

The augmentation category map induced by exact Lagrangian cobordisms

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To a Legendrian knot, one can associate an A_∞ category, the augmentation category. An exact Lagrangian cobordism between two Legendrian knots gives a functor of the augmentation categories of the two knots. We study this functor and establish a long exact sequence relating the corresponding cohomology of morphisms of the two ends. As applications, we prove that the functor between augmentation categories is injective on the level of equivalence classes of objects and find new obstructions to the existence of exact Lagrangian cobordisms in terms of linearized contact homology and ruling polynomials.

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1 Introduction

Let Λ_\pm be Legendrian submanifolds in the standard contact manifold $(\mathbb{R}^3, \xi = \ker \alpha)$, where $\alpha = dz - y dx$. An exact Lagrangian cobordism Σ from Λ_- to Λ_+ is a 2-dimensional surface in the symplectization of \mathbb{R}^3 that has cylindrical ends over Λ_+ and Λ_- with certain properties. See [Figure 1](#) for a schematic picture and [Definition 2.1](#) for a detailed description.

Lagrangian cobordism is a natural relation between Legendrian submanifolds and is crucial in the definition of the functorial property of the Legendrian contact homology differential graded algebra (DGA). For a Legendrian knot Λ in $(\mathbb{R}^3, \xi = \ker \alpha)$, the Legendrian contact homology DGA is a powerful invariant of Λ that was introduced by Eliashberg [21] and Chekanov [7] in the spirit of symplectic field theory; see Eliashberg, Givental and Hofer [22]. The underlying algebra $\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)])$ is a unital graded algebra freely generated by Reeb chords of Λ and a basis of $H_1(\Lambda)$ over a field \mathbb{F} , where $H_1(\Lambda)$ is the singular homology of Λ with \mathbb{Z} coefficients. The differential ∂ is defined by a count of rigid holomorphic disks in $\mathbb{R} \times \mathbb{R}^3$ with boundary on the Lagrangian submanifold $\mathbb{R} \times \Lambda$. The DGA $(\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)]), \partial)$ is invariant up to stable tame isomorphism under Legendrian isotopy of Λ . Ekholm, Honda and

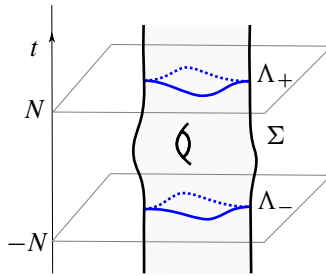


Figure 1: A Lagrangian cobordism Σ from Λ_- to Λ_+ lies in the symplectization of \mathbb{R}^3 , which is $(\mathbb{R}_t \times \mathbb{R}^3, d(e^t \alpha))$. The vertical direction is the t direction and each horizontal plane is \mathbb{R}^3 . Two Legendrian submanifolds Λ_+ and Λ_- sit inside different copies of \mathbb{R}^3 with t coordinates N and $-N$, respectively.

Kálmán [19] showed that an exact Lagrangian cobordism Σ from Λ_- to Λ_+ gives a DGA map

$$\phi_\Sigma: (\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\Sigma)]), \partial) \rightarrow (\mathcal{A}(\Lambda_-; \mathbb{F}[H_1(\Sigma)]), \partial),$$

which is defined by a count of rigid holomorphic disks in $\mathbb{R} \times \mathbb{R}^3$ as well, but with boundary on Σ . Here \mathbb{F} can be any field if the cobordism Σ is spin. If the condition is not satisfied, the field \mathbb{F} is assumed to be \mathbb{Z}_2 .

Remark When Σ is spin, the boundary Legendrian knots Λ_+ and Λ_- get induced spin structure from the spin structure of Σ . This condition makes the moduli spaces of the holomorphic disks used in the DGA differentials and the DGA map equipped with a coherent orientation (following Ekholm, Etnyre and Sullivan [18]). In particular, when the dimension of a moduli space is 0, one can associate each rigid holomorphic disk in the moduli space with a sign. Therefore, we can count the disks with sign and get coefficients in any field \mathbb{F} . Otherwise, it is only reasonable to count the disks mod 2, which means ignoring the orientation. For the rest of the paper, we focus on the case where Σ is spin. If one is working on a nonspin cobordism, one can omit our description of orientation and get the corresponding statements for $\mathbb{F} = \mathbb{Z}_2$.

A fundamental question about Lagrangian cobordisms is: given two Legendrian knots Λ_+ and Λ_- , does there exist an exact Lagrangian cobordism Σ between them? In order to answer this question, we need to investigate the properties of Lagrangian cobordisms and obtain a relationship between Legendrian knots Λ_+ and Λ_- . If the two given Legendrian knots do not satisfy the desired relationship, there does not exist a cobordism between them. In this way, we can find obstructions to the existence of

exact Lagrangian cobordisms. Chantraine [4] first gave a relationship between the Thurston–Bennequin numbers of two Legendrian knots:

$$(1) \quad \text{tb}(\Lambda_+) - \text{tb}(\Lambda_-) = -\chi(\Sigma).$$

This question was explored further in many works, including Bourgeois, Sabloff and Traynor [3], Sabloff and Traynor [34], Baldwin and Sivek [1], Cornwell, Ng and Sivek [11] and Chantraine, Dimitroglou Rizell, Ghiggini and Golovko [6] using generating families, normal rulings and Floer theory.

We approach the question through studying the relationship between the augmentation category of the Legendrian knots that are connected by an exact Lagrangian cobordism. Analogous to the derived Fukaya category of exact Lagrangian compact submanifolds introduced in Nadler and Zaslow [29], the augmentation category is an A_∞ category of Legendrian knots in $(\mathbb{R}^3, \ker \alpha)$. Bourgeois and Chantraine first introduced a nonunital A_∞ category in [2] and then Ng, Rutherford, Sivek, Shende and Zaslow introduced a unital version in [33]. We will focus on the latter one.

For a fixed DGA $(\mathcal{A}(\Lambda), \partial)$ of a Legendrian knot Λ , the augmentation category $\text{Aug}_+(\Lambda)$ consists of objects, morphisms and A_∞ operations. The objects in the category are augmentations ϵ of the Legendrian contact homology DGA, ie DGA maps $\epsilon: (\mathcal{A}(\Lambda), \partial) \rightarrow (\mathbb{F}, 0)$. For any two objects ϵ_1 and ϵ_2 , the morphism space $\text{Hom}_+(\epsilon_1, \epsilon_2)$ is a vector space over the field \mathbb{F} generated by Reeb chords from Λ to Λ' , where Λ' is a positive Morse perturbation of Λ . The A_∞ operations are composition maps $\{m_n \mid n \geq 1\}$ that satisfy certain relations. These relations allow us to take cohomology of the $\text{Hom}_+(\epsilon_1, \epsilon_2)$ space with respect to m_1 , denoted by $H^* \text{Hom}_+(\epsilon_1, \epsilon_2)$. From [33], we know that up to A_∞ equivalence, the augmentation category $\text{Aug}_+(\Lambda)$ is an invariant of Legendrian knots under Legendrian isotopy.

We will show that an exact Lagrangian cobordism Σ from a Legendrian knot Λ_- to a Legendrian knot Λ_+ gives a DGA map ϕ_Σ from the DGA $\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\Lambda_+)])$ to the DGA $\mathcal{A}(\Lambda_-; \mathbb{F}[H_1(\Lambda_-)])$. By [33], this DGA map induces an A_∞ -category map $f: \text{Aug}_+(\Lambda_-) \rightarrow \text{Aug}_+(\Lambda_+)$. As a result, the augmentation category Aug_+ acts functorially under Lagrangian cobordisms as well. For each augmentation ϵ_- of $\mathcal{A}(\Lambda_-)$, the cobordism Σ induces an augmentation ϵ_+ of $\mathcal{A}(\Lambda_+)$ by composing with the DGA map ϕ_Σ , ie

$$\epsilon_+ = \epsilon_- \circ \phi_\Sigma.$$

The augmentation category map f sends an object ϵ_- of $\text{Aug}_+(\Lambda_-)$ to the object ϵ_+ of $\text{Aug}_+(\Lambda_+)$. For any two objects ϵ_-^1 and ϵ_-^2 in $\text{Aug}_+(\Lambda_-)$, the category map f sends the morphism $\text{Hom}_+(\epsilon_-^1, \epsilon_-^2)$ to the morphism $\text{Hom}_+(\epsilon_+^1, \epsilon_+^2)$, where ϵ_+^1 and ϵ_+^2 are the augmentations induced by Σ .

We investigate properties of this A_∞ -category map through the Floer theory of a pair of exact Lagrangian cobordisms (see Chantraine, Dimitroglou Rizell, Ghiggini and Golovko [6]), which is an analog of the construction of Ekholm [15] for a pair of Lagrangian fillings in the spirit of symplectic field theory (see Eliashberg, Givental and Hofer [22]). Let Σ be an exact Lagrangian cobordism from Λ_- to Λ_+ . Perturb Σ using a positive Morse function F and get a new exact Lagrangian cobordism Σ' . In [6], Chantraine, Dimitroglou Rizell, Ghiggini and Golovko constructed a chain complex for this pair of exact Lagrangian cobordisms $\Sigma \cup \Sigma'$, called the Cthulhu chain complex. The generators of this chain complex are the union of double points of $\Sigma \cup \Sigma'$ and Reeb chords on the cylindrical ends from Σ to Σ' . Indeed, the second part agrees with the union of Hom_+ spaces in the augmentation category of the Legendrian submanifolds on two ends. The differential of this chain complex is defined by a count of rigid holomorphic disks with boundary on $\Sigma \cup \Sigma'$ as well. From [6], the Cthulhu chain complex is acyclic, which implies the following long exact sequence:

Theorem 1.1 (see Corollary 5.2) *Let Σ be an exact Lagrangian cobordism with Maslov number 0 from Λ_- to Λ_+ . If ϵ_-^i , for $i = 1, 2$, is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ_+^i is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ , then we have the following long exact sequence:*

$$\begin{aligned} \dots \rightarrow H^k(\Sigma, \Lambda_-) \rightarrow H^k \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \rightarrow H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \\ \rightarrow H^{k+1}(\Sigma, \Lambda_-) \rightarrow \dots \end{aligned}$$

If $\epsilon_-^1 = \epsilon_-^2 = \epsilon_-$, we can identify $H^k \text{Hom}_+(\epsilon, \epsilon)$ with the linearized contact homology $\text{LCH}_{1-k}^\epsilon(\Lambda)$ by Ng, Rutherford, Shende, Sivek and Zaslow [33, Section 5.2]. The long exact sequence above can be rewritten as

$$\dots \rightarrow H^k(\Sigma, \Lambda_-) \rightarrow \text{LCH}_{1-k}^{\epsilon_+}(\Lambda_+) \rightarrow \text{LCH}_{1-k}^{\epsilon_-}(\Lambda_-) \rightarrow H^{k+1}(\Sigma, \Lambda_-) \rightarrow \dots$$

Computing the Euler characteristics of the exact triangle, we have

$$\text{tb}(\Lambda_+) - \text{tb}(\Lambda_-) = -\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of the surface Σ . This result was previously shown by Chantraine in [4].

Combine Theorem 1.1 with the augmentation category map induced by exact Lagrangian cobordisms and we have the following theorem.

Theorem 1.2 (see Theorem 5.4) *Let Σ be an exact Lagrangian cobordism with Maslov number 0 from a Legendrian knot Λ_- to a Legendrian knot Λ_+ . Assume*

neither Λ_+ nor Λ_- are empty. For $i = 1, 2$, if ϵ_-^i is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ_+^i is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ , the map

$$i^0: H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \rightarrow H^0 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2)$$

in the long exact sequence in [Theorem 1.1](#) is an isomorphism. Moreover, we have that

$$H^* \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \cong H^* \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \oplus \mathbb{F}^{-\chi(\Sigma)}[1],$$

where $\mathbb{F}^{-\chi(\Sigma)}[1]$ denotes the vector space $\mathbb{F}^{-\chi(\Sigma)}$ in degree 1 and $\chi(\Sigma)$ is the Euler characteristic of the surface Σ .

This relation was shown for positive braid closures in Menke [\[28\]](#). [Theorem 1.2](#) shows that this is true for general Legendrian knots.

When $\epsilon_-^1 = \epsilon_-^2$, we restate [Theorem 1.2](#) in terms of linearized contact homology as follows:

Corollary 1.3 (see [Corollary 5.5](#)) *Let Σ be an exact Lagrangian cobordism with Maslov number 0 from a Legendrian knot Λ_- to a Legendrian knot Λ_+ . Assume neither Λ_+ nor Λ_- is empty. If ϵ_- is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ_+ is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ , then*

$$\text{LCH}_*^{\epsilon_+}(\Lambda_+) \cong \text{LCH}_*^{\epsilon_-}(\Lambda_-) \oplus \mathbb{F}^{-\chi(\Sigma)}[0],$$

where $\mathbb{F}^{-\chi(\Sigma)}[0]$ denotes the vector space $\mathbb{F}^{-\chi(\Sigma)}$ in degree 0.

Therefore, if there exists an exact Lagrangian cobordism Σ from Λ_- to Λ_+ , the Poincaré polynomials of the linearized contact homology of Λ_+ and Λ_- agree in all degrees except 0. In degree 0 their coefficients differ by $-\chi(\Sigma)$. This is a stronger obstruction to the existence of the exact Lagrangian cobordism than the Thurston–Bennequin number relation [\(1\)](#).

For instance, [Figure 2](#) shows two Legendrian knots Λ_1 and Λ_2 of smooth knot types 4_1 and 6_1 , respectively. There is a topological cobordism between 4_1 and 6_1 with genus 1. The Thurston–Bennequin numbers of Λ_1 and Λ_2 are -3 and -5 , respectively, and thus satisfy the Thurston–Bennequin number relation [\(1\)](#). Therefore, there possibly exists an exact Lagrangian cobordism from Λ_2 to Λ_1 with genus 1. However, the Poincaré polynomials of the linearized contact homology for Λ_1 and Λ_2 are $t^{-1} + 2t$ and $2t^{-1} + 3t$, respectively. Thus, we have the following proposition.

Proposition 1.4 (see [Proposition 5.6](#)) *There does not exist an exact Lagrangian cobordism with Maslov number 0 from Λ_2 to Λ_1 , where Λ_1 and Λ_2 are as shown in [Figure 2](#).*

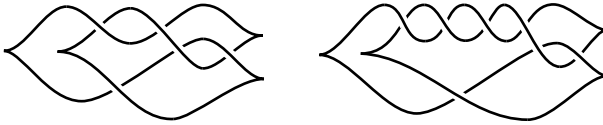


Figure 2: Legendrian knot Λ_1 of knot type 4_1 (left) and Legendrian knot Λ_2 of knot type 6_1 (right)

Various long exact sequences similar to that in [Theorem 1.1](#) have been explored. Sabloff and Traynor [34] gave a long exact sequence using generating families. Chantraine, Dimitroglou Rizell, Ghiggini and Golovko gave three long exact sequences in [6] in the same spirit as this paper but use different Morse functions to perturb the cobordism. The way we construct the pair of cobordisms allows us to have more control over the behavior of the Morse function. This turns out to be a key point toward proving the following surprising theorem.

Theorem 1.5 (see [Theorems 5.14](#) and [5.15](#)) *Let Σ be an exact Lagrangian cobordism with Maslov number 0 from a Legendrian knot Λ_- to a Legendrian knot Λ_+ . Then the A_∞ -category map $f: \mathcal{A}ug_+(\Lambda_-) \rightarrow \mathcal{A}ug_+(\Lambda_+)$ induced by the exact Lagrangian cobordism Σ is injective on the level of equivalence classes of objects. In addition, the corresponding cohomology category map $\tilde{f}: H^* \mathcal{A}ug_+(\Lambda_-) \rightarrow H^* \mathcal{A}ug_+(\Lambda_+)$ is faithful. In particular, when $\chi(\Sigma) = 0$, the functor \tilde{f} is fully faithful.*

By Ng, Rutherford, Shende, Sivek and Zaslow [33], for a Legendrian knot with a single basepoint, two augmentations are equivalent if and only if they are isomorphic as DGA maps. This theorem tells us that the number of augmentations of Λ_- is smaller than or equal to the number of augmentations of Λ_+ up to equivalence. However, in general, it is hard to count the number of augmentations of Λ up to equivalence. Ng, Rutherford, Shende and Sivek [32] introduced a new way to count the augmentations, called the homotopy cardinality, which is related to the ruling polynomial. Recall that the ruling polynomial is defined by $R_\Lambda(z) = \sum_R z^{-\chi(R)}$, where the sum is over all normal rulings R of Λ (see Chekanov [8] for the detailed definition). This invariant is much easier to compute than the augmentation equivalence class. Using [Theorem 1.5](#), we have the following corollary.

Corollary 1.6 (see [Corollary 5.17](#)) *Suppose there exists a spin exact Lagrangian cobordism with Maslov number 0 from a Legendrian knot Λ_- to a Legendrian knot Λ_+ . Then the ruling polynomials R_{Λ_-} and R_{Λ_+} satisfy*

$$R_{\Lambda_-}(q^{1/2} - q^{-1/2}) \leq q^{-\chi(\Sigma)/2} R_{\Lambda_+}(q^{1/2} - q^{-1/2})$$

for any q that is a power of a prime number.

This corollary gives a new obstruction to the existence of exact Lagrangian cobordisms. In particular, we have a new and simpler proof to the fact given by Chantraine [5] that there does not exist an exact Lagrangian cobordism from the Legendrian $m(9_{46})$ knot shown in Figure 18 to the Legendrian unknot. This fact is crucial to prove that Lagrangian concordance is not a symmetric relation.

Another important step toward proving the injectivity and faithfulness in Theorem 1.5 is to understand the differential map of the Cthulhu chain complex better. Analogous to a result for Legendrian submanifolds in Ekholm, Etnyre and Sabloff [16], we give a bijective correspondence between rigid holomorphic disks with boundary on a 2-copy of Σ and rigid holomorphic disks with boundary on Σ together with Morse flow lines. With this in hand, we can decompose the Cthulhu chain complex in various ways and recover the three long exact sequences in Chantraine, Dimitroglou Rizell, Ghiggini and Golovko [6].

Outline In Section 2, we review the Chekanov–Eliashberg DGA of a Legendrian submanifold and the DGA map induced by a Lagrangian cobordism. In Section 3, we introduce the augmentation category for a Legendrian submanifold and describe the A_∞ -category map induced by an exact Lagrangian cobordism. In Section 4, we review the Floer theory of Lagrangian cobordisms. Finally, using the techniques in Section 4, we prove the main result Theorem 1.1 in Section 5 and discuss its applications.

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2 Legendrian contact homology DGA and exact Lagrangian cobordisms

2.1 The Legendrian contact homology DGA

In this section, we review the Legendrian contact homology DGA from the geometric perspective of [19] and the combinatorial perspective of [33, Section 2.2.1]. We refer readers to [7; 23; 30] for a more detailed introduction.

Let Λ be a Legendrian submanifold in the standard contact space $(\mathbb{R}^3, \xi = \ker \alpha)$, where $\alpha = dz - y dx$. For simplicity when defining the degree, we assume throughout the paper that Λ has rotation number 0.

Let $(\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)]), \partial)$ denote the Legendrian contact homology DGA of Λ , which is also called the Chekanov–Eliashberg DGA. The underlying algebra $\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)])$ is a noncommutative unital graded algebra over a field \mathbb{F} generated by

$$\{c_1, \dots, c_m, t_1, t_1^{-1}, \dots, t_M, t_M^{-1}\}$$

with relations $t_i t_i^{-1} = 1$ for $i = 1, \dots, M$. Here c_1, \dots, c_m are Reeb chords of Λ and $\{t_1, \dots, t_M\}$ is a basis of the singular homology $H_1(\Lambda)$. The grading of a Reeb chord c is defined as

$$|c| = \text{CZ}(\gamma_c) - 1,$$

where γ_c is a capping path for c and CZ is the Conley–Zehnder index introduced in [17]. See [13, Section 4.1] for the way to choose a capping path γ_c for a Reeb chord of a Legendrian link. The grading of a Reeb chord depends on the choice of capping paths, but the difference between two Reeb chords’ gradings is independent of the choice of capping paths. Furthermore, set the grading of t_i to be zero for $i = 1, \dots, M$, and then extend the definition of degree to $\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)])$ through the relation $|ab| = |a| + |b|$.

To define the differential ∂ , we need a *cylindrical almost complex structure* J on $(\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha))$, ie

- J is compatible with the symplectic form $d(e^t \alpha)$;
- J is invariant under the action of \mathbb{R}_t ;
- $J(\partial_t) = \partial_z$ and $J(\xi) = \xi$.

For a generic choice of cylindrical almost complex structure J , the differential ∂ is defined by counting rigid J –holomorphic disks in $(\mathbb{R}_t \times \mathbb{R}^3, d(e^t \alpha))$ with boundary on $\mathbb{R} \times \Lambda$. See Figure 3 for an example. For Reeb chords a, b_1, \dots, b_m of Λ , let $\mathcal{M}(a; b_1, \dots, b_m)$ denote the moduli space of J –holomorphic disks

$$u: (D_{m+1}, \partial D_{m+1}) \rightarrow (\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda)$$

such that

- D_{m+1} is a 2–dimensional unit disk with $m + 1$ boundary points p, q_1, \dots, q_m removed and the points p, q_1, \dots, q_m are labeled in a counterclockwise order;
- u is asymptotic to $[0, \infty) \times a$ at p ;
- u is asymptotic to $(-\infty, 0] \times b_i$ at q_i .

Let $\widetilde{\mathcal{M}}(a; b_1, \dots, b_m)$ denote the quotient of $\mathcal{M}(a; b_1, \dots, b_m)$ by vertical translation of \mathbb{R}_t . When $\dim \widetilde{\mathcal{M}}(a; b_1, \dots, b_m) = 0$, the disk $u \in \mathcal{M}(a; b_1, \dots, b_m)$ is called *rigid*. The gradings of corresponding Reeb chords satisfy

$$|a| - |b_1| - \dots - |b_m| = 1.$$

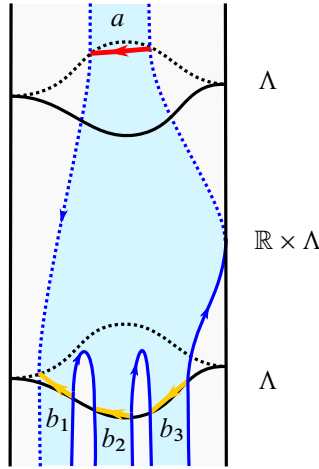


Figure 3: An example of a J -holomorphic disk with boundary on $\mathbb{R} \times \Lambda$. The arrows on the Reeb chords indicate the orientations of the Reeb chords, while the arrows on the disk boundary indicate the orientation inherited from the unit disk boundary with counterclockwise orientation through u .

For the image of the boundary segment from q_i to q_{i+1} under u , one can close it up on $\mathbb{R} \times \Lambda$ in a particular way as described in [19, Section 3.2] and take the homology class of this curve in $H_1(\Lambda)$, denoted by τ_i . Here we use $q_0 = q_{m+1} = p$. Moreover, if Λ is spin, all the relevant moduli spaces of J -holomorphic disks admit a coherent orientation. Hence, one can associate a sign $s(u)$ to each rigid J -holomorphic disk u . In this way, associate the rigid J -holomorphic disk u with a monomial

$$w(u) = s(u)\tau_0 b_1 \tau_1 \cdots b_m \tau_m.$$

We call the homology classes τ_i , for $i = 1, \dots, m$, the *coefficients* of $w(u)$. The *differential* on Reeb chords is defined by counting rigid J -holomorphic disks:

$$\partial a = \sum_{\dim \tilde{\mathcal{M}}(a; b_1, \dots, b_m) = 0} \sum_{u \in \mathcal{M}(a; b_1, \dots, b_m)} w(u).$$

Let $\partial \tau_i = \partial \tau_i^{-1} = 0$ for $i = 1, \dots, M$ and extend the differential to $\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)])$ through the Leibniz rule

$$\partial(xy) = (\partial x)y + (-1)^{|x|}x(\partial y).$$

An implicit condition for J -holomorphic disks is the positive energy constraint. For a Reeb chord c , define the action of c by

$$a(c) = \int_c \alpha,$$

which is the length of the Reeb chord c . The energy $E(u)$ of a J -holomorphic disk $u \in \mathcal{M}(a; b_1, \dots, b_m)$ satisfies

$$E(u) = \alpha(a) - \alpha(b_1) - \dots - \alpha(b_m).$$

Therefore, to make each J -holomorphic disk have positive energy, we must have

$$\alpha(b_1) + \dots + \alpha(b_m) < \alpha(a).$$

There is an equivalent definition from the combinatorial perspective. Project Λ onto the xy -plane to get the Lagrangian projection $\pi_{xy}(\Lambda)$ of Λ . After possibly perturbing Λ , we can assume that there is a one-to-one correspondence between the double points of $\pi_{xy}(\Lambda)$ and the Reeb chords of Λ . Suppose Λ is an M -component Legendrian link. Decorate the diagram with an orientation and a set of *minimum basepoints* $\{*_1, \dots, *_M\}$, ie

- there is exactly one point in $\{*_1, \dots, *_M\}$ on each component of Λ , and
- the set $\{*_1, \dots, *_M\}$ does not include any end points of Reeb chords of Λ .

The graded algebra $\mathcal{A}(\Lambda, *_1, \dots, *_M)$ is a noncommutative unital graded algebra over a field \mathbb{F} generated by

$$\{c_1, \dots, c_m, t_1, t_1^{-1}, \dots, t_M, t_M^{-1}\} \text{ with relations } \{t_i t_i^{-1} = 1 \mid i = 1, \dots, M\},$$

where c_1, \dots, c_m are double points of $\pi_{xy}(\Lambda)$ and t_1, \dots, t_M correspond to the basepoints $*_1, \dots, *_M$. The grading is defined the same as above. For the unit disk D_{m+1} as defined above, consider $\Delta(a; b_1, \dots, b_m)$, the space of orientation-preserving smooth immersions up to parametrization

$$u: (D_{m+1}, \partial D_{m+1}) \rightarrow (\mathbb{R}^2, \pi_{xy}(\Lambda))$$

with the following properties:

- u can be extended to the unit disk $\overline{D_{m+1}}$ continuously.
- $u(p) = a$ and the neighborhood of a in the image of u is a single positive quadrant (see Figure 4).
- $u(q_i) = b_i$ and the neighborhood of b_i in the image of u is a single negative quadrant for $1 \leq i \leq m$ (see Figure 4).

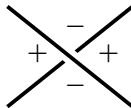


Figure 4: At each crossing, the quadrants labeled with a + sign are called *positive quadrants* while the other two are called *negative quadrants*.

If, when traversing $\overline{\partial D_{m+1}}$ counterclockwise from a , one encounters Reeb chords and basepoints in a sequence s_1, \dots, s_l , then we associate u with a monomial $w(u) = s(u)w(s_1) \cdots w(s_l)$, where:

- $s(u)$ is the sign associated to the disk u induced from the moduli space coherent orientation.
- If s_i is a Reeb chord b_j , then $w(s_i) = b_j$.
- If s_i is a basepoint $*_j$ for some $j = 1, \dots, M$, then $w(s_i) = t_j$ if the orientation of the boundary agrees with the orientation of the link and $w(s_i) = t_j^{-1}$ otherwise.

Define the differential on generators as follows:

$$\partial a = \sum_{|a| - \sum |b_i| = 1} \sum_{u \in \Delta(a; b_1, \dots, b_m)} w(u) \quad \text{and} \quad \partial t_j = \partial t_j^{-1} = 0 \quad \text{for } j = 1, \dots, M.$$

This can be extended to the whole DGA through the Leibniz rule.

For all the definitions of DGAs (\mathcal{A}, ∂) above, the differential ∂ has degree -1 and satisfies $\partial^2 = 0$ [7; 23]. Up to stable tame isomorphism, the Legendrian contact homology DGA is an invariant of Λ under Legendrian isotopy. In this sense of equivalence, the combinatorial definition does not depend on the choice of basepoints [31].

However, the homology of the DGA is hard to compute in general. Let us introduce augmentations of a DGA and use that to deduce linearized contact homology, which is much easier to compute. Let (\mathcal{A}, ∂) be a DGA over a field \mathbb{F} of a Legendrian link Λ with basepoints. A graded augmentation of \mathcal{A} is a DGA map

$$\epsilon: (\mathcal{A}, \partial) \rightarrow (\mathbb{F}, 0),$$

where $(\mathbb{F}, 0)$ is a chain complex that is \mathbb{F} in degree 0 and is 0 in other degrees. In other words, a graded augmentation is an algebra map $\epsilon: \mathcal{A} \rightarrow \mathbb{F}$ such that $\epsilon(1) = 1$, $\epsilon \circ \partial = 0$ and $\epsilon(a) = 0$ if $|a| \neq 0$.

Given a graded augmentation ϵ , define $\mathcal{A}^\epsilon := \mathcal{A} \otimes \mathbb{F} / (t_i = \epsilon(t_i))$. Notice that the differential ∂ descends to \mathcal{A}^ϵ since $\partial(t_i) = 0$. Elements in \mathcal{A}^ϵ are summands of words of Reeb chords. Let C be a free \mathbb{F} -module generated by Reeb chords. We can decompose \mathcal{A}^ϵ in terms of word length as $\mathcal{A}^\epsilon = \bigoplus_{n \geq 0} C^{\otimes n}$. Let \mathcal{A}_+^ϵ be the part of \mathcal{A}^ϵ containing the words with length at least 1, ie $\mathcal{A}_+^\epsilon = \bigoplus_{n \geq 1} C^{\otimes n}$. Consider a new differential $\partial^\epsilon: \mathcal{A}^\epsilon \rightarrow \mathcal{A}^\epsilon$ given by

$$\partial^\epsilon := \phi_\epsilon \circ \partial \circ \phi_\epsilon^{-1},$$

where $\phi_\epsilon: \mathcal{A}^\epsilon \rightarrow \mathcal{A}^\epsilon$ is an automorphism defined by $\phi_\epsilon(a) = a + \epsilon(a)$. Observe that ∂^ϵ preserves \mathcal{A}_+^ϵ and does not decrease the minimal length of a word. Thus, it descends to a differential on $\mathcal{A}_+^\epsilon / (\mathcal{A}_+^\epsilon)^2 \cong C$. The homology of (C, ∂^ϵ) is called the

linearized contact homology of Λ with respect to ϵ , denoted by $LCH_*^\epsilon(\Lambda)$. The chain complex (C, ∂^ϵ) is called the *linearized contact homology chain complex*.

2.2 Exact Lagrangian cobordisms

We now review the DGA map induced by an exact Lagrangian cobordism [19] with coefficients in general fields (following the orientation convention of [18]). In other words, an exact Lagrangian cobordism Σ from Λ_- to Λ_+ gives a DGA map

$$\phi_\Sigma: \mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\Sigma)]) \rightarrow \mathcal{A}(\Lambda_-; \mathbb{F}[H_1(\Sigma)]).$$

As required in Section 3, we restrict to the case where Λ_+ and Λ_- are Legendrian knots with a single basepoint, denoted by $*_+$ and $*_-$, respectively. We modify the DGA map such that the coefficients only depend on the basepoints but not depend on the cobordism, ie we get a DGA map

$$\phi_\Sigma: \mathcal{A}(\Lambda_+, *_+) \rightarrow \mathcal{A}(\Lambda_-, *_-).$$

Definition 2.1 Suppose Λ_\pm are Legendrian submanifolds in $(\mathbb{R}^3, \ker \alpha)$, where $\alpha = dz - y dx$. An exact Lagrangian cobordism Σ from Λ_- to Λ_+ is a 2-dimensional surface in $(\mathbb{R} \times \mathbb{R}^3, \omega = d(e^t \alpha))$ (see Figure 1) such that for some big number $N > 0$,

- $\Sigma \cap ((N, \infty) \times \mathbb{R}^3) = (N, \infty) \times \Lambda_+$,
- $\Sigma \cap ((-\infty, -N) \times \mathbb{R}^3) = (-\infty, -N) \times \Lambda_-$, and
- $\Sigma \cap ([-N, N] \times \mathbb{R}^3)$ is compact.

Moreover, there exists a smooth function $g: \Sigma \rightarrow \mathbb{R}$ such that

$$e^t \alpha|_{T\Sigma} = dg$$

and g is constant when $t \leq -N$ and $t \geq N$. The function g is called a *primitive* of Σ .

For a spin exact Lagrangian cobordism Σ from Λ_- to Λ_+ , the Legendrian submanifolds Λ_\pm inherit induced spin structures. Hence Λ_\pm have $\mathbb{F}[H_1(\Lambda_\pm)]$ -coefficients DGAs $(\mathcal{A}(\Lambda_\pm; \mathbb{F}[H_1(\Lambda_\pm)]), \partial)$, respectively, as described in Section 2.1. Ekholm, Honda and Kálmán in [19] showed that an exact Lagrangian cobordism Σ induces a DGA map from $\mathcal{A}(\Lambda_+)$ to $\mathcal{A}(\Lambda_-)$ with $\mathbb{F}[H_1(\Sigma)]$ coefficients. In order to see that, first, we need to view the DGAs of Λ_\pm as DGAs with $\mathbb{F}[H_1(\Sigma)]$ coefficients. Notice that the inclusion $H_1(\Lambda_\pm) \hookrightarrow H_1(\Sigma)$ induces a canonical inclusion map $\mathbb{F}[H_1(\Lambda_\pm)] \hookrightarrow \mathbb{F}[H_1(\Sigma)]$ of the group ring coefficients, which makes it natural to consider the DGAs of Λ_\pm with $\mathbb{F}[H_1(\Sigma)]$ coefficients. Specifically, the new DGA $\mathcal{A}(\Lambda_\pm; \mathbb{F}[H_1(\Sigma)])$ is generated by Reeb chords of Λ_\pm and elements in $H_1(\Sigma)$ over \mathbb{F} . The differential is defined by the original differential in $\mathcal{A}(\Lambda_\pm; \mathbb{F}[H_1(\Lambda_\pm)])$ composed with the inclusion map $H_1(\Lambda_\pm) \hookrightarrow H_1(\Sigma)$.

Second, construct a DGA map with $\mathbb{F}[H_1(\Sigma)]$ coefficients. Consider an almost complex structure J that is compatible with the symplectic form ω and is cylindrical on both ends. In other words, J matches the cylindrical almost complex structures on both cylindrical ends. Fix a generic choice of such an almost complex structure J . For Reeb chords a of Λ_+ and b_1, \dots, b_m of Λ_- , define $\mathcal{M}(a; b_1, \dots, b_m)$ to be the moduli space of the J -holomorphic disks

$$u: (D_{m+1}, \partial D_{m+1}) \rightarrow (\mathbb{R} \times \mathbb{R}^3, \Sigma)$$

such that

- D_{m+1} is a 2-dimensional unit disk with $m + 1$ boundary points p, q_1, \dots, q_m removed and the points p, q_1, \dots, q_m are arranged in a counterclockwise order;
- u is asymptotic to $[N, \infty) \times a$ at p ;
- u is asymptotic to $(-\infty, -N] \times b_i$ at q_i .

When $\dim \mathcal{M}(a; b_1, \dots, b_m) = 0$, the disk $u \in \mathcal{M}(a; b_1, \dots, b_m)$ is called *rigid*. The gradings of corresponding Reeb chords satisfy

$$|a| - |b_1| - \dots - |b_m| = 0.$$

For the image of the boundary segment from q_i to q_{i+1} , one can close up in a similar way as the one in the definition of the DGA differential and take the homology class in $H_1(\Sigma)$, denoted by τ_i . If Σ is spin, all the relevant moduli spaces of J -holomorphic disks admit a coherent orientation. In particular, each rigid J -holomorphic disk obtains a sign, denoted by $s(u)$. Associate a monomial $w(u)$ to the J -holomorphic disk u as

$$w(u) = s(u)\tau_0 b_1 \tau_1 \cdots b_m \tau_m.$$

The homology classes τ_i , for $i = 1, \dots, m$, are called the *coefficients* of $w(u)$. The DGA map is defined by counting rigid J -holomorphic disks with boundary on Σ :

$$\phi(a) = \sum_{\dim \mathcal{M}(a; b_1, \dots, b_m) = 0} \sum_{u \in \mathcal{M}(a; b_1, \dots, b_m)} w(u).$$

We can extend the morphism to $\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\Sigma)])$ by setting $\phi(t) = t$ for any generator t in $H_1(\Sigma)$ and applying the Leibniz rule.

In order to modify the coefficients of the DGA map ϕ , let us consider $H_1(\Sigma)$ more precisely. To simplify the description, we restrict Σ to $[-N, N] \times \mathbb{R}^3$ and denote it by Σ as well. According to Poincaré duality, $H^1(\Sigma) \cong H_1(\Sigma, \Lambda_+ \cup \Lambda_-)$. In particular, for any loop α in Σ with ends on $\Lambda_+ \cup \Lambda_-$ which is an element in $H_1(\Sigma, \Lambda_+ \cup \Lambda_-)$ there is an element θ_α in $H^1(\Sigma)$ such that for any oriented loop γ on Σ , the intersection

number of α and γ is $\theta_\alpha(\gamma)$. Thus, in order to know the homology class of a curve γ in $H_1(\Sigma)$, we only need to count the intersection number of each generator curve of $H_1(\Sigma, \Lambda_+ \cup \Lambda_-)$ with γ .

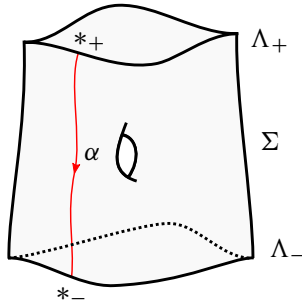


Figure 5: Curve α on a cobordism

Consider a connected exact Lagrangian cobordism Σ from a Legendrian knot Λ_- to a Legendrian knot Λ_+ (see Remark 5.3 for the reason that we assume that Σ is connected). Choose basepoints $*_+$ and $*_-$ for Λ_+ and Λ_- , respectively. There exists a curve α on Σ from $*_+$ to $*_-$ with exactly one intersection with Λ_+ and Λ_- , respectively. An example is shown in Figure 5. Let V^* denote the subgroup of $H^1(\Sigma)$ that is generated by the Poincaré dual of curve α . The dual space V in $H_1(\Sigma)$ is isomorphic to \mathbb{Z} .

Now we can modify the DGA map ϕ described above to be a map from $\mathcal{A}(\Lambda_+; \mathbb{F}[V])$ to $\mathcal{A}(\Lambda_-; \mathbb{F}[V])$. First, restrict the generators of $\mathcal{A}(\Lambda_\pm)$ to Reeb chords of Λ_\pm and a basis of V . Second, project the coefficients τ_i of the monomial $w(u)$ from $H_1(\Sigma)$ to V . Therefore, the DGA map works in $\mathbb{F}[V]$ coefficients. Indeed, the definitions of $\mathcal{A}(\Lambda_\pm; \mathbb{F}[V])$ match the definition of $\mathcal{A}(\Lambda_\pm, *_\pm)$, respectively. Hence a connected exact Lagrangian cobordism Σ induces a DGA map with $\mathbb{F}[V]$ coefficients from the DGA of Λ_+ with a single basepoint to the DGA of Λ_- with a single basepoint:

$$\phi: (\mathcal{A}(\Lambda_+, *_+), \partial) \rightarrow (\mathcal{A}(\Lambda_-, *_-), \partial).$$

This DGA map does depend on the choice of the curve α connecting the two basepoints.

3 The augmentation category

3.1 A_∞ categories

In this section, we give a lightning review of A_∞ algebras and A_∞ categories, following [33]. See [26; 25] for a more detailed introduction.

Definition 3.1 [26, Section 3.1] An A_∞ algebra over a field \mathbb{F} is a \mathbb{Z} -graded vector space A endowed with degree- $(2-n)$ maps $m_n: A^{\otimes n} \rightarrow A$ such that

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0.$$

The most important things we need among these complicated relations are:

- m_1 is a differential on A (ie $m_1^2 = 0$).
- m_2 is associative after passing to the homology with respect to m_1 .

An A_∞ algebra can be achieved nicely through the following construction. Let $\bar{T}(C) = \bigoplus_{n \geq 1} C^{\otimes n}$ be a graded vector space over \mathbb{F} equipped with a *codifferential* b , ie

- b has degree 1,
- $b^2 = 0$,
- $b = \bigoplus b_n$, where b_n is a map $C^{\otimes n} \rightarrow C$, and
- b satisfies the co-Leibniz rule

$$\Delta b = (1 \otimes b + b \otimes 1)\Delta,$$

where $\Delta(a_1 \otimes \dots \otimes a_n) = \sum_{i=1}^n (a_1 \otimes \dots \otimes a_i) \otimes (a_{i+1} \otimes \dots \otimes a_n)$.

Let $C^\vee := C[-1]$ and let $s: C \rightarrow C^\vee$ be the canonical degree-1 identification map $a \mapsto a$. Define maps $m_n: (C^\vee)^{\otimes n} \rightarrow C^\vee$ such that the following diagram commutes for all n :

$$\begin{array}{ccc} C^{\otimes n} & \xrightarrow{b_n} & C \\ s^{\otimes n} \downarrow & & \downarrow s \\ (C^\vee)^{\otimes n} & \xrightarrow{m_n} & C^\vee \end{array}$$

Then C^\vee is an A_∞ algebra with the m_n as A_∞ operations [36; 37]. One can check that the degree of m_n is $2 - n$.

Example 3.2 If a Legendrian contact homology DGA $(\mathcal{A}(\Lambda), \partial)$ has an augmentation ϵ , the conjugated differential ∂^ϵ is a differential of $\mathcal{A}_+^\epsilon = \bigoplus_{n \geq 1} C^{\otimes n} = \bar{T}(C)$, where C is the vector space over a field \mathbb{F} generated by Reeb chords of Λ . We define δ^ϵ to be the adjoint of ∂^ϵ on $\bar{T}(C^*) = \bigoplus_{n \geq 1} (C^*)^{\otimes n}$, where C^* is the dual of C . More specifically,

$$\delta^\epsilon(b_m^* \otimes \dots \otimes b_1^*) = \sum_a \text{Coeff}_{b_1 b_2 \dots b_m}(\partial^\epsilon(a)).$$

It is not hard to check that δ^ϵ is a codifferential of $\bar{T}(C^*)$. Hence one can use the construction above to construct an A_∞ algebra $(C^*)^\vee$.

Definition 3.3 [10] An A_∞ category over a field \mathbb{F} is a category where, for any two objects ϵ_1 and ϵ_2 , the morphism is a graded vector space $\text{Hom}(\epsilon_1, \epsilon_2)$. Moreover, for any objects $\epsilon_1, \epsilon_2, \dots, \epsilon_{n+1}$, there exists a degree- $(2-n)$ map

$$m_n: \text{Hom}(\epsilon_n, \epsilon_{n+1}) \otimes \cdots \otimes \text{Hom}(\epsilon_1, \epsilon_2) \rightarrow \text{Hom}(\epsilon_1, \epsilon_{n+1})$$

satisfying

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0.$$

As noted before, the first A_∞ operation m_1 is a differential for $\text{Hom}(\epsilon_1, \epsilon_2)$ with degree 1. Denote its cohomology by $H^* \text{Hom}(\epsilon_1, \epsilon_2)$. Moreover, we have that m_2 descends to an associative map on the cohomology level:

$$m_2: H^* \text{Hom}(\epsilon_2, \epsilon_3) \otimes H^* \text{Hom}(\epsilon_1, \epsilon_2) \rightarrow H^* \text{Hom}(\epsilon_1, \epsilon_3)$$

for any objects $\epsilon_1, \epsilon_2, \epsilon_3$.

An A_∞ morphism between two A_∞ categories $f: \mathcal{A} \rightarrow \mathcal{B}$ maps the object ϵ of \mathcal{A} to $f(\epsilon)$ of \mathcal{B} and for any objects $\epsilon_1, \epsilon_2, \dots, \epsilon_{n+1}$ of \mathcal{A} , there exists a map

$$f_n: \text{Hom}(\epsilon_n, \epsilon_{n+1}) \otimes \cdots \otimes \text{Hom}(\epsilon_1, \epsilon_2) \rightarrow \text{Hom}(f(\epsilon_1), f(\epsilon_{n+1}))$$

satisfying the A_∞ relations [26]. In particular, the first map f_1 , called *the category map on the level of morphisms*, maps the morphism $\text{Hom}(\epsilon_1, \epsilon_2)$ of \mathcal{A} to the morphism $\text{Hom}(f(\epsilon_1), f(\epsilon_2))$ of \mathcal{B} . From the A_∞ relations, we know that:

- The functor f_1 , the category map on the level of morphisms, commutes with m_1 and thus f_1 descends to a map on cohomology

$$f^*: H^* \text{Hom}(\epsilon_1, \epsilon_2) \rightarrow H^* \text{Hom}(f(\epsilon_1), f(\epsilon_2)).$$

- For any $a \in \text{Hom}(\epsilon_2, \epsilon_3)$ and $b \in \text{Hom}(\epsilon_1, \epsilon_2)$, we have

$$f^*(m_2([a], [b])) = m_2(f^*[a], f^*[b]),$$

ie, the composition map m_2 commutes with f^* when passing to the cohomology level.

An A_∞ morphism between two A_∞ categories $f: \mathcal{A} \rightarrow \mathcal{B}$ induces a functor on the cohomology categories, $\tilde{f}: H^* \mathcal{A} \rightarrow H^* \mathcal{B}$. It behaves the same as f on the object level. On the level of morphisms $\tilde{f} = f^*$. The functor \tilde{f} is *faithful* if f^* is injective and is *fully faithful* if f^* is an isomorphism for any morphism in $H^* \mathcal{A}$.

3.2 The augmentation category

In this section, we briefly review the augmentation category $\mathcal{A}ug_+(\Lambda)$, following [33].

Let Λ be an oriented Legendrian knot in $(\mathbb{R}^3, \ker \alpha)$ endowed with a single basepoint $*$. Denote its Legendrian contact homology DGA by (\mathcal{A}, ∂) . Given a field \mathbb{F} , the objects of the augmentation category $\text{Aug}_+(\Lambda)$ are augmentations of (\mathcal{A}, ∂) to \mathbb{F} ,

$$\epsilon: \mathcal{A} \rightarrow \mathbb{F}.$$

In order to describe the morphism $\text{Hom}_+(\epsilon_1, \epsilon_2)$ for any two objects ϵ_1 and ϵ_2 , we need to study the DGA of a 2-copy of Λ , denoted by $\Lambda^{(2)}$.

By the Weinstein tubular neighborhood theorem, we can identify a neighborhood of Λ with a neighborhood of the zero section in the 1-jet space $J^1(\Lambda) = T^*(\Lambda) \times \mathbb{R}$ through a contactomorphism. The contact form in $J^1(\Lambda)$ is $\alpha = dz - p dq$, where q is the coordinate on Λ and p is the coordinate in the cotangent direction. For any C^1 small function $f: \Lambda \rightarrow \mathbb{R}$, the 1-jet $j^1 f = \{(q, f'(q), f(q)) \mid q \in \Lambda\}$ is a Legendrian knot in $J^1(\Lambda)$ and thus is a Legendrian knot in \mathbb{R}^3 . Now choose a particular Morse function $f: \Lambda \rightarrow (0, \delta)$ such that

- δ is smaller than the minimum length of Reeb chords of Λ ,
- the Morse function f has exactly 1 local maximum point at x and 1 local minimum point at y , and
- around the basepoint $*$, the three points $*$, x , y show up in order when traveling along the link (see Figure 6).

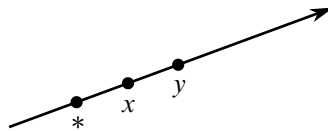


Figure 6: A neighborhood of the basepoint $*$ on Λ . The arrow indicates the orientation of Λ .

Decorate $j^1 f$ with a basepoint in the same location and with the same orientation as Λ . Now $\Lambda \cup j^1 f$ is a 2-copy of Λ , denoted by $\Lambda^{(2)}$. Label $\Lambda^{(2)}$ from top (higher z coordinate) to bottom (lower z coordinate) by Λ^1 and Λ^2 . An example of the 2-copy of the trefoil with a single basepoint is shown in Figure 7.

The Legendrian contact homology DGA $(\mathcal{A}(\Lambda^{(2)}), \partial^{(2)})$ of $\Lambda^{(2)}$ can be recovered from the data carried by the DGA $(\mathcal{A}(\Lambda), \partial)$ of Λ . Recall that $\mathcal{A}(\Lambda)$ is generated by the set \mathcal{R} of Reeb chords $\{a_1, \dots, a_m\}$ and the set $\mathcal{T} = \{t, t^{-1}\}$ that corresponds to the basepoint as stated in Section 2.1. Similarly, divide the set of generators of $\mathcal{A}(\Lambda^{(2)})$ into two parts $\mathcal{R}^{(2)}$ and $\mathcal{T}^{(2)}$. It is obvious that $\Lambda^{(2)}$ has two basepoints, and thus we write $\mathcal{T}^{(2)}$ as $\{(t^1)^{\pm 1}, (t^2)^{\pm 1}\}$. As for the set of Reeb chords $\mathcal{R}^{(2)}$, we divide it into four parts: $\mathcal{R}^{(2)} = \bigcup_{i,j=1,2} \mathcal{R}^{ij}$, where \mathcal{R}^{ij} is the set of Reeb chords to Λ_j from Λ_i .

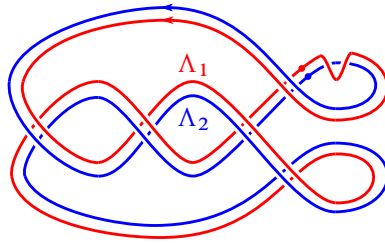


Figure 7: The Lagrangian projection of a 2-copy of the trefoil with a single basepoint

Observe that Reeb chords of $\Lambda^{(2)}$ come from two sources:

- Each critical point x or y of the Morse function f gives one Reeb chord in \mathcal{R}^{12} , denoted by x^{12} and y^{12} , respectively. We call these Reeb chords *Morse Reeb chords*.
- Each Reeb chord a_l of Λ gives four Reeb chords of $\Lambda^{(2)}$, denoted by $a_l^{ij} \in \mathcal{R}^{ij}$, where $i, j = 1, 2$ and $l = 1, \dots, m$. We call these Reeb chords *non-Morse Reeb chords*.

It is obvious that a^{ii} and t^i , for $i = 1, 2$, inherit the grading from a and t in $\mathcal{A}(\Lambda)$, respectively. We can choose a family of capping paths such that $|a^{ij}| = |a|$ for any Reeb chord a of Λ . Under this choice of capping paths γ , one can show that $\text{CZ}(\gamma_{x^{12}}) = \text{Ind}_f(x)$ for any Morse Reeb chord x^{12} through a computation similar to that in [16]. Hence we have $|x^{12}| = 0$ and $|y^{12}| = -1$.

In order to describe the differential $\partial^{(2)}$, we encode the generators in matrices. Let A_l , for $1 \leq l \leq m$, and X, Y, Δ be 2×2 matrices given by

$$A_l = \begin{pmatrix} a_l^{11} & a_l^{12} \\ a_l^{21} & a_l^{22} \end{pmatrix}, \quad X = \begin{pmatrix} 1 & x^{12} \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & y^{12} \\ 0 & 0 \end{pmatrix}, \quad \Delta = \begin{pmatrix} t^1 & 0 \\ 0 & t^2 \end{pmatrix}.$$

The differential $\partial^{(2)}$ is defined on generators as follows by applying it entry-by-entry to these matrices:

$$\begin{aligned} \partial^{(2)} A_l &= \Phi(\partial a_l) + Y A_l - (-1)^{|a_l|} A_l Y, \\ \partial^{(2)} X &= \Delta^{-1} Y \Delta X - X Y, \\ \partial^{(2)} Y &= Y^2, \\ \partial^{(2)} \Delta &= 0, \end{aligned}$$

where $\Phi: \mathcal{A} \rightarrow \text{Mat}(2, \mathcal{A}(\Lambda^{(2)}))$ is a ring homomorphism given by $\Phi(a_l) = A_l$ and $\Phi(t) = \Delta X$.

Given two augmentations ϵ^1 and ϵ^2 of (\mathcal{A}, ∂) , we get an augmentation ϵ of $(\Lambda^{(2)}, \partial^{(2)})$ by sending $a_l^{ii} \mapsto \epsilon^i(a_l)$ and $t^{ii} \mapsto \epsilon^i(t)$ and sending everything else to 0. Therefore

$\partial_\epsilon^{(2)} = \phi_\epsilon \circ \partial^{(2)} \circ \phi_\epsilon^{-1}$ is a differential of $\mathcal{A}^{(2)} = \mathcal{A}(\Lambda^{(2)})/(t^{ii} = \epsilon(t^{ii}))$. Both the morphism $\text{Hom}_+(\epsilon_1, \epsilon_2)$ and the first A_∞ operation m_1 are defined from $(\mathcal{A}^{(2)}, \partial_\epsilon^{(2)})$ through the construction stated in Section 3.1. For $i, j = 1, 2$, let C^{ij} denote the free graded \mathbb{F} algebra generated by \mathcal{R}^{ij} , which is a subalgebra of $\mathcal{A}^{(2)}$. Notice that C^{12} and C^{21} are closed under $\partial_\epsilon^{(2)}$ since ϵ vanishes on the components in C^{11} and C^{22} of the image of $\partial_\epsilon^{(2)}$. Hence C^{12} and C^{21} are subcomplexes of $(\mathcal{A}^{(2)}, \partial_\epsilon^{(2)})$. Define the morphism $\text{Hom}_+(\epsilon_1, \epsilon_2)$ between objects ϵ_1 and ϵ_2 to be $(C^{12})^\vee$. To simplify the notation, we write $(a_l^{12})^\vee$ as a_l^\vee , $(x^{12})^\vee$ as x^\vee and $(y^{12})^\vee$ as y^\vee . Therefore, their gradings satisfy $|a_l^\vee| = |a_l| + 1$, $|x^\vee| = 1$ and $|y^\vee| = 0$. The first A_∞ operation m_1 is defined by the adjoint of $\partial_\epsilon^{(2)}$, ie for any Reeb chord $c \in \mathcal{R}$,

$$m_1(c^\vee) = \sum_{a \in \mathcal{R}} \text{Coeff}_c(\partial_\epsilon^{(2)} a) a^\vee.$$

As noted before, m_1 is a differential for $\text{Hom}_+(\epsilon_1, \epsilon_2)$. The corresponding cohomology is denoted by $H^* \text{Hom}_+(\epsilon_1, \epsilon_2)$. Similarly, define $\text{Hom}_-(\epsilon_2, \epsilon_1)$ to be $(C^{21})^\vee$. Take the cohomology of $\text{Hom}_-(\epsilon_2, \epsilon_1)$ with respect to m_1 , denoted by $H^* \text{Hom}_-(\epsilon_2, \epsilon_1)$.

Remark One may find the notational convention of $\text{Hom}_-(\epsilon_2, \epsilon_1)$ unnatural. However, the notations are consistent in the sense that both $\text{Hom}_+(\epsilon, \epsilon')$ and $\text{Hom}_-(\epsilon, \epsilon')$ are generated by Reeb chords from the component with the augmentation ϵ' to the component with the augmentation ϵ .

The $\text{Hom}_+(\epsilon_1, \epsilon_2)$ space and the $\text{Hom}_-(\epsilon_1, \epsilon_2)$ space are closely related. Recall that the generators of $\text{Hom}_+(\epsilon_1, \epsilon_2)$ naturally correspond to the Reeb chords in \mathcal{R}^{12} , which consist of non-Morse Reeb chords and Morse Reeb chords. Note that the lengths of Morse Reeb chords are smaller than the lengths of non-Morse Reeb chords. Due to the positive energy constraint, there does not exist any holomorphic disk that has a positive puncture at a Morse Reeb chord and a negative puncture at a non-Morse Reeb chord. Therefore, the graded vector subspace of $\text{Hom}_+(\epsilon_1, \epsilon_2)$ generated by non-Morse Reeb chords is closed under m_1 , and thus is a subcomplex. Indeed, this subcomplex agrees with $(\text{Hom}_-(\epsilon_1, \epsilon_2), m_1)$. From [27], for a Legendrian knot Λ with a single basepoint, any two augmentations ϵ_1 and ϵ_2 agree on the generator t that corresponds to the basepoint. As a result, by [33, Proposition 5.2], the quotient chain complex that is generated by $\{x^\vee, y^\vee\}$ is the Morse cochain complex induced by the Morse function f . Therefore we have the following long exact sequence:

$$(2) \quad \cdots \rightarrow H^{i-1}(\Lambda) \rightarrow H^i \text{Hom}_-(\epsilon_1, \epsilon_2) \rightarrow H^i \text{Hom}_+(\epsilon_1, \epsilon_2) \rightarrow H^i(\Lambda) \rightarrow \cdots.$$

Furthermore, given that both $\text{Hom}_+(\epsilon_1, \epsilon_2)$ and $\text{Hom}_-(\epsilon_1, \epsilon_2)$ are vector spaces over the field \mathbb{F} , combining the universal coefficient theorem with the Sabloff duality in

[33, Section 5.1.2], we have

$$(3) \quad H^k \text{Hom}_-(\epsilon_1, \epsilon_2) \cong H^{-k} (\text{Hom}_-(\epsilon_1, \epsilon_2)^\dagger) \cong H^{2-k} \text{Hom}_+(\epsilon_2, \epsilon_1).$$

For a chain complex C , the chain complex C^\dagger is obtained by dualizing the underlying vector space and differential of C and then negating the gradings.

For the other A_∞ operators m_n , one needs to consider an n -copy of Λ , denoted by $\Lambda^{(n)}$. Construct a DGA $(\mathcal{A}^n, \partial_\epsilon^{(n)})$ of $\Lambda^{(n)}$ that is analogous to $(\mathcal{A}^2, \partial_\epsilon^{(2)})$. Define m_n to be the adjoint of ∂_ϵ^n as in Example 3.2. See [33] for more details.

By [33], the augmentation category described above does not depend on the choice of the Morse function f . Moreover, up to A_∞ category equivalence, the augmentation category is invariant of Legendrian knot under Legendrian isotopy.

A key property of $\text{Aug}_+(\Lambda)$ is that $\text{Aug}_+(\Lambda)$ is a *strictly unital* A_∞ category, with the *units* given by

$$e_\epsilon = -y^\vee \in \text{Hom}_+(\epsilon, \epsilon),$$

ie

- $m_1(e_\epsilon) = 0$;
- for any ϵ_1, ϵ_2 and any $c \in \text{Hom}_+(\epsilon_1, \epsilon_2)$, we have $m_2(c, e_{\epsilon_1}) = m_2(e_{\epsilon_2}, c) = c$;
- any higher composition involving e_ϵ is 0.

As a result, the corresponding cohomology category $H^* \text{Aug}_+(\Lambda)$ is a unital category, which makes it natural to talk about the equivalence relation of objects in $\text{Aug}_+(\Lambda)$.

Definition 3.4 Two objects ϵ_1 and ϵ_2 are *equivalent* in $\text{Aug}_+(\Lambda)$ if they are isomorphic in the cohomology category $H^* \text{Aug}_+(\Lambda)$, ie if there exist $[\alpha] \in H^0 \text{Hom}_+(\epsilon_1, \epsilon_2)$ and $[\beta] \in H^0 \text{Hom}_+(\epsilon_2, \epsilon_1)$ such that $m_2([\alpha], [\beta]) = [e_{\epsilon_2}] \in H^0 \text{Hom}_+(\epsilon_2, \epsilon_2)$ and $m_2([\beta], [\alpha]) = [e_{\epsilon_1}] \in H^0 \text{Hom}_+(\epsilon_1, \epsilon_1)$:

$$e_{\epsilon_1} \left(\curvearrowright \epsilon_1 \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} \epsilon_2 \left(\curvearrowright \right) e_{\epsilon_2}$$

By [33], for a Legendrian knot with a single basepoint, two augmentations are equivalent if and only if they are isomorphic as DGA maps.

Suppose Σ is a connected exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+ . It induces a DGA map ϕ from the DGA $(\mathcal{A}(\Lambda_+), \partial)$ with a single basepoint to a DGA $(\mathcal{A}(\Lambda_-), \partial)$ with a single basepoint. By [33, Proposition 3.29], this DGA map ϕ induces a unital A_∞ category morphism f from $\text{Aug}_+(\Lambda_-)$ to $\text{Aug}_+(\Lambda_+)$. The category map sends an augmentation ϵ_- of Λ_-

to $\epsilon_+ = \epsilon_- \circ \phi$, which is an augmentation of Λ_+ . The family of maps $\{f_n\}$ is constructed through a family of DGA morphisms of n -copies:

$$\begin{aligned} f^{(n)}: (\mathcal{A}^{(n)}(\Lambda_+), \partial^{(n)}) &\mapsto (\mathcal{A}^{(n)}(\Lambda_-), \partial^{(n)}), \\ \Delta &\mapsto \Delta, \\ Y &\mapsto Y, \\ X &\mapsto \Delta^{-1} \cdot \Phi_- \circ f(t), \\ \Phi_+(a) &\mapsto \Phi_- \circ f(a) \quad \text{for } a \in \mathcal{A}(\Lambda_+). \end{aligned}$$

Let ϵ_- be the augmentation of $(\mathcal{A}^{(n)}(\Lambda_+), \partial^{(n)})$ that sends $a_i^{ii} \mapsto \epsilon_-^i(a_i)$, $t^{ii} \mapsto \epsilon_-^i(t)$ and sends everything else to 0. Define the map f_{n-1} to be the adjoint of $f_{\epsilon_-}^{(n)}$, where

$$f_{\epsilon_-}^{(n)} = \phi_{\epsilon_-} \circ f^{(n)} \circ \phi_{\epsilon_-}^{-1}.$$

In particular, f_1 can be written as

$$\begin{aligned} f_1: \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) &\rightarrow \text{Hom}_+(\epsilon_+^1, \epsilon_+^2), \\ y_-^\vee &\mapsto y_+^\vee, \\ c^\vee &\mapsto \sum_{a \in \mathcal{A}(\Lambda_+)} \text{Coeff}_c(f_{\epsilon_-}^{(2)}(a))a^\vee \quad \text{for } c \in \mathcal{A}(\Lambda_-), \\ x_-^\vee &\mapsto x_+^\vee + \sum_{a \in \mathcal{A}(\Lambda_+)} \text{Coeff}_t(f_{\epsilon_-}^{(2)}(a))a^\vee. \end{aligned} \tag{4}$$

When computing $\text{Coeff}_b(f_{\epsilon_-}^{(2)}(a))$, where b is either a Reeb chord $c \in \mathcal{A}(\Lambda_-)$ or $t \in \mathcal{T}$, one considers all the terms of $f(a)$ including b . If a term of $f(a)$ including b can be written as pbq , where p and q are words of pure Reeb chords of Λ_- , this term contributes $\text{Coeff}_{pbq}(f(a))\epsilon_-^1(p)\epsilon_-^2(q)$ to $\text{Coeff}_b(f_{\epsilon_-}^{(2)}(a))$. Therefore we have

$$\text{Coeff}_b(f_{\epsilon_-}^{(2)}(a)) = \sum_{p,q} \text{Coeff}_{pbq}(f(a))\epsilon_-^1(p)\epsilon_-^2(q).$$

Remark According to [33, Proposition 3.29], the condition for a DGA map to induce a unital A_∞ category morphism is that the DGA map is compatible with the weak link gradings in the sense of [33, Definition 3.19]. In our case, where both Λ_+ and Λ_- are single component Legendrian knots with a single basepoint, this condition is trivially satisfied.

4 Floer theory for Lagrangian cobordisms

In this section, we give a brief introduction to the Floer theory of a pair of exact Lagrangian cobordisms, following [6]. Let Σ^i , for $i = 1, 2$, be exact Lagrangian

cobordisms from Λ_-^i to Λ_+^i in $(\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha))$, where $\alpha = dz - y dx$. A schematic picture is shown in Figure 8. The union of the cobordisms $\Sigma^1 \cup \Sigma^2$ is cylindrical over $\Lambda_+^1 \cup \Lambda_+^2$ (resp. $\Lambda_-^1 \cup \Lambda_-^2$) on the positive end (resp. negative end). Viewing $\Sigma^1 \cup \Sigma^2$ as a Lagrangian cobordism from the Legendrian link $\Lambda_-^1 \cup \Lambda_-^2$ to the Legendrian $\Lambda_+^1 \cup \Lambda_+^2$, we obtain a chain complex generated by Reeb chords of $\Lambda_-^1 \cup \Lambda_-^2$ and $\Lambda_+^1 \cup \Lambda_+^2$. On the other hand, if we lift the exact Lagrangian cobordism $\Sigma^1 \cup \Sigma^2$ to a Legendrian manifold in $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$, we have its Legendrian contact homology DGA, which is generated by double points of $\Sigma^1 \cup \Sigma^2$. One can construct the Cthulhu chain complex $\text{Cth}(\Sigma^1, \Sigma^2)$ as a mix of the two chain complexes above. It is generated by some Reeb chords on the cylindrical ends and intersection points of Σ^1 and Σ^2 . Moreover, this chain complex has trivial cohomology, ie $H^* \text{Cth}(\Sigma^1, \Sigma^2) = 0$.

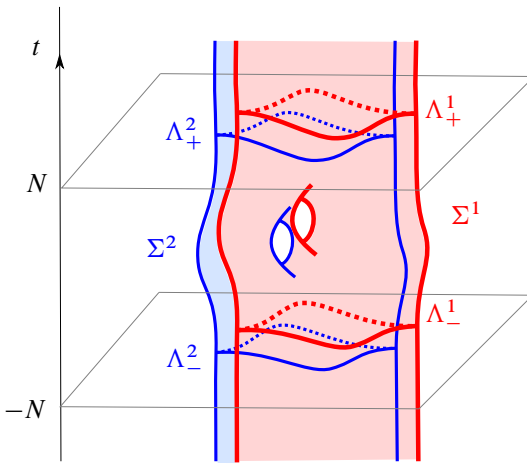


Figure 8: Pair of Lagrangian cobordisms in $(\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha))$

For simplicity in defining gradings, we assume that Σ^i , for $i = 1, 2$, has trivial Maslov number throughout this paper.

4.1 The graded vector space

Assume that both Σ^1 and Σ^2 are cylindrical outside $[-N, N] \times \mathbb{R}^3$, where N is a positive number. The underlying vector space is a direct sum of three parts:

$$\text{Cth}(\Sigma^1, \Sigma^2) = C(\Lambda_+^1, \Lambda_+^2) \oplus \text{CF}(\Sigma^1, \Sigma^2) \oplus C(\Lambda_-^1, \Lambda_-^2).$$

The top level $C(\Lambda_+^1, \Lambda_+^2)$ (resp. bottom level $C(\Lambda_-^1, \Lambda_-^2)$) is an \mathbb{F} -module generated by Reeb chords to Λ_+^1 (resp. Λ_-^1) from Λ_+^2 (resp. Λ_-^2) that are lying on the slice of $t = N$ (resp. $t = -N$). The middle level $\text{CF}(\Sigma^1, \Sigma^2)$ is an \mathbb{F} -module generated by intersection points of Σ^1 and Σ^2 , which are all contained in $(-N, N) \times \mathbb{R}^3$.

Grading To define the degree, first fix a capping path γ_c for each generator c . For a Reeb chord a in $C(\Lambda_+^1, \Lambda_+^2)$ or $C(\Lambda_-^1, \Lambda_-^2)$, define the degree $|a|$ by

$$|a| = \text{CZ}(\gamma_a) - 1,$$

which matches the definition of degree when viewing a as a generator in the Legendrian contact homology DGA of $\Lambda_+^1 \cup \Lambda_+^2$ or $\Lambda_-^1 \cup \Lambda_-^2$. For an intersection point $x \in \text{CF}(\Sigma^1, \Sigma^2)$, define the degree $|x|$ by

$$|x| = \text{CZ}(\gamma_x),$$

following [35]. One can also see [6, Section 4.2] for details. Note that for a Reeb chord in $C(\Lambda_+^1, \Lambda_+^2)$, its degree in $\text{Cth}(\Sigma^1, \Sigma^2)$ will not necessarily coincide with its degree in $C(\Lambda_+^1, \Lambda_+^2)$. It is shifted as we will see later.

Action For $i = 1, 2$, suppose g_i is a primitive of the exact Lagrangian cobordism Σ^i , and hence g_i is constant when $t < -N$ or $t > N$. Note that primitive functions are well defined up to a overall shift by a constant. Thus we may assume that the primitives g_i are both zero on $\Sigma^i \cup ((-\infty, -N) \times \mathbb{R}^3)$ for $i = 1, 2$. The action of generators is defined under this choice of primitives.

For Reeb chords $a^+ \in C(\Lambda_+^1, \Lambda_+^2)$ and $a^- \in C(\Lambda_-^1, \Lambda_-^2)$, define the action \mathfrak{a} by

$$\mathfrak{a}(a^+) = g_2(a^+) - g_1(a^+) + \int_{a^+} e^N \alpha$$

and

$$\mathfrak{a}(a^-) = g_2(a^-) - g_1(a^-) + \int_{a^-} e^{-N} \alpha = \int_{a^-} e^{-N} \alpha.$$

The last part is due to the special choice of primitives. For double points x of $\Sigma^1 \cup \Sigma^2$, the action $\mathfrak{a}(x)$ is defined by $\mathfrak{a}(x) = g_2(x) - g_1(x)$.

4.2 The differential

Remark 4.1 Throughout this paper, we restrict ourselves to the case where all intersection generators have positive actions since that is the case for the special pair of cobordisms constructed in Section 5.1. In general, the differential could include one more map from $\text{CF}(\Sigma^1, \Sigma^2)$ to $C(\Lambda_-^1, \Lambda_-^2)$, which is called the Nessie map. However, by [6, Proposition 9.1], the positive energy condition of the holomorphic disks counted by the Nessie map requires the corresponding intersections in $\text{CF}(\Sigma^1, \Sigma^2)$ to have negative actions. Therefore, in our special case, we can exclude the Nessie map and get the differential as an upper triangle as below.

With the assumption in Remark 4.1, we define the differential under the decomposition

$$\text{Cth}(\Sigma^1, \Sigma^2) = C(\Lambda_+^1, \Lambda_+^2) \oplus \text{CF}(\Sigma^1, \Sigma^2) \oplus C(\Lambda_-^1, \Lambda_-^2)$$

by a degree-1 map of the form

$$d = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & 0 & d_{--} \end{pmatrix}.$$

To describe the differential explicitly, we need to study the holomorphic disks with boundary on $\Sigma^1 \cup \Sigma^2$. Fix a generic domain-dependent almost complex structure J that is compatible with the symplectic form on $\mathbb{R} \times \mathbb{R}^3$ and the cylindrical ends in the sense of [6, Section 3.1.5]. Suppose that the induced cylindrical almost complex structure on the positive end $(\Sigma^1 \cup \Sigma^2) \cap ([N, \infty) \times \mathbb{R}^3)$ is J_+ and on the negative end $(\Sigma^1 \cup \Sigma^2) \cap ((-\infty, -N] \times \mathbb{R}^3)$ is J_- . The differential $d_{\pm\pm}$ of $C(\Lambda_{\pm}^1, \Lambda_{\pm}^2)$ counts rigid J_{\pm} -holomorphic disks with boundary on $\mathbb{R} \times (\Lambda_{\pm}^1 \cup \Lambda_{\pm}^2)$, respectively, as described in Section 2.1. The corresponding moduli space is denoted by $\mathcal{M}_{J_{\pm}}(a^{\pm}; \mathbf{p}^{\pm}, b^{\pm}, \mathbf{q}^{\pm})$, where a^{\pm} and b^{\pm} are Reeb chords to Λ_{\pm}^1 from Λ_{\pm}^2 while \mathbf{p}^{\pm} and \mathbf{q}^{\pm} are words of pure Reeb chords of Λ_{\pm}^1 and Λ_{\pm}^2 , respectively. We also write $\tilde{\mathcal{M}}_{J_{\pm}}(a^{\pm}; \mathbf{p}^{\pm}, b^{\pm}, \mathbf{q}^{\pm})$ to denote the moduli space $\mathcal{M}_{J_{\pm}}(a^{\pm}; \mathbf{p}^{\pm}, b^{\pm}, \mathbf{q}^{\pm})$ modulo the action of \mathbb{R} in the t direction.

For the remaining maps in the differential, we need to describe J -holomorphic disks with boundary on $\Sigma^1 \cup \Sigma^2$, where J is the chosen domain-dependent almost complex structure. The punctures of these J -holomorphic disks can be either Reeb chords or intersection points. For generators c_0, c_1, \dots, c_m in $\text{Cth}(\Sigma^1, \Sigma^2)$, let $\mathcal{M}_J(c_0; c_1, \dots, c_m)$ denote the moduli space of the J -holomorphic disks

$$u: (D_{m+1}, \partial D_{m+1}) \rightarrow (\mathbb{R} \times \mathbb{R}^3, \Sigma^1 \cup \Sigma^2)$$

with the following properties:

- D_{m+1} is a 2-dimensional unit disk with $m + 1$ boundary points q_0, q_1, \dots, q_m removed and the points q_0, q_1, \dots, q_m are arranged in a counterclockwise order.
- If c_0 is a Reeb chord, the image of u is asymptotic to $[N, \infty) \times c_0$ near q_0 . If c_0 is an intersection point, then $\lim_{z \rightarrow q_0} u(z) = c_0$ and u maps the incoming segment (resp. outgoing segment) of the boundary to Σ^2 (resp. Σ^1).
- For $i > 0$, if c_i is a Reeb chord, the image of u is asymptotic to $(-\infty, -N] \times c_i$ near q_i . If c_i is an intersection point, then $\lim_{z \rightarrow q_i} u(z) = c_i$ and u maps the incoming segment (resp. outgoing segment) of the boundary to Σ^1 (resp. Σ^2).

The four types of moduli spaces used in the differential d of the Cthulhu chain complex are shown in Figure 9. For any one of these four moduli spaces \mathcal{M} , we say a disk $u \in \mathcal{M}$ is *rigid* if $\dim \mathcal{M} = 0$. All the moduli spaces of holomorphic disks introduced above admit a coherent orientation since both Σ^1 and Σ^2 are spin. Therefore, one can associate each rigid holomorphic disk $u \in \mathcal{M}$ with a sign and thus can count the number of rigid holomorphic disks in \mathcal{M} with sign.

Let each a_i^\pm be a Reeb chord to Λ_\pm^1 from Λ_\pm^2 and each x_i a double point of $\Sigma^1 \cup \Sigma^2$. The bold letters $\mathbf{p}^\pm, \mathbf{q}^\pm$ are words of pure Reeb chords of Λ_\pm^1 and Λ_\pm^2 , respectively. Here we assume, for $i = 1, 2$, that ϵ_i^- is an augmentation of $\mathcal{A}(\Lambda_i^-)$ and ϵ_i^+ is the augmentation of $\mathcal{A}(\Lambda_i^+)$ induced by Σ_i . The differential is defined as follows:

$$\begin{aligned}
 d_{++}(a_i^+) &= \sum_{\dim \widetilde{\mathcal{M}}_{J_+}(a_j^+; \mathbf{p}^+, a_i^+, \mathbf{q}^+) = 0} |\widetilde{\mathcal{M}}_{J_+}(a_j^+; \mathbf{p}^+, a_i^+, \mathbf{q}^+)| \epsilon_+^1(\mathbf{p}^+) \epsilon_+^2(\mathbf{q}^+) a_j^+, \\
 d_{--}(a_i^-) &= \sum_{\dim \widetilde{\mathcal{M}}_{J_-}(a_j^-; \mathbf{p}^-, a_i^-, \mathbf{q}^-) = 0} |\widetilde{\mathcal{M}}_{J_-}(a_j^-; \mathbf{p}^-, a_i^-, \mathbf{q}^-)| \epsilon_-^1(\mathbf{p}^-) \epsilon_-^2(\mathbf{q}^-) a_j^-, \\
 d_{00}(x_i) &= \sum_{\dim \mathcal{M}_J(x_j; \mathbf{p}^-, x_i, \mathbf{q}^-) = 0} |\mathcal{M}_J(x_j; \mathbf{p}^-, x_i, \mathbf{q}^-)| \epsilon_-^1(\mathbf{p}^-) \epsilon_-^2(\mathbf{q}^-) x_j, \\
 d_{0-}(a_i^-) &= \sum_{\dim \mathcal{M}_J(x_j; \mathbf{p}^-, a_i^-, \mathbf{q}^-) = 0} |\mathcal{M}_J(x_j; \mathbf{p}^-, a_i^-, \mathbf{q}^-)| \epsilon_-^1(\mathbf{p}^-) \epsilon_-^2(\mathbf{q}^-) x_j, \\
 d_{+0}(x_i) &= \sum_{\dim \mathcal{M}_J(a_j^+; \mathbf{p}^-, x_i, \mathbf{q}^-) = 0} |\mathcal{M}_J(a_j^+; \mathbf{p}^-, x_i, \mathbf{q}^-)| \epsilon_-^1(\mathbf{p}^-) \epsilon_-^2(\mathbf{q}^-) a_j^+, \\
 d_{+-}(a_i^-) &= \sum_{\dim \mathcal{M}_J(a_j^+; \mathbf{p}^-, a_i^-, \mathbf{q}^-) = 0} |\mathcal{M}_J(a_j^+; \mathbf{p}^-, a_i^-, \mathbf{q}^-)| \epsilon_-^1(\mathbf{p}^-) \epsilon_-^2(\mathbf{q}^-) a_j^+,
 \end{aligned}$$

where $|\mathcal{M}|$ denotes the number of rigid holomorphic disks in the moduli space \mathcal{M} counted with sign. Note that the definition of differential depends on the choice of augmentations ϵ_1^- and ϵ_2^- , whose existence are essential to the Floer theory.

A holomorphic disk counted by the differential must satisfy the rigidity condition and the positive energy condition. We will describe these conditions in detail.

The rigidity condition Let us interpret the condition $\dim \mathcal{M}_J(c_1; \mathbf{p}, c_2, \mathbf{q}) = 0$ in terms of $|c_i|$ for $i = 1, 2$, where c_i can be either a Reeb chord or an intersection point while \mathbf{p} and \mathbf{q} are words of pure Reeb chords in degree 0. Instead of deriving a formula for the dimension of a moduli space, we use the idea of the wrapped Floer homology to find the relation between $|c_1|$ and $|c_2|$. Recall that both Σ^1 and Σ^2 are cylindrical outside $[-N, N] \times \mathbb{R}^3$. Consider a nondecreasing function $\sigma(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such

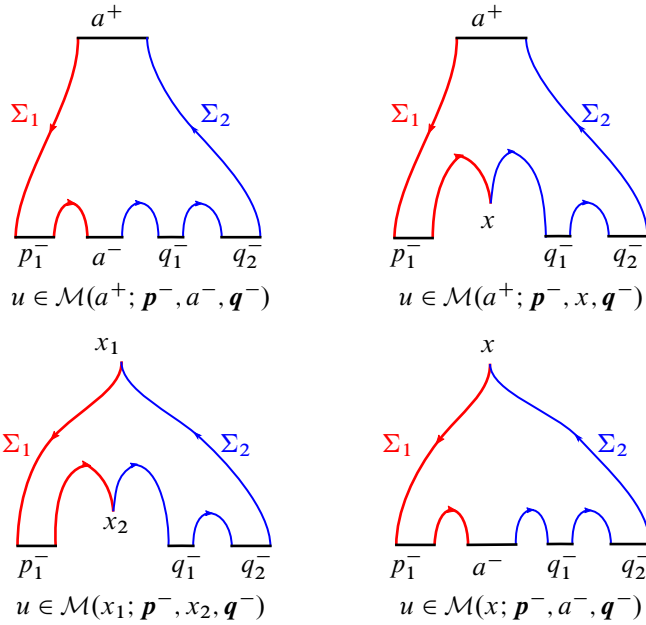


Figure 9: A sketch of the J -holomorphic disks in the differential d . Here a^\pm are Reeb chords to Λ_\pm^1 from Λ_\pm^2 , respectively, and x, x_1, x_2 are double points of $\Sigma^1 \cup \Sigma^2$. In these examples, p^- is a word of one pure Reeb chord p_1^- of Λ_-^1 while q^- is a word $q_1^-q_2^-$ of two pure Reeb chords of Λ_-^2 . The arrows denote the orientation inherited from the boundary of the unit disk.

that $\sigma'(t) = 0$ when $t \leq N$ and $\sigma'(t) = 1$ when $t \geq N'$, where N' is a number bigger than N . Note that $X_H = -\sigma(|t|)\partial_z$ is a Hamiltonian vector field with its time- s flow denoted by Φ_H^s . Flow Σ^1 through X_H and get a new cobordism $\Phi_H^s(\Sigma^1)$, which is another exact Lagrangian cobordism according to Section 5.1. Observe that $\Phi_H^s(\Sigma^1)$ wraps Σ^1 on both ends in the negative Reeb chord direction. Hence for a large enough number s , each Reeb chord c to Λ_+^1 (resp. Λ_-^1) from Λ_+^2 (resp. Λ_-^2) corresponds to a transversally double point \check{c} of $\Phi_H^s(\Sigma^1) \cup \Sigma^2$ in $N < t < N'$ (resp. $-N' < t < -N$). Moreover, if c is a Reeb chord in $C(\Lambda_+^1, \Lambda_+^2)$, we have

$$|\check{c}| = CZ(\gamma_{\check{c}}) = CZ(\gamma_c) + 1 = |c| + 2.$$

If c is Reeb chord in $C(\Lambda_-^1, \Lambda_-^2)$,

$$|\check{c}| = CZ(\gamma_{\check{c}}) = CZ(\gamma_c) = |c| + 1.$$

Each double point x of $\Sigma^1 \cup \Sigma^2$ naturally corresponds to a double point \check{x} of $\Phi_H^s(\Sigma^1) \cup \Sigma^2$ in $-N < t < N$ with gradings satisfying $|\check{x}| = |x|$.

Remark The difference in grading correspondence between the Reeb chords in $C(\Lambda_+^1, \Lambda_+^2)$ and the Reeb chords in $C(\Lambda_-^1, \Lambda_-^2)$ can be understood better in a special case where Σ^1 is a pushoff of Σ^2 through a positive Morse function $F: \Sigma^2 \rightarrow \mathbb{R}_{>0}$. In other words, in a Weinstein neighborhood of Σ^2 , the cobordism Σ^1 is the graph of dF for some positive Morse function $F: \Sigma^2 \rightarrow \mathbb{R}_{>0}$. In this case, the cobordism $\Phi_h^s(\Sigma^1)$ is a pushoff of Σ^2 through another Morse function \tilde{F} as well. By the canonical Floer theory [24], we can choose a family of capping paths so that $\text{CZ}(\gamma_x) = \text{Ind}_{\tilde{F}}(x)$ for any intersection point x of $\Phi_h^s(\Sigma^1)$ and Σ^2 . Similarly, for any Morse Reeb chord c in $\Lambda_+^1 \cup \Lambda_+^2$, we can further require that $\text{Ind}_{f_+}(c) = \text{CZ}(\gamma_c)$, where $f_+ = F|_{\Lambda_+^2}$. Notice that $\text{Ind}_{\tilde{F}}(\check{c}) = \text{Ind}_{f_+}(c) + 1$. Therefore

$$|\check{c}| = \text{CZ}(\gamma_{\check{c}}) = \text{Ind}_{\tilde{F}}(\check{c}) = \text{Ind}_{f_+}(c) + 1 = \text{CZ}(\gamma_c) + 1 = |c| + 2.$$

For a Morse Reeb chord c in $\Lambda_-^1 \cup \Lambda_-^2$, the indices satisfy $\text{Ind}_{\tilde{F}}(\check{c}) = \text{Ind}_{f_-}(c)$, where $f_- = F|_{\Lambda_-^2}$. Hence $|\check{c}| = \text{CZ}(\gamma_{\check{c}}) = \text{Ind}_{\tilde{F}}(\check{c}) = \text{Ind}_{f_-}(c) = \text{CZ}(\gamma_c) = |c| + 1$. A schematic figure is shown in Figure 10.

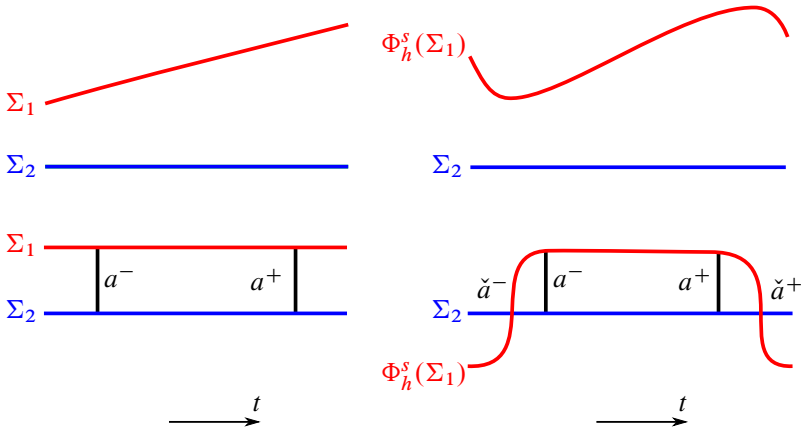


Figure 10: Comparison between Σ_1 and $\Phi_h^s(\Sigma_1)$ in terms of front projection (top) and Lagrangian projection (bottom). The indices satisfy $|\check{a}^+| = |a^+| + 2$ and $|\check{a}^-| = |a^-| + 1$.

So far, we have shown that generators of $\text{Cth}(\Sigma^1, \Sigma^2)$ can be identified with intersection points of $\Phi_H^s(\Sigma^1)$ and Σ^2 , which are generators of $\text{Cth}(\Phi_H^s(\Sigma^1), \Sigma^2)$. Moreover, by [6, Proposition 8.2], the Cthulhu chain complexes $\text{Cth}(\Sigma^1, \Sigma^2)$ and $\text{Cth}(\Phi_H^s(\Sigma^1), \Sigma^2)$ are identified on the level of complexes as well. Note that the generators of $\text{Cth}(\Phi_H^s(\Sigma^1), \Sigma^2)$ do not contain any Reeb chords and hence we have $\text{Cth}(\Phi_H^s(\Sigma^1), \Sigma^2) = (\text{CF}(\Phi_H^s(\Sigma^1), \Sigma^2), d_{00})$. Lift $\Phi_H^s(\Sigma^1) \cup \Sigma^2$ to a Legendrian submanifold L in $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$. Note that $(\text{CF}(\Phi_H^s(\Sigma^1), \Sigma^2), d_{00})$ is the dual of the

linearized contact homology chain complex of L as introduced in Section 2.1. Hence if $\dim \mathcal{M}(c_1; \mathbf{p}, c_2, \mathbf{q}) = 0$, there are rigid holomorphic disks that have a positive puncture at \check{c}_1 and a negative puncture at \check{c}_2 , which implies $|\check{c}_1| - |\check{c}_2| = 1$. We can get the corresponding grading relation between c_1 and c_2 . In particular, let a^\pm be Reeb chords in $C(\Lambda_\pm^1, \Lambda_\pm^2)$ and x_1, x_2 be intersection points of Σ^1 and Σ^2 while \mathbf{p} and \mathbf{q} are words of pure Reeb chords in degree 0 of Λ_-^1 and Λ_-^2 , respectively. We have:

- If $\dim \mathcal{M}_J(x_1; \mathbf{p}, x_2, \mathbf{q}) = 0$, then $|c_1| - |c_2| = 1$.
- If $\dim \mathcal{M}_J(x_1; \mathbf{p}, a^-, \mathbf{q}) = 0$, then $|c_1| - |a^-| = 2$.
- If $\dim \mathcal{M}_J(a^+; \mathbf{p}, x_1, \mathbf{q}) = 0$, then $|a^+| - |x_1| = -1$.
- If $\dim \mathcal{M}_J(a^+; \mathbf{p}, a^-, \mathbf{q}) = 0$, then $|a^+| - |a^-| = 0$.

As a result, the Cthulhu chain complex can be written as

$$\text{Cth}^k(\Sigma^1, \Sigma^2) = C^{k-2}(\Lambda_+^1, \Lambda_+^2) \oplus \text{CF}^k(\Sigma^1, \Sigma^2) \oplus C^{k-1}(\Lambda_-^1, \Lambda_-^2).$$

Under this decomposition, the differential

$$d = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & 0 & d_{--} \end{pmatrix}$$

has degree 1 as we expected.

The positive energy condition We interpret the positive energy condition of a holomorphic disk $u \in \mathcal{M}(c_0; c_1, \dots, c_m)$ in terms of the action of c_i , where $i = 0, \dots, m$. Following [14], we define the energy $E(u)$ of a holomorphic disk

$$u: (D^2, \partial D^2) \rightarrow (\mathbb{R} \times \mathbb{R}^3, \Sigma^1 \cup \Sigma^2)$$

by $E(u) = E_\omega(u) + E_\alpha(u)$, where the ω -energy is given by

$$E_\omega(u) = \int_{u^{-1}([-N, N] \times \mathbb{R}^3)} u^*(\omega) + \int_{u^{-1}((-\infty, -N) \times \mathbb{R}^3)} u^*(e^{-N}d\alpha) + \int_{u^{-1}((N, \infty) \times \mathbb{R}^3)} u^*(e^N d\alpha).$$

Write $u = (t, v)$, where $t: D^2 \rightarrow \mathbb{R}$ and $v: D^2 \rightarrow \mathbb{R}^3$. Define the α -energy $E_\alpha(u)$ by

$$\sup_{\phi_-} \left(\int_{u^{-1}((-\infty, -N) \times \mathbb{R}^3)} \phi_-(t) dt \wedge (v^* \alpha) \right) + \sup_{\phi_+} \left(\int_{u^{-1}((N, \infty) \times \mathbb{R}^3)} \phi_+(t) dt \wedge (v^* \alpha) \right),$$

where ϕ_+ and ϕ_- range over all compactly supported smooth functions such that

$$\int_{-\infty}^{-N} \phi_-(t) dt = e^{-N} \quad \text{and} \quad \int_N^{\infty} \phi_+(t) dt = e^N,$$

respectively. By Stokes' theorem, for any holomorphic disk $u \in \mathcal{M}(c_0; c_1, \dots, c_m)$, we have

$$E_\omega(u) = \mathfrak{a}(c_0) - \sum_{l=1}^m \mathfrak{a}(c_l).$$

A holomorphic disk has positive ω -energy, that is, $E_\omega(u) > 0$, which implies that $\mathfrak{a}(c_0) > \sum_{l=1}^m \mathfrak{a}(c_l)$.

Under the assumption in Remark 4.1, the differential d of the Cthulhu chain complex is of the form of an upper triangle. By [6, Section 6], we have $d^2 = 0$ and thus d is a differential map. Moreover, from [6, Section 8], the induced cohomology $H^* \text{Cth}(\Sigma^1, \Sigma^2)$ is an invariant under compactly supported Hamiltonian isotopies. Push Σ^1 along the negative z direction until Σ^1 is far below Σ^2 and then there is no Reeb chord to Σ^1 from Σ^2 nor intersection point between Σ^1 and Σ^2 . It is obvious that the cohomology is trivial, ie $H^* \text{Cth}(\Sigma^1, \Sigma^2) = 0$.

5 Main result

In Section 5.1, we perturb an exact Lagrangian cobordism using a Morse function and obtain a pair of Lagrangian cobordisms. In Section 5.2, we apply the Floer theory to this pair of cobordisms and get the long exact sequence in Theorem 1.1. In Section 5.3, we describe the rigid holomorphic disks counted by d_{+-} , which is a part of the differential map of the Cthulhu chain complex, in terms of holomorphic disks with boundary on Σ and Morse flow lines. This is useful when identifying f_1 , ie the category map on the level of morphisms, with d_{+-} . In Section 5.4, we extend the method in Section 5.3 to describe the differential d of the Cthulhu chain complex and recover the long exact sequences in [6]. Finally, we use the identification in Section 5.3 between f_1 and d_{+-} to prove Theorem 1.5 in Section 5.5.

5.1 Construction of the pair of cobordisms

First let us describe the neighborhood of a Lagrangian cobordism. Let Σ be an exact Lagrangian cobordism from Λ_- to Λ_+ in $(\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha))$, where Λ_- and Λ_+ are Legendrian links. By the Weinstein Lagrangian neighborhood theorem, there is a symplectomorphism

$$\psi: \text{nbhd}(\Sigma) \subset (\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha)) \rightarrow (T^* \Sigma, d\theta),$$

where θ is the negative Liouville form $\theta = -\sum p_i dq_i$ of $T^* \Sigma$ with coordinates $((q_1, q_2), (p_1, p_2))$. Specifically, on the $(\pm\infty-)$ boundary $\mathbb{R} \times \Lambda_\pm$, the symplectomorphism ψ is given by a composition $\psi_1 \circ \psi_0$ of two symplectomorphisms. As

mentioned before, there is a contactomorphism from a tubular neighborhood of Λ_{\pm} in \mathbb{R}^3 to a neighborhood of the zero section of $J^1(\Lambda_{\pm})$. Composing with the identity map on \mathbb{R}_t , we get the symplectomorphism ψ_0 from the neighborhood of $\mathbb{R} \times \Lambda_{\pm}$ in $\mathbb{R} \times \mathbb{R}^3$ to $\mathbb{R} \times J^1(\Lambda_{\pm})$. The second part ψ_1 is given by

$$\psi_1: \text{nbhd}(\Sigma) \subset ((\mathbb{R} \times J^1(\Lambda_{\pm})), d(e^t \alpha)) \rightarrow (T^*(\mathbb{R}_{>0} \times \Lambda_{\pm}), d\theta),$$

$$(t, (q, p, z)) \mapsto ((e^t, q), (z, e^t p)).$$

For a Morse function $F: \Sigma \rightarrow \mathbb{R}_{\geq 0}$ such that the determinant of the Hessian matrix is small enough, the graph of dF is a Lagrangian submanifold in $T^*\Sigma$. Pull it back to $\mathbb{R} \times \mathbb{R}^3$ and denote $\psi^{-1}(\text{graph}(dF))$ by Σ' .

Now we show that Σ' is an exact Lagrangian submanifold as well. Notice that $V_{(q,p)} := dF|_q$ is a Hamiltonian vector field in $T^*\Sigma$ since $\iota_V d\theta = -d\tilde{F}$, where $\tilde{F} = F \circ \pi$ and π is the natural projection $\pi: T^*(\Sigma) \rightarrow \Sigma$. In order to extend $\psi_*^{-1}(V)$ to a Hamiltonian vector field in $\mathbb{R} \times \mathbb{R}^3$, we choose a smooth cutoff function $\gamma: T^*(\Sigma) \rightarrow \mathbb{R}$ such that $\gamma(q, p) = 1$ in a tubular neighborhood of the zero section containing the graph of dF and $\gamma(q, p) = 0$ outside a slightly bigger tubular neighborhood of the zero section. Pull the Hamiltonian vector field of $\gamma \cdot \tilde{F}$ back through ψ and extend to a Hamiltonian vector field X_H in $\mathbb{R} \times \mathbb{R}^3$. For a suitable neighborhood of Σ in $\mathbb{R} \times \mathbb{R}^3$, we have

$$\iota_{X_H} d(e^t \alpha)|_{\text{nbhd}(\Sigma)} = \psi^*(\iota_V d\theta) = \psi^*(-d\tilde{F}) = d(-\tilde{F} \circ \psi)|_{\text{nbhd}(\Sigma)}.$$

Hence its Hamiltonian H is the same as $-\tilde{F} \circ \psi$ around Σ . Denote the time- s flow of X_H by ϕ_H^s and thus $\Sigma' = \phi_H^1(\Sigma)$. We can compute the 1-form on Σ' :

$$(5) \quad \begin{aligned} \phi_H^{1*} e^t \alpha &= e^t \alpha + \int_0^1 \frac{d}{ds} \phi_H^{s*} (e^t \alpha) ds \\ &= e^t \alpha + \int_0^1 \phi_H^{s*} (\iota_{X_H} d(e^t \alpha) + d(\iota_{X_H} e^t \alpha)) ds \\ &= e^t \alpha + \int_0^1 \phi_H^{s*} (dH + d(e^t \alpha(X_H))) ds \\ &= e^t \alpha + d \left(\int_0^1 (H + e^t \alpha(X_H)) \circ \phi_H^s ds \right). \end{aligned}$$

Thus Σ' is exact. Moreover, if Σ has a primitive g , then Σ' has a primitive

$$g + \int_0^1 (H + e^t \alpha(X_H)) \circ \phi_H^s ds.$$

We are going to construct a particular Morse function for Σ such that the image of the Morse function has cylindrical ends as well and thus is an exact Lagrangian cobordism.

Suppose Σ is cylindrical outside $[-N + \delta, N - \delta] \times \mathbb{R}^3$, where $0 < \delta < 1$. Choose Morse functions $g_{\pm}: \Lambda_{\pm} \rightarrow (0, \frac{1}{2})$ and $G: \Sigma \cap ([-N, N] \times \mathbb{R}^3) \rightarrow (0, 1)$ such that

$$G|_{\Sigma \cap \{t \in [-N, -N + \delta] \cup [N - \delta, N]\}} = e^t.$$

Define a smooth nondecreasing function $\rho: \mathbb{R}_{>0} \rightarrow [0, 1]$ such that $\rho(s) = 0$ for $s \leq 1$ and $\rho(s) = 1$ for $s \geq e^{\delta}$.

For $0 < \eta < e^{-2}$, define a Morse function $F^{\eta}: \Sigma \rightarrow \mathbb{R}_{>0}$ to be

$$F^{\eta}(t, q) := \begin{cases} \eta^{2N} g_{-}(q)s & \text{if } t < -N, \\ (\rho(e^N s)(\eta^N - \eta^{2N} g_{-}(q)) + \eta^{2N} g_{-}(q))s & \text{if } -N \leq t \leq -N + \delta, \\ \eta^N G & \text{if } -N + \delta < t < N - \delta, \\ (\eta^N + \rho(e^{-N + \delta} s)\eta^N g_{+}(q))s & \text{if } N - \delta \leq t \leq N, \\ (\eta^N + \eta^N g_{+}(q))s & \text{if } t > N, \end{cases}$$

where $s = e^t$. One can check that F^{η} has the following properties:

- The Morse function F^{η} is increasing with respect to t when $t \leq -N + \delta$ or $t \geq N - \delta$. This implies that the critical points of F^{η} and the critical points of G are in one-to-one correspondence and are all contained in $\Sigma \cap ([-N, N] \times \mathbb{R}^3)$.
- The Morse function F^{η} is bounded by $2\eta^N e^N$ on $\Sigma \cap ([-N, N] \times \mathbb{R}^3)$.
- Write $F^{\eta}|_{\{N\} \times \Lambda_{+}}$ as $f_{+}^{\eta} e^N$ and $F^{\eta}|_{\{-N\} \times \Lambda_{-}}$ as $f_{-}^{\eta} e^{-N}$, respectively. The graph of dF^{η} on $(-\infty, -N) \times \Lambda_{-}$ is the same as $(-\infty, -N) \times \text{graph}(df_{-}^{\eta})$ and the graph of dF^{η} on $(N, \infty) \times \Lambda_{+}$ is the same as $(N, \infty) \times \text{graph}(df_{+}^{\eta})$.

Push $(\Sigma, \Lambda_{+}, \Lambda_{-})$ off through F^{η} and obtain a copy of $(\Sigma, \Lambda_{+}, \Lambda_{-})$, labeled by $(\Sigma^1, \Lambda_{+}^1, \Lambda_{-}^1)$. Label the original $(\Sigma, \Lambda_{+}, \Lambda_{-})$ by $(\Sigma^2, \Lambda_{+}^2, \Lambda_{-}^2)$. Thus Σ^1 is a pushoff of Σ^2 through F^{η} and Λ_{+}^1 (resp. Λ_{-}^1) is a pushoff of Λ_{+}^2 (resp. Λ_{-}^2) through f_{+}^{η} (resp. f_{-}^{η}).

5.2 The long exact sequence

Now we apply the Floer theory to the pair of cobordisms $\Sigma^1 \cup \Sigma^2$ constructed in Section 5.1 and get a long exact sequence. Combining the long exact sequence with the augmentation category map induced by the exact Lagrangian cobordism, we obtain an obstruction to the existence of the exact Lagrangian cobordisms.

Recall that the grading for generators in the Cthulhu chain complex depends on the choice of capping paths. According to the canonical Floer theory [24], we can choose a family of capping paths such that the Conley–Zehnder index of any double point x of $\Sigma^1 \cup \Sigma^2$ satisfies $\text{CZ}(\Gamma_x) = \text{Ind}_{F^{\eta}(x)}$. Now we apply the Floer theory to the pair of Lagrangian cobordisms $\Sigma^1 \cup \Sigma^2$ and have the following theorem.

Theorem 5.1 Let Σ^i , for $i = 1, 2$, be the cobordisms from Λ_-^i to Λ_+^i as constructed in Section 5.1. Suppose ϵ_-^i is an augmentation of $\mathcal{A}(\Lambda_-^i)$ and ϵ_+^i is the augmentation of $\mathcal{A}(\Lambda_+^i)$ induced by Σ^i . Fix a suitable domain-dependent almost complex structure on $\mathbb{R} \times \mathbb{R}^3$ that is compatible with the symplectic form and cylindrical ends. For η small enough, the Cthulhu chain complex is

$$\text{Cth}^k(\Sigma^1, \Sigma^2) = C^{k-2}(\Lambda_+^1, \Lambda_+^2) \oplus \text{CF}^k(\Sigma^1, \Sigma^2) \oplus C^{k-1}(\Lambda_-^1, \Lambda_-^2).$$

Under this decomposition, the differential is

$$d = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & 0 & d_{--} \end{pmatrix}.$$

Moreover:

- (1) The map d_{00} is the Morse codifferential induced by F^η , ie the chain complex $(\text{CF}^k(\Sigma^1, \Sigma^2), d_{00})$ is the Morse cochain complex $(C_{\text{Morse}}^k F^\eta, d_{F^\eta})$ induced by F^η .
- (2) The chain complex $(C^{k-2}(\Lambda_+^1, \Lambda_+^2), d_{++})$ is equal to $(\text{Hom}_+^{k-1}(\epsilon_+^1, \epsilon_+^2), m_1)$ while the chain complex $(C^{k-1}(\Lambda_-^1, \Lambda_-^2), d_{--})$ equals $(\text{Hom}_+^k(\epsilon_-^1, \epsilon_-^2), m_1)$.

Proof First, we need to show that each intersection point $x \in \text{CF}^k(\Sigma^1, \Sigma^2)$ has a positive action, which is the condition for the differential to have the form above by Remark 4.1.

Let g_i be a primitive of Σ^i for $i = 1, 2$. According to the computation (5), we have

$$g_1 = g_2 + \int_0^1 (H + e^t \alpha(X_H)) \circ \phi_H^s ds,$$

where $H = -\tilde{F}^\eta \circ \psi$. It is not hard to check that $g_1 = g_2$ on $\Sigma \cap ((-\infty, -N) \times \mathbb{R}^3)$. Therefore we can assume $g_1 = g_2 = 0$ on $\Sigma \cap ((-\infty, -N) \times \mathbb{R}^3)$ and use g_1 and g_2 as primitives to define actions. The action of each intersection point x is given by

$$\alpha(x) = g_2(x) - g_1(x) = \int_0^1 (H + e^t \alpha(X_H)) \circ \phi_H^s ds.$$

Notice that the vector field X_H vanishes at the intersection point x . Hence

$$\alpha(x) = - \int_0^1 H \circ \phi_H^s ds = -H = F^\eta \circ \psi(x) > 0.$$

Next, we are going to show that for η small enough, the rigid holomorphic disks that contribute to d_{00} do not include any pure Reeb chords as negative punctures. Let x and y be two double points of $\Sigma^1 \cup \Sigma^2$. For any rigid holomorphic disk

$u \in \mathcal{M}(x; \mathbf{p}, y, \mathbf{q})$, where \mathbf{p} and \mathbf{q} are words of pure Reeb chords of $\Lambda_-^1 \cup \Lambda_-^2$, we have the energy estimate

$$E_\omega(u) \leq \mathfrak{a}(x) - \mathfrak{a}(y) - \mathfrak{a}(\mathbf{p}) - \mathfrak{a}(\mathbf{q}).$$

Since u has positive energy, we have

$$\mathfrak{a}(\mathbf{p}) + \mathfrak{a}(\mathbf{q}) \leq F^\eta(\psi(x)) - F^\eta(\psi(y)) \leq F^\eta(\psi(x)).$$

Therefore, for η small enough that the maximum of the Morse function F^η is smaller than the minimum action of pure Reeb chords of Λ_-^1 and Λ_-^2 , the moduli space that contributes to d_{00} is of the form $\mathcal{M}(x; y)$. By [16, Lemma 6.11], the boundary of a rigid holomorphic disk with two punctures at intersection points converges to a rigid Morse flow line, which implies $d_{00} = d_{F^\eta}$. Furthermore, the gradings satisfy $|x| = \text{CZ}(\gamma_x) = \text{Ind}_{F^\eta(x)}$. Therefore $(\text{CF}^k(\Sigma^1, \Sigma^2), d_{00}) = (C_{\text{Morse}}^k F^\eta, d_{F^\eta})$.

Recall that there is a natural identity map with degree 1 from $C^{k-1}(\Lambda_\pm^1, \Lambda_\pm^2)$ to $\text{Hom}_+^k(\epsilon_\pm^1, \epsilon_\pm^2)$, respectively. Moreover, the definitions of $d_{\pm\pm}$ and m_1 match as well. Hence we have $(C^{k-1}(\Lambda_-^1, \Lambda_-^2), d_{--}) = (\text{Hom}_+^k(\epsilon_-^1, \epsilon_-^2), m_1)$. On the other hand, we have $(C^{k-2}(\Lambda_+^1, \Lambda_+^2), d_{++}) = (\text{Hom}_+^{k-1}(\epsilon_+^1, \epsilon_+^2), m_1)$. \square

For the remainder of the paper, we fix a small enough η and write $F^\eta, f_+^\eta, f_-^\eta$ as F, f_+, f_- , respectively. According to the Floer theory in Section 4, we have $H^k(\text{Cth}(\Sigma^1, \Sigma^2), d) = 0$, where

$$\text{Cth}^k(\Sigma^1, \Sigma^2) = \text{Hom}_+^{k-1}(\epsilon_+^1, \epsilon_+^2) \oplus C_{\text{Morse}}^k F \oplus \text{Hom}_+^k(\epsilon_-^1, \epsilon_-^2)$$

and

$$d = \begin{pmatrix} m_1 & d_{+0} & d_{+-} \\ 0 & d_F & d_{0-} \\ 0 & 0 & m_1 \end{pmatrix}.$$

Consider the chain map $\Psi = d_{+-} + d_{0-}$:

$$\Psi: (\text{Hom}_+^k(\epsilon_-^1, \epsilon_-^2), m_1) \rightarrow (\text{Hom}_+^k(\epsilon_+^1, \epsilon_+^2) \oplus C_{\text{Morse}}^{k+1} F, d'), \quad \text{where } d' = \begin{pmatrix} m_1 & d_{+0} \\ 0 & d_F \end{pmatrix}.$$

Notice that the mapping cone of Ψ has trivial homology. Therefore,

$$H^k(\text{Hom}_+(\epsilon_-^1, \epsilon_-^2)) \cong H^k \text{Cone}(d_{+0}).$$

Hence we have the following long exact sequence:

$$\begin{aligned} \cdots \rightarrow H^k(C_{\text{Morse}} F, d_F) &\rightarrow H^k \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \rightarrow H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \\ &\rightarrow H^{k+1}(C_{\text{Morse}} F, d_F) \rightarrow \cdots \end{aligned}$$

Moreover, notice that in the construction in Section 5.1, the gradient flows of F flow in from the bottom and out of the top. Hence we have

$$H^k(C_{\text{Morse}}F, d_F) = H^k(\Sigma, \Lambda_-).$$

Corollary 5.2 *Let Σ be an exact Lagrangian cobordism with Maslov number 0 from Λ_- to Λ_+ . For $i = 1, 2$, if ϵ_-^i is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ_+^i is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ , then we have the following long exact sequence:*

$$(6) \quad \cdots \rightarrow H^k(\Sigma, \Lambda_-) \rightarrow H^k \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \rightarrow H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow H^{k+1}(\Sigma, \Lambda_-) \rightarrow \cdots.$$

Remark When the Maslov number of Σ is d which is not 0 the method above works as well. The only difference is that the grading of generators in the Cthulhu chain complex is defined mod d . Thus, the long exact sequence (6) holds with gradings mod d .

If $\epsilon_-^1 = \epsilon_-^2 = \epsilon_-$, by [33, Section 5.2], we have the identification

$$H^k \text{Hom}_+(\epsilon, \epsilon) \cong \text{LCH}_{1-k}^\epsilon(\Lambda),$$

where $\text{LCH}_k^\epsilon(\Lambda)$ is the linearized contact homology of Λ . The long exact sequence (6) can be rewritten in terms of linearized contact homology:

$$\cdots \rightarrow H^k(\Sigma, \Lambda_-) \rightarrow \text{LCH}_{1-k}^{\epsilon_+}(\Lambda_+) \rightarrow \text{LCH}_{1-k}^{\epsilon_-}(\Lambda_-) \rightarrow H^{k+1}(\Sigma, \Lambda_-) \rightarrow \cdots.$$

Furthermore, if Λ_- is empty, then Σ is an exact Lagrangian filling of Λ_+ and ϵ_+ is an augmentation of $\mathcal{A}(\Lambda_+)$ induced by the Lagrangian filling. The long exact sequence (6) gives

$$H^k(\Sigma) \cong H^k \text{Hom}_+(\epsilon_+, \epsilon_+) \cong \text{LCH}_{1-k}^{\epsilon_+}(\Lambda_+),$$

which is the Seidel isomorphism (following [33]). This theorem was conjectured by Seidel [35] and was proved by Dimitroglou Rizell [13].

If Λ_+ , instead, is empty and $\mathcal{A}(\Lambda_-)$ has an augmentation ϵ_- , the long exact sequence (6) tells us that

$$H^k \text{Hom}_+(\epsilon_-, \epsilon_-) \cong H^{k+1}(\Sigma, \Lambda_-) = \begin{cases} \mathbb{F} & \text{if } k = 1, \\ \mathbb{F}^{1-\chi(\Sigma)} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

However, by Sabloff duality (3),

$$\dim H^0 \text{Hom}_+(\epsilon_-, \epsilon_-) = \dim H^2 \text{Hom}_-(\epsilon_-, \epsilon_-) = \dim H^2 \text{Hom}_+(\epsilon_-, \epsilon_-) = 0.$$

This is a contradiction since the unit $e_{\epsilon_-} = -y^\vee$ is always in $H^0 \text{Hom}_+(\epsilon_-, \epsilon_-)$ and is not 0. Thus if Λ_+ is empty, then Λ_- does not admit any augmentation. This result was previously known [6; 12].

Remark 5.3 For the rest of the paper, we will focus on the case where Λ_+ and Λ_- are single component knots. Given the fact that there does not exist a compact Lagrangian manifold in $\mathbb{R} \times \mathbb{R}^3$ and Λ_- does not admit a cap (since Λ_- has an augmentation), we know that any cobordism Σ from Λ_- to Λ_+ must be connected.

Combining the long exact sequence (6) with the augmentation category map induced by the exact Lagrangian cobordism Σ , we have the following theorem.

Theorem 5.4 *Let Σ be an exact Lagrangian cobordism with Maslov number 0 from a Legendrian knot Λ_- to a Legendrian knot Λ_+ . For $i = 1, 2$, assume ϵ_-^i is an augmentation of $\mathcal{A}(\Lambda_-)$ with a single basepoint and ϵ_+^i is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ . Then the map*

$$i^0: H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \rightarrow H^0 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2)$$

in the long exact sequence (6) is an isomorphism. Moreover, we have that

$$(7) \quad H^* \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \cong H^* \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \oplus \mathbb{F}^{-\chi(\Sigma)}[1],$$

where $\mathbb{F}^{-\chi(\Sigma)}[1]$ denotes the vector space $\mathbb{F}^{-\chi(\Sigma)}$ in degree 1 and $\chi(\Sigma)$ is the Euler characteristic of the surface Σ .

Proof By Remark 5.3, we have

$$H^k(\Sigma, \Lambda_-) = \begin{cases} \mathbb{F}^{-\chi(\Sigma)} & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The long exact sequence (6) shows that for $k > 1$ or $k < 0$, the map

$$i^k: H^k \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \rightarrow H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2)$$

in the long exact sequence induces an isomorphism

$$H^k \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \cong H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2).$$

When $k = 0$ or 1 , we have

$$\begin{aligned} 0 \rightarrow H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \xrightarrow{i^0} H^0 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow \mathbb{F}^{-\chi(\Sigma)} \\ \rightarrow H^1 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \rightarrow H^1 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow 0. \end{aligned}$$

To finish the proof, we need to show

$$\dim(H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2)) \geq \dim(H^0 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2)).$$

Once this inequality holds, the fact that $i^0: H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \rightarrow H^0 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2)$ is injective implies that it is an isomorphism. Note that the long exact sequence (6) is over the field \mathbb{F} . It follows that

$$H^1 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \cong H^1 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \oplus \mathbb{F}^{-\chi(\Sigma)}.$$

To prove the inequality, we exchange the positions of ϵ^1 and ϵ^2 in the long exact sequence (6) and get

$$\begin{aligned} \dots \rightarrow H^k(\Sigma, \Lambda_-) \rightarrow H^k \text{Hom}_+(\epsilon_+^2, \epsilon_+^1) \\ \rightarrow H^k \text{Hom}_+(\epsilon_-^2, \epsilon_-^1) \rightarrow H^{k+1}(\Sigma, \Lambda_-) \rightarrow \dots, \end{aligned}$$

which implies

$$H^2 \text{Hom}_+(\epsilon_+^2, \epsilon_+^1) \cong H^2 \text{Hom}_+(\epsilon_-^2, \epsilon_-^1).$$

By Sabloff duality (3), we have

$$\dim(H^0 \text{Hom}_-(\epsilon_\pm^1, \epsilon_\pm^2)) = \dim(H^2 \text{Hom}_+(\epsilon_\pm^2, \epsilon_\pm^1)).$$

Thus $\dim(H^0 \text{Hom}_-(\epsilon_+^1, \epsilon_+^2)) = \dim(H^0 \text{Hom}_-(\epsilon_-^1, \epsilon_-^2))$.

Since Λ_+ and Λ_- are both Legendrian knots with a single basepoint, we have the long exact sequence (2) for Λ_+ and Λ_- :

$$\begin{aligned} 0 \rightarrow H^0 \text{Hom}_-(\epsilon_\pm^1, \epsilon_\pm^2) \rightarrow H^0 \text{Hom}_+(\epsilon_\pm^1, \epsilon_\pm^2) \\ \rightarrow H^0(\Lambda_\pm) \xrightarrow{\delta_\pm} H^1 \text{Hom}_-(\epsilon_\pm^1, \epsilon_\pm^2) \rightarrow \dots. \end{aligned}$$

From this long exact sequence, we have

$$\dim(H^0 \text{Hom}_+(\epsilon_\pm^1, \epsilon_\pm^2)) = \dim(H^0 \text{Hom}_-(\epsilon_\pm^1, \epsilon_\pm^2)) + \dim(\ker \delta_\pm).$$

Thus, to prove $\dim(H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2)) \geq \dim(H^0 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2))$, we only need to show $\dim(\ker \delta_+) \geq \dim(\ker \delta_-)$.

Recall that the cobordism Σ from Λ_- and Λ_+ induces an A_∞ -category map

$$f: \text{Aug}_+(\Lambda_-) \rightarrow \text{Aug}_+(\Lambda_+)$$

in the way described in Section 3.2. In particular, we get the functor f_1 of augmentation categories on the level of morphisms:

$$f_1: \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow \text{Hom}_+(\epsilon_+^1, \epsilon_+^2).$$

This map descends to the cohomology level as

$$f^*: H^* \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow H^* \text{Hom}_+(\epsilon_+^1, \epsilon_+^2).$$

Notice that f_1 sends $\text{Hom}_-(\epsilon_-^1, \epsilon_-^2)$ to $\text{Hom}_-(\epsilon_+^1, \epsilon_+^2)$. Hence f_1 induces a map between the cohomology of the quotient chain complexes, denoted by f^* as well.

The following diagram commutes:

$$\begin{array}{ccccccc}
 0 \rightarrow & H^0 \text{Hom}_-(\epsilon_+^1, \epsilon_+^2) & \rightarrow & H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) & \rightarrow & H^0(\Lambda_+) & \xrightarrow{\delta_+} & H^1 \text{Hom}_-(\epsilon_+^1, \epsilon_+^2) & \rightarrow & \dots \\
 & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & & \\
 0 \rightarrow & H^0 \text{Hom}_-(\epsilon_-^1, \epsilon_-^2) & \rightarrow & H^0 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) & \rightarrow & H^0(\Lambda_-) & \xrightarrow{\delta_-} & H^1 \text{Hom}_-(\epsilon_-^1, \epsilon_-^2) & \rightarrow & \dots
 \end{array}$$

Thus $f^*(\ker \delta_-) \subset \ker \delta_+$. Furthermore, notice that f_1 sends the generator $(y^-)^\vee \in C_{\text{Morse}}^0(\Lambda_-)$ to the corresponding $(y^+)^\vee \in C_{\text{Morse}}^0(\Lambda_+)$, and $C_{\text{Morse}}^{-1}(\Lambda_+) = 0$. Hence f^* is injective on $H^0(\Lambda_-)$, which implies

$$\dim(\ker \delta_+) \geq \dim(\ker \delta_-). \quad \square$$

If $\epsilon_-^1 = \epsilon_-^2 = \epsilon_-$ and ϵ_- comes from a Lagrangian filling L_- , then ϵ_+ also comes from the filling L_+ , which is a concatenation of Σ and L_- . By Seidel’s isomorphism (following [33]), we have $\text{Hom}_+^k(\epsilon_\pm, \epsilon_\pm) \cong H^k(L_\pm)$, which implies that

$$H^k \text{Hom}_+(\epsilon_+, \epsilon_+) \cong H^k \text{Hom}_+(\epsilon_-, \epsilon_-) \quad \text{for } k \neq 1$$

and when $k = 1$,

$$H^1 \text{Hom}_+(\epsilon_+, \epsilon_+) \cong H^1 \text{Hom}_+(\epsilon_-, \epsilon_-) \oplus \mathbb{F}^{-\chi(\Sigma)}.$$

Theorem 5.4 is a generalization of Seidel’s isomorphism. Equation (7) holds even if ϵ_- does not come from a Lagrangian filling or ϵ_-^1 and ϵ_-^2 are not the same.

If the two augmentations are the same, we can identify the cohomology of Hom_+ space with the linearized contact homology by [33]:

$$\text{Hom}_+^k(\epsilon, \epsilon) \cong \text{LCH}_{1-k}^\epsilon(\Lambda).$$

Now we restate **Theorem 5.4** in terms of linearized contact homology.

Corollary 5.5 *Let Σ be an exact Lagrangian cobordism with Maslov number 0 from a Legendrian knot Λ_- to a Legendrian knot Λ_+ . Assume ϵ_- is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ_+ is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ . Then*

$$\text{LCH}_*^{\epsilon_+}(\Lambda_+) \cong \text{LCH}_*^{\epsilon_-}(\Lambda_-) \oplus \mathbb{F}^{-\chi(\Sigma)}[0],$$

where $\mathbb{F}^{-\chi(\Sigma)}[0]$ denotes the vector space $\mathbb{F}^{-\chi(\Sigma)}$ in degree 0.

Therefore, if there exists an exact Lagrangian cobordism Σ from Λ_- to Λ_+ , the Poincaré polynomial of the linearized contact homology of Λ_+ agrees with that of Λ_- in all degrees except 0. In degree 0 their coefficients differ by $-\chi(\Sigma)$. This gives a strong and computable obstruction to the existence of exact Lagrangian cobordisms. One can check the Poincaré polynomials of the linearized contact homology for any two Legendrian knots with small crossings through the atlas in [9]. If they do not

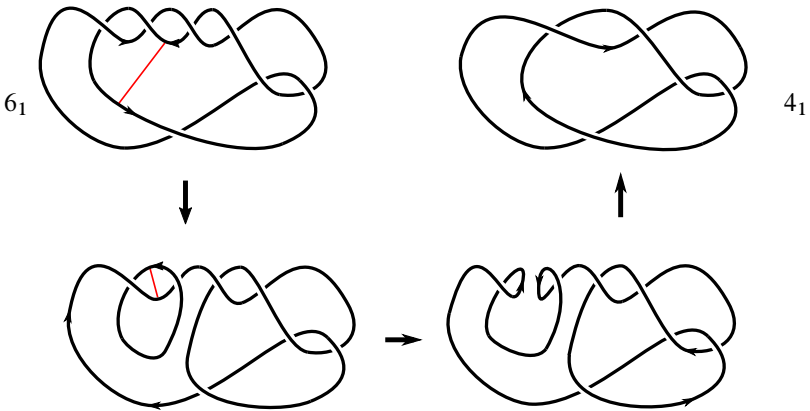


Figure 11: The topological cobordism between 6_1 and 4_1 can be achieved by two saddle moves along the red (straight) lines followed by an isotopy.

satisfy the relation given in [Corollary 5.5](#), there does not exist an exact Lagrangian cobordism between them.

For example, let Λ_1 and Λ_2 be the Legendrian knots with maximum Thurston–Bennequin number of smooth knot types 4_1 and 6_1 , respectively (as shown in [Figure 2](#)). There is a topological cobordism between 4_1 and 6_1 with genus 1 as shown in [Figure 11](#). Moreover, the Thurston–Bennequin numbers of Λ_1 and Λ_2 are -3 and -5 , respectively, which satisfy the Thurston–Bennequin number relation (1). Thus there possibly exists an exact Lagrangian cobordism from Λ_2 to Λ_1 with genus 1. However, we have the following proposition:

Proposition 5.6 *There does not exist an exact Lagrangian cobordism from Λ_2 to Λ_1 with Maslov number 0.*

Proof The Poincaré polynomials of the linearized contact homology for Λ_1 and Λ_2 are $t^{-1} + 2t$ and $2t^{-1} + 3t$, respectively. As a result of [Corollary 5.5](#), there does not exist an exact Lagrangian cobordism from Λ_2 to Λ_1 with Maslov number 0. \square

5.3 Geometric description of the differential map

Let Σ be an exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+ . For $i = 1, 2$, assume ϵ_-^i is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ_+^i is the augmentation induced by Σ . So far we have two maps from $\text{Hom}_+(\epsilon_-^1, \epsilon_-^2)$ to $\text{Hom}_+(\epsilon_+^1, \epsilon_+^2)$. One is the geometric map d_{+-} in the differential of the Cthulhu chain complex $\text{Cth}(\Sigma^1, \Sigma^2)$ defined by counting rigid holomorphic disks with boundary on $\Sigma^1 \cup \Sigma^2$. The other map is the augmentation category map induced by Σ on the level of morphisms f_1 , defined algebraically in [Section 3.2](#). In this section, we

will show that, with a choice of Morse function F on the cobordism Σ , the maps d_{+-} and f_1 are the same. To do that, we describe the two maps separately and then compare their images on each generator of $\text{Hom}_+(\epsilon_-^1, \epsilon_-^2)$.

In order to describe d_{+-} , we want to interpret rigid holomorphic disks with boundary on $\Sigma^1 \cup \Sigma^2$ in terms of rigid holomorphic disks with boundary on Σ together with negative gradient flows of a Morse function. This is analogous to a result in [16], which gives a correspondence between rigid holomorphic disks with boundary on a 2-copy of a Legendrian submanifold L and rigid holomorphic disks with boundary on L together with negative gradient flows of a Morse function. Now let us describe the result in [16] in detail.

Let L be a Legendrian submanifold in the contact manifold $(P \times \mathbb{R}, \ker(dz - \theta))$, where $(P, d\theta)$ is an exact symplectic $2n$ -dimensional manifold. Instead of considering holomorphic disks in the symplectization of $P \times \mathbb{R}$ with boundary on $\mathbb{R} \times L$, according to [13], we can consider holomorphic disks in P with boundary on $\pi(L)$, where π is the projection $P \times \mathbb{R} \rightarrow P$. See [16, Section 2.2.3] for the detailed definition of holomorphic disks with boundary on $\pi(L)$. As the points on $\pi(L)$ and points on L are in natural correspondence except that the double points of $\pi(L)$ correspond to the Reeb chords of L , we refer to the holomorphic disks as *J-holomorphic disks with boundary on L* as in [16], where J is a generic almost complex structure on P . Choose a Morse–Smale pair (f, g) , where f is a Morse function $L \rightarrow \mathbb{R}$ and g is a Riemannian metric on L , such that (f, g, J) is *adjusted to L* in the sense of [16, Section 6.3]. Push L off through the Morse function f and get a 2-copy of L , denoted by $2L$. In order to describe rigid holomorphic disks with boundary on $2L$, we need to introduce the generalized disks determined by (f, g, J) . A *generalized disk* is a pair (u, γ) , where

- $u \in \mathcal{M}$ is a J -holomorphic disk with boundary on L ;
- γ is a negative gradient flow of f with one end on the boundary of u and the other end at a critical point p of the Morse function f ;
- the boundary of u and γ intersect transversely.

The point p is called a *negative Morse puncture* if the flow line γ flows toward p , and is called a *positive Morse puncture* if γ flows away from p . The formal dimension $\dim(u, \gamma)$ is defined by

$$\dim(u, \gamma) = \begin{cases} \dim \mathcal{M} + 1 + \text{Ind}_f(p) - n & \text{if } p \text{ is a positive Morse puncture,} \\ \dim \mathcal{M} + 1 - \text{Ind}_f(p) & \text{if } p \text{ is a negative Morse puncture.} \end{cases}$$

The generalized disk (u, γ) is called *rigid* if $\dim(u, \gamma) = 0$.

The rigid holomorphic disks with boundary on a 2–copy of L can be described as below in terms of whether their punctures are Morse Reeb chords or non-Morse Reeb chords (as defined in Section 3.2).

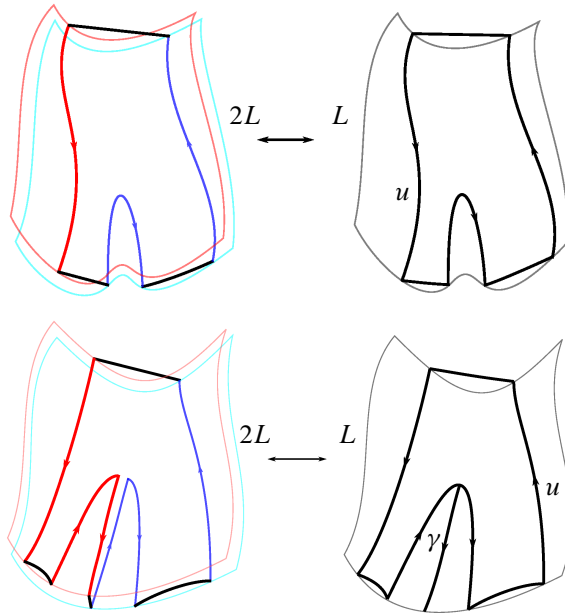


Figure 12: The correspondences in Lemma 5.7. The arrows indicate the orientations of holomorphic disks and the negative gradient flow line.

Lemma 5.7 [16, Theorem 3.6] *Let (f, g, J) be a triple as described above that is adjusted to the Legendrian submanifold L . Push L off through the Morse function f and get a 2–copy $2L$. There are the following bijective correspondences:*

- Rigid holomorphic disks with boundary on $2L$ that have one positive puncture and one negative puncture at non-Morse mixed Reeb chords and the other punctures at pure Reeb chords are in one-to-one correspondence with rigid holomorphic disks with boundary on L as shown in Figure 12 (top).
- Rigid holomorphic disks with boundary on $2L$ that have exactly one puncture at a Morse Reeb chord are in one-to-one correspondence with rigid generalized disks (u, γ) determined by (f, g, J) as shown in Figure 12 (bottom).
- Rigid holomorphic disks with boundary on $2L$ that have two punctures at Morse Reeb chords are in one-to-one correspondence with rigid negative gradient flows of the Morse function f .

In order to get an analogous description for a 2-copy of Σ , we need a result in [19] to relate rigid holomorphic disks with boundary on a cobordism Σ to rigid holomorphic disks with boundary on some Legendrian submanifold L_Σ .

Let us first construct the Legendrian submanifold L_Σ . Suppose Σ is an exact Lagrangian submanifold in $(\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha))$ from Λ_- to Λ_+ . Assume it is cylindrical outside $[-N + \delta, N - \delta] \times \mathbb{R}^3$, where δ is a small positive number. Under the symplectomorphism

$$\psi: (\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha)) \rightarrow (T^*(\mathbb{R}_{>0} \times \mathbb{R}), d\theta), \quad (t, x, y, z) \mapsto ((e^t, x), (z, e^t y)),$$

the cobordism Σ can be viewed as a cobordism in $(T^*(\mathbb{R}_{>0} \times \mathbb{R}), d\theta)$, where θ is the negative Liouville form of the cotangent bundle. Let $a_- = e^{-N}$ and $a_+ = e^N$. There is a small $\epsilon > 0$ such that Σ is cylindrical outside $T^*([a_- + \epsilon, a_+ - \epsilon] \times \mathbb{R})$. Chopping off the ends of Σ , we get a cobordism in $T^*([a_-, a_+] \times \mathbb{R})$ with the canonical symplectic form. Lift it to a Legendrian submanifold $\bar{\Sigma}$ in the 1-jet space $J^1([a_-, a_+] \times \mathbb{R}) = T^*([a_-, a_+] \times \mathbb{R}) \times \mathbb{R}$. The Legendrian $\bar{\Sigma}$ can be parametrized near the positive boundary $J^1((a_+ - \epsilon, a_+] \times \mathbb{R})$ as

$$(s, x_+(q), z_+(q), sy_+(q), sz_+(q) + B_+) = j^1(sz_+(q) + B_+)$$

for some constant B_+ , where $s = e^t$ and $(t, q) \in \Sigma \cap ((\log(a_+ - \epsilon), \log a_+] \times \mathbb{R}^3) = (\log(a_+ - \epsilon), \log a_+] \times \Lambda_+$. Here $sz_+(q) + B_+$ may not be a function of (s, x) . However, consider $\{(x_+(q), z_+(q)) \mid q \in \Lambda_+\}$, which is the front projection of Λ to the xz -plane. The cusps divide the front diagram of Λ_+ into pieces. Note that on each piece $z_+(p)$ is a perfect function of $x_+(p)$ and at each cusp, the two functions from different pieces match at the cusp. Therefore, we can write the parametrization as $j^1(sz_+(q) + B_+)$. Similarly, near the negative boundary $J^1([a_-, a_- + \epsilon) \times \mathbb{R})$, the Legendrian $\bar{\Sigma}$ can be parametrized as

$$(s, x_-(q), z_-(q), sy_-(q), sz_-(q) + B_-) = j^1(sz_-(q) + B_-),$$

where $(s, q) \in [a_-, a_- + \epsilon) \times \Lambda_-$ and B_- is a constant.

However, notice that $\bar{\Sigma}$ does not have any Reeb chords. Therefore, we consider the Morse Legendrian $\bar{\Sigma}^{\text{Mo}}$, which is a Legendrian submanifold in $J^1([a_-, a_+] \times \mathbb{R})$ that agrees with $\bar{\Sigma}$ on $J^1((a_- + \epsilon, a_+ - \epsilon) \times \mathbb{R})$. But near the (\pm) -boundary, the Morse Legendrian can be parametrized as $j^1((A_\pm \mp (s - a_\pm)^2)z_\pm(q))$, ie

$$(8) \quad (s, x_\pm(q), \mp 2(s - a_\pm)z_\pm(q), (A_\pm \mp (s - a_\pm)^2)y_\pm(q), (A_\pm \mp (s - a_\pm)^2)z_\pm(q)),$$

where A_\pm are positive constants. The key property of the Morse Legendrian is that the Reeb chords of $\bar{\Sigma}^{\text{Mo}}$ on the (\pm) -boundary are in bijective correspondence with the Reeb chords of Λ_\pm , respectively.

There are isotopies from $sz_{\pm}(q) + B_{\pm}$ to $(A_{\pm} \mp (s - a_{\pm})^2)z_{\pm}(q)$, respectively, which each induce a diffeomorphism from $\bar{\Sigma}$ to the Morse Legendrian $\bar{\Sigma}^{\text{Mo}}$. Extend $\bar{\Sigma}^{\text{Mo}}$ to a Legendrian submanifold L_{Σ} in $J^1(\mathbb{R}_{>0} \times \mathbb{R})$ by adding

$$j^1((A_+ - (s - a_+)^2)z_+(q))$$

with $(s, q) \in (a_+, \infty) \times \Lambda_+$ to the positive boundary and adding

$$j^1((A_- + (s - a_-)^2)z_-(q))$$

with $(s, q) \in (0, a_-) \times \Lambda_-$ to the negative boundary. In other words, when $s < a_- + \epsilon$ or $s > a_+ - \epsilon$, we can parametrize L_{Σ} as (8). Note that

$$L_{\Sigma} \cap J^1((a_- + \epsilon, a_+ - \epsilon) \times \mathbb{R}) = \bar{\Sigma}^{\text{Mo}}.$$

Moreover, according to [20], there is a natural bijective correspondence between rigid holomorphic disks with boundary on L_{Σ} and rigid holomorphic disks with boundary on $\bar{\Sigma}^{\text{Mo}}$. Combining this with a result in [19], we know that rigid holomorphic disks with boundary on an exact Lagrangian cobordism Σ are in one-to-one correspondence with rigid holomorphic disks with boundary on L_{Σ} that have positive (resp. negative) punctures at Reeb chords lying in the slice $s = a_+$ (resp. $s = a_-$). The proof of this result can be applied directly to the case of immersed exact Lagrangian submanifolds with cylindrical ends, where we only consider the rigid holomorphic disks with punctures on Reeb chords but no double points. Hence we have the following result for a 2-copy of Σ , denoted by $\Sigma \cup \Sigma'$.

Lemma 5.8 *Let Σ and Σ' be exact Lagrangian cobordisms from Λ_- to Λ_+ and from Λ'_- to Λ'_+ , respectively. The Morse Legendrian $L_{\Sigma \cup \Sigma'}$ constructed above is a union of L_{Σ} and $L_{\Sigma'}$. Moreover, rigid holomorphic disks with boundary on $\Sigma \cup \Sigma'$ that have positive (resp. negative) punctures at Reeb chords of $\Lambda_+ \cup \Lambda'_+$ (resp. $\Lambda_- \cup \Lambda'_-$) are in one-to-one correspondence with rigid holomorphic disks with boundary on $L_{\Sigma} \cup L_{\Sigma'}$ that have positive (resp. negative) punctures at the Reeb chords lying in the slice $s = a_+$ (resp. $s = a_-$).*

Note that the rigid holomorphic disks with boundary on $\Sigma \cup \Sigma'$ considered in Lemma 5.8 are not all the rigid holomorphic disks since we did not talk about holomorphic disks with punctures at double points. The disks we considered are the ones counted by d_{+-} .

In order to apply Lemma 5.7 to $L_{\Sigma \cup \Sigma'}$ and get the analogous correspondences for exact Lagrangian cobordisms, we need to view $L_{\Sigma'}$ as the 1-jet of a function

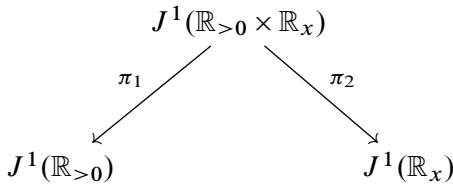
$$\tilde{F}: L_{\Sigma} \rightarrow \mathbb{R}$$

in the neighborhood of L_{Σ} and show that \tilde{F} is Morse. To describe the function

easily, pull it back to a function $\Sigma \rightarrow \mathbb{R}$, denoted by \tilde{F} as well. Note that $\tilde{F} = F$ on $\Sigma \cap T^*((a_- + \epsilon, a_+ - \epsilon) \times \mathbb{R})$.

Now let us focus on the part $s \in (a_+ - \epsilon, \infty)$. Denote $L_\Sigma \cap J^1((a_+ - \epsilon, \infty) \times \mathbb{R})$ by $\partial_+(L_\Sigma)$ and denote $\Sigma \cap T^*((a_+ - \epsilon, \infty) \times \mathbb{R})$ by $\partial_+(\Sigma)$. One can check that the Reeb chords from $\partial_+(L_\Sigma)$ to $\partial_+(L_{\Sigma'})$ are in bijective correspondence with the Reeb chords from Λ_+ to Λ'_+ by a property of the Morse Legendrian. As a result, the only critical points of \tilde{F} on $\partial_+(\Sigma)$ are (s, q) , where $s = a_+$ and $f'_+(q) = 0$.

Let π_1 and π_2 be the natural projections as follows:



First project $\partial_+(L_\Sigma)$ and $\partial_+(L_{\Sigma'})$ to $J^1(\mathbb{R}_x)$. We have

$$\pi_2(\partial_+(L_\Sigma)) = (x_+(q), (A_+ - (s - a_+)^2)y_+(q), (A_+ - (s - a_+)^2)z_+(q))$$

and

$$\pi_2(\partial_+(L_{\Sigma'})) = (x'_+(q), (A_+ - (s - a_+)^2)y'_+(q), (A_+ - (s - a_+)^2)z'_+(q)).$$

Thus for fixed $s \in (a_+ - \epsilon, \infty)$, we have $\tilde{F}(s, q) = (A_+ - (s - a_+)^2)f_+(q)$, where $f_+ = F|_{\{a_+\} \times \Lambda_+}$. Second, project $\partial_+(L_\Sigma)$ and $\partial_+(L_{\Sigma'})$ to $J^1(\mathbb{R}_{>0})$. We have

$$\pi_1(\partial_+(L_\Sigma)) = (s, -2(s - a_+)z_+(q), (A_+ - (s - a_+)^2)z_+(q)),$$

and

$$\pi_1(\partial_+(L_{\Sigma'})) = (s, -2(s - a_+)z'_+(q), (A_+ - (s - a_+)^2)z'_+(q)).$$

For a fixed $q \in \Lambda_+$, the only nondegenerate singularity of $\tilde{F}(s, q)$ is a_+ . In particular, it is a local maximum since $z'_+(q) > z_+(q)$, which comes from the fact that $f_+ > 0$ by the construction in Section 5.1. Therefore, we have

$$\text{Ind}_{\tilde{F}}(a_+, q) = \text{Ind}_{f_+}(q) + 1.$$

Similarly, on the negative side, denote $F|_{\{a_-\} \times \Lambda_-}$ as f_- . The critical points of \tilde{F} on $\Sigma \cap T^*((-\infty, a_- + \epsilon) \times \mathbb{R})$ agree with the critical points of f_- that lie in the slice $s = a_-$. Moreover, the indices satisfy $\text{Ind}_{\tilde{F}}(a_-, q) = \text{Ind}_{f_-}(q)$. Hence \tilde{F} is a Morse function.

Choose a Riemannian metric g on Σ and a generic almost complex structure J on $\mathbb{R} \times \mathbb{R}^3$ that is adjusted to cylindrical ends such that (\tilde{F}, g, J) is adjusted to L_Σ . Now we can apply Lemma 5.7 to the 2-copy $L_\Sigma \cup L_{\Sigma'}$.

Define a *generalized disk* to be a pair (u, γ) consisting of a J -holomorphic disk u with boundary on Σ as defined in Section 2.2 and a negative gradient flow line γ of \tilde{F} with one end on the boundary of u and one end at a critical point p of \tilde{F} such that the boundary of u intersects transversely with the negative gradient flow γ . The point p is called a *negative Morse puncture* if the flow line γ flows toward p , and is called a *positive Morse puncture* if γ flows away from p . The formal dimension $\dim(u, \gamma)$ is defined by

$$(9) \quad \dim(u, \gamma) = \begin{cases} \dim \mathcal{M} + 1 + \text{Ind}_f(p) - 2 & \text{if } p \text{ is a positive Morse puncture,} \\ \dim \mathcal{M} + 1 - \text{Ind}_f(p) & \text{if } p \text{ is a negative Morse puncture.} \end{cases}$$

The generalized disk (u, γ) is called *rigid* if $\dim(u, \gamma) = 0$. We have the following result, which is analogous to Lemma 5.7.

Theorem 5.9 *Let Σ be an exact Lagrangian cobordism in $(\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha))$ from Λ_- to Λ_+ that is cylindrical outside $[-N + \delta, N - \delta] \times \mathbb{R}^3$. Let $F: \Sigma \rightarrow \mathbb{R}_{>0}$ be a positive Morse function. Push Σ off through F and get a new cobordism Σ' .*

Denote $F|_{\{N\} \times \Lambda_+}$ by f_+ and $F|_{\{-N\} \times \Lambda_-}$ by f_- . Define a new Morse function $\tilde{F}: \Sigma \rightarrow \mathbb{R}$ satisfying the following properties:

- The Morse function \tilde{F} satisfies $\tilde{F} = F$ on $\Sigma \cap ([-N + \delta, N - \delta] \times \mathbb{R}^3)$.
- On $\Sigma \cap ((N - \delta, \infty) \times \mathbb{R}^3)$, all the critical points of the Morse function \tilde{F} lie in $\Sigma \cap (\{N\} \times \mathbb{R}^3) = \{N\} \times \Lambda_+$ and agree with the critical points of f_+ . Moreover, at each critical point c , we have $\text{Ind}_{\tilde{F}} c = \text{Ind}_{f_+} c + 1$.
- On $\Sigma \cap ((-\infty, -N + \delta) \times \mathbb{R}^3)$, all the critical points of the Morse function \tilde{F} lie in $\Sigma \cap (\{-N\} \times \mathbb{R}^3) = \{-N\} \times \Lambda_-$ and agree with the critical points of f_- . Moreover, at each critical point c , we have $\text{Ind}_{\tilde{F}} c = \text{Ind}_{f_-} c$.

The Riemannian metric g and almost complex structure J are chosen as above. Then we can describe the rigid holomorphic disks with boundary on $\Sigma \cup \Sigma'$ that have punctures on Reeb chords of $\Lambda_+ \cup \Lambda'_+$ and $\Lambda_- \cup \Lambda'_-$ in terms of whether the Reeb chords are Morse or non-Morse as defined in Section 3.2:

- (1) Rigid holomorphic disks with boundary on $\Sigma \cup \Sigma'$ that have two punctures at non-Morse mixed Reeb chords are in one-to-one correspondence with rigid holomorphic disks with boundary on Σ . See Figure 13 (top).
- (2) Rigid holomorphic disks with boundary on $\Sigma \cup \Sigma'$ that have exactly one puncture at a Morse Reeb chord are in one-to-one correspondence with rigid generalized disks (u, γ) determined by (\tilde{F}, g, J) . See Figure 13 (bottom).
- (3) Rigid holomorphic disks with boundary on $\Sigma \cup \Sigma'$ with two punctures at Morse Reeb chords are in one-to-one correspondence with rigid negative gradient flows of the Morse function \tilde{F} from a critical point on Λ_+ to a critical point on Λ_- .

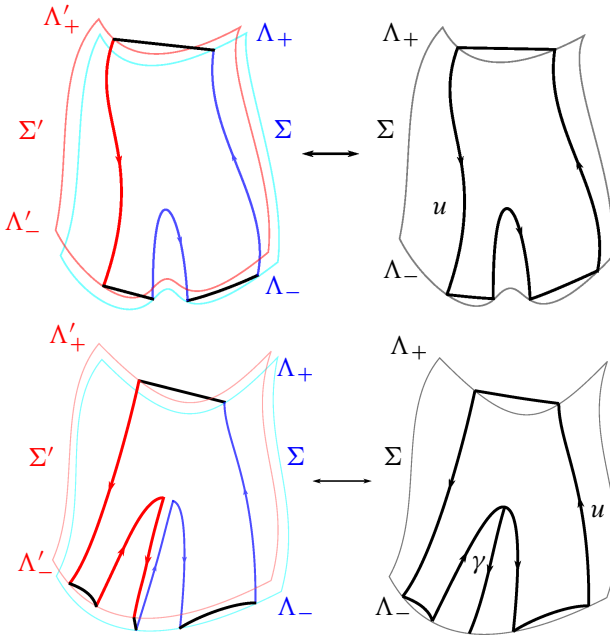


Figure 13: The correspondences in Theorem 5.9. The arrows denote the orientations of holomorphic disks and the negative gradient flow line.

Proof According to Lemma 5.8, rigid holomorphic disks with boundary on $\Sigma \cup \Sigma'$ that have two punctures at mixed Reeb chords correspond to rigid holomorphic disks with boundary on $L_\Sigma \cup L_{\Sigma'}$ that have positive (resp. negative) boundary at mixed Reeb chords lying in the slice $s = a_+$ (resp. $s = a_-$). By Lemma 5.7, these disks with boundary on $L_\Sigma \cup L_{\Sigma'}$ are in one-to-one correspondence with holomorphic disks with boundary on L_Σ that have positive (resp. negative) boundary at the Reeb chords lying in the slice $s = a_+$ (resp. $s = a_-$) together with Morse flow lines of \tilde{F} . If it is a rigid Morse flow line of \tilde{F} on L_Σ , it flows from a critical point on Λ_+ to a critical point on Λ_- . Pull it back to a flow line Σ and get the correspondence (3). If it is a rigid holomorphic disk with boundary on L_Σ that has positive (resp. negative) boundary at the Reeb chords lying in the slice $s = a_+$ (resp. $s = a_-$), by [19], it corresponds to a rigid holomorphic disk with boundary on Σ , which is the correspondence (1). Otherwise, it is a rigid generalized disk (u, γ) determined by (\tilde{F}, g, J) . From the construction of \tilde{F} , one can note that all the critical points of \tilde{F} on $\Sigma \cap (\{N\} \times \mathbb{R}^3)$ are of index 1 or 2 while all the critical points of \tilde{F} on $\Sigma \cap (\{-N\} \times \mathbb{R}^3)$ are of index 0 or 1. By the dimension formula (9), the generalized disk (u, γ) is rigid if and only if u is a rigid holomorphic disk. Each rigid holomorphic disk u with boundary on L_Σ that has positive (resp. negative) boundary at the Reeb chords lying in the

slice $s = a_+$ (resp. $s = a_-$) in turn corresponds to a rigid holomorphic disk with boundary on Σ . Pulling γ back to Σ , we get a rigid generalized disk on Σ determined by (\tilde{F}, g, J) . Hence we get the correspondence (2). \square

Recall f_1 is defined algebraically as follows. The exact Lagrangian cobordism Σ from a Legendrian knot Λ_- to a Legendrian knot Λ_+ induces a DGA map ϕ between the DGAs with a single basepoint by counting rigid holomorphic disks with boundary on Σ :

$$\phi: (\mathcal{A}(\Lambda_+), \partial) \rightarrow (\mathcal{A}(\Lambda_-), \partial),$$

as described in Section 2.2. This DGA map ϕ induces an A_∞ -category map

$$f: \text{Aug}_+(\Lambda_-) \rightarrow \text{Aug}_+(\Lambda_+)$$

in the way described in Section 3.2. Restricting the category map on the level of morphisms, we have

$$f_1: \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow \text{Hom}_+(\epsilon_+^1, \epsilon_+^2).$$

See calculation (4) for the explicit formula.

Theorem 5.10 *With a choice of Morse function $F: \Sigma \rightarrow \mathbb{R}$, we have $d_{+-} = f_1$.*

Proof We show $d_{+-} = f_1$ by checking their images on generators of $\text{Hom}_+(\epsilon_-^1, \epsilon_-^2)$. Recall that $\text{Hom}_+(\epsilon_-^1, \epsilon_-^2)$ is generated by the elements in $\text{Hom}_-(\epsilon_-^1, \epsilon_-^2)$ that correspond to non-Morse Reeb chords and the elements in $T = \{x_-^\vee, y_-^\vee\}$ that correspond to Morse Reeb chords, respectively.

First consider the element b^\vee in $\text{Hom}_-(\epsilon_-^1, \epsilon_-^2)$. Notice that Morse Reeb chords are much shorter than non-Morse Reeb chords. The energy restriction ensures that $d_{+-}(b^\vee)$ does not include any element in T . Therefore d_{+-} sends b^\vee to $a^\vee \in \text{Hom}_-(\epsilon_+^1, \epsilon_+^2)$ by counting rigid holomorphic disks $u \in \mathcal{M}(a^{12}; \mathbf{p}^{11}, b^{12}, \mathbf{q}^{22})$ with boundary on $\Sigma^1 \cup \Sigma^2$, where \mathbf{p}^{11} and \mathbf{q}^{22} are words of pure Reeb chords of Λ_-^1 and Λ_-^2 , respectively. According to the correspondence (1) in Theorem 5.9, these disks correspond to rigid holomorphic disks $u \in \mathcal{M}(a; \mathbf{p}, b, \mathbf{q})$ with boundary on Σ (as shown in Figure 14), which are the disks counted by f_1 . Notice that both d_{+-} and f_1 send b^\vee to $|\mathcal{M}(a; \mathbf{p}, b, \mathbf{q})| \epsilon_-^1(\mathbf{p}) \epsilon_-^2(\mathbf{q}) a^\vee$, where $|\mathcal{M}(a; \mathbf{p}, b, \mathbf{q})|$ is the number of rigid disks in \mathcal{M} counted with sign. Hence the definition of d_{+-} matches the definition of f_1 on $\text{Hom}_-(\epsilon_-^1, \epsilon_-^2)$.

In order to simplify the map d_{+-} , we can choose a Morse function F such that the negative gradient flow of F flows from $*_+$ directly to $*_-$ without going through any critical points. We can further require that the negative gradient flow of F behave the same in a collar neighborhood of the flow line from $*_+$ to $*_-$ as shown in Figure 15.

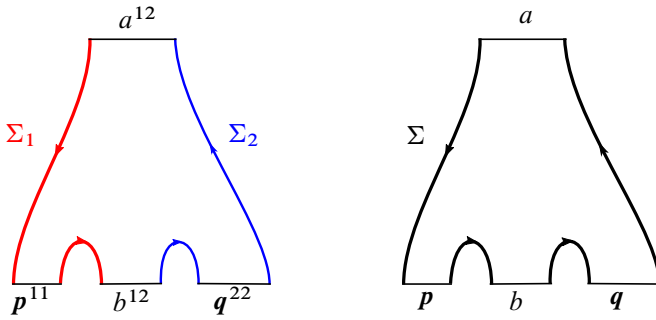


Figure 14: The disk on the left is counted by d_{+-} while the disk on the right is counted by ϕ .

As x_{\pm} and y_{\pm} sit right beside $*_{\pm}$, the negative gradient flow lines of \tilde{F} flow from x_+ and y_+ directly to x_- and y_- , respectively.

For the element $c^{\vee} \in T$, the map d_{+-} counts the rigid holomorphic disks in $\Sigma_1 \cup \Sigma_2$ that have a negative puncture at the Morse Reeb chord c . For the rigid disk that has a positive puncture at a Morse Reeb chord as well, according to the correspondence (3) in **Theorem 5.9**, it corresponds to a rigid Morse flow line of \tilde{F} . The indices of \tilde{F} on y_- , x_- , y_+ and x_+ are 0, 1, 1 and 2, respectively. Therefore $d_{+-}(x_+^{\vee})$ has x_+^{\vee} as a term and $d_{+-}(y_+^{\vee})$ has y_+^{\vee} as a term. If the rigid disk has a positive puncture at a non-Morse mixed Reeb chord a^{12} , we denote it by $u \in \mathcal{M}(a^{12}; p^{11}, c^{12}, q^{22})$. By the correspondence (2) in **Theorem 5.9**, it corresponds to a rigid generalized disk (u, γ) , where $u \in \mathcal{M}(a; p, q)$ is a holomorphic disk with boundary on Σ and γ is a Morse flow of \tilde{F} that flows toward c (see **Figure 16**). Due to the dimension formula (9) of generalized disks, no rigid disk has a negative puncture at y_- since $\text{Ind}_{\tilde{F}} y_- = 0$ but $\dim \mathcal{M} \geq 0$. Hence $d_{+-}(y_-^{\vee}) = y_+^{\vee}$, which matches the definition of f_1 on y_k^- .

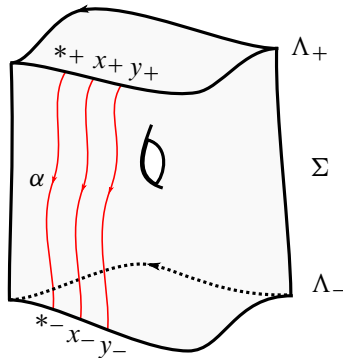


Figure 15: An example of Morse flows of F

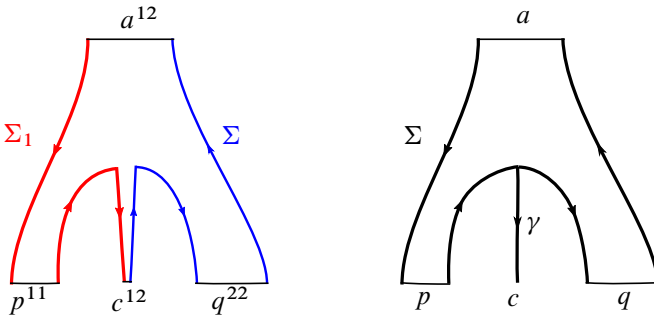


Figure 16: The disk on the left is counted by d_{+-} while the disk on the right is counted by ϕ .

For the element x_-^\vee , we know that $f_1(x_+^\vee)$ counts the element a^\vee if t shows up in the image of the DGA map $\phi(a)$. In other words, there exists a rigid holomorphic disk $u \in \mathcal{M}(a; \mathbf{p}, \mathbf{q})$ with boundary on Σ , where \mathbf{p} and \mathbf{q} are words of pure Reeb chords of Λ_- , such that u has a nontrivial intersection number with α , where α is the curve from the basepoint $*_+$ to $*_-$. Each rigid holomorphic disk u contributes to $f_1(x_+^\vee)$ a term of a^\vee with coefficient $s(u, \alpha)\epsilon_-^1(\mathbf{p})\epsilon_-^2(\mathbf{q})a^\vee$, where $s(u, \alpha)$ is the intersection number of the boundary of u and α . We can make the Morse function F satisfy the property that the negative gradient flow line γ of F from x_+ to x_- is parallel to α and of the same orientation. For each intersection point p_i of the boundary of u and γ , denote the part of γ from p_i to c by γ_i . By the correspondence (3) in Theorem 5.9, the rigid generalized disk (u, γ_i) corresponds to a rigid holomorphic disk in $\mathcal{M}(a^{12}; \mathbf{p}^{11}, c^{12}, \mathbf{q}^{22})$ with boundary on $\Sigma^1 \cup \Sigma^2$, and hence contributes to $d_{+-}(x_-^\vee)$ with a term $s(u, \gamma_i)\epsilon_-^1(\mathbf{p}^{11})\epsilon_-^2(\mathbf{q}^{22})a^\vee$, where $s(u, \gamma_i)$ is the sign of the intersection. Summing over all the intersections of the boundary of u and γ , the rigid holomorphic disk u contributes $s(u, \gamma)\epsilon_-^1(\mathbf{p})\epsilon_-^2(\mathbf{q})a^\vee$ to $d_{+-}(x_-^\vee)$, where $s(u, \gamma)$ is the intersection number of the boundary of u and γ . Therefore, we have $d_{+-} = f_1$ on x_-^\vee . \square

5.4 Aside

In this section, we describe the differential map of the Cthulhu chain complex in terms of holomorphic disks with boundary on Σ and Morse flow lines. This allows us to recover the long exact sequences in [6]. The theorem in this section is stated without rigorous proof. But it will not be used elsewhere in the paper.

In Section 5.3, we only need to describe the rigid disks with boundary on $\Sigma^1 \cup \Sigma^2$ that have punctures at Reeb chords. Hence we only have correspondences for those types of disks. However, the method should work for all the rigid holomorphic disks

with boundary on $\Sigma^1 \cup \Sigma^2$ including the disks counted by d_{+0} and d_{0-} . We state the following theorem without proof.

Theorem 5.11 *Let Σ be an exact Lagrangian cobordism from Λ_- to Λ_+ and $\Sigma^1 \cup \Sigma^2$ be a 2-copy of Σ as constructed in Section 5.1. For $i = 1, 2$, assume ϵ_-^i is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ_+^i is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ . For η small enough, the Cthulhu chain complex can be decomposed into five parts:*

$$\text{Cth}^k(\Sigma^1, \Sigma^2) = \text{Hom}_-^{k-1}(\epsilon_+^1, \epsilon_+^2) \oplus C_{\text{Morse}}^{k-1} f_+ \oplus C_{\text{Morse}}^k F \oplus \text{Hom}_-^k(\epsilon_-^1, \epsilon_-^2) \oplus C_{\text{Morse}}^k f_-.$$

Under this decomposition, the differential can be written as

$$d = \begin{pmatrix} m_1 & d_{+f_+} & d_{+F} & d_{+-} & d_{+f_-} \\ 0 & d_{f_+} & d_{f_+F} & 0 & d_{f_+f_-} \\ 0 & 0 & d_F & 0 & d_{Ff_-} \\ 0 & 0 & 0 & m_1 & d_{-f_-} \\ 0 & 0 & 0 & 0 & d_{f_-} \end{pmatrix}.$$

Moreover:

- (1) *The holomorphic disks counted by d_{+F} and d_{+f_-} are in one-to-one correspondence with rigid generalized disks on Σ determined by (\tilde{F}, g, J) .*
- (2) *The holomorphic disks counted by d_{+-} are in one-to-one correspondence with rigid holomorphic disks with boundary on Σ .*
- (3) *The holomorphic disks counted by d_{f_+F} , $d_{f_+f_-}$ and d_{Ff_-} are in one-to-one correspondence with the rigid Morse flow lines of \tilde{F} .*

This theorem is similar to the conjectural analytic Lemma 4.11 in [15], which describes the correspondence in the case of exact Lagrangian fillings.

We next discuss how to recover the three long exact sequences in [6] from this chain complex.

- (1) Decompose the Cthulhu chain complex as

$$\text{Hom}_-^{k-1}(\epsilon_+^1, \epsilon_+^2) \oplus (C_{\text{Morse}}^{k-1} f_+ \oplus C_{\text{Morse}}^k F) \oplus \text{Hom}_+^k(\epsilon_-^1, \epsilon_-^2).$$

Notice that the chain complex

$$\left(C_{\text{Morse}}^{k-1} f_+ \oplus C_{\text{Morse}}^k F, \begin{pmatrix} d_{f_+} & d_{f_+F} \\ 0 & d_F \end{pmatrix} \right)$$

can be identified with the Morse cochain complex $(C_{\text{Morse}}^k \bar{F}, d_{\bar{F}})$ induced by a Morse function \bar{F} , where \bar{F} agrees with \tilde{F} near Λ_+ and agrees with F elsewhere. Hence

$$H^*(C_{\text{Morse}}^k \bar{F}, d_{\bar{F}}) = H^k(\Sigma, \Lambda_+ \cup \Lambda_-).$$

Therefore, we have the following long exact sequence:

$$\begin{aligned} \cdots \rightarrow H^k(\Sigma, \Lambda_+ \cup \Lambda_-) &\rightarrow H^k \text{Hom}_-(\epsilon_+^1, \epsilon_+^2) \rightarrow H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \\ &\rightarrow H^{k+1}(\Sigma, \Lambda_+ \cup \Lambda_-) \rightarrow \cdots, \end{aligned}$$

which is Theorem 1.5 of [6].

(2) To obtain another long exact sequence, view the Cthulhu chain complex as a direct sum of $\text{Hom}_-^{k-1}(\epsilon_+^1, \epsilon_+^2)$, $C_{\text{Morse}}^{k-1}f_+ \oplus C_{\text{Morse}}^k F \oplus \text{Hom}_-^k(\epsilon_-^1, \epsilon_-^2)$ and the Morse cochain complex $C_{\text{Morse}}^k f_-$. We have

$$\begin{aligned} \cdots \rightarrow H^{k-1}(\Lambda_-) &\rightarrow H^k(\Sigma, \Lambda_+ \cup \Lambda_-) \oplus H^k \text{Hom}_-(\epsilon_-^1, \epsilon_-^2) \rightarrow H^k \text{Hom}_-(\epsilon_+^1, \epsilon_+^2) \\ &\rightarrow H^k(\Lambda_-) \rightarrow \cdots, \end{aligned}$$

which is Theorem 1.6 in [6].

(3) Rewrite the Cthulhu chain complex as

$$\begin{aligned} \text{Cth}^k(\Sigma^1, \Sigma^2) &= \text{Hom}_-^{k-1}(\epsilon_+^1, \epsilon_+^2) \oplus \text{Hom}_-^k(\epsilon_-^1, \epsilon_-^2) \oplus C_{\text{Morse}}^{k-1}f_+ \oplus C_{\text{Morse}}^k F \oplus C_{\text{Morse}}^k f_- \\ &= \text{Hom}_-^{k-1}(\epsilon_+^1, \epsilon_+^2) \oplus \text{Hom}_-^k(\epsilon_-^1, \epsilon_-^2) \oplus C_{\text{Morse}}^k \tilde{F} \end{aligned}$$

with the differential

$$d = \begin{pmatrix} m_1 & * & * \\ 0 & m_1 & * \\ 0 & 0 & d_{\tilde{F}} \end{pmatrix}.$$

We have the long exact sequence

$$\begin{aligned} \cdots \rightarrow H^k \text{Hom}_-(\epsilon_-^1, \epsilon_-^2) &\rightarrow H^k \text{Hom}_-(\epsilon_+^1, \epsilon_+^2) \\ &\rightarrow H^{k+1}(\Sigma, \Lambda_+) \rightarrow H^{k+1} \text{Hom}_-(\epsilon_-^1, \epsilon_-^2) \rightarrow \cdots, \end{aligned}$$

which is Theorem 1.4 in [6].

One may get other long exact sequences from the Cthulhu chain complex above. One example is

$$\begin{aligned} \cdots \rightarrow H^k \text{Hom}_-(\epsilon_-^1, \epsilon_-^2) &\rightarrow H^k \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \\ &\rightarrow H^k(\Sigma) \rightarrow H^{k+1} \text{Hom}_-(\epsilon_-^1, \epsilon_-^2) \rightarrow \cdots, \end{aligned}$$

which is obtained by decomposing the Cthulhu chain complex as a direct sum of $\text{Hom}_+^{k-1}(\epsilon_+^1, \epsilon_+^2)$, $\text{Hom}_-^k(\epsilon_-^1, \epsilon_-^2)$ and $C_{\text{Morse}}^k F \oplus C_{\text{Morse}}^k f_-$.

5.5 Injectivity

Theorem 5.10 implies that d_{+-} is a chain map and hence it gives the Cthulhu chain complex a stronger algebraic structure. In this section, we use this algebraic information to deduce that the augmentation category map induced by the exact Lagrangian cobordism Σ is injective on the level of equivalence classes of objects. And its induced map on the cohomology category $H^* \text{Aug}_+$ is faithful.

Notice that $d_{+-} = f_1$ implies that d_{+-} is a chain map and thus induces maps $d_{+-}^k: H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow H^k \text{Hom}_+(\epsilon_+^1, \epsilon_+^2)$ for $k \in \mathbb{Z}$. Then we have the following theorem deduced from the double cone structure of the Cthulhu chain complex. We would like to thank the referee for pointing out this theorem.

Theorem 5.12 *Let Σ be an exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+ with Maslov number 0. For $i = 1, 2$, assume ϵ_-^i is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ_+^i is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ . With the same choice of Morse function as in **Theorem 5.10**, we have the following statement.*

For fixed $k \in \mathbb{Z}$, the map

$$i^k: H^k \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \rightarrow H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2)$$

in the long exact sequence (6) is injective (resp. surjective) if and only if the map

$$d_{+-}^k: H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow H^k \text{Hom}_+(\epsilon_+^1, \epsilon_+^2)$$

is surjective (resp. injective).

Proof We will first prove that i^k is surjective if and only if d_{+-}^k is injective for fixed k .

Consider the Cthulhu chain complex as a mapping cone of $\Phi: \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow \text{Cone}(d_{+0})$, where $\Phi = d_{+-} + d_{0-}$. The trivial cohomology of the Cthulhu chain complex implies that Φ induces isomorphisms

$$\Phi^k: H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow H^k \text{Cone}(d_{+0}) \quad \text{for } k \in \mathbb{Z}.$$

The following diagram commutes:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^k \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) & \xrightarrow{i_k} & H^k \text{Cone}(d_{+0}) & \xrightarrow{j_k} & H^{k+1}(C_{\text{Morse}} F) \longrightarrow \dots \\ & & & & \uparrow \cong & \nearrow d_{0-}^k & \\ & & & & H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) & & \end{array}$$

Comparing this to the long exact sequence (6), we know that $i^k = (\Phi^k)^{-1} \circ i_k$. Notice that d_{+-} is a chain map and thus $d_{+0} \circ d_{0-} = 0$. It is not hard to show that $\Phi^k = d_{+-}^k + d_{0-}^k$. Thus

$$H^k \text{Cone}(d_{+0}) \cong d_{+-}^k (H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2)) \oplus d_{0-}^k (H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2)).$$

Since we are working over the field \mathbb{F} , we have the following relation on dimensions:

$$\begin{aligned} \dim(H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2)) &= \dim(H^k \text{Cone}(d_{+0})) \\ &= \dim(d_{+-}^k (H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2))) + \dim(d_{0-}^k (H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2))). \end{aligned}$$

Therefore $\dim(d_{+-}^k (H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2))) \leq \dim(H^k \text{Hom}_+(\epsilon_-^1, \epsilon_-^2))$ and the equality holds if and only if $d_{0-}^k = 0$, which is equivalent to the condition that i^k is surjective. Hence d_{+-}^k is injective if and only if i^k is surjective.

The proof of the statement that i^k is injective if and only if d_{+-}^k is surjective is basically the same if we consider the Cthulhu chain complex as a mapping cone of $\Psi: \text{Cone}(d_{0-}) \rightarrow \text{Hom}_+(\epsilon_+^1, \epsilon_+^2)$, where $\Psi = d_{+0} + d_{+-}$. □

Thanks to Theorem 5.10, we know that d_{+-} agrees with f_1 . Theorem 5.4 shows that i^0 is both injective and surjective. Therefore we have the following corollary.

Corollary 5.13 *Let f^* denote the induced map of f_1 on cohomology. Then f^* restricted on degree-0 cohomology,*

$$f^*: H^0 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2),$$

is an isomorphism.

Theorem 5.14 *Let Σ be an exact Lagrangian cobordism with Maslov number 0 from a Legendrian knot Λ_- to a Legendrian knot Λ_+ . The A_∞ -category map $f: \text{Aug}_+(\Lambda_-) \rightarrow \text{Aug}_+(\Lambda_+)$ induced by the exact Lagrangian cobordism Σ is injective on the level of equivalence classes of objects.*

In other words, for $i = 1, 2$, assume ϵ_-^i is an augmentation of $\mathcal{A}(\Lambda_-)$ with a single basepoint and ϵ_+^i is the augmentation of $\mathcal{A}(\Lambda_+)$ with a single basepoint induced by Σ . If ϵ_+^1 and ϵ_+^2 are equivalent in $\text{Aug}_+(\Lambda_+)$, then ϵ_-^1 and ϵ_-^2 are equivalent in $\text{Aug}_+(\Lambda_-)$.

Proof Since ϵ_+^1 and ϵ_+^2 are equivalent in $\text{Aug}_+(\Lambda_+)$, there exist

$$[\alpha_+] \in H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2) \quad \text{and} \quad [\beta_+] \in H^0 \text{Hom}_+(\epsilon_+^2, \epsilon_+^1)$$

such that

$$[m_2(\alpha_+, \beta_+)] = [e_{\epsilon_+^2}] \in H^0 \text{Hom}_+(\epsilon_+^2, \epsilon_+^2)$$

and

$$[m_2(\beta_+, \alpha_+)] = [e_{\epsilon_+^1}] \in H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^1).$$

Corollary 5.13 shows that $f^*: H^0 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2)$ is an isomorphism. Hence there exists $[\alpha_-] \in H^0 \text{Hom}_+(\epsilon_-^1, \epsilon_-^2)$ such that

$$f^*([\alpha_-]) = [\alpha_+] \in H^0 \text{Hom}_+(\epsilon_+^1, \epsilon_+^2).$$

Similarly, there exists $[\beta_-] \in H^0 \text{Hom}_+(\epsilon_-^2, \epsilon_-^1)$ such that

$$f^*([\beta_-]) = [\beta_+] \in H^0 \text{Hom}_+(\epsilon_+^2, \epsilon_+^1).$$

Moreover, we have

$$\begin{aligned} f^*[m_2(\alpha_-, \beta_-)] &= m_2(f^*([\alpha_-]), f^*([\beta_-])) \\ &= m_2([\alpha_+], [\beta_+]) \\ &= [e_{\epsilon_+^2}] \in H^0 \text{Hom}_+(\epsilon_+^2, \epsilon_+^2). \end{aligned}$$

Notice that f sends $y_-^\vee \in \text{Hom}_+^0(\epsilon_-^2, \epsilon_-^2)$ to $y_+^\vee \in \text{Hom}_+^0(\epsilon_+^2, \epsilon_+^2)$ and hence we have $f^*[e_{\epsilon_-^2}] = [e_{\epsilon_+^2}]$. By **Corollary 5.13**, the map

$$f^*: H^0 \text{Hom}_+(\epsilon_-^2, \epsilon_-^2) \rightarrow H^0 \text{Hom}_+(\epsilon_+^2, \epsilon_+^2)$$

is an isomorphism. Hence $[m_2(\alpha_-, \beta_-)] = [e_{\epsilon_-^2}] \in H^0 \text{Hom}_+(\epsilon_-^2, \epsilon_-^2)$. Similarly, $[m_2(\beta_-, \alpha_-)] = [e_{\epsilon_-^1}] \in H^0 \text{Hom}_+(\epsilon_-^1, \epsilon_-^1)$. Therefore ϵ_-^1 and ϵ_-^2 are equivalent in $\text{Aug}_+(\Lambda_-)$. \square

In addition, the exact Lagrangian cobordism Σ described above also induces a category functor on the cohomology category

$$\tilde{f}: H^* \text{Aug}_+(\Lambda_-) \rightarrow H^* \text{Aug}_+(\Lambda_+),$$

as described in **Section 3.1**.

We have the following statement.

Theorem 5.15 *Let Σ be an exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+ with Maslov number 0. The corresponding cohomology category map $\tilde{f}: H^* \text{Aug}_+(\Lambda_-) \rightarrow H^* \text{Aug}_+(\Lambda_+)$ induced by Σ is faithful. Moreover, if $\chi(\Sigma) = 0$, this functor is fully faithful.*

Proof Notice that the category map \tilde{f} restricted on the level of morphisms is

$$f^*: H^* \text{Hom}_+(\epsilon_-^1, \epsilon_-^2) \rightarrow H^* \text{Hom}_+(\epsilon_+^1, \epsilon_+^2).$$

The long exact sequence (6) tells us that the i^k are surjective for all $k \in \mathbb{Z}$. By Theorem 5.12, we know that f^* is injective. Thus \tilde{f} is faithful.

In particular, if $\chi(\Sigma) = 0$, Theorem 5.4 implies that the i^k are isomorphisms for all $k \in \mathbb{Z}$. Therefore, by Theorem 5.12, the map f^* is an isomorphism, which implies that \tilde{f} is fully faithful. \square

As a result of Theorem 5.14, there is an induced map from the equivalence classes of augmentations of Λ_- to the equivalence classes of augmentations of $\text{Aug}_+(\Lambda_+)$. Thus the number of equivalence classes of augmentations of $\mathcal{A}(\Lambda_-)$ is less than or equal to the number of equivalence classes of augmentations of $\mathcal{A}(\Lambda_+)$. However, the equivalence classes of augmentations is difficult to count in general. Ng, Rutherford, Shende and Sivek [32] introduced another way to count objects: the homotopy cardinality of $\pi_{\geq 0} \text{Aug}_+(\Lambda; \mathbb{F}_q)^*$, where \mathbb{F}_q is a finite field. This can be computed using ruling polynomials.

The homotopy cardinality is defined by

$$\begin{aligned} \pi_{\geq 0} \text{Aug}_+(\Lambda; \mathbb{F}_q)^* &= \sum_{[\epsilon] \in \text{Aug}_+(\Lambda; \mathbb{F}_q)/\sim} \frac{1}{|\text{Aut}(\epsilon)|} \cdot \frac{|H^{-1} \text{Hom}_+(\epsilon, \epsilon)| \cdot |H^{-3} \text{Hom}_+(\epsilon, \epsilon)| \cdots}{|H^{-2} \text{Hom}_+(\epsilon, \epsilon)| \cdot |H^{-4} \text{Hom}_+(\epsilon, \epsilon)| \cdots}, \end{aligned}$$

where $[\epsilon]$ is the equivalence class of ϵ in the augmentation category $\text{Aug}_+(\Lambda)$ and $|\text{Aut}(\epsilon)|$ is the number of invertible elements in $H^0 \text{Hom}_+(\epsilon, \epsilon)$.

Corollary 5.16 *Let Σ be a spin exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+ with Maslov number 0. Then for any finite field \mathbb{F}_q , we have*

$$\pi_{\geq 0} \text{Aug}_+(\Lambda_-; \mathbb{F}_q)^* \leq \pi_{\geq 0} \text{Aug}_+(\Lambda_+; \mathbb{F}_q)^*.$$

Proof Assume $[\epsilon_-]$ is an equivalence class in $\text{Aug}_+(\Lambda_-; \mathbb{F}_q)$ and $[\epsilon_+]$ is the induced equivalence class in $\text{Aug}_+(\Lambda_+; \mathbb{F}_q)$. Theorem 5.4 implies

$$H^k \text{Hom}_+(\epsilon_-, \epsilon_-) \cong H^k \text{Hom}_+(\epsilon_+, \epsilon_+) \quad \text{for } k < 1.$$

In particular, we have $H^0 \text{Hom}_+(\epsilon_-, \epsilon_-) \cong H^0 \text{Hom}_+(\epsilon_+, \epsilon_+)$, which implies that $|\text{Aut}(\epsilon_-)| = |\text{Aut}(\epsilon_+)|$.

Notice that $\text{Aug}_+(\Lambda_+; \mathbb{F}_q)$ may have more equivalence classes than $\text{Aug}_+(\Lambda_-; \mathbb{F}_q)$. Therefore, we have

$$\pi_{\geq 0} \text{Aug}_+(\Lambda_-; \mathbb{F}_q)^* \leq \pi_{\geq 0} \text{Aug}_+(\Lambda_+; \mathbb{F}_q)^*. \quad \square$$

From [32, Corollary 2], this cardinality can be related to the ruling polynomial in the following way:

$$\pi_{\geq 0} \text{Aug}_+(\Lambda; \mathbb{F}_q)^* = q^{\text{tb}(\Lambda)/2} R_\Lambda(q^{1/2} - q^{-1/2}).$$

Recall that a *normal ruling* R is a decomposition of the front projection of Λ into embedded disks connected by switches that satisfy certain requirements (see details in [8]). The *ruling polynomial* is defined by

$$R_\Lambda(z) = \sum_R z^{\#(\text{switches}) - \#(\text{disks})}.$$

Corollary 5.17 *Suppose there is a spin exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+ with Maslov number 0. Then the ruling polynomials R_{Λ_-} and R_{Λ_+} satisfy*

$$R_{\Lambda_-}(q^{1/2} - q^{-1/2}) \leq q^{-\chi(\Sigma)/2} R_{\Lambda_+}(q^{1/2} - q^{-1/2})$$

for any q that is a power of a prime number.

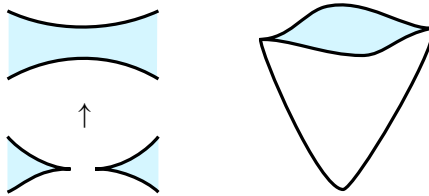


Figure 17: The relation between rulings of Legendrian submanifolds that are related by a pinch move (left) or a minimum cobordism (right)

When Σ is decomposable, ie consists of pinch moves and minimum cobordisms [19], there is a map from the rulings of Λ_- to rulings of Λ_+ . For each pinch move or minimal cobordism, any normal ruling of the bottom knot gives a normal ruling of the top knot, as shown in Figure 17. Moreover, different rulings of the bottom knot give different rulings of the top knot. Therefore the ruling polynomials of Λ_+ and Λ_- satisfy the relation in Corollary 5.17. This corollary shows that the result is true even if the cobordism is not decomposable.

One can check the atlas in [9] for the ruling polynomials of small crossing Legendrian knots. This corollary gives a new and easily computable obstruction to the existence of exact Lagrangian cobordisms. We can use Corollary 5.17 to give a new proof of the follow theorem which is a result in [5] and was reproved in [11].

Theorem 5.18 [5] *Lagrangian concordance is not a symmetric relation.*

Proof Consider the Legendrian knot Λ of smooth knot type $m(9_{46})$ with maximum Thurston–Bennequin number and the Legendrian unknot Λ_0 as shown in Figure 18. There is an exact Lagrangian concordance from the Λ_0 to Λ by doing a pinch move at the red (straight) line in Figure 18 and Legendrian isotopy. However, there does not exist an exact Lagrangian concordance from Λ to Λ_0 since the ruling polynomial of Λ is 2 while the ruling polynomial of Λ_0 is 1. \square

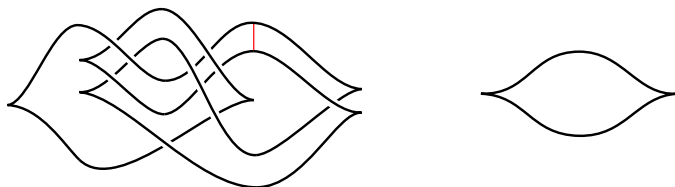


Figure 18: Front projections of the Legendrian knot Λ of knot type $m(9_{46})$ (left) and the Legendrian unknot Λ_0 (right)

References

- [1] **J A Baldwin, S Sivek**, *Invariants of Legendrian and transverse knots in monopole knot homology*, preprint (2014) [arXiv](#) To appear in J. Symplectic Geom.
- [2] **F Bourgeois, B Chantraine**, *Bilinearized Legendrian contact homology and the augmentation category*, J. Symplectic Geom. 12 (2014) 553–583 [MR](#)
- [3] **F Bourgeois, J M Sabloff, L Traynor**, *Lagrangian cobordisms via generating families: construction and geography*, Algebr. Geom. Topol. 15 (2015) 2439–2477 [MR](#)
- [4] **B Chantraine**, *Lagrangian concordance of Legendrian knots*, Algebr. Geom. Topol. 10 (2010) 63–85 [MR](#)
- [5] **B Chantraine**, *Lagrangian concordance is not a symmetric relation*, Quantum Topol. 6 (2015) 451–474 [MR](#)
- [6] **B Chantraine, G Dimitroglou Rizell, P Ghiggini, R Golovko**, *Floer homology and Lagrangian concordance*, from “Proceedings of the Gökova Geometry–Topology Conference” (S Akbulut, D Auroux, T Önder, editors), GGT, Gökova (2015) 76–113 [MR](#)

- [7] **Y Chekanov**, *Differential algebra of Legendrian links*, Invent. Math. 150 (2002) 441–483 [MR](#)
- [8] **Y V Chekanov**, *Invariants of Legendrian knots*, from “Proceedings of the International Congress of Mathematicians, II” (T Li, editor), Higher Ed. Press, Beijing (2002) 385–394 [MR](#)
- [9] **W Chongchitmate, L Ng**, *An atlas of Legendrian knots*, Exp. Math. 22 (2013) 26–37 [MR](#)
- [10] **G Civan, P Koprowski, J Etnyre, J M Sabloff, A Walker**, *Product structures for Legendrian contact homology*, Math. Proc. Cambridge Philos. Soc. 150 (2011) 291–311 [MR](#)
- [11] **C Cornwell, L Ng, S Sivek**, *Obstructions to Lagrangian concordance*, Algebr. Geom. Topol. 16 (2016) 797–824 [MR](#)
- [12] **G Dimitroglou Rizell**, *Exact Lagrangian caps and non-uniruled Lagrangian submanifolds*, Ark. Mat. 53 (2015) 37–64 [MR](#)
- [13] **G Dimitroglou Rizell**, *Lifting pseudo-holomorphic polygons to the symplectisation of $P \times \mathbb{R}$ and applications*, Quantum Topol. 7 (2016) 29–105 [MR](#)
- [14] **T Ekholm**, *Rational symplectic field theory over \mathbb{Z}_2 for exact Lagrangian cobordisms*, J. Eur. Math. Soc. 10 (2008) 641–704 [MR](#)
- [15] **T Ekholm**, *Rational SFT, linearized Legendrian contact homology, and Lagrangian Floer cohomology*, from “Perspectives in analysis, geometry, and topology” (I Itenberg, B Jöricke, M Passare, editors), Progr. Math. 296, Springer (2012) 109–145 [MR](#)
- [16] **T Ekholm, J B Etnyre, J M Sabloff**, *A duality exact sequence for Legendrian contact homology*, Duke Math. J. 150 (2009) 1–75 [MR](#)
- [17] **T Ekholm, J Etnyre, M Sullivan**, *The contact homology of Legendrian submanifolds in \mathbb{R}^{2n+1}* , J. Differential Geom. 71 (2005) 177–305 [MR](#)
- [18] **T Ekholm, J Etnyre, M Sullivan**, *Orientations in Legendrian contact homology and exact Lagrangian immersions*, Internat. J. Math. 16 (2005) 453–532 [MR](#)
- [19] **T Ekholm, K Honda, T Kálmán**, *Legendrian knots and exact Lagrangian cobordisms*, J. Eur. Math. Soc. 18 (2016) 2627–2689 [MR](#)
- [20] **T Ekholm, T Kálmán**, *Isotopies of Legendrian 1–knots and Legendrian 2–tori*, J. Symplectic Geom. 6 (2008) 407–460 [MR](#)
- [21] **Y Eliashberg**, *Invariants in contact topology*, Doc. Math. (1998) 327–338 [MR](#)
- [22] **Y Eliashberg, A Givental, H Hofer**, *Introduction to symplectic field theory*, Geom. Funct. Anal. (2000) 560–673 [MR](#)
- [23] **J B Etnyre, L L Ng, J M Sabloff**, *Invariants of Legendrian knots and coherent orientations*, J. Symplectic Geom. 1 (2002) 321–367 [MR](#)

- [24] **A Floer**, *Morse theory for Lagrangian intersections*, J. Differential Geom. 28 (1988) 513–547 [MR](#)
- [25] **E Getzler**, **JDS Jones**, *A_∞ -algebras and the cyclic bar complex*, Illinois J. Math. 34 (1990) 256–283 [MR](#)
- [26] **B Keller**, *Introduction to A -infinity algebras and modules*, Homology Homotopy Appl. 3 (2001) 1–35 [MR](#)
- [27] **C Levenson**, *Augmentations and rulings of Legendrian knots*, J. Symplectic Geom. 14 (2016) 1089–1143 [MR](#)
- [28] **M Menke**, *On the augmentation categories of positive braid closures*, preprint (2015) [arXiv](#)
- [29] **D Nadler**, **E Zaslow**, *Constructible sheaves and the Fukaya category*, J. Amer. Math. Soc. 22 (2009) 233–286 [MR](#)
- [30] **LL Ng**, *Computable Legendrian invariants*, Topology 42 (2003) 55–82 [MR](#)
- [31] **L Ng**, **D Rutherford**, *Satellites of Legendrian knots and representations of the Chekanov–Eliashberg algebra*, Algebr. Geom. Topol. 13 (2013) 3047–3097 [MR](#)
- [32] **L Ng**, **D Rutherford**, **V Shende**, **S Sivek**, *The cardinality of the augmentation category of a Legendrian link*, preprint (2015) [arXiv](#) To appear in Math. Res. Lett.
- [33] **L Ng**, **D Rutherford**, **V Shende**, **S Sivek**, **E Zaslow**, *Augmentations are sheaves*, preprint (2015) [arXiv](#)
- [34] **JM Sabloff**, **L Traynor**, *Obstructions to Lagrangian cobordisms between Legendrians via generating families*, Algebr. Geom. Topol. 13 (2013) 2733–2797 [MR](#)
- [35] **P Seidel**, *Fukaya categories and Picard–Lefschetz theory*, Eur. Math. Soc., Zürich (2008) [MR](#)
- [36] **JD Stasheff**, *Homotopy associativity of H -spaces, I*, Trans. Amer. Math. Soc. 108 (1963) 275–292 [MR](#)
- [37] **JD Stasheff**, *Homotopy associativity of H -spaces, II*, Trans. Amer. Math. Soc. 108 (1963) 293–312 [MR](#)

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