Errata to Relative Thom spectra via operadic Kan extensions

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In the paper "Relative Thom Spectra Via Operadic Kan Extensions" there were minor errors in Lemma 6, Proposition 8 and the proof of Theorem 1. The following Theorem, Lemma and Proposition serve to replace them.

Theorem 1 *Suppose $Y \xrightarrow{i} X \xrightarrow{q} B$ is a fiber sequence of reduced $\mathbb{E}_n$-monoidal Kan complexes for $n > 1$ with i and q both maps of $\mathbb{E}_n$-algebras. Let $f\colon X \to BGL_1(R)$ be a morphism of $\mathbb{E}_n$-monoidal Kan complexes for $n > 1$. Then there is a a morphism of $\mathbb{E}_{n-1}$-algebras $B \to BGL_1(M(f \circ i))$ whose associated Thom spectrum is equivalent to Mf.*

Proof Note that $M(f \circ i)$ is an $\mathbb{E}_n$-algebra, so $BGL_1(M(f \circ i))$ is an $(n-1)$-fold loop space, so we cannot hope for the desired map to be more structured than this. By Lemmas 5 and 2 the $\mathbb{E}_{n-1}$-monoidal left Kan extension of $X \xrightarrow{f} BGL_1(\mathbb{S}) \hookrightarrow \mathcal{S}$ along $q\colon X \to B$ exists and takes the unique 0-simplex of B to the $\mathbb{E}_n$-algebra $M(f \circ i)$. By Proposition 3, this Kan extension factors as a morphism of $\mathbb{E}_{n-1}$-monoidal Kan complexes through $BGL_1(M(f \circ i))$. Taking the Thom spectrum of the induced morphism $B \to BGL_1(M(f \circ i))$ produces $M(f \circ i)/(\Omega B)$ as a Thom spectrum over $M(f \circ i)$. Moreover, taking the colimit of the functor $B \to BGL_1(M(f \circ i)) \hookrightarrow LMod_{M(f\circ i)}$ is equivalent to taking the colimit of the underlying spectra, by Corollary 4.2.3.7 of [1]. However, taking the colimit in spectra is equivalent to forming the left operadic Kan extension of $B \to \mathcal{S}$ along the map $B \to *$. By Lemma 7 and Corollary 3.1.4.2 of [1] we have that the left operadic Kan extension along $X \to B$ followed by the left operadic Kan extension along $B \to *$ is equivalent to the left operadic Kan extension along $X \to *$ (i.e. Kan extensions compose). In other words, the $\mathbb{E}_{n-1}$-$M(f \circ i)$-module $M(f \circ i)/(\Omega B)$ has an underlying spectrum equivalent to the colimit of $X \to BGL_1(\mathbb{S})$ which is of course Mf. Thus the iterated Kan extension which produces $M(f \circ i) = \mathbb{S}/\Omega Y$ and then quotients it by the action of ΩB is equivalent to the one-step Kan extension producing $\mathbb{S}/\Omega X \simeq Mf$ with an "action" of the trivial $\mathbb{E}_{n-1}$-space. Hence Mf is produced as a Thom spectrum over $M(f \circ i)$. □

The following replaces Lemma 6 in the original paper. Therein we claimed to compute the colimit in $LMod(M(f \circ i))$, when we should have been computing the colimit in $\mathcal{S}$. This issue is corrected here.

Lemma 2 *Let $Y \xrightarrow{i} X \xrightarrow{q} B$ be a fiber sequence of $\mathbb{E}_n$-monoidal Kan complexes. The $\mathbb{E}_n$-monoidal left Kan extension of an $\mathbb{E}_n$-monoidal morphism $f\colon X \to BGL_1(\mathbb{S}) \hookrightarrow \mathcal{S}$ along $q\colon X \to B$ is computed by taking the colimit of the composition*

$$fib(X \to B) \simeq Y \to X \to BGL_1(\mathbb{S}) \hookrightarrow \mathcal{S}.$$

Proof Following the notation given in Definition 3.1.2.2 and the construction in Remark 3.1.3.15 of [1], we have a correspondence of ∞-operads given by

$$\mathcal{M}^{\otimes} \simeq (X^{\otimes} \times \Delta^1) \coprod_{X^{\otimes} \times \{1\}} B^{\otimes} \to \mathcal{F}in_* \times \Delta^1.$$

In other words, there is a family of ∞-operads indexed by Δ^1 which looks like $X^{\otimes}$ (the ∞-operad associated to X as an $\mathbb{E}_n$-monoidal Kan complex) at one end and $B^{\otimes}$ at the other end. Formula $(*)$ of Definition 3.1.2.2 of [1] states that the value of the desired Kan extension at a 0-simplex $\sigma \in B$ is given by the colimit diagram:

$$((\mathcal{M}^{\otimes}_{act})_{/\sigma} \times_{\mathcal{M}^{\otimes}} X^{\otimes})^{\triangleright} \to (\mathcal{M}^{\otimes})^{\triangleright}_{/\sigma} \to \mathcal{M}^{\otimes} \to \mathcal{T}$$

where the morphism $(\mathcal{M}^{\otimes})^{\triangleright}_{/\sigma} \to \mathcal{M}^{\otimes}$ takes the cone point to σ. In other words, the value of the Kan extension at σ is computed by taking the colimit over the diagram in $\mathcal{M}^{\otimes}$ of objects (and active morphisms) living over σ. As the simplicial set $\mathcal{M}^{\otimes}$ is nothing more than the mapping cylinder of the morphism of $\mathbb{E}_n$-monoidal Kan complexes $X^{\otimes} \to B^{\otimes}$, we have the result. □

The following replaces Proposition 8 in the original paper. Similarly to the last error, the mistake in the original paper was to lift from $\mathcal{S}$ to $LMod(M(f \circ i))$ prematurely. In what follows, we show that the map to $\mathcal{S}$ factors *through* $BGL_1(M(f \circ i))$ and hence through $LMod(M(f \circ i))$. This latter fact was incorrectly assumed in the original.

Proposition 3 *Let $Y \xrightarrow{i} X \xrightarrow{q} B$ be a fiber sequence of reduced, connected $\mathbb{E}_n$-monoidal Kan complexes. The left operadic Kan extension of an $\mathbb{E}_n$-morphism $f\colon X \to BGL_1(\mathbb{S}) \to \mathcal{S}$ along the $\mathbb{E}_n$-morphism $q\colon X \to B$ factors as a morphism of $\mathbb{E}_{n-1}$-monoidal Kan complexes through $BGL_1(M(f \circ i))$.*

Proof Note that the left operadic Kan extension along q takes the unique zero simplex of B to $M(f \circ i)$ by Lemma 2. Since B is an $\mathbb{E}_n$-monoidal Kan complex it is also an $\mathbb{E}_n$-monoidal quasicategory with monoidal unit 1_B corresponding to the base point of B. Moreover all the morphisms of B are also 1_B-module isomorphisms. In other words, $LMod_{1_B} \simeq BGL_1(1_B) \simeq B$ as $\mathbb{E}_{n-1}$-monoidal quasicategories (also cf. Corollary 4.2.4.9 of [1]). Hence it must be that this Kan extension, being an $\mathbb{E}_n$-monoidal functor, induces an $\mathbb{E}_{n-1}$-monoidal functor $BGL_1(1_B) \simeq B \to BGL_1(M(f \circ i))$. □

Remark 4 We can think of the identification $B \simeq BGL_1(1_B)$ as a construction of the delooping of ΩB by taking the base point component of $Pic(LMod_{\Omega B})$. In other words as a quasicategory B can be thought of as the maximal $\mathbb{E}_{n-1}$-monoidal Kan complex on the object $\Omega B \in LMod_{\Omega B}$.

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