Algebraic & Geometric Topology 17 (2017) 1375–1405

Fully irreducible automorphisms of the free group via Dehn twisting in $\sharp_k(S^2 \times S^1)$

FUNDA GÜLTEPE

By using a notion of a geometric Dehn twist in $\sharp_k(S^2 \times S^1)$, we prove that when projections of two $\mathbb{Z}$–splittings to the free factor complex are far enough from each other in the free factor complex, Dehn twist automorphisms corresponding to the $\mathbb{Z}$–splittings generate a free group of rank 2. Moreover, every element from this free group either is conjugate to a power of one of the Dehn twists or is a fully irreducible outer automorphism of the free group. We also prove that, when the projections of $\mathbb{Z}$–splittings are sufficiently far away from each other in the intersection graph, the group generated by the Dehn twists has automorphisms that are either conjugate to Dehn twists or atoroidal fully irreducible.

20F28, 20F65, 57M07

1 Introduction

Due to their dynamical properties, *fully irreducible* outer automorphisms are important to understanding the dynamics and geometric structure of $\mathrm{Out}(F_k)$ and its subgroups; see Levitt and Lustig [22], Clay and Pettet [5] and Behrstock, Bestvina and Clay [1]. Just like pseudo-Anosov surface homeomorphisms, fully irreducibles are characterized to be the class of automorphisms no power of which fixes a conjugacy class of a proper free factor of F_k. Since their dynamical properties and their role in $\mathrm{Out}(F_k)$ are similar to those of pseudo-Anosov mapping classes for the mapping class group, to construct fully irreducibles it is natural to seek ways similar to those of pseudo-Anosov constructions. In this work we will provide such a construction using *Dehn twist automorphisms*, by composing powers of Dehn twists from the free group of rank 2 that they generate. This is inspired by the work of Thurston on pseudo-Anosov mapping classes of the mapping class group of a surface; see Thurston [30]. Yet in our proof we use a ping-pong method similar to that used by Hamidi-Tehrani [12] to generalize Thurston's result to Dehn twists along multicurves.

Finding free groups of rank 2 generated by outer automorphisms and constructing fully irreducible automorphisms by composing (possibly powers of) other automorphisms and is not new to the study of $\mathrm{Out}(F_k)$. For instance, Clay and Pettet in [5] constructed

Published: 17 July 2017 DOI: 10.2140/agt.2017.17.1375

fully irreducibles by composing elements of a free group of rank 2 which was generated by powers of two Dehn twist automorphisms. However, the powers of the Dehn twists used to generate the free group were not uniform but depended on the twists; one needed to take a different power for each pair of Dehn twists to obtain a free group.

In their work, Clay and Pettet studied Dehn twists algebraically, as outer automorphisms of the free group, and they used algebraic tools to study them. As a result their construction produced the nonuniform powers of twists. In this paper, our goal is to construct fully irreducible automorphisms by studying Dehn twist automorphisms. To obtain a certain type of uniformity on the way, we first change the model for $\mathrm{Out}(F_k)$ from the 1–dimensional one to the 3–dimensional one: $M = \sharp_k (S^2 \times S^1)$. This way, we are able to understand Dehn twists geometrically, using essential embedded tori in M. This approach results in a more geometric construction of fully irreducibles.

More specifically, we will prove the following theorem using *geometric Dehn twists*.

Theorem 1.1 *Let T_1 and T_2 be two $\mathbb{Z}$–splittings of the free group F_k with rank $k > 2$ and α_1 and α_2 be two corresponding free factors in the free factor complex FF_k of the free group F_k. Let D_1 be a Dehn twist fixing α_1 and D_2 a Dehn twist fixing α_2, corresponding to T_1 and T_2, respectively. Then there exists a constant $N = N(k)$ such that whenever $d_{\mathrm{FF}_k}(\alpha_1, \alpha_2) \geq N$,*

(1) *$\langle D_1, D_2 \rangle \simeq F_2$, and*

(2) *all elements of $\langle D_1, D_2 \rangle$ which are not conjugate to the powers of D_1 and D_2 in $\langle D_1, D_2 \rangle$ are fully irreducible.*

Now we would like to give the definitions necessary to understand the statement of Theorem 1.1 and explain the ideas used in its proof.

Splittings and $\mathrm{Out}(F_k)$–complexes A *Dehn twist automorphism* is an element of $\mathrm{Out}(F_k)$ defined by using $\mathbb{Z}$–splittings of F_k either as an amalgamated free product (eg $F_k = A *_{\langle c\rangle} B$) or as an HNN extension of the free group (eg $F_k = A*_{\langle c\rangle}$). More precisely, it is induced by the following automorphisms corresponding to each type of $\mathbb{Z}$–splitting:

$$A *_{\langle c\rangle} B\colon \begin{array}{ll} a \mapsto a & \text{for } a \in A, \\ b \mapsto cbc^{-1} & \text{for } b \in B \end{array} \qquad \text{and} \qquad A*_{\langle tct^{-1}=c'\rangle}\colon \begin{array}{ll} a \mapsto a & \text{for } a \in A, \\ t \mapsto tc. & \end{array}$$

Given a $\mathbb{Z}$–splitting of F_k as $F_k = A_1 *_{\langle c_1\rangle} B_1$ at least one of A_1, B_1 is a proper free factor. In the HNN extension case $F_k = A_1 *_{\langle c_1\rangle}$, the stable letter is a proper free factor. By Bass–Serre theory, each $\mathbb{Z}$–splitting of F_k gives rise to a tree whose quotient with respect to the action of the free group is a single edge. The edge stabilizer is $\mathbb{Z}$, and the vertex stabilizers in the amalgamated case are A_1 and B_1 while in the HNN case the vertex stabilizer is A_1. We will coarsely *project* each splitting onto the vertex which is a proper free factor. In the amalgamated case we consider the Dehn twist automorphism corresponding to the $\mathbb{Z}$–splitting which fixes this free factor and in the HNN case the Dehn twist automorphism will be the one fixing the vertex stabilizer. We study the action of this Dehn twist on the *free factor complex* FF_k of the free group F_k of rank k and we determine under which conditions the compositions of the Dehn twist automorphisms give fully irreducible automorphisms. The free factor complex is a simplicial complex whose vertices are conjugacy classes of proper free factors of F_k and the adjacency between two vertices corresponding to two free factors A and B is given whenever $A < B$ or $B < A$. This complex was first introduced by Hatcher and Vogtmann in [15] as a *curve complex* analog for $\mathrm{Out}(F_k)$ and in this work we will use its geometric properties due to its hyperbolicity, which are given in [2] by Bestvina and Handel.

There are several geometrically distinct hyperbolic simplicial complexes $\mathrm{Out}(F_k)$ acts on by simplicial automorphisms which are considered to be analogs to the curve complex for the mapping class group. Contrary to the case with the curve complex and the action of the mapping class group on it, it is not always possible to identify fully irreducible elements with respect to the way they act on a curve complex analog. For example, an element of $\mathrm{Out}(F_k)$ might act hyperbolically on a curve complex analog yet it may not be fully irreducible. In this work the free factor complex was used since loxodromic action of an automorphism on the free factor complex completely characterizes being fully irreducible for a free group automorphism. Thus, to identify fully irreducibles in a group generated by two Dehn twists, it is enough to have a loxodromic action.

By *geometric* Dehn twist we mean the following: For each equivalence class of a $\mathbb{Z}$–splitting, by Lemma 2.6, there is an associated homotopy class of a torus in M. More specifically, an amalgamated free product gives a separating torus in M whereas an HNN extension corresponds to a nonseparating torus. Hence each Dehn twist automorphism corresponds to a Dehn twist along the torus given by the $\mathbb{Z}$–splitting. The Dehn twist along a torus will be called a geometric Dehn twist.

Dehn twists and their almost fixed sets To prove the main theorem we use what we call a *ping-pong argument for an elliptic-type subgroup* since Dehn twists have fixed points in the free factor complex. To set up such an argument one needs to construct so-called ping-pong sets. Thus we need to know first that the points of the free factor complex which are not moved too far away by a power of a Dehn twist are manageable. More precisely, let $\phi \in \mathrm{Out}(F_k)$ and let

$$F_C(\phi) = \{x \in \mathrm{FF}_k : \exists n \neq 0 \text{ such that } d(x, \phi^n(x)) \leq C\}$$

be the almost fixed set corresponding to $\langle\phi\rangle$ in FF_k. The following theorem is the main ingredient in the elliptic-type ping-pong argument.

Theorem 1.2 *Let T be a $\mathbb{Z}$–splitting of the free group F_k with $k > 2$ and D_T denote a corresponding Dehn twist. Then, for all sufficiently large constants C, there exists a $C' = C'(C, k)$ such that the diameter of the almost fixed set $F_C(D_T)$ corresponding to $\langle D_T\rangle$ is bounded above by C'.*

We will drop the reference to the automorphism from the notation for the almost fixed set whenever it is clear from the context.

Relative twisting and distances along paths Now, to prove that the almost fixed sets of Dehn twists have bounded diameter, one needs to be able to calculate distances between points in the free factor complex effectively. However we cannot assume that there is a geodesic between two points in the free factor complex which is appropriate for our calculation purposes since we do not know what these geodesics are. But it is known that the *folding paths* in outer space give rise to geodesics in outer space and their projections to the free factor complex are quasigeodesics. To prove Theorem 1.2, we prove that there is a folding path whose projection to the free factor complex is at a bounded distance from the given free factor. To achieve this one would need an analog of the *annulus projection* and to be able to calculate distances on an annulus complex. Then using a version of the *bounded geodesic image theorem* of Masur and Minsky [24] one would conclude that whenever the number of twists is more than the universal constant given in that theorem, the quasigeodesic between a point and its twisted image has a vertex which does not intersect the core curve of the annulus. However, we do not have the main tool needed, which is an analog for annulus projection, since the subfactor projection is not defined for free factors of rank 1; see Bestvina and Feighn [3] and Taylor [29].

To calculate distances between points related to a rank-1 free factor without using a projection onto that free factor we refer to a theorem of Clay and Pettet. In [6] they give a pairing $\mathrm{tw}_a(G, G')$ called the *relative twisting number* between two graphs $G, G' \in \mathrm{CV}_k$ relative to some nontrivial $a \in F_k$ and it is defined using the *Guirardel core*. Using this pairing, they obtain a condition on the graphs $G, G' \in \mathrm{CV}_k$ that, when satisfied, enables them to construct a connecting geodesic between them, traveling through the thin part of CV_k.

Relative twisting along tori in $\sharp_k(S^2 \times S^1)$ We have used the interpretation of the *relative twisting number* pairing $\mathrm{tw}_a(G, G')$ for two spheres relative to an element of the free group, which is in our case the generator of the core (longitudinal) curve of a torus. Then the relative twist is a number which calculates distances between two spheres which are intersecting the same torus along its core curve. Mimicking annulus projection, the relative twisting number might be interpreted as the number of intersections between *projections* of some spheres in M onto a torus (yet we do not make a formal definition of such a projection). With the relative twisting number we are able to calculate a lower bound for the twisting number between a sphere and its Dehn-twisted image, relative to a torus hence in some sense relative to a rank-1 free factor (related to its core curve). Afterwards, a lemma of Clay and Pettet [6] guarantees the existence of a geodesic between the corresponding points in outer space along which the core curve gets short. Using a lemma of Bestvina and Feighn [2], we project this geodesic to the free factor complex and using the distance calculations we show easily that the almost fixed set of a Dehn twist automorphism has a bounded diameter. This completes the preparation for ping-pong with elliptic-type groups as it is given by Kapovich and Weidmann in [19]. Now we have ping-pong sets that we have control over.

The main argument, which also finishes the proof of our main result, is encoded in the following theorem.

Theorem 1.3 *Let G be a group acting on a δ–hyperbolic metric space X by isometries and ϕ_1, $\phi_2 \in G$. Suppose $C > 100\delta$ and the almost fixed sets $\mathcal{X}_C(\phi_1)$ and $\mathcal{X}_C(\phi_2)$ of $\langle\phi_1\rangle$ and $\langle\phi_2\rangle$, respectively, have diameters bounded above by a constant C'. Then there exists a constant C_1 such that whenever $d_X(\mathcal{X}_C(\phi_1), \mathcal{X}_C(\phi_2)) \geq C_1$,*

(1) *$\langle\phi_1, \phi_2\rangle \simeq F_2$, and*

(2) *every element of $\langle\phi_1, \phi_2\rangle$ which is not conjugate to the powers of ϕ_1 and ϕ_2 in $\langle\phi_1, \phi_2\rangle$ acts loxodromically in X.*

Finally, we *project* Dehn twists to the *intersection graph* $\mathcal{P}_k$, which is a simplicial complex with vertex set given by marked roses up to equivalence. There are two instances connecting vertices of $\mathcal{P}_k$. The first one is that whenever two roses share an edge with the same label, corresponding vertices are connected by an edge in $\mathcal{P}_k$. The second one is obtained whenever there is a marked surface with one boundary component such that the element of the fundamental group represented by the boundary crosses each edge of both roses twice. This simplicial complex is closely related to the *intersection graph* introduced by Kapovich and Lustig in [17], and it is proven to be hyperbolic by Mann in [23].

A fully irreducible automorphism is called *geometric* if it is induced by a pseudo-Anosov homeomorphism of a surface with one boundary component. A fully irreducible automorphism is *atoroidal* if no positive power of it preserves the conjugacy class of a nontrivial element of F_k. Moreover, only nongeometric fully irreducible automorphisms are atoroidal by a theorem of Bestvina and Handel in [4]. The important feature of the intersection graph for us is that the atoroidal fully irreducibles act loxodromically on this graph; see Mann [23].

We obtain the following theorem.

Theorem 1.4 *Let T_1 and T_2 be two $\mathbb{Z}$–splittings of F_k with $k > 2$ with corresponding free factors α_1 and α_2, and let D_1 and D_2 be two Dehn twists corresponding to T_1 and T_2, respectively. Then there exists a constant $N_2 = N_2(k)$ such that $\langle D_1, D_2\rangle \simeq F_2$ whenever $d_{\mathcal{P}_k}(\sigma(\alpha_1), \sigma(\alpha_2)) \geq N_2$, and all elements from this group which are not conjugate to the powers of D_1 and D_2 in $\langle D_1, D_2\rangle$ are atoroidal fully irreducible.*

Acknowledgements I would like to thank Ilya Kapovich for asking the question which led to the writing of this paper and many talks while I was writing it. I am especially indebted to Chris Leininger for his constant support and for suggesting the current shorter proof for the main theorem. I am very grateful for Matt Clay for giving me permission to use his proof in this paper. Many thanks also to Mark C Bell for reading the preprint and making many useful suggestions. Lastly I would like to thank the anonymous referee whose suggestions and corrections improved the paper substantially.

2 Preliminaries

2.1 Sphere systems and normal tori

The manifold $\sharp_k(S^2 \times S^1)$ is a reducible, connected 3–manifold which can be described as follows. We remove the interiors of $2k$ disjoint 3–balls from the 3–sphere S^3 and identify the resulting 2–sphere boundary components in pairs by orientation-reversing diffeomorphisms, creating k many $S^2 \times S^1$ summands. $\mathrm{Out}(F_k)$ is isomorphic to the mapping class group of $\sharp_k(S^2 \times S^1)$ up to twists about 2–spheres in $\sharp_k(S^2 \times S^1)$; see [21]. From now on we will let $M = \sharp_k(S^2 \times S^1)$.

Associated to M is a rich algebraic structure coming from the essential 2–spheres that M contains. A *sphere system* is a collection of isotopy classes of disjoint and nontrivial 2–spheres in M no two of which are isotopic.

We call a collection Σ of disjointly embedded essential, nonisotopic 2–spheres in M a *maximal sphere system* if every complementary component of Σ in M is a 3–punctured 3–sphere.

A fixed maximal sphere system Σ in M gives a description of the universal cover $\widetilde{M}$ of M as follows. Let $\mathbb{P}$ be the set of 3–punctured 3–spheres in M given by a maximal sphere system Σ and regard M as obtained from copies of P in $\mathbb{P}$ by identifying pairs of boundary spheres. Note that both boundary spheres in a pair might be contained in a single P, in which case the image of P in M is a once-punctured $S^2 \times S^1$. To construct $\widetilde{M}$, begin with a single copy of P and attach copies of P in $\mathbb{P}$ inductively along boundary spheres, as determined by unique path lifting. Repeating this process gives a description of $\widetilde{M}$ as a treelike union of copies of P. We remark that $\widetilde{M}$ is homeomorphic to the complement of a Cantor set in S^3.

To be able to define a concept of geometric Dehn twist we need to use the one-to-one correspondence between the equivalence classes of $\mathbb{Z}$–splittings of F_k and homotopy classes of *essential* tori in M. This important correspondence is given in Lemma 2.6.

For us, a torus in M is an embedding of a 2–torus in M so that the image of its fundamental group in $\pi_1(M)$ is a cyclic group isomorphic to $\mathbb{Z}$. Moreover, we consider only the tori which do not bound a solid torus in M and we call such a torus *essential*. There are two types of essential tori in M, depending on the type of the splitting of the free group they correspond to. Namely, for an amalgamated free product we have a separating torus in M and a nonseparating one for an HNN extension of the free group. Two examples can be seen in Figure 1.

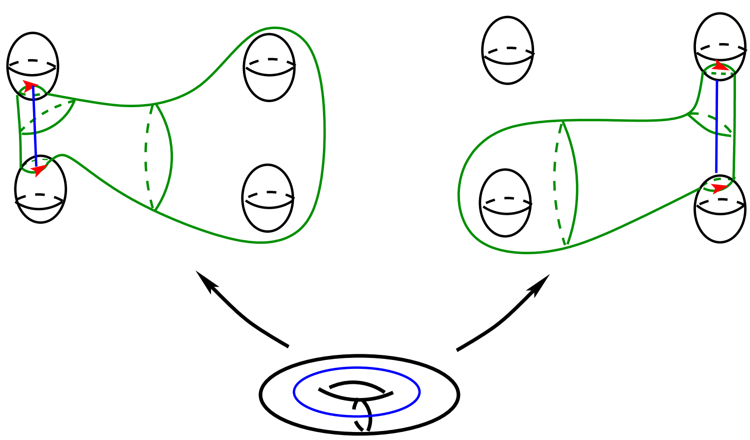

Figure 1: Embeddings with the given identifications correspond to a separating torus (left) and a nonseparating torus (right) in $\sharp_4(S^2 \times S^1)$.

Given a homotopy class of a torus, it is necessary for our purposes to identify a representative which intersects the spheres in a given maximal sphere system of M minimally. To this end, the *normal form* for tori is defined. Following Hatcher's normal form for sphere systems in [13], a normal form for tori is defined in [11] so that the intersection of the normal torus with each complementary 3–punctured 3–sphere is a disk, a cylinder or a pants piece. By [11], if a torus τ is in *normal form* with respect to a maximal sphere system Σ, the number of components of the intersection between τ and any S in Σ is minimal among all the representatives of the homotopy class $\boldsymbol{\tau}$.

In this work we will implicitly use the following existence theorem from [11].

Theorem 2.1 *Every embedded essential torus in M is homotopic to a normal torus and the homotopy process does not increase the intersection number with any sphere of a given maximal sphere system Σ.*

2.2 Geometric models, complexes and projections

Given the free group F_k on k letters, the associated *outer space* of the marked metric graphs which are homotopy equivalent to F_k was introduced by Culler and Vogtmann in [7]. We will denote by CV_k the *projectivized* outer space, in which the graphs will all have total volume 1. A marked metric graph is an equivalence class of a pair consisting of a metric graph Γ and a marking, which is a homotopy equivalence with a rose. Outer space might be thought as an analog to Teichmüller space for the mapping class group. For the details we refer the reader to [7] and [31].

The *free factor complex* of a free group is defined first by Hatcher and Vogtmann in [15] as a simplicial complex whose vertices are conjugacy classes of proper free factors and adjacency is determined by inclusion. It is hyperbolic by [2].

We will use the coarse projection $\pi\colon \mathrm{CV}_k \to \mathrm{FF}_k$ defined as follows. For each proper subgraph Γ_0 of a marked graph G that contains a circle, its image in FF_k is the conjugacy class of the smallest free factor containing Γ_0. Now by [2, Lemma 3.1], for two such proper subgraphs Γ_1 and Γ_2, we have $d_{\mathrm{FF}_k}(\pi(\Gamma_1), \pi(\Gamma_2)) \leq 4$. Then for $G \in \mathrm{CV}_k$ we define

$$\pi(G) := \{\pi(\Gamma) \mid \Gamma \text{ is a proper, connected, noncontractible subgraph of } G\}.$$

We will denote the induced map $\mathrm{CV}_k \to \mathrm{FF}_k$ also by π, which is clearly a *coarse* projection in that the diameter of each $\pi(G)$ is bounded by 4. Another hyperbolic $\mathrm{Out}(F_k)$–graph we refer to is the *free bases graph* FB_k given by Kapovich and Rafi in [18]. For $k \geq 3$, this graph has vertices the free bases of F_k up to equivalence (two bases are equivalent if their Cayley graphs are F_k–equivariantly isometric), and whenever two bases representing the vertices have a common element, these vertices are connected by an edge. What is useful for us is that FB_k and FF_k are quasi-isometric.

The *intersection graph* $\mathcal{P}_k$ has vertex set consisting of marked roses up to equivalence. Two roses are connected by an edge if either they have a common edge with the same label, or there is a marked surface with one boundary component and the representative of this component crosses each edge of both roses twice. There is a Lipschitz map between FB_k and $\mathcal{P}_k$, constructed by thinking of each basis of F_k as its corresponding rose marking and observing that $\mathcal{P}_k$ shares the edges and vertices of FB_k and has some additional edges between roses.

2.3 Tori in M and $\mathbb{Z}$–splittings of the free group

In this section we will establish the correspondence between an equivalence class of a $\mathbb{Z}$–splitting and a homotopy class of a torus in M. Consider an embedded essential torus τ in M. There is a simplicial tree associated to this torus. To obtain this simplicial tree we take a neighborhood of each lift in the set of lifts $\widetilde{\tau}$ of τ and we take a vertex for each complementary component. Two complementary components are adjacent if they bound the neighborhood of the same lift. We will denote this tree by $T_{\boldsymbol{\tau}}$ and as correspondence between this tree and the torus τ we will understand the F_k–equivariant map $\widetilde{M} \to T_\tau$ which sends each complementary component of a neighborhood of a lift to a vertex and shrinks each such neighborhood to an edge. The tree constructed this way is referred to as the *Bass–Serre* tree corresponding to τ. Recall that by Bass–Serre theory, the action of F_k on T_τ gives a single-edged graph of groups decomposition of F_k, and hence a $\mathbb{Z}$–splitting of the free group F_k.

The next lemma gives the existence of an equivalence class of a $\mathbb{Z}$–splitting for each homotopy class of a torus in M and its proof is based on the notion of the *ends* of $\widetilde{M}$.

An end of a topological space is a point of the so-called *Freudenthal compactification* of the space. More precisely:

Definition 2.2 Let X be a topological space. For a compact set K, let $C(K)$ denote the set of components of the complement $X - K$. For L compact with $K \subset L$, we have a natural map $C(L) \to C(K)$. These compact sets define a *directed system* under inclusion. The set of ends $E(X)$ of X is defined to be the inverse limit of the sets $C(K)$.

The space $\widetilde{M}$ is noncompact and it has infinitely many ends. We denote the set of ends of $\widetilde{M}$ by $E(\widetilde{M})$. It is homeomorphic to a Cantor set; in particular, it is compact. For a maximal sphere system Σ in M, the set $E(T_\Sigma)$ of ends of the Bass–Serre tree T_Σ of Σ is identified with the set $E(\widetilde{M})$. By analyzing this set we were able to prove the following.

Lemma 2.3 *Let T_{τ_1} and T_{τ_2} be two Bass–Serre trees corresponding to two tori τ_1 and τ_2, respectively. If τ_1 and τ_2 are homotopic, $T_{\tau_1} = T_{\tau_2}$ and hence τ_1 and τ_2 have equivalent $\mathbb{Z}$–splittings.*

Proof Let τ be an embedded essential torus. We claim that each lift of τ is 2–sided. If it is not, then there is a nontrivial (nonhomotopic to a point) loop which intersects a lift once and connects an end of $\widetilde{M}$ to itself. By the loop theorem this loop bounds a disk in $\widetilde{M}$. Then the projection of this disk to M bounds a disk in M, which means that the torus bounds a solid torus in M. This contradicts the fact that τ is essential. Hence each lift divides $\widetilde{M}$ into two disjoint parts.

A transverse orientation on the torus gives an orientation on the spheres on each disjoint part and hence there is a labeling on the valence-2 vertices corresponding to these spheres. It is clear that each lift $L = S^1 \times \mathbb{R}$ of τ defines a decomposition of the set of ends of T_Σ into two sets L^+ and L^-, where $\mathsf{L}^+ \cap \mathsf{L}^-$ consists of two endpoints corresponding to the axis of the lift L. For each torus, let us consider all the endpoints corresponding to the axes of all lifts and eliminate them from the set of ends $E(\widetilde{M})$ of $\widetilde{M}$. Let us denote the remaining set $\widetilde{E}(\widetilde{M})$.

Now, for each lift L, we have a partition $(\mathsf{L}^+, \mathsf{L}^-)$ of the set $\widetilde{E}(\widetilde{M})$. Since lifts are disjoint, for any two lifts L_1 and L_2 we have either $\mathsf{L}_1^+ \subset \mathsf{L}_2^+$ or $\mathsf{L}_1^+ \subset \mathsf{L}_2^-$.

We construct a tree corresponding to the set of partitions as follows. For each partition $(\mathsf{L}^+, \mathsf{L}^-)$ we take a vertex. Given a pair of partitions $(\mathsf{L}_1^+, \mathsf{L}_1^-)$ and $(\mathsf{L}_2^+, \mathsf{L}_2^-)$ with $\mathsf{L}_1^+ \subset \mathsf{L}_2^+$ or $\mathsf{L}_1^+ \subset \mathsf{L}_2^-$, we connect the corresponding vertices with an edge whenever there is no collection of ends $(\mathsf{Z}^+, \mathsf{Z}^-)$ satisfying $\mathsf{L}_1^+ \subset \mathsf{Z}^+ \subset \mathsf{L}_2^+$ or $\mathsf{L}_1^+ \subset \mathsf{Z}^- \subset \mathsf{L}_2^-$. For each maximal subset of $\widetilde{E}(\widetilde{M})$ which is not separated by any lift, we take a vertex. Since the partitions of ends do not intersect, we have a tree. We will denote this tree by $T_{\mathcal{P}}$.

Since for each lift we have a partition of the ends, there is an isomorphism between the tree $T_{\mathcal{P}}$ given by the partitions and the Bass–Serre tree T_τ. To see this we first introduce another set of vertices in T_τ given by midpoints of edges and we map these "edge-midpoint" vertices of T_τ to the set of partitions, which are the components of $\widetilde{\tau}$. The image of the vertices of T_τ given by the components of $\widetilde{M} - \widetilde{\tau}$ are collections of lifts having the following property: assuming that we select the notation so that $\mathsf{L}_1^+ \subset \mathsf{L}_2^+$ for the two lifts L_1 and L_2, there is no L_3 in the collection for which $\mathsf{L}_1^+ \subset \mathsf{L}_3^+ \subset \mathsf{L}_2^+$ or $\mathsf{L}_1^+ \subset \mathsf{L}_3^- \subset \mathsf{L}_2^+$.

Now we claim that for a homotopy of embedded tori in M, the initial and final tori determine the same partition of the ends in $\widetilde{E}(\widetilde{M})$, and hence they have the same partition tree, and the same Bass–Serre tree as a result.

To see this, let τ_1 be homotopic to τ_2. To see that the lifts of τ_1 and the lifts of τ_2 give the same partition of ends we need to show that if two endpoints are separated by a component L of $\widetilde{\tau}_1$, and L is homotopic to a component L' of $\widetilde{\tau}_2$, then they are separated by L' too.

Let p and q be two endpoints separated by L. Fix an arc between them that crosses $L = S^1 \times \mathbb{R}$ in one point. During the homotopy, although no longer embedded, L moves in $\widetilde{M}$. In particular, it does not touch any endpoint. So assuming that the homotopy is transverse to the arc, its inverse image in $S^1 \times \mathbb{R} \times I$ consists of circles and arcs properly embedded in $S^1 \times \mathbb{R} \times I$. Note that if the homotopy could cross an endpoint of the arc, then an arc of the inverse image could fail to be properly embedded in $S^1 \times \mathbb{R} \times I$. But this does not happen since the homotopy between the two tori induces a homotopy between normal representatives of each tori, corresponding to a fixed maximal sphere system in M. By [11], such a homotopy is *normal* at each stage hence cannot cross an endpoint.

Back to the inverse image of the homotopy between the two tori, since only one endpoint of the inverse image of the arc is in L, there must be an odd number of endpoints in L' (ie the arc crosses L' an odd number of times) and therefore L' also separates p and q. □

Recall that given a $\mathbb{Z}$–splitting of F_k, an associated *Dehn twist automorphism* of F_k is defined in the following two ways:

$$A *_{\langle c\rangle} B\colon \begin{array}{ll} a \mapsto a & \text{for } a \in A, \\ b \mapsto cbc^{-1} & \text{for } b \in B \end{array} \qquad \text{and} \qquad A*_{\langle tct^{-1}=c'\rangle}\colon \begin{array}{ll} a \mapsto a & \text{for } a \in A, \\ t \mapsto tc. & \end{array}$$

On the left is the definition when the $\mathbb{Z}$–splitting is given by an amalgamated product $F_k = A *_{\langle c\rangle} B$ and on the right is the definition when the $\mathbb{Z}$–splitting is an HNN extension $A*_{\langle c\rangle}$ of the free group F_k. Note that the Dehn twist automorphism in the amalgamated case is defined up to conjugacy since it is possible to reverse the roles of A and B.

Before we give the last lemma in this section, we give two theorems relating $\mathbb{Z}$–splittings to free splittings which will be used in the proof.

Theorem 2.4 (Shenitzer [26]) *Suppose that a free group $\mathbb{F}$ is an amalgamated free product $\mathbb{F} = A *_B C$, where B is cyclic. Then B is a free factor of A or a free factor of C.*

Theorem 2.5 (Swarup [28]) *Suppose that a free group $\mathbb{F}$ is an HNN extension $\mathbb{F} = A*_B$, where $B \neq 1$ is cyclic. Then A has a free product structure $A = A_0 * A_1$ in such a way that one of the following symmetric alternatives hold, where t is the stable letter:*

(1) *$B \subset A_0$ and there exists $a \in A$ such that $t^{-1}Bt = a^{-1}A_1a$, or*

(2) *$t^{-1}Bt \subset A_0$ and there exists $a \in A$ such that $B = a^{-1}A_1a$.*

Finally, the following lemma gives the converse relationship between a torus and a $\mathbb{Z}$–splitting and hence explains why we are interested in tori in M. The proof is due to Matt Clay.

Lemma 2.6 *Given a $\mathbb{Z}$–splitting Z and an associated Dehn twist automorphism, there is a torus τ in M unique up to homotopy such that $T_Z = T_\tau$, where T_Z and T_τ are the corresponding Bass–Serre trees.*

Proof In this proof we will build a homotopy class of a torus from a sphere and a loop. First we use Theorems 2.4 and 2.5, which relate a $\mathbb{Z}$–splitting of F_k to a free splitting of F_k. Then, to relate the free splitting to a homotopy class of a sphere, we use a theorem originally due to Kneser [20]. This theorem is later developed by Grushko [9], and most recently by Stallings [27], and these are the versions we will be referring to. We treat the amalgamated product and HNN-extension cases separately. The amalgamated case has schematic pictures Figure 2 and Figure 3 associated to the proof.

Case 1 We first consider the case of an amalgamated free product $F_k = A *_{\langle b\rangle} B$. By Shenitzer's theorem, Theorem 2.4, $\langle b\rangle$ is either a free factor of A or a free factor of B. Hence there is a free splitting $F_k = A * B_0$, where $B = \langle b\rangle * B_0$, or $F_k = A_0 * B$ with $A = A_0 * \langle b\rangle$. Let us assume the former, and let $S \subset M$ be an embedded (separating) sphere representing this splitting. We fix a basepoint $* \in M$ and assume it lies on S. As $b \in A$, there is an embedded loop $\gamma \subset M$ that represents $b \in F$ and only intersects S at $*$. For small ϵ, the boundary of the closed ϵ–neighborhood of $S \cup \gamma$ consists of two components: an embedded sphere isotopic to S and an embedded essential torus τ_γ.

Every torus can be written as a sphere and a loop attached to it. Hence it is clear from the construction that the splitting of F_k associated to τ_γ is the original splitting. However, there are some choices made in the construction of τ_γ and it must be shown that different choices result in homotopic tori. It is clear that changing S or γ in the construction by a homotopy results in a change of τ_γ by a homotopy.

Now since Shenitzer's theorem, Theorem 2.4, gives many possible splittings differing by automorphisms of B fixing $\langle b\rangle$ (hence Nielsen automorphisms that fix b), we need to consider two different complementary free factors B_0 and B_1 of A such that $\langle b\rangle * B_0 = \langle b\rangle * B_1 = B$ and show that the tori obtained after we add the loop to corresponding spheres are homotopic, even when the spheres themselves are not. For this, let S_0 and S_1 be the spheres representing the splittings $A * B_0$ and $A * B_1$, respectively, and τ_0 and τ_1 be the tori as constructed above using these spheres. We assume that γ intersects S_0 only at the fixed basepoint $* \in M$.

We first treat the special case that B_1 is obtained from B_0 by replacing a generator x in B_0 by xb. Fix a basis for F_k consisting of a basis for A and a basis for B_0, where x is one of the generators for B_0. This corresponds to a sphere system in M which decomposes as $\Sigma_A \cup \Sigma_{B_0}$; the sphere S_0 separates the two sets Σ_A and Σ_{B_0}. In terms of these sphere systems, we can describe a homeomorphism that takes S_0 to (a sphere isotopic to) S_1.

Denote by Σ_γ the ordered set of spheres (all in Σ_A) pierced by γ starting from the basepoint. Cut M open along the sphere β corresponding to the generator x and via a homotopy push the boundary sphere β^- through the spheres in Σ_γ in order, dragging S_0 along. After regluing β^+ and β^-, the image of S_0 is S_1 and the sphere β now corresponds to xb. By shrinking β^- and S_0, we can assume that the homotopy is the identity on τ_0 and γ. Thus, we have a homeomorphism taking S_0 to S_1, taking $S_0 \cup \gamma$ to $S_1 \cup \gamma$ and which is the identity on τ_0. As a homeomorphism takes a regular neighborhood to a regular neighborhood, τ_0 is homotopic to τ_1.

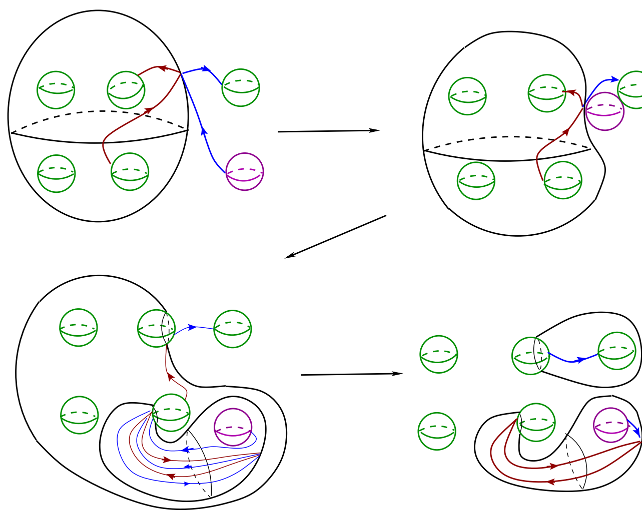

Figure 2: The homotopy which slides the pink sphere along the red loop γ representing b, where $\pi_1(M) = \langle a, b, c\rangle$. In the first picture, a sphere (black) and the b loop (red) are given, where the base point is on the sphere. c is depicted in blue in the picture.

To see a schematic picture of this homotopy we refer the reader to Figure 2. In the first picture, the black sphere (an example of S_0) and a neighborhood of the red loop give a torus (τ_0), and this torus is homotopic to the torus obtained from the last picture by taking a neighborhood of the black sphere (now an example of S_1) and the red loop.

A similar argument works if we replace x by xb^{-1}, bx or $b^{-1}x$.

The general case now follows as we can transform B_0 to B_1 by a finite sequence of the above transformations plus changes of basis that do not affect the associated spheres.

Indeed, by [32, Theorem 4.1], the subgroup of automorphisms of B that fix $b \in B$ is generated by the Nielsen automorphisms that fix b.

Last, we consider the possibility $F_k = A_0 * \langle b \rangle * B_0$, where $A = \langle A_0, b\rangle$ and $B = \langle B_0, b\rangle$. Let S_A and S_B be the spheres representing the splittings $A * B_0$ and $A_0 * B$, respectively, fix loops γ_A and γ_B representing b, and consider the neighborhoods s_A of $S_A \cup \gamma_A$ and s_B of $S_B \cup \gamma_B$. Since these neighborhoods each give a torus and a sphere, we have tori τ_A and τ_B that both represent the splitting $A *_{\langle b\rangle} B$. In this case, as a component of $M - (S_A \cup S_B)$ is $S^1 \times S^2$ with two balls removed, it is easy to see that τ_A and τ_B are homotopic. Indeed, let us model $S^1 \times S^2$ as the region between the spheres of radius 1 and 2 in $\mathbb{R}^3$ after identifying the boundary spheres. Remove a ball of radius $\frac{1}{4}$ at each of the points $\left(0, 0, \frac{3}{2}\right)$ and $\left(0, 0, -\frac{3}{2}\right)$. For γ_A we can choose the intersection with the positive z–axis; for γ_B we can choose the intersection with the negative z–axis. Then clearly the torus obtained from the intersection with the xy–plane is homotopic to both τ_A and τ_B. For a simple example see Figure 3.

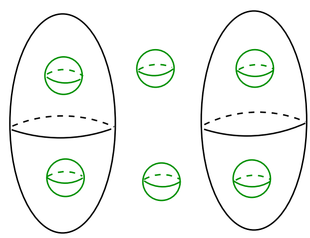

Figure 3: In this example, $F_3 = \langle a\rangle * \langle b\rangle * \langle c\rangle$ and $A = \langle A_0, b\rangle$ with $A_0 = \langle a\rangle$ and $B = \langle b, B_0\rangle$ with $B_0 = \langle c\rangle$. Spheres S_A and S_B are displayed in black.

Case 2 We now consider the case of an HNN-extension $F_k = A *_{\langle b\rangle}$. By Swarup's theorem, Theorem 2.5, there is a free factorization $A = A_0 * \langle t^{-1} b t\rangle$ for some $t \in F_k$ such that A_0 is a corank-1 free factor of F_k and such that $b \in A_0$. Let $S \subset M$ be an embedded (nonseparating) sphere representing the splitting $F_k = A_0 *_{\{1\}}$. We fix a basepoint $p \in M$ and assume it lies on S. As $b \in A_0$, there is an embedded loop $\gamma \subset M$ that represents $b \in F_k$ and only intersects S at p. Further, both ends of γ are on the same side of S and so a neighborhood of $s = S \cup \gamma$ gives a torus. Let τ be this neighborhood of s, and as in Case 1, it is clear that the splitting associated to τ is the original splitting.

Given another torus $\tau' \subset M$ that represents the same splitting, we can compress τ' to a union of a sphere and a loop $s' = S' \cup \gamma'$ such that S' represents a splitting of the form $A_1 *_{\{1\}}$, where $A = A_1 * \langle t b t^{-1}\rangle$. Then, as in Case 1, there is a sequence of transformations taking A_0 to A_1 that do not change the homotopy type of the corresponding torus. □

3 Geometric intersection and relative twisting using ends of $\widetilde{M}$

3.1 Intersection criterion and the relative twisting number

The Guirardel core $\mathcal{C}$ is a way of assigning a closed, connected CAT(0) complex to a pair of splittings which counts the number of times the corresponding Bass–Serre trees intersect. Guirardel's version unifies several notions of intersection number in the literature, including the one for two splittings of finitely generated groups given by Scott and Swarup [25]. For two F_k–trees T_0 and T_1, the core is roughly the main part of the diagonal action of F_k on $T_0 \times T_1$. For the details we refer the reader to [10] and [1].

Definition 3.1 Let T be a tree and p a point in it. A *direction* is a connected component of $T - p$. Given two trees T_0 and T_1, a *quadrant* is a product $\delta \times \delta'$ of two directions $\delta \subset T_0$ and $\delta' \subset T_1$.

We fix a basepoint $* = (*_0, *_1)$ in $T_0 \times T_1$ and we say that a quadrant Q is *heavy* if there exists a sequence $\{g_n\}$ in F_k such that $g_n(*) \in Q$ for every n and $d_{T_i}(*_i, g_n(*_i)) \to \infty$ as $n \to \infty$. A quadrant is *light* if it is not heavy.

Definition 3.2 Let T_0 and T_1 be two F_k–trees. The Guirardel core $\mathcal{C}$ is defined as

$$\mathcal{C} = \mathcal{C}(T_0 \times T_1) = (T_0 \times T_1) - \cup_I Q,$$

where I is over all of the light quadrants.

Let p be a point in T_0. Then $\mathcal{C}_p = \{x \in T_1 \mid (p, x) \in \mathcal{C}\}$ is a subtree of T_1 called the *slice of the core* above the point p. The slice which is a subtree of T_0 is defined similarly.

Given two trees T_0 and T_1, we define the *Guirardel intersection number* between them by

$$i(T_0, T_1) = \mathrm{vol}(\mathcal{C}/F_k),$$

where the right-hand side is the volume of the action of F_k on the Guirardel core $\mathcal{C}(T_0 \times T_1)$ for the product measure on $T_0 \times T_1$. Note that for simplicial trees T_0 and T_1 this volume is the number of 2–cells in $\mathcal{C}/F_k$, which will be our case.

Given two homotopy classes of sphere systems $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ in M, the intersection number between them is the number $i(\boldsymbol{S_1}, \boldsymbol{S_2})$ of components of the intersection of normal representatives of S_1 and S_2 in $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$, respectively, when the intersections are transversal [13]. This is called the *geometric* intersection number.

The work of Horbez in [16] relates the geometric intersection number between two sphere systems to the Guirardel intersection number between corresponding trees. Each sphere in $\widetilde{M}$ corresponds to an edge in the associated Bass–Serre tree. By taking ϵ–neighborhoods of each edge, one obtains for each geometric intersection between two spheres a square in the product of the Bass–Serre trees. If the intersection is essential, this square is in the Guirardel core. By the definition of the Guirardel core, a square which is in the core is in a heavy quadrant. As a consequence there are 4 unbounded disjoint regions in $\widetilde{M}$ corresponding to such a square. On the other hand, each sphere $\widetilde{S}$ gives two disjoint sets $E^+(\widetilde{S})$ and $E^-(\widetilde{S})$ of ends of $\widetilde{M}$. Thus whenever two spheres $\widetilde{S}_1$ and $\widetilde{S}_2$ intersect essentially in $\widetilde{M}$ there are four disjoint sets of ends $E^+(\widetilde{S}_1)\cap E^+(\widetilde{S}_2)$, $E^+(\widetilde{S}_1)\cap E^-(\widetilde{S}_2)$, $E^-(\widetilde{S}_1)\cap E^+(\widetilde{S}_2)$ and $E^-(\widetilde{S}_1)\cap E^-(\widetilde{S}_2)$ of $\widetilde{M}$ in the complement of the intersection circle, each matching with the corresponding unbounded region.

Finally we have the following definition which we will use in the next section.

Definition 3.3 We will call the existence of four disjoint sets of ends of $\widetilde{M}$ in the complement of an intersection circle between two spheres the *intersection criterion* for that intersection circle.

According to the discussion above, an intersection circle between two spheres is essential if and only if we have the intersection criterion satisfied for that intersection circle.

Now we will define the *relative twisting number* for two intersecting sphere systems Σ_1 and Σ_2.

Given an axis of an element in $\widetilde{M}$, there are two ends of $\widetilde{M}$ which are fixed by this axis.

Definition 3.4 A sphere $\widetilde{S}$ is said to intersect an axis $\mathfrak{a}$ whenever the two ends of $\widetilde{M}$ determined by $\mathfrak{a}$ are separated by the two disjoint sets $E^+(\widetilde{S})$ and $E^-(\widetilde{S})$ of ends corresponding to $\widetilde{S}$.

Now we will define the relative twisting number between Σ_1 and Σ_2 relative to an element $a \in F_k$. This number is meaningful when both sphere systems intersect an axis $\mathfrak{a}$ of a in $\widetilde{M}$. The definition of geometric relative twisting of [6] is for two points in outer space, which are simple sphere systems in our setting. For our purposes we will translate their definition to one which is stated in terms of ends of $\widetilde{M}$.

Before we give the definition given in [6], we set up some notation first. Let Σ_1 and Σ_2 be two simple sphere systems in M and T_{Σ_1} and T_{Σ_2} the corresponding Bass–Serre trees. Let $e_{\widetilde{S}_1}$ and $e_{\widetilde{S}_2}$ be two edges in T_{Σ_1} and T_{Σ_2} corresponding to spheres $S_1\in\Sigma_1$ and $S_2\in\Sigma_2$, respectively. For an element $a\in F_k$, let us call $T^a_{\Sigma_1}$ and $T^a_{\Sigma_2}$ the sets of edges of T_{Σ_1} and T_{Σ_2}, respectively, whose elements intersect a fixed axis $\mathfrak{a}$ of a. Also denote by $a^k e_{\widetilde{S}_1}$ the k^{th} iterate of $e_{\widetilde{S}_1}$ under the action of $\langle a\rangle$ on T_{Σ_1} along an axis $\mathfrak{a}$.

Definition 3.5 [6] For an element $a\in F_k$, the relative twisting number $\mathrm{tw}_a(\Sigma_1,\Sigma_2)$ of Σ_1 and Σ_2 relative to a is defined to be

$$\mathrm{tw}_a(\Sigma_1,\Sigma_2):=\max_{\substack{e_{\widetilde{S}_1}\subset T^a_{\Sigma_1}\\ e_{\widetilde{S}_2}\subset T^a_{\Sigma_2}}}\{k\mid a^k e_{\widetilde{S}_1}\times e_{\widetilde{S}_2}\in\mathcal{C}\text{ and }e_{\widetilde{S}_1}\times e_{\widetilde{S}_2}\in\mathcal{C}\}.$$

Using the fact that a geometric intersection between two spheres is a square in the Guirardel core and hence gives a separation of ends of $\widetilde{M}$ into 4 nonempty disjoint sets, we tailor the definition of [6] above to the one below to suit our needs.

Definition 3.6 For $i\in\{1,2\}$, let $\widetilde{S}_i$ be two spheres and $E^{\mp}(\widetilde{S}_i)$ be the set of ends of $\widetilde{M}$ separated by these spheres. Assume that both spheres intersect an axis of $\boldsymbol{a}\in F_k$. Then the relative twisting number $\mathrm{tw}_a(\widetilde{S}_1,\widetilde{S}_2)$ of $\widetilde{S}_1$ and $\widetilde{S}_2$ relative to a is defined by

$$\begin{aligned}\mathrm{tw}_a(\widetilde{S}_1,\widetilde{S}_2):=\max\bigl\{k\in\mathbb{Z}\mid E^{\mp}(\widetilde{S}_i)\cap E^{\mp}(a^k\widetilde{S}_j)\neq\varnothing\\ \text{whenever }E^{\mp}(\widetilde{S}_i)\cap E^{\mp}(\widetilde{S}_j)\neq\varnothing\text{ and }\{i,j\}\in\{1,2\}\bigr\}.\end{aligned}$$

4 Dehn twist along a torus: the geometric picture

4.1 Definition of a Dehn twist along a torus

We will now give the definition of the Dehn twist homeomorphism about a torus in M, a description of the action of such a homeomorphism on spheres in M, and a description of the action on F_k.

To define a Dehn twist along an embedded torus τ, we will take a parametrized tubular neighborhood of the torus in M.

Definition 4.1 Let $\tau\colon\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}\times[0,1]\to M$ be an embedding such that the image $\tau(\{0\}\times\mathbb{R}/\mathbb{Z}\times\{0\})$ bounds a disk in M. Denote by τ the associated torus $\tau(\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}\times\{0\})$. The *geometric Dehn twist* D_τ along the torus τ is the homeomorphism of M that is the identity on the complement of the image of the map τ and for which a point $p=\tau(x,y,t)$ is sent to $\tau(x+t,y,t)$.

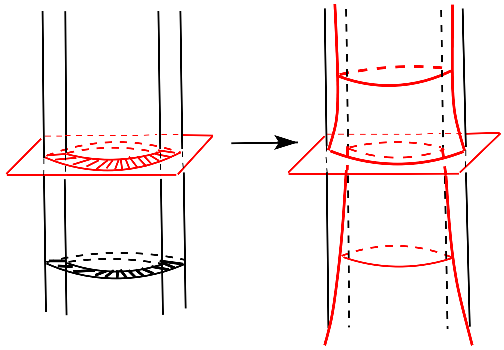

Figure 4: The image of the intersection annulus under a Dehn twist along the thick black torus

Here, the direction of the associated torus which bounds a disk in M will be called the *meridional* direction and the other one will be called the *longitudinal* direction. For a geometric description of the twist, we refer the reader to Figure 4.

Now, an ambiguity might arise in the definition of a geometric Dehn twist when it comes to determining a longitudinal curve for the parametrization. But two such choices differ by a map $x \mapsto x + ny$ for some integer n and a twist along the meridional direction. By work of Laudenbach [21], the meridional direction does not give a nontrivial homeomorphism since twists along the meridional direction correspond to twists along 2–spheres in M. It is known that such mapping classes act trivially on F_k, and hence they are in the kernel of the homomorphism $\mathrm{Map}(M) \to \mathrm{Out}(F_k)$; see [14]. Hence the induced outer automorphism $D_{\tau *}$ from the geometric Dehn twist D_τ is independent of the parametrization of the neighborhood of the torus (the image of the map τ).

Now, assume that we have a loop ξ intersecting a torus τ transversely. Then the image $D_\tau(\xi)$ of the loop under the geometric twist D_τ is obtained as follows: We surger the loop at the intersection point and insert a loop β^+ or β^- representing a generator of $\pi_1(\tau)$ in $\pi_1(M)$, depending on which side of the torus the intersection point is. Hence the induced automorphism conjugates ξ with one of β^+ or β^- and fixes the other. If τ is nonseparating, then the stable letter is multiplied by ξ.

This coincides with the action of $D_{\tau *}$ on the corresponding splitting: When the splitting is an amalgamated product of the form $A *_{\langle c \rangle} B$, the factor A is fixed whereas B is conjugated by the generator of the fundamental group of torus τ in M. (The roles of A and B might be changed.) When we have $A *_{\langle tct^{-1} = c' \rangle}$, the Dehn twist automorphism $D_{\tau *}$ fixes A and t is multiplied by c.

As a summary we have:

Lemma 4.2 *Let* τ *be an embedded torus and* D_τ *the associated geometric Dehn twist. Then* $D_{\tau *} = D_Z$, *where* $D_{\tau *}$ *is the Dehn twist automorphism induced by the homeomorphism* D_τ *and* D_Z *is the Dehn twist automorphism given by the* $\mathbb{Z}$–*splitting* Z *associated to the torus* τ.

From Lemmata 4.2 and 2.3, we easily deduce the following.

Proposition 4.3 *Let* τ_1 *and* τ_2 *be two homotopic tori. Then up to conjugacy we have* $D_{\tau_1 *} = D_{\tau_2 *}$, *where* $D_{\tau_1 *}$ *and* $D_{\tau_2 *}$ *are the Dehn twist automorphisms induced by the geometric Dehn twists* D_{τ_1} *and* D_{τ_2}.

Proof Since homotopic tori give equivalent splittings by Lemma 2.3, the Dehn twist automorphisms are equal by definition. Then by Lemma 4.2 the corresponding Dehn twists induced by the geometric Dehn twists are equal. □

For our purposes, we need to find a lower bound on $\mathrm{tw}_a(G, D_T^n(G))$ for a given simple sphere system G and a $\mathbb{Z}$–splitting T. Now we have $D_T = D_\tau$, where τ is the essential embedded torus given by the $\mathbb{Z}$–splitting T for which the Dehn twist is defined and a is the generator of the image of the fundamental group of τ in $\pi_1(M)$ under the map induced from the embedding $\iota\colon \tau \to M$. Now we take a maximal sphere system Σ containing G and homotope τ to be normal with respect to this sphere system. Then, by [11], τ intersects the spheres of Σ minimally. Now we take a lift of the torus which has an axis a conjugate of a. The relative twisting number counts the number of iterates of a sphere which intersect the image of another sphere under a Dehn twist along an axis. Hence we have

$$\mathrm{tw}_a(G, D_T^n(G)) = \mathrm{tw}_\tau(G, D_\tau^n(G)).$$

5 Lower bound on the relative twisting number

In this section we will prove that the relative twisting number of a simple sphere system and its Dehn-twisted image has a lower bound, which is linear with respect to the power of the Dehn twist. We first have the following introductory lemma. Recall that a *simple* sphere system in M is one which has the complementary components in M simply connected.

Given a sphere S and a torus τ an intersection circle of $S \cap \tau$ which does not bound a disk in τ is called a *meridian* in τ.

Lemma 5.1 *Given an embedded essential torus τ and a simple sphere system G, there exists a sphere $S \in G$ such that at least one intersection between S and τ is a meridian in τ.*

Proof We first complete G to a maximal sphere system and homotope τ to be normal with respect to this maximal sphere system. Then τ intersects minimally every sphere of G it intersects [11]. Assume that no isotopy class of spheres in G intersects τ in a way that the intersection is a meridian in τ. On the other hand, since τ is essential, the image of the fundamental group of τ in $\pi_1(M)$ is nontrivial. Hence the core curve of τ exists and by the assumption it must be contained in a complementary component of G in M. But this is not possible since G is simple. □

Theorem 5.2 *Let T be a $\mathbb{Z}$–splitting of the free group F_k, let τ be the associated torus, and let D_τ be the Dehn twist along τ. For $G \in \mathrm{CV}_k$ and $n \geq 2$,*

$$\mathrm{tw}_\tau(G, D_\tau^n(G)) \geq n-1.$$

Proof We will prove this theorem by taking G as a simple sphere system instead of a marked metric graph. Let Σ be a maximal sphere system completing G and homotope τ to be the normal with respect to Σ.

Now, let $S \in G$ be such that S and τ intersect in a way that $\mu = S \cap \tau$ is a meridian in τ. Existence of a meridional intersection circle is given by Lemma 5.1. As before, let D_τ be the Dehn twist along the normal torus τ. More precisely, let $N(\tau)$ be a tubular neighborhood of τ in which D_τ is supported. Let $\tilde{\tau}$ be the full preimage of τ in $\widetilde{M}$ and $\tilde{\tau}_0 \in \tilde{\tau}$. Denote also by $N(\tilde{\tau})$ the full preimage of $N(\tau)$ and by $N(\tilde{\tau}_0)$ the component containing $\tilde{\tau}_0$. Let $\widetilde{S}$ be the full preimage of S and $\widetilde{S}_0$ a component such that $\widetilde{S}_0 \cap \tilde{\tau}_0 = \tilde{\mu}_0$, a lift of μ. Let us denote by a the generator of $\pi_1(\tau)$ corresponding to $\tilde{\tau}_0$, and hence we have a covering transformation $a\colon \widetilde{M} \to \widetilde{M}$ which stabilizes $\tilde{\tau}_0$. Let Δ be the region between $\widetilde{S}_0$ and $a\widetilde{S}_0$ which is the fundamental domain of $\langle a \rangle$ on $\widetilde{M}$. Set $\widetilde{S}_{0,j} = a^j \widetilde{S}_0$. Then $a^j \Delta$ is the region bounded by $\widetilde{S}_{0,j}$ and $\widetilde{S}_{0,j+1}$.

Since $\mathrm{tw}_\tau(G, D_\tau^n(G)) \geq \mathrm{tw}_\tau(S, D_\tau^n(S))$, it is sufficient to prove that

$$\mathrm{tw}_\tau(S, D_\tau^n(S)) \geq n-1.$$

Recall that we denote by $E(\widetilde{M})$ the ends of $\widetilde{M}$. As discussed in Section 2.2, there is a pair of ends of $\widetilde{M}$ fixed by a and $\tilde{\tau}_0$ separates the remaining set of ends into two disjoint sets E_0^+ and E_0^-. Since $\tilde{\tau}_0$ is separating in $\widetilde{M}$, there is a ray ℓ which connects an end $\mathrm{e}^+ \in E_0^+$ to an end $\mathrm{e}^- \in E_0^-$, intersecting $\tilde{\tau}_0$ only once. Observe that since τ is essential, there is always such a ray which is disjoint from $\tilde{\tau} - \tilde{\tau}_0$.

Let X^+ and X^- be the components of $\widetilde{M}-N(\widetilde{\tau})$ whose closures meet $N(\widetilde{\tau}_0)$. Since D_τ is the identity on $M-N(\tau)$ we choose a lift $\widetilde{D}_\tau$ which is the identity on X^- and a translation on X^+. Without loss of generality we may assume this is translation by a. Hence $\widetilde{D}_\tau(\mathsf{e}^-)=\mathsf{e}^-$ and $\widetilde{D}^m_\tau(\mathsf{e}^+)=\mathsf{a}^m\mathsf{e}^+$.

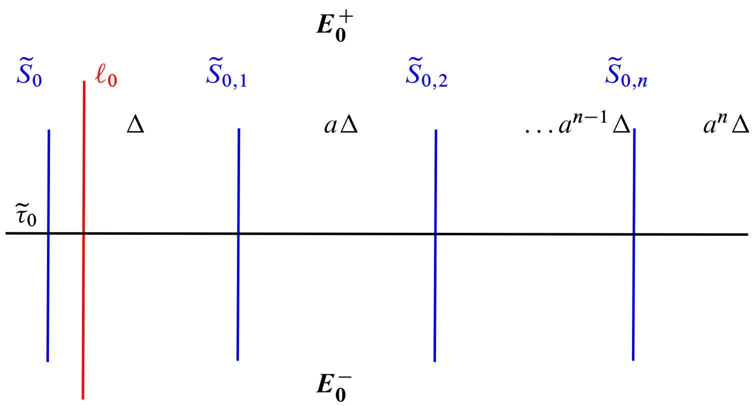

Figure 5: Fundamental domains, sets of ends separated by $\widetilde{\tau}_0$ and an arc ℓ_0 connecting them

Let $E^-_{0,0}\subset E^-_0$ and $E^+_{0,0}\subset E^+_0$ be two disjoint sets of ends in Δ. Now, as described above, in this fundamental domain there is a line ℓ_0 which connects an end e_0^- in $E^-_{0,0}$ to an end e_0^+ in $E^+_{0,0}$ intersecting $\widetilde{\tau}$ once in $\widetilde{\tau}_0$. (See Figure 5.) Now, after n times Dehn twisting along $\widetilde{\tau}$, the image ray $\widetilde{D}^n_\tau(\ell_0)$ will connect the point e_0^- in $E^-_{0,0}$ to the point $\widetilde{D}^n_\tau(\mathsf{e}_0^+)=\mathsf{a}^n\mathsf{e}_0^+=\mathsf{e}_n^+$ in $E^+_{0,n-1}=\widetilde{D}^n_\tau(E^+_{0,0})$. (See schematic picture Figure 6.)

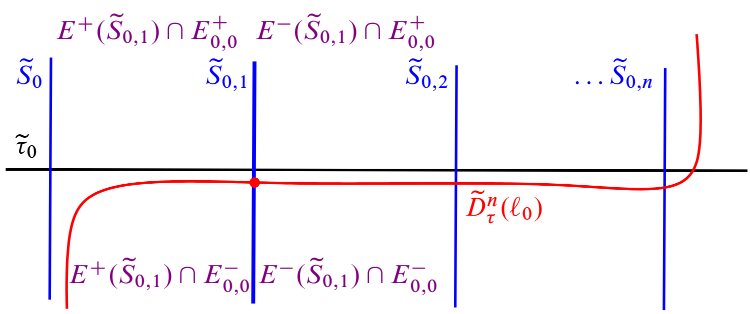

Figure 6: The image of ℓ_0 under Dehn twisting and sets of ends corresponding to the intersection $\widetilde{D}^n_\tau(\widetilde{S}_0)\cap\widetilde{S}_{0,1}$

On the other hand, let us denote by $E^+(\widetilde{S}_{0,s})$ and $E^-(\widetilde{S}_{0,s})$ the two disjoint sets of ends corresponding to the sphere $\widetilde{S}_{0,s}$ for $s\in\{1,\dots,n\}$. Now, without loss of generality, assume that $\mathsf{e}_0^- \in E^-(\widetilde{S}_{0,1})$. Then, since the image ray $\widetilde{D}_\tau^n(\ell_0)$ still intersects $\widetilde{\tau}_0$ only once, and intersects neither $\widetilde{S}_0$ (this would create a bigon) nor any other lift of the torus, $\widetilde{D}_\tau^n(\mathsf{e}_0^+) = \mathsf{e}_\mathsf{n}^+ \in E^+(\widetilde{S}_{0,1})$. So for $s = 1$ we have 4 disjoint, nonempty sets of ends

$$E^+(\widetilde{S}_{0,1}) \cap E_{0,0}^-, \quad E^+(\widetilde{S}_{0,1}) \cap E_{0,0}^+, \quad E^-(\widetilde{S}_{0,1}) \cap E_{0,0}^-, \quad E^-(\widetilde{S}_{0,1}) \cap E_{0,0}^+.$$

By the intersection criterion, this shows that $\widetilde{D}_\tau^n(\ell_0)$ intersects $\widetilde{S}_{0,1}$. Since

$$E^+(\widetilde{S}_{0,1}) \subset E^+(\widetilde{S}_{0,n}) \quad \text{and} \quad E^-(\widetilde{S}_{0,n}) \subset E^-(\widetilde{S}_{0,0}),$$

we similarly have the nonempty disjoint sets of ends

$$E^+(\widetilde{S}_{0,n}) \cap E_{0,n-1}^-, \quad E^+(\widetilde{S}_{0,n}) \cap E_{0,n-1}^+, \quad E^-(\widetilde{S}_{0,n}) \cap E_{0,n-1}^-, \quad E^-(\widetilde{S}_{0,n}) \cap E_{0,n-1}^+.$$

Again by the intersection criterion this means that $\widetilde{D}_\tau^n(\ell_0)$ intersects $\widetilde{S}_{0,n}$ as well. As a result, $\widetilde{D}_\tau^n(\ell_0)$ intersects all iterates $\widetilde{S}_{0,s}$ of $\widetilde{S}_0$, where $s \in \{1, \dots, n\}$.

Hence $\widetilde{D}_\tau^n(\widetilde{S}_0)$ intersects all n iterates of $\widetilde{S}_0$. More precisely,

$$\widetilde{D}_\tau^n(\widetilde{S}_0) \cap \widetilde{S}_{0,j} \neq \varnothing \quad \text{for } j = 1, \dots, n \text{ with } n \geq 2,$$

and thus we conclude that

$$\mathrm{tw}_\tau(S, D_\tau^n(S)) \geq n - 1. \qquad \square$$

6 The almost fixed set

In this section we will prove the following theorem which says that there is an upper bound on the diameter of the almost fixed set of a Dehn twist and this upper bound depends only on the rank of the free group.

Theorem 1.2 *Let T be a $\mathbb{Z}$–splitting of F_k with $k > 2$ and D_T denote the corresponding Dehn twist. Then, for all sufficiently large constants C, there exists a $C' = C'(C, k)$ such that the diameter of the almost fixed set*

$$F_C = \{x \in \mathrm{FF}_k : \exists n \neq 0 \text{ such that } d(x, D_T^n(x)) \leq C\}$$

corresponding to $\langle D_T \rangle$ is bounded above by C'.

6.1 Finding the folding line

For G_1 and G_2 in CV_k, let $m(G_1, G_2)$ be the infimum of the set of maximal slopes of all *change of marking* maps (maps linear on edges) $f\colon G_1 \to G_2$. Then we define a function $d_L\colon \mathrm{CV}_k \times \mathrm{CV}_k \to \mathbb{R}_{\geq 0}$ by

$$d_L(G_1, G_2) = \log m(G_1, G_2).$$

Despite being nonsymmetric, since this is its only failure to be a distance, d_L is referred to as the *Lipschitz metric* on CV_k.

For an interval $I \subset \mathbb{R}$ the folding lines $\tilde{g}\colon I \to \mathrm{CV}_k$ are the paths connecting any given two points in the interior of CV_k, obtained as follows:

For $G_1, G_2 \in \mathrm{CV}_k$ let $f\colon G_1 \to G_2$ be a change of marking map whose Lipschitz constant realizes the maximal slope. We find a path based at G_1 which is contained in an open simplex of unprojectivized outer space and parametrize it by arclength. Then we concatenate this path with another geodesic path outside the open simplex obtained by the folding process. The resulting path $g\colon [0, d_L(G_1, G_2)] \to \mathrm{CV}_k$ is a geodesic by Francaviglia and Martino [8] and is called the *folding line*.

For a given F_k–tree Γ and an element $a \in F_k$, let $\ell_\Gamma(a)$ denote the minimal translation length of a in Γ. To prove Theorem 1.2, we use the following result from [6].

Theorem 6.1 [6] *Suppose $G, G' \in \mathrm{CV}_k$ with $d = d_L(G, G')$ such that $\mathrm{tw}_a(G, G')$ is at least $n+2$ for some $a \in F_k$. Then there is a geodesic (folding line) $g\colon [0, d] \to \mathrm{CV}_k$ such that $g(0) = G$ and $g(d) = G'$ and for some $t \in [0, d]$, we have $\ell_{g(t)}(a) \le 1/n$. In other words, $g([0, d]) \cap \mathrm{CV}_k^{1/n} \neq \varnothing$.*

6.2 Converting short length to distance

Let π be the coarse projection $\pi\colon \mathrm{CV}_k \to \mathrm{FF}_k$.

Lemma 6.2 [2, Lemma 3.3] *Let $a \in F_k$ be a simple class and G a point in CV_k so that the loop corresponding to a in G intersects some edge $\le m$ times. Then*

$$d_{\mathrm{FF}_k}(\alpha, \pi(G)) \le 6m + 13,$$

where α is the smallest free factor containing the conjugacy class of a.

Using this lemma we prove the following:

Lemma 6.3 *Let α be a free factor containing the conjugacy class of an element $a \in F_k$ and G a point in CV_k, and suppose $\ell_G(a) \le m$. Then there is a constant B and a number A depending only on the rank of the free group such that*

$$d_{\mathrm{FF}_k}(\alpha, \pi(G)) \le Am + B.$$

Proof Let e be the edge of G with the greatest length. Hence $\ell(e) \ge 1/(3k+3)$. Then α crosses e less than $(3k+3)m$ times. Therefore $d_{\mathrm{FF}_k}(\alpha, \pi(G)) \le 6(3k+3)m + 13$ by Lemma 6.2. Here $A = 6(3k+3)$ clearly depends only on the rank of the free group. □

6.3 Proof of Theorem 1.2

Given $x \in F_C$, let us assume that $G \in \mathrm{CV}_k$ is a point which is projected to x. We will write $\pi(G) = x$. Let τ be the essential embedded torus in M corresponding to the given $\mathbb{Z}$–splitting T.

Let n be such that $d_{\mathrm{FF}_k}(G, D^n_T(G)) \le C$. Up to replacing C by a constant we can assume that $n \ge 4$. By Theorem 5.2, we have $\mathrm{tw}_\tau(G, D^n_\tau(G)) \ge n-1$. Since $\mathrm{tw}_\tau(G, D^n_\tau(G)) = \mathrm{tw}_a(G, D^n_T(G))$, with $a \in F_k$ representing the core curve of the torus τ, we use the theorem of Clay and Pettet, Theorem 6.1, to deduce that there is a folding path $G_t \colon [0,d] \to \mathrm{CV}_k$ such that $G_0 = G$ and $G_d = D^n_\tau(G)$ and such that $\ell_{G_t}(a) \le 1/(n-3)$ for some $t \in [0,d]$.

Now, by [2] and [18], the projection $\pi(\{G_t\})$ of the folding path $\{G_t\}_t$ onto FF_k is a quasigeodesic in FF_k between $\pi(D^n_\tau(G))$ and $\pi(G) = x$. Let α be the smallest free factor containing a. Since $n \ge 4$ we use Lemma 6.3 to deduce that

$$d_{\mathrm{FF}_k}(\alpha, \pi(G_t)) \le \frac{A}{n-3} + 13,$$

where A is the same as it is given in the lemma. Since $\pi(\{G_t\})$ is uniformly Hausdorff-close to a geodesic and $d_{\mathrm{FF}_k}\big(\pi(G), \pi(D^n_\tau(G))\big) \le C$, by the triangle inequality we have

$$d_{\mathrm{FF}_k}(\pi(G), \alpha) \le \frac{A}{n-3} + C + H + 13 \le A + C + H + 13,$$

where $H = H(k)$ is the distance between the geodesic and the unparametrized quasi-geodesic $\pi(\{G_t\})$. Hence we have

$$C' = 2\,(A + C + H + 13). \qquad \square$$

7 Constructing fully irreducibles

In this section we will prove the main theorem of this paper.

Theorem 1.1 *Let T_1 and T_2 be two $\mathbb{Z}$–splittings of the free group F_k with rank $k > 2$ and α_1 and α_2 be two corresponding free factors in the free factor complex FF_k of the free group F_k. Let D_1 be a Dehn twist fixing α_1 and D_2 a Dehn twist fixing α_2, corresponding to T_1 and T_2, respectively. Then there exists a constant $N = N(k)$ such that whenever $d_{\mathrm{FF}_k}(\alpha_1, \alpha_2) \ge N$,*

1. *$\langle D_1, D_2\rangle \simeq F_2$, and*
2. *all elements of $\langle D_1, D_2\rangle$ which are not conjugate to the powers of D_1 and D_2 in $\langle D_1, D_2\rangle$ are fully irreducible.*

We will start with some basic definitions and lemmata that are standard for δ–hyperbolic spaces.

Let (X, d) be a metric space. For $x, y, z \in X$, the *Gromov product* $(y, z)_x$ is defined as

$$(y, z)_x := \tfrac{1}{2}(d(y, x) + d(z, x) - d(y, z)).$$

If (X, d) is a δ–hyperbolic space, the initial segments of length $(y, z)_x$ of any two geodesics $[x, y]$ and $[x, z]$ stay close to each other. In other words, these geodesics are in 2δ–neighborhoods of each other. Hence the Gromov product measures how long two geodesics stay close together. This characterization of δ–hyperbolicity will be used in our work as the definition of being δ–hyperbolic for a metric space.

The Gromov product $(y, z)_x$ also approximates the distance between x and the geodesic $[y, z]$ within 2δ:

$$(y, z)_x \le d(x, [y, z]) \le (y, z)_x + 2\delta.$$

Definition 7.1 A path $\sigma\colon I \to X$ is called a (λ, ϵ)–quasigeodesic if σ is parametrized by arclength and if, for any $s_1, s_2 \in I$, we have

$$|s_1 - s_2| \le \lambda d(\sigma(s_1), \sigma(s_2)) + \epsilon.$$

If the restriction of σ to any subsegment $[a, b] \subset I$ of length at most ℓ is a $(\lambda, \mathcal{L})$–quasigeodesic, then we call σ an ℓ–local $(\lambda, \mathcal{L})$–quasigeodesic.

Let X be a geodesic metric space and $Y \subset X$. We say that Y is c–quasiconvex if for all $y_1, y_2 \in Y$ the geodesic segment $[y_1, y_2]$ lies in the c–neighborhood of Y.

For any $x \in X$ we call $p_x \in Y$ an *ϵ–quasiprojection* of x onto Y if

$$d(x, p_x) \le d(x, Y) + \epsilon.$$

The lemma below shows that in hyperbolic spaces, quasiprojections onto quasiconvex sets are *quasiunique*.

Lemma 7.2 *Let X be a δ–hyperbolic metric space and let $Y \subset X$ be c–quasiconvex. Let $x \in X$ and let p_x and $p_{x'}$ be two ϵ–quasiprojections of x onto Y. Then*

$$d(p_x, p_{x'}) \le 2c + 4\delta + 2\epsilon.$$

For a δ–hyperbolic geodesic G–space X, consider the almost fixed set $X_C(g)$ corresponding to a subgroup $\langle g \rangle$ for $g \in G$. Then the *quasiconvex hull* $\mathcal{X}_C(g)$ of $X_C(g)$ is defined to be the union of all geodesics connecting any two points of $X_C(g)$. From now on we will work with quasiconvex hulls of almost fixed sets. The following is standard for δ–hyperbolic spaces.

Lemma 7.3 [19, Lemma 3.9] *$\mathcal{X}_C(g)$ is g–invariant and 4δ–quasiconvex.*

The following lemma appears as Lemma 3.12 in [19].

Lemma 7.4 *Let (X,d) be a δ–hyperbolic space and let $[x_p,x_q]$ be a geodesic segment in X. Let $p,q\in X$ be such that x_p is a projection of p on $[x_p,x_q]$ and such that x_q is a projection of q on $[x_p,x_q]$. Then if $d(x_p,x_q)>100\delta$, the path $[p,x_p]\cup[x_p,x_q]\cup[x_q,q]$ is a $(1,30\delta)$–quasigeodesic.*

In the proof of Theorem 1.3 we use Lemma 7.4, which assumes that the projections onto almost fixed sets exist. However, given a geodesic metric space X and $Y\subset X$ a subset which is not necessarily closed in X, the closest point projection onto Y may not exist. To fix this we will use quasiprojections which exist quasiuniquely when there is quasiconvexity, by Lemma 7.2.

Now we state and prove the following theorem, which essentially proves Theorem 1.1.

Theorem 1.3 *Let G be a group acting on a δ–hyperbolic metric space X by isometries and $\phi_1,\phi_2\in G$. Suppose $C>100\delta$ and the almost fixed sets $\mathcal{X}_C(\phi_1)$ and $\mathcal{X}_C(\phi_2)$ of $\langle\phi_1\rangle$ and $\langle\phi_2\rangle$, respectively, have diameters bounded above by a constant C'. Then there exists a constant C_1 such that, whenever $d_X(\mathcal{X}_C(\phi_1),\mathcal{X}_C(\phi_2))\ge C_1$,*

(1) *$\langle\phi_1,\phi_2\rangle\simeq F_2$, and*
(2) *every element of $\langle\phi_1,\phi_2\rangle$ which is not conjugate to the powers of ϕ_1 and ϕ_2 in $\langle\phi_1,\phi_2\rangle$ acts loxodromically in X.*

Proof Let $p_1\in\mathcal{X}_C(\phi_1)$ and $p_2\in\mathcal{X}_C(\phi_2)$ be two points such that

$$d_X(p_1,p_2)=d_X(\mathcal{X}_C(\phi_1),\mathcal{X}_C(\phi_2)).$$

To prove the theorem we will pick a random word ω and construct a ping-pong argument involving the sets $\mathcal{X}_C(\phi_1)$ and $\mathcal{X}_C(\phi_2)$. The goal is to show that the iterates of $[p_1,p_2]$ under ω give a local quasigeodesic, hence a quasigeodesic. Without loss of generality, for $g\in\langle\phi_1\rangle$, we will start with proving that the path $[p_2,p_1]\cup[p_1,gp_1]\cup g[p_1,p_2]$ is a quasigeodesic.

Let $\pi(p_2)$ and $\pi(gp_2)$ be quasiprojections of the points p_2 and gp_2 on the geodesic segment $[p_1,gp_1]$. Then p_1 and $\pi(p_2)$ are both 4δ–quasiprojections. This is true for $\pi(gp_2)$ and gp_1 also. Since the difference is negligible we will assume p_1 and gp_1 are closest point projections.

Now we prove that $d(p_1, gp_1) \geq C$. To see this suppose that $d_X(p_1, gp_1) < C$. We take a point x on the geodesic segment $[p_1, p_2]$ such that $d(x, p_1) < \epsilon$, where $\epsilon = \frac{1}{2}(C - d(p_1, gp_1))$. Then we have

$$d(x, gx) \leq 2\epsilon + d(p_1, gp_1) = C,$$

which contradicts the assumption that p_1 is the closest point of $\mathcal{X}_C(\phi_1)$ to $\mathcal{X}_C(\phi_2)$. Since we proved that $d(p_1, gp_1) \geq C$, we apply Lemma 7.4 to conclude that the path $[p_2, p_1] \cup [p_1, gp_1] \cup g[p_1, p_2]$ is a quasigeodesic.

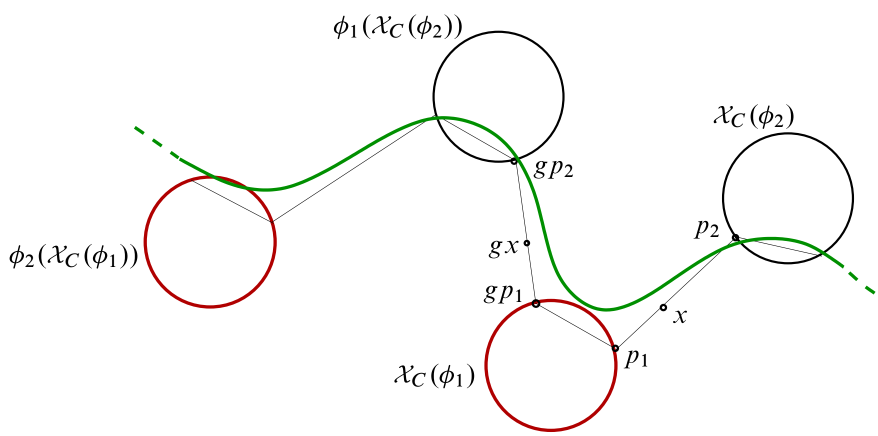

Figure 7: The ping-pong sets and a quasigeodesic between iterated points

Given two points $p_1 \in \mathcal{X}_C(\phi_1)$ and $p_2 \in \mathcal{X}_C(\phi_2)$ as above, a geodesic connecting them is called a *bridge* and it is not unique. However, by [19, Lemma 5.2], when two sets are sufficiently far apart it is *almost unique*. Hence we first assume that

$$d(p_1, p_2) \geq C.$$

Since $C > 100\delta$ this is sufficient to have a quasiunique bridge. Now we take a word $\omega = \phi_2^{m_l}\phi_1^{s_l}\cdots\phi_2^{m_1}\phi_1^{s_1}$ and consider the iterates of the quasiunique bridge $[p_1, p_2]$ under ω.

It is known that by the hyperbolicity of X, given $(\lambda, \mathcal{L})$, there exists $\ell > 0$ such that an ℓ–local $(\lambda, \mathcal{L})$–quasigeodesic is a $(\lambda', \mathcal{L}')$–quasigeodesic, where $\lambda' = \lambda'(\lambda, \mathcal{L}, \ell)$ and $L' = L'(\lambda, L, \ell)$. Since we have $(1, 30\delta)$–quasigeodesic pieces, we have such an ℓ which satisfies $\ell = \ell(30\delta)$. Hence we let

$$d(p_1, p_2) \geq C_1 := \max\{100\delta, \ell\}.$$

Now consisting of previously given quasigeodesic pieces, the path

$$\gamma := [p_1, p_2] \cup [p_1, \phi_1^{s_1} p_1] \cup [\phi_1^{s_1} p_1, \phi_1^{s_1} p_2] \cup [\phi_1^{s_1} p_2, \phi_2^{m_1} \phi_1^{s_1} p_2] \cup \cdots \cup [\omega p_1, \omega p_2]$$

is an ℓ–local $(1, 100\delta)$–quasigeodesic, and as a result, it is a quasigeodesic.

In particular, for any word ω in $\langle \phi_1, \phi_2 \rangle$ we have $d(\omega(x), x) \geq |\omega|$, where $|\omega|$ denotes the syllable length, up to conjugation. Now, it follows that $\langle \phi_1, \phi_2 \rangle$ is free. Since the path which is obtained by iterating any segment between two almost fixed sets under ω is a quasigeodesic, ω is loxodromic. □

Proof of Theorem 1.1 Consider the action of $\mathrm{Out}(F_k)$ on the free factor complex FF_k. By Theorem 1.2, for a sufficiently large constant $C = C(k)$ there exists a $C' = C'(C)$ such that the diameter of the almost fixed set of a Dehn twist is bounded above by C'. Let C_1 be the constant from Theorem 1.3.

Now assume that D_1 and D_2 are the Dehn twists so that $d_{\mathrm{FF}_k}(\alpha_1, \alpha_2) \geq 2C' + C_1$, where α_1 and α_2 are the projections of the given $\mathbb{Z}$–splittings T_1 and T_2 to FF_k, respectively. Since Theorem 1.2 implies $\mathrm{diam}\{F_C\} \leq A + C + H + 13 = C'$, we have

$$d(F_C(D_1), F_C(D_2)) \geq C_1.$$

Hence Theorem 1.3 applies to $\langle D_1, D_2 \rangle$ with $N = 2\,\mathrm{diam}\{F_C\} + C_1 = 2C' + C_1$. It is clear that $N = N(k)$. As a result, since loxodromically acting elements in the free factor complex are fully irreducible, every element from the group $\langle D_1, D_2 \rangle$ is either conjugate to powers of the twists or fully irreducible. □

8 Constructing atoroidal fully irreducibles

In this section we prove the following theorem which produces *atoroidal* fully irreducibles. Recall $\mathrm{FF}_k^{(1)}$ is the 1–skeleton of the free factor complex (free factor graph) and that $\sigma\colon \mathrm{FF}_k^{(1)} \to \mathcal{P}_k$ is a coarse surjective map and that both graphs have the same vertex sets.

Theorem 1.4 *Let T_1 and T_2 be two $\mathbb{Z}$–splittings of F_k with $k > 2$ with corresponding free factors α_1 and α_2, and let D_1 and D_2 be two Dehn twists corresponding to T_1 and T_2, respectively. Then there exists a constant $N_2 = N_2(k)$ such that $\langle D_1, D_2 \rangle \simeq F_2$ whenever $d_{\mathcal{P}_k}(\sigma(\alpha_1), \sigma(\alpha_2)) \geq N_2$, and all elements from this group which are not conjugate to the powers of D_1 and D_2 in $\langle D_1, D_2 \rangle$ are atoroidal fully irreducible.*

Proof To see that $\langle D_1, D_2 \rangle \simeq F_2$ we use Theorem 1.3. To do that, we need to show that given a $\mathbb{Z}$–splitting T and a constant C there is another constant depending only on C which bounds from above the diameter of the almost fixed set $\mathcal{P}_C$ of a Dehn twist corresponding to T.

Let x be a point in $\mathcal{P}_C$ and $\sigma\pi(G) = x$ for some $G \in \mathrm{CV}_k$. Then, as before, we use the theorem of Clay and Pettet, Theorem 6.1, to obtain a folding line $\{G_t\}_t$ between G and $D_\tau^n(G)$ along which a is short, where a is the generator of the fundamental group of the torus τ in M. Then, since distances in the intersection graph are shorter, the lemma of Bestvina and Feighn, Lemma 6.2, applies, and

$$d_{\mathcal{P}_k}(\sigma\alpha, \sigma\pi(G)) \le d_{\mathrm{FF}_k}(\alpha, \pi(G)) \le 6m + 13.$$

Hence we can convert the short length to distance in the intersection graph. Thus there exists constants A and B such that

$$d_{\mathcal{P}_k}(\alpha, \pi(G_t)) \le Am + B,$$

where G_t is the point along which a is short. Then the rest of the proof follows the same since the image of the folding line in $\mathcal{P}_k$ is a quasigeodesic and it is Hausdorff-close to a geodesic by [18] and [23]. Hence the diameter of $\mathcal{P}_C$ is uniformly bounded above by a constant, and Theorem 1.3 applies to $\langle D_1, D_2 \rangle$ with $N_2 = 2\,\mathrm{diam}\{\mathcal{P}_C\} + C_1$.

Since in $\mathcal{P}_k$ loxodromically acting automorphisms are atoroidal fully irreducible [23], an element of $\langle D_1, D_2 \rangle$ which is not conjugate to the powers of D_1 and D_2 in $\langle D_1, D_2 \rangle$ is *atoroidal* fully irreducible. □

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Received: 28 January 2015 Revised: 19 May 2016