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An algebraic model for commutative $H\mathbb{Z}$–algebras

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We show that the homotopy category of commutative algebra spectra over the Eilenberg–Mac Lane spectrum of an arbitrary commutative ring R is equivalent to the homotopy category of E_∞–monoids in unbounded chain complexes over R. We do this by establishing a chain of Quillen equivalences between the corresponding model categories. We also provide a Quillen equivalence to commutative monoids in the category of functors from the category of finite sets and injections to unbounded chain complexes.

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1 Introduction

Let R be an arbitrary commutative ring. In Shipley [29] it was shown that the model category of algebra spectra over the Eilenberg–Mac Lane spectrum, HR, is connected to the model category of differential graded R–algebras via a chain of Quillen equivalences. In this paper we extend this result to the case of commutative HR–algebra spectra. As a guiding example we consider the function spectrum $F(X, HR)$ from a space X to the Eilenberg–Mac Lane spectrum of a commutative ring R. As R is commutative, $F(X, HR)$ is a commutative HR–algebra spectrum whose homotopy groups are the cohomology groups of the space X with coefficients in R:

$$\pi_{-n} F(X, HR) \cong H^n(X; R).$$

The singular cochains on X with coefficients in R, denoted by $S^*(X; R)$, give a chain model of the cohomology of X by regrading. We set

$$C_{-*}(X; R) := S^*(X; R).$$

Note that for $R = \mathbb{F}_p$ the Steenrod operations on $H^*(X; R)$ can be constructed from the $\cup_i$–products. These are chain homotopies that measure the failure of the cup-product to produce a strictly graded commutative product of cochains. Thus, in general, one cannot expect to find a model of the singular cochains of a space that is a differential graded commutative R–algebra. Instead, one must work with E_∞–algebra structures.

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See also Cenkl [3, Theorem 2]. A notable exception are rational cochains of a space with the Sullivan cochains as a strictly differential graded commutative model.

We establish a chain of Quillen equivalences between commutative HR–algebra spectra, $C(HR\text{–mod})$, and differential graded E_∞–R–algebras, $E_\infty\mathsf{Ch}_R$:

$$\begin{array}{ccccccc} C(HR\text{–mod}) & \underset{U}{\overset{Z}{\rightleftarrows}} & C(\mathsf{Sp}^\Sigma(s\mathsf{mod}_R)) & \underset{\Phi^*N}{\overset{L_N}{\leftrightarrows}} & C(\mathsf{Sp}^\Sigma(\mathsf{ch}_R)) & \underset{C_0}{\overset{i}{\rightleftarrows}} & C(\mathsf{Sp}^\Sigma(\mathsf{Ch}_R)) \\ & & & & & & L_\varepsilon \uparrow \downarrow R_\varepsilon \\ & & & & E_\infty\mathsf{Ch}_R & \underset{\mathrm{Ev}_0}{\overset{F_0}{\rightleftarrows}} & E_\infty(\mathsf{Sp}^\Sigma(\mathsf{Ch}_R)) \end{array}$$

Here, our intermediary categories include symmetric spectra (Sp^Σ) over the categories of simplicial R–modules ($s\mathsf{mod}_R$), nonnegatively graded chain complexes over R (ch_R), and unbounded chain complexes over R (Ch_R). The functors will be introduced in the sections below.

The fact that there is such an equivalence should not be surprising, but to our knowledge, no explicit chain of Quillen equivalences can be found in the literature.

In the context of infinite loop space theory, E_∞–ring spectra, and their units, the theory of $\mathcal{I}$–spaces is important; see Sagave and Schlichtkrull [22]. Here $\mathcal{I}$ is the category of finite sets and injections and $\mathcal{I}$–spaces are functors from $\mathcal{I}$ to simplicial sets. More generally, functor categories from $\mathcal{I}$ to categories of modules feature as FI–modules in the work of Church, Ellenberg and Farb [6] and others. We relate symmetric spectra in unbounded chain complexes over R via a chain of Quillen equivalences to the category of unbounded $\mathcal{I}$–chain complexes and prove that commutative monoids in this category, $C(\mathsf{Ch}_R^{\mathcal{I}})$, provide an alternative model for commutative HR–algebra spectra. In fact, there is a chain of Quillen equivalences between $C(HR\text{–mod})$ and $E_\infty(\mathsf{Ch}_R^{\mathcal{I}})$, the E_∞–monoids in unbounded $\mathcal{I}$–chain complexes over R, that passes via $E_\infty(\mathsf{Sp}^\Sigma(\mathsf{Ch}_R))$ and $E_\infty\mathsf{Ch}_R$. The rigidification result of Pavlov and Scholbach [20, Theorem 3.4.4] for symmetric spectra implies that the model category $E_\infty(\mathsf{Ch}_R^{\mathcal{I}})$ is Quillen equivalent to the one of commutative monoids in $\mathsf{Ch}_R^{\mathcal{I}}$, that is, $C(\mathsf{Ch}_R^{\mathcal{I}})$. Taking these results together we obtain a chain of Quillen equivalences between commutative HR–algebra spectra and commutative monoids in $\mathcal{I}$–chain complexes over R. See Theorem 9.5. We expect that our comparison result makes it possible to find explicit commutative $\mathcal{I}$–chain models for certain commutative HR–algebras and there is ongoing work on this by Richter, Sagave and Schulz with applications to logarithmic structures on commutative ring spectra in mind.

If $R = \mathbb{Q}$ is the field of rational numbers we can extend our chain of Quillen equivalences and obtain a comparison (Corollary 8.4) between commutative $H\mathbb{Q}$–algebra spectra and differential graded commutative $\mathbb{Q}$–algebras.

Mike Mandell showed in [16, Theorem 7.11] that for every commutative ring R the homotopy categories of E_∞–HR–algebra spectra and of E_∞–monoids in the category of unbounded R–chain complexes are equivalent. He also claims in loc. cit. that he can improve this equivalence of homotopy categories to an actual chain of Quillen equivalences. He suggests using the methods of Schwede and Shipley [27], but only associative monoids are treated there.

Our approach is different from Mandell's because we work in the setting of symmetric spectra. The idea to integrate the symmetric groups into the monoidal structure to construct a symmetric monoidal category of spectra is due to Jeff Smith. Our arguments heavily rely on combinatorial and monoidal features of the category of symmetric spectra in the categories of simplicial sets, simplicial R–modules, nonnegatively graded chain complexes (ch_R) and unbounded chain complexes (Ch_R).

The structure of the paper is as follows: We recall some basic facts and some model categorical features of symmetric spectra in Section 2. In Section 3 we recall results from Pavlov and Scholbach [19; 20] that establish model structures on commutative ring spectra in the cases that arise as intermediate steps in our chain of Quillen equivalences and we also recall their rigidification result. We sketch how to use methods from Chadwick and Mandell [4] for an alternative proof. The Quillen equivalence between commutative HR–algebra spectra and commutative symmetric ring spectra in simplicial R–modules can be found in Section 4 as Theorem 4.1. The Quillen equivalence between the latter model category and commutative symmetric ring spectra in nonnegatively graded chain complexes is based on the Dold–Kan correspondence and is stated as Theorem 6.6 in Section 6. There is a natural inclusion functor $i\colon \mathsf{ch} \to \mathsf{Ch}$ and the Quillen equivalence between commutative symmetric ring spectra in ch and in Ch (see Corollary 7.3) is based on this functor. In Section 8 we establish a Quillen equivalence between E_∞–monoids in symmetric spectra in unbounded chain complexes and E_∞–monoids in unbounded chain complexes. The link with E_∞–monoids and commutative monoids in the diagram category of chain complexes indexed by the category of finite sets and injections is worked out in Section 9.

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2 Background

In the following we will consider model category structures that are transferred by an adjunction. Given an adjunction

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

where $\mathcal{C}$ is a model category and $\mathcal{D}$ is a bicomplete category, we call a model structure on $\mathcal{D}$ *right-induced* if the weak equivalences and fibrations in $\mathcal{D}$ are determined by the right adjoint functor R.

We use the general setting of symmetric spectra as in [11]. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a bicomplete closed symmetric monoidal category and let K be an object of $\mathcal{C}$. A symmetric sequence in $\mathcal{C}$ is a family of objects $X(n) \in \mathcal{C}$ with $n \in \mathbb{N}_0$ such that the n^{th} level $X(n)$ carries an action of the symmetric group Σ_n. Symmetric sequences form a category $\mathcal{C}^{\Sigma}$ whose morphisms are given by families of Σ_n–equivariant morphisms $f(n)$, $n \geqslant 0$. For every $r \geqslant 0$ there is a functor $G_r\colon \mathcal{C} \to \mathcal{C}^{\Sigma}$ with

$$G_r(C)(n) = \begin{cases} \Sigma_n \times C & \text{for } n = r, \\ \varnothing & \text{for } n \neq r, \end{cases}$$

where $\varnothing$ denotes the initial object of $\mathcal{C}$. Here $\Sigma_n \times C = \bigsqcup_{\Sigma_n} C$ carries the Σ_n–action that permutes the summands.

We consider the symmetric sequence $\mathsf{Sym}(K)$ whose n^{th} level is $K^{\otimes n}$. Here we follow the usual convention that $K^{\otimes 0}$ is the unit $\mathbf{1}$. The category $\mathcal{C}^{\Sigma}$ inherits a symmetric monoidal structure from $\mathcal{C}$: for X, $Y \in \mathcal{C}^{\Sigma}$ we set

$$(X \odot Y)(n) = \bigsqcup_{p+q=n} \Sigma_n \times_{\Sigma_p \times \Sigma_q} X(p) \otimes Y(q).$$

It is straightforward to show (see for instance [11, Section 7]) that $\mathsf{Sym}(K)$ is a commutative monoid in $\mathcal{C}^{\Sigma}$.

The category of *symmetric spectra* (in $\mathcal{C}$ with respect to K), $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)$, is the category of right $\mathsf{Sym}(K)$–modules in $\mathcal{C}^{\Sigma}$. Explicitly, a symmetric spectrum is a family of Σ_n–objects $X(n) \in \mathcal{C}$ together with Σ_n–equivariant maps

$$X(n) \otimes K \to X(n+1)$$

for all $n \geqslant 0$ such that the composites

$$X(n) \otimes K^{\otimes p} \to X(n+1) \otimes K^{\otimes p-1} \to \cdots \to X(n+p)$$

are $\Sigma_n \times \Sigma_p$–equivariant for all $n, p \geqslant 0$. Morphisms in $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)$ are morphisms of symmetric sequences that are compatible with the right $\mathsf{Sym}(K)$–module structure.

There is an evaluation functor Ev_n that maps an $X \in \mathsf{Sp}^\Sigma(\mathcal{C}, K)$ to $X(n) \in \mathcal{C}$. This functor has a left adjoint

$$F_n\colon \mathcal{C} \to \mathsf{Sp}^\Sigma(\mathcal{C}, K)$$

such that $F_n(C)(m)$ is the initial object for $m < n$ and

$$F_n(C)(m) \cong \Sigma_m \times_{\Sigma_{m-n}} C \otimes K^{\otimes m-n} \quad \text{if } m \geqslant n.$$

Note that $F_n(C) \cong G_n(C) \odot \mathsf{Sym}(K)$.

Symmetric spectra form a symmetric monoidal category $(\mathsf{Sp}^\Sigma(\mathcal{C}, K), \wedge, \mathsf{Sym}(K))$ such that for $X, Y \in \mathsf{Sp}^\Sigma(\mathcal{C}, K)$,

$$X \wedge Y = X \odot_{\mathsf{Sym}(K)} Y.$$

Here $X \odot_{\mathsf{Sym}(K)} Y$ denotes the coequalizer of the diagram

$$X \odot \mathsf{Sym}(K) \odot Y \rightrightarrows X \odot Y$$

where we use the right action of $\mathsf{Sym}(K)$ on X and we use the right action of $\mathsf{Sym}(K)$ on Y after applying the twist-map in the symmetric monoidal structure on $\mathcal{C}^\Sigma$.

A crucial map is

(1) $$\lambda\colon F_1 K \to F_0 \mathbf{1};$$

it is given as the adjoint to the identity map $K \to \mathrm{Ev}_1 F_0 \mathbf{1} = K$.

We recall the basics about model category structures on symmetric spectra from [11]: If $\mathcal{C}$ is a closed symmetric monoidal model category which is left proper and cellular and if K is a cofibrant object of $\mathcal{C}$, then there is a *projective model structure on the category* $\mathsf{Sp}^\Sigma(\mathcal{C}, K)$ [11, Theorem 8.2], $\mathsf{Sp}^\Sigma(\mathcal{C}, K)_{\mathrm{proj}}$, such that the fibrations and weak equivalences are levelwise fibrations and weak equivalences in $\mathcal{C}$ and such that the cofibrations are determined by the left lifting property with respect to the class of acyclic fibrations.

This model structure has a Bousfield localization with respect to the set of maps

$$\{\zeta_n^{QC}\colon F_{n+1}(QC \otimes K) \to F_n(QC) \mid n \geqslant 0\},$$

where $Q(-)$ is a cofibrant replacement and C runs through the domains and codomains of the generating cofibrations of $\mathcal{C}$. The map ζ_n^{QC} is adjoint to the inclusion map into the component of $F_n(QC)(n+1)$ corresponding to the identity permutation. We call the Bousfield localization of $\mathsf{Sp}^\Sigma(\mathcal{C}, K)_{\mathrm{proj}}$ at this set of maps the *stable model structure on* $\mathsf{Sp}^\Sigma(\mathcal{C}, K)$ and denote it by $\mathsf{Sp}^\Sigma(\mathcal{C}, K)^s$.

As we are interested in commutative monoids in symmetric spectra, we use positive variants of the above mentioned model structures: Let $\mathsf{Sp}^\Sigma(\mathcal{C}, K)^+_{\mathrm{proj}}$ be the model structure where fibrations are maps that are fibrations in each level $n \geqslant 1$ and weak equivalences are levelwise weak equivalences for positive levels. The cofibrations are again determined by their lifting property and they turn out to be isomorphisms in level zero (compare [17, Section 14]). By adapting the localizing set and considering only positive n, we get the positive stable model structure on $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)$ and we denote it by $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)^{s,+}$.

Remark 2.1 We consider several examples of categories $\mathcal{C}$ with different choices of objects $K \in \mathcal{C}$. Despite the name, the stable model structure on $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)$ does not have to define a stable model category in the sense that the category is pointed with a homotopy category that carries an invertible suspension functor. Proposition 9.1 for instance makes this explicit in the case when K is the unit of the symmetric monoidal structure on $\mathcal{C}$.

3 Model structures on algebras over an operad over $\mathsf{Sp}^{\Sigma}(\mathcal{C})$ for $\mathcal{C} = \mathsf{ch}, \mathsf{sAb}, \mathsf{Ch}$

From now on we restrict to the case $R = \mathbb{Z}$ in order to ease notation. The proofs work in general.

Establishing right-induced model structures for commutative monoids in model categories is hard. Sometimes it is not possible, for instance there is no right-induced model structure on differential graded commutative rings, because the free functor does not respect acyclicity. However, if the underlying model category is nice enough, then such model structures can be established. In broader generality, one might ask whether algebras over operads possess a right-induced model structure. In our setting we will apply the results of Pavlov and Scholbach. They show in [19, Theorem 5.10] and [20, Theorem 3.4.1] that for a tractable, pretty small, left proper, h–monoidal, flat symmetric monoidal model category $\mathcal{C}$ the category of $\mathcal{O}$–algebras in $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)^{s,+}$ has a right-induced model structure. Here $\mathcal{O}$ is an operad in $\mathcal{C}$. See loc. cit. for an explanation of the assumptions. These conditions are satisfied for the model categories of simplicial abelian groups and both nonnegatively graded and unbounded chain complexes. Hence, using their results, we obtain:

Theorem 3.1 *The category of $\mathcal{O}$–algebras in $\mathsf{Sp}^{\Sigma}(\mathcal{C}, K)^{s,+}$ has a right-induced model structure for $\mathcal{C} = \mathsf{sAb}, \mathsf{ch}, \mathsf{Ch}$, any K and any operad $\mathcal{O}$ in $\mathcal{C}$.*

We follow the convention that an E_∞–operad $\mathcal{P}$ in Ch (or ch, sAb) is a symmetric unital operad whose augmentation induces a weak equivalence to the operad that describes commutative monoids. For the sake of brevity we call algebras over an E_∞–operad *E_∞–monoids*. Pavlov and Scholbach also prove a rigidification theorem

[19, Theorem 7.5; 20, Theorem 3.4.4]. We apply this to the case of E_∞–monoids and in this case it provides a Quillen equivalence between the model category of E_∞–monoids in $\mathsf{Sp}^\Sigma(\mathcal{C}, K)^{s,+}$ and commutative monoids in $\mathsf{Sp}^\Sigma(\mathcal{C}, K)^{s,+}$. Related rectification results in the setting of spaces instead of chain complexes are due to [8] and [22]. Berger and Moerdijk obtain general results about rectifications of homotopy algebra structures in [2].

Other approaches to model structures for commutative monoids in symmetric spectra and rigidification results can be found for instance in work by John Harper [9], David White [31], and Steven Chadwick and Michael Mandell [4].

In the following we sketch an alternative proof of the existence of a positive stable right-induced model structure for the category of symmetric spectra in the category of unbounded chain complexes, $\mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$, where $\mathbb{Z}[1]$ denotes the chain complex which is concentrated in chain degree one with chain group $\mathbb{Z}$. This proof uses a modification of the methods used by Chadwick and Mandell [4]. A similar proof works for the categories of symmetric spectra in simplicial abelian groups, $\mathsf{Sp}^\Sigma(\mathsf{sAb}, \widetilde{\mathbb{Z}}(\mathbb{S}^1))$, with $K = \widetilde{\mathbb{Z}}(\mathbb{S}^1)$ the reduced free abelian simplicial group generated by the simplicial 1–sphere, and for symmetric spectra in the category of nonnegatively graded chain complexes, $\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1])$.

A reader who is just interested in the application of these results is invited to resume reading in Section 4.

Theorem 3.2 *Let $\mathcal{O}$ be an operad in Ch. Then the category $\mathcal{O}(\mathsf{Sp}^\Sigma(\mathsf{Ch}))$ of $\mathcal{O}$–algebras over $\mathsf{Sp}^\Sigma(\mathsf{Ch})$ is a model category with fibrations and weak equivalences created in the positive stable model structure on $\mathsf{Sp}^\Sigma(\mathsf{Ch})$.*

Theorem 3.3 *Let $\phi\colon \mathcal{O} \to \mathcal{O}'$ be a map of operads. The induced adjoint functors*

$$\mathcal{O}(\mathsf{Sp}^\Sigma(\mathsf{Ch})) \underset{R_\phi}{\overset{L_\phi}{\rightleftarrows}} \mathcal{O}'(\mathsf{Sp}^\Sigma(\mathsf{Ch}))$$

form a Quillen adjunction. This is a Quillen equivalence if $\phi(n)\colon \mathcal{O}(n) \to \mathcal{O}'(n)$ is a (nonequivariant) weak equivalence for each n.

In particular, if ε is the augmentation from any E_∞–operad to the commutative operad, then it induces a Quillen equivalence between the categories of E_∞–monoids and of commutative monoids in $\mathsf{Sp}^\Sigma(\mathsf{Ch})$.

The proofs of both of these theorems use the following statement, which is a translation of [17, Lemma 15.5] to $\mathsf{Sp}^\Sigma(\mathsf{Ch})$ with a slight generalization based on [4, Remark 8.3(i)]. As a model for $E\Sigma_n$ in the category $\mathsf{Sp}^\Sigma(\mathsf{Ch})$ we take F_0 applied to the normalization of the free simplicial abelian group generated by the nerve of the translation category of the symmetric group Σ_n.

Proposition 3.4 *Let X and Z be objects in $\mathsf{Sp}^\Sigma(\mathsf{Ch})$.*

(1) *Let K be a chain complex, assume X has a Σ_i–action, and let $n > 0$. Then the quotient map*

$$q\colon E\Sigma_{i+} \wedge_{\Sigma_i} ((F_n K)^{\wedge i} \wedge X) \to ((F_n K)^{\wedge i} \wedge X)/\Sigma_i$$

is a level homotopy equivalence.

(2) *For any positive cofibrant object X and any Σ_i–equivariant object Z,*

$$q\colon E\Sigma_{i+} \wedge_{\Sigma_i} (Z \wedge X^{\wedge i}) \to (Z \wedge X^{\wedge i})/\Sigma_i$$

is a π_*–isomorphism.

Proof First, the proof of [17, Lemma 15.5] easily translates to the setting of $\mathsf{Sp}^\Sigma(\mathsf{Ch})$ from $\mathsf{Sp}^\Sigma(\mathcal{S})$ considered there. The key point is that if $q \geqslant ni$, then $E\Sigma_i \times \Sigma_q \to \Sigma_q$ is a $(\Sigma_i \times \Sigma_{q-ni})$–equivariant homotopy equivalence. As mentioned in [4, Remark 8.3(i)], the proof of the first statement in [17, Lemma 15.5] still works when X has a Σ_i–action because the Σ_i–action remains free on Σ_q (or $\mathcal{O}(q)$ in the explicit case there). Similarly the second statement here follows by the same cellular filtration of X as in [17, Lemma 15.5]. □

The proofs of both of the theorems above also require the following definition and statement of properties.

Definition 3.5 A chain map $i\colon A \to B$ in Ch is an *h–cofibration* if each homomorphism $i_n\colon A_n \to B_n$ has a section (or splitting). These are the cofibrations in a model structure on Ch; see [5, Example 3.4], [24, Proposition 4.6.2], or [18, Theorem 18.3.1]. We say a map $i\colon X \to Y$ in $\mathsf{Sp}^\Sigma(\mathsf{Ch})$ is an *h–cofibration* if each level $i_n\colon X_n \to Y_n$ is an h–cofibration as a chain map.

Below we refer to Σ_n–equivariant h–cofibrations. These are Σ_n–equivariant maps for which the underlying nonequivariant map is an h–cofibration. We use the following properties of h–cofibrations below.

Proposition 3.6 (1) *The generating cofibrations and acyclic cofibrations in Ch are h–cofibrations.*

(2) *Sequential colimits and pushouts preserve h–cofibrations.*

(3) *If f and g are two h–cofibrations in Ch, then their pushout product $f \square g$ is also an h–cofibration.*

(4) *If f is an h–cofibration in Ch, then $F_i f$ is an h–cofibration in $\mathsf{Sp}^\Sigma(\mathsf{Ch})$.*

(5) *For every Σ_n–equivariant object Z, subgroup H of Σ_n, Σ_n–equivariant h–cofibration f, and $i \geqslant n$, the map $Z \wedge_H F_i(f)$ is an h–cofibration.*

We write $\mathcal{O}I$ and $\mathcal{O}J$ for the sets of maps in $\mathcal{O}(\mathsf{Sp}^\Sigma(\mathsf{Ch}))$ obtained by applying the free $\mathcal{O}$–algebra functor to the generating cofibrations I and generating acyclic cofibrations J from [29]. Since $\mathsf{Sp}^\Sigma(\mathsf{Ch})$ is a combinatorial model category and the free functor $\mathcal{O}$ commutes with filtered direct limits, to prove Theorem 3.2 it is enough to prove the following lemma by [26, Lemma 2.3].

Lemma 3.7 *Every sequential composition of pushouts in $\mathcal{O}(\mathsf{Sp}^\Sigma(\mathsf{Ch}))$ of maps in $\mathcal{O}J$ is a stable equivalence.*

Proof of Lemma 3.7 This follows as in [4, 8.7–8.10]. Chadwick and Mandell consider pushouts of algebras over an operad $\mathcal{O}$ for three different symmetric monoidal categories of spectra simultaneously (including $\mathsf{Sp}^\Sigma(\mathcal{S})$); all of their arguments hold as well for $\mathsf{Sp}^\Sigma(\mathsf{Ch})$ using the properties of h–cofibrations listed in Proposition 3.6 and the generalization of [17, Lemma 15.5] given in Proposition 3.4(2). □

Proof of Theorem 3.3 This follows as in [4, Theorem 8.2] again using Proposition 3.6 and Proposition 3.4 . □

4 Commutative $H\mathbb{Z}$–algebras and $\mathsf{Sp}^\Sigma(\mathsf{sAb})$

In this section we consider the Quillen equivalence between $H\mathbb{Z}$–module spectra and $\mathsf{Sp}^\Sigma(\mathsf{sAb})$ and show that it also induces an equivalence on the associated categories of commutative monoids. Recall the functor Z from $H\mathbb{Z}$–modules to $\mathsf{Sp}^\Sigma(\mathsf{sAb})$ from [29] which is given by $Z(M) = \widetilde{\mathbb{Z}}(M) \wedge_{\widetilde{\mathbb{Z}}H\mathbb{Z}} H\mathbb{Z}$ where $\widetilde{\mathbb{Z}}$ is the free abelian group on the nonbasepoint simplices on each level. The right adjoint of Z is given by recognizing that $\mathsf{Sym}(\widetilde{\mathbb{Z}}(\mathbb{S}^1))$, the unit in $\mathsf{Sp}^\Sigma(\mathsf{sAb})$, is isomorphic to $\widetilde{\mathbb{Z}}(\mathbb{S}) \cong H\mathbb{Z}$. The right adjoint is labeled U for underlying. In [29, Proposition 4.3], the pair (Z, U) was shown to induce a Quillen equivalence on the standard model structures. Since Z is strong symmetric monoidal, (Z, U) also induces an adjunction between the commutative monoids. We use the right-induced model structure on commutative monoids in $\mathsf{Sp}^\Sigma(\mathsf{sAb})$ and $H\mathbb{Z}$–module spectra [20, Theorem 3.4.1].

Theorem 4.1 *The functors Z and U induce a Quillen equivalence between commutative $H\mathbb{Z}$–algebra spectra and commutative symmetric ring spectra over* sAb*:*

$$Z\colon C(H\mathbb{Z}\text{–mod}) \rightleftarrows C(\mathsf{Sp}^\Sigma(\mathsf{sAb})) :U$$

Proof It follows from [29, Proof of Proposition 4.3] that U preserves and detects all weak equivalences and fibrations since weak equivalences and fibrations are determined on the underlying category of symmetric spectra in pointed simplicial sets, $\mathsf{Sp}^\Sigma(\mathcal{S}_*)$.

To show that (Z, U) is a Quillen equivalence, by [17, Lemma A.2(iii)] it is enough to show that for all cofibrant commutative $H\mathbb{Z}$ algebras A, the map $A \to UZA$ is a stable equivalence. If A were in fact cofibrant as an $H\mathbb{Z}$ module spectrum, this would follow from the Quillen equivalence on the module level [29]. In the standard model structure on commutative algebra spectra though, cofibrant objects are not necessarily cofibrant as modules. The positive flat model (or R–model) structures from [28, Theorem 3.2] were developed for just this reason. In Lemma 4.2 we show that for positive flat cofibrant commutative $H\mathbb{Z}$ algebras B, the map $B \to UZB$ is a stable equivalence. It follows from Lemma 4.2 that $A \to UZA$ is a stable equivalence for all standard (positive) cofibrant commutative $H\mathbb{Z}$ algebras A, since such A are also positive flat cofibrant by [28, Proposition 3.5]. See also [19, Theorem 8.10] for an alternative approach to this theorem. □

As discussed in the proof above, we next consider the flat model (or R–model) structures from [28, Theorem 3.2]; see also [25, III, Sections 2 and 3].

Lemma 4.2 *For positive flat cofibrant commutative $H\mathbb{Z}$ algebras B, the map $B \to UZB$ is a stable equivalence.*

Proof The crucial property for positive flat cofibrant ($H\mathbb{Z}$–cofibrant) commutative monoids is that they are also (absolute) flat cofibrant as underlying modules. Thus, if B is a positive flat cofibrant commutative $H\mathbb{Z}$–algebra, then it is also an (absolute) flat cofibrant $H\mathbb{Z}$–module by [28, Corollary 4.3]. (In fact B is also a positive flat cofibrant $H\mathbb{Z}$–module by [28, Corollary 4.1], but we do not use that here.) Since the Quillen equivalence in [29, Proposition 4.3] is with respect to the standard model structures [29, Proposition 2.9], we next translate to that setting. Consider a cofibrant replacement $p\colon cB \to B$ in the standard model structure on $H\mathbb{Z}$–modules; the map p is a trivial fibration and hence a level equivalence. Consider the commuting diagram:

$$\begin{array}{ccc} cB & \xrightarrow{\;p\;} & B \\ \downarrow & & \downarrow \\ UZcB & \longrightarrow & UZB \end{array}$$

The left map is a stable equivalence by [29, Proposition 4.3]. In Lemma 4.3 below we show that Z takes level equivalences between flat cofibrant objects to level equivalences. By [28, Proposition 2.8], cB is flat cofibrant, so it follows that the bottom map is also a stable equivalence. Thus, the right map is a stable equivalence as well. □

Lemma 4.3 *The functor Z takes level equivalences between flat cofibrant objects to level equivalences.*

Proof Here we will consider Z as a composite of two functors and we will always work over symmetric spectra in pointed simplicial sets, $\mathsf{Sp}^{\Sigma}(\mathcal{S}_*)$, by forgetting from sAb to $\mathcal{S}_*$ wherever necessary. The first component is $\widetilde{\mathbb{Z}}$ from $H\mathbb{Z}$–modules to $\widetilde{\mathbb{Z}}H\mathbb{Z}$–modules, and the second component is the extension of scalars functor μ_* associated to the ring homomorphism $\mu\colon \widetilde{\mathbb{Z}}H\mathbb{Z} \to H\mathbb{Z}$ induced by recognizing $H\mathbb{Z}$ as isomorphic to $\widetilde{\mathbb{Z}}\mathbb{S}$ and using the monad structure on $\widetilde{\mathbb{Z}}$.

First, note that $\widetilde{\mathbb{Z}}$ is applied to each level and preserves level equivalences as a functor from simplicial sets to simplicial abelian groups. The functor $\widetilde{\mathbb{Z}}$ also preserves flat cofibrations, and hence flat cofibrant objects. The generating flat cofibrations ($H\mathbb{Z}$–cofibrations) are of the form $H\mathbb{Z} \otimes M$ where M is the class of monomorphisms of symmetric sequences. Since $\widetilde{\mathbb{Z}}$ is strong symmetric monoidal, these maps are taken to maps of the form $\widetilde{\mathbb{Z}}(H\mathbb{Z}) \otimes \widetilde{\mathbb{Z}}(M)$. Since $\widetilde{\mathbb{Z}}$ preserves monomorphisms, these are contained in the generating flat ($\widetilde{\mathbb{Z}}H\mathbb{Z}$–) cofibrations, which are of the form $\widetilde{\mathbb{Z}}H\mathbb{Z} \otimes M$.

Next, note that restriction of scalars, μ^*, preserves level equivalences and level fibrations since they are determined as maps on the underlying flat (S–) model structure; see the paragraph above [28, Theorem 2.6] and [28, Proposition 2.2]. It follows by adjunction that μ_* preserves the flat cofibrations and level equivalences between flat cofibrant objects. □

Remark 4.4 In the proof of Theorem 4.1 we use a reduction argument that allows us to establish the desired Quillen equivalence by checking that the unit map of the adjunction is a weak equivalence on flat cofibrant objects in the flat model structure on commutative $H\mathbb{Z}$–algebras. This approach avoids a discussion of a flat model structure on commutative symmetric ring spectra in simplicial abelian groups.

5 Dold–Kan correspondence for commutative monoids

The classical Dold–Kan correspondence is an equivalence of categories between the category of simplicial abelian groups, sAb, and the category of nonnegatively graded chain complexes of abelian groups, ch. In this section we establish a Quillen equivalence between categories of commutative monoids in symmetric sequences of simplicial abelian groups, $C(\mathsf{sAb}^{\Sigma})$, and nonnegatively graded chain complexes, $C(\mathsf{ch}^{\Sigma})$, carrying positive model structures. In the special case of pointed commutative monoids in symmetric sequences of simplicial modules and nonnegatively graded chain complexes, such a Quillen equivalence is established in [21, Theorem 6.5].

In the next section we extend this equivalence from symmetric sequences to symmetric spectra. We first define the relevant model structures on the categories of symmetric sequences in simplicial abelian groups, sAb^{Σ}, and chain complexes, ch^{Σ}.

Definition 5.1 • Let $f\colon A \to B$ be a morphism in ch^Σ. Then f is a positive weak equivalence, if $H_*(f)(\ell)$ is an isomorphism for positive levels $\ell > 0$. It is a positive fibration, if $f(\ell)$ is a fibration in the projective model structure on nonnegatively graded chain complexes for all $\ell > 0$.

- A morphism $g\colon C \to D$ in sAb^Σ is a positive fibration if $g(\ell)$ is a fibration of simplicial abelian groups in positive levels and it is a positive weak equivalence if $g(\ell)$ is a weak equivalence for all $\ell > 0$.

In both cases, the positive cofibrations are determined by their left lifting property with respect to positive acyclic fibrations. Positive cofibrations are cofibrations that are isomorphisms in level zero. One can check directly that the above definitions give model category structures or use Hirschhorn's criterion [10, Theorem 11.6.1] and restrict to the diagram category whose objects are natural numbers greater than or equal to one and then use the trivial model structure in level zero with cofibrations being isomorphisms and weak equivalences and fibrations being arbitrary. The generating cofibrations are maps of the form $G_r(i)$ for r positive and such that i is a generating cofibration in chain complexes (simplicial modules). The generating acyclic cofibrations are maps of the form $G_r(j)$ for r positive and where j is a generating acyclic cofibration in chain complexes (simplicial modules).

We also get the corresponding right-induced model structures on commutative monoids:

Definition 5.2 An f in $C(\mathsf{ch}^\Sigma)(A, B)$ is a positive weak equivalence (fibration) if the map on underlying symmetric sequences, $U(f)$ in $\mathsf{ch}^\Sigma(U(A), U(B))$, is a positive weak equivalence (fibration). Similarly, g in $C(\mathsf{sAb}^\Sigma)(C, D)$ is a positive weak equivalence (fibration) if the map on underlying symmetric sequences, $U(g) \in \mathsf{sAb}^\Sigma(U(C), U(D))$ is a positive weak equivalence (fibration).

In [21, Corollary 5.8, Definition 6.2] these model structures were established for *pointed* commutative monoids in symmetric sequences of simplicial modules and nonnegatively graded chain complexes. An object A in $C(\mathsf{ch}^\Sigma)$ or $C(\mathsf{sAb}^\Sigma)$ is called pointed, if its zeroth level is the unit of the monoidal structure of the base category. We recall the key points of the argument in the proof below. This also makes it clear that the results of [21] can be adapted to the setting of Definition 5.1.

Lemma 5.3 *The structures defined in Definition 5.2 yield cofibrantly generated model categories where the generating cofibrations are $C(G_r(i))$ and the generating acyclic cofibrations are $C(G_r(j))$ with i, j as above and r positive.*

Proof Adjunction gives us that the maps with the right lifting property with respect to all $C(G_r(j))$, $r > 0$, are precisely the positive fibrations and the ones with the RLP with respect to all $C(G_r(i))$, $r > 0$, are the positive acyclic fibrations. Performing the small object argument based on the $C(G_r(j))$ for all positive r yields a factorization of any map as a positive acyclic cofibration and a fibration whereas the small object argument based on the $C(G_r(i))$ for positive r gives the other factorization. □

Let $\underline{\mathbb{Z}}$ denote the constant simplicial abelian group with value $\mathbb{Z}$. In the positive model structures cofibrant objects are commutative monoids whose zeroth level is isomorphic to $\underline{\mathbb{Z}}$ in $C(\mathsf{sAb}^\Sigma)$ or to $\mathbb{Z}[0]$ in $C(\mathsf{ch}^\Sigma)$. In particular, such objects are pointed in the sense of [21, Definition 5.1].

Let Γ denote the functor from nonnegatively graded chain complexes to simplicial abelian groups that is the inverse of the normalization functor. We can extend Γ to a functor from ch^Σ to sAb^Σ by applying Γ in every level. As the category of symmetric sequences of abelian groups is an abelian category, the pair (N, Γ) is still an equivalence of categories.

In the following we extend the result [21, Theorem 6.5] in the pointed setting, to the setting of positive model structures.

Theorem 5.4 *Let $C(\mathsf{sAb}^\Sigma)$ and $C(\mathsf{ch}^\Sigma)$ carry the positive model structures. Then the normalization functor $N\colon C(\mathsf{sAb}^\Sigma) \to C(\mathsf{ch}^\Sigma)$ is the right adjoint in a Quillen equivalence and its left adjoint is denoted L_N.*

Proof A left adjoint L_N to N is constructed in [21, Lemma 6.4]. As positive fibrations and weak equivalences are defined via the forgetful functors to sAb^Σ and ch^Σ, the functor N is a right Quillen functor and N also detects weak equivalences. Every object is fibrant, so we have to show that the unit of the adjunction

$$\eta\colon A \to NL_N(A)$$

is a weak equivalence for all cofibrant $A \in C(\mathsf{ch}^\Sigma)$. But cofibrant objects are pointed and for these it is shown in [21, Proof of Theorem 6.5] that the unit map is a weak equivalence. □

6 Extension to commutative ring spectra

We will show that the pair (L_N, N) gives rise to a Quillen equivalence $(L_N, \phi^* N)$ on the level of commutative symmetric ring spectra.

Lemma 6.1 *The Quillen pair* (L_N, N) *satisfies*

$$L_N(\mathsf{Sym}\, X_*) \cong \mathsf{Sym}(\Gamma(X_*))$$

for all nonnegatively graded chain complexes X_*.

Proof We can identify $\mathsf{Sym}(C_*)$ with the free commutative monoid generated by $G_1 X_*$, $C(G_1 X_*)$. Then, by definition of L_N, we obtain

$$L_N(C(G_1 X_*)) \cong C(\Gamma(G_1 X_*)) \cong C(G_1 \Gamma(X_*)) \cong \mathsf{Sym}(\Gamma(X_*)). \qquad \square$$

Let $\mathcal{C}$ be a category and let A be an object of $\mathcal{C}$. Then we denote by $A \downarrow \mathcal{C}$ the category of objects under A.

Corollary 6.2 *Let* $C(\mathsf{ch}^\Sigma)$ *and* $C(\mathsf{sAb}^\Sigma)$ *carry the positive model category structures and consider the induced model structures on the categories under a specific object. Then the model categories* $\mathsf{Sym}(\mathbb{Z}[0]) \downarrow C(\mathsf{ch}^\Sigma)$ *and* $\mathsf{Sym}(\underline{\mathbb{Z}}) \downarrow C(\mathsf{sAb}^\Sigma)$ *are Quillen equivalent.*

Proof By Lemma 6.1 we know that

$$L_N\, \mathsf{Sym}(\mathbb{Z}[0]) \cong \mathsf{Sym}(\underline{\mathbb{Z}}).$$

A direct calculation shows that $N(\mathsf{Sym}(\underline{\mathbb{Z}}))$ is isomorphic to $\mathsf{Sym}(\mathbb{Z}[0])$. Therefore the Quillen equivalence (L_N, N) passes to a Quillen adjunction on the under categories. As the classes of fibrations, weak equivalences and cofibrations in the under categories are determined by the ones in the ambient category, this adjunction is a Quillen equivalence. $\square$

Note that there is an isomorphism of categories between the category of commutative monoids in $\mathsf{Sp}^\Sigma(\mathsf{sAb}, \tilde{\mathbb{Z}}(\mathbb{S}^1))$ and the category $\mathsf{Sym}(\tilde{\mathbb{Z}}(\mathbb{S}^1)) \downarrow C(\mathsf{sAb}^\Sigma)$. A similar isomorphism of categories compares commutative monoids in $\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1])$ and objects in $\mathsf{Sym}(\mathbb{Z}[1]) \downarrow C(\mathsf{ch}^\Sigma)$. We can extend the Quillen equivalence from Corollary 6.2 to these under categories. Recall from [29, page 358] that $\mathcal{N}$ is the symmetric sequence in chain complexes with $N(\tilde{\mathbb{Z}}(\mathbb{S}^\ell))$ in level ℓ. We denote by $\mathbb{1}$ the unit of the symmetric monoidal category ch^Σ. This is the symmetric sequence with $\mathbb{Z}[0]$ in level zero and zero in all positive levels.

Proposition 6.3 *The functors* $(L_N, \Phi^* N)$ *induce a Quillen equivalence on the model categories* $\mathsf{Sym}(\mathbb{Z}[1]) \downarrow C(\mathsf{ch}^\Sigma)$ *and* $\mathsf{Sym}(\tilde{\mathbb{Z}}(\mathbb{S}^1)) \downarrow C(\mathsf{sAb}^\Sigma)$ *where* $C(\mathsf{ch}^\Sigma)$ *and* $C(\mathsf{sAb}^\Sigma)$ *carry the positive model structures. Here,* Φ^* *is a suitable change-of-rings functor.*

Proof As $\Gamma(\mathbb{Z}[1])$ is isomorphic to $\widetilde{\mathbb{Z}}(\mathbb{S}^1)$ we obtain with Lemma 6.1 that

$$L_N(\mathsf{Sym}(\mathbb{Z}[1])) \cong \mathsf{Sym}(\widetilde{\mathbb{Z}}(\mathbb{S}^1)).$$

Therefore, if A is an object in $\mathsf{Sym}(\mathbb{Z}[1]) \downarrow C(\mathsf{ch}^\Sigma)$, then $L_N(A)$ is an object of $\mathsf{Sym}(\widetilde{\mathbb{Z}}(\mathbb{S}^1)) \downarrow C(\mathsf{sAb}^\Sigma)$. We consider the functors

$$\begin{array}{ccc} \mathsf{Sym}(\mathbb{Z}[1]) \downarrow C(\mathsf{ch}^\Sigma) & \xrightarrow{\quad L_N \quad} & \mathsf{Sym}(\widetilde{\mathbb{Z}}(\mathbb{S}^1)) \downarrow C(\mathsf{sAb}^\Sigma) \\ & {\scriptstyle \Phi^*} \nwarrow \quad \swarrow {\scriptstyle N} & \\ & \mathcal{N} \downarrow C(\mathsf{ch}^\Sigma) & \end{array}$$

where $\Phi\colon \mathsf{Sym}(\mathbb{Z}[1]) \to \mathcal{N}$ is induced by the shuffle transformation (see [29, page 358]) and Φ^* is the associated change-of-rings map. Note that $NL_N \,\mathsf{Sym}\,\mathbb{Z}[1] \cong \mathcal{N}$. Both functors N and Φ^* preserve and detect level and stable weak equivalences [29, Proof of Proposition 4.4], therefore they preserve and detect positive weak equivalences and hence it suffices to show that

$$A \to \Phi^* N L_N A$$

is a weak equivalence in the model category $\mathsf{Sym}(\mathbb{Z}[1]) \downarrow C(\mathsf{ch}^\Sigma)$ for all cofibrant objects $\alpha\colon \mathsf{Sym}(\mathbb{Z}[1]) \to A$. There is a map of commutative monoids $\gamma\colon \mathbb{1} \to \mathsf{Sym}(\mathbb{Z}[1])$ which is given by the identity in level zero and by the zero map in higher levels. Let γ^* be the associated change-of-rings functor:

$$\begin{array}{ccccc} \mathbb{1} & \xrightarrow{\gamma} & \mathsf{Sym}(\mathbb{Z}[1]) & \xrightarrow{\alpha} & A \\ & & {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \eta_A} \\ & & \mathcal{N} & \xrightarrow{NL_N\alpha} & NL_N A \end{array}$$

Note that $\eta_A \circ \alpha \circ \gamma = NL_N\alpha \circ \Phi \circ \gamma$. As A is cofibrant, we know that the map $\alpha(0)\colon \mathbb{Z}[0] = \mathsf{Sym}(\mathbb{Z}[1])(0) \to A(0)$ is an isomorphism. Therefore $\gamma^*(A)$ is positively cofibrant as an object in $C(\mathsf{ch}^\Sigma)$. Hence the map

$$\gamma^*(A) \to \gamma^* \Phi^* N L_N(A)$$

is a positive weak equivalence in $C(\mathsf{ch}^\Sigma)$, ie a level equivalence in all positive levels (it is also a weak equivalence in level zero). As γ^* is the identity on objects and only changes the module structure we get that

$$A \to \Phi^* N L_N(A)$$

is a level equivalence in $\mathsf{Sym}(\mathbb{Z}[1]) \downarrow C(\mathsf{ch}^\Sigma)$. □

Remark 6.4 With the positive model structure, $C(\mathsf{ch}^\Sigma)$ is not left proper. Consider for instance the map $CG_r(0) = \mathbb{1} \to CG_r(\mathbb{Z}[0])$. This map is a cofibration for positive r in the positive model structure. On the other hand, take the projection map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$. This yields a map π in $C(\mathsf{ch}^\Sigma)$ from the initial object $\mathbb{1}$ to $\mathbb{1}/2\mathbb{1}$ (where the latter object is concentrated in level zero with value $\mathbb{Z}/2\mathbb{Z}[0]$). As we work in the positive model structure, this map is actually a weak equivalence. If we push out π along the cofibration $\mathbb{1} \to CG_r(\mathbb{Z}[0])$ we get

$$g\colon CG_r(\mathbb{Z}[0]) \to CG_r(\mathbb{Z}[0]) \odot \mathbb{1}/2\mathbb{1}.$$

In level r this is the chain map

$$\begin{aligned} g(r)\colon G_r(\mathbb{Z}[0])(r) &\cong \mathbb{Z}[\Sigma_r] \otimes \mathbb{Z}[0] \cong \mathbb{Z}[\Sigma_r][0] \\ &\to \mathbb{Z}[\Sigma_r] \otimes \mathbb{Z}[0] \otimes \mathbb{Z}/2\mathbb{Z}[0] \cong \mathbb{Z}/2\mathbb{Z}[\Sigma_r][0]. \end{aligned}$$

Therefore we do not get an isomorphism for positive r and the pushout of the weak equivalence π is not a weak equivalence.

We want to transfer our results to a comparison of commutative monoids in symmetric spectra of simplicial abelian groups and nonnegatively graded chain complexes where we consider the positive stable model structure.

Lemma 6.5 *Cofibrant objects in $C(\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1]))$ in the positive stable model structure are cofibrant in $C(\mathsf{ch}^\Sigma)$.*

Proof We can express the map $\mathbb{1} \to \mathsf{Sym}(\mathbb{Z}[1])$ as

$$\mathbb{1} \cong C(G_1(0)) \to C(G_1(\mathbb{Z}[1])) = \mathsf{Sym}(\mathbb{Z}[1]).$$

Therefore the unit of $\mathsf{Sym}(\mathbb{Z}[1])$ is $C(G_1(i))$ with $i\colon 0 \to \mathbb{Z}[1]$ and hence it is a cofibration and therefore the initial object $\mathsf{Sym}(\mathbb{Z}[1])$ of $C(\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1]))$ is cofibrant in $C(\mathsf{ch}^\Sigma)$.

As usual, let $\mathbb{S}^n$ denote the chain complex whose only nontrivial chain group is $\mathbb{Z}$ in degree n and let $\mathbb{D}^n$ denote the chain complex with $\mathbb{D}^n_n = \mathbb{D}^n_{n-1} = \mathbb{Z}$ and $\mathbb{D}^n_i = 0$ for all $i \neq n$, $n-1$ whose only nontrivial boundary map is the identity. The cofibrant generators of the positive stable model structure are the maps

$$\mathsf{Sym}(\mathbb{Z}[1]) \odot G_m(\mathbb{S}^{n-1}) \xrightarrow{\mathsf{Sym}(\mathbb{Z}[1]) \odot G_m(i_n)} \mathsf{Sym}(\mathbb{Z}[1]) \odot G_m(\mathbb{D}^n), \tag{2}$$

where i_n is the cofibration of chain complexes $i_n\colon \mathbb{S}^{n-1} \to \mathbb{D}^n$ and $m \geqslant 1$. The $\odot$–product is the coproduct in the category $C(\mathsf{ch}^\Sigma)$ and thus the map $\mathsf{Sym}(\mathbb{Z}[1]) \odot G_m(i_n)$

is the coproduct of the identity map on $\mathsf{Sym}(\mathbb{Z}[1])$ and the map $G_m(i_n)$ and hence a cofibration in $C(\mathsf{ch}^\Sigma)$.

Coproducts of generators as in (2) are cofibrations in $C(\mathsf{ch}^\Sigma)$ as well, because the coproduct in $C(\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1]))$ is given by the $\odot_{\mathsf{Sym}(\mathbb{Z}[1])}$–product.

Every cofibrant object is a retract of a cell-object and these are sequential colimits of pushout diagrams of the form

$$\begin{array}{ccc} X & \longrightarrow & A^{(n)} \\ {\scriptstyle f}\downarrow & & \vdots \\ Y & \cdots\cdots\to & A^{(n+1)} \end{array}$$

where f is a coproduct of maps like in (2) and $A^{(n)}$ is inductively constructed such that $A^{(0)}$ is $\mathsf{Sym}(\mathbb{Z}[1]))$. We can inductively assume that X, Y and $A^{(n)}$ are cofibrant in $C(\mathsf{ch}^\Sigma)$. The pushout in $C(\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1]))$ is the pushout in $C(\mathsf{ch}^\Sigma)$ and hence the pushout $A^{(n+1)}$ is cofibrant in $C(\mathsf{ch}^\Sigma)$ as well. Sequential colimits and retracts of cofibrant objects are cofibrant. □

Theorem 6.6 *The Quillen pair* $(L_N, \Phi^* N)$ *induces a Quillen equivalence between* $C(\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1]))$ *and* $C(\mathsf{Sp}^\Sigma(\mathsf{sAb}, \widetilde{\mathbb{Z}}(\mathbb{S}^1)))$ *with the model structures that are right-induced from the positive stable model structures on the underlying categories of symmetric spectra.*

Proof We have to show that the unit of the adjunction

$$A \to \Phi^* N L_N A$$

is a stable equivalence for all cofibrant $A \in C(\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1]))$. Lemma 6.5 ensures that A is cofibrant as an object in $C(\mathsf{ch}^\Sigma)$. Both A and $\Phi^* N L_N A$ receive a unit map from $\mathsf{Sym}(\mathbb{Z}[1])$. As in the proof of Proposition 6.3 we get that

$$\gamma^* A \to N L_N \gamma^* A$$

is a level equivalence in $C(\mathsf{ch}^\Sigma)$ and therefore the map $A \to \Phi^* N L_N A$ is a level equivalence in $C(\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1]))$ and hence a stable equivalence. □

7 Comparison of spectra in bounded and unbounded chain complexes

Recall that ch denotes the category of nonnegatively graded chain complexes and Ch is the category of unbounded chain complexes of abelian groups. There is a canonical inclusion functor $i\colon \mathsf{ch} \to \mathsf{Ch}$ and a good truncation functor $C_0\colon \mathsf{Ch} \to \mathsf{ch}$ which assigns to an unbounded chain complex X_* the nonnegatively graded chain complex $C_0(X_*)$ with

$$C_0(X_*)_m = \begin{cases} X_m & \text{for } m > 0, \\ \text{cycles}(X_0) & \text{for } m = 0. \end{cases}$$

We denote the induced functors on the corresponding categories of symmetric spectra again by i and C_0. In this section we consider the Quillen equivalence

$$i\colon \mathsf{Sp}^\Sigma(\mathsf{ch}) \rightleftarrows \mathsf{Sp}^\Sigma(\mathsf{Ch}) :C_0$$

and show that it extends to a Quillen equivalence of categories of commutative monoids. The original Quillen equivalence is established in [29, Proposition 4.9] for the usual stable model structures. Here we consider instead the positive stable model structures from [17, Section 14] and then consider the right-induced model structures on commutative monoids where f is a weak equivalence or fibration if it is an underlying positive weak equivalence or fibration. Note that the weak equivalences of the stable model structure agree with the weak equivalences of the positive stable model structure in $\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1])$ and $\mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$. For this reason the positive and stable model structures are Quillen equivalent; see also [17, Proposition 14.6]. It follows that the Quillen equivalence induced by i and C_0 on the usual stable model structures also induces a Quillen equivalence on the positive stable model structures.

Proposition 7.1 *The adjoint functors i and C_0 form a Quillen equivalence between the positive stable model structures on $\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1])$ and $\mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$.*

Corollary 7.2 *Let f be a positive stably fibrant replacement functor in $\mathsf{Sp}^\Sigma(\mathsf{Ch})$ and let $\eta\colon X \to C_0 i X$ be the unit of the adjunction. The composite $X \to C_0 i X \to C_0 f i X$ is a stable equivalence for all objects X in $\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1])$.*

Proof It follows from the proof of Proposition 7.1 that the derived unit of the adjunction is a weak equivalence whenever X is positive cofibrant. Since positive trivial fibrations are positive levelwise weak equivalences and a positive cofibrant replacement $cX \to X$ is a positive trivial fibration, we only need to show that $C_0 f i$ preserves positive levelwise equivalences. The inclusion i preserves positive levelwise equivalences and f preserves stable equivalences. Any stable equivalence between positive stably fibrant objects is a positive levelwise equivalence, so fi preserves positive levelwise equivalences. Since C_0 preserves positive levelwise equivalences between positive stably fibrant objects, the corollary follows. □

Corollary 7.3 *The adjoint functors i and C_0 induce a Quillen equivalence between the commutative monoids in $\mathsf{Sp}^\Sigma(\mathsf{ch}, \mathbb{Z}[1])$ and $\mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$.*

Proof Since the weak equivalences and fibrations are determined on the underlying positive stable model structures, C_0 still preserves fibrations and weak equivalences between positive stably fibrant objects. By [12, Lemma 4.1.7] it is then enough to check the derived composite $C_0 i$ is a stable equivalence for all cofibrant commutative monoids. This is shown for all objects in Corollary 7.2. The fibrant replacement functor for commutative monoids will be different, but the properties used in the proof of that corollary still hold, so we conclude. □

8 Quillen equivalence between E_∞–monoids in Ch and $\mathsf{Sp}^\Sigma(\mathsf{Ch})$

We fix a cofibrant E_∞–operad $\mathcal{O}$ in Ch (in the model structure on operads as in [30, Section 2, Remark 2]) and we consider the operad $F_0\mathcal{O}$ in symmetric spectra in chain complexes.

Let Ch carry the projective model structure and let $E_\infty\mathsf{Ch}$ denote the category of $\mathcal{O}$–algebras in Ch with its right-induced model structure [30, Section 4, Theorem 4]. This model structure exists because Ch is a cofibrantly generated monoidal model category, it satisfies the monoid axiom [29, Corollary 3.4] and $\mathcal{O}$ is cofibrant. Alternatively, we could work with Mandell's model structure on E_∞–monoids in Ch using the operad of the chains on the linear isometries operad [15]. See also [1] for general existence results of model structures for categories of algebras over operads.

Similarly, $\mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$ with the stable model structure is a cofibrantly generated monoidal model category satisfying the monoid axiom [29, Corollary 3.4], and as the set of generating acyclic cofibrations for the positive stable model structure on $\mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$ is a subset of the ones for the stable structure, the positive stable model category also satisfies the monoid axiom. We consider two model structures for $E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$, the E_∞–monoids in $\mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$:

- We denote by $E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])^{s,+}$ the model structure in which the forgetful functor to the positive stable model category structure on $\mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$ determines the fibrations and weak equivalences.
- Let $E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])^s$ denote the model category whose fibrations and weak equivalences are determined by the forgetful functor to the stable model structure on $\mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$.

Proposition 8.1 *The model structure $E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])^{s,+}$ is Quillen equivalent to the model structure $E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])^s$.*

Proof We consider the adjunction

$$(E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])^s) \underset{R}{\overset{L}{\leftrightarrows}} (E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])^{s,+}),$$

where R and L are both the identity functor. If p is a fibration in the a positive stable fibration in $E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$. Therefore R preserves fibrations. As the weak equivalences in both model structures agree, R is a right Quillen functor and it preserves and reflects weak equivalences. Hence the unit of the adjunction is a weak equivalence. □

In the following we use Hovey's comparison result [11, Theorem 9.1]: Tensoring with $\mathbb{Z}[1]$ induces a Quillen autoequivalence on the category of unbounded chain complexes, so we get that the pair (F_0, Ev_0) induces a Quillen equivalence

$$\mathsf{Ch} \underset{\mathrm{Ev}_0}{\overset{F_0}{\rightleftarrows}} \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])^s.$$

We can then transfer this Quillen equivalence to the corresponding categories of E_∞–monoids: Both F_0 and Ev_0 are strong symmetric monoidal functors. Fix a cofibrant E_∞–operad $\mathcal{O}$ in Ch as above. As $\mathrm{Ev}_0 \circ F_0$ is the identity, Ev_0 maps $F_0\mathcal{O}$–algebras in $E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$ to $\mathcal{O}$–algebras in unbounded chain complexes.

Theorem 8.2 *The functors (F_0, Ev_0) induce a Quillen equivalence*

$$F_0\colon E_\infty\mathsf{Ch} \rightleftarrows E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])^s :\mathrm{Ev}_0.$$

Proof The proof follows Hovey's proof of [11, Theorem 5.1]. It is easy to see that Ev_0 reflects weak equivalences between stably fibrant objects: If $f\colon X \to Y$ is such a map and $f(0)$ is a weak equivalence, then $f(\ell)$ is a weak equivalence for all $\ell \geqslant 0$, because X and Y are fibrant and $(-) \otimes \mathbb{Z}[1]$ is a Quillen equivalence.

In our case $(-) \otimes \mathbb{Z}[1]$ is an equivalence of categories with inverse the functor $\mathrm{Hom}(\mathbb{Z}[1], -)$, where $\mathrm{Hom}(-,-)$ is the internal homomorphism bifunctor.

Therefore, for any X in $E_\infty\mathsf{Ch}$, we have that F_0X is stably fibrant because

$$(F_0X)_n = X \otimes \mathbb{Z}[n] \cong \mathrm{Hom}(\mathbb{Z}[1], X \otimes \mathbb{Z}[n+1])$$

and as every object in Ch is fibrant, F_0X is always fibrant in the projective model structure on $E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$.

Let A be a cofibrant object in $E_\infty\mathsf{Ch}$. We have to show that

$$\eta\colon A \to \mathrm{Ev}_0 W(F_0A)$$

is a weak equivalence, for $W(-)$ the fibrant replacement in $E_\infty \mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])$. But we saw that F_0A is fibrant and $A \to \mathrm{Ev}_0F_0A = A$ is the identity map, thus η is a weak equivalence. See also [19, Theorem 8.10] for an alternative approach to this theorem. □

Observe that all of the Quillen equivalences that we have established so far did not use any particular properties of $\mathbb{Z}$. We can therefore generalize our results as follows.

Corollary 8.3 *Let R be a commutative ring with unit. There is a chain of Quillen equivalences between the model category of commutative HR–algebra spectra and E_∞–monoids in the category of unbounded R–chain complexes.*

For $R = \mathbb{Q}$ we can strengthen the result:

Corollary 8.4 *There is a chain of Quillen equivalences between the model category of commutative $H\mathbb{Q}$–algebra spectra and differential graded commutative $\mathbb{Q}$–algebras.*

Proof It is well known that the category of differential graded commutative algebras and E_∞–monoids in $\mathsf{Ch}(\mathbb{Q})$ possess a right-induced model category structure and that there is a Quillen equivalence between them. For a proof of these facts see for instance [14, Section 7.1.4]. □

Remark 8.5 Note that the proof of Theorem 8.2 applies in broader generality: If $\mathcal{O}$ is an arbitrary operad in the category of chain complexes such that right-induced model structures on $\mathcal{O}$–algebras in Ch and on $F_0(\mathcal{O})$–algebras in $\mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])^s$ exist, then the pair (F_0, Ev_0) yields a Quillen equivalence between the model category of $\mathcal{O}$–algebras in Ch and the model category of $F_0(\mathcal{O})$–algebras in $\mathsf{Sp}^\Sigma(\mathsf{Ch}, \mathbb{Z}[1])^s$.

9 Symmetric spectra and $\mathcal{I}$–chain complexes

Let $\mathcal{I}$ denote the skeleton of the category of finite sets and injective maps with objects the sets $\boldsymbol{n} = \{1, \dots, n\}$ for $n \geqslant 0$ with the convention that $\boldsymbol{0} = \varnothing$. The set of morphisms $\mathcal{I}(\boldsymbol{p}, \boldsymbol{n})$ consists of all injective maps from $\boldsymbol{p}$ to $\boldsymbol{n}$. In particular, this set is empty if n is smaller than p. The category $\mathcal{I}$ is a symmetric monoidal category under disjoint union of sets.

For any category $\mathcal{C}$ we consider the diagram category $\mathcal{C}^{\mathcal{I}}$ of functors from $\mathcal{I}$ to $\mathcal{C}$. If $(\mathcal{C}, \otimes, e)$ is symmetric monoidal, then $\mathcal{C}^{\mathcal{I}}$ inherits a symmetric monoidal structure: For $A, B \in \mathcal{C}^{\mathcal{I}}$ we set

$$(A \boxtimes B)(\boldsymbol{n}) = \mathrm{colim}_{\boldsymbol{p} \sqcup \boldsymbol{q} \to \boldsymbol{n}} A(\boldsymbol{p}) \otimes B(\boldsymbol{q}).$$

For details about $\mathcal{I}$–diagrams see [22]. The following fact is folklore; it was pointed out to Shipley by Jeff Smith in 2006 at the Mittag-Leffler Institute.

Proposition 9.1 *Let $\mathcal{C}$ be any closed symmetric monoidal category with unit e. Then the category $\mathsf{Sp}^\Sigma(\mathcal{C}, e)$ is isomorphic to the diagram category $\mathcal{C}^{\mathcal{I}}$.*

Proof Let $X \in \mathsf{Sp}^\Sigma(\mathcal{C}, e)$. Then $X(n) \in \mathcal{C}^{\Sigma_n}$ and we have Σ_n–equivariant maps $X(n) \cong X(n) \otimes e \to X(n+1)$, such that the composite

$$\sigma_{n,p}\colon X(n) \cong X(n) \otimes e^{\otimes p} \to X(n+1) \otimes e^{\otimes p-1} \to \cdots \to X(n+p)$$

is $\Sigma_n \times \Sigma_p$–equivariant for all $n, p \geqslant 0$.

We send X to $\phi(X) \in \mathcal{C}^{\mathcal{I}}$ with $\phi(X)(\boldsymbol{n}) = X(n)$. If $i = i_{p,n-p} \in \mathcal{I}(\boldsymbol{p}, \boldsymbol{n})$ is the standard inclusion, then we let $\phi(i)\colon \phi(X)(\boldsymbol{p}) \to \phi(X)(\boldsymbol{n})$ be $\sigma_{p,n-p}$. Every morphism $f \in \mathcal{I}(\boldsymbol{p}, \boldsymbol{n})$ can be written as $\xi \circ i$ where i is the standard inclusion and $\xi \in \Sigma_n$. For such ξ, the map $\phi(\xi)$ is given by the Σ_n–action on $X(n) = \phi(X)(\boldsymbol{n})$.

If $f = \xi' \circ i$ is another factorization of f into the standard inclusion followed by a permutation, then ξ and ξ' differ by a permutation $\tau \in \Sigma_n$ which maps all j with $1 \leqslant j \leqslant p$ identically, ie τ is of the form $\tau = \mathrm{id}_{\boldsymbol{p}} \oplus \tau'$ with $\tau' \in \Sigma_{n-p}$. As the structure maps $\sigma_{p,n-p}$ are $\Sigma_p \times \Sigma_{n-p}$–equivariant, the induced map $\phi(f) = \phi(\xi') \circ \phi(i)$ agrees with $\phi(\xi) \circ \phi(i)$.

The inverse of ϕ, denoted by ψ, sends A, an $\mathcal{I}$–diagram in $\mathcal{C}$, to the symmetric spectrum $\psi(A)$ whose n^{th} level is $\psi(A)(n) = A(\boldsymbol{n})$. The Σ_n–action on $\psi(A)(n)$ is given by the corresponding morphisms $\Sigma_n \subset \mathcal{I}(\boldsymbol{n}, \boldsymbol{n})$ and the structure maps of the spectrum are defined as

$$\psi(A)(n) \otimes e^{\otimes p} = A(\boldsymbol{n}) \otimes e^{\otimes p} \xrightarrow{\cong} A(\boldsymbol{n}) \xrightarrow{A(i_{n,p})} A(\boldsymbol{n}+\boldsymbol{p}) = \psi(A)(n+p).$$

The functors ϕ and ψ are well-defined and inverse to each other. □

Lemma 9.2 *The functors ϕ and ψ are strong symmetric monoidal.*

Proof Consider two free objects $F_s C_*$ and $F_t D_*$ in $\mathsf{Sp}^\Sigma(\mathcal{C}, e)$ for two chain complexes C_* and D_*. We know in general [11, Section 7] that

$$F_s C_* \wedge F_t D_* \cong F_{s+t}(C_* \otimes D_*). \tag{3}$$

Note that as an object in $\mathcal{C}^{\mathcal{I}}$ we have for $\boldsymbol{n} \in \mathcal{I}$

$$\phi(F_s C_*)(\boldsymbol{n}) = \mathbb{Z}\Sigma_n \otimes_{\mathbb{Z}\Sigma_{n-s}} C_*$$

for $n \geqslant s$ and zero otherwise. This coincides with the value of the free $\mathcal{I}$–diagram on $\boldsymbol{n}$,

$$F_s^{\mathcal{I}}(C_*)(\boldsymbol{n}) = \mathbb{Z}\mathcal{I}(\boldsymbol{s}, \boldsymbol{n}) \otimes C_*,$$

and in fact this yields an isomorphism of functors. Similarly, $\psi(F_s^{\mathcal{I}}(C_*)) \cong F_s C_*$.

As the symmetric monoidal product in $\mathcal{C}^{\mathcal{I}}$ is given by left Kan extension along the exterior product using the monoidal structure of $\mathcal{C}$ we get

$$F_s^{\mathcal{I}}(C_*) \boxtimes F_t^{\mathcal{I}}(D_*) \cong F_{s+t}^{\mathcal{I}}(C_* \otimes D_*). \tag{4}$$

From (3) we obtain that

$$\psi(F_s^{\mathcal{I}}(C_*)) \wedge \psi(F_t^{\mathcal{I}}(D_*)) \cong \psi(F_{s+t}^{\mathcal{I}}(C_* \otimes D_*)) \cong \psi(F_s^{\mathcal{I}}(C_*) \boxtimes F_t^{\mathcal{I}}(D_*))$$

and (4) yields

$$\phi(F_s C_*) \boxtimes \phi(F_t D_*) \cong \phi(F_{s+t}(C_* \otimes D_*)) \cong \phi(F_s C_* \wedge F_t D_*).$$

The used isomorphisms are associative and compatible with the symmetry isomorphisms. Every object in $\mathsf{Sp}^{\Sigma}(\mathcal{C}, e)$ and $\mathcal{C}^{\mathcal{I}}$ can be written as a colimit of free objects and as $\mathcal{C}$ is closed, the general case follows from the free case. □

Remark 9.3 In [20, Proposition 3.3.9] Pavlov and Scholbach describe explicitly (for a well-behaved symmetric monoidal model category $\mathcal{C}$) how the unstable and stable model structures on $\mathsf{Sp}^{\Sigma}(\mathcal{C}, e)$ transfer to $\mathcal{C}^{\mathcal{I}}$ under the above mentioned isomorphism of categories. If $\mathcal{C}$ is Ch, their assumptions are satisfied.

Note that the weak equivalences in $\mathsf{Ch}^{\mathcal{I}}$ have an explicit description: they are the maps that become weak equivalences after applying a corrected homotopy colimit [7, Definition 5.1]. This is the homotopy colimit of the diagram where every node is functorially replaced by a cofibrant object first. To see this, consider Dugger's Bousfield localizations of diagram categories in [7, Section 5]. As the cofibrations and the fibrant objects in his model structure in [7, Theorem 5.2] agree with ours, an argument due to Joyal [13, Proposition E.1.10] ensures that we have the same class of weak equivalences as well.

Taking a cofibrant E_∞–operad $\mathcal{O}$ in Ch then ensures that $\mathcal{O}$–algebras in $\mathsf{Sp}^{\Sigma}(\mathsf{Ch}, \mathbb{Z}[0])^s$ and in $\mathsf{Ch}^{\mathcal{I}}$ carry a model category structure such that the forgetful functor determines fibrations and weak equivalences

Since tensoring with the unit $\mathbb{Z}[0]$ is isomorphic to the identity, we can repeat all of the arguments in the previous section with $\mathbb{Z}[1]$ replaced by $\mathbb{Z}[0]$. Thus we also obtain that the model category $E_\infty \mathsf{Sp}^{\Sigma}(\mathsf{Ch}, \mathbb{Z}[0])^s$ is Quillen equivalent to the model category of E_∞–monoids in Ch. Summarizing:

Theorem 9.4 *There is a chain of Quillen equivalences*

$$E_\infty \mathsf{Sp}^{\Sigma}(\mathsf{Ch}, \mathbb{Z}[1])^s \underset{\mathrm{Ev}_0}{\overset{F_0}{\leftrightarrows}} E_\infty \mathsf{Ch} \underset{\mathrm{Ev}_0}{\overset{F_0}{\rightleftarrows}} E_\infty \mathsf{Sp}^{\Sigma}(\mathsf{Ch}, \mathbb{Z}[0])^s$$

and the rightmost model category is isomorphic to $E_\infty \mathsf{Ch}^{\mathcal{I}}$.

Last but not least we can connect commutative HR–algebras to commutative $\mathcal{I}$–chain complexes. The positive stable model structure on $\mathsf{Sp}^\Sigma(\mathsf{Ch}(R), R[0])$ satisfies the assumptions of [19, Theorem 5.10] and hence commutative monoids and E_∞–monoids in $\mathsf{Sp}^\Sigma(\mathsf{Ch}(R), R[0])^{s,+}$ carry model category structures and there is a Quillen equivalence between them [20, Theorem 3.4.1, Theorem 3.4.4]. This yields that the model categories of commutative $\mathcal{I}$–chain complexes, $C(\mathsf{Ch}(R)^{\mathcal{I},+})$, and E_∞ $\mathcal{I}$–chain complexes, $E_\infty(\mathsf{Ch}(R)^{\mathcal{I},+})$ are Quillen equivalent, if we take the model structure that is right-induced from the positive model structure on $\mathsf{Ch}(R)^{\mathcal{I},+}$.

Theorem 9.5 *There is a chain of Quillen equivalences between the model categories of commutative HR–algebra spectra, $C(HR\text{–mod})$, and commutative monoids in the category $\mathsf{Ch}(R)^{\mathcal{I}}$ where the latter carries the right-induced model structure from the positive model structure on $\mathsf{Ch}(R)^{\mathcal{I}}$, $\mathsf{Ch}(R)^{\mathcal{I},+}$.*

We close with an important example of a commutative $\mathcal{I}$–chain complex. Consider a chain complex C_* together with a 0–cycle, ie with a map $\eta\colon \mathbb{Z}[0] \to C_*$. The assignment $\boldsymbol{n} \mapsto C_*^{\otimes n}$ defines a functor sym from $\mathcal{I}$ to the category of unbounded chain complexes (namely $\mathsf{Sym}(C_*)$). Schlichtkrull shows in [23] that sym is the algebraic analogue of the symmetric product in the category of spaces.

References

[1] **C Berger**, **I Moerdijk**, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. 78 (2003) 805–831 MR

[2] **C Berger**, **I Moerdijk**, *Resolution of coloured operads and rectification of homotopy algebras*, from "Categories in algebra, geometry and mathematical physics" (A Davydov, M Batanin, M Johnson, S Lack, A Neeman, editors), Contemp. Math. 431, Amer. Math. Soc. (2007) 31–58 MR

[3] **B Cenkl**, *Cohomology operations from higher products in the de Rham complex*, Pacific J. Math. 140 (1989) 21–33 MR

[4] **S G Chadwick**, **M A Mandell**, *E_n genera*, Geom. Topol. 19 (2015) 3193–3232 MR

[5] **J D Christensen**, **M Hovey**, *Quillen model structures for relative homological algebra*, Math. Proc. Cambridge Philos. Soc. 133 (2002) 261–293 MR

[6] **T Church**, **J S Ellenberg**, **B Farb**, *FI-modules and stability for representations of symmetric groups*, Duke Math. J. 164 (2015) 1833–1910 MR

[7] **D Dugger**, *Replacing model categories with simplicial ones*, Trans. Amer. Math. Soc. 353 (2001) 5003–5027 MR

[8] **P G Goerss**, **M J Hopkins**, *Moduli spaces of commutative ring spectra*, from "Structured ring spectra" (A Baker, B Richter, editors), London Math. Soc. Lecture Note Ser. 315, Cambridge Univ. Press (2004) 151–200 MR

[9] **J E Harper**, *Homotopy theory of modules over operads in symmetric spectra*, Algebr. Geom. Topol. 9 (2009) 1637–1680 MR

[10] **P S Hirschhorn**, *Model categories and their localizations*, Mathematical Surveys and Monographs 99, Amer. Math. Soc. (2003) MR

[11] **M Hovey**, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra 165 (2001) 63–127 MR

[12] **M Hovey**, **B Shipley**, **J Smith**, *Symmetric spectra*, J. Amer. Math. Soc. 13 (2000) 149–208 MR

[13] **A Joyal**, *The theory of quasi-categories and its applications*, lecture notes (2008) Available at `http://tinyurl.com/joyalquasi`

[14] **J Lurie**, *Higher algebra*, unpublished manuscript (2014) Available at `http://math.harvard.edu/~lurie/papers/HA.pdf`

[15] **M A Mandell**, *Flatness for the E_∞ tensor product*, from "Homotopy methods in algebraic topology" (J P C Greenlees, R R Bruner, N Kuhn, editors), Contemp. Math. 271, Amer. Math. Soc. (2001) 285–309 MR

[16] **M A Mandell**, *Topological André–Quillen cohomology and E_∞ André–Quillen cohomology*, Adv. Math. 177 (2003) 227–279 MR

[17] **M A Mandell**, **J P May**, **S Schwede**, **B Shipley**, *Model categories of diagram spectra*, Proc. London Math. Soc. 82 (2001) 441–512 MR

[18] **J P May**, **K Ponto**, *More concise algebraic topology: localization, completion, and model categories*, Univ. of Chicago Press (2012) MR

[19] **D Pavlov**, **J Scholbach**, *Admissibility and rectification of colored symmetric operads*, preprint (2014) arXiv

[20] **D Pavlov**, **J Scholbach**, *Symmetric operads in abstract symmetric spectra*, preprint (2014) arXiv

[21] **B Richter**, *On the homology and homotopy of commutative shuffle algebras*, Israel J. Math. 209 (2015) 651–682 MR

[22] **S Sagave**, **C Schlichtkrull**, *Diagram spaces and symmetric spectra*, Adv. Math. 231 (2012) 2116–2193 MR

[23] **C Schlichtkrull**, *The homotopy infinite symmetric product represents stable homotopy*, Algebr. Geom. Topol. 7 (2007) 1963–1977 MR

[24] **R Schwänzl**, **R M Vogt**, *Strong cofibrations and fibrations in enriched categories*, Arch. Math. (Basel) 79 (2002) 449–462 MR

[25] **S Schwede**, *An untitled book project about symmetric spectra* Available at `http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf`

[26] **S Schwede**, **B E Shipley**, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. 80 (2000) 491–511 MR

[27] **S Schwede**, **B Shipley**, *Equivalences of monoidal model categories*, Algebr. Geom. Topol. 3 (2003) 287–334 MR

[28] **B Shipley**, *A convenient model category for commutative ring spectra*, from "Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K–theory" (P Goerss, S Priddy, editors), Contemp. Math. 346, Amer. Math. Soc. (2004) 473–483 MR

[29] **B Shipley**, *$H\mathbb{Z}$–algebra spectra are differential graded algebras*, Amer. J. Math. 129 (2007) 351–379 MR

[30] **M Spitzweck**, *Operads, algebras and modules in general model categories*, preprint (2001) arXiv

[31] **D White**, *Model structures on commutative monoids in general model categories*, preprint (2014) arXiv

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