1,f+1,s-1,w}$. The motivic May spectral sequence converges to $\operatorname{Ext}_{\mathcal{A}_{**}}(H_{**}, H_{**})$.

To be consistent with the work of Dugger and Isaksen [12; 26], we write the grading of an element in the May spectral sequence in the form (m, f, s, w).

Proof The E_2 page of the motivic May spectral sequence is identified by Ravenel in [40, Theorem A1.3.9] in terms of the derived functors of the cotensor product $H_{**}\square_{\mathcal{A}_{**}}-$. In this case, the natural isomorphism $\operatorname{Hom}_{\mathcal{A}_{**}}(H_{**}, -) \cong H_{**}\square_{\mathcal{A}_{**}}-$

identifies the Cotor groups with the Ext groups in the statement of the proposition. The second isomorphism follows formally from the result over $\mathbb{C}$ established by Dugger and Isaksen in [12, Proposition 5.2(b)] by the flatness of $\mathbb{F}_2[\tau,u]/(u^2)$ over $\mathbb{F}_2[\tau]$. □

A description of the motivic May spectral sequence E_2 page over $\mathbb{C}$ is given by Dugger and Isaksen in [12, Section 5] up to the 36 stem, from which one obtains a description of the motivic May spectral sequence E_2 page over $\mathbb{F}_q$ when $q \equiv 1 \bmod 4$ using the previous proposition. One must simply add u to the list of generators of the E_2 page given in [12, Table 1] and the relation $u^2 = 0$.

7.2 The E_2 page of MASS over $\mathbb{F}_q$ when $q \equiv 3 \bmod 4$

For a finite field $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$, the E_2 page of the MASS can be identified in a range using the ρ–Bockstein spectral sequence (ρ–BSS) which was introduced by Hill in [18]. Here $\rho = [-1]$ is the nonzero class in $H^{1,1}(\mathbb{F}_q) \cong \mathbb{F}_q^\times/2$, since -1 is not a square in $\mathbb{F}_q^\times$. We briefly describe the construction of the ρ–BSS and refer the reader to Dugger and Isaksen [13] or Ormsby [38; 37] for more details.

Let $\mathcal{C}$ be the cobar construction corresponding to the Hopf algebroid

$$\big(\mathbb{F}_2[\tau,\rho]/(\rho^2), \mathcal{A}_{**}(\mathbb{F}_q)\big).$$

The filtration of $\mathcal{C}$ given by $0 \subseteq \rho\mathcal{C} \subseteq \mathcal{C}$ determines a spectral sequence, which in this case is just the long exact sequence associated to the short exact sequence of complexes

$$0 \to \rho\mathcal{C} \to \mathcal{C} \to \mathcal{C}/\rho\mathcal{C} \to 0.$$

Note that $\rho\mathcal{C}$ and $\mathcal{C}/\rho\mathcal{C}$ are both isomorphic to the cobar construction over $\mathbb{C}$. Hence we have the following long exact sequence:

$$\cdots \to \rho\,\mathrm{Ext}^{i,(*,*)}(\mathbb{C}) \to \mathrm{Ext}^{i,(*,*)}(\mathbb{F}_q) \to \mathrm{Ext}^{i,(*,*)}(\mathbb{C}) \xrightarrow{d_1} \rho\,\mathrm{Ext}^{i+1,(*,*)}(\mathbb{C}) \to \cdots$$

In spectral sequence notation, the E_1 page is given by

$$E_1^{\epsilon,f,(s,w)} \cong \begin{cases} \mathrm{Ext}^{f,(s,w)}(\mathbb{C}) & \text{if } \epsilon = 0, \\ \rho\,\mathrm{Ext}^{f,(s+1,w+1)}(\mathbb{C}) & \text{if } \epsilon = 1, \\ 0 & \text{otherwise,} \end{cases}$$

with differential $d_1\colon E_1^{\epsilon,f,(s,w)} \to E_1^{\epsilon+1,f+1,(s-1,w)}$. The differential d_1 satisfies the Leibniz rule, so it suffices to identify the differential on irreducible elements. We identify all differentials up to the 20 stem by hand in the following proposition; these calculations have been verified by computer calculations.

Proposition 7.5 *In the ρ–BSS for $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$, every irreducible element x of $\mathrm{Ext}(\mathbb{C})$ in stem $s \leq 19$ other than τ has $d_1(x) = 0$. Also, $d_1(\tau) = \rho h_0$ and $d_1([\tau g]) = \rho h_2 e_0$. Here $[\tau g]$ is the irreducible element of $\mathrm{Ext}(\mathbb{C})$ in stem 20, weight 11 and filtration 4.*

Proof The differential d_1 vanishes on all irreducible classes in $\mathrm{Ext}(\mathbb{C})$ up to stem 20 for degree reasons except for possibly τ, f_0 and $[\tau g]$. The class τ cannot survive the ρ–BSS, since if it did, it would contribute a nonzero element to $\mathrm{Ext}^{0,0,-1}(\mathbb{F}_q) \cong \mathrm{Hom}_{\mathcal{A}}^{0,-1}(H^{**}, H^{**})$, which is trivial. We conclude $d_1(\tau) = \rho h_0$, because this is the only possible nonzero value for $d_1(\tau)$.

The two possibilities for $d_1(f_0)$ are 0 and $\rho h_1 e_0$. Since $h_1 f_0 = 0$ in $\mathrm{Ext}(\mathbb{C})$, we must have $d_1(h_1 f_0) = h_1 d_1(f_0) = 0$; hence $d_1(f_0)$ is annihilated by h_1. But as $\rho h_1 e_0$ is not annihilated by h_1, we must have $d_1(f_0) = 0$.

The only possible nonzero value for $d_1([\tau g])$ is $\rho h_2 e_0$. From the relation $h_0[\tau g] = \tau h_2 e_0$, we calculate $d_1(\tau h_2 e_0) = \rho h_0 h_2 e_0$ and $d_1(h_0[\tau g]) = h_0 d_1([\tau g])$. Hence $h_0 d_1([\tau g]) = h_0 \rho h_2 e_0$, from which the result follows. □

Example 7.6 Since $d_1(h_1) = 0$, we conclude $d_1(\tau h_1) = \rho h_0 h_1 = 0$, as $h_0 h_1$ vanishes in $\mathrm{Ext}(\mathbb{C})$. Hence there is a class $[\tau h_1] \in \mathrm{Ext}^{1,(1,0)}(\mathbb{F}_q)$ which is irreducible.

With this analysis of the ρ–BSS for $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$, the structure of $\mathrm{Ext}(\mathbb{F}_q)$ as a graded abelian group up to stem 21 follows immediately and we may further identify all irreducible elements in this range. The results of this proposition were verified by computer calculation by Fu and Wilson [16].

Proposition 7.7 *When $q \equiv 3 \bmod 4$, the irreducible elements of $\mathrm{Ext}(\mathbb{F}_q)$ up to stem $s = 21$ are given in Table 2.*

Proof The structure of $\mathrm{Ext}(\mathbb{F}_q)$ as an abelian group follows directly from the ρ–BSS and the differentials calculated in Proposition 7.5. We now explain why the tabulated elements comprise all of the irreducible elements in this range. If $y \in H^{**}(\rho\mathcal{C}) \cong \rho\,\mathrm{Ext}(\mathbb{C})$, then we may write $y = \rho \cdot x$ with $x \in H^{**}(\mathcal{C}/\rho\mathcal{C}) \cong \mathrm{Ext}(\mathbb{C})$. So long as $x \neq 1$ and $d_1(x) = 0$, the element y is reducible. By Proposition 7.5 we conclude the only irreducible elements arising from $\rho\,\mathrm{Ext}(\mathbb{C})$ in this range are ρ, $[\rho\tau]$ and $[\rho\tau g]$.

Now consider an element x of $H^{**}(\mathcal{C}/\rho\mathcal{C}) \cong \mathrm{Ext}(\mathbb{C})$ which survives the ρ–BSS, that is, $d_1(x) = 0$. Then x is irreducible in $\mathrm{Ext}(\mathbb{F}_q)$ if and only if for any factorization $x = a \cdot b$ in $\mathrm{Ext}(\mathbb{C})$ with $d_1(a) = d_1(b) = 0$ it follows $a = 1$ or $b = 1$. This observation identifies all of the remaining irreducible elements in $\mathrm{Ext}(\mathbb{F}_q)$ in the range $s \leq 21$. □

element	filtration (f,s,w)	element	filtration (f,s,w)	element	filtration (f,s,w)
ρ	$(0,-1,-1)$	$[\tau c_0]$	$(3,8,4)$	$[\tau P c_0]$	$(7,16,8)$
$[\rho\tau]$	$(0,-1,-2)$	Ph_1	$(5,9,5)$	e_0	$(4,17,10)$
$[\tau^2]$	$(0,0,-2)$	$[\tau P h_1]$	$(5,9,4)$	P^2h_1	$(9,17,9)$
h_0	$(1,0,0)$	Ph_2	$(5,11,6)$	$[\tau P^2 h_1]$	$(9,17,8)$
h_1	$(1,1,1)$	$[\tau h_0 h_3^2]$	$(3,14,7)$	f_0	$(4,18,10)$
$[\tau h_1]$	$(1,1,0)$	d_0	$(4,14,8)$	P^2h_2	$(9,19,10)$
h_2	$(1,3,2)$	$[\tau h_0^2 d_0]$	$(6,14,7)$	c_1	$(3,19,11)$
$[\tau h_2^2]$	$(2,6,3)$	h_4	$(1,15,8)$	$[\tau c_1]$	$(3,19,10)$
h_3	$(1,7,4)$	$[\tau h_0^7 h_4]$	$(8,15,7)$	$[\rho\tau g]$	$(4,19,10)$
$[\tau h_0^3 h_3]$	$(4,7,3)$	Pc_0	$(7,16,9)$	$[\tau^2 g]$	$(4,20,10)$
c_0	$(3,8,5)$				

Table 2: The irreducible elements of $\mathrm{Ext}(\mathbb{F}_q)$ with $q \equiv 3 \bmod 4$ in stem $s \leq 21$

Remark 7.8 Although Proposition 7.7 lists all of the irreducible elements in $\mathrm{Ext}(\mathbb{F}_q)$ when $q \equiv 3 \bmod 4$ in a range, there are hidden products in the ρ–BSS. For example, the product $[\tau h_2^2]\cdot h_1 = \rho c_0$ is hidden in the ρ–BSS. We obtained this product by computer calculation, however the arguments by Dugger and Isaksen in [13, Lemma 6.2] can be used to obtain some products by hand.

7.3 The Adams spectral sequence for $H\mathbb{Z}[p^{-1}]$

We begin with the motivic Adams spectral sequence for $X = H\mathbb{Z}[p^{-1}]$ over a finite field $\mathbb{F}_q$ of characteristic p, as defined in Definition 5.2. In Propositions 7.10 and 7.11 we identify the differentials for $\mathfrak{M}_{\mathbb{F}_q}(H\mathbb{Z}[p^{-1}])$, which converges to $\pi_{**}(H\mathbb{Z}[p^{-1}]_2^\wedge) \cong H_{**}(\mathbb{F}_q;\mathbb{Z})_2^\wedge$. We accomplish this by working backwards from our knowledge of the target group $H^{**}(\mathbb{F}_q;\mathbb{Z})_2^\wedge$, which is isomorphic to $H^*_{\mathrm{et}}(\mathbb{F}_q;\mathbb{Z}_2(*))$ as a consequence of the Beilinson–Lichtenbaum conjecture. Soulé's calculation of $H^*_{\mathrm{et}}(\mathbb{F}_q;\mathbb{Z}_2(*))$ in [44, Paragraphe IV.2] then gives

$$\pi_{s,w}(H\mathbb{Z}[p^{-1}]) \cong \begin{cases} \mathbb{Z}_\ell & \text{if } s = w = 0,\\ \mathbb{Z}/(q^w-1)_2^\wedge & \text{if } s=-1 \text{ and } w \geq 1,\\ 0 & \text{otherwise.}\end{cases}$$

Although the spectrum $H\mathbb{Z}[p^{-1}]$ is cellular by the Hopkins–Morel theorem proven by Hoyois [21, Section 8.1], it is unclear if it is of finite type. Instead of relying on Proposition 5.10 for convergence, we establish a weak equivalence of the H–nilpotent completion of $H\mathbb{Z}[p^{-1}]$ with $H\mathbb{Z}_2^\wedge$.

Lemma 7.9 *Let $\mathbb{F}_q$ be a finite field of characteristic $p \neq 2$. The H–nilpotent completion of $H\mathbb{Z}[p^{-1}]$ is weakly equivalent to $H\mathbb{Z}_2^\wedge$.*

Proof We will show that the tower $H\mathbb{Z}/2 \leftarrow H\mathbb{Z}/2^2 \leftarrow H\mathbb{Z}/2^3 \leftarrow \cdots$ under $H\mathbb{Z}[p^{-1}]$ is an H–nilpotent resolution under $H\mathbb{Z}[p^{-1}]$ (as defined by Bousfield in [6, Definition 5.6]). It will then follow that the homotopy limit of this tower is weakly equivalent to the H–nilpotent completion of $H\mathbb{Z}[p^{-1}]$; that is, $H\mathbb{Z}_2^\wedge \cong H\mathbb{Z}[p^{-1}]_H^\wedge$ by the observations of Dugger and Isaksen in [12, Section 7.7], which shows Bousfield's result [6, Proposition 5.8] holds in the motivic stable homotopy category.

The spectrum $H\mathbb{Z}[p^{-1}]$ is the homotopy colimit of the diagram $H\mathbb{Z} \xrightarrow{p\cdot} H\mathbb{Z} \xrightarrow{p\cdot} \cdots$. From the triangle $H\mathbb{Z} \xrightarrow{2^\nu\cdot} H\mathbb{Z} \to H\mathbb{Z}/2^\nu$, we obtain, after inverting p, a triangle $H\mathbb{Z}[p^{-1}] \xrightarrow{2^\nu\cdot} H\mathbb{Z}[p^{-1}] \to H\mathbb{Z}/2^\nu$ since $p \neq 2$ and $H\mathbb{Z}/2^\nu \xrightarrow{p\cdot} H\mathbb{Z}/2^\nu$ is a homotopy equivalence. Consider the following cofibration sequence of towers:

$$\begin{array}{ccccccc}
H\mathbb{Z}[p^{-1}] & \xleftarrow{2\cdot} & H\mathbb{Z}[p^{-1}] & \xleftarrow{2\cdot} & H\mathbb{Z}[p^{-1}] & \longleftarrow & \cdots \\
{=}\downarrow & & \downarrow 2\cdot & & \downarrow 2^2\cdot & & \\
H\mathbb{Z}[p^{-1}] & \xleftarrow{=} & H\mathbb{Z}[p^{-1}] & \xleftarrow{=} & H\mathbb{Z}[p^{-1}] & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{pt} & \longleftarrow & H\mathbb{Z}/2 & \longleftarrow & H\mathbb{Z}/2^2 & \longleftarrow & \cdots
\end{array}$$

It is clear that $H\mathbb{Z}/2^\nu$ is H–nilpotent for all $\nu \geq 1$. For any H–nilpotent spectrum N we show that the induced map $\mathrm{colim}_\nu\, \mathcal{SH}_{\mathbb{F}_q}(H\mathbb{Z}/2^\nu, N) \to \mathcal{SH}_{\mathbb{F}_q}(H\mathbb{Z}[p^{-1}], N)$ is an isomorphism following the proof of Bousfield [6, Lemma 5.7]. This isomorphism holds if and only if

$$\mathrm{colim}\bigl\{\mathcal{SH}_{\mathbb{F}_q}\bigl(H\mathbb{Z}[\tfrac{1}{p}], N\bigr) \xrightarrow{2\cdot} \mathcal{SH}_{\mathbb{F}_q}\bigl(H\mathbb{Z}[\tfrac{1}{p}], N\bigr)\bigr\} \cong \mathcal{SH}_{\mathbb{F}_q}\bigl(H\mathbb{Z}[\tfrac{1}{p}], N\bigr)[\tfrac{1}{2}]$$

vanishes for all H–nilpotent N. This follows by an inductive proof with the following filtration of the H–nilpotent spectra given in [6, Lemma 3.8]. Take C_0 to be the collection of spectra $H \wedge X$ for X any spectrum, and let C_{m+1} be the collection of the spectra N for which either N is a retract of an element of C_m or there is a triangle $X \to N \to Z$ with X and Z in C_m.

If $N = H \wedge X$, it is clear that $\mathcal{SH}_{\mathbb{F}_q}(H\mathbb{Z}[p^{-1}], N) \xrightarrow{2} \mathcal{SH}_{\mathbb{F}_q}(H\mathbb{Z}[p^{-1}], N)$ is the zero map, which establishes the base case. If the claim holds for N in filtration C_m, the claim holds for N in filtration C_{m+1} by a standard argument. The claim now follows. □

Proposition 7.10 *The mod 2 motivic Adams spectral sequence for $X = H\mathbb{Z}[p^{-1}]$ over $\mathbb{F}_q$ when $q \equiv 1 \bmod 4$ has E_1 page given by*

$$E_1 \cong \mathbb{F}_2[\tau, u, h_0]/(u^2),$$

where $h_0 \in E_1^{1,(0,0)}$.

Write ν_2 for the 2–adic valuation and $\epsilon(q)$ for $\nu_2(q-1)$. For all $r \geq 1$ the differentials d_r vanish on $u\tau^j$ and h_0^j. If $r < \epsilon(q)+\nu_2(j)$ the differentials $d_r\tau^j$ vanish and we have

$$d_{\epsilon(q)+\nu_2(j)}\tau^j = u\tau^{j-1}h_0^{\epsilon(q)+\nu_2(j)}.$$

In particular, the differential d_1 is trivial, so $E_2 \cong E_1$.

Proof We build the following H^{**}–Adams resolution of $H\mathbb{Z}[p^{-1}]$ utilizing the triangles constructed in Lemma 7.9:

$$\begin{array}{ccccccc} H\mathbb{Z}[p^{-1}] & \xleftarrow{2\cdot} & & H\mathbb{Z}[p^{-1}] & \xleftarrow{2\cdot} & & H\mathbb{Z}[p^{-1}] \longleftarrow \cdots \\ & \searrow^{j_0} & & \nearrow_{\partial_0}^{\bullet} \quad \searrow^{j_1} & & \nearrow_{\partial_1}^{\bullet} & \\ & & H & & H & & \end{array} \tag{7-1}$$

The spectrum $H\mathbb{Z}[p^{-1}]$ is cellular, and so the motivic Adams spectral sequence for $X = H\mathbb{Z}[p^{-1}]$ converges to $\pi_{**}(H\mathbb{Z}[p^{-1}]^\wedge_H)$ by Proposition 5.10. Lemma 7.9 shows that $\pi_{**}(H\mathbb{Z}[p^{-1}]^\wedge_H) \cong \pi_{**}(H\mathbb{Z}^\wedge_2)$, so the spectral sequence converges:

$$E_2^{f,(s,w)} \Rightarrow H^{-s,-w}(\mathbb{F}_q;\mathbb{Z})^\wedge_2.$$

The groups $H^{s,w}(\mathbb{F}_q;\mathbb{Z})^\wedge_2$ are isomorphic to the groups $H^s_{\mathrm{et}}(\mathbb{F}_q;\mathbb{Z}_2(w))$ which were calculated by Soulé in [44, Paragraphe IV.2]. If $q \equiv 1 \bmod 4$,

$$H^{-s,-w}(\mathbb{F}_q;\mathbb{Z})^\wedge_2 \cong \begin{cases} \mathbb{Z}_\ell & \text{if } s = w = 0, \\ \mathbb{Z}/(q^w-1)^\wedge_2 & \text{if } s = -1 \text{ and } w \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7-2}$$

Note that $\nu_2(q^w-1) = \epsilon(q)+\nu_2(w)$ for all natural numbers w. The formulas for the differentials on τ^j are the only choice to give $H^{**}(\mathbb{F}_q;\mathbb{Z})^\wedge_2$ as the E_∞ term. □

Proposition 7.11 *The mod 2 motivic Adams spectral sequence for $X = H\mathbb{Z}[p^{-1}]$ over $\mathbb{F}_q$ when $q \equiv 3 \bmod 4$ has E_1 page given by*

$$E_1 \cong \mathbb{F}_2[\tau,\rho,h_0]/(\rho^2),$$

where $h_0 \in E_1^{1,(0,0)}$.

For all $r \geq 1$ the differentials d_r vanish on $\rho\tau^j$ and h_0^j. For odd natural numbers j, we calculate $d_1(\tau^j) = \rho\tau^{j-1}h_0$. Write $\lambda(q)$ for $\nu_2(q^2-1)$. If $r < \lambda(q)+\nu_2(n)$ the differentials $d_r\tau^{2n}$ vanish and

$$d_{\lambda(q)+\nu_2(n)}\tau^{2n} = \rho\tau^{2n-1}h_0^{\lambda(q)+\nu_2(n)}.$$

Proof The proof of the previous proposition goes through, except the target groups $H^{-s,-w}(\mathbb{F}_q;\mathbb{Z})^\wedge_2$ force different differentials in the spectral sequence when $q \equiv 3 \bmod 4$. Soulé's calculation in (7-2) shows the order of $H^{1,1}(\mathbb{F}_q;\mathbb{Z})^\wedge_2$ is $\nu_2(q-1) = 1$, so we conclude $d_1(\tau) = \rho h_0$. As we have $\nu_2(q^{2j}-1) = \lambda(q) + \nu_2(j)$ for all natural numbers j, the claimed formulas for the differentials on τ^{2n} hold. □

Corollary 7.12 *In the MASS of* $\mathbb{1}$ *over a finite field* $\mathbb{F}_q$ *with* $q \equiv 1 \bmod 4$, *the differentials* $d_r(\tau^j)$ *vanish when* $r < \epsilon(q) + \nu_2(j)$ *and*

$$d_{\epsilon(q)+\nu_2(j)}\tau^j = u\tau^{j-1}h_0^{\epsilon(q)+\nu_2(j)}.$$

In the MASS of $\mathbb{1}$ *over a finite field* $\mathbb{F}_q$ *with* $q \equiv 3 \bmod 4$, *the differentials* $d_r([\tau^2]^n)$ *vanish when* $r < \lambda(q) + \nu_2(n)$ *and*

$$d_{\lambda(q)+\nu_2(n)}[\tau^2]^n = [\rho\tau][\tau^2]^{n-1}h_0^{\lambda(q)+\nu_2(n)}.$$

Proof The unit map $\mathbb{1} \to H\mathbb{Z}[p^{-1}]$ induces a map of motivic Adams spectral sequences $\mathfrak{M}(\mathbb{1}) \to \mathfrak{M}(H\mathbb{Z}[p^{-1}])$. On the E_2 page, observe that when $q \equiv 1 \bmod 4$ the classes τ and u map to τ and u, respectively. When $q \equiv 3 \bmod 4$, the classes $[\tau^2]$, ρ, $[\rho\tau]$ map to $[\tau^2]$, ρ, $[\rho\tau]$, respectively. The identification of the differentials in the MASS for $H\mathbb{Z}[p^{-1}]$ in Propositions 7.10 and 7.11 then force the differentials stated in the corollary. □

Example 7.13 When $q \equiv 3 \bmod 4$, the Massey product $\langle \rho, \rho, h_0 \rangle$ in the mod 2 motivic Adams spectral sequence for $H\mathbb{Z}[p^{-1}]$ is $\rho\tau$. Since we have $\rho^2 = 0$ and $d_1(\tau) = \rho h_0$, it follows that $0 + \rho\tau$ is in the Massey product. It is straightforward to verify that the indeterminacy is trivial.

7.4 Stable stems over $\mathbb{F}_q$

We now begin an analysis of the differentials in the MASS to identify the two-complete stable stems over $\mathbb{F}_q$. To assist the reader with the computations presented below, Figures 1 and 3 in Section 9 display E_2 page charts of the MASS over $\mathbb{F}_q$. Throughout this section, $\mathbb{F}_q$ is a finite field with q elements where q is odd, and we write $\widehat{G}$ for the two-completion of an abelian group G.

Corollary 6.8 shows that $\widehat{\pi}^s_n$ is a summand of $\widehat{\pi}_{n,0}(\mathbb{F}_q)$ for all $n \geq 0$. We will soon see that for small values of $n \geq 0$ we have $\widehat{\pi}_{n,0}(\mathbb{F}_q) \cong \widehat{\pi}^s_n \oplus \widehat{\pi}^s_{n+1}$. However this pattern fails when $n = 19$ and $q \equiv 1 \bmod 4$.

Lemma 7.14 *For a finite field* $\mathbb{F}_q$ *with* q *odd, there is an isomorphism* $\pi_{0,0}(\mathbb{F}_q) \cong \pi^s_0 \oplus \pi^s_1$.

Proof The stem $\pi_{0,0}(\mathbb{F}_q)$ is isomorphic to the Grothendieck–Witt group $\mathrm{GW}(\mathbb{F}_q)$ by Morel [32]. The isomorphism $\mathrm{GW}(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ was established by Scharlau in [42, Chapter 2, Section 3.3]. Recall that $\pi_0^s \cong \mathbb{Z}$ and $\pi_1^s \cong \mathbb{Z}/2$. Hence we conclude $\pi_{0,0}(\mathbb{F}_q) \cong \pi_0^s \oplus \pi_1^s$. □

Morel's calculation of $\pi_{0,0}(\mathbb{F}_q)$ shows that $2 = (1-\epsilon) + \rho\eta$, hence multiplication by 2 in $\pi_{**}(\mathbb{F}_q)$ is detected in the mod 2 motivic Adams spectral sequence by the class $h_0 + \rho h_1$ in $\mathrm{Ext}(\mathbb{F}_q)$. This is needed to solve the extension problems when passing from the Adams spectral sequence E_∞ page to the stable stems.

Proposition 7.15 *When* $q \equiv 1 \bmod 4$ *and* $0 \le n \le 18$*, there is an isomorphism* $\widehat{\pi}_{n,0}(\mathbb{F}_q) \cong \widehat{\pi}_n^s \oplus \widehat{\pi}_{n+1}^s$.

Proof Lemma 7.14 takes care of the case when $n = 0$. We now focus on $0 < n \le 18$ where the mod 2 MASS over $\mathbb{F}_q$ converges to the groups $\widehat{\pi}_{n,0}(\mathbb{F}_q)$ by Propositions 5.10 and 5.14.

The irreducible elements of $\mathrm{Ext}(\mathbb{F}_q)$ in this range are given in Table 1. All differentials d_r for $r \ge 2$ vanish on h_0, h_1, h_3, c_0, Ph_1, d_0, Pc_0, P^2h_1 for degree reasons. As $\widehat{\pi}_{3,0}(\mathbb{F}_q)$ must contain $\widehat{\pi}_3^s \cong \mathbb{Z}/8$ as a summand by Corollary 6.8, we conclude $d_2(\tau^2 h_2) = \tau^2 d_2(h_2) = 0$. The only possible nonzero value for $d_2(h_2)$ is uh_1^3. If $d_2(h_2) = uh_1^3$, then $d_2(\tau^2 h_2) = u\tau^2 h_1^3$ would be nonzero by the product structure of $\mathrm{Ext}(\mathbb{F}_q)$ in Proposition 7.1, a contradiction. Hence $d_2(h_2) = 0$.

The nonzero Massey product $Ph_2 = \langle h_3, h_0^4, h_2\rangle$ has no indeterminacy, because $h_3 E_2^{4,(3,2)} + E_2^{4,(7,4)} h_2 = 0$. Since $\widehat{\pi}_{11}^s \cong \mathbb{Z}/8$ is a summand of $\widehat{\pi}_{11,0}$, the differential $d_2(Ph_2)$ must vanish. The nonzero Massey product $P^2h_2 = \langle h_3, h_0^4, h_2\rangle$ has no indeterminacy, because $h_3 E_2^{8,(11,6)} + E_2^{4,(7,4)} Ph_2 = 0$. Since $d_2(Ph_2) = 0$, the topological result of Moss [36, Theorem 1.1(ii)] implies $d_2(P^2h_2) = 0$.

The comparison map $\mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\overline{\mathbb{F}}_p)$ shows that $d_2(h_4)$ and $d_3(h_0h_4)$ must be nonzero, as these differentials are nonzero in $\mathfrak{M}(\overline{\mathbb{F}}_p)$ by Corollary 6.3 and calculations of Isaksen [26, Table 8] over $\mathbb{C}$. The only possible choice for $d_2(h_4)$ is $h_0h_3^2$, but $d_3(h_0h_4)$ is either h_0d_0 or $h_0d_0 + uh_1d_0$. In order to have $\widehat{\pi}_{14}^s \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ as a summand of $\widehat{\pi}_{14,0}$, we must have $d_3(h_0h_4) = h_0d_0$. A similar argument establishes $d_2(e_0) = h_1^2 d_0$ and $d_2(f_0) = h_0^2 e_0$. Note that $d_4(h_0^3 h_4) = 0$ for degree reasons.

The elements in weight 0 are all of the form $\tau^j x$ or $u\tau^{j-1}x$ where x is not a multiple of τ and of weight j. The differentials of the elements in weight 0 are now readily identified by using the Leibniz rule from Corollary 7.12. Since $\widehat{\pi}_n^s$ is a summand of $\widehat{\pi}_{n,0}(\mathbb{F}_q)$ for all $n \ge 0$, we see that there are no hidden 2–extensions for $0 < n \le 18$. □

In the proof of the following proposition, we provide some technical details in footnotes for the convenience of the reader. We follow the convention of Dugger and Isaksen [12] and write the grading of an element in the motivic May spectral sequence as (m, s, f, w) where m is the May filtration, s is the stem, f is the Adams filtration and w is the motivic weight. However, we continue to write the grading in the MASS as $(f, (s, w))$.

Proposition 7.16 *When $q \equiv 1 \bmod 4$, there are isomorphisms*

$$\hat{\pi}_{19,0}(\mathbb{F}_q) \cong (\mathbb{Z}/8 \oplus \mathbb{Z}/2) \oplus \mathbb{Z}/4 \quad \textit{and} \quad \hat{\pi}_{20,0}(\mathbb{F}_q) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2.$$

In particular, when $q \equiv 1 \bmod 8$ we find $d_2([\tau g])$ is trivial, and when $q \equiv 5 \bmod 8$ we calculate $d_2([\tau g]) = uh_0h_2e_0$.

Proof When $q \equiv 1 \bmod 4$, it is possible that $d_2([\tau g])$ is $uh_0h_2e_0$. We analyze this differential using Massey products obtained from the May spectral sequence. We show that $\langle \tau, h_1^4, h_4\rangle = \{[\tau g]\}$ in the E_2 page of the MASS using Massey products in the May spectral sequence and the May convergence theorem in Isaksen [26, Theorem 2.2.1].

At the E_4 page of the May spectral sequence we calculate $d_4(b_{21}^2) = h_1^4h_4$ and $d_4(0) = \tau h_1^4$, as $\tau h_1^4 = 0$; hence $[\tau g] = \tau b_{21}^2 \in \langle \tau, h_1^4, h_4\rangle$ in the May spectral sequence. There are no crossing differentials, so the May convergence shows $[\tau g] \in \langle \tau, h_1^4, h_4\rangle$ in the MASS.[1]

The indeterminacy $\tau E_2^{4,(20,12)} + E_2^{3,(5,3)}h_4$ in the MASS is trivial, so we conclude $\langle \tau, h_1^4, h_4\rangle = \{[\tau g]\}$.

We now identify $d_2([\tau g])$ using the following formula of Moss [36, Theorem 1.1(ii)]:

$$d_2(\langle \tau, h_1^4, h_4\rangle) \subseteq \langle d_2(\tau), h_1^4, h_4\rangle + \langle \tau, 0, h_4\rangle + \langle \tau, h_1^4, h_0h_3^2\rangle. \tag{7-3}$$

The Massey product $\langle \tau, 0, h_4\rangle$ contains 0 and has no indeterminacy.[2]

To calculate $\langle \tau, h_1^4, h_0h_3^2\rangle$ we again use the May spectral sequence and the May convergence theorem. We calculate this Massey product at the E_2 page using $d_2(h_2b_{20}) = \tau h_1^4$ and $h_1^4h_0h_3^2 = 0$ and see that $0 \in \langle \tau, h_1^4, h_0h_3^2\rangle$. There are no crossing differentials, so 0 is in this Massey product in the MASS.[3]

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[1] In this case, we must check if there are crossing differentials d_t for $t \geq 5$. To see $E_4^{*,5,3,3} = 0$ over $\mathbb{F}_q$, we check $E_4^{*,5,3,3} = 0$ and $E_4^{*,6,3,4} = 0$ over $\mathbb{C}$ using the chart in [12, Appendix C]. All that is in $(*, 5, 3, 3)$ is h_1b_{20}, but this does not survive to E_4. And nothing is in $(*, 6, 3, 4)$ even at the E_2 page.

To see $E_5^{*,20,4,12}$ is trivial over $\mathbb{F}_q$, observe that all that is in $E_4^{*,20,4,12}$ over $\mathbb{C}$ is b_{21}^2, which does not survive to the E_5 page. The group $E_4^{*,21,4,13}$ over $\mathbb{C}$ is trivial. A potential contribution from $h_0h_3^3$ or $h_0h_2^2h_4$ is ruled out by weight reasons, and because they do not survive to the E_4 page from the differentials $d_2(h_0(1))$ and $d_2(h_0b_{22})$.

[2] Here $0 = d_2(h_1^4)$ is in grading $E_2^{6,(3,4)}$, so the indeterminacy is $\tau E_3^{6,(19,12)} + E_3^{5,(4,3)}h_4$. The degree of $h_1^2e_0$ is $6, (19, 12)$, but it does not survive to the E_3 page. The group $E_3^{5,(4,3)}$ is trivial by checking the E_2 page.

[3] Note that $a_{01} = h_1b_{20}$ is in degree $(5, 5, 3, 3)$ and a_{12} is in degree $(8, 19, 6, 12)$. Then for a_{01} crossing differentials occur in $(?, 5, 3, 3)$, which is trivial from the fourth page on. For a_{12} crossing

The indeterminacy for $\langle \tau, h_1^4, h_0h_3^2\rangle$ in the MASS is $\tau E_2^{6,(19,12)} + E_2^{3,(5,3)}h_0h_3^2$, which is trivial. The group $E_2^{6,(19,12)}$ is generated by $h_1^2e_0$, which is annihilated by τ, while $E_2^{3,(5,3)}$ is trivial.

We now handle the Massey product $\langle d_2(\tau), h_1^4, h_4\rangle$, which depends on the base field. Let us suppose that $q \equiv 1 \bmod 8$ so that $d_2(\tau) = 0$ by Corollary 7.12. If a_{12} is in the E_1 page of the MASS with $d_1(a_{12}) = h_1^4h_4$, then the Massey product contains $0 \cdot h_4 + 0 \cdot a_{12} = 0$. It is straightforward to check that the indeterminacy $0 \cdot E_2^{4,(20,12)} + E_2^{5,(4,3)} \cdot h_4$ is trivial. We conclude $d_2([\tau g]) = 0$ when $q \equiv 1 \bmod 8$.

When $q \equiv 5 \bmod 8$, Corollary 7.12 establishes $d_2(\tau) = uh_0^2$. We identify the Massey product $\langle uh_0^2, h_1^4, h_4\rangle$ using the May spectral sequence and the May convergence theorem. At the E_4 page of the May spectral sequence we have $d_4(b_{21}^2) = h_1^4h_4$ and $uh_0^2h_1^4 = 0$. Hence $uh_0^2b_{21}^2 + 0h_4 = uh_0b_{21}h_2h_0(1) = uh_0h_2e_0$ is in the Massey product under consideration. It is straightforward to verify that there are no crossing differentials in this case.[4]

The indeterminacy of $\langle uh_0^2, h_1^4, h_4\rangle$ in the MASS is $uh_0^2E_2^{4,(20,12)} + E_2^{5,(4,3)}h_4$, which is trivial. Thus the May convergence theorem shows the Massey product is exactly $\{uh_0h_2e_0\}$ and we conclude $d_2([\tau g]) = uh_0h_2e_0$ if $q \equiv 5 \bmod 8$.

We now analyze the differentials in the MASS in the 19 and 20 stems. Since $[\tau g]$ has weight 11, the class $\tau^{11}[\tau g]$ is in $E^{4,(20,0)}$. If $q \equiv 1 \bmod 8$, we calculate $d_2(\tau^{11}[\tau g]) = \tau^{11}uh_0h_2e_0 = u\tau^{10}h_0^2[\tau g]$. If $q \equiv 5 \bmod 8$, then $d_2(\tau^{11}[\tau g]) = u\tau^{10}h_0^2[\tau g]$. This resolves all differentials in the 19 and 20 stems, so the calculation of the 19 stem follows.

As $\hat{\pi}_{20}^s \cong \mathbb{Z}/8$ must be a summand of $\hat{\pi}_{20,0}(\mathbb{F}_q)$, we conclude there is a hidden extension from $u\tau^{11}h_2^2h_4 = u\tau^{11}h_3^3$ to $\tau^{12}h_2e_0$. The calculation of the 20 stem now follows. □

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differentials occur in degree $(m', 19, 6, 12)$ with $m' \geq 8$. The only thing in this filtration, stem and weight is $h_1^2e_0$, which has May filtration 10. But note that both h_1^2 and e_0 are permanent cycles, so that $h_1^2e_0$ is as well. So there are no crossing differentials in this case.

[4]As $a_{01} = 0$ in $E_4^{9,4,5,3}$ and $a_{12} = b_{12}^2$, we must check two conditions: (1) whenever $m' \geq 9$ and $m' - 5 < t$ that d_t is trivial on $E_t^{m',4,5,3}$ and (2) whenever $m' \geq 8$ and $m' - 4 < t$ that d_t is trivial on $E_t^{m',20,4,12}$. Condition (1) is easily verified as $E_4^{*,4,5,3} = 0$ over $\mathbb{C}$ and $E_4^{*,5,5,4} = 0$ over $\mathbb{C}$ as well. We conclude $E_4^{*,4,5,3} = 0$ over $\mathbb{F}_q$ as only these two groups can contribute to this graded piece. We remark that uh_1^5 does not contribute any terms, since to get the weight correct one needs to multiply by τ which annihilates the element. For condition (2), we will check that for all $t \geq 6$ the differentials vanish on $E_t^{(*,20,4,12)}$. This graded piece contains b_{21}^2 at the E_4 page, but it does not survive to $E_5 = E_6$. The only other possible elements in this group arise from elements in $E_t^{*,21,4,13}$ over $\mathbb{C}$ which we have seen is trivial at the E_4 page. This verifies the hypotheses of May's convergence theorem.

Remark 7.17 Note that over $\mathbb{F}_q$ with $q \equiv 5 \bmod 8$ the map $\mathbb{L}c\{g\}$ is detected by $u\tau^{11}h_3^3$, which is in Adams filtration 3. But over $\overline{\mathbb{F}}_q$, the map $\mathbb{L}c\{g\}$ is in Adams filtration 4.

Proposition 7.18 *When $q \equiv 3 \bmod 4$ and $0 \leq n \leq 18$, there is an isomorphism $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}_n^s \oplus \hat{\pi}_{n+1}^s$.*

Proof The case $n = 0$ is resolved by Lemma 7.14, so we now consider $0 < n \leq 18$, where we may use the motivic Adams spectral sequence as in Proposition 7.15.

The differentials d_r for $r \geq 2$ vanish on the following generators for degree reasons:

$$[\rho\tau],\quad \rho,\quad h_0,\quad h_1,\quad h_3,\quad [\tau h_2^2],\quad [\tau c_0],\quad [\tau P h_1],\quad d_0,\quad [\tau P c_0],\quad [\tau P^2 h_1].$$

Since $\hat{\pi}_1^s \cong \mathbb{Z}/2$ is a summand of $\hat{\pi}_{1,0}(\mathbb{F}_q)$, we must have $d_r([\tau h_1]) = 0$ for all $r \geq 2$. Since $\hat{\pi}_3^s \cong \mathbb{Z}/8$ is a summand of $\hat{\pi}_{3,0}(\mathbb{F}_q)$, we must have $d_2(h_2) = 0$. An argument similar to that given for Proposition 7.15 establishes

$$d_2(h_4) = h_0 h_3^2, \quad d_2(e_0) = h_1^2 d_0, \quad d_2(f_0) = h_0^2 e_0$$

by comparison to $\mathfrak{M}(\overline{\mathbb{F}}_q)$. Also, we determine $d_r([\tau c_1]) = 0$ for $r \geq 2$ by comparing with $\mathfrak{M}(\overline{\mathbb{F}}_q)$, as the class $[\tau c_1]$ must be a permanent cycle.

The one exceptional case is $d_3(h_0 h_4)$. Here we must have $d_3(h_0 h_4) = h_0 d_0 + \rho h_1 d_0$ in order for $\hat{\pi}_{14}^s = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ to be a summand of $\hat{\pi}_{14,0}(\mathbb{F}_q)$.

The elements in weight 0 are all of the form $[\tau^2]^i x$ or $[\rho\tau][\tau^2]^{i-1}x$ where x is not a multiple of τ^2 and weight $2i$, or of the form $\rho[\tau^2]^i x$ if x is not a multiple of τ^2 and of weight $2i+1$. The differentials of the elements in weight 0 are now determined by using the Leibniz rule. Since $\lambda(q) = \nu_2(q^2-1) \geq 3$, we have $d_2(\tau^2) = 0$. This is sufficient to ensure that for elements x in stems $s \leq 19$ there are no nontrivial differentials of the form $d_r([\tau^2]^i x) = \rho\tau^{2i-1}h_0^r x$ when $[\tau^2]^i x$ has weight 0. This resolves all differentials in weight 0 for stems $s \leq 19$ and there are no hidden 2–extensions in this range. Hence for $0 < n \leq 18$ there is an isomorphism $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}_n^s \oplus \hat{\pi}_{n+1}^s$. □

Remark 7.19 When $q \equiv 3 \bmod 4$, it is unclear whether $d_2([\tau^2 g]) = [\rho\tau g]$ or $d_2([\tau^2 g]) = 0$. This is all that obstructs the identification of the stems $\hat{\pi}_{19,0}(\mathbb{F}_q)$ and $\hat{\pi}_{20,0}(\mathbb{F}_q)$ in this case.

7.5 Base change for finite fields

Proposition 7.20 *Let $q = p^\nu$, where p is an odd prime. For a field extension $f\colon \mathbb{F}_q \to \mathbb{F}_{q^i}$ with i odd, the induced maps*

$$\mathbb{L}f^*\colon H^{**}(\mathbb{F}_q) \to H^{**}(\mathbb{F}_{q^i}) \quad \text{and} \quad \mathbb{L}f^*\colon \mathcal{A}^{**}(\mathbb{F}_q) \to \mathcal{A}^{**}(\mathbb{F}_{q^i})$$

are isomorphisms.

Proof The claim follows by checking on étale cohomology. The map on cohomology is determined by $H^1_{\text{et}}(\mathbb{F}_q;\mu_2) \to H^1_{\text{et}}(\mathbb{F}_{q^i};\mu_2)$, which is just the induced map $\mathbb{F}_q^\times/2 \to \mathbb{F}_{q^i}^\times/2$. So long as i is odd, this map is an isomorphism. □

Corollary 7.21 *For $q = p^\nu$ with p an odd prime, the induced map $\mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\mathbb{F}_{q^i})$ is an isomorphism of spectral sequences whenever i is odd.*

Proposition 7.22 *Let $q = p^\nu$ with p an odd prime. Let $\widetilde{\mathbb{F}}_q$ denote the union of the field extensions $\mathbb{F}_{q^i}$ over $\mathbb{F}_q$ with i odd. The field extension $f\colon \mathbb{F}_q \to \widetilde{\mathbb{F}}_q$ induces isomorphisms $\mathbb{L}f^*\colon H^{**}(\mathbb{F}_q) \to H^{**}(\widetilde{\mathbb{F}}_q)$ and $\mathbb{L}f^*\colon \mathcal{A}^{**}(\mathbb{F}_q) \to \mathcal{A}^{**}(\widetilde{\mathbb{F}}_q)$. Hence the map $\mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\widetilde{\mathbb{F}}_q)$ is an isomorphism of spectral sequences.*

Proof This follows by a colimit argument using Proposition 7.20. □

Corollary 7.23 *For any integers s and $w \geq 0$, there is an isomorphism $\hat{\pi}_{s,w}(\mathbb{F}_q) \cong \hat{\pi}_{s,w}(\widetilde{\mathbb{F}}_q)$.*

Proposition 7.24 *Let $q = p^\nu$, where p is an odd prime. For a field extension $f\colon \mathbb{F}_q \to \mathbb{F}_{q^i}$ with i even, the map $f^*\colon H^{1,*}(\mathbb{F}_q) \to H^{1,*}(\mathbb{F}_{q^j})$ is trivial, and the map $f^*\colon H^{0,*}(\mathbb{F}_q) \to H^{0,*}(\mathbb{F}_{q^j})$ is injective.*

Proof The map is determined by $\mathbb{L}f^*\colon H^{1,1}(\mathbb{F}_q) \to H^{1,1}(\mathbb{F}_{q^i})$, which is just the map $\mathbb{F}_q^\times/2 \to \mathbb{F}_{q^i}^\times/2$. However, any nonsquare $x \in \mathbb{F}_q^\times$ will be a square in $\mathbb{F}_{q^i}^\times$ when i is even. □

Corollary 7.25 *Let $q = p^\nu$ with p an odd prime. For a field extension $f\colon \mathbb{F}_q \to \mathbb{F}_{q^i}$ with i even, the induced map $\mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\mathbb{F}_{q^i})$ kills the class u (respectively ρ and $[\rho\tau]$) and all of their multiples at the E_2 page.*

Proof The induced map of cobar complexes is determined from Proposition 7.24 and shows the class u (respectively ρ and $[\rho\tau]$) is killed under base change. □

8 Implementation of motivic Ext group calculations

The computer calculations used in this paper were done with the program available from Fu and Wilson [16]. The program is written in Python and calculates $\operatorname{Ext}(F)$ when F is $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{F}_q$ by producing a minimal resolution of $H^{**}(F)$ by $\mathcal{A}^{**}(F)$ modules in a range. With this complex in hand, the program then produces its dual and calculates cohomology in each degree.

To calculate a free resolution of $H^{**}(F)$ of $\mathcal{A}^{**}(F)$ modules, we first need the program to efficiently perform calculations in $\mathcal{A}^{**}(F)$. The mod 2 motivic Steenrod algebra is generated by the squaring operations Sq^i and the cup products $\alpha \cup -$ for $\alpha \in H^{**}(F)$. These generators satisfy Adem relations, which are recorded in [22, Section 5.1] by Hoyois, Kelly and Østvær and in [49, Theorem 10.2] by Voevodsky. Additionally, one needs the commutation relations $\mathrm{Sq}^{2i}\tau = \tau\,\mathrm{Sq}^{2i} + \tau\rho\,\mathrm{Sq}^{2i-1}$ for $i > 0$ and $\mathrm{Sq}^{2i+1}\tau = \tau\,\mathrm{Sq}^{2i+1} + \rho\,\mathrm{Sq}^{2i} + \rho^2\,\mathrm{Sq}^{2i-1}$ for $i \geq 0$ which are obtained from the Cartan formula. With these relations, the program can calculate the canonical form of any element of $\mathcal{A}^{**}$, that is, as a sum of monomials $\alpha \cdot \mathrm{Sq}^I$ where $\alpha \in H^{**}(F)$ and I is an admissible sequence.

With the algebra of $\mathcal{A}^{**}(F)$ available to the program, it then proceeds to calculate a minimal resolution of $H^{**}(F)$ by $\mathcal{A}^{**}(F)$ modules. This is where a great deal of computational effort is spent. To clarify what a minimal resolution is in practice, let $\prec$ denote the order on $\mathbb{Z} \times \mathbb{Z}$ given by $(m_1, n_1) \prec (m_2, n_2)$ if and only if $m_1 + n_1 < m_2 + n_2$, or $m_1 + n_1 = m_2 + n_2$ and $n_1 < n_2$. The reader is encouraged to compare this definition with the definition of McCleary in [30, Definition 9.3] and consult Bruner [9] for detailed calculations of a minimal resolution for the Adams spectral sequence of topology.

Definition 8.1 A resolution of $H^{**}(F)$ by $\mathcal{A}^{**}(F)$ modules $H^{**}(F) \leftarrow P^\bullet$ is a minimal resolution if the following conditions are satisfied:

(1) Each module P^i is equipped with ordered basis $\{h_i(j)\}$ such that if $j \leq k$ then $\deg h_i(j) \preceq \deg h_i(k)$.

(2) $\mathrm{im}(h_i(k)) \notin \mathrm{im}(\langle h_i(j) \mid j < k\rangle)$.

(3) $\deg h_i(k)$ is minimal with respect to degree in the order $\prec$ over all elements in $P^{i-1} \setminus \mathrm{im}(\langle h_i(j) \mid j < k\rangle)$.

The computer program calculates the first n maps and modules in a minimal resolution up to bidegree $(2n, n)$. With this, it then calculates the dual of the resolution by applying the functor $\mathrm{Hom}_{\mathcal{A}^{**}(F)}(-, H^{**}(F))$ to the resolution $P^\bullet$. With the cochain complex $\mathrm{Hom}_{\mathcal{A}^{**}(F)}(P^\bullet, H^{**}(F))$ in hand, the program calculates cohomology in each degree, that is, $\mathrm{Ext}^{f,(s+f,w)}(\mathbb{F}_q)$.

Because the program calculates an explicit resolution of $H^{**}(F)$, the products of elements in $\mathrm{Ext}(F)$ can be obtained from the composition product; see McCleary [30, Theorem 9.5].

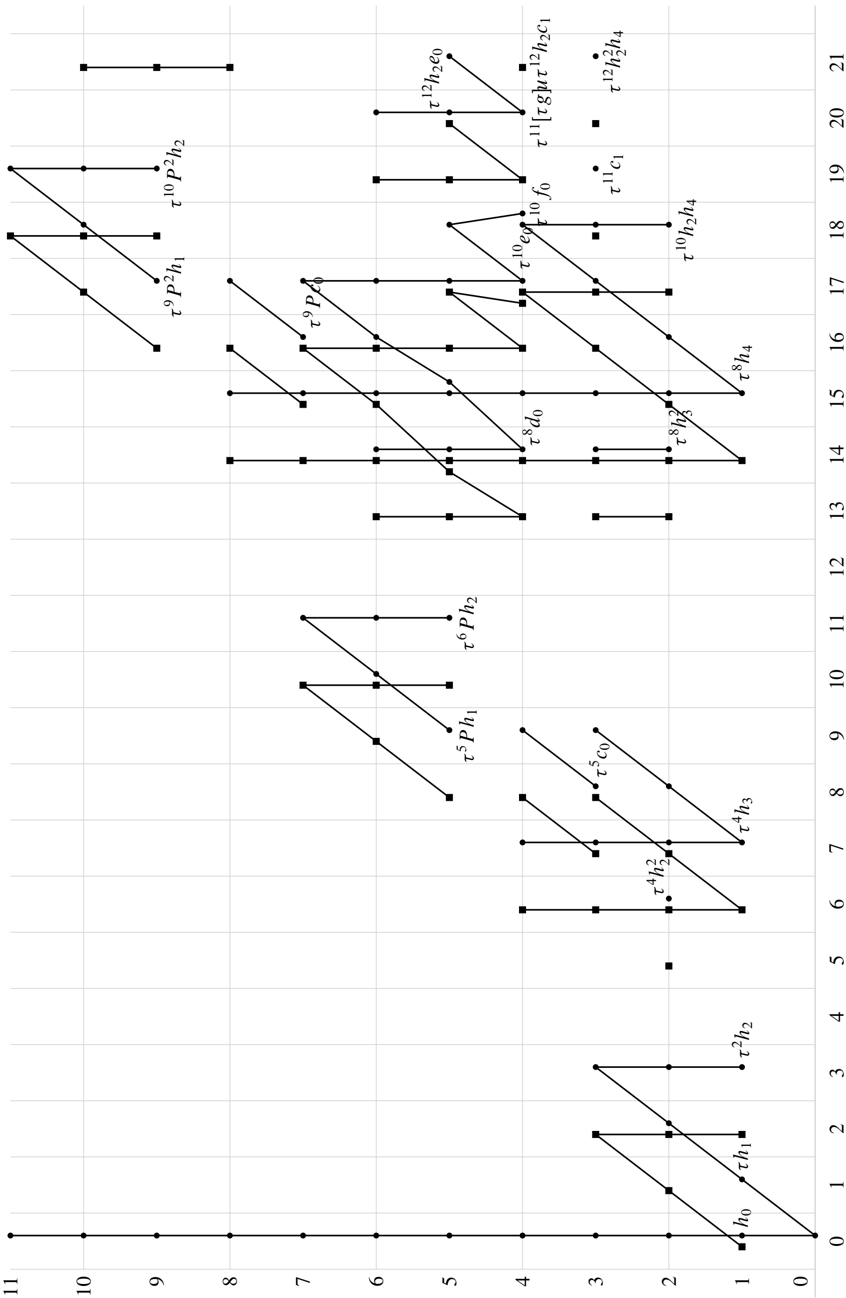

Figure 1: E_2 page of MASS for $\mathbb{F}_q$ with $q \equiv 1 \bmod 4$, weight 0

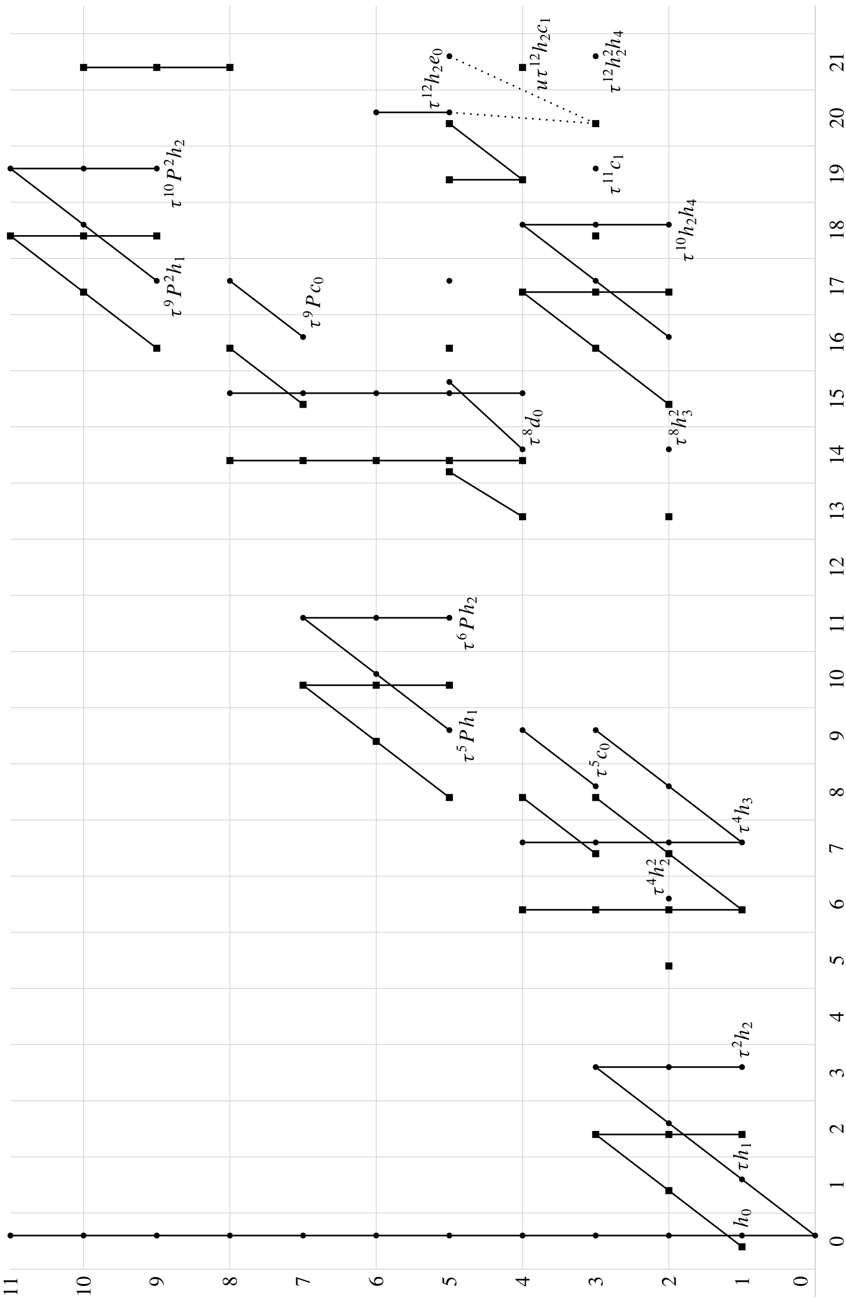

Figure 2: E_∞ page of MASS for $\mathbb{F}_q$ with $q \equiv 1 \bmod 4$, weight 0

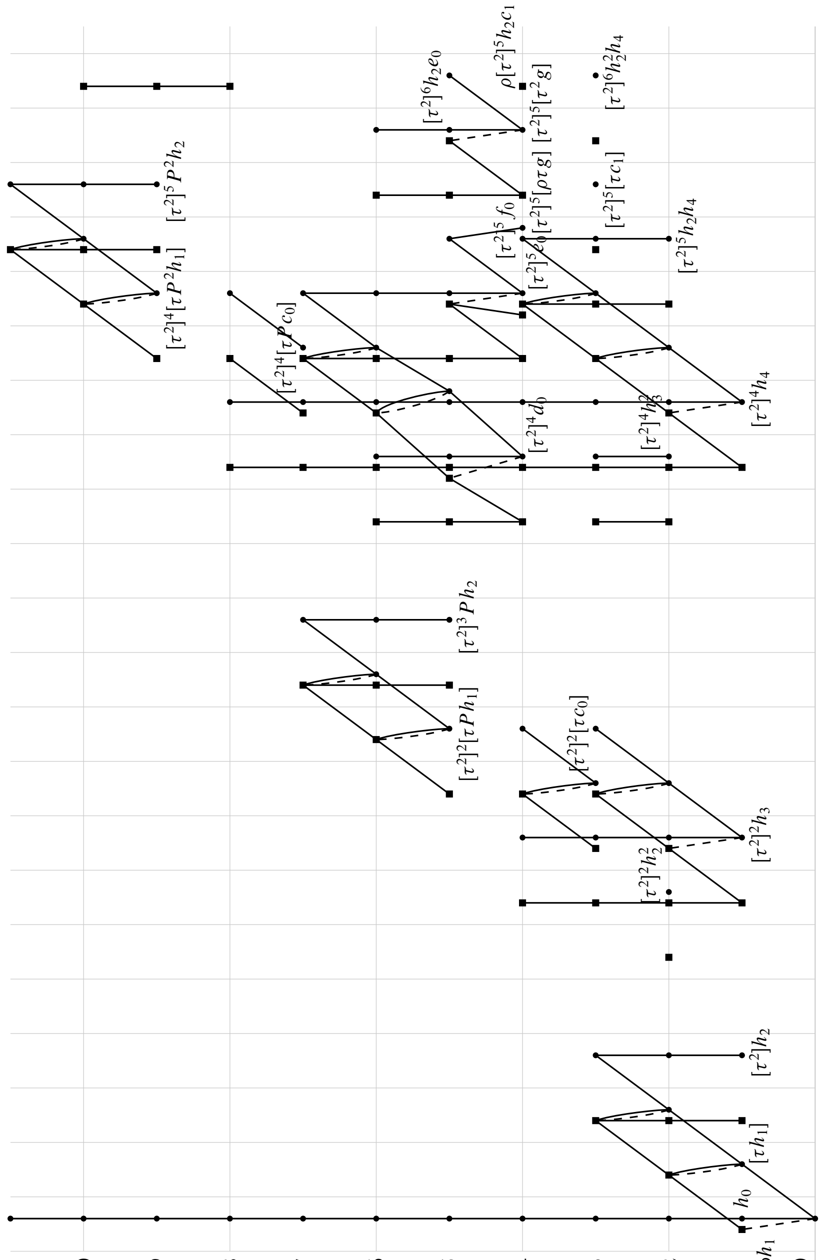

Figure 3: E_2 page of MASS for $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$, weight 0

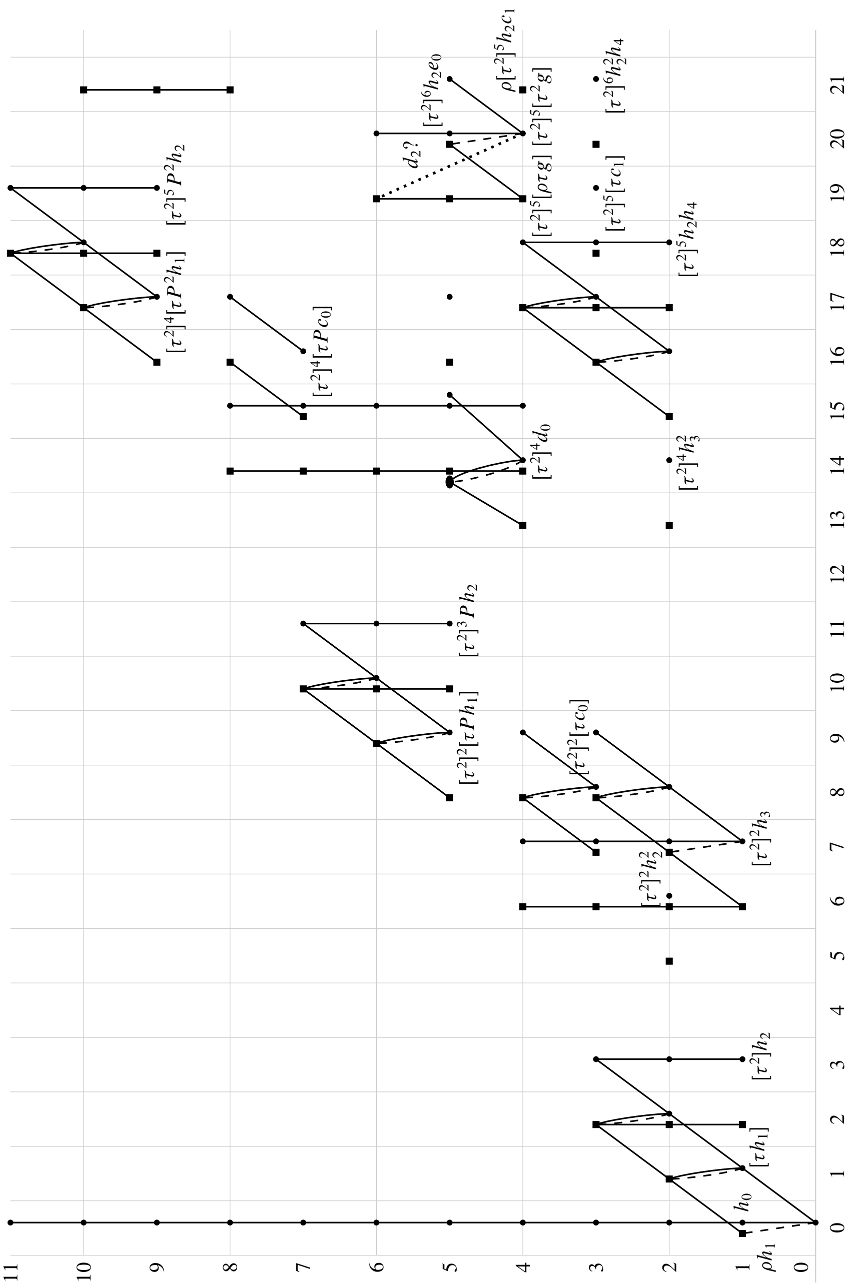

Figure 4: E_∞ page of MASS for $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$, weight 0

9 Charts

The weight 0 part of the E_2 page of the mod 2 MASS over $\mathbb{F}_q$ is depicted in Figures 1 and 3 according to the case $q \equiv 1 \bmod 4$ or $q \equiv 3 \bmod 4$. The weight 0 part of the E_∞ page of the mod 2 MASS over $\mathbb{F}_q$ can be found in Figures 2 and 4.

In each chart, a circular or square dot in grading (s, f) represents a generator of the $\mathbb{F}_2$ vector space in the graded piece of the spectral sequence. The square dots are used to indicate that the given element is divisible by u, ρ or $\rho\tau$, depending on the case. Circular dots denote elements which are not divisible by u, ρ or $\rho\tau$. In Figure 4, there is an oval dot which corresponds to the class with representative $\tau^8 \rho h_1 d_0 \equiv \tau^8 h_0 d_0$, as the class $\rho h_1 d_0 + h_0 d_0$ is killed.

We indicate the product of a given class by h_0 with a solid, vertical line. In the case $q \equiv 3 \bmod 4$, multiplication by ρh_1 plays an important role, so nonzero products by ρh_1 are indicated by dashed vertical lines. In particular, when $q \equiv 3 \bmod 4$, multiplication by 2 in $\hat{\pi}_{**}(\mathbb{F}_q)$ is detected by multiplication by $h_0 + \rho h_1$. The lines of slope 1 indicate multiplication by τh_1 or $[\tau h_1]$ depending on the case. We caution the reader that the product structure displayed in this chart was obtained by computer calculation and not all products were established by hand in this paper. For example, the products in the 8 stem by h_0 are hidden in the May spectral sequence.

Dotted lines are used in two separate instances in these charts. The first use is in Figure 2, where dotted lines indicate hidden extensions by h_0 and τh_1. The other instance is in Figure 4 to indicate an unknown d_2 differential.

Additional charts obtained from the program of Fu and Wilson [16] may be found at the website `http://math.rutgers.edu/~wilson47/image_viewer/`.

References

[1] **J F Adams**, *A finiteness theorem in homological algebra*, Proc. Cambridge Philos. Soc. 57 (1961) 31–36 MR

[2] **J F Adams**, *Stable homotopy and generalised homology*, University of Chicago Press (1974) MR

[3] **A Ananyevskiy**, **M Levine**, **I Panin**, *Witt sheaves and the η–inverted sphere spectrum*, preprint (2015) arXiv

[4] **J Ayoub**, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique, I*, Astérisque 314, Société Mathématique de France, Paris (2007) MR

[5] **B A Blander**, *Local projective model structures on simplicial presheaves*, K–Theory 24 (2001) 283–301 MR

[6] **A K Bousfield**, *The localization of spectra with respect to homology*, Topology 18 (1979) 257–281 MR

[7] **A K Bousfield**, **D M Kan**, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics 304, Springer (1972) MR

[8] **R R Bruner**, *Ext in the nineties*, from "Algebraic topology" (M C Tangora, editor), Contemp. Math. 146, Amer. Math. Soc., Providence, RI (1993) 71–90 MR

[9] **R R Bruner**, *An Adams spectral sequence primer*, notes (2009) Available at http://math.wayne.edu/~rrb/papers/adams.pdf

[10] **R R Bruner**, *The cohomology of the mod 2 Steenrod algebra*, unpublished manuscript (2016) Available at http://math.wayne.edu/~rrb/papers/cohom.pdf

[11] **D Dugger**, **D C Isaksen**, *Motivic cell structures*, Algebr. Geom. Topol. 5 (2005) 615–652 MR

[12] **D Dugger**, **D C Isaksen**, *The motivic Adams spectral sequence*, Geom. Topol. 14 (2010) 967–1014 MR

[13] **D Dugger**, **D Isaksen**, *Low-dimensional Milnor–Witt stems over* $\mathbb{R}$, Ann. K-Theory 2 (2017) 175–210 MR

[14] **B I Dundas**, **M Levine**, **P A Østvær**, **O Röndigs**, **V Voevodsky**, *Motivic homotopy theory*, Springer (2007) MR

[15] **B I Dundas**, **O Röndigs**, **P A Østvær**, *Motivic functors*, Doc. Math. 8 (2003) 489–525 MR

[16] **K Fu**, **G M Wilson**, *Motivic Adams spectral sequence program*, Python code (2015) Available at https://github.com/glenwilson/MassProg

[17] **T Geisser**, *Motivic cohomology over Dedekind rings*, Math. Z. 248 (2004) 773–794 MR

[18] **M A Hill**, *Ext and the motivic Steenrod algebra over* $\mathbb{R}$, J. Pure Appl. Algebra 215 (2011) 715–727 MR

[19] **P S Hirschhorn**, *Model categories and their localizations*, Mathematical Surveys and Monographs 99, Amer. Math. Soc., Providence, RI (2003) MR

[20] **M Hovey**, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra 165 (2001) 63–127 MR

[21] **M Hoyois**, *From algebraic cobordism to motivic cohomology*, J. Reine Angew. Math. 702 (2015) 173–226 MR

[22] **M Hoyois**, **S Kelly**, **P A Østvær**, *The motivic Steenrod algebra in positive characteristic*, preprint (2013) arXiv To appear in J. Eur. Math. Soc.

[23] **P Hu**, *S–modules in the category of schemes*, Mem. Amer. Math. Soc. 767, Amer. Math. Soc., Providence, RI (2003) MR

[24] **P Hu**, **I Kriz**, **K Ormsby**, *Remarks on motivic homotopy theory over algebraically closed fields*, J. K-Theory 7 (2011) 55–89 MR

[25] **D C Isaksen**, *Classical and motivic Adams charts*, preprint (2014) arXiv

[26] **D C Isaksen**, *Stable stems*, preprint (2014) arXiv

[27] **J F Jardine**, *Motivic symmetric spectra*, Doc. Math. 5 (2000) 445–553 MR

[28] **S O Kochman**, *Stable homotopy groups of spheres: a computer-assisted approach*, Lecture Notes in Mathematics 1423, Springer (1990) MR

[29] **M Levine**, *A comparison of motivic and classical stable homotopy theories*, J. Topol. 7 (2014) 327–362 MR

[30] **J McCleary**, *A user's guide to spectral sequences*, 2nd edition, Cambridge Studies in Advanced Mathematics 58, Cambridge University Press (2001) MR

[31] **J Milnor**, *The Steenrod algebra and its dual*, Ann. of Math. 67 (1958) 150–171 MR

[32] **F Morel**, *An introduction to $\mathbb{A}^1$–homotopy theory*, from "Contemporary developments in algebraic K–theory" (M Karoubi, A O Kuku, C Pedrini, editors), ICTP Lect. Notes XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste (2004) 357–441 MR

[33] **F Morel**, *The stable $\mathbb{A}^1$–connectivity theorems*, K–Theory 35 (2005) 1–68 MR

[34] **F Morel**, *$\mathbb{A}^1$–algebraic topology over a field*, Lecture Notes in Mathematics 2052, Springer (2012) MR

[35] **F Morel**, **V Voevodsky**, *$\mathbf{A}^1$–homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. 90 (1999) 45–143 MR

[36] **R M F Moss**, *Secondary compositions and the Adams spectral sequence*, Math. Z. 115 (1970) 283–310 MR

[37] **K M Ormsby**, *Motivic invariants of p–adic fields*, J. K-Theory 7 (2011) 597–618 MR

[38] **K M Ormsby**, **P A Østvær**, *Motivic Brown–Peterson invariants of the rationals*, Geom. Topol. 17 (2013) 1671–1706 MR

[39] **K M Ormsby**, **P A Østvær**, *Stable motivic π_1 of low-dimensional fields*, Adv. Math. 265 (2014) 97–131 MR

[40] **D C Ravenel**, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics 121, Academic Press, Orlando, FL (1986) MR

[41] **O Röndigs**, **M Spitzweck**, **P A Østvær**, *The first stable homotopy groups of motivic spheres*, preprint (2016) arXiv

[42] **W Scharlau**, *Quadratic and Hermitian forms*, Grundl. Math. Wissen. 270, Springer (1985) MR

[43] **J-P Serre**, *Local fields*, Graduate Texts in Mathematics 67, Springer (1979) MR

[44] **C Soulé**, *K–théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. 55 (1979) 251–295 MR

[45] **M Spitzweck**, *A commutative $\mathbb{P}^1$–spectrum representing motivic cohomology over Dedekind domains v3*, preprint (2013) arXiv

[46] **R M Switzer**, *Algebraic topology—homotopy and homology*, Grundl. Math. Wissen. 212, Springer (1975) MR

[47] **V Voevodsky**, *$\mathbf{A}^1$–homotopy theory*, from "Proceedings of the International Congress of Mathematicians, I", Documenta Mathematica, Bielefeld (1998) 579–604 MR

[48] **V Voevodsky**, *Motivic cohomology with $\mathbf{Z}/2$–coefficients*, Publ. Math. Inst. Hautes Études Sci. 98 (2003) 59–104 MR

[49] **V Voevodsky**, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. 98 (2003) 1–57 MR

[50] **V Voevodsky**, *Motivic Eilenberg–Maclane spaces*, Publ. Math. Inst. Hautes Études Sci. 112 (2010) 1–99 MR

[51] **G Wang**, **Z Xu**, *The algebraic Atiyah–Hirzebruch spectral sequence of real projective spectra*, preprint (2016) arXiv

[52] **G M Wilson**, *Motivic stable stems over finite fields*, PhD thesis, Rutgers University–New Brunswick (2016) MR Available at http://search.proquest.com/docview/1844381634

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