Embedding calculus knot invariants are of finite type

RYAN BUDNEY
JAMES CONANT
ROBIN KOYTCHEFF
DEV SINHA

We show that the map on components from the space of classical long knots to the nth stage of its Goodwillie–Weiss embedding calculus tower is a map of monoids whose target is an abelian group and which is invariant under clasper surgery. We deduce that this map on components is a finite type-$(n-1)$ knot invariant. We compute the E^2–page in total degree zero for the spectral sequence converging to the components of this tower: it consists of $\mathbb{Z}$–modules of primitive chord diagrams, providing evidence for the conjecture that the tower is a universal finite-type invariant over the integers. Key to these results is the development of a group structure on the tower compatible with connected sum of knots, which in contrast with the corresponding results for the (weaker) homology tower requires novel techniques involving operad actions, evaluation maps and cosimplicial and subcubical diagrams.

55P65, 57M25

1 Introduction

We connect three current threads in studying knots and the moduli space of all knots: the Goodwillie–Weiss embedding calculus, Budney's operad actions and Vassiliev's theory of finite-type invariants. This work also connects two fundamental results on commutativity which are over fifty years old. In 1949, H Schubert [25] established and applied the fact that connected sum of knots is commutative. In 1947, Steenrod [30] gave explicit formulae exhibiting commutativity of cup product, in the course of defining the cohomology operations which bear his name. We establish and use compatibility of these classical results as we develop a group structure on the components of the Goodwillie–Weiss tower.

One of our main results is that the Goodwillie–Weiss tower for knots yields additive invariants of finite type. Such a result is modest compared to the conjecture made by Budney, Conant, Scannell and Sinha [4] that the tower gives all such invariants. We provide evidence for this conjecture through calculation of the E^2–term of the spectral

sequence for the components of this tower. The current results are meaningful first steps, and establishing them requires new combinations of tools in the areas of operad actions, clasper surgery, cosimplicial and cubical diagrams, and evaluation maps.

A large part of the work is "getting off the ground", showing that the Goodwillie–Weiss tower yields abelian group-valued invariants compatible with connected sum of knots. Because the knot invariants are defined as an induced map on π_0, such invariants are a priori only set-valued. A number of authors have already studied multiplicative structures on the closely related Goodwillie–Weiss tower for the homotopy fiber of the map from embeddings to immersions, namely Sinha [27], Turchin [31], Dwyer and Hess [8] and Boavida and Weiss [1], the latter three giving deloopings. The techniques of Turchin, and Boavida and Weiss, can be adapted to imply that the components of stages of the tower for classical framed knots is an abelian group (while the result of Dwyer and Hess applies to the limit and Sinha's result only implies an abelian monoid structure in the limit). Thus, including the present results, there are now five different monoid structures on the components of the inverse limit of the tower for framed classical knots, four of which are group structures, which are all conjecturally equivalent. For our present application, we need compatibility with connected sum of knots, which our approach here provides, as does the approach of Boavida and Weiss, which appeared after our work. Our techniques involve modeling the totalization tower of a cosimplicial space using cubical diagrams, which is of independent interest. Once we establish the group structure and compatibility with connected sum, we use the Habiro surgery criterion for finite type to establish the main result that the nth stage in the tower defines a type-$(n-1)$ invariant.

Our work has many predecessors. The first is [4], in which we establish that the third stage of the tower is a universal type-2 invariant. Here the group structure and the finite-type result were straightforward from ad hoc arguments. The more interesting aspect of this work was the development of geometry to explicitly distinguish components in the tower, yielding a new interpretation of the type-2 knot invariant by counting collinearities of points. This geometric approach was continued by Flowers [9].

The second predecessor is Volić's thesis, which establishes that the Goodwillie–Weiss tower for the space level rational homology of knots is a universal rational finite-type invariant. The map to the homology tower factors through the tower we consider (the "homotopy tower"), which thus encodes such a universal rational invariant as well. The question of whether the tower gives a universal invariant over the integers is of considerable interest, since Vassiliev invariants with values in arbitrary $\mathbb{Z}$–modules are not well understood. A universal invariant is known to exist over the integers (see for example Habiro's ψ_n map, used in Theorem 6.1 of this paper), but it is not computable nor does it directly relate to the combinatorics of weight systems. Identification of the

tower as a universal invariant would likely resolve questions about the combinatorics and geometry of finite-type invariants and, in particular, the open question of whether weight systems "integrate".

Our techniques are distinct from those of Volić in three ways. In our result the type-n invariants are potentially realized at the $n+1^{\text{st}}$ stage, while in the rational homology tower they are realized precisely at stage $2n$. Volić uses the Vassiliev derivative approach to establish his finite-type result, while we use Habiro's clasper surgery. Finally, Volić's result proceeds by extending the theory of Bott–Taubes integration to the homology tower, while our approach through the homotopy tower invites new techniques from geometric and algebraic topology.

We extend the abelian group structures to the spectral sequence level, which is crucial for analysis since studying components of an arbitrary tower of spaces can potentially lead to unending ad hoc calculations. To do so we employ C_1–operad actions, which necessitates their development throughout our work despite the fact that our main theorems are at the level of components. With group structure in hand, we immediately establish convergence at each finite stage and then use results of the second author [5] to identify the E^2 term in degree zero as the $\mathbb{Z}$–modules of primitive chord diagrams.

Based on this, we conjecture that the map from the knot space to the tower sends linear combinations of knots given by resolving a singular knot to the corresponding elements of E^2. We also conjecture that the spectral sequence collapses and, together, these two conjectures would imply that weight systems over the integers all "integrate" to finite-type invariants. Unlike Vassiliev's original spectral sequence, this spectral sequence does not involve a subtle limit process (see Giusti [10]) but instead is simply the spectral sequence of a tower of fibrations. It thus is more amenable to tools from algebraic topology such as generalized Hopf invariants.

The paper is organized as follows. Section 2 gives needed general background on compactified configuration spaces and on cubical and cosimplicial diagrams. Section 3 recalls the resulting mapping space and cosimplicial models we prefer to use, interchangeably, for the nth stage of the Taylor tower for $\text{Emb}^{\text{fr}}(\mathbb{R}, \mathbb{R}^3)$. Some readers may prefer to only look back at these sections as needed, in particular for the "infinitesimal transformation" map between the spatial and operadic mapping space models.

In Section 4, we construct homotopy-commutative multiplications on the nth stage of the Taylor tower for $\text{Emb}^{\text{fr}}(\mathbb{R}, \mathbb{R}^3)$. This shows that π_0 of each stage of the Taylor tower is an abelian monoid. Section 5 contains two main results, the first being that the projections in the tower are surjective on π_0. This is then used to show that π_0 of each stage is a group.

In Section 6, we show that π_0 of the map from the space of knots to its $n+1^{\text{st}}$ Goodwillie–Weiss approximation is invariant under clasper surgery and thus of type n. In Section 7, we show that the homotopy spectral sequence for the tower is a spectral sequence of abelian groups (in particular in total degrees zero and one), and we identify the E^2 term.

2 Compactification of configuration spaces

2.1 Basic definition

We briefly review the simplicial compactification of configuration space, defined in [26; 28]. For any manifold M — not necessarily compact, and possibly with boundary — let $C_n(M)$ denote the configuration space of n points in M. It is the space of distinct ordered n–tuples of points in M.

Definition 2.1 For an M embedded in some Euclidean space $\mathbb{R}^d$, let $C_n\langle M \rangle$ denote the closure of the image of

$$C_n(M) \hookrightarrow M^n \times (S^{d-1})^{\binom{n}{2}}, \quad (x_1, \ldots, x_n) \mapsto \left((x_1, \ldots, x_n), \left(\frac{x_i - x_j}{|x_i - x_j|} \right)_{i<j} \right).$$

As shown in [26], this is independent of embedding and is a quotient of $C_n[M]$, the Fulton–MacPherson compactification of $C_n(M)$. If M is noncompact, then $C_n\langle M \rangle$ as defined above as well as the Fulton–MacPherson compactification are not compact but are more accurately described as completions. Informally, we refer to the projection to M^n as the "spatial" information, while information which distinguishes points with the same projection to M is "infinitesimal". If M is one-dimensional and connected, then $C_n\langle M \rangle$ is in general not connected. For such M, by abuse we use $C_n\langle M \rangle$ to denote the connected component where the n points are in (cyclic) order.

2.2 Framings and tangent data

We need framed configurations. For any manifold M, define $C_n^{\text{fr}}\langle M \rangle$ as the pullback

$$
\begin{array}{ccc}
C_n^{\text{fr}}\langle M \rangle & \longrightarrow & (\text{Fr}\, M)^n \\
\downarrow & & \downarrow \\
C_n\langle M \rangle & \longrightarrow & M^n
\end{array}
$$

where $\text{Fr}\, M \to M$ is the unit frame bundle of the tangent bundle of M. If M is parallelizable and d–dimensional, then $C_n^{\text{fr}}\langle M \rangle$ is homeomorphic to $C_n\langle M \rangle \times O(d)^n$.

Let $C'_n\langle M \rangle$ be defined similarly, with the unit tangent bundle *STM* in place of the unit frame bundle.

2.3 Distinguished boundary points

Suppose M has two distinguished points y_0 and y_1 in its boundary. Then, as in [4; 28], let $C_n\langle M, \partial \rangle$ denote the subspace of $C_{n+2}\langle M \rangle$ where the first and last points are located at y_0 and y_1, by abuse omitting dependence on these points from notation.

Further, if there are distinguished tangent vectors $v_0 \in TM|_{y_0}$ and $v_1 \in TM|_{y_1}$, let $C'_n\langle M, \partial \rangle$ be the subspace of $C'_{n+2}\langle M \rangle$ where (p_1, v_1) and (p_{n+2}, v_{n+2}) are (y_0, v_0) and (y_1, v_1), respectively. By fixing framings at y_0 and y_1, define $C^{\mathrm{fr}}_n\langle M, \partial \rangle$ similarly.

Define $C_n\langle \mathbb{I}, \partial \rangle$ by taking the two endpoints to be the distinguished points. The fact that $C_n\langle \mathbb{I}, \partial \rangle$ is the n–simplex is the main rationale for calling this compactification "simplicial". Define $C'_n\langle \mathbb{I}^d \rangle$ by taking $\{y_0, y_1\}$ to be $\partial \mathbb{I} \times \{(0, \ldots, 0)\}$ and $v_0 = v_1 = (1, 0, \ldots, 0)$ (so our knots will "proceed from left to right") and similarly define $C^{\mathrm{fr}}_n\langle \mathbb{I}^d, \partial \rangle$ by using the identity element in $O(d)$ for framings at those boundary points (using the standard parallelization of $\mathbb{I}^d \subset \mathbb{R}^d$).

2.4 Quotients by translation and scaling, and insertion maps

There are maps between products of $C_n\langle \mathbb{R}^d \rangle$ and $C_n\langle \mathbb{I}^d, \partial \rangle$ defined by "inserting an infinitesimal configuration into a point of another configuration". Let $\widetilde{C}_n(\mathbb{R}^d) := C_n(\mathbb{R}^d)/(\mathbb{R}^d \rtimes \mathbb{R}_+)$ be the quotient of configuration space by translations and positive scalings of all n points.

Definition 2.2 Define $\widetilde{C}_n\langle \mathbb{R}^d \rangle$ as the closure of the image of the map

$$\widetilde{C}_n(\mathbb{R}^d) \xrightarrow{\ e\ } (S^{d-1})^{\binom{n}{2}}, \quad (x_1, \ldots, x_n) \mapsto \left(\frac{x_i - x_j}{|x_i - x_j|} \right)_{i < j},$$

which is injective except on collinear configurations.

We let v_{ij} denote the projection of $\widetilde{C}_n\langle \mathbb{R}^d \rangle$ to the $(i, j)^{\mathrm{th}}$ factor of S^{d-1}.

The similarly defined $\widetilde{C}_n\langle \mathbb{I}^d \rangle$ is homeomorphic to $\widetilde{C}_n\langle \mathbb{R}^d \rangle$, so both $C_n\langle \mathbb{R}^d \rangle$ and $C_n\langle \mathbb{I}^d \rangle$ naturally surject onto $\widetilde{C}_n\langle \mathbb{R}^d \rangle$.

We proceed directly to the framed setting in defining insertion maps. Let $\widetilde{C}^{\mathrm{fr}}_n\langle \mathbb{R}^d \rangle := \widetilde{C}_n\langle \mathbb{R}^d \rangle \times O(d)^n$, the framed version of the "infinitesimal configuration space".

For every m, n, and $i \in \{1, \ldots, n\}$, we define a map $\circ_i$ which, informally, inserts a configuration of m points (with framings) into the ith point of a configuration of n

points (with framings). In the resulting configuration of $m+n-1$ points, the m points form an "infinitesimal configuration". Precisely, in coordinates we define

$$\circ_i \colon C_n^{\mathrm{fr}} \langle \mathbb{I}^d, \partial \rangle \times C_m^{\mathrm{fr}} \langle \mathbb{I}^d \rangle \to C_{m+n-1}^{\mathrm{fr}} \langle \mathbb{I}^d, \partial \rangle$$

as follows. First, suppressing the dependence on i, we let

$$\hat{j} = \begin{cases} j & \text{if } j \le i, \\ i & \text{if } i \le j \le i+m-1, \\ j+m-1 & \text{if } j > i+m-1. \end{cases}$$

Now define $\circ_i$ as sending

$$((x_j)_{j=1}^n, (u_{jk})_{j<k}, (\alpha_j)_{j=1}^n) \times ((y_j)_{j=1}^m, (v_{jk})_{j<k}, (\beta_j)_{j=1}^m)$$
$$\mapsto ((z_j)_{j=1}^{m+n-1}, (w_{jk})_{j<k}, (\gamma_j)_{j=1}^{m+n-1}),$$

where $z_j = x_{\hat{j}}$,

$$w_{jk} = \begin{cases} \alpha_i v_{(j-i+1)(k-i+1)} & \text{if } i \le j, k \le i+m-1, \\ u_{\hat{j}\hat{k}} & \text{otherwise}, \end{cases}$$

and

$$\gamma_j = \begin{cases} \alpha_i \beta_{j-i+1} & \text{if } i \le j \le i+m-1, \\ \alpha_{\hat{j}} & \text{otherwise}. \end{cases}$$

One must check that such maps send subspaces of $(\mathbb{R}^d)^n \times (S^{d-1})^{\binom{n}{2}}$ to each other appropriately. Here we only cite a similar check, namely [26, Proposition 6.6], using the description of the $C_n \langle \mathbb{R}^d \rangle$ as a subspace of $(\mathbb{R}^d)^n \times (S^{d-1})^{\binom{n}{2}}$ given in [26, Theorem 5.14]. The results are not dependent on the y_j coordinates of $C_m^{\mathrm{fr}} \langle \mathbb{I}^d \rangle$, which means that these insertion maps factor through the projection to $\widetilde{C}_m^{\mathrm{fr}} \langle \mathbb{R}^d \rangle$ on that factor.

3 The models

We study the space of framed knots $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}, \mathbb{R}^3)$. A framed knot is a smooth embedding of $\mathbb{R}$ into $\mathbb{R}^3$ together with a smooth map $\mathbb{R} \to O(3)$ whose first (column) vector is the unit derivative map. Embeddings take $\mathbb{I} = [-1, 1] \subset \mathbb{R}$ into $\mathbb{I}^3 \subset \mathbb{R}^3$, and on $\mathbb{R} \setminus \mathbb{I}$ are standard, given by $t \mapsto (t, 0, 0)$. The framing is required to be constant at the identity on $\mathbb{R} \setminus \mathbb{I}$.

We primarily use a mapping space model for the nth stage of the Goodwillie–Weiss tower for the space of framed knots $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}, \mathbb{R}^d)$. But, for both spectral sequence calculations and for clarity through comparison, we use a cosimplicial model as well. For each of these, there are a few variants depending on the choice of compactifications of configuration spaces $C_n(M)$. The Fulton–MacPherson (Axelrod–Singer)

compactification $C_n[M]$ is a smooth manifold with corners. In this paper, we use instead the simplicial compactification $C_n\langle M \rangle$, developed in the previous section. It is not a manifold with corners, but it has the advantage that one component of $C_n\langle \mathbb{I} \rangle$ is the n–simplex. This is needed to define a cosimplicial model, and the corresponding mapping space model is defined in terms of the face poset of the simplex rather than that of the associahedron.

3.1 Cosimplicial models

A cosimplicial space is a functor to $\mathcal{T}op$ from the category Δ with one object for each ordered set $[n] = (0, \dots, n)$ and morphisms given by order-preserving maps. We rely on the standard set of generating morphisms, denoting coface maps by d^i and codegeneracy maps by s^i. We let Δ_n be the full subcategory of Δ containing the first $n + 1$ objects.

Goodwillie–Weiss embedding calculus leads to forming a cosimplicial space from the spaces $C_\bullet^{\mathrm{fr}}\langle \mathbb{I}^d, \partial \rangle$, by using the first vector in the framing as the "doubling direction". This was first done in the unframed setting in [28, Corollary 4.22]. In more detail, we have the following:

Definition 3.1 The spatial cosimplicial model $C_\bullet^{\mathrm{fr}}\langle \mathbb{I}^d, \partial \rangle$ has n^{th} entry $C_n^{\mathrm{fr}}\langle \mathbb{I}^d, \partial \rangle$.

The codegeneracy $s_i \colon C_n^{\mathrm{fr}}\langle \mathbb{I}^d, \partial \rangle \to C_{n-1}^{\mathrm{fr}}\langle \mathbb{I}^d, \partial \rangle$ is the extension to compactifications of the projection which forgets the ith point.

The coface d^i is given by $d^i(x) = x \circ_i \mu$, which "doubles" the ith point by inserting into its position the infinitesimal two-point configuration μ rotated by the ith framing.

So here d^0 and d^{n+1} "double" the "extra" points which are located in the middle of the left and right faces of $\mathbb{I}^d$.

Definition 3.2 The operadic cosimplicial model $\widetilde{C}_\bullet^{\mathrm{fr}}\langle \mathbb{R}^d \rangle$ has n^{th} entry $\widetilde{C}_n^{\mathrm{fr}}\langle \mathbb{R}^d \rangle$.

The codegeneracy $s_i \colon \widetilde{C}_n^{\mathrm{fr}}\langle \mathbb{R}^d \rangle \to \widetilde{C}_{n-1}^{\mathrm{fr}}\langle \mathbb{R}^d \rangle$ is the extension to compactifications modulo translation and scaling of the projection which forgets the ith point.

For $1 \leq i \leq n$, the coface d^i is given by $d^i(x) = x \circ_i \mu$, which "doubles" the ith point by inserting into its position the infinitesimal two-point configuration μ rotated by the ith framing. The coface $d^0(x)$ equals $\mu \circ_2 x$, while the coface $d^{n+1}(x)$ equals $\mu \circ_1 x$.

Thus, in the operadic model, the first and last coface maps insert configurations into two-point configuration, which has the effect of "adding a point at infinity". Along

with the obvious passing to quotients by translation and scaling, this is the difference between the spatial and operadic models. We discuss the relative advantages of these models as well as why we need both when we discuss their associated mapping space models below. The operadic model is so-called because it fits into a framework by which operads with multiplication produce cosimplicial objects, following Gerstenhaber and Voronov, and McClure and Smith [18].

The fourth author showed [27; 28] that the homotopy-invariant totalizations of similar cosimplicial spaces for unframed knots, which we will use and denote by $C'_\bullet\langle \mathbb{I}^d, \partial \rangle$ and $\widetilde{C}'_\bullet\langle \mathbb{R}^d \rangle$, give models for the Goodwillie–Weiss tower. The framed version of this construction was studied by Salvatore [23, Section 3].

By needing to use a homotopy-invariant totalization to model the towers, some control of geometry and combinatorics is lost. A standard approach to cosimplicial spaces through (sub)cubical diagrams, reviewed in the next section, retains both combinatorics and geometry by sacrificing some symmetry. In particular the resulting spatial model is compatible with the evaluation map, also known as a Gauss map, from the knot space.

3.2 Mapping space models

Our mapping space models are defined as homotopy limits of subcubical diagrams of compactified configuration spaces. A subcubical diagram is a functor from $\mathcal{P}_v[n]$, the poset of nonempty subsets of $[n]$, which is the face poset of the n–simplex. Because the cosimplicial and subcubical diagram categories both involve ordered sets, there is an immediate relationship between them. In general Δ admits a canonical functor from any category defined through finite ordered sets, as follows:

Definition 3.3 Let C be a category whose objects are given by ordered finite sets and whose morphisms are subsets of the order-preserving maps between those sets. Define $\mathcal{G}_C\colon C \to \Delta$ to be the functor which sends an ordered finite set S to $[\#S - 1]$ and which sends an order-preserving map $S \to T$ to the composite

$$[\#S - 1] \cong S \to T \cong [\#T - 1],$$

where the isomorphisms are order-preserving.

For $C = \mathcal{P}_v[n]$, we abbreviate $\mathcal{G}_{\mathcal{P}_v[n]}$ to $\mathcal{G}_n\colon \mathcal{P}_v[n] \to \Delta_n$.

The functor $\mathcal{G}_n$ was used [28; 22] to use cubical diagrams to model cosimplicial spaces. The resulting diagrams use all of the coface maps "multiple times", while the codegeneracy maps are ignored.

Definition 3.4 The spatial mapping space model $\mathrm{AM}_n^{\mathrm{fr}}$ is the homotopy limit of the composite functor $C_\bullet^{\mathrm{fr}}\langle \mathbb{I}^3, \partial \rangle \circ \mathcal{G}_n \colon \mathcal{P}_v[n] \to \mathcal{T}op$.

The operadic mapping space model $\widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$ is the homotopy limit of the composite functor $\widetilde{C}_\bullet^{\mathrm{fr}}\langle \mathbb{R}^3 \rangle \circ \mathcal{G}_n \colon \mathcal{P}_v[n] \to \mathcal{T}op$.

We are henceforth focusing on classical knots, in three dimensions, so we are suppressing the ambient dimension from notation. The mapping space model AM_n of [4] is defined similarly to $\mathrm{AM}_n^{\mathrm{fr}}$, but pulled back from $C_\bullet'\langle \mathbb{I}^3, \partial \rangle$ and so with tangent vectors instead of frames.

Using the definition of homotopy limit through nerves of under-categories, the homotopy limit of a subcubical diagram is given by a collection of maps from simplices. Since the structure maps d^i are injective, an element φ of $\mathrm{AM}_n^{\mathrm{fr}}$ is determined by a map $\Delta^n \to C_n^{\mathrm{fr}}\langle \mathbb{I}^3, \partial \rangle$, so $\mathrm{AM}_n^{\mathrm{fr}}$ is a subspace of $\mathrm{Map}(\Delta^n, C_n^{\mathrm{fr}}\langle \mathbb{I}^3, \partial \rangle)$. Because the faces of the simplex map to configurations which are degenerate in an "aligned" manner we sometimes refer to this as the subspace of aligned maps. Explicitly, if some (consecutive) collection of t_i in $\vec{t} = (t_1, \ldots, t_n) \in \Delta^n$ are equal, then the corresponding points in the configuration $\varphi(\vec{t})$ have "collided" in $\mathbb{I}^3$, their framings ($\alpha_i \in O(3)$) are all equal, and the first column of α_i is the direction of collision of these points.

To interrelate cosimplicial and mapping space models, a main technical result is [28, Theorem 6.7], which establishes that $\mathcal{G}_n$ is left cofinal. So if $X^\bullet$ is a cosimplicial space then the homotopy limit of the subcubical diagram $X^\bullet \circ \mathcal{G}_n$ is equivalent to the homotopy limit of the restriction of $X^\bullet$ to Δ_n. By work of Bousfield and Kan [2], this is homotopy equivalent to $\widetilde{\mathrm{Tot}}^n X^\bullet$, the nth stage in the homotopy invariant totalization tower. In [28] this is used to establish the validity of the cosimplicial models, building from that of the mapping space models.

3.2.1 Evaluation maps Our main results make use of evaluation maps, which naturally connect with the spatial mapping space model. By functoriality for embeddings of compactified configuration spaces [26, Corollary 4.8], we have that an embedding $f \colon \mathbb{I} \to \mathbb{I}^3$ will extend to a map from $\Delta^n = C_n\langle \mathbb{I}, \partial \rangle$ to $C_n\langle \mathbb{I}^3, \partial \rangle$. For a framed embedding with framing $\alpha \in \mathrm{Map}(\mathbb{I}, O(3))$ to go along with the embedding f, this map is given as follows:

Definition 3.5 Define $\mathrm{ev}_n \colon \mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}, \mathbb{R}^3) \to \mathrm{AM}_n^{\mathrm{fr}} \subset \mathrm{Map}(\Delta^n, C_n\langle \mathbb{I}^3, \partial \rangle \times O(3)^n)$ as sending an embedding f and framing α to the map which sends

$$(-1 \le t_1 \le \cdots \le t_n \le 1) \mapsto \big((f(t_1), \ldots, f(t_n)), \alpha(t_1), \ldots, \alpha(t_n)\big).$$

Intuitively, ev_n samples the knot and its framing at n points in the domain.

One of the main results of [28], namely Theorem 5.4, immediately extends to the framed setting to establish that ev_n agrees in the homotopy category with the canonical map from the embedding space $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}, \mathbb{R}^d)$ to the nth stage of the Taylor tower, $T_n \mathrm{Emb}$.

In dimensions $d > 3$ it is known that the connectivity of ev_n increases with n, and hence the tower converges to the embedding space. Thus, the homotopy theory of evaluation maps captures the nature of knots in these dimensions. For $d = 3$, it is not known if ev_n is even 0–connected (that is, surjective) much less injective in the limit. We show below that on components ev_n defines abelian group-valued invariants of finite type $n - 1$.

3.2.2 Translation between mapping space models In addition to the quotienting by translation and scaling, AM_n and $\widetilde{\mathrm{AM}}_n$ differ by the first and last coface maps, which in the former case "add a point to the configuration on the left or right face of $\mathbb{I}^3$" while in the latter case such points are added "at infinity to the left or right". This means that the obvious quotient maps do not define a map between these models. We require a map between them, as the spatial model is the target of the evaluation map as we just saw and the operadic model is more convenient for defining multiplications below.

Observe that the evaluation of a long knot can be extended to include times greater than 1 or less than -1, in which case the corresponding configuration points are standard along the x–axis. We achieve something similar for arbitrary elements of AM_n to obtain maps $\widetilde{\varphi}\colon C_n\langle \mathbb{R}\rangle \to C_n\langle \mathbb{R}^3, \partial\rangle$.

Definition 3.6 For $t_1 \leq t_2 \leq \cdots \leq t_n$ with all $t_i \in \mathbb{R}$, we let

$$\widehat{t_i} = \begin{cases} -1 & \text{if } t_i \leq -1, \\ t_i & \text{if } -1 \leq t_i \leq 1, \\ 1 & \text{if } t_i \geq 1. \end{cases}$$

Then we define $\widetilde{\varphi}(t_1 \leq t_2 \leq \cdots \leq t_n)$ to be the configuration which is the union of the following:

(1) One point at $(t_i, 0, 0)$ for each $t_i \leq -1$ or $t_i \geq 1$.

(2) The configuration obtained by taking $\varphi(\widehat{t_1} \leq \cdots \leq \widehat{t_n})$ and applying the projection which forgets points x_i for which $t_i \leq -1$ or $t_i \geq 1$.

Moreover, after we compose with the quotient $C_n\langle \mathbb{R}^3, \partial\rangle \to \widetilde{C}_n\langle \mathbb{R}^3, \partial\rangle$, the map $\widetilde{\varphi}$ extends to a map from $C_n\langle[-\infty, \infty]\rangle$ to $\widetilde{C}_n\langle \mathbb{R}^3, \partial\rangle$ since the limit as some t_i goes to $-\infty$ or ∞ will have all vectors v_{ij} approaching $(1, 0, 0)$ or $(-1, 0, 0)$, as all points in the resulting configurations not in $\mathbb{I}^3$ lie along the x–axis.

Let h be an order-preserving homeomorphism between $[-1, 1]$ and $[-\infty, \infty]$. By our usual abuse of notation, let h also denote the induced map on collections of points.

Definition 3.7 Let $\iota\colon \mathrm{AM}_n \to \widetilde{\mathrm{AM}}_n$ be the map which sends φ to the composite

$$\Delta^n \xrightarrow{\ h\ } C_n\langle[-\infty, \infty]\rangle \xrightarrow{\ \widetilde{\varphi}\ } \widetilde{C}_n\langle\mathbb{R}^3, \partial\rangle.$$

Proposition 3.8 *The map ι is a homotopy equivalence.*

Proof To compare AM_n and $\widetilde{\mathrm{AM}}_n$ it is simplest to map them both to a third space which is easily seen to be equivalent. Let $\widetilde{\mathrm{AM}}_n^{\mathrm{hs}}$, where "hs" stands for hemispherical, be the subspace of $\mathrm{Map}(\Delta^n, \widetilde{C}_n\langle\mathbb{I}^3\rangle)$ defined by the same conditions as for $\widetilde{\mathrm{AM}}_n$ for all faces except the $t_1 = -1$ and $t_n = 1$ faces. On those faces, instead of the vectors v_{1j} and v_{kn} being $(1, 0, 0)$ as in the definition of $\widetilde{\mathrm{AM}}_n$, for $\widetilde{\mathrm{AM}}_n^{\mathrm{hs}}$ those vectors are simply required to lie in the hemisphere with nonnegative x–coordinate. (We could also define this as a homotopy limit.)

Consider the diagram

$$
\begin{array}{ccc}
& \mathrm{AM}_n & \\
{\scriptstyle q}\ \Big\downarrow {\scriptstyle \wr} & & \overset{\textstyle \iota}{\searrow} \\
\widetilde{\mathrm{AM}}_n^{\mathrm{hs}} & \underset{i}{\xleftarrow{\ \cong\ }} & \widetilde{\mathrm{AM}}_n
\end{array}
$$

where q is the map induced by composition of $\Delta^n \to C_n\langle\mathbb{I}^3, \partial\rangle$ with the quotient map to $\widetilde{C}_n\langle\mathbb{I}^3\rangle$ and i is the inclusion. Both of these are homotopy equivalences. The diagram commutes up to homotopy, interpolating between q and $i \circ \iota$ by first extending $\widetilde{\varphi}$ to $[-x, x]$. Then use a continuous family of homeomorphisms h_x of $\mathbb{I}$ with $[-x, x]$ to define a map ι_x which serves as a homotopy. It is then elementary that the composite of i with the homotopy inverse of q serves as a homotopy inverse to ι. $\qquad\square$

The main idea in the definition of ι, and in particular the construction of $\widetilde{\varphi}$, is "focusing" on the interval $\mathbb{I} = [-1, 1]$ by modifying configurations in $\mathbb{R}$ with points outside the interval to have points at the endpoints of the interval instead. Corresponding points in the configuration are replace by standard points along the x–axis. We will use similar ideas in the development of multiplicative structures on AM_n and $\widetilde{\mathrm{AM}}_n$.

4 Multiplicative structures

In this section we first define an action of the little intervals operad $\mathcal{C}_1$ on the framed knot space and $\mathrm{AM}_n^{\mathrm{fr}}$, which are compatible via the evaluation map. Then we construct a multiplication which is homotopy commutative on $\widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$. We show that these actions are all compatible up to homotopy.

Definition 4.1 The little intervals operad $\mathcal{C}_1$ has as its nth entry $\mathcal{C}_1(n)$ the space of n disjoint subintervals of $\mathbb{I}$, which we topologize as a subspace of $\mathbb{I}^{2n}$ through the endpoints of the intervals.

Given an subinterval $L \subseteq \mathbb{I}$ we by abuse let L also denote the orientation-preserving affine-linear transformation which sends $\mathbb{I}$ to L.

For use with operad actions on knot spaces, we let $\widehat{L}$ denote the map $\mathbb{I}^3 \to \mathbb{I}^3$ which applies L to the first coordinate and then shrinks the second and third coordinates by the same scaling factor (but doesn't translate them).

4.1 The little intervals action on the knot space

If $\mathcal{L} = \bigcup L_i$ is a union of k disjoint little intervals, its action on a k–tuple of embeddings $f_i \colon \mathbb{R} \to \mathbb{R}^3$ yields the embedding which at time t has value

$$\mathcal{L} \cdot (f_1, \ldots, f_k)(t) = \begin{cases} \widehat{L}_i \circ f_i \circ L_i^{-1}(t) & \text{if } t \in L_i \text{ for some } i, \\ (t, 0, 0) & \text{otherwise.} \end{cases}$$

That is, the embeddings are "shrunk and placed in succession" according to $\mathcal{L}$. The action on $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}, \mathbb{R}^3)$ is similar, with the framings unchanged by the shrinking. We will view this action as a case of "insertion into the trivial embedding, with standard framing".

4.2 The spatial little intervals action on aligned maps

Defining a $\mathcal{C}_1$ action on AM_n is more involved, guided by wanting the evaluation map ev_n to be compatible with the action. We want to take a configuration in the interval and evaluate points in the various L_i on different elements of AM_n. But typically fewer than n points will be in each L_i, so we adjust accordingly. We first define restriction of an aligned map to some interval L.

Definition 4.2 For $L \subset \mathbb{I}$, define L^{-1} on some $\vec{t}$ in the interval by applying to each t_i the piecewise-linear map which is L^{-1} on the image of L, sends points to the left of L to -1, and sends points to the right of L to 1.

Define the restriction of $\varphi \in \mathrm{AM}_n$ to L, denoted by $\varphi|_L$, by applying φ to $L^{-1}(\vec{t})$ and then applying projection maps to forget all of the points in the resulting configurations whose indices j do not correspond to a t_j in the interior of L.

This is not continuous, as the different projection maps depending on the number of t_i in L produce points in different configuration spaces, but it is an essential auxiliary construction.

Definition 4.3 Define the action of a union of little intervals $\mathcal{L} = \bigcup L_i$ on a collection of maps $\varphi_i \in \mathrm{AM}_n$ as the map in AM_n which takes $\vec{t} \in \Delta^n$ to the union of all of the $\hat{L}_i \circ \varphi_i|_{L_i}$ applied to $\vec{t}$ along with a point in the configuration at $(t_j, 0, 0)$ for each t_j which is not in the interior of any L_i. If for such a t_j we have, say, $t_j = t_{j+1}$, then we set the "direction of collision from $(t_j, 0, 0)$ to $(t_{j+1}, 0, 0)$", denoted by $v_{i,i+1}$ when we have used coordinates, to be the positive x–axis direction.

The $\mathcal{C}_1$–action on $\mathrm{AM}_n^{\mathrm{fr}}$ is defined in the same manner, where the framings are unchanged because each interval shrinks an aligned map equally in all directions.

This action varies continuously as t_j approaches an endpoint e of some L_i because the limit from either side of this endpoint is the configuration with the corresponding point x_j at $(e, 0, 0)$. Checking continuity elsewhere is immediate, as is checking the usual conditions required for an action of the operad $\mathcal{C}_1$. The definition was arranged so that this action is compatible with the $\mathcal{C}_1$ action on the knot space via the evaluation map.

Proposition 4.4 *The map $\mathrm{ev}_n \colon \mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}, \mathbb{R}^3) \to \mathrm{AM}_n^{\mathrm{fr}}$ is a map of $\mathcal{C}_1$–spaces.*

For this paper we only need that this is a map of H–spaces. We will also need that the action is compatible with the structure maps in the Goodwillie–Weiss tower, which will be a main focus of Section 5.

4.3 A spatial-infinitesimal single little interval action

In his work on operad actions on knot spaces [3], the first author extensively uses the fact that embeddings of $\mathbb{R} \times D^2$ into $\mathbb{R} \times D^2$ can be composed. Our main idea in establishing homotopy commutativity of the multiplication(s) on $\mathrm{AM}_n^{\mathrm{fr}}$ is to set up composition. In order to do so, we produce products where the spatial points are along the evaluation map of the unknot, and use infinitesimal composition for the essential part of the multiplication. Given an interval L and $\varphi \in \widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$, we produce an aligned map with infinitesimal configuration at the point $(L(0), 0, 0)$ together with some points along $\mathbb{I} \times (0, 0)$. For continuity, we need points near the boundary and, say, inside of L to be pulled towards $L(0)$.

Let $L^\circ = L\big(\big[-\tfrac{1}{2}, \tfrac{1}{2}\big]\big)$, which is the "core" of L. Let $e_L \colon \mathbb{I} \to \mathbb{I}$ be a monotone continuous map which sends L° to the point $L(0)$ and which is the identity outside of L. By abuse use the same notation e_L for the induced map on collections of points in the interval.

As in previous constructions, we define the map $L \cdot \varphi$ piecewise on Δ^n according to the partition defined by which indices of t_i occur in $\vec{t} \cap L^\circ$. Let i and j be the indices

of the leftmost and rightmost t_k in L°, which we will consider as functions of L° and $\vec{t}$. For an element $\varphi \in \widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$, let $\varphi|_L$ be defined similarly as in Definition 4.2 for the AM_n setting, though this now produces equivalence classes of configurations modulo translation and scaling, with framings.

Definition 4.5 With notation as above, for $\varphi \in \widetilde{\mathrm{AM}}_n$ define $L \cdot \varphi \in \mathrm{AM}_n$ by

$$(L \cdot \varphi)(\vec{t}) = \big(e_L(t_1, \ldots, t_{i-1}, L(0), t_{j+1}, \ldots, t_n) \times (0,0)\big) \circ_i \varphi|_{L^\circ}(\vec{t}).$$

For a fixed L, it is clear that the output varies continuously with φ and with all $\vec{t}$ which have the same indices occurring in $\vec{t} \cap L^\circ$. To check that the various pieces fit together to a continuous function, suppose that some t_i is equal to $L\big(-\frac{1}{2}\big)$, the left endpoint of L°. The formulae on the two pieces of Δ^n that meet at $t_i = L\big(-\frac{1}{2}\big)$ are

$$\big(e_L(t_1, t_{i-1}, L(0), t_{j+1}, \ldots, t_n) \times (0,0)\big) \circ_i \varphi((L^\circ)^{-1}(t_i, t_{i+1}, \ldots, t_j))$$

and, using that $e_L(t_i) = e_L\big(L\big(-\frac{1}{2}\big)\big) = e_L(L(0)) = L(0)$,

$$\big(e_L(t_1, \ldots, t_{i-1}, L(0), L(0), t_{j+1}, \ldots, t_n) \times (0,0)\big) \circ_{i+1} \varphi((L^\circ)^{-1}(t_{i+1}, \ldots, t_j)).$$

The key point is that, by definition of the doubling maps in $\widetilde{C}_\bullet^{\mathrm{fr}}\langle \mathbb{I}^3, \partial \rangle$, the map φ sends a point in $\partial \mathbb{I}$ to a point "that looks infinitely far away from the images of the interior points in $\mathbb{I}$". Here it is essential that we are starting with elements of the operadic mapping space model rather than the spatial model.

To elaborate, using our standard coordinates on these compactifications, $v_{ik} = (1, 0, 0)$ for any $k \in \{i+1, \ldots, j\}$ in either of the two configurations above, and every other $v_{k\ell}$ for $1 \le k \le \ell \le n$ is the same in these two configurations as well. Furthermore, the projections to $(\mathbb{I}^3)^n$ of the two expressions agree, so they are the same configuration in $C_n^{\mathrm{fr}}\langle \mathbb{I}^3, \partial \rangle$.

Checking continuity between any other two pieces of Δ^n is similar, as is checking continuity if we vary both $\vec{t}$ and L.

If we compose $L \cdot \varphi$ with the projection to $\widetilde{C}_n\langle \mathbb{R}^3 \rangle$, the resulting map satisfies the conditions of being in $\widetilde{\mathrm{AM}}_n$. We need the spatial information in the next section.

4.4 A commutative multiplication on infinitesimal aligned maps

Now we define a multiplication on $\widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$ determined by a choice of two intervals $L_1, L_2 \in \mathcal{C}_1(1)$, one entry in the operad of "overlapping intervals". Here, when two intervals overlap, they are also ordered. Informally, this order says which interval is "on top". Since $\mathcal{C}_1(1)$ is connected, this product will be homotopy-commutative.

To see that homotopy-commutativity of connected sum holds for knots coherently as part of an operad action, it is technically necessary to "thicken" the knots, by studying embeddings of $\mathbb{R} \times D^2$ in $\mathbb{R} \times D^2$. In [3], the first author shows that such embedding spaces have an action of the overlapping intervals operad and, moreover, uses this action to determine the homotopy type of the space of classical knots. As we work with evaluation maps between compactified configuration spaces, our substitute for $\mathbb{R} \times D^2$ is the following:

Definition 4.6 Let $C_n^{\mathrm{it}}\langle \mathbb{I} \times (0,0), \partial \rangle$, the space of infinitesimally thickened configurations in an interval, be the subspace of $C_n^{\mathrm{fr}}\langle \mathbb{I}^3, \partial \rangle$ whose image under the projection to $(\mathbb{I}^3)^n$ lies in $(\mathbb{I} \times (0,0))^n$.

So far, $L \cdot \varphi$ is a map $C_n\langle \mathbb{I}, \partial \rangle \to C_n^{\mathrm{fr}}\langle \mathbb{I}^3, \partial \rangle$ whose image lies in the subspace $C_n^{\mathrm{it}}\langle \mathbb{I} \times (0,0), \partial \rangle$. We can view $C_n\langle \mathbb{I}, \partial \rangle$ as a subspace of $C_n^{\mathrm{it}}\langle \mathbb{I} \times (0,0), \partial \rangle$ in an obvious way with identity framings at every point. We will now extend the domain of $L \cdot \varphi$ from $C_n\langle \mathbb{I}, \partial \rangle$ to all of $C_n^{\mathrm{it}}\langle \mathbb{I} \times (0,0), \partial \rangle$.

For any $c \in C_n^{\mathrm{it}}\langle \mathbb{I} \times (0,0), \partial \rangle$, let $\vec{t}(c) = (t_1 < \cdots < t_m)$ be the set of *distinct* points in $p(c)$ (so $m \le n$). Then c can be written as

(1) $$c = \big(\cdots \big((\vec{t}(c) \times (0,0)) \circ_{m_1} c_1 \big) \circ_{m_2} \cdots \big) \circ_{m_k} c_k$$

for some $c_i \in \widetilde{C}_{n_i}\langle \mathbb{I}^3 \rangle$. This expression is unique once we require that $m_i \ge m_{i-1} + n_{i-1}$, so that the underlying points of insertion are distinct.

Definition 4.7 Define the extension of $L \cdot \varphi$ to $C_n^{\mathrm{it}}\langle \mathbb{I} \times (0,0), \partial \rangle$ as

$$(L \cdot \varphi)(c) = \big(\cdots \big((L \cdot \varphi)(\vec{t}(c)) \circ_{m_1} c_1 \big) \circ_{m_2} \cdots \big) \circ_{m_k} c_k.$$

Here the $L \cdot \varphi$ on the right-hand side is as in Definition 4.5, using the identification of $C_n\langle \mathbb{I}, \partial \rangle$ with Δ^n.

The key point now is checking continuity when points enter or exit the infinitesimal configurations c_i. The argument is just as for continuity in Definition 4.5 but with $(1,0,0)$ replaced by the tangent vector, which is the first vector in the framing, at $\varphi(t_{m_i})$.

Now both the input and output of the map $L \cdot \varphi$ can be regarded as elements in $C_n^{\mathrm{it}}\langle \mathbb{I} \times (0,0), \partial \rangle$. Thus two such maps can be composed, and we denote the composition by $\circ$.

Definition 4.8 Given two little intervals L_1, $L_2 \in \mathcal{C}_1(1)$, define the product of elements φ, $\psi \in \widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$ as

$$\mu_{L_1, L_2}(\varphi, \psi) \colon \vec{t} \mapsto ((L_2 \cdot \psi) \circ (L_1 \cdot \varphi))(\vec{t}),$$

where we take the equivalence class in $\widetilde{C}_n^{\mathrm{fr}}\langle \mathbb{I}^3 \rangle$ of the right-hand side.

Let $L_- = [-1, 0]$ and $L_+ = [0, 1]$. Abbreviate μ_{L_-, L_+} as μ, and abbreviate $\mu(\varphi, \psi)$ as $\varphi \cdot \psi$.

Theorem 4.9 $\varphi \cdot \psi$ *is homotopic to* $\psi \cdot \varphi$.

Proof The map $(L, \varphi) \mapsto L \cdot \varphi$ varies continuously with L. Because $\mathcal{C}_1(1)$ is connected, any two multiplications induced by choices of (L_1, L_2) — in particular (L_-, L_+) and (L_+, L_-) — are homotopic. $\qquad\square$

4.5 A compatible little intervals action on infinitesimal aligned maps

We now compare the multiplications on the spatial model $\mathrm{AM}_n^{\mathrm{fr}}$ and the infinitesimal model $\widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$. We first present a simpler multiplication μ' on $\widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$, which is homotopic to the one defined above. The multiplication μ' avoids the use of the maps e_L which pull points towards L, which are needed homotopy-commutativity. The multiplication μ' also has the advantage of extending to an action of the little intervals operad.

Definition 4.10 Let μ' be the map

$$\mu' \colon \mathcal{C}_1(2) \times (\widetilde{\mathrm{AM}}_n^{\mathrm{fr}})^2 \to \widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$$

defined by

$$\mu'_{L_1, L_2}(\varphi, \psi)(\vec{t})$$
$$:= \Big(\big((t_1, \ldots, t_{i-1}, L_1(0), t_{j+1}, \ldots, t_{k-1}, L_2(0), t_{\ell+1}, \ldots, t_n) \times (0, 0) \big) \circ_i \varphi|_{L_1}(\vec{t}) \Big)$$
$$\circ_k \psi|_{L_2}(\vec{t}),$$

where $t_i, \ldots, t_j$ are the points in L_1 and $t_k, \ldots, t_\ell$ are the points in L_2. Similarly, for any $k \geq 1$, define a map

$$(2) \qquad\qquad \mathcal{C}_1(k) \times (\widetilde{\mathrm{AM}}_n^{\mathrm{fr}})^k \to \widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$$

which sends $((L_1, \ldots, L_k), (\varphi_1, \ldots, \varphi_k))$ to the result of inserting each $\varphi_i|_{L_i}$ into the point $(L_i(0), 0, 0)$.

It is straightforward to verify that the maps (2) define a $\mathcal{C}_1$–action on $\widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$, using the fact that $\widetilde{C}_n \langle \mathbb{I}^3 \rangle$ records only directions of collision, as well as again using the fact that in the operadic models configuration points which corresponding to times t_i which are equal to ± 1 are "at infinity".

Proposition 4.11 *For disjoint intervals* L_1 *and* L_2, *the multiplication* $\mu_{L_1, L_2}(\varphi, \psi)$ *is homotopic to the multiplication* $\mu'_{L_1, L_2}(\varphi, \psi)$.

Proof Since configurations of points along the x–axis are all equivalent in $\widetilde{C}_n\langle\mathbb{I}^3\rangle$, the resulting aligned maps differ only in the subsets of $\mathbb{I}$ on which they are constant. The aligned map resulting from μ' sends all t_j between L_1 and L_2 to the same configuration point, while the same is true for μ and L_1° and L_2°. These are related by a homotopy which "reparametrizes the domains" of φ and ψ from L_1° and L_2° to L_1 and L_2, respectively. $\qquad\square$

Recall from Definition 3.7 our transformation ι from the spatial to the operadic mapping space models, which is an equivalence.

Proposition 4.12 *The $\mathcal{C}_1$–actions on $\mathrm{AM}_n^{\mathrm{fr}}$ and $\widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$ are compatible. That is, the diagram*

$$
\begin{array}{ccc}
\mathcal{C}_1(k)\times(\mathrm{AM}_n^{\mathrm{fr}})^k & \longrightarrow & \mathrm{AM}_n^{\mathrm{fr}} \\
{\scriptstyle \iota^k}\big\downarrow & & \big\downarrow{\scriptstyle \iota} \\
\mathcal{C}_1(k)\times(\widetilde{\mathrm{AM}}_n^{\mathrm{fr}})^k & \longrightarrow & \widetilde{\mathrm{AM}}_n^{\mathrm{fr}}
\end{array}
$$

commutes up to homotopy, where the top horizontal map is the action given in Definition 4.3 and the bottom horizontal map is the action that generalizes the multiplication μ' given in Definition 4.10.

Proof Recall that the $\mathcal{C}_1$ action on $\mathrm{AM}_n^{\mathrm{fr}}$ is defined by taking the union of the images of $\widehat{L}_i\circ\varphi_i|_{L_i}$, along with points along the x–axis. We will perform a homotopy with three steps, parametrized by $s\in[0,3]$, from the composite through the upper-right corner to the composite through the lower-left corner.

The first step, as s varies in $[0,1]$, is a straightforward homotopy from $(L_1,\ldots,L_k)\cdot(\varphi_1,\ldots,\varphi_k)$ to $(L_1^\circ,\ldots,L_k^\circ)\cdot(\varphi_1,\ldots,\varphi_k)$, where as before $L^\circ:=L\big(\big[-\tfrac{1}{2},\tfrac{1}{2}\big]\big)$. The configuration produced by an aligned map in the image at $s=1$ is the union of configurations resulting from all the $\varphi_i|_{L_i^\circ}$, together with points along the x–axis. Roughly, we are ensuring that each aligned map is standard on some interval between each pair of intervals L_i and L_{i+1}, so that we can push apart and shrink configurations continuously.

Next define $L_i^\times:=L_i\setminus L_i^\circ$. In the second step, as s varies in $[1,2]$, we scale the image of each $\varphi_i|_{L_i^\circ}$ to an infinitesimal configuration at $(L_i(0),0,0)$. In the notation of Section 4.1, we follow each aligned map by $(\widehat{L}_i^\circ)^{-1}\circ\widehat{J}_s\circ\widehat{L}_i^\circ$, where we define $J_s:=[s-2,2-s]$ for $s\in[1,2]$. This can be done at the level of representatives in $C_n^{\mathrm{fr}}\langle\mathbb{I}^3,\partial\rangle$ coming from the top horizontal map. The configurations become infinitesimal only at $s=2$, so continuity requires us to simultaneously pull the images of $L_i^\times$ toward $(L_i(0),0,0)$, from occupying $L_i^\times\times\{(0,0)\}$ at $s=1$ to occupying all of $L_i\times\{(0,0)\}$ at $s=2$.

At $s = 2$, the elements in $\widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$ produce infinitesimal configurations $\varphi_i|_{L_i^\circ}$, together with points on the x–axis between them (where the distances between them are *not* recorded by the operadic compactification $\widetilde{C}_n^{\mathrm{fr}}\langle \mathbb{I}^3\rangle$). This aligned map looks almost like the composite through the lower-left corner. The only difference is that this aligned map is constant on each component complementary to the L_i°, rather than each component complementary to the L_i. The last stage of the homotopy thus simply requires "reparametrizing the domain" of each φ_i from L_i° to all of L_i, as in the proof of Proposition 4.11. □

Because ι is a homotopy equivalence, we have the following:

Corollary 4.13 *The C_1 action on $\mathrm{AM}_n^{\mathrm{fr}}$ induces a homotopy-commutative multiplication.*

We have not used that the ambient dimension is three, so these results hold for knots in higher-dimensional Euclidean spaces as well. Similar results were proven in [27] for knots modulo immersions (that is, the homotopy fiber of the inclusion of embeddings into immersions) and would work similarly for framed knots, but only for the limit of the $\mathrm{AM}_n^{\mathrm{fr}}$. Turchin [31], and more recently Boavida and Weiss [1], have established versions of this theorem, along with a group structure as in Theorem 5.16 below, for the stages of the tower for knots modulo immersions. Dwyer and Hess have similar results for the limit [8]. We were not able to show that the structure studied in Turchin's paper is compatible with connected sum, which necessitated the present approach.

Remark 4.14 At this point we can explain the connection between Schubert's elementary geometric result that connected sum of knots is commutative [25] and Steenrod's deep, formal work on commutativity of cup product and operations in cohomology [30]. This connection was implied to us by the work of McClure and Smith [19; 20], as applied in this setting by Sinha [27], whose product structures on totalizations of cosimplicial spaces are related to Steenrod's formulae for higher cup products [19].

In different notation from Definition 4.3, the product of two aligned maps f and g is

$$(t_1, \ldots, t_n) \mapsto \bigcup_{t_i \leq 0 \leq t_{i+1}} \hat{f}(t_1, \ldots, t_i) * \hat{g}(t_{i+1}, \ldots, t_n).$$

Here the union refers to a decomposition of the domain, the n–simplex; $\hat{f}$ and $\hat{g}$ are obtained from f and g by appending times and rescaling (as we did frequently this section); and $*$ indicates a "stacking product" of configurations in $\mathbb{R}^d$.

This is formally similar to the standard formula for cup product

$$\varphi \cup \psi (\sigma \colon [v_1, \ldots, v_n] \to X) = \sum \varphi(\sigma|_{[v_1, \ldots, v_i]}) \cdot \psi(\sigma|_{[v_{i+1}, \ldots, v_n]}).$$

Here φ and ψ are cochains, σ is a chain, and brackets $[\,,]$ around some variables refers to the simplex given as convex linear combinations of those variables. This sum is zero except for one term, but, as McClure and Smith show, it is the correct sum to write down for purposes of generalization.

Recall that Theorem 4.9 gives a homotopy from $\mu(f, g)$ to the multiplication defined by Definition 4.8 with $(L_1, L_2) = ([-1, 0], [0, 1])$. By choosing a path in $\mathcal{C}_1(1)$ from $([-1, 0], [0, 1])$ to $([0, 1], [-1, 0])$ (and applying a homotopy from Proposition 4.11) we ultimately get a homotopy from $\mu(f, g)$ to $\mu(g, f)$. We choose the following path, which in the overlapping intervals setting the first interval always lies above the second: start with $([-1, 0], [0, 1])$; grow the second interval to obtain $([-1, 0], [-1, 1])$; then translate $[-1, 0]$ to $[0, 1]$; finally shrink $[-1, 1]$ to $[-1, 0]$ to obtain the pair $([0, 1], [-1, 0])$.

If we apply the formulae of Definition 4.8 for the products of f and g governed by this path of 1–disks, we see a formal analogue for Steenrod's formula for cup-one, namely

$$\varphi \cup_1 \psi(\sigma) = \sum_{i<j} \varphi(\sigma|_{[v_1,\ldots,v_i,v_j,\ldots,v_n]}) \cdot \psi(\sigma|_{[v_i,\ldots,v_j]}).$$

The main difference is that the product rather than using an underlying commutative ring uses operad insertion maps. McClure and Smith show this to be an appropriate extension of Steenrod's formula to the operadic setting.

5 Maps and layers in the tower, and abelian group structure

Our goal is to show that each stage of the Goodwillie–Weiss tower for knots has an abelian group structure compatible with connected sum. This also would follow by adapting recent work of Boavida and Weiss [1], which appeared after the present work, where in Theorem 10.2 they exhibit the stages of a closely related tower as the fiber of a two-fold loop map of spaces whose multiplications are compatible with connected sum.

Though we are primarily interested in mapping space models, we use the cosimplicial models and language around them as a starting point and key organizational tool. Cosimplicial structures are also essential for the study of spectral sequences below. We need develop a variety of models for the maps in the totalization tower of a cosimplicial space, which are of independent interest.

5.1 Maps in the tower through projection and restriction

In the cosimplicial realm the totalization tower is a basic object of study, dual in a sense to the skeletal filtration of a simplicial complex. When using the functors $\mathcal{G}_n$ to pull back a subcubical diagram from a cosimplicial space, the maps from $\widetilde{\mathrm{Tot}}^n$ to $\widetilde{\mathrm{Tot}}^{n-1}$ are induced by inclusions $\mathcal{P}_\nu[n-1] \to \mathcal{P}_\nu[n]$.

In the mapping space models, this inclusion of indexing categories explicitly gives rise to the following:

Definition 5.1 The restriction projection $p_n \colon \mathrm{AM}^{\mathrm{fr}}_n \to \mathrm{AM}^{\mathrm{fr}}_{n-1}$ sends a map $\varphi \in \mathrm{AM}^{\mathrm{fr}}_n$ to the composite

$$\Delta^{n-1} \xrightarrow{\ d\ } \Delta^n \xrightarrow{\ \varphi\ } C^{\mathrm{fr}}_n \langle \mathbb{I}^3, \partial \rangle \xrightarrow{\ s\ } C^{\mathrm{fr}}_{n-1} \langle \mathbb{I}, \partial \rangle,$$

where d and s are the images of a (face, degeneracy) pair whose composite is the identity. As any two such choices of a pair (d, s) yield homotopic projections $\mathrm{AM}^{\mathrm{fr}}_n \to \mathrm{AM}^{\mathrm{fr}}_{n-1}$, we take d to be the map $d_n \colon (t_1, \ldots, t_{n-1}) \mapsto (t_1, \ldots, t_{n-1}, 1)$ and s to be the map s_n that forgets the last configuration point and framing.

The restriction projection map $\widetilde{p}_n \colon \widetilde{\mathrm{AM}}^{\mathrm{fr}}_n \to \widetilde{\mathrm{AM}}^{\mathrm{fr}}_{n-1}$ and nonframed versions are defined analogously.

Then p_n is our first model for the standard map $\widetilde{\mathrm{Tot}}^n C^{\mathrm{fr}}_\bullet \langle \mathbb{I}^d, \partial \rangle$ to $\widetilde{\mathrm{Tot}}^{n-1} C^{\mathrm{fr}}_\bullet \langle \mathbb{I}^d, \partial \rangle$, and its main use is the following:

Proposition 5.2 *The restriction projection $p_n \colon \mathrm{AM}^{\mathrm{fr}}_n \to \mathrm{AM}^{\mathrm{fr}}_{n-1}$ is a map of $\mathcal{C}_1-$ algebras.*

Proof We need to check that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{C}_1(k) \times (\mathrm{AM}^{\mathrm{fr}}_n)^k & \longrightarrow & \mathrm{AM}^{\mathrm{fr}}_n \\
\downarrow & & \downarrow \\
\mathcal{C}_1(k) \times (\mathrm{AM}^{\mathrm{fr}}_{n-1})^k & \longrightarrow & \mathrm{AM}^{\mathrm{fr}}_{n-1}
\end{array}
$$

The composite through the upper-right corner is the map

$$(t_1, \ldots, t_{n-1}) \mapsto s_n \left(\bigcup_{i=1}^k \widehat{L}_i \circ \varphi_i \circ L_i^{-1}(t_1, \ldots, t_{n-1}, 1) \cup \bigcup \{(t_j, 0, 0)\} \right),$$
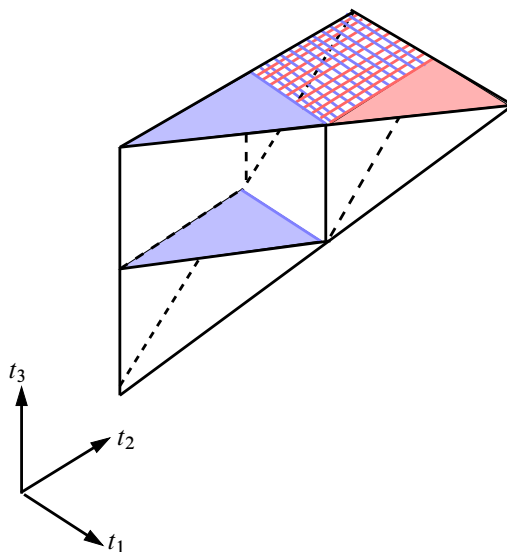
Figure 1: A partial illustration of how the projection $AM_n \to AM_{n-1}$ preserves the $\mathcal{C}_1$–action for $n = 3$, $k = 2$, with intervals $(L_-, L_+) = ([-1, 0], [0, 1])$ acting on φ, $\psi \in AM_n$. The product $(L_-, L_+) \cdot (\varphi, \psi)$ is a map of the whole simplex. The restriction to each light-blue triangular face is (after rescaling in the domain and range, and after forgetting the last configuration point) the projection of φ to AM_2; the blue edge of this triangle is in turn the restriction to the face $t_2 = 1$. Similarly the restriction to the light-red triangular face is the projection of ψ to AM_2; the red edge of this triangle is in turn the restriction to the face $t_1 = 0$. The map on the top face of the whole simplex is the projection of $(L_-, L_+) \cdot (\varphi, \psi)$, which is indeed obtained as a product of the projections of φ and ψ to AM_2.

while the composite through the lower-left corner is the map

$$(t_1, \ldots, t_{n-1}) \mapsto \bigcup_{i=1}^{k} \widehat{L}_i \circ s_n \circ (\varphi_i|_{d_n \Delta^{n-1}}) \circ L_i^{-1}(t_1, \ldots, t_{n-1}) \cup \bigcup \{(t_j, 0, 0)\}.$$

These maps agree. In either expression, each φ_i is applied to the same configuration, in particular with the same number of t_i equal to 1. As for the resulting configuration in $\mathbb{R}^3$, in the first expression one forgets an extra framed point at $(1, 0, 0)$. In the second expression, one forgets an extra framed point at $(L_i(1), 0, 0)$ for each $i = 1, \ldots, k$, yielding the same result. $\qquad\square$

A similar argument shows that the projection $AM_n \to AM_{n-1}$ is a map of $\mathcal{C}_1$–algebras.

5.2 Maps in the tower through (only) projection

For purposes of connecting our multiplications with those coming from cosimplicial structure, we need another model for the maps and layers in the totalization tower, one which is closely related to p_n in that it uses projection. While the map p_n is defined through including $\mathcal{P}_v[n-1]$ in $\mathcal{P}_v[n]$, we now proceed by "fibering $\mathcal{P}_v[n]$ over $\mathcal{P}_v[n-1]$". Consider the functor i_n from $\mathcal{P}_v[n]$ to $\mathcal{P}_v[n-1]$ which modifies a subset S by identifying $n-1$ and n — that is, changing occurrences of n in S to $n-1$.

Lemma 5.3 *The homotopy limit of $X^\bullet \circ \mathcal{G}_{n-1} \circ i_n$ is homotopy equivalent to $\widetilde{\mathrm{Tot}}^{n-1} X^\bullet$.*

The nth degeneracy $s^n\colon X^n \to X^{n-1}$ extends to a map

$$\widehat{s}^n\colon X^\bullet \circ \mathcal{G}_n \to X^\bullet \circ \mathcal{G}_{n-1} \circ i_n.$$

On homotopy limits, the map induced by $\widehat{s}^n$ agrees with the standard map from $\widetilde{\mathrm{Tot}}^n X^\bullet \to \widetilde{\mathrm{Tot}}^{n-1} X^\bullet$.

Proof For the first statement, it suffices to show that the functor i_n is left cofinal [2, Theorem XI.9.2], so that the category $P_v[n] \times_{P_v[n-1]} (P_v[n-1] \downarrow S)$ has contractible nerve for every $S \in P_v[n-1]$. But this category has a final object, namely the object corresponding to $S \cup n \in P_v[n]$.

It follows from the cosimplicial identities that s^n induces a natural transformation of functors $\widehat{s}^n\colon X^\bullet \circ \mathcal{G}_n \to X^\bullet \circ \mathcal{G}_{n-1} \circ i_n$, where the map on each object is given by either its last degeneracy or the identity map.

Again, the standard map $\widetilde{\mathrm{Tot}}^n X^\bullet \to \widetilde{\mathrm{Tot}}^{n-1} X^\bullet$ is induced by the inclusion functor $j_n\colon \mathcal{P}_v[n-1] \to \mathcal{P}_v[n]$. Note that the composite $i_n \circ j_n$ is the identity. Consider the following diagram to complete the proof:

$$
\begin{array}{ccc}
\mathrm{holim}(X^\bullet \circ \mathcal{G}_n) & & \\
{\scriptstyle -\circ j_n}\Big\downarrow & \overset{\widehat{s}^n}{\searrow} & \\
\mathrm{holim}(X^\bullet \circ \mathcal{G}_{n-1}) & \underset{-\circ j_n}{\overset{\simeq}{\longleftarrow}} & \mathrm{holim}(X \circ \mathcal{G}_{n-1} \circ i_n)
\end{array}
$$

The left-hand vertical map is the map $\widetilde{\mathrm{Tot}}^n X^\bullet \to \widetilde{\mathrm{Tot}}^{n-1} X^\bullet$, since $X^\bullet \circ \mathcal{G}_n \circ j_n = X^\bullet \circ \mathcal{G}_{n-1}$. Precomposing $X^\bullet \circ \mathcal{G}_{n-1} \circ i_n$ with j_n induces the horizontal map because $i_n \circ j_n = \mathrm{id}$. This horizontal map is an equivalence because, on these homotopy limits, "precomposing with j_n" is the right-inverse to the equivalence given by "precomposing with i_n". $\square$

By abuse, we let $\hat{s}^n$ also denote the map it induces on homotopy limits. For purposes of distinction we let $\mathrm{AM}^{\mathrm{fr}}_{n-1}(\Delta^n)$ denote $C^{\mathrm{fr}}_{\bullet}\langle\mathbb{I}^3,\partial\rangle\circ\mathcal{G}_{n-1}\circ i_n$, which by the above is a model for $\mathrm{AM}^{\mathrm{fr}}_{n-1}$. (The notation indicates that we model $\mathrm{AM}^{\mathrm{fr}}_{n-1}$ by maps of Δ^n.) The fact that $\hat{s}^n\colon \mathrm{AM}^{\mathrm{fr}}_n \to \mathrm{AM}^{\mathrm{fr}}_{n-1}(\Delta^n)$ and p_n both model the structure maps in the totalization tower implies that they are homotopic. We choose $\hat{s}^n$ for more detailed study of the fibers of these maps, starting with the observation that $\hat{s}^n$ is a fibration in both the $\widetilde{\mathrm{AM}}^{\mathrm{fr}}_n$ and $\mathrm{AM}^{\mathrm{fr}}_n$ settings, since the projection or identity maps which define it entrywise are fibrations. (See [4, Lemma 3.5] for an explicit proof in this case of a standard result about enriched model structures on diagram categories in general.)

Definition 5.4 Let L_n be the fiber of $\hat{s}^n\colon \mathrm{AM}^{\mathrm{fr}}_n\to\mathrm{AM}^{\mathrm{fr}}_{n-1}(\Delta^n)$, based at the evaluation map of the unknot. That is, L_n is the space of aligned maps where when one forgets the last point in each configuration in the image one obtains a standard configuration along the x–axis parametrized by the points in the domain simplex. Let $\widetilde{L}_n$ be the fiber of $\hat{s}^n\colon \widetilde{\mathrm{AM}}^{\mathrm{fr}}_n \to \widetilde{\mathrm{AM}}^{\mathrm{fr}}_{n-1}(\Delta^n)$, which then sits over the constant map at the infinitesimal configuration where all x_{ij} for $i < j$ are equal to $(1,0,0)$.

We will write $L_n(\Delta^n)$ or $\widetilde{L}_n(\Delta^n)$ if we want to emphasize the model we are using for this fiber.

We used both the AM_n and $\widetilde{\mathrm{AM}}_n$ models in the previous section since the former supports an evaluation map and a $\mathcal{C}_1$ structure and the latter has a commutative multiplication. We use both L_n and $\widetilde{L}_n$ in similar fashion here, which means we also need a comparison.

Proposition 5.5 *The map $\iota\colon \mathrm{AM}_n \to \widetilde{\mathrm{AM}}_n$ restricts to a map from L_n to $\widetilde{L}_n$ which is an equivalence and preserves multiplication up to homotopy.*

Proposition 5.6 *The inclusion of L_n into AM_n is a map of $\mathcal{C}_1$–spaces.*

The proofs of all of these are straightforward from the definitions, checking that previous definitions' arguments, such as Proposition 4.12, are compatible with the condition of being a standard configuration but for the last coordinate.

We can use the equivalence $\mathrm{AM}^{\mathrm{fr}}_{n-1}(\Delta^n) \to \mathrm{AM}^{\mathrm{fr}}_{n-1}$ defined by restriction to the last face of Δ^n to define a $\mathcal{C}_1$–structure on $\mathrm{AM}^{\mathrm{fr}}_{n-1}(\Delta^n)$. Combining Propositions 5.2 and 5.6 and composing with the homotopy inverse $\mathrm{AM}^{\mathrm{fr}}_{n-1} \to \mathrm{AM}^{\mathrm{fr}}_{n-1}(\Delta^n)$, which is then also $\mathcal{C}_1$, gives the following:

Corollary 5.7 $L_n \to \mathrm{AM}^{\mathrm{fr}}_n \to \mathrm{AM}^{\mathrm{fr}}_{n-1}(\Delta^n)$ *is a fibration sequence of $\mathcal{C}_1$ spaces.*

5.3   The layers of a totalization tower via cubical diagrams

Following Goodwillie, we use cubical diagrams to develop loop structures on layers in the totalization tower of a cosimplicial space. Some of this material is treated in [22]. We have already seen $\mathcal{P}_v[n]$ and $\mathcal{P}_v[n-1]$ related by inclusion and "fibering". We next relate them through a Mayer–Vietoris decomposition.

Just as the standard map $\widetilde{\mathrm{Tot}}^n X^\bullet \to \widetilde{\mathrm{Tot}}^{n-1} X^\bullet$ can be defined by an inclusion $\Delta_{n-1} \subset \Delta_n$, it can also be defined by the canonical inclusion $\mathcal{P}_v[n-1] \subset \mathcal{P}_v[n]$ through the equivalence given by the functor $\mathcal{G}_n$. We will analyze the fibers of these maps—that is, the layers in the totalization tower—in two different ways in this section. First we use a sort of Mayer–Vietoris decomposition of the category $\mathcal{P}_v[n]$.

Definition 5.8   • Let $\mathcal{P}_{\neq n} \subset \mathcal{P}_v[n]$ be the full subcategory given by all nonempty subsets of $[n]$ except the singleton $\{n\}$.

   • Let $\mathcal{P}_{n\in}$ be the (cubical) poset of all subsets of $[n]$ containing n.

   • Let $\mathcal{P}_{n+}$ be the (subcubical) poset of subsets of $[n]$ containing n and at least one other element.

The inclusion $\iota\colon \mathcal{P}_v[n-1] \hookrightarrow \mathcal{P}_{\neq n}$ is left cofinal, so the map induced on homotopy limits is an equivalence. We can thus replace $\mathrm{holim}_{\mathcal{P}_v[n]} X^\bullet \circ \mathcal{G}_n \to \mathrm{holim}_{\mathcal{P}_v[n-1]} X^\bullet \circ \mathcal{G}_n$ by an alternate model for the maps in the tower, namely

$$\mathrm{holim}_{\mathcal{P}_v[n]} X^\bullet \circ \mathcal{G}_n \to \mathrm{holim}_{\mathcal{P}_{\neq n}} X^\bullet \circ \mathcal{G}_n.$$

The poset $\mathcal{P}_v[n]$ can be written as the union of $\mathcal{P}_{\neq n}$ and $\mathcal{P}_{n\in}$ along $\mathcal{P}_{n+}$, yielding the following square:

(3)
$$
\begin{array}{ccc}
\mathcal{P}_v[n] & \longleftarrow\!\!\!\!\supset & \mathcal{P}_{n\in} \\
\uparrow & & \uparrow \\
\mathcal{P}_{\neq n} & \longleftarrow\!\!\!\!\supset & \mathcal{P}_{n+}
\end{array}
$$

Applying $\mathrm{holim}_{(-)} X^\bullet \circ \mathcal{G}_n$ to the diagram above, we get a pullback square of fibrations [11, Proposition 0.2]. Thus, to study the fiber(s) of the map from $\widetilde{\mathrm{Tot}}^n$ to $\widetilde{\mathrm{Tot}}^{n-1}$, which up to homotopy is the left-hand column of the induced map of homotopy limits of this square, it suffices to study the right-hand column. We say fiber(s) because in our application we study unbased and sometimes disconnected spaces.

Since $\mathcal{P}_{n+}$ is just the cube $\mathcal{P}_{n\in}$ with its initial object removed, the map on homotopy limits induced by the right vertical arrow is just the map from the initial object, at $\{n\}$, to the homotopy limit of the rest of the diagram, which is subcubical. We conclude

that the fiber(s) are the total fiber(s) of the cube $\mathcal{P}_{n\in}$ over different possible basepoints in the unbased case. In the based setting the original map is k–connected if and only if the n–cube $X^\bullet \circ \mathcal{G}_n|_{\mathcal{P}_{n\in}}$ is k–Cartesian.

5.4 Loop structure on layers via retracts of cubes

Lemma 5.9 *Let $f\colon C_\bullet \to D_\bullet$ be a map of cubical diagrams and let $r\colon D_\bullet \to C_\bullet$ be a retraction objectwise. Then the total fiber of the cube $C_\bullet \to D_\bullet$ is the loopspace of the total fiber of $D_\bullet \to C_\bullet$.*

Proof Consider the square of cubes, which is itself a cubical diagram:

$$\begin{array}{ccc} C_\bullet & \xrightarrow{\ f\ } & D_\bullet \\ {\scriptstyle\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ C_\bullet & \xrightarrow[\ \mathrm{id}\ ]{} & C_\bullet \end{array}$$

We find the total fiber of these cubes in two ways, first internally to the $C_\bullet$ and $D_\bullet$ subcubes followed by taking horizontal then vertical fibers. This yields the total fiber of $C_\bullet \to D_\bullet$. If we first take internal fibers, then vertical, then horizontal, we see loops on the total fiber of $D_\bullet \to C_\bullet$. $\qquad\square$

Cosimplicial identities imply that the codegeneracy maps of a cosimplicial space can be used to define retractions of cubes. For example, at the first two levels of a cosimplicial space the codegeneracy map is a retract for either coface map. Since the first layer in the Tot tower is the total fiber of $\mathcal{P}_{1\in}$, which is just a coface map $X^0 \to X^1$, this lemma shows that is loops on the fiber of the codegeneracy $X^1 \to X^0$. More generally we have the following:

Definition 5.10
- For an inclusion of ordered sets $i\colon S \hookrightarrow S'$, we define the dual surjection $i^!\colon S' \to S$ to be the order-preserving retraction which sends each element of S' to the maximal value of S possible among such retractions.
- Let $\mathcal{P}_{n\in}^!$ be the category whose objects are subsets of $[n]$ containing n and where morphisms are all the dual surjections.
- For brevity, let $\mathcal{G}_n^!\colon \mathcal{P}_{n\in}^! \to \Delta_n$ denote the functor $\mathcal{G}_{\mathcal{P}_{n\in}^!}$ (defined in Definition 3.3).

Proposition 5.11 *For a cosimplicial space $X^\bullet$, the homotopy limit of $X^\bullet \circ \mathcal{G}_n|_{\mathcal{P}_{n\in}}$ and thus the fiber of $\widetilde{\mathrm{Tot}}^n X^\bullet \to \widetilde{\mathrm{Tot}}^{n-1} X^\bullet$ is homotopy equivalent to $\Omega^n \mathrm{holim} X^\bullet \circ \mathcal{G}_n^!$.*

The proof of this proposition in [22, Proposition 9.4.10] is essentially correct, though it makes reference to a diagram which is generally not possible to construct for an arbitrary cosimplicial space and in particular for those we use. We only use a subdiagram of that used in [22] which can always be constructed.

Proof We interpolate between $\mathcal{G}_{\mathcal{P}_{n\in}}$ and $\mathcal{G}_{\mathcal{P}^!_{n\in}}$, which share the underlying objects, namely subsets of $[n]$ which contain n. But $\mathcal{P}_{n\in}$ has basic morphisms (of which all others are composites) of $S \mapsto S \cup i$, while $\mathcal{P}^!_{n\in}$ has basic morphisms with some $i, i+1 \mapsto i+1$ and identity otherwise. For $j = 0, 1, \ldots, n$, define $\mathcal{P}_{n\in}(j)$ as having this same set of objects but with generating morphisms $S \subset S \cup i$ for $i \le j$ and $i, i+1 \mapsto i+1$ and identity otherwise for $i > j$.

We view $X^\bullet \circ \mathcal{G}_{\mathcal{P}_{n\in}(j+1)}$ as a map of cubes $C_\bullet \to D_\bullet$, where $C_\bullet$ is the restriction to subsets which do not contain j and $D_\bullet$ is the restriction to those which do, and the map between cubes is defined by all of the $S \mapsto S \cup j$ maps. Then $X^\bullet \circ \mathcal{G}_{\mathcal{P}_{n\in}(j)}$ is a retract $D_\bullet \to C_\bullet$, with morphisms defined by sending j to the next element in the ordering. We deduce from Lemma 5.9 that the total fiber of $X^\bullet \circ \mathcal{G}_{\mathcal{P}_{n\in}(j+1)}$ is the loopspace of the total fiber of $X^\bullet \circ \mathcal{G}_{\mathcal{P}_{n\in}(j)}$. Thus fib $X^\bullet \circ \mathcal{G}_{\mathcal{P}_{n\in}(n)} \simeq \Omega^n$ fib $X^\bullet \circ \mathcal{G}_{\mathcal{P}_{n\in}(0)}$. Since $\mathcal{G}_n|_{\mathcal{P}_{n\in}}$ is $\mathcal{G}_{\mathcal{P}_{n\in}(n)}$ while $\mathcal{G}_{\mathcal{P}^!_{n\in}}$ is $\mathcal{G}_{\mathcal{P}_{n\in}(0)}$, we obtain the result. $\qquad\square$

Remark 5.12 It is a tautology that the nth layer in the totalization tower of a fibrant cosimplicial space is an n–fold loopspace. But in our setting we could not use standard fibrant replacement to produce a multiplication on the entries of the tower which is compatible with connected sum. Cubical diagram models for the entries and the layers of the totalization tower give a workable alternative to fibrant replacement, which is broadly applicable.

5.5 Surjectivity on components of maps in the tower

In order to inductively establish a group structure on components of stages in the tower, we need a surjectivity result.

Theorem 5.13 *The restriction-projection map $\widetilde{p}_n\colon \widetilde{\mathrm{AM}}_n \to \widetilde{\mathrm{AM}}_{n-1}$ induces a surjection on π_0.*

We prove this unframed version first and then use it to prove the desired framed version, as the end of the proof we give here breaks down in the framed setting.

Proof of Theorem 5.13 We extend techniques from the previous subsection. Recall from Section 3 the cosimplicial space $\widetilde{C}'_\bullet \langle \mathbb{I}^3 \rangle$, which we abbreviate here as C. This

functor defines both our cosimplicial model and, through pullback by the functors $\mathcal{G}_n$, our operadic mapping space model.

We use the square of posets (3) above. We are led to the map on homotopy limits induced by $\mathcal{P}_{n+} \hookrightarrow \mathcal{P}_{n\in}$, which if surjective on components implies the same for the map $\widetilde{\mathrm{AM}}_n \to \widetilde{\mathrm{AM}}_{n-1}$, induced by the left-vertical map in the square (3).

The initial object in $\mathcal{P}_{n\in}$ is $C(\{n\}) = \widetilde{C}'_0\langle \mathbb{R}^3 \rangle$, a point. Thus the map on homotopy limits induced by $\mathcal{P}_{n+} \hookrightarrow \mathcal{P}_{n\in}$ is surjective on components if and only if the homotopy limit over $\mathcal{P}_{n+}$ is connected. As before we reindex this diagram, using the isomorphism of $\mathcal{P}_{n+}$ with $\mathcal{P}_v[n-1]$ which sends a set containing n to the set obtained by removing n.

We prove the connectedness of this homotopy limit by induction on n. The case $n = 1$ is immediate, as $\widetilde{C}'_1\langle \mathbb{R}^3 \rangle = S^2$ is connected. For the induction step, we exhibit the homotopy limit $\mathcal{P}_v[n-1]$ as a fibration over a connected space with a connected fiber. Consider the reindexed pushout square (3), with n replaced by $n-1$ everywhere. The inclusion $\mathcal{P}_v[n-2] \hookrightarrow \mathcal{P}_{\neq n-1}$ is left cofinal, so the left-hand column of the reindexed (3) gives a fibration whose base, the homotopy limit of D, which we can take over $\mathcal{P}_v[n-2]$, is connected by induction.

The square of holims induced by (3) is a pullback, so it suffices to establish connectedness of the fiber of the map induced by the right-hand column, taken over the component to which the connected space $\operatorname{holim}_{\mathcal{P}_{\neq n-1}} D$ maps. Here we choose basepoints for D by choosing the basepoint $(1, 0, 0)$ in $S^2 = \widetilde{C}'_1\langle \mathbb{I}^3 \rangle$.

This induced square of holims is of based spaces, we can describe the fiber of this map as the total fiber of the based cube $D(\mathcal{P}_{n-1})$. Apply Proposition 5.11 to deduce that this total fiber is Ω^{n-1} tfib $D^!$.

To show that Ω^{n-1} tfib $D^!$ is connected or, equivalently, that $D^!$ is $(n-1)$–Cartesian, we use a Blakers–Massey theorem. This is an $(n-1)$–cube of spaces $\widetilde{C}'_i\langle \mathbb{I}\rangle$ for $i \leq n-1$ of configurations in $\mathbb{I}^3$ up to scaling and translation, with a tangent vector at each point. The maps forget points and corresponding tangent vectors. We replace this by a homotopy equivalent cube of spaces $C'_i\langle \mathbb{I}^3 \rangle$ or even $C'_i(\mathbb{I}^3)$ of configurations in $\mathbb{I}^3$ with a tangent vector at each point. Every map in this cube is a fibration, so we can take the fiber in one direction. The resulting $(n-2)$–cube, which we call $\varphi D^!$, has entries $\mathbb{I}^3 - f([i])$, where the deleted points are images of a fixed embedding $f\colon [n] \hookrightarrow \mathbb{I}^3$. The maps in the cube are inclusions of open submanifolds and are thus cofibrations. Moreover, $\varphi D^!$ is a pushout cube, so it is strongly co-Cartesian.

Each inclusion $\mathbb{I}^3 - f([i + 1]) \hookrightarrow \mathbb{I}^3 - f([i])$ is a 2–connected map. The Blakers–Massey theorem [11, Theorem 2.3] (see also [22, Theorem 6.2.1]) applies to give that the total fiber of $\varphi D^!$ is n–connected. Thus its nth loopspace is connected, which yields the result. $\qquad\square$

We will say more about this total fiber — and thus the layers in the tower — both below in this section and in Section 7 when we make spectral sequence calculations.

In the proof just given, the analogous $(n-1)$–cube with frames instead of tangent vectors is not $(n-1)$–Cartesian, which is why we prove the framed version separately now.

Theorem 5.14 *In the framed setting, the map* $\widetilde{p}_n\colon \widetilde{\mathrm{AM}}_n^{\mathrm{fr}} \to \widetilde{\mathrm{AM}}_{n-1}^{\mathrm{fr}}$ *induces a surjection on* π_0.

Proof As $\widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$ is a subspace of $\mathrm{Map}(\Delta^n, \widetilde{C}_n\langle \mathbb{I}^3 \rangle \times O(3)^n)$, we consider the diagram:

$$
\begin{array}{ccc}
\Delta^n \longrightarrow \widetilde{C}_n\langle \mathbb{I}^3 \rangle \times O(3)^n \longrightarrow \widetilde{C}_n\langle \mathbb{I}^3 \rangle \times (S^2)^n \\
\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \\
\Delta^{n-1} \longrightarrow \widetilde{C}_{n-1}\langle \mathbb{I}^3 \rangle \times O(3)^{n-1} \longrightarrow \widetilde{C}_{n-1}\langle \mathbb{I}^3 \rangle \times (S^2)^{n-1}
\end{array}
$$

Suppose $\Phi \in \widetilde{\mathrm{AM}}_{n-1}^{\mathrm{fr}}$. Let φ be the image of Φ under $\widetilde{\mathrm{AM}}_{n-1}^{\mathrm{fr}} \to \widetilde{\mathrm{AM}}_{n-1}$, which essentially composes a map to $\widetilde{C}_n\langle \mathbb{I}^3 \rangle \times O(3)^n$ with the projection onto $\widetilde{C}_n\langle \mathbb{I}^3 \rangle \times (S^2)^n$, using the first vector in each frame. By Theorem 5.13, there is a $\psi \in \widetilde{\mathrm{AM}}_n$ whose image in $\widetilde{\mathrm{AM}}_{n-1}$ is in the same component as φ. We lift ψ to a map $\Delta^n \to \widetilde{C}_n\langle \mathbb{I}^3 \rangle \times O(3)^{n-1}$, using Φ to define the map to the $O(3)^{n-1}$ factor.

It remains to lift this map to one additional factor of $O(3)$. Compatibility with the cosimplicial structure maps determine the map $\Delta^n \to \widetilde{C}_n\langle \mathbb{I}^3 \rangle \times O(3)^n$ on precisely two faces of Δ^n, where the map to the additional factor of $O(3)$ must agree with the map on another factor. Away from these faces, there are no constraints on the map to the additional factor of $O(3)$. Thus topologically the problem is to extend this map from $D^{n-1} \subset \partial D^n$ to D^n, which is immediate. $\qquad\square$

5.6 Group structure

Lemma 5.15 *The multiplication on* $\widetilde{L}_n$ *obtained by restricting the multiplication* μ' *of Definition 4.10 is homotopic to the one coming from the description of* $\widetilde{L}_n$ *as the* n–*fold loopspace of the total fiber of an* n–*cube of configuration spaces in Proposition 5.11.*

This compatibility is key in proving the following. Recall the multiplication μ from Definition 4.8.

Theorem 5.16 $\pi_0(\widetilde{\mathrm{AM}}_n^{\mathrm{fr}})$ *is an abelian group with the multiplication* μ *(or* μ'*) on* $\widetilde{\mathrm{AM}}_n^{\mathrm{fr}}$.

Proof By Theorem 4.9, $\pi_0(\widetilde{\mathrm{AM}}_n^{\mathrm{fr}})$ is an abelian monoid. We prove that this monoid is a group by induction on n.

The case $n = 0$ is clear, since $\widetilde{\mathrm{AM}}_0^{\mathrm{fr}} = *$. Suppose $n \geq 1$ and $\widetilde{\mathrm{AM}}_{n-1}^{\mathrm{fr}}$ is a group. From Lemma 5.15, the fiber $\widetilde{L}_n$ is a loopspace when $n \geq 1$, so that its components form a group. Moreover, $\pi_0(\widetilde{L}_n) \to \pi_0(\widetilde{\mathrm{AM}}_n)$ is a map of monoids. Thus applying π_0 to the fiber sequence $\widetilde{L}_n \to \widetilde{\mathrm{AM}}_n^{\mathrm{fr}} \to \widetilde{\mathrm{AM}}_{n-1}^{\mathrm{fr}}$ gives an exact sequence of monoids. By Theorem 5.14, $\pi_0(\widetilde{\mathrm{AM}}_n^{\mathrm{fr}} \to \widetilde{\mathrm{AM}}_{n-1}^{\mathrm{fr}})$ is surjective. The lemma now follows from the elementary fact that, if $G \to H \to K \to 0$ is an exact sequence of monoids with G and K groups, then H is a group. $\qquad\square$

Corollary 5.17 *The homotopy fibers of $\mathrm{AM}_n^{\mathrm{fr}}$ over varying components of $\mathrm{AM}_{n-1}^{\mathrm{fr}}$ are homotopy equivalent.*

Proof of Lemma 5.15 Explicitly, $\widetilde{L}_n(\Delta^n)$ is the subspace of maps $\Delta^n \to \widetilde{C}_n'\langle \mathbb{I}^3 \rangle$ in $\widetilde{\mathrm{AM}}_n$ whose projection by forgetting the last point in the configuration yields the chosen constant map. This implies that such maps are themselves constant on the $t_n = t_{n-1}$ and $t_n = t_{n+1}(= 1)$ faces of Δ^n. We show that μ' and the multiplication from the n–fold loopspace structure both agree with another multiplication, which we now define.

Consider the map $\Delta^{n-1} \times \mathbb{I} \to \Delta^n$ given by

$$(t_1, \ldots, t_{n-1}) \times t \mapsto (t_1, \ldots, t_{n-1}, (1-t)t_{n-1} + t).$$

By precomposing by this map, we can view an element of $\widetilde{L}_n(\Delta^n)$ as a loop in $\mathrm{Map}(\Delta^{n-1}, \widetilde{C}_n'\langle \mathbb{I}^3 \rangle)$, based at the constant standard map. As we use configuration spaces modulo translations and scalings, these loops begin and end at the exact same map. Loop concatenation then defines a multiplication μ_Ω on $\widetilde{L}_n(\Delta^n)$, which of course has homotopy inverses.

We first prove that μ_Ω is homotopic to the multiplication μ' of Definition 4.10 by "homotoping away the mixed terms of μ'". Consider the function $r\colon \Delta^n \times \mathbb{I} \to \Delta^n$ which linearly interpolates between a configuration of times $\vec{t} = (t_1, \ldots, t_n)$ and the "constant configuration" at t_n, but only until all the t_i are contained in $[-1, 0]$ or $[0, 1]$. Let r_s denote its restriction to $\Delta^n \times s$. This function r is not continuous at points with $t_n = 0$. But because a product $\mu(\varphi, \psi)$ of elements $\varphi, \psi \in \widetilde{L}_n(\Delta^n)$ is constant along $t_n = 0$, the map given by

$$\vec{t} \mapsto \begin{cases} \varphi(r_s(\vec{t})) & \text{if } t_n \leq 0, \\ \psi(r_s(\vec{t})) & \text{if } t_n \geq 0, \end{cases}$$

is ultimately continuous. Moreover, for any s the resulting map is again an element of $\widetilde{\mathrm{AM}}_n$ and in fact $\widetilde{L}_n(\Delta^n)$. More specifically, in the image of the product $\mu'(\varphi, \psi)$ in $\widetilde{C}_n\langle \mathbb{I}^d\rangle \subset \prod_{i<j}(S^{d-1})$, the vectors labeled by pairs i, j with $t_i \leq 0$ and $t_j \geq 0$ are all of the form $(1, 0, 0)$, and this will be preserved throughout the homotopy.

Thus the "mixed stacking terms" in the product are not essential and up to homotopy the product can be "decoupled" to a map which at each fixed time uses only one of φ and ψ.

The multiplication when $s = 1$ agrees with μ_Ω, as the map $\Delta^{n-1} \times I \to \Delta^n$ that we used to define μ_Ω sends $\Delta^{n-1} \times [-1, 0]$ to $\{t_n \leq 0\}$ and $\Delta^{n-1} \times [0, 1]$ to $\{t_n \geq 0\}$. Thus μ' restricted to $\widetilde{L}_n(\Delta^n)$ is homotopic to μ_Ω, which completes the first half of our proof.

Next, Proposition 5.11 in the case of $\widetilde{\mathrm{AM}}_n$ implies that $\widetilde{L}_n(\Delta^n)$ is homotopy equivalent to $\Omega^n \operatorname{tfib}(\widetilde{C}'_\bullet\langle \mathbb{I}^3\rangle \circ \mathcal{G}^!_n)$, where $\widetilde{C}'_\bullet\langle \mathbb{I}^3\rangle \circ \mathcal{G}^!_n$ is a cube of configuration spaces with structure maps which forget points. This is established in the proof of Proposition 5.11 through a series of equivalences

$$\operatorname{tfib}(\widetilde{C}'_\bullet\langle \mathbb{I}^3\rangle \circ \mathcal{G}_n) \simeq \Omega \operatorname{tfib}(\widetilde{C}'_\bullet\langle \mathbb{I}^3\rangle \circ \mathcal{G}_{\mathcal{P}_n \in (n-1)}) \simeq \cdots \simeq \Omega^i \operatorname{tfib}(\widetilde{C}'_\bullet\langle \mathbb{I}^3\rangle \circ \mathcal{G}_{\mathcal{P}_n \in (n-i)}).$$

The multiplication from the single loopspace structure on $\Omega \operatorname{tfib}(\widetilde{C}'_\bullet\langle \mathbb{I}^3\rangle \circ \mathcal{G}_{\mathcal{P}_n \in (n-1)})$ agrees up to homotopy with the others coming from i–fold loopspace structure, including $i = n$. (That is, all the equivalences above, except the first one, are loop maps.) We will complete the proof by showing that this single loopspace multiplication coincides with μ_Ω.

By definition $\operatorname{tfib}(\widetilde{C}'_\bullet\langle \mathbb{I}^3\rangle \circ \mathcal{G}_{\mathcal{P}_n \in (n-1)})$ is the total fiber of an $(n-1)$–cube of fibers, where each fibration forgets the last point. An element of loops on this total fiber is a loop of maps $\alpha \colon I^{n-1} \to \operatorname{fib}(\widetilde{C}'_n\langle \mathbb{I}^3\rangle \to \widetilde{C}'_{n-1}\langle \mathbb{I}^3\rangle)$, where

- each map α is constant on the face $\{t_i = 1\}$ for every i, and

- on each of the remaining $n - 1$ faces, the image of α is in the image of the appropriate doubling map.

On the other hand, an element of $\widetilde{L}_n(\Delta^n)$ is a loop of maps $\alpha \colon \Delta^{n-1} \to \widetilde{C}'_n\langle \mathbb{I}^3\rangle$, where

- the image of each α in $\widetilde{C}'_{n-1}\langle \mathbb{I}^3\rangle$ is the standard constant map,

- α is constant on the face $\{t_{n-1} = 1\}$, and

- on each of the remaining $n - 1$ faces, the image of α is in the image of the appropriate doubling map.

Consider a homeomorphism from I^{n-1} to Δ^{n-1} which identifies $\bigcup_{i=1}^{n}\{t_i = 1\} \subset I^{n-1}$ with $\{t_{n-1} = 1\} \subset \Delta^{n-1}$. Via this homeomorphism, we get a homeomorphism between loops on this total fiber and $\widetilde{L}_n(\Delta^n)$ which is compatible with the multiplications on each. □

5.7 Summary

Putting results together the results, the sequences $L_n \to \mathrm{AM}_n^{\mathrm{fr}} \to \mathrm{AM}_{n-1}^{\mathrm{fr}}$ in the Goodwillie–Weiss tower approximating classical framed knots satisfy the following:

(1) They are fibration sequences of $\mathcal{C}_1$–spaces (Definition 4.3 and Corollary 5.7).

(2) $\mathrm{AM}_n^{\mathrm{fr}}$ receives a multiplication-preserving (in fact, $\mathcal{C}_1$–action-preserving) evaluation map from the knot space, with connected sum as its multiplication (Proposition 4.4).

(3) At the level of components, all multiplications are commutative (Corollary 4.13) and have inverses (Theorem 5.16 and Proposition 4.12). Moreover, $\mathrm{AM}_n^{\mathrm{fr}} \to \mathrm{AM}_{n-1}^{\mathrm{fr}}$ is surjective on components (Theorem 5.14).

6 The homotopy tower is a finite-type invariant

In this section we show that $\pi_0(\mathrm{ev}_n)\colon \pi_0\,\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}, \mathbb{R}^3) \to \pi_0\,\mathrm{AM}_n^{\mathrm{fr}}$, which we've shown to be an abelian group-valued invariant compatible with connected sum of knots, is a finite-type invariant of type $n-1$. The main tool is a theorem of Habiro [15], which states that two classical knots share finite-type invariants of degree $\leq n-1$ if and only if they differ by a series of C_n–moves.

We describe these moves in a way that will facilitate our proof. Let E_2 be a copy of $D^2 \times I = D^2 \times [-1, 1]$, with two properly embedded subarcs which clasp in the center as in Figure 2. Iteratively form E_n from E_{n-1} by replacing a regular neighborhood of the top left arc of E_n by a copy of E_2. Thus E_n is $D^2 \times I$ with n properly embedded arcs. (So technically each E_n is a pair of spaces.)

Now a basic C_n–move on a knot K is given by finding an embedding e of E_n into $\mathbb{R}^3$ which meets the knot K as the given collection of arcs, and sliding another subarc of K across the central disk $e(D^2 \times \{0\})$ of the embedded E_n as in Figure 3. In this figure, the almost vertical strand in each left (resp. right) picture is a subarc of the knot which is isotopic to the front (resp. back) half of the boundary of this central disk.

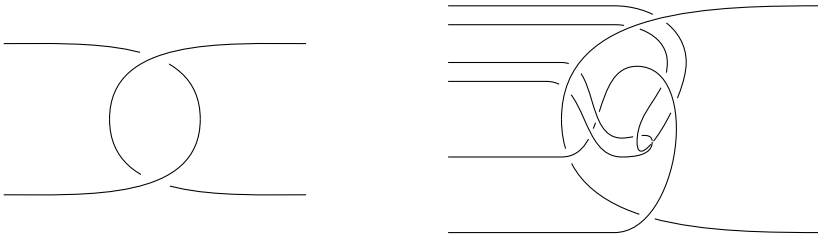The basic tool we need is the following theorem, which follows directly from work of Habiro [15]:

Figure 2: The basic clasp E_2 and the iteratively constructed E_4. The ambient cylinders are not drawn.

Theorem 6.1 *Suppose that v is an additive invariant of (unframed) knots. It is invariant under C_n–moves if and only if it is a finite-type invariant of degree $n-1$.*

Proof The C_n moves constructed here are an alternate formulation of clasper surgery, and in fact are very close to Habiro's original formulation in his master's thesis. The fact that C_n–moves give the same equivalence relation as arbitrary capped clasper surgery is well known, but we will sketch the argument here. (See eg [21] at the top of page 124.) The first step is to prove that C_n–moves correspond to "linear claspers", where all nodes are directly adjacent to a leaf, while the second step is to show that any capped tree clasper surgery may be represented by surgery on a sequence of linear claspers.
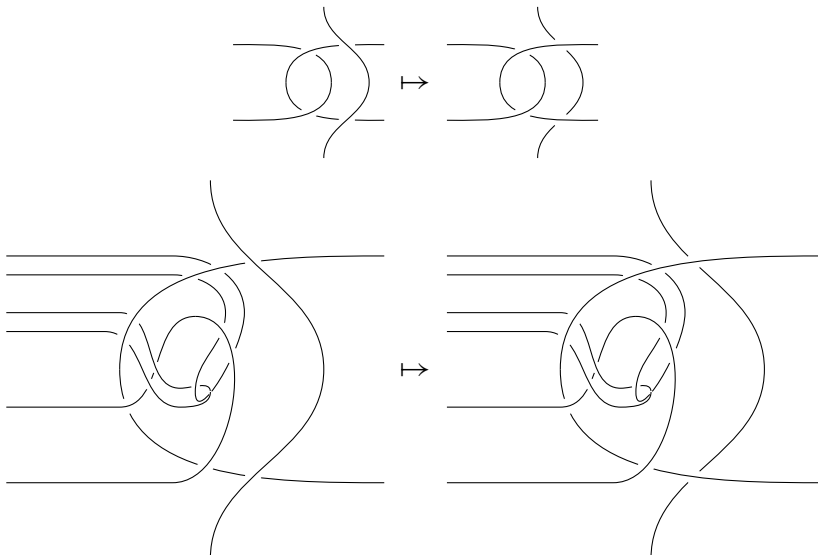

Figure 3: A C_2–move and a C_4–move

Step one can be explicitly proven relatively easily by using the duality between claspers and capped grope cobordism of [7]. One explicitly builds a capped grope cobordism between the before and after knots in a C_n–move. Moreover every capped grope cobordism corresponding to a linear tree can be isotoped to a standard picture of a capped grope embedded in 3–space with a band attached that pierces each cap once. This standard picture is visibly represents the C_n–move.

Step two follows because the topological IHX relation allows us to reduce to the case of linear claspers. (See [7, Corollary 31(a)] or [6, Theorem 7].) The point is that surgery on a capped tree clasper can be replaced by surgery on a sequence of two claspers, each corresponding to the other two trees in an IHX relation. Any tree type can be reduced to linear trees by judicious applications of the IHX relation, as illustrated in the proof of [7, Corollary 31(a)]. So the C_n–moves introduced here are equivalent to the C_n–moves in [15].

The monoid of knots modulo the equivalence relation of n–equivalence is a finitely generated abelian group [14; 15]. Theorem 6.17 of [15] states that the natural projection ψ_{n-1} from knots to this abelian group is a universal additive finite-type invariant of degree $n-1$. So if ν is an additive invariant of knots which also is invariant under C_n moves, it induces a homomorphism on the group of knots modulo n–equivalence. It thus factors as a composition of a degree $n-1$ invariant with a group homomorphism. It is therefore a degree $n-1$ invariant itself. □

In order to move to the framed case, we need the following lemma. For any integer k, let fr_k be the map from unframed knots to framed knots which adds a k–framing.

Lemma 6.2 *Let U_1 represent the $+1$–framed unknot. Let ν be an additive framed knot invariant taking values in an abelian group. Then*

$$\nu \circ \mathrm{fr}_k = k\nu(U_1) + \nu \circ \mathrm{fr}_0 .$$

Proof One can push the twisting of the framing onto a standard subarc of the knot to see that $\mathrm{fr}_k(K) = U_k \# \mathrm{fr}_0(K)$, where U_k is a k–framed unknot. Then one separates each of the twists and uses the fact that ν is additive. □

Corollary 6.3 *Suppose that ν is an additive invariant of framed knots. If it is invariant under C_n–moves then it is a finite-type invariant of degree $n-1$ for $n \geq 2$.*

Proof Note that $\nu \circ \mathrm{fr}_0$ is an additive invariant of unframed knots, and that it is invariant under C_n moves, since ν is. Therefore $\nu \circ \mathrm{fr}_0$ is finite-type of type $n-1$ by Theorem 6.1. On the other hand, by Lemma 6.2, $\nu(K) = \mathrm{fr}(K)\nu(U_1) + \nu \circ \mathrm{fr}_0(K)$,

where $\mathrm{fr}(K)$ is the framing number. The invariant fr is known to be of type 1, so we have a linear combination of a type-$(n-1)$ and type-1 invariant, which is therefore of type $n-1$. $\qquad\square$

This corollary gives us the main tool we need to show that $\pi_0(\mathrm{ev}_n)$ is of finite type. We also use deformations of evaluation maps of knots.

Definition 6.4 A σ–*deformation* of a knot K is a map obtained by precomposing the adjoint of the evaluation map $\Delta^n \times \mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}, \mathbb{R}^3) \to C_n\langle \mathbb{I}^3, \partial \rangle \times O(3)^n$ with a section $\sigma \colon \Delta^n \to \Delta^n \times \mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}, \mathbb{R}^3)$ of this trivial bundle which maps into the component of K in the embedding space.

Theorem 6.5 *The map* $\pi_0(\mathrm{ev}_n)$ *is invariant under* C_n–*moves, and therefore is a type-$(n-1)$ invariant.*

Proof Let K be a framed knot, and let K' be the knot after the C_n–move has been applied. Our strategy is to find σ–deformations of K and K' which we can then show are homotopic. These σ–deformations will have the property that none of the configuration points in $\mathrm{ev}_n(\sigma(\vec{t}))$ meet the central disk whenever strictly fewer than n of those configuration points in $\mathrm{ev}_n(\sigma(\vec{t}))$ lie on the subarcs of E_n. This implies that the σ–deformations are homotopic, since we can just push the arc across the central disk of E_n without ever introducing collisions of configuration points. This is by design for points in Δ^n for which $n-1$ points or fewer are inside E_n. If there are n points in E_n, that means there is no point left over to lie in the exterior arc that we are homotoping.

Both of our σ–deformations will isotop only the copies of E_n in K and K'. Consider a nested copy of E_i inside E_n. It consists of an arc α_i clasping with a copy of E_{i-1}. Let D_i be the (pair of) space(s) given by the intersection of E_i with a small neighborhood $D^2 \times (-\delta, \delta)$ of the central disk $D^2 \times \{0\}$, where δ is large enough so that there is a homeomorphism of pairs $E_i \cong D_i$. The isotopies of E_i will slide D_i along E_i, ie they will take D_i into $D^2 \times (a, b)$ for some $(a, b) \subset [-1, 1]$. Roughly, (a, b) will be a small interval near either -1 or 1 according as which of these endpoints is closer to the center of mass of the configuration points. More specifically, a set of configuration points in the arc α_i will pull D_i toward that side of $D^2 \times [-1, 1]$ in a manner that increases as the minimum distance of these points to the midpoint of the arc α_i decreases. Configuration points in the copy of E_{i-1} exert a similar tug to their end of E_i in a manner which increases as they get closer to the midpoints of their arcs. However the tug of a point is halved in magnitude when you pass to E_{i-1}. This has
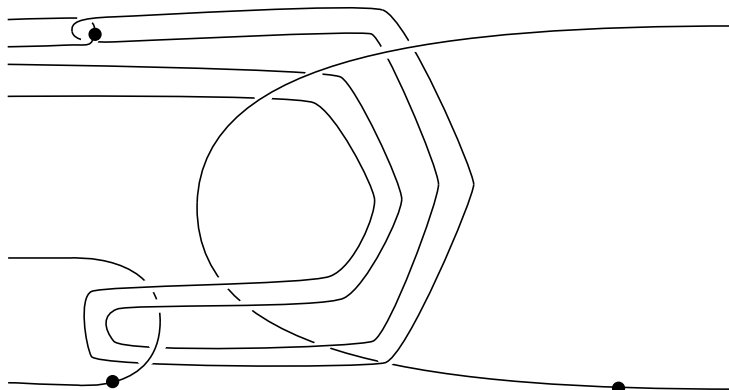
Figure 4: An example of the reembedding of E_4 resulting from a point $(t_1, \dots, t_4) \in C_4\langle \mathbb{I}, \partial \rangle$, whose image under ev_4 is also shown. (The image of one of the t_i is not in E_4.) The two left-hand configuration points drag their respective arcs toward their ends of their cylinders. The right-hand configuration point is partially balancing their tug on the central clasp. If the right-hand point moved toward the middle of its arc, the central clasp of E_4 would get tugged close to the right-hand end of the cylinder, although the other configuration points would still be close to the left-hand end. A point added to the arc without a point on it would bring E_4 closer to the original embedding as it got closer to the midpoint of the arc.

been set up so that a point at the midpoint of α_i will always exert a tug that equals or exceeds the collective tug of E_{i-1}.

We also set things up so that if E_{i-1} has fewer than $i-1$ configuration points on it then configuration points in α_i can never get more than ε away from their end disk $D^2 \times 0$. The point is that any configuration points in α_i will have a greater tug on the clasp than E_{i-1} when one is at the midpoint of α_i. We just ensure that this tug is strong enough to pull α_i ε–close to its end disk. (We can come up with a uniform ε in this way since there is a discrete gap between the maximal tug that E_{i-1} can exert and the tug it exerts at less than full occupancy.)

Similarly, arrange that if the arc α_i has no configuration points then no configuration points in E_{i-1} can get more than ε away from $D^2 \times 1$ inside E_i.

Ultimately, we have an isotopy of E_n parametrized by $C_n\langle \mathbb{I}, \partial \rangle$ or, equivalently, a reembedding of E_n for each $(t_1, \dots, t_n) \in C_n\langle \mathbb{I}, \partial \rangle$. See Figure 4 for an example.

With such a family of reembeddings, we claim that no configuration point ever passes through the $D^2 \times \frac{1}{2}$ disk of E_n, provided that strictly fewer than n configuration points

are present inside E_n. Let the arcs in E_n be called α_1, α_2, ... arranged in order of decreasing depth.

We know that there is some arc α_i which is not occupied by a configuration point. The points in E_{i-1} stay within ε of the end disk and cannot cross the center disk of E_n. Thus any points that do cross the center must lie on one of the arcs $\alpha_{i+1}, \ldots, \alpha_n$. Now consider such an arc α_j with $j > i$. It links with E_{j-1} which has less than full occupancy. Points in α_j cannot get further than ε away from their end disk, and cannot cross the center disk of E_n, so we are done! □

7 The homotopy spectral sequence for the tower and finite-type knot theory

In this section we further develop the spectral sequence for the homotopy groups and in particular the components of the stages in the Goodwillie–Weiss tower for classical knots and its relationship — both established and conjectural — with finite-type knot theory. Such analysis for knots in higher-dimensional Euclidean space, which are connected, has been covered elsewhere, starting in [28]. We see that at the E^2 stage the entries of the spectral sequence are exactly what one would expect if the tower is to serve as a universal additive finite-type invariant. We in particular see similar structures to what Goodwillie and Weiss [13, Section 5] originally saw in higher dimensions, but can also compare that to newer results on the combinatorics of finite-type invariants [5].

A priori the spectral sequence of a totalization tower, or any other tower of fibrations, is difficult to discern in degree zero. Not only is π_0 only a set-valued functor, but homotopy groups can differ over different components. We saw in Section 5 however that this tower has additional algebraic structure, which leads to the following:

Theorem 7.1 *The spectral sequence for the homotopy groups, and in particular components, of $\mathrm{AM}_n^{\mathrm{fr}}$ as a stage in the Goodwillie–Weiss tower is a spectral sequence of abelian groups which converges.*

Proof By Corollary 5.7 and Theorem 5.16 the fiber sequence

$$(4) \qquad\qquad L_n \to \mathrm{AM}_n^{\mathrm{fr}} \to \mathrm{AM}_{n-1}^{\mathrm{fr}}$$

is a fibration sequence of group-like $\mathcal{C}_1$–spaces. It is thus loops on the fibration sequence defined on their classifying spaces. Its long exact sequence in homotopy groups is then a degree shift of that for the classifying spaces, starting with π_1 of the classifying spaces being π_0 of these spaces. These exact sequences can be spliced in the usual way to obtain a spectral sequence, which by Theorem 5.16 is one of abelian groups.

Convergence follows from the fact that this tower is finite when we truncate it at $\mathrm{AM}_n^{\mathrm{fr}}$, and the groups are finitely generated by Proposition 7.2 below along with the fact that homotopy groups of spheres are finitely generated. □

We can now analyze the spectral sequence in further detail. By Proposition 5.11, L_n is equivalent to Ω^n of the total fiber of the n–cube $C_\bullet^{\mathrm{fr}}\langle \mathbb{I}^3, \partial \rangle \circ \mathcal{G}_n^!$. Unraveling definitions, this is an n–cube whose entries are spaces of configurations of at most n points, together with at most n frames, where each map in the cube forgets a point and a frame. We can express this cube as an entrywise product of the cubes $S \mapsto C_{\underline{n}\setminus S}\langle \mathbb{I}^3, \partial \rangle$ and $S \mapsto O(3)^{\underline{n}\setminus S}$.

The total fiber of the product is the product of the total fibers, and for $n \geq 2$ the total fiber of the cube of powers of $O(3)$ is a point. Thus it suffices to consider the total fiber of $S \mapsto C_{\underline{n}\setminus S}\langle \mathbb{I}^3, \partial \rangle$. Furthermore, we can switch to open configuration spaces, for which these forgetting maps are well known to be fibrations. We take fibers in one direction and consider the resulting $(n-1)$–cube instead. Here we see the entries as $\mathbb{I}^3$ with a finite set of points removed, and maps which are inclusions. Up to homotopy, this is a cube of spaces $\bigvee_T S^2$ indexed by subsets $T \subset \underline{n-1}$, where each map projects off a wedge factor. Call this cube $\mathcal{P}(\underline{n-1})(\bigvee S^2)$.

By Hilton's theorem [16], the homotopy groups of a wedge $\bigvee_n S^2$ is a direct sum of homotopy groups of higher-dimensional spheres. To elaborate, let $\mathcal{L}_n$ be the free graded Lie algebra (working over the integers for the rest of this section) on n odd-graded generators in degree one. Let $\mathcal{B}_{n,\mathrm{full}}$ be a basis for $\mathcal{L}_n$, choosing these consistently as n varies. For example, we could use a graded version of Hall bases.

Hilton's theorem states that $\pi_*\big(\bigvee_n S^2\big)$ is a direct sum $\bigoplus_{W \in \mathcal{B}_{n,\mathrm{full}}} \pi_* S^{|W|-1}$, where W is the degree or word length of W. The theorem is functorial if we use bases for free Lie algebras of different ranks which extend one another, since the Whitehead products used to define the elements of homotopy are functorial. Because the projection maps between wedge products of S^2 are split, these different bases split off. An immediate inductive calculation of the homotopy groups of the total fiber (as an iterated fiber of fibers) shows that what is left for homotopy groups is indexed by a basis $\mathcal{B}_n$ of the submodule of the $\mathcal{L}_n$ spanned by brackets in which all generators occur.

Proposition 7.2 *The spectral sequence for the homotopy groups (including π_0) of* AM_n *has as* $E^1_{-p,*}$ *the module* $\bigoplus_{W \in \mathcal{B}_{p-1}} \pi_* S^{|W|-1}$.

By Lemma 5.15, the abelian group structure in the spectral sequence agrees with the usual abelian group structure on the homotopy groups of spheres.

We now focus on total degree zero. Let $\mathcal{A}_n^I$ be the $\mathbb{Z}$–module of chord diagrams on a line segment with n chords, modulo the usual four-term relation 4T and the relation SEP, which sets every separated (ie nonprimitive) diagram to zero. Alternatively, $\mathcal{A}_n^I$ is the $\mathbb{Z}$–module of trivalent diagrams, modulo antisymmetry, the IHX relation and SEP. See [5] for more details.

Theorem 7.3 *The group $E_{-n,n}^2$ is isomorphic to $\mathcal{A}_{n-1}^I$.*

Proof By Proposition 7.2 the group $E_{-n,n}^1$ is isomorphic to the submodule of the free Lie algebra on $n-1$ generators generated by $(n-1)$–fold brackets where each generator appears exactly once. This module is $\mathcal{L}ie(n-1)$, the $n-1^{\text{st}}$ space in the Lie operad, which well known to be free of rank $(n-1)!$.

Next, we consider the 1-line of the E^1 page. Under the identification of Proposition 7.2 these groups decompose into a free summand and two-torsion. The free summands are indexed by n–fold brackets in the free Lie algebra on $n-1$ generators, again in which all generators occur. The two-torsion summands occur as composites of $S^n \xrightarrow{\eta} S^{n-1}$ with $(n-1)$–fold Whitehead products from $S^{n-1} \to \mathcal{P}(\underline{n-1})(\bigvee S^2)$. This summand is thus isomorphic to $\mathcal{L}ie(n-1) \otimes \mathbb{Z}/2$.

The differential d_1 must be zero on the torsion summand. We claim that on the free summand the differential is the integral version of the differential defined in [24]. There, in Theorem 2.1, through the tower of fibrations

$$(5) \qquad C_n(\mathbb{R}^d) \to C_{n-1}(\mathbb{R}^d) \to \cdots \to C_0(\mathbb{R}^d)$$

the well-known rational homotopy Lie algebra of the configuration space $C_n(\mathbb{R}^d)$ is calculated as generated by classes b_{ij} in degree $d-1$. Under the map from the total fiber of $\mathcal{P}(\underline{n-1})(\bigvee S^2)$ to $C_n(\mathbb{R}^3)$ the basis for the free Lie algebra on generators, say x_i, sends a bracket to a corresponding brackets in the b_{in}. Because the projections in the tower (5) are split, these brackets are integral generators of free summands as well. (In fact, one can use the splitting of the tower (5) and the Hilton–Milnor theorem to express homotopy groups of configuration spaces as a direct sum of homotopy groups of spheres.) The formulas for the differential given in [24] are given in terms of these integral generators, so they hold for the spectral sequence over the integers as well.

In [5], the cokernel of the rational d^1 is computed to be $\mathcal{A}_{n-1}^I \otimes \mathbb{Q}$. While the result is stated rationally (which is where the conjecture was made), all of the calculations involve only integer coefficients. □

At the E^2–level the components of AM_n thus look like they should receive a universal additive finite type-$(n-1)$ invariant over the integers, which was established for $n = 3$ as the main result of [4].

We elaborate our conjecture as follows. Because the map ev_n from the knot space to this Goodwillie–Weiss tower, sometimes called the "homotopy tower", factors the map to the variant that Volić considers in [32], sometimes called the "rational homology tower", we already knew that it encodes every rational finite type-$\frac{n}{2}$ invariant. By Theorem 6.5, we now know that any invariant which factors through $\pi_0(\mathrm{ev}_n)$ is of type at most $n-1$. Standard finite-type theory then gives a map from $\mathcal{A}_{n-1}^I$ to $\pi_0(\mathrm{AM}_n)$, as follows. For a diagram with $n-1$ chords, choose a knot K with $n-1$ double points prescribed by the chords; by the well known Vassiliev skein relation, K can be rewritten as an alternating sum of 2^{n-1} knots where all the singularities are resolved. This alternating sum can then be interpreted inside the abelian group $\pi_0(\mathrm{AM}_n)$, which defines the map. Although the choice of K is not determined up to isotopy, any two choices will have the same type-$(n-1)$ invariants, and will yield the same image under $\pi_0(\mathrm{ev}_n)$. We conjecture that this is an isomorphism at E^2, which collapses to E^∞. This would imply by Theorem 7.3 that all weight systems lift to finite-type invariants over the integers. That is, it would establish $\pi_0(\mathrm{ev}_\infty)$ as a refinement of the Kontsevich integral, defined over the integers.

In the framed setting we have some additional low-dimensional calculations. Namely, $\mathrm{AM}_1^{\mathrm{fr}} \simeq \Omega \mathrm{SO}(3)$, implying its components are isomorphic to $\mathbb{Z}/2$. The evaluation map calculates the parity of the framing, which is indeed the only additive type-1 invariant. Next, Theorem 3.6 of [4] states that AM_2 is contractible. The subcubical diagram which defines $\mathrm{AM}_2^{\mathrm{fr}}$ fibers over that which defines AM_2, with the maps in this fibering built from the standard fibration of $\mathrm{SO}(3)$ over S^2 as the unit tangent bundle. The fiber is a subcubical model for $\Omega S^1 \simeq \mathbb{Z}$ (pulled back from the cosimplicial model for ΩS^1 via $\mathcal{G}_2$), and the map from the framed knot space to $\mathrm{AM}_2^{\mathrm{fr}}$ classifies framing number.

More generally, the subcubical diagram which defines $\mathrm{AM}_n^{\mathrm{fr}}$ fibers over that which defines AM_n with fiber given by a n–subcubical diagram which models ΩS^1. Because $\pi_0(\mathrm{AM}_n^{\mathrm{fr}})$ projects surjectively onto $\pi_0(\mathrm{AM}_2^{\mathrm{fr}})$, compatibly with the identification of this fiber with ΩS^1, on components this yields a splitting $\pi_0(\mathrm{AM}_n^{\mathrm{fr}}) \cong \pi_0(\mathrm{AM}_n) \times \mathbb{Z}$. (Note that this would follow immediately from elementary results in Goodwillie–Weiss calculus if the splitting of spaces of framed knots can be made functorial.) Thus these Goodwillie–Weiss models reflect the fact that the usual and framed finite-type theories differ only by the framing number invariant.

One key step towards establishing this conjecture would be the collapse of the spectral sequence, which is now of a tower of fibrations amenable to tools from algebraic topology. This is in contrast to Vassiliev's approach, where the limiting process of unstable spectral sequences is not well understood. (See [10] for this limiting process in a piecewise-linear setting.) The Goodwillie–Weiss tower is built from maps of spaces,

in particular the sequences of (4) and the diagrams which define AM_n, which have been analyzed to great effect for knots in higher dimensions [17].

Another key intermediate step would be the surjectivity on components of ev_n. This statement would follow from deep connectivity results of Goodwillie and Klein [12] if those applied in codimension two, but it may be approachable more directly in this case.

We suspect, however, that direct analysis of the invariants which arise from $\pi_0(\mathrm{ev}_n)$ will be most fruitful. They have already led to new geometric insight in degrees three and four [4; 9]. Sinha and Walter's Hopf invariants [29] can fully be applied by the calculations of Proposition 7.2. They seem to lead to Goussarov–Polyak–Viro formulae, which is a promising sign.

References

[1]   **P Boavida de Brito**, **M S Weiss**, *Spaces of smooth embeddings and configuration categories*, preprint (2015) arXiv

[2]   **A K Bousfield**, **D M Kan**, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics 304, Springer, Berlin (1972) MR

[3]   **R Budney**, *Little cubes and long knots*, Topology 46 (2007) 1–27 MR

[4]   **R Budney**, **J Conant**, **K P Scannell**, **D Sinha**, *New perspectives on self-linking*, Adv. Math. 191 (2005) 78–113 MR

[5]   **J Conant**, *Homotopy approximations to the space of knots, Feynman diagrams, and a conjecture of Scannell and Sinha*, Amer. J. Math. 130 (2008) 341–357 MR

[6]   **J Conant**, **R Schneiderman**, **P Teichner**, *Jacobi identities in low-dimensional topology*, Compos. Math. 143 (2007) 780–810 MR

[7]   **J Conant**, **P Teichner**, *Grope cobordism of classical knots*, Topology 43 (2004) 119–156 MR

[8]   **W Dwyer**, **K Hess**, *Long knots and maps between operads*, Geom. Topol. 16 (2012) 919–955 MR

[9]   **G Flowers**, *Satanic and Thelemic circles on knots*, J. Knot Theory Ramifications 22 (2013) art. id. 1350017 MR

[10]   **C D Giusti**, *Plumbers' knots and unstable Vassiliev theory*, PhD thesis, University of Oregon (2010) MR Available at `http://search.proquest.com/docview/749774372`

[11]   **T G Goodwillie**, *Calculus, II: Analytic functors*, *K*–Theory 5 (1992) 295–332 MR

[12]   **T G Goodwillie**, **J R Klein**, *Multiple disjunction for spaces of Poincaré embeddings*, J. Topol. 1 (2008) 761–803 MR

[13]   **T G Goodwillie**, **M Weiss**, *Embeddings from the point of view of immersion theory, II*, Geom. Topol. 3 (1999) 103–118 MR

[14]   **M N Goussarov**, *Interdependent modifications of links and invariants of finite degree*, Topology 37 (1998) 595–602 MR

[15]   **K Habiro**, *Claspers and finite type invariants of links*, Geom. Topol. 4 (2000) 1–83 MR

[16]   **P J Hilton**, *On the homotopy groups of the union of spheres*, J. London Math. Soc. 30 (1955) 154–172 MR

[17]   **P Lambrechts**, **V Turchin**, **I Volić**, *The rational homology of spaces of long knots in codimension > 2*, Geom. Topol. 14 (2010) 2151–2187 MR

[18]   **J E McClure**, **J H Smith**, *A solution of Deligne's Hochschild cohomology conjecture*, from "Recent progress in homotopy theory" (D M Davis, J Morava, G Nishida, W S Wilson, N Yagita, editors), Contemp. Math. 293, Amer. Math. Soc., Providence, RI (2002) 153–193 MR

[19]   **J E McClure**, **J H Smith**, *Multivariable cochain operations and little n–cubes*, J. Amer. Math. Soc. 16 (2003) 681–704 MR

[20]   **J E McClure**, **J H Smith**, *Cosimplicial objects and little n–cubes, I*, Amer. J. Math. 126 (2004) 1109–1153 MR

[21]   **J-B Meilhan**, **A Yasuhara**, *On C_n–moves for links*, Pacific J. Math. 238 (2008) 119–143 MR

[22]   **B A Munson**, **I Volić**, *Cubical homotopy theory*, New Mathematical Monographs 25, Cambridge University Press (2015) MR

[23]   **P Salvatore**, *Knots, operads, and double loop spaces*, Int. Math. Res. Not. 2006 (2006) art. id. 13628 MR

[24]   **K P Scannell**, **D P Sinha**, *A one-dimensional embedding complex*, J. Pure Appl. Algebra 170 (2002) 93–107 MR

[25]   **H Schubert**, *Die eindeutige Zerlegbarkeit eines Knotens in Primknoten*, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1949 (1949) 57–104 MR

[26]   **D P Sinha**, *Manifold-theoretic compactifications of configuration spaces*, Selecta Math. 10 (2004) 391–428 MR

[27]   **D P Sinha**, *Operads and knot spaces*, J. Amer. Math. Soc. 19 (2006) 461–486 MR

[28]   **D P Sinha**, *The topology of spaces of knots: cosimplicial models*, Amer. J. Math. 131 (2009) 945–980 MR

[29] **D Sinha**, **B Walter**, *Lie coalgebras and rational homotopy theory, II: Hopf invariants*, Trans. Amer. Math. Soc. 365 (2013) 861–883 MR

[30] **N E Steenrod**, *Products of cocycles and extensions of mappings*, Ann. of Math. 48 (1947) 290–320 MR

[31] **V Turchin**, *Delooping totalization of a multiplicative operad*, J. Homotopy Relat. Struct. 9 (2014) 349–418 MR

[32] **I Volić**, *Finite type knot invariants and the calculus of functors*, Compos. Math. 142 (2006) 222–250 MR

Mathematics and Statistics, University of Victoria
PO Box 1700 STN CSC, Victoria, BC V8W 2Y2, Canada

Department of Mathematics, University of Tennessee
227 Ayres Hall, 1403 Circle Dr, Knoxville, TN 37996, United States

Department of Mathematics and Statistics, University of Massachusetts-Amherst
Leaderless Graduate Research Tower, Amherst, MA 01003, United States

Department of Mathematics, University of Oregon
Eugene, OR 97403, United States

ryan.budney@gmail.com, jconant@utk.edu, koytcheff@math.umass.edu, dps@uoregon.edu