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A refinement of Betti numbers and homology in the presence of a continuous function, I

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We propose a refinement of the Betti numbers and the homology with coefficients in a field of a compact ANR X, in the presence of a continuous real-valued function on X. The refinement of Betti numbers consists of finite configurations of points with multiplicities in the complex plane whose total cardinalities are the Betti numbers, and the refinement of homology consists of configurations of vector spaces indexed by points in the complex plane, with the same support as the first, whose direct sum is isomorphic to the homology. When the homology is equipped with a scalar product, these vector spaces are canonically realized as mutually orthogonal subspaces of the homology.

The assignments above are in analogy with the collections of eigenvalues and generalized eigenspaces of a linear map in a finite-dimensional complex vector space. A number of remarkable properties of the above configurations are discussed.

55N35; 46M20, 57R19

1 Introduction

The results of this paper and its subsequent part II, mostly obtained in collaboration with Stefan Haller, provide a shorter version of some results in [3], still unpublished, extend their generality based on the involvement of the topology of Hilbert cube manifolds and refine them as configurations of complex numbers and of vector spaces.

Precisely, for a fixed field κ and $r \geq 0$, one proposes a refinement of the Betti numbers $b_r(X)$ of a compact ANR X[1] and a refinement of the homology $H_r(X)$ with coefficients in the field κ in the presence of a continuous function $f\colon X \to \mathbb{R}$.

The refinements consists of finite configurations of points with multiplicity located in the plane $\mathbb{R}^2 = \mathbb{C}$, denoted by δ^f_r, equivalently of monic polynomials with complex coefficients $P^f_r(z)$, of degree the Betti numbers $b_r(X)$, and finite configurations of κ–vector spaces denoted by $\hat{\delta}^f_r$ with the same support and direct sum of all vector spaces isomorphic to $H_r(X)$; see Theorem 4.1. The points of the configurations δ^f_r,

[1] See the definition of an ANR in Section 2.2.

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equivalently the zeros of the polynomials $P^f_r(z)$, are complex numbers $z=a+ib\in\mathbb{C}$ with both a,b critical values;[2] see Theorem 4.1. The two configurations are related by $\dim\hat\delta^f_r=\delta^f_r$.

We show the following:

(1) The assignment $f\rightsquigarrow P^f_r(z)$ is continuous when f varies in the space of continuous maps equipped with the compact open topology; see Theorem 4.2.

(2) For an open and dense subset of continuous maps (defined on X, an ANR satisfying some mild properties) the points of the configurations δ^f_r or the zeros of the polynomials $P^f_r(z)$ have multiplicity one; see Theorem 4.1.

(3) When X is a closed topological n–manifold, the Poincaré duality between the Betti numbers β_r and β_{n-r} gets refined to a Poincaré duality between configurations δ^f_r and δ^f_{n-r}, and the Poincaré duality between $H_r(X)$ and $H_{n-r}(X)^*$ to a Poincaré duality between configurations $\hat\delta^f_r$ and $(\hat\delta^f_{n-r})^*$; see Theorem 4.3.

(4) For each point of the configuration δ^f_r, equivalently zero z of the polynomial $P^f_r(z)$, the assigned vector space $\hat\delta^f_r(z)$ has dimension the multiplicity of z and is a quotient of vector subspaces $\hat\delta^f_r(z)=\mathbb{F}_r(z)/\mathbb{F}'_r(z)$, $\mathbb{F}'_r(z)\subseteq\mathbb{F}_r(z)\subseteq H_r(X)$. When $\kappa=\mathbb{R}$ or $\mathbb{C}$ and $H_r(X)$ is equipped with a Hilbert space structure $\hat\delta^f_r(z)$ identifies canonically to a subspace $\boldsymbol{H}_r(z)$ of $H_r(X)$ such that $\boldsymbol{H}_r(z)\perp\boldsymbol{H}_r(z')$ for $z\neq z'$ and $\bigoplus_z\boldsymbol{H}_r(z)=H_r(X)$; see Theorem 4.1. This provides an additional structure (direct sum decomposition of $H_r(X)$, which in view of Theorem 4.1, for a generic f, has all components of dimension 1).

We refer to the system $(H_r(X),P^f_r(z),\hat\delta^f_r)$ as the *r–homology spectral package* of (X,f), in analogy with the spectral package of (V,T), where V is a vector space and T a linear endomorphism, which consists of the characteristic polynomial $P^T(z)$ with its roots z_i, the eigenvalues of T and with their corresponding generalized eigenspaces V_{z_i}.

In case X is the underlying space of a closed oriented Riemannian manifold (M^n,g) and $\kappa=\mathbb{R}$ or $\mathbb{C}$, the vector space $H_r(M^n)$, via the identification with the harmonic r–forms, has a structure of a Hilbert space. The configuration $\hat\delta^f_r$, for f generic, provides a base in the space of harmonic forms.

All these results are collected in the main theorems below, Theorems 4.1–4.3, which were partially established in [3], not yet in print, but under more restrictive hypotheses like "X homeomorphic to a simplicial complex" or "f a tame map". In this paper, we removed these hypotheses using results on Hilbert cube manifolds reviewed in Section 2.3, and complete them with additional results.

[2] See Section 2.2 below for the definition of regular and critical value.

It is worth noting that the points of the configurations δ^f_r located above and on the diagonal in the plane $\mathbb{R}^2$ determine and are determined by the closed r–bar codes in the level persistence of f, while those below the diagonal are determined by and determine the open $(r-1)$–bar codes in the level persistence as observed in [3]. The algorithms proposed by Carlsson, de Silva and Morozov [4] and the author and Dey [2] can be used for their calculation.

Similar refinements hold for angle-valued maps and will be discussed in part II. In this case, the homology has to be replaced by either the Novikov homology of (X, ξ_f) which in our work is a finitely generated free module over the ring of Laurent polynomials $\kappa[t^{-1}, t]$ or, in case κ is $\mathbb{R}$ or $\mathbb{C}$, by the L_2–homology of the infinite cyclic cover defined by $\xi_f \in H^1(X : \mathbb{Z})$, determined by f. In this case, the L_2–homology is regarded as a Hilbert module over the von Neumann algebra associated to the group $\mathbb{Z}$, $\boldsymbol{H}_r(z)$ are Hilbert submodules and $\delta^f_r(x)$ is the von Neumann dimension of $\boldsymbol{H}_r(z)$. Note that the L_2–Betti numbers are actually the Novikov–Betti numbers of (X, ξ_f) (which agree with the rank of the corresponding free module).

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2 Preliminary definitions

2.1 Configurations

Let X be a topological space. A *finite configuration of points in X* is a map

$$\delta\colon X \to \mathbb{Z}_{\geq 0}$$

with finite support.

A *finite configuration of vector spaces indexed by points in X* is a map with finite support

$$\hat{\delta}\colon X \to \mathrm{Vect}$$

(ie $\hat{\delta}(x) = 0$ for all but finitely many $x \in X$), where Vect denotes the collection of κ–vector spaces.

For N a positive integer, denote by $\mathcal{C}_N(X)$ the set of configurations of points in X with total cardinality N:

$$\mathcal{C}_N(X) := \Big\{\delta\colon X \to \mathbb{Z}_{\geq 0} \mid \sum_{x \in X} \delta(x) = N\Big\}.$$

For V a finite-dimensional κ–vector space, denote by $\mathcal{P}(V)$ the set of subspaces of V and by $\mathcal{C}_V(X)$ the set

$$\mathcal{C}_V(X):=\Big\{\bar\delta\colon X\to\mathcal{P}(V)\,\Big|\,\sharp\{x\in X\mid\bar\delta(x)\neq 0\}<\infty,\ \bar\delta(x)\cap\sum_{y\neq x}\bar\delta(y)=0,\ \sum_{x\in X}\bar\delta(x)=V\Big\}.$$

Here $\sharp$ denotes cardinality of the set in braces.

Consider the map

$$e\colon \mathcal{C}_V(X)\to\mathcal{C}_{\dim V}(X)$$

defined by

$$e(\bar\delta)(x)=\dim\bar\delta(x),$$

and call the configuration $e(\bar\delta)$ the *dimension* of $\bar\delta$.

Both sets $\mathcal{C}_N(X)$ and $\mathcal{C}_V(X)$ can be equipped with natural topology (the *collision topology*). One way to describe these topologies is to specify for each δ or $\hat\delta$ a system of *fundamental neighborhoods*. If δ has as support the set of points $\{x_1,x_2,\dots,x_k\}$, a fundamental neighborhood $\mathcal{U}$ of δ is specified by a collection of k disjoint open neighborhoods $U_1,\dots,U_k$ of $x_1,\dots,x_k$, and consists of $\big\{\delta'\in\mathcal{C}_N(X)\mid\sum_{x\in U_i}\delta'(x)=\delta(x_i)\big\}$. Similarly if $\bar\delta$ has as support the set of points $\{x_1,x_2,\dots,x_k\}$ with $\bar\delta(x_i)=V_i\subseteq V$, a fundamental neighborhood $\mathcal{U}$ of $\bar\delta$ is specified by a collection of k disjoint open neighborhoods $U_1,U_2,\dots,U_k$ of $x_1,\dots,x_k$, and consists of

$$\Big\{\bar\delta'\in\mathcal{C}_V(X)\mid x\in U_i\Rightarrow\bar\delta'(x)\subset V_i,\ \bigoplus_{x\in U_i}\bar\delta'(x)=V_i\Big\}.$$

Clearly e is continuous.

When κ is an infinite field, the topology of $\mathcal{C}_V(X)$ has too many connected components to be useful unless the geometry forces the possible values of the configurations to be at most countable.

When $\kappa=\mathbb{R}$ or $\mathbb{C}$ and V is a Hilbert space, it is natural to consider the subset of $\mathcal{C}_V^O(X)\subset\mathcal{C}_V(X)$ consisting of configurations whose vector spaces $\bar\delta(x)$ are mutually orthogonal. In this case for $\bar\delta$ with support the set of points $\{x_1,x_2,\dots,x_k\}$ and $\bar\delta(x_i)=V_i\subseteq V$, one can consider a fundamental neighborhood $\mathcal{U}$ of $\bar\delta$ that is specified by a collection of k disjoint open neighborhoods $U_1,U_2,\dots,U_k$ of $x_1,\dots,x_k$ and open neighborhoods $O_1,O_2,\dots,O_k$ of V_i in $G_{\dim V_i}(V)$, and consists of

$$\Big\{\bar\delta'\in\mathcal{C}_V^O(X)\mid\bigoplus_{x\in U_i}\hat\delta'(x)\in O_i\Big\}.$$

Here $G_k(V)$ denotes the Grassmannian of k–dimensional subspaces of V.

With respect to this topology e is continuous, surjective and proper, with fiber above δ, the subset of $G_{n_1}(V) \times G_{n_2}(V) \times \cdots \times G_{n_k}(V)$ consisting of $(V'_1, V'_2, \ldots, V'_k)$, $V'_i \in G_{n_i}(V)$ mutually orthogonal, where $n_i = \dim V_i$. This set is compact and is actually an algebraic variety.

Remark (1) $\mathcal{C}_N(X) = X^N/\Sigma_N$ is the so-called N–symmetric product, and if X is a metric space with distance D then the collision topology is the topology defined by the distance $\underline{D}$ on X^N/Σ_N induced from the distance on X^N given by $D(x_1, x_2, \ldots, x_N; y_1, y_2, \ldots, y_N) := \sup_{i=1,\ldots,N}\{D(x_i, y_i)\}$.

(2) If $X = \mathbb{R}^2 = \mathbb{C}$ then $\mathcal{C}_N(X)$ identifies to the set of monic polynomials with complex coefficients. To the configuration δ whose support consists of the points $z_1, z_2, \ldots, z_k$ with $\delta(z_i) = n_i$, one associates the monic polynomial $P^f(z) = \prod_i (z - z_i)^{n_i}$. Then $\mathcal{C}_N(X)$ and $\mathbb{C}^N$ are identified as metric spaces.

(3) The space $\mathcal{C}_V(X)$ and thus $\mathcal{C}_V(\mathbb{R}^2)$ can be equipped with a complete metric which induces the collision topology but this will not be used here.

2.2 Tame maps

Recall that a metrizable space X is an ANR if any closed subset A of a metrizable space B with A homeomorphic to X has a neighborhood U which retracts to A; see [7, Chapter 3]. Recall also that any space homeomorphic to a locally finite simplicial complex, a finite-dimensional topological manifold or an infinite-dimensional manifold (ie a paracompact Hausdorff space locally homeomorphic to the Hilbert space l_2 or the Hilbert cube I^∞) is an ANR; see [7].

Convention All maps $f\colon X \to \mathbb{R}$ in this paper are continuous proper maps defined on an ANR X, hence if such maps exists, X is locally compact. From now on the words "proper continuous" should always be assumed to precede the word "map" even if not specified.

The following concepts are consistent with the familiar terminology in topology:

- A map $f\colon X \to \mathbb{R}$ is *weakly tame* if for any $t \in \mathbb{R}$, the level $f^{-1}(t)$ is an ANR. Therefore, for any bounded or unbounded closed interval $I = [a,b]$, $a, b \in \mathbb{R} \sqcup \{\infty, -\infty\}$, $f^{-1}(I)$ is an ANR. Indeed if $I = [a,b]$, in view of the hypothesis that $f^{-1}(a)$ and $f^{-1}(b)$ are ANRs and of the definition of ANR, there exists an open set $U \subset X \setminus f^{-1}(a,b)$ which retracts to $f^{-1}(a) \sqcup f^{-1}(b)$. Then $U \cup f^{-1}[a,b]$ is an open set in X which retracts to $f^{-1}(I)$. Since X is an ANR this suffices to conclude that $f^{-1}(I)$is an ANR; see [7]. A similar argument can be used for $I = (-\infty, a]$ or $I = [b, \infty)$.

- The number $t \in \mathbb{R}$ is a *regular value* if there exists $\epsilon > 0$ such that for any $t' \in (t-\epsilon, t+\epsilon)$, the open set $f^{-1}(t-\epsilon, t+\epsilon)$ retracts by deformation to $f^{-1}(t')$. A number t which is not a regular value is a *critical value*. In view of the hypothesis on f a map (ie X locally compact and f proper), the requirement on t in the definition of *weakly tame* is satisfied for any regular value t. Informally, the critical values are the values t for which the topology of the level ($=$ homotopy type) changes. One denotes by $\mathrm{Cr}(f)$ the collection of critical values of f.

- The map f is called *tame* if it is weakly tame and, in addition,

  (a) the set of critical values $\mathrm{Cr}(f) \subset \mathbb{R}$ is discrete, and

  (b) $\epsilon(f) := \inf\{|c-c'| : c, c' \in \mathrm{Cr}(f), c \neq c'\}$ satisfies $\epsilon(f) > 0$.

If X is compact then (a) implies (b).

- An ANR which has the tame maps dense in the set of all maps with respect to the fine C_0–topology is called a *good* ANR.

There exist compact ANRs (actually compact homological n–manifolds; see [6]) with no codimension-one subsets which are ANRs, hence compact ANRs which are not *good*.

The reader should be aware of the following rather obvious facts.

Observation 2.1 (1) If f is a weakly tame map then $f^{-1}([a,b])$ is a compact ANR and has the homotopy type of a finite simplicial complex (see [8]) and therefore has finite-dimensional homology with respect to any field κ.

(2) If X is a locally finite simplicial complex and f is a simplicial map, then f is weakly tame with the set of critical values discrete. Critical values are among the values of f on vertices. If in addition X is compact then f is tame.

(3) If X is homeomorphic to a finite simplicial complex then the set of tame maps is dense in the set of all continuous maps with the C_0–topology (ie compact open topology). The same remains true if X is a compact Hilbert cube manifold, defined in the next section. In particular all these spaces are good ANRs.

For the needs of this paper, weaker than usual concepts of regular or critical values and tameness, relative to homology with coefficients in the field κ, suffice. They are introduced in Section 3.

2.3 Compact Hilbert cube manifolds

Recall the following:

- The *Hilbert cube* Q is the infinite product $Q = I^\infty = \prod_{i \in \mathbb{Z}_{\geq 1}} I_i$ with $I_i = [0,1]$. The topology of Q is given by the distance $d(\bar{u}, \bar{v}) = \sum_i |u_i - v_i|/2^i$ with $\bar{u} = \{u_i \in I, i \in \mathbb{Z}_{\geq 1}\}$ and $\bar{v} = \{v_i \in I, i \in \mathbb{Z}_{\geq 1}\}$.
- The space Q is a compact ANR and so is any $X \times Q$ for any compact ANR X.
- A *compact Hilbert cube manifold* is a compact Hausdorff space locally homeomorphic to the Hilbert cube Q.

For $f\colon X \to \mathbb{R}$ and $F\colon X \times Q \to \mathbb{R}$, denote by $\bar{f}_Q\colon X \times Q \to \mathbb{R}$ and $F_k\colon X \times Q \to \mathbb{R}$ the maps defined by

$$\bar{f}_Q(x, \bar{u}) = f(x) \quad \text{and} \quad F_k(x, \bar{u}) = F(x, u_1, u_2, \dots, u_k, 0, 0, \dots).$$

Observation 2.2 In view of the definition of $\bar{f}_Q$ and of the metric on Q, observe the following:

(1) If $f\colon X \to \mathbb{R}$ is a tame map, so is $\bar{f}_Q$.

(2) If X is compact then the sequence of maps F_n is uniformly convergent to the map F when $n \to \infty$.

The following are basic results about compact Hilbert cube manifolds whose proof can be found in [5].

Theorem 2.3 (1) (R Edwards) *If X is a compact ANR then $X \times Q$ is a compact Hilbert cube manifold.*

(2) (T Chapman) *Any compact Hilbert cube manifolds is homeomorphic to $K \times Q$ for some finite simplicial complex K.*

(3) (T Chapman) *If $\omega\colon X \to Y$ is a homotopy equivalence between two finite simplicial complexes with Whitehead torsion $\tau(\omega) = 0$ then the there exists a homeomorphism $\omega'\colon X \times Q \to Y \times Q$ such that ω' and $\omega \times \mathrm{id}_Q$ are homotopic. As a consequence of Observation 2.4 below, two compact Hilbert cube manifolds which are homotopy equivalent become homeomorphic after product with $\mathbb{S}^1$.*

Observation 2.4 (folklore) If ω is a homotopy equivalence between two finite simplicial complexes then $\omega \times \mathrm{id}_{\mathbb{S}^1}$ has the Whitehead torsion $\tau(\omega \times \mathrm{id}_{\mathbb{S}^1}) = 0$.

As a consequence of the above statements we have the following proposition.

Proposition 2.5 *Any compact Hilbert cube manifold M is a good ANR.*

Proof A map $f\colon M \to \mathbb{R}$, M a compact Hilbert cube manifold, is called *special* if there exists a finite simplicial complex K, a map $g\colon K \to \mathbb{R}$ and a homeomorphism $\theta\colon M \to K \times Q$ such that $\bar{g} \cdot \theta = f$, and a special map is PL[3] if in addition g is PL. By Observation 2.2 any map $f\colon M \to \mathbb{R}$ is $\epsilon/2$–close to a special map. Since any continuous real-valued map defined on a simplicial complex K is $\epsilon/2$–close to a PL map then any special map on M is $\epsilon/2$–close to a special PL map. Consequently f is ϵ–close to a special PL map which is tame in view of Observations 2.1 and 2.2. This implies that the set of tame maps is dense in the set of all continuous maps. □

3 The configurations δ^f_r and $\hat{\delta}^f_r$

In this paper we fix a field κ, and for a space X denote by $H_r(X)$ the homology of X with coefficients in the field κ. Let $f\colon X \to \mathbb{R}$ be a map. As in the previous section, f is proper continuous and X is a locally compact ANR. One defines

(1) the sublevel $X_a := f^{-1}(-\infty, a])$,
(2) the superlevel $X^b := f^{-1}([b, \infty))$,
(3) $\mathbb{I}^f_a(r) := \mathrm{img}(H_r(X_a) \to H_r(X)) \subseteq H_r(X)$,
(4) $\mathbb{I}^b_f(r) := \mathrm{img}(H_r(X^b) \to H_r(X)) \subseteq H_r(X)$,
(5) $\mathbb{F}^f_r(a, b) = \mathbb{I}^f_a(r) \cap \mathbb{I}^b_f(r) \subseteq H_r(X)$.

Clearly one has the following observation.

Observation 3.1 (1) For $a' \le a$ and $b \le b'$, one has $\mathbb{F}^f_r(a', b') \subseteq \mathbb{F}^f_r(a, b)$.
(2) For $a' \le a$ and $b \le b'$, one has $\mathbb{F}^f_r(a', b) \cap \mathbb{F}^f_r(a, b') = \mathbb{F}^f_r(a', b')$.
(3) $\sup_{x\in X} |f(x) - g(x)| < \epsilon$ implies $\mathbb{F}^g(a - \epsilon, b + \epsilon) \subseteq \mathbb{F}^f_r(a, b)$.

Note that we also have the following proposition.

Proposition 3.2 *If f is a map as above then* $\dim \mathbb{F}^f_r(a, b) < \infty$.

Proof If X is compact, there is nothing to prove since $H_r(X)$ has finite dimension. Suppose X is not compact. In view of Observation 3.1(1), it suffices to check the statement for $a > b$. If f is weakly tame, in view of Observation 2.1 X_a, X^b and

[3] PL stands for piecewise linear.

$X_a \cap X^b$ are ANRs, with $X_a \cap X^b$ compact and $X = X_a \cup X^b$, hence the Mayer–Vietoris long exact sequence in homology is valid. Denote by $i_a(r)\colon H_r(X_a) \to H_r(X)$ and $i^b(r)\colon H_r(X^b) \to H_r(X)$ the inclusion-induced linear maps and observe that $\mathbb{F}_r(a,b) := \mathbb{I}_a \cap \mathbb{I}^b \subseteq i_a(r)(\ker(i_a(r) - i^b(r)))$. In view of the Mayer–Vietoris sequence in homology, $\ker(i_a(r) - i^b(r))$ is isomorphic to a quotient of the vector space of $H_r(X_a \cap X^b)$, hence of finite dimension, and the result holds.

If f is not weakly tame, one argue as follows. It is known that any X a locally compact ANR is proper homotopy dominated with respect to any open cover by some locally finite simplicial complex K; see [1].[4] Choose such a cover, for example $f^{-1}(n-1, n+1)_{n\in\mathbb{Z}}$ and such a homotopy domination $X \xrightarrow{i} K \xrightarrow{\pi} X$ for this cover. Choose $g\colon K \to \mathbb{R}$ a proper simplicial approximation of $f \cdot \pi$ (hence tame) and $a' > a$ and $b' < b$ such that $i(X_a^f) \subset K_{a'}^g$ and $i(X_f^b) \subset K_g^{b'}$. Then $\mathbb{F}_r^f(a,b)$ is isomorphic to a subspace of $\mathbb{F}_r^g(a',b')$. Since the dimension of $\mathbb{F}_r^g(a',b')$ is finite, so is the dimension of $\mathbb{F}_r^f(a,b)$. □

Definition 3.3 We say a real number t is a *homologically regular value* if there exists $\epsilon(t) > 0$ such that for any $0 < \epsilon < \epsilon(t)$ the inclusions $\mathbb{I}^f_{t-\epsilon}(r) \subseteq \mathbb{I}^f_t(r) \subseteq \mathbb{I}^f_{t+\epsilon}(r)$ and $\mathbb{I}_f^{t-\epsilon}(r) \supseteq \mathbb{I}_f^t(r) \supseteq \mathbb{I}_f^{t+\epsilon}(r)$ are equalities, and a *homologically critical value* if it is not a homologically regular value.

Denote by $\mathrm{CR}(f)$ the set of all homologically critical values. If f is weakly tame then $\mathrm{CR}(f) \subseteq \mathrm{Cr}(f)$.

Proposition 3.4 *If $f\colon X \to \mathbb{R}$ is a map (hence X is ANR and f is proper) then $\mathrm{CR}(f)$ is discrete.*

Proof As pointed out above in the proof of Proposition 3.2, one can find a proper simplicial map $g\colon K \to \mathbb{R}$ and a proper homotopy domination $\alpha\colon K \to X$ such that $|f \cdot \alpha - g| < M$. If so, for any $a < b$ with $a, b \in \mathbb{R}$, one has $\dim(\mathbb{I}^f_b(r)/\mathbb{I}^f_a(r)) \le \dim(\mathbb{I}^g_{b+M}(r)/\mathbb{I}^g_{a-M}(r)) \le \dim(H_r(g^{-1}([a-M, b+M]), g^{-1}(a-M)) < \infty$, which implies that there are only finitely many changes in $\mathbb{I}^f_t(r)$ for t with $a \le t \le b$, Similar arguments show that there are only finitely many changes of $\mathbb{I}^t_f(r)$ for t with $a \le t \le b$. This suffices to have $\mathrm{CR}(f) \cap [a,b]$ a finite set for any $a < b$, hence $\mathrm{CR}(f)$ discrete. □

Definition 3.5 Define $\tilde{\epsilon}(f) := \inf |c' - c''|$ where $c', c'' \in \mathrm{CR}(f)$ and $c' \neq c''$, and call f *homologically tame* (with respect to κ) if $\tilde{\epsilon}(f) > 0$.

Clearly tame maps are homologically tame with respect to any field κ, and $\tilde{\epsilon}(f) > \epsilon(f)$.

[4] As a replacement for an argument based on an incorrect reference, the above argument and reference were proposed by the referee.

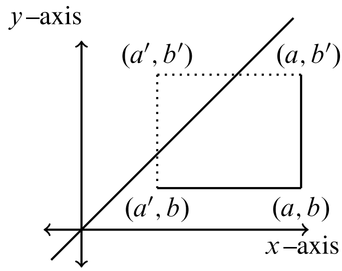

Figure 1: The box $B:=(a',a]\times[b,b')\subset\mathbb{R}^2$

Consider the sets of the form $B=(a',a]\times[b,b')$ with $a'<a, b<b'$ and refer to B as a *box*; see Figure 1.

To a box B we assign the quotient of subspaces

$$\mathbb{F}_r^f(B):=\mathbb{F}_r^f(a,b)/(\mathbb{F}_r^f(a',b)+\mathbb{F}_r^f(a,b')),$$

and define

$$F_r^f(a,b):=\dim\mathbb{F}_r^f(a,b),\quad F_r^f(B):=\dim\mathbb{F}_r^f(B).$$

In view of Observation 3.1(2), one has

$$F_r^f(B):=F_r^f(a,b)+F^f(a',b')-F_r^f(a',b)-F^f(a,b').$$

It will also be convenient to define

$$(\mathbb{F}_r^f)'(B):=\mathbb{F}_r^f(a',b)+\mathbb{F}_r^f(a,b')\subseteq\mathbb{F}_r^f(a,b),$$

in which case

$$\mathbb{F}_r^f(B)=\mathbb{F}_r^f(a,b)/(\mathbb{F}_r^f)'(B).$$

We denote by $\pi^B_{ab,r}$ the obvious projection

$$\pi^B_{ab,r}\colon\mathbb{F}_r^f(a,b)\to\mathbb{F}_r^f(B). \tag{1}$$

To ease the writing, when no risk of ambiguity, one drops f from the notation.

If $\kappa=\mathbb{R}$ or $\mathbb{C}$ and $H_r(X)$ is equipped with an inner product (nondegenerate positive definite hermitian scalar product), one denotes by $\boldsymbol{H}_r(B)$ the orthogonal complement of $\mathbb{F}'_r(B)=(\mathbb{F}_r(a',b)+\mathbb{F}(a,b'))$ inside $\mathbb{F}_r(a,b)$, which is a finite-dimensional Hilbert space, and one has

$$\boldsymbol{H}_r(B)\subseteq\mathbb{F}_r(a,b)\subseteq H_r(X).$$

Proposition 3.6 *Let $a''<a'<a$, $b<b'$ and B_1, B_2 and B the boxes $B_1=(a'',a']\times[b,b'')$, $B_2=(a',a]\times[b,b')$ and $B=(a'',a]\times[b,b')$; see Figure 2 (left).*

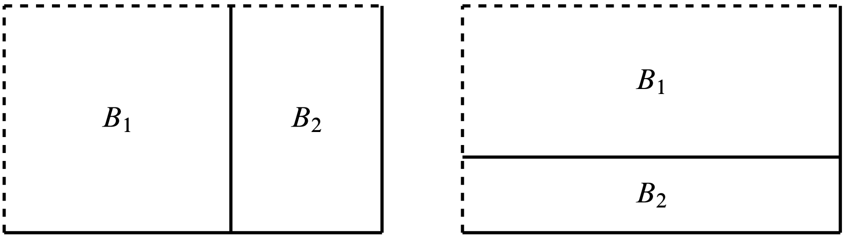

Figure 2

(a) *The inclusions* $B_1 \subset B$ *and* $B_2 \subset B$ *induce the linear maps*

$$i^{B}_{B_1,r}\colon \mathbb{F}_r(B_1) \to \mathbb{F}_r(B), \tag{2}$$

$$\pi^{B_2}_{B,r}\colon \mathbb{F}_r(B) \to \mathbb{F}_r(B_2) \tag{3}$$

such that the following sequence is exact:

$$0 \to \mathbb{F}_r(B_1) \xrightarrow{i^{B}_{B_1,r}} \mathbb{F}_r(B) \xrightarrow{\pi^{B_2}_{B,r}} \mathbb{F}_r(B_2) \to 0.$$

(b) *If* $H_r(X)$ *is equipped with a scalar product then*

$$\boldsymbol{H}_r(B_1) \perp \boldsymbol{H}_r(B_2) \quad \text{and} \quad \boldsymbol{H}_r(B) = \boldsymbol{H}_r(B_1) \oplus \boldsymbol{H}_r(B_2).$$

Proposition 3.7 *Let* $a' < a$, $b < b' < b''$ *and* B_1, B_2 *and* B *the boxes* $B_1 = (a', a] \times [b', b'')$, $B_2 = (a', a] \times [b, b')$ *and* $B = (a', a] \times [b, b'')$; *see Figure 2 (right).*

(a) *The inclusions* $B_1 \subset B$ *and* $B_2 \subset B$ *induce the linear maps*

$$i^{B}_{B_1,r}\colon \mathbb{F}_r(B_1) \to \mathbb{F}_r(B), \tag{4}$$

$$\pi^{B_2}_{B,r}\colon \mathbb{F}_r(B) \to \mathbb{F}_r(B_2) \tag{5}$$

such that the following sequence is exact:

$$0 \to \mathbb{F}_r(B_1) \xrightarrow{i^{B}_{B_1,r}} \mathbb{F}_r(B) \xrightarrow{\pi^{B_2}_{B,r}} \mathbb{F}_r(B_2) \to 0.$$

(b) *If* $\kappa = \mathbb{R}$ *or* $\mathbb{C}$ *and* $H_r(X)$ *is equipped with a scalar product then*

$$\boldsymbol{H}_r(B_1) \perp \boldsymbol{H}_r(B_2) \quad \text{and} \quad \boldsymbol{H}_r(B) = \boldsymbol{H}_r(B_1) \oplus \boldsymbol{H}_r(B_2).$$

Proof Item (a) in both Propositions 3.6 and 3.7 follows from Observation 3.1(1) and (2). To conclude item (b) note that $\boldsymbol{H}_r(B_2)$ as a subspace of $\mathbb{F}_r(a'', b)$ in Proposition 3.6 and as a subspace of $\mathbb{F}_r(a, b'')$ in Proposition 3.7 is orthogonal to a subspace which contains $\boldsymbol{H}_r(B_1)$. $\square$

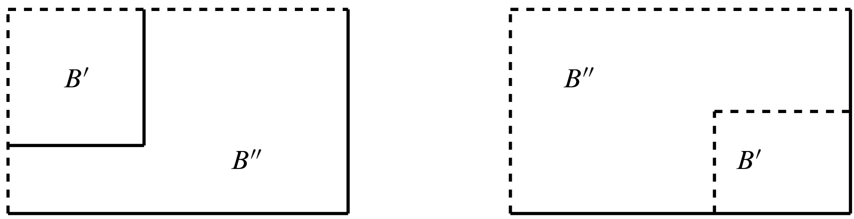

Figure 3

In view of Propositions 3.6 and 3.7, one has the following observation.

Observation 3.8 (1) If B' and B'' are two boxes with $B' \subseteq B''$ and B' is located in the upper left corner of B'' (see Figure 3 (left)) then the inclusion induces the canonical injective linear maps $i^{B''}_{B',r}: \mathbb{F}_r(B') \to \mathbb{F}_r(B'')$.

(2) If B' and B'' are two boxes with $B' \subseteq B''$ and B' is located in the lower right corner of B'' (see Figure 3 (right)) then the inclusion induces the canonical surjective linear maps $\pi^{B'}_{B'',r}: \mathbb{F}_r(B'') \to \mathbb{F}_r(B')$.

(3) If B is a finite disjoint union of boxes $B = \bigsqcup B_i$ then $\mathbb{F}_r(B)$ is isomorphic to $\bigoplus_i \mathbb{F}_r(B_i)$; the isomorphism is not canonical.

(4) If in addition $\kappa = \mathbb{R}$ or $\mathbb{C}$ and $H_r(X)$ is a Hilbert space then $\boldsymbol{H}_r(B) = \bigoplus_i \boldsymbol{H}_r(B_i)$.

In view of this observation, define $B(a,b:\epsilon) = (a-\epsilon, a] \times [b, b+\epsilon)$ and

$$\hat{\delta}^f_r(a,b) := \varinjlim_{\epsilon \to 0} \mathbb{F}_r(B(a,b;\epsilon)).$$

The limit refers to the direct system $\mathbb{F}_r(B(a,b;\epsilon')) \to \mathbb{F}_r(B(a,b;\epsilon''))$ whose arrows are the surjective linear maps induced by the inclusion of $B(a,b;\epsilon')$ as the lower right corner of $B(a,b;\epsilon'')$ for $\epsilon' < \epsilon''$.

Define also

$$\delta^f_r(a,b) := \lim_{\epsilon \to 0} F_r(B(a,b;\epsilon)).$$

Clearly one has $\dim \hat{\delta}^f_r(a,b) = \delta^f_r(a,b)$. Denote by $\operatorname{supp} \delta^f_r$ the set

$$\operatorname{supp} \delta^f_r := \{(a,b) \in \mathbb{R}^2 \mid \delta^f_r(a,b) \neq 0\}.$$

Observation 3.9 For any (a,b), $a,b \in \mathbb{R}$, the direct system stabilizes and $\hat{\delta}^f_r(a,b) = \mathbb{F}^f(B(a,b;\epsilon))$ for some ϵ small enough. Moreover $\delta^f_r(a,b) \neq 0$ implies that $a,b \in \mathrm{CR}(f)$. In particular $\operatorname{supp} \delta^f_r$ is a discrete subset of $\mathbb{R}^2$. If f is homologically tame then for any (a,b) with $a,b\in \mathrm{CR}(f)$, we have $\hat{\delta}^f_r(a,b)=\mathbb{F}^f(B(a,b;\epsilon))$ for any ϵ, $0<\epsilon<\tilde{\epsilon}(f)$.

Recall that for a box $B=(a',a]\times[b,b')$, we have denoted the canonical projection on $\mathbb{F}_r(B)=\mathbb{F}(a,b)/\mathbb{F}'(B)$ by $\pi^B_{ab,r}\colon \mathbb{F}_r(a,b)\to\mathbb{F}_r(B)$, and for $B'=(a'',a]\times[b,b'')$, $a''\le a'<a$, $b''\ge b'>b$, we have denoted by $\pi^B_{B',r}\colon \mathbb{F}_r(B')\to\mathbb{F}_r(B)$ the canonical surjective linear map between quotient spaces induced by $\mathbb{F}'(B')\subseteq\mathbb{F}'(B)\subseteq\mathbb{F}(a,b)$. Clearly

$$\pi^B_{ab,r}=\pi^B_{B',r}\cdot\pi^{B'}_{ab,r}.$$

Consider the surjective linear map

$$\pi_r(a,b)\colon \mathbb{F}(a,b)\to \varinjlim_{\epsilon\to 0}\mathbb{F}(B(a,b;\epsilon))=\hat{\delta}^f_r(a,b),$$
$$\pi_r(a,b):=\varinjlim_{\epsilon\to 0}\pi^{B(a,b;\epsilon)}_{ab,r}.$$

Definition 3.10 A *special splitting* is a linear map

$$s_r(a,b)\colon \hat{\delta}^f_r(a,b)\to\mathbb{F}_r(a,b)$$

which satisfies $\pi_r(a,b)\cdot s_r(a,b)=\mathrm{id}$. In particular, in view of Observation 3.1, for any $\alpha>a$ and $\beta<b$, we have $\mathrm{img}(s_r(a,b))\subseteq\mathbb{F}_r(\alpha,\beta)$.

We denote by $i_r(a,b)$ the composition of $s_r(a,b)$ with the inclusion $\mathbb{F}_r(a,b)\subseteq H_r(X)$.

The diagram

$$\begin{tikzcd}
H_r(X) & \mathbb{F}_r(a,b) \arrow[l, "\supseteq"'] \arrow[r, "\pi_r(a,b)"'] \arrow[d, "\pi^B_{ab,r}"'] & \hat{\delta}^{\tilde f}_r(a,b) \arrow[ll, bend right, "i_r(a,b)"'] \arrow[l, bend right, "s_r(a,b)"'] \arrow[dl, "i^B_r(a,b)"] \\
\mathbb{F}_r(B_1) \arrow[r, "i^B_{B',r}"'] & \mathbb{F}_r(B) \arrow[r, "\pi^{B_2}_{B,r}"'] & \mathbb{F}_r(B_2)
\end{tikzcd} \tag{6}$$

reviews for the reader the linear maps considered so far. In this diagram suppose $B=(\alpha',\alpha]\times[\beta,\beta')$ with $a\in(\alpha',\alpha]$ and $b\in[\beta,\beta')$ and $B=B_1\sqcup B_2$ as in Figure 2 (left). In view of Observations 3.8 and 3.9, one has the following.

Observation 3.11 (1) If $(a,b)\in B_2$ then $\pi^{B_2}_{B,r}\cdot i^B_r(a,b)$ is injective.

(2) If $(a,b)\in B_1$ then $\pi^{B_2}_{B,r}\cdot i^B_r(a,b)$ is zero.

Choose special splittings $\{s_r(a,b) \mid (a,b) \in \operatorname{supp}(\delta_r^{\tilde f})\}$, and consider the sum

$$I_r = \sum_{(a,b)\in \operatorname{supp}(\delta_r^{\tilde f})} i_r(a,b) : \bigoplus_{(a,b)\in \operatorname{supp}(\delta_r^{\tilde f})} \hat\delta_r^f(a,b) \to H_r(X),$$

and for a finite or infinite box B the sum

$$I_r^B = \sum_{(a,b)\in \operatorname{supp}(\delta_r^{\tilde f})\cap B} i_r^B(a,b) : \bigoplus_{(a,b)\in \operatorname{supp}(\delta_r^{\tilde f})\cap B} \hat\delta_r^f(a,b) \to \mathbb{F}_r(B).$$

For $\Sigma \subseteq \operatorname{supp}(\delta_r^f)$ denote by $I_r(\Sigma)$ the restriction of I_r to $\bigoplus_{(a,b)\in\Sigma} \hat\delta_r^f(a,b)$ and for $\Sigma \subseteq \operatorname{supp}(\delta_r^f) \cap B$ denote by $I_r^B(\Sigma)$ the restriction of I_r^B to $\bigoplus_{(a,b)\in\Sigma} \hat\delta_r^f(a,b)$. Note the following.

Observation 3.12 For $B = B_1 \sqcup B_2$ as in Figure 2 and $\Sigma \subseteq \operatorname{supp} \delta_r^{\tilde f}$ with $\Sigma = \Sigma_1 \sqcup \Sigma_2$, $\Sigma_1 \subseteq B_1$ and $\Sigma_2 \subseteq B_2$, the diagram

$$\begin{tikzcd}
\mathbb{F}_r(B_1) \arrow[rr] & & \mathbb{F}_r(B) \arrow[rr] & & \mathbb{F}_r(B_2) \\
\bigoplus_{(a,b)\in\Sigma_1} \hat\delta_r^{\tilde f}(a,b) \arrow[u, "I_r^{B_1}(\Sigma_1)"'] \arrow[rr] & & \bigoplus_{(a,b)\in\Sigma} \hat\delta_r^{\tilde f}(a,b) \arrow[u, "I_r^{B}(\Sigma)"'] \arrow[rr] & & \bigoplus_{(a,b)\in\Sigma_2} \hat\delta_r^{\tilde f}(a,b) \arrow[u, "I_r^{B_2}(\Sigma_2)"']
\end{tikzcd}$$

is commutative. In particular if $I_r^{B_1}(\Sigma_1)$ and $I_r^{B_2}(\Sigma_2)$ are injective then so is $I_r^B(\Sigma)$.

If $\kappa = \mathbb{R}$ or $\mathbb{C}$ and $H_r(X)$ is equipped with a Hilbert space structure, then the inverse of the restriction of $\pi_r(a,b)$ to the orthogonal complement of $\ker(\pi_r(a,b))$ provides a *canonical special splitting* $s_r(a,b)$. For these canonical special splittings, one denotes by $\hat{\hat\delta}_r^f$ the assignment

$$\hat{\hat\delta}_r^f(a,b) = \boldsymbol{H}_r(a,b) := \operatorname{img} s_r(a,b).$$

Then if X is compact in view of Observation 3.8(4) the assignment $\hat{\hat\delta}_r^f$ is a configuration $\mathcal{C}^O_{H_r(X)}(\mathbb{R}^2)$. The configuration $\hat{\hat\delta}_r^f(a,b)$ has the configuration $\delta_r^f \in \mathbb{C}^{\dim H_r(X)}$ as its dimension.

Let f be a map, and for any $(a,b) \in \mathbb{R}^2$ choose a special splitting $s_r(a,b)$: $\hat\delta_r^f(a,b) \to H_r(X)$.

Observation 3.13 (1) For any $\Sigma \subseteq \operatorname{supp}(\delta_r^f)$ (resp. $\Sigma \subseteq \operatorname{supp}(\delta_r^f) \cap B$), the linear maps $I_r(\Sigma)$ (resp. $I_r^B(\Sigma)$) are injective.

(2) For any box $B = (a',a] \times [b,b')$ the set $\delta_r^f \cap B$ is finite.

(3) For any box B, the linear map I_r^B is an isomorphism.

(4) If X compact, $m < \inf f$ and $M > \sup f$ then $H_r(X) = \mathbb{F}_r((m,M]\times[m,M))$ and I_r is an isomorphism. Therefore, for any special splittings, the collection of subspaces $\mathrm{img}(i_r(a,b))$ provide a configuration of subspaces of $H_r(X)$ hence and element in $\mathcal{C}_{H_r(X)}(\mathbb{R}^2)$.

Proof (1) If $\Sigma \subset B$ then in view of Observations 3.11 and 3.12, the injectivity of $I_r^B(\Sigma)$ implies the ineffectiveness of $I_r^{B'}(\Sigma)$ for any box $B' \supseteq B$, as well as the injectivity of $I_r(\Sigma)$. To check the injectivity of $I_r^B(\Sigma)$, one proceeds as follows:

- If the cardinality of Σ is one, then the statement follows from Observation 3.11.
- If all elements (α_i, β_i), $i = 1, \dots, k$, of Σ have the same first component $\alpha_i = a$, the statement follows by induction on k. One writes the box $B = B_1 \sqcup B_2$ as in Figure 2 (left) such that B_2 contains one element of Σ, say (α_1, β_1), and B_1 contains the remaining $k-1$ elements. The injectivity follows from Observation 3.12 in view of the injectivity of $I_r^{B_2}(\Sigma \cap B_2)$ and of $I_r^{B_1}(\Sigma \cap B_1)$, assumed by the induction hypothesis.
- In general, one writes Σ as the disjoint union $\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \cdots \sqcup \Sigma_k$ such that each Σ_i contains all points of Σ with the same first component a_i, and $a_k > a_{k-1} > \cdots > a_2 > a_1$. One proceeds again by induction on k. One decomposes the box B as in Figure 2 (right), $B = B_1 \sqcup B_2$ such that $\Sigma_1 \subset B_2$ and $(\Sigma \setminus \Sigma_1) \subset B_1$. The injectivity of $I_r^B(\Sigma)$ follows then using Observation 3.12 from the injectivity of $I_r^{B_2}(\Sigma_1)$ and the induction hypothesis which assumes the injectivity of $I_r^{B_1}(\Sigma \cap B_1)$.

(2) In view of (1), any subset of $\mathrm{supp}(\delta_r^f) \cap B$ with $B = (a', a]\times[b, b')$ has cardinality smaller than $\dim \mathbb{F}_r(a,b)$, which by Proposition 3.2 is finite. Hence Σ is finite.

(3) The injectivity of I_r^B is ensured by (1). The surjectivity follows from the equality of the dimension of the source and of the target implied by Observations 3.8 and 3.9.

(4) This follows from definitions and from (3). □

In case X is not compact, for the needs of part II of this paper it is useful to extend Observation 3.13(3) to the situation of an infinite box $B(a,b;\infty) := (-\infty, a]\times[b,\infty)$, and evaluate the image of I_r, which might not be a finite-dimensional space. For this purpose we introduce the following:

(1) $\mathbb{I}^f_{-\infty}(r) = \bigcap_{a\in\mathbb{R}} \mathbb{I}^f_a(r)$ and $\mathbb{I}_f^{\infty}(r) = \bigcap_{b\in\mathbb{R}} \mathbb{I}_f^b(r)$,

(2) $\mathbb{F}^f_r(-\infty, b) := \mathbb{I}^f_{-\infty}(r) \cap \mathbb{I}^b_f(r)$ and $\mathbb{F}^f_r(a,\infty) := \mathbb{I}^f_a(r) \cap \mathbb{I}^{\infty}_f(r)$,

(3) $(\mathbb{F}^f)'_r(B(a,b;\infty)) := \mathbb{F}^f_r(-\infty,b) + \mathbb{F}^f_r(a,\infty)$,

(4) $\mathbb{F}^f_r(B(a,b;\infty)) := \mathbb{F}^f_r(a,b)/(\mathbb{F}^f)'_r(B(a,b;\infty))$.

Observation 3.14 (1) In view of the finite-dimensionality of $\mathbb{F}_r(a,b)$, one has the following:

(i) For any a, there exists $b(a)$ such that
$$\mathbb{F}_r(a,b(a)) = \mathbb{F}_r(a,b') = \mathbb{F}_r(a,\infty)$$
provided that $b' \geq b(a)$.

(ii) For any b, there exists $a(b))$ such that
$$\mathbb{F}_r(-\infty,b) = \mathbb{F}_r(a',b) = \mathbb{F}_r(a(b),b)$$
provided that $a' \leq a(b)$.

(2) In view of (1), for $a' < a(b)$ and $b' > b(a)$, the canonical projections
$$\mathbb{F}_r(B(a,b;\infty)) \to \mathbb{F}_r((a',a]\times[b,b')) \to \mathbb{F}_r((a(b),a]\times[b,b(a)))$$
are isomorphisms.

Observation 3.15 (addendum to Observation 3.13(3)) The maps
$$\bigoplus_{(a',b')\in\operatorname{supp}(\delta^f_r)\cap B(a,b;\infty)} i_r^{B(a,b;\infty)}(a',b')\colon \bigoplus_{(a',b')\in\operatorname{supp}\delta^f_r\cap B(a,b;\infty)} \hat{\delta}^f_r(a',b') \to \mathbb{F}_r(B(a,b;\infty)),$$
$$\bigoplus_{(a,b)\in\operatorname{supp}(\delta^f_r)} i_r(a,b)\colon \bigoplus_{(a,b)\in\operatorname{supp}(\delta^f_r)} \hat{\delta}^f_r(a,b) \to H_r(X)/(\mathbb{I}^f_{-\infty}(r)+\mathbb{I}^\infty_f(r))$$
are isomorphisms.

Proof The first isomorphism follows from Observations 3.13 and 3.14.

For the second, note that for $k<k'$ (for simplicity in writing we drop f and r from the notation)
$$(\mathbb{I}_{-\infty}\cap\mathbb{I}^{-k'}+\mathbb{I}_{k'}\cap\mathbb{I}^{\infty})\cap\mathbb{I}^{-k}\cap\mathbb{I}_k = \mathbb{I}_{-\infty}\cap\mathbb{I}^{-k}+\mathbb{I}_k\cap\mathbb{I}^{\infty}$$
and that
$$H_r(X) = \varinjlim_{k\to\infty} \mathbb{F}_r(k,-k) = \varinjlim_{k\to\infty} \mathbb{I}^{-k} = \varinjlim_{k\to\infty} \mathbb{I}_k.$$
Then in view of stabilization properties,
$$\varinjlim \frac{\mathbb{F}(k,-k)}{\mathbb{I}_{-\infty}\cap\mathbb{I}^{-k}+\mathbb{I}_k\cap\mathbb{I}^{\infty}} = \frac{H_r(X)}{\mathbb{I}_{-\infty}+\mathbb{I}^{\infty}}. \qquad \square$$

Let $D(a,b;\epsilon) := (a-\epsilon,a+\epsilon]\times[b-\epsilon,b+\epsilon)$. If $x=(a,b)$, one also writes $D(x;\epsilon)$ for $D(a,b;\epsilon)$.

Proposition 3.16 (see [3, Proposition 5.6]) *Let $f\colon X\to\mathbb{R}$ be a tame map and $\epsilon<\epsilon(f)/3$. For any map $g\colon X\to\mathbb{R}$ which satisfies $\|f-g\|_\infty<\epsilon$ and $a,b\in\mathrm{Cr}(f)$ critical values, one has*

$$\sum_{x\in D(a,b;2\epsilon)}\delta^g_r(x)=\delta^f_r(a,b), \tag{7}$$

$$\operatorname{supp}\delta^g_r\subset\bigcup_{(a,b)\in\operatorname{supp}\delta^f_r}D(a,b;2\epsilon). \tag{8}$$

If in addition $H_r(X)$ is equipped with a Hilbert space structure ($\kappa=\mathbb{R}$ or $\mathbb{C}$), the above statement can be strengthened to

$$x\in D(a,b;2\epsilon)\ \Rightarrow\ \hat\delta^g_r(x)\subseteq\hat\delta^f_r(a,b),\qquad \bigoplus_{x\in D(a,b;2\epsilon)}\hat\delta^g_r(x)=\hat\delta^f_r(a,b). \tag{9}$$

Proposition 3.16 implies that in an ϵ–neighborhood of a tame map f (with respect to the $\|\cdot\|_\infty$ norm) any other map g has the support of δ^g_r in a 2ϵ–neighborhood of the support of δ^f_r and in case X compact is of cardinality counted with multiplicities equal to $\dim H_r(X)$.

Proof of Proposition 3.16 See [3]. Consider a collection of real numbers

$$C:=\{\cdots<c_i<c_{i+1}<c_{i+2}<\cdots\mid i\in\mathbb{Z}\}$$

which satisfies the following properties:

(1) $\mathrm{Cr}(f)\subseteq C$,

(2) $c_{i+1}-c_i>\epsilon(f)$,

(3) $\lim_{i\to\infty}c_i=\infty$,

(4) $\lim_{i\to-\infty}c_i=-\infty$.

Next, one establishes two intermediate results.

Lemma 3.17 *For f as in Proposition 3.16 and $c_i,c_j\in C$, one has*

$$\begin{aligned}\hat\delta^f_r(c_i,c_j)&=\mathbb{F}^f_r((c_{i-1},c_i]\times[c_j,c_{j+1}))\\&=\mathbb{F}^f_r(c_i,c_j)/\mathbb{F}^f_r(c_{i-1},c_j)+\mathbb{F}^f_r(c_i,c_{j+1}),\end{aligned}\tag{10}$$

and therefore

$$\begin{aligned}\delta^f_r(c_i,c_j)&=F^f_r((c_{i-1},c_i]\times[c_j,c_{j+1}))\\&=F^f_r(c_{i-1},c_{j+1})+F^f_r(c_i,c_j)-F^f_r(c_{i-1},c_j)-F^f_r(c_i,c_{j+1}).\end{aligned}\tag{11}$$

Proof It is known (see [7], for example) that X a closed subset of Y and X, Y ANRs implies that X is a neighborhood deformation retract [7]. Then in view of the tameness of f, for any $\epsilon', \epsilon'' \in (0, \epsilon(f))$ one has

$$\begin{aligned} \mathbb{F}_r^f(c_i, c_j) &= \mathbb{F}_r^f(c_i+\epsilon', c_j) = \mathbb{F}_r^f(c_{i+1}-\epsilon'', c_j) = \mathbb{F}_r^f(c_{i+1}-\epsilon'', c_{j-1}+\epsilon''), \\ \mathbb{F}_r^f(c_i, c_j) &= \mathbb{F}_r^f(c_i, c_j-\epsilon') = \mathbb{F}_r^f(c_i, c_{j-1}+\epsilon'') = \mathbb{F}_r^f(c_{i+1}-\epsilon', c_{j-1}+\epsilon''). \end{aligned} \tag{12}$$

Since $\epsilon < \epsilon(f)$, in view of the definition of $\hat{\delta}_r^f$ one has

$$\begin{aligned} \hat{\delta}_r^f(c_i, c_j) &= \mathbb{F}_r^f((c_i-\epsilon, c_i]\times[c_j, c_j+\epsilon)) \\ &= \mathbb{F}_r^f(c_i, c_j)/\mathbb{F}_r^f(c_i-\epsilon, c_j) + \mathbb{F}_r^f(c_i, c_j+\epsilon). \end{aligned} \tag{13}$$

Combining (13) with (12) one obtains the equality (10):

$$\hat{\delta}_r^f(c_i, c_j) = \mathbb{F}_r^f(c_i, c_j)/\mathbb{F}_r^f(c_{i-1}, c_j) + \mathbb{F}_r^f(c_i, c_{j+1}).$$

Since $\mathbb{F}^f(c_{i-1}, c_j) \cap \mathbb{F}^f(c_i, c_{j+1}) = \mathbb{F}^f(c_{i-1}, c_{j+1})$ one has

$$\begin{aligned} &\dim(\mathbb{F}_r^f(c_{i-1}, c_j) + \mathbb{F}_r^f(c_i, c_{j+1})) \\ &\qquad = \dim \mathbb{F}_r^f(c_{i-1}, c_j) + \dim \mathbb{F}_r^f(c_i, c_{j+1} - \dim \mathbb{F}^f(c_{i-1}, c_{j+1}) \end{aligned}$$

and the equality (11) follows. □

To simplify the notation, the index r in the following lemma will be dropped.

Lemma 3.18 *Suppose f is tame. Let $a = c_i$, $b = c_j$, $c_i, c_j \in C$ and $\epsilon < \epsilon(f)/3$. If g is a continuous map with $\|f-g\|_\infty < \epsilon$, then*

$$\begin{aligned} \mathbb{F}_r^g(a-2\epsilon, b+2\epsilon) &= \mathbb{F}_r^f(c_{i-1}, c_{j+1}), \\ \mathbb{F}_r^g(a+2\epsilon, b-2\epsilon) &= \mathbb{F}_r^f(c_i, c_j), \\ \mathbb{F}_r^g(a+2\epsilon, b+2\epsilon) &= \mathbb{F}_r^f(c_i, c_{j+1}), \\ \mathbb{F}_r^g(a-2\epsilon, b-2\epsilon) &= \mathbb{F}_r^f(c_{i-1}, c_j). \end{aligned} \tag{14}$$

Proof Since $\|f-g\|_\infty < \epsilon$, in view of Observation 3.1(3) one has

$$\begin{aligned} \mathbb{F}_r^f(a-3\epsilon, b+3\epsilon) &\subseteq \mathbb{F}_r^g(a-2\epsilon, b+2\epsilon) \subseteq \mathbb{F}_r^f(a-\epsilon, b+\epsilon), \\ \mathbb{F}_r^f(a+\epsilon, b-\epsilon) &\subseteq \mathbb{F}_r^g(a+2\epsilon, b-2\epsilon) \subseteq \mathbb{F}_r^f(a+3\epsilon, b-3\epsilon), \\ \mathbb{F}_r^f(a+\epsilon, b+3\epsilon) &\subseteq \mathbb{F}_r^g(a+2\epsilon, b+2\epsilon) \subseteq \mathbb{F}_r^f(a+3\epsilon, b+\epsilon), \\ \mathbb{F}_r^f(a-3\epsilon, b-\epsilon) &\subseteq \mathbb{F}_r^g(a-2\epsilon, b-2\epsilon) \subseteq \mathbb{F}_r^f(a-\epsilon, b-3\epsilon). \end{aligned} \tag{15}$$

Since $3\epsilon < \epsilon(f)$, one has

$$\begin{aligned} \mathbb{F}^f(a-3\epsilon, b+3\epsilon) &= \mathbb{F}^f(a-\epsilon, b+\epsilon), \\ \mathbb{F}^f(a+\epsilon, b-\epsilon) &= \mathbb{F}^f(a+3\epsilon, b-3\epsilon), \\ \mathbb{F}^f(a+\epsilon, b+3\epsilon) &= \mathbb{F}^f(a+3\epsilon, b+\epsilon), \\ \mathbb{F}^f(a-3\epsilon, b-\epsilon) &= \mathbb{F}^f(a-\epsilon, b-3\epsilon), \end{aligned} \tag{16}$$

which imply that in (15) the inclusion $\subseteq$ is actually equality.

Note that in view of the equalities (12), for $\epsilon', \epsilon'' < \epsilon(f)$ one has

$$\begin{aligned} \mathbb{F}^f(c_{i-1}, c_{j+1}) &= \mathbb{F}^f(a-\epsilon', b+\epsilon''), \\ \mathbb{F}^f(c_i, c_j) &= \mathbb{F}^f(a+\epsilon', b-\epsilon''), \\ \mathbb{F}^f(c_i, c_{j+1}) &= \mathbb{F}^f(a+\epsilon', b+\epsilon''), \\ \mathbb{F}^f(c_{i-1}, c_j) &= \mathbb{F}^f(a-\epsilon', b-\epsilon''). \end{aligned} \tag{17}$$

Then (15) and (17) imply (14) and hence the statement of Lemma 3.18. $\square$

Next observe that Lemma 3.18 gives (for $a = c_i$, $b = c_j$ with $c_i, c_j \in C$) the equality

$$\mathbb{F}^g((a-2\epsilon, a+2\epsilon] \times [b-2\epsilon, b+2\epsilon)) = \mathbb{F}^f((c_{i-1}, c_i] \times [c_j, c_{j+1})).$$

This combined with Lemma 3.17 implies $\mathbb{F}^g((a-2\epsilon,a+2\epsilon]\times[b-2\epsilon,b+2\epsilon)) = \hat{\delta}^f(a,b)$, which combined with Observation 3.13 implies the inclusion (7) and the equality (9), not only for critical values but for any $a, b \in C$.

To check inclusion (8) observe the following:

(a) $\|f-g\|_\infty < \epsilon$ implies $X_a^f \subseteq X_{a+\epsilon}^g \subseteq X_{a+2\epsilon}^f$ and $X_f^b \subseteq X_g^{b-\epsilon} \subseteq X_f^{b-2\epsilon}$, and when $a, b \in C$,

$$\mathbb{F}^f(a,b) \subseteq \mathbb{F}^g(a+\epsilon, b-\epsilon) \subseteq \mathbb{F}^f(a+2\epsilon, b-2\epsilon). \tag{18}$$

(b) When $\epsilon < \epsilon(f)/3$, the inclusions (18) imply

$$\mathbb{F}^f(a,b) = \mathbb{F}^g(a+\epsilon, b-\epsilon) = \mathbb{F}^f(a+2\epsilon, b-2\epsilon)$$

which in view of Observation 3.15 implies

$$\begin{aligned} \sum_{x \in (-\infty,a]\times(b,\infty)\cap \operatorname{supp} \delta_r^f} \delta_r^f(x) &= \sum_{y \in (-\infty,a+\epsilon]\times(b-\epsilon,\infty)\cap \operatorname{supp} \delta_r^g} \delta_r^g(y) \\ &= \sum_{x \in (-\infty,a+2\epsilon]\times(b-2\epsilon,\infty)\cap \operatorname{supp} \delta_r^f} \delta_r^f(x). \end{aligned} \tag{19}$$

Since $\mathbb{R}^2 = \bigcup_{i\in\mathbb{Z}} B(c_i, c_{-i}; \infty)$, (19) and (7) rule out the existence of an $x \in \mathrm{supp}(\delta_r^g)$ away from $\bigcup_{x\in\mathrm{supp}(\delta_r^f)} D(x; 2\epsilon)$, finishing the proof of Proposition 3.16. □

Let K be a compact ANR and $f\colon X \to \mathbb{R}$ be a map. Denote by

$$\bar{f}_K; X \times K \to \mathbb{R}$$

the composition $f \cdot \pi_K$ with $\pi_K\colon X \times K \to X$ the first factor projection. If f is weakly tame then so is $\bar{f}_K$ and the set of critical values of f and of $\bar{f}_K$ are the same. Moreover in view of the Künneth theorem about the homology of the cartesian product of two spaces one can make the following observation.

Observation 3.19 (1) $\mathbb{F}_r^{\bar{f}_K}(a,b) = \bigoplus_{0\le k\le r} \mathbb{F}_k^f(a,b) \otimes H_{r-k}(K)$, and therefore

(2) $\hat{\delta}_r^{\bar{f}_K}(a,b) = \bigoplus_{0\le k\le r} \hat{\delta}_k^f(a,b) \otimes H_{r-k}(K)$, and

(3) $\hat{\delta}_r^{\bar{f}_K}(a,b) = \hat{\delta}_k^f(a,b)$ when K is acyclic.

Note that the embedding $I\colon C(X;\mathbb{R}) \to C(X \times K;\mathbb{R})$ defined by $I(f) = \bar{f}_K$ is an isometry when both spaces are equipped with the distance $\|\cdot\|_\infty$. Note also that when K is acyclic one has $\delta_r^f = \delta_r^{I(f)}$ and $\hat{\delta}_r^f = \hat{\delta}_r^{I(f)}$ provided that $H_r(X)$ is identified with $H_r(X \times K)$.

4 The main results

Theorem 4.1 (topological results) *Suppose X is compact and $f\colon X \to \mathbb{R}$ a map.*[5]

(1) *$\delta_r^f(x) \neq 0$ with $x = (a,b)$ implies that both $a, b \in \mathrm{CR}(f)$.*

(2) *$\sum_{x\in\mathbb{R}^2} \delta_r^f(x) = \dim H_r(X)$ and $\bigoplus_{x\in\mathbb{R}^2} \hat{\delta}_r^f(x) = H_r(X)$. In particular, we have $\delta_r^f \in \mathcal{C}_{\dim H_r(X)}(\mathbb{R}^2)$.*

(3) *If $H_r(X)$ is equipped with a Hilbert space structure then $\hat{\hat{\delta}}^f \in \mathcal{C}^O_{H_r(X)}(\mathbb{R}^2)$.*

(4) *If X is homeomorphic to a finite simplicial complex or a compact Hilbert cube manifold then for an open and dense set of maps f in the space of continuous maps with compact open topology, $\delta_r^f(x) = 0$ or 1.*

Statements (1) and (3) formulated in terms of bar codes (see [2]) were verified first in [3] under the hypothesis that f is a tame map.

[5] This means X is also ANR and f continuous.

Theorem 4.2 (stability) *Suppose X is a compact ANR.*

(1) *The assignment $f \rightsquigarrow \delta^f_r$ provides a continuous map from the space of real-valued maps $C(X;\mathbb{R})$ equipped with the compact open topology to the space of configurations $\mathcal{C}_{b_r}(\mathbb{R}^2) = \mathbb{C}^{b_r}$ and $b_r = \dim H_r(X)$, equipped with the collision topology (also regarded as the space of monic polynomials of degree b_r). Moreover, with respect to the canonical metric $\underline{D}$ on the space of configurations, which induces the collision topology, one has*

$$\underline{D}(\delta^f, \delta^g) < 2D(f,g).$$

Recall that $D(f,g) := \|f-g\|_\infty = \sup_{x\in X} |f(x)-g(x)|$.

(2) *If $\kappa = \mathbb{R}$ or $\mathbb{C}$ then the assignment $f \rightsquigarrow \hat{\hat{\delta}}^f_r$ is continuous with respect to both collision topologies. (The continuity with respect to the first implies that with respect to the second.)*

Theorem 4.2(1) was first established in [3] under the hypothesis X homeomorphic to a finite simplicial complex.

Theorem 4.3 (Poincaré duality) (1) *Suppose X is a closed smooth κ–orientable manifold*[6] *of dimension n, and f a continuous map. Then $\delta^f_r(a,b) = \delta^f_{n-r}(b,a)$.*

(2) *In addition any collection of isomorphisms $H_r(X) \to H_r(X)^*$ induce the isomorphisms of the configuration $\hat{\delta}^f_r$ and $\hat{\delta}^f_{n-r} \cdot \tau$ with $\tau(a,b) = (b,a)$.*

Item (1) of the above theorem was established in [3] for f a tame map.

4.1 Proof of Theorem 4.1

Items (1)–(3) are contained in Observation 3.13 and Observation 3.9.

We first prove item (4). In view of Theorem 4.2, whose proof does not involve Theorem 4.1, it suffices to establish only the density in the space of all continuous functions of tame maps f with δ^f_r taking values only 0 and 1.

We say that a tame map $f\colon X \to \mathbb{R}$ satisfies Property G if the following holds.

Property G There exists a finite sequence of real numbers

$$a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b$$

[6]The results probably remain true as stated for topological manifolds based essentially on the same arguments, but being unable to find appropriate references we formulate them under the hypothesis of smoothness.

such that

(1) $\mathbb{I}^f_a(r) = 0$ and $\mathbb{I}^f_b(r) = H_r(X)$,

(2) for any $i \geq 1$, $\dim(\mathbb{I}^f_{a_i}/\mathbb{I}^f_{a_{i-1}}) \leq 1$.

The verification of Theorem 4.1(4) is based on the Observations 4.4 and 4.5.

Observation 4.4 For any tame map f which satisfies Property G, the configuration δ^f_r takes only the values 0 and 1.

If f has Property G then it satisfies $\dim(\mathbb{I}^f_{a_i}/\mathbb{I}^f_{a_{i-1}}) \leq 1$ for $a_i = c_i, i = 1, \dots, n$; since for $\alpha < \beta$ with no critical value in the open interval (α, β) and β a regular value, the inclusion $X^f_\alpha \subset X^f_\beta$ induces isomorphism in homology and for any $a' \leq a \leq b \leq b'$, $\dim(\mathbb{I}^f_b(r)/\mathbb{I}^f_a(r)) \leq \dim(\mathbb{I}^f_{b'}(r)/\mathbb{I}^f_{a'}(r))$.

If so, then for any two consecutive critical values $c_{i-1} < c_i$ and any other critical value c_j, the inclusion $\mathbb{F}_r(c_{i-1}, c_j) \subseteq \mathbb{F}_r(c_i, c_j)$ has cokernel of dimension at most one, which by (10) in Lemma 3.17 implies that δ^f_r takes only the values 0 and 1. Based on this observation, if X is a compact smooth manifold (possibly with boundary), any Morse function $f\colon X \to \mathbb{R}$ which takes different values of different critical points has Property G.

Indeed if $\{\cdots < c_i < c_{i+1} < \cdots\}$ is the collection of all critical values, $X^f_{c_{i+1}}$ is homotopy equivalent to a space obtained from $X^f_{c_i}$ by adding a closed disk D^k along $\partial D^k = S^{k-1}$ or $\partial D^k_+ = D^{k-1}$, which insures that Property G is satisfied. Since the set of such Morse functions is dense in the space of all continuous functions equipped with the C_0–topology, item (4) is verified (once Theorem 4.2 is established).

If X is a compact Hilbert cube manifold, then is homeomorphic to $M \times Q$ with M a compact smooth manifold (possible with boundary), and any continuous map $f\colon X \to \mathbb{R}$ is arbitrarily closed to $\bar{f}_Q$, with $f\colon M \to \mathbb{R}$ a Morse function. This observation establishes item (4) for compact Hilbert cube manifolds.

If X is a finite simplicial complex, one needs the following observation.

Observation 4.5 If X is a finite simplicial complex and $a < b$, one can construct a map $h\colon X \to \mathbb{R}$ simplicial on the barycentric subdivision of X with the following properties:

(1) $a < h(x) < b$;

(2) h takes different values on the barycenters of different simplices;

(3) the value of h on the barycenter of a simplex σ is strictly larger than the values of h on the barycenter of any of its faces.

Proof The construction is straightforward. Such a map satisfies Property G, since adding a simplex to a finite simplicial complex might change the dimension of the homology with at most one unit, and for any α, X^h_α retracts by deformation to the simplicial complex generated by the barycenters on which h takes value smaller or equal to α.

For $f\colon X \to \mathbb{R}$ a simplicial map, X a finite simplicial complex with critical values $\{\cdots < c_{i-1} < c_i < \cdots\}$, if for some i we have $\dim(\mathbb{I}^f_{c_i}/\mathbb{I}^f_{c_{i-1}}) \geq 2$, one chooses $\epsilon < \epsilon(f)/2$ and a subdivision of X which makes $f^{-1}(c_i \pm \epsilon/2)$ and $f^{-1}(c_i))$, and thus $f^{-1}([c_i - \epsilon/2, c_i + \epsilon/2])$ and $f^{-1}([c_i, c_i + \epsilon])$, subcomplexes. One takes the barycentric subdivision of this subdivision and replaces f by g, the simplicial map for the new triangulation. We define the map g to take the same value as f on the barycenters of simplices not contained in $f^{-1}(c_i)$, and as h constructed using Observation 4.5 for $a = c_I - \epsilon/2$, $b = c_i + \epsilon/2$ on the barycenters of simplices contained in $f^{-1}(c_i)$. The map g gets as possible critical values, in addition to the critical values of f the critical values of $h = g|_{f^{-1}(c_i)}$. We leave the reader to check that g satisfies Property G in view of the fact that h does and $\epsilon < \epsilon(f)$. Clearly g differs from f by less than ϵ as it follows from construction.

Since simplicial maps (for some subdivision) are dense in the space of continuous maps and any simplicial map is arbitrarily close to one that satisfies Property G, item (4) follows. $\square$

4.2 Stability: proof of Theorem 4.2

The stability theorem is a consequence of Proposition 3.16. In order to explain this we begin with a few observations:

(1) Consider the space of maps $C(X, \mathbb{R})$, X a compact ANR, equipped with the compact open topology which is induced from the metric

$$D(f, g) := \sup_{x \in X} |f(x) - g(x)| = \|f - g\|_\infty.$$

This metric is complete.

(2) Observe that if $f, g \in C(X, \mathbb{R})$, then for any $t \in [0, 1]$,

$$h_t := tf(x) + (1-t)g(x) \in C(X; \mathbb{R})$$

is continuous, and for any $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$ one has the equality

$$D(f, g) = \sum_{0 \leq i < N} D(h_{t_{i+1}}, h_{t_i}). \tag{20}$$

(3) If X is a simplicial complex let $\mathcal{U} \subset C(X, \mathbb{R})$ denote the subset of PL maps. Then

(i) $\mathcal{U}$ is a dense subset in $C(X, \mathbb{R})$;

(ii) if $f, g \in \mathcal{U}$ then $h_t \in \mathcal{U}$, hence $\epsilon(h_t) > 0$, hence for any $t \in [0,1]$ there exists $\delta(t) > 0$ such that $t', t'' \in (t - \delta(t), t + \delta(t))$ implies $D(h_{t'}, h_t) < \epsilon(h_t)/3$.

These two statements are not hard to check. Recall the following:

- f is PL on X if with respect to some subdivision of X f is simplicial (ie the restriction of f to each simplex is linear), and
- for any two PL maps f, g, there exists a common subdivision of X which makes f and g simultaneously simplicial, hence h_t is a simplicial map for any t.

Item (i) follows from the fact that continuous maps can be approximated with arbitrary accuracy by PL maps and item (ii) follows from the continuity in t of the family h_t and from the compactness of X.

(4) Consider $\mathcal{C}_{b_r}(\mathbb{R}^2) = \mathbb{C}^{b_r}$, $b_r = \dim(H_r(X)$, with the canonical metric $\underline{D}$ which is complete. Since any map in $\mathcal{U}$ is tame, in view of Proposition 3.16, $f, g \in \mathcal{U}$ with $D(f, g) < \epsilon(f)/3$ implies

$$\underline{D}(\delta^f_r, \delta^g_r) \leq 2D(f, g). \tag{21}$$

To prove Theorem 4.2, first check that inequality (21) extends to all $f, g \in \mathcal{U}$. To do that we start with $f, g \in \mathcal{U}$ and consider the homotopy h_t, $t \in [0,1]$ defined above.

Choose a sequence $0 < t_1 < t_3 < \cdots < t_{2N-1} < 1$ such that for $i = 1, \dots, (2N-1)$, the intervals $(t_{2i-1} - \delta(t_{2i-1}), t_{2i-1} + \delta(t_{2i-1}))$ cover $[0,1]$ and

$$(t_{2i-1}, t_{2i-1} + \delta(t_{2i-1})) \cap (t_{2i+1} - \delta(t_{2i+1}), t_{2i+1}) \neq \varnothing.$$

This is possible in view of the compactness of $[0,1]$.

Take $t_0 = 0, t_{2N} = 1$ and $t_{2i} \in (t_{2i-1}, t_{2i-1} + \delta(t_{2i-1})) \cap (t_{2i+1} - \delta(t_{2i+1})$. To simplify the notation, abbreviate h_{t_i} to h_i.

In view of item (3)(ii) and item (4) (inequality (21)), one has

$$\begin{aligned} |t_{2i-1} - t_{2i}| < \delta(t_{2i-1}) \quad &\text{implies} \quad \underline{D}(\delta^{h_{2i-1}}, \delta^{h_{2i}}) < 2D(h_{2i-1}, h_{2i}), \\ |t_{2i} - t_{2i+1}| < \delta(t_{2i+1}) \quad &\text{implies} \quad \underline{D}(\delta^{h_{2i}}, \delta^{h_{2i+1}}) < 2D(h_{2i}, h_{2i+1}). \end{aligned}$$

Then we have

$$\underline{D}(\delta^f, \delta^g) \leq \sum_{0 \leq i < 2N-1} \underline{D}(\delta^{h_i}, \delta^{h_{i+1}}) \leq 2 \sum_{0 \leq i < 2N-1} D(h_i, h_{i+1}) = D(f, g).$$

In view of the density of $\mathcal{U}$ and the completeness of the metrics on $C(X;\mathbb{R})$ and $\mathcal{C}_{b_r}(\mathbb{R}^2)$, inequality (21) extends to the entire $C(X;\mathbb{R})$ when X is a simplicial complex. Indeed, the assignment $\mathcal{U} \ni f \rightsquigarrow \delta_r^f \in C_{b_r}(\mathbb{R}^2)$ preserves the Cauchy sequences.

Next we verify (21) for $X = K \times Q$, K a simplicial complex and Q the Hilbert cube. For this purpose we write $Q := I^k \times Q^{\infty-k}$ and say that $f\colon K \times Q \to \mathbb{R}$ is an $(\infty{-}k)$–PL map if $f = \bar{g}_{Q^{\infty-k}}$ (see Section 2.3 for the definition of $\bar{g}_{Q^{\infty-k}}$) with $g\colon K \times I^k \to \mathbb{R}$ a PL map. Clearly an $(\infty{-}k)$–PL map is an $(\infty{-}k')$–PL map for $k' \geq k$.

Denote by $C_{\mathrm{PL}}(K \times Q;\mathbb{R})$ the set of maps in $C(K \times Q;\mathbb{R})$ which are $(\infty{-}k)$–PL for some k.

In view of Observation 2.2, $C_{\mathrm{PL}}(K \times Q;\mathbb{R})$ is dense in $C(K \times Q;\mathbb{R})$. To conclude that (21) holds for $K \times Q$, it suffices to check the inequality for $f_1 = (\bar{g}_1)_{Q^{\infty-k}}$, $f_2 = (\bar{g}_2)_{Q^{\infty-k}} \in C_{\mathrm{PL}}(K \times Q;\mathbb{R})$. The inequality holds since, in view of Observation 3.19, we have $\delta^{f_i} = \delta^{g_i}$.

Since by Theorem 2.3 any compact Hilbert cube manifold is homeomorphic to $K \times Q$ for some finite simplicial complex K, inequality (21) holds for X any compact Hilbert cube manifold. Since for any X a compact ANR, by Theorem 2.3, $X \times Q$ is a Hilbert cube manifold, $I\colon C(X;\mathbb{R}) \to C(X \times Q;\mathbb{R})$ defined by $I(f) = \bar{f}_Q$ is an isometric embedding and $\delta^f = \delta^{\bar{f}_Q}$, (21) holds for any X a compact ANR.

Both parts of Theorem 4.2 follow from inequality (21) and Proposition 3.16(9).

4.3 Poincaré duality: proof of Theorem 4.3

Before we proceed to the proof of Theorem 4.3, the following elementary observation on linear algebra, used also in part II, will be useful.

For the commutative diagram

$$E := \begin{array}{ccc} C & \xrightarrow{\gamma_2} & A_2 \\ \downarrow{\scriptstyle \gamma_1} & & {\scriptstyle \alpha_2}\downarrow \\ A_1 & \xrightarrow{\alpha_1} & B \end{array}$$

define

$$\ker(E) := \ker(C \xrightarrow{\gamma} A_1 \times_B A_2),$$
$$\operatorname{coker}(E) := \operatorname{coker}(A_1 \oplus_C A_2 \xrightarrow{\alpha} B)$$

with

$A_1 \times_B A_2 = \{(a_1, a_2) \in A_1 \times A_2 \mid \alpha_1(a_1) = \alpha_2(a_2)\}$,

$A_1 \oplus_C A_2 = A_1 \oplus A_2 \big/ \big\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 = \beta_1(c), a_2 = -\beta_2(c) \text{ for some } c \in C\big\}$

and with $\gamma(c) = (\gamma_1(c), \gamma_2(c))$ and $\alpha(a_1, a_2) = \alpha_1(a_1) + \alpha_2(a_2)$.

If one denotes by E^* the dual diagram

$$E^* := \begin{tikzcd} C^* & A_2^* \arrow[l, "\gamma_2^*"'] \\ A_1^* \arrow[u, "\gamma_1^*"] & B^* \arrow[l, "\alpha_1^*"] \arrow[u, "\alpha_2^*"'] \end{tikzcd}$$

then we have a canonical isomorphism

$$\ker(E) = (\operatorname{coker}(E^*))^*. \tag{22}$$

Note the following.

Proposition 4.6 *If in the diagram E either all arrows are injective and α is injective or all arrows are surjective and γ is surjective, then*

$$\dim(\operatorname{coker} E) = \dim C + \dim B - \dim A_1 - \dim A_2.$$

The proof is a straightforward calculation of dimensions.

For the proof of extended Poincaré duality claimed by Theorem 4.3 it is useful to provide an alternative definition of $\mathbb{F}_r(B)$ for a box B.

For this purpose introduce the quotient space

$$\mathcal{G}_r(a,b) = H_r(X)/(\mathbb{I}_a(r) + \mathbb{I}^b(r)).$$

Consider a box $B = (a', a] \times [b, b')$ and denote by $\mathcal{G}(B)$ and $\mathcal{F}(B)$ the diagrams

$$\mathcal{G}(B) := \begin{tikzcd} \mathcal{G}_r(a', b') \arrow[r] \arrow[d] & \mathcal{G}_r(a, b') \arrow[d] \\ \mathcal{G}_r(a', b) \arrow[r] & \mathcal{G}_r(a, b) \end{tikzcd} \qquad \mathcal{F}(B) := \begin{tikzcd} \mathbb{F}_r(a', b') \arrow[r] \arrow[d] & \mathbb{F}_r(a, b') \arrow[d] \\ \mathbb{F}_r(a', b) \arrow[r] & \mathbb{F}_r(a, b) \end{tikzcd}$$

whose arrows are induced by the inclusions $\mathbb{I}_{a'}(r) \subseteq \mathbb{I}_a(r)$ and $\mathbb{I}^{b'}(r) \subseteq \mathbb{I}^b(r)$. Let

$$\mathcal{G}_r^f(B) := \ker \mathcal{G}(B)$$

and recognize that
$$\mathbb{F}^f_r(B) = \operatorname{coker}\mathcal{F}(B).$$
Note that the hypotheses of Proposition 4.6 are verified, (1) for $\mathcal{G}(B)$ and (2) for $\mathcal{F}(B)$, and $\mathcal{G}_r(B)$ identifies to $\ker(\mathcal{G}(B))$ and $\mathbb{F}_r(B)$ to $\operatorname{coker}(\mathcal{F}(B))$.

Since $\mathcal{G}_r(a',b)\times_{\mathcal{G}_r(a,b)}\mathcal{G}_r(a,b') = H_r(X)/((\mathbb{I}_{a'}(r)+\mathbb{I}^b(r))\cap(\mathbb{I}_a(r)+\mathbb{I}^{b'}(r)))$, the vector space $\mathcal{G}_r(B)$ is canonically isomorphic to
$$\big((\mathbb{I}_{a'}(r)+\mathbb{I}^b(r))\cap(\mathbb{I}_a(r)+\mathbb{I}^{b'}(r))\big)/(\mathbb{I}_{a'}(r)+\mathbb{I}^{b'}(r)). \tag{23}$$
Similarly, since $\mathbb{F}_r(a',b)\oplus_{\mathbb{F}_r(a',b')}\mathbb{F}_r(a,b')) = (\mathbb{I}_{a'}(r)\cap\mathbb{I}^b(r)+\mathbb{I}_a(r)\cap\mathbb{I}^{b'}(r))$, the vector space $\mathbb{F}_r(B)$ is canonically isomorphic to
$$\mathbb{I}_a(r)\cap\mathbb{I}^b(r)/(\mathbb{I}_{a'}(r)\cap\mathbb{I}^b(r)+\mathbb{I}_a(r)\cap\mathbb{I}^{b'}(r)).$$
The obvious inclusion $\mathbb{I}_a(r)\cap\mathbb{I}^b(r)\subseteq(\mathbb{I}_{a'}(r)+\mathbb{I}^b(r))\cap(\mathbb{I}_a(r)+\mathbb{I}^{b'}(r))$ induces the linear map
$$\begin{aligned}\mathbb{F}_r(B) = \mathbb{I}_a(r)\cap\mathbb{I}^b(r)/(\mathbb{I}_{a'}(r)\cap\mathbb{I}^b(r)+\mathbb{I}_a(r)\cap\mathbb{I}^{b'}(r))&\\ \to(\mathbb{I}_{a'}(r)+\mathbb{I}^b(r))\cap(\mathbb{I}_a(r)+\mathbb{I}^{b'}(r))/(\mathbb{I}_{a'}(r)+\mathbb{I}^{b'}(r)&=\mathcal{G}_r(B).\end{aligned}$$

Proposition 4.7 *For any map $f\colon X\to\mathbb{R}$ and any box B the canonical linear map $\mathbb{F}_r(B)\to\mathcal{G}_r(B)$ defined above is an isomorphism: $\mathbb{F}^f_r(B)=\mathcal{G}^f_r(B)$.*

Proof Note that the injectivity is straightforward. Indeed, suppose
$$\mathbb{I}_a(r)\cap\mathbb{I}^b(r)\ni x = x_1+x_2$$
with $x_1\in\mathbb{I}_{a'}(r)$ and $x_2\in\mathbb{I}^{b'}(r)$. Then $x_1 = x-x_2\in\mathbb{I}^b(r)$ and $x_2\in(\mathbb{I}_a(r)\cap\mathbb{I}^{b'}(r))$.

To check the surjectivity, start with $x = x_1+y_1 = x_2+y_2$ such that $x_1\in\mathbb{I}_{a'}$, $y_1\in\mathbb{I}^b$, $x_2\in\mathbb{I}_a$, $y_2\in\mathbb{I}^{b'}$. Then $x-x_1-y_2$ is equivalent to x in $\mathcal{G}_r(B)$. But $x-x_1-y_2 = y_1-y_2 = x_2-x_1$ hence it belongs to $\mathbb{I}^b$ and to $\mathbb{I}^a$. □

Let $f\colon M^n\to\mathbb{R}$ be a map, M^n a κ–orientable closed topological manifold and a,b regular values such that the restriction of f to $f^{-1}(a-\epsilon,a+\epsilon)$ and $f^{-1}(b-\epsilon,b+\epsilon)$ for a small enough positive ϵ are topological submersions. This makes $f^{-1}(a)$ and $f^{-1}(b)$ codimension-one topological submanifolds of M.

Let $i_a\colon M_a\to M$, $i^b\colon M^b\to M$, $j_a\colon M\to(M,M_a)$, $j^b\colon M\to(M,M^b)$ denote the obvious inclusions, $i_a(k), i^b(k), j_a(k), j^b(k)$ denote the inclusion induced linear maps for homology in degree k, and $r_a(k), r^b(k), s_a(k), s^b(k)$ denote the inclusion induced linear maps in cohomology (with coefficients in the field κ), as indicated in diagrams (24) and (25) below. Poincaré duality provides the commutative diagrams (24) and (25) with all vertical arrows isomorphisms:

$$
\begin{tikzcd}
H_r(M_a) \arrow[r, "i_a(r)"] \arrow[d] & H_r(M) \arrow[r, "j_a(r)"] \arrow[d] & H_r(M, M_a) \arrow[d] \\
H^{n-r}(M, M^a) \arrow[r, "s^a(n-r)"] \arrow[d] & H^{n-r}(M) \arrow[r, "r^a(n-r))"] \arrow[d] & H^{n-r}(M^a) \arrow[d] \\
(H_{n-r}(M, M^a))^* \arrow[r, "(j^a(n-r))^*"] & (H_{n-r}(M))^* \arrow[r, "(i^a(n-r))^*"] & (H_{n-r}(M^a))^*
\end{tikzcd}
\tag{24}
$$

$$
\begin{tikzcd}
H_r(M^b) \arrow[r, "i^b(r)"] \arrow[d] & H_r(M) \arrow[r, "j^b(r)"] \arrow[d] & H_r(M, M^b) \arrow[d] \\
H^{n-r}(M, M_b) \arrow[r, "s_b(n-r)"] \arrow[d] & H^{n-r}(M)) \arrow[r, "r_b(n-r)"] \arrow[d] & H^{n-r}(M_b) \arrow[d] \\
(H_{n-r}(M, M_b))^* \arrow[r, "(j_b(n-r))^*"] & (H_{n-r}(M))^* \arrow[r, "(i_b(n-r))^*"] & (H_{n-r}(M_b))^*
\end{tikzcd}
\tag{25}
$$

As a consequence of these two diagrams, observe that Poincaré duality provides a canonical isomorphism

$$\mathbb{F}^f_r(a,b) = (\mathcal{G}^f_{n-r}(b,a))^*. \tag{26}$$

Indeed, observe the following:

- $\mathbb{F}_r(a,b) = \ker(j_a(r), j^b(r))$ by the exactness of the first rows in diagrams (24) and (25). Precisely $\ker(j_a(r), j^b(r)) = \ker j_a(r) \cap j^b(r) = \mathbb{I}_a(r) \cap \mathbb{I}^b(r)$.

- $\ker(j_a(r), j^b(r)) \equiv \ker(r^a(n-r), r_b(n-r))$ by the isomorphism of the upper vertical arrows in these diagrams.

- $\ker(r^a(n-r), r_b(n-r)) \equiv \ker\big((i^a(n-r))^*, (i_b(n-r))^*\big)$ by the isomorphism of the lower vertical arrow in these diagrams.

The isomorphisms above are induced by Poincaré duality and cohomology in terms of homology; their composition is still referred to as Poincaré duality.

- $\ker\big((i^a(n-r))^*, (i_b(n-r))^*\big) = \big(\operatorname{coker}\big(i^a(n-r) + i_b(n-r)\big)\big)^* = (\mathcal{G}^f_{n-r}(b,a))^*$ by standard finite-dimensional linear algebra duality.

Putting together these equalities one obtains (26).

Suppose M is a closed κ–orientable smooth manifold and $f\colon M \to \mathbb{R}$ a smooth map which is locally polynomial (ie in the neighborhood of any point, in some local coordinates, is a polynomial). Such a map is tame. For $(a,b) \in \mathbb{R}^2$ choose ϵ small enough so that the intervals $(a-\epsilon, a), (a, a+\epsilon)$ as well as $(a-\epsilon, a), (a, a+\epsilon)$ are contained in the set of regular values (in the sense of differential calculus). Such a choice is possible in view of the tameness of f.

To establish the result as stated for such a map we proceed as follows.

In view of the tameness of f,

$$\hat{\delta}^f_r(a,b) = \mathbb{F}^f_r((a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon)). \tag{27}$$

By definition,

$$\mathbb{F}^f_r((a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon)) = \operatorname{coker} \mathcal{F}_r((a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon)). \tag{28}$$

By Proposition 4.7,

$$\operatorname{coker} \mathcal{F}_r((a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon)) = \ker\big(\mathcal{G}_r((a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon))\big). \tag{29}$$

By equality (22),

$$\begin{aligned}\ker\big(\mathcal{G}_r((a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon))\big)& \\ &= \big(\operatorname{coker}\big(\mathcal{G}_r((a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon))^*\big)\big)^*.\end{aligned} \tag{30}$$

By equality (26),

$$\begin{aligned}\big(\operatorname{coker}\big(\mathcal{G}_r((a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon))^*\big)\big)^*& \\ &= \big(\operatorname{coker}\big(\mathcal{F}_{n-r}((b-\epsilon, b+\epsilon] \times [a-\epsilon, a+\epsilon))\big)\big)^*.\end{aligned} \tag{31}$$

In view of the equality $\mathbb{F}^f_r(B) = \operatorname{coker} \mathcal{F}(B)$,

$$\begin{aligned}\big(\operatorname{coker}\big(\mathcal{F}_{n-r}((b-\epsilon, b+\epsilon] \times [a-\epsilon, a+\epsilon))\big)\big)^*& \\ &= \big(\mathbb{F}_{n-r}((b-\epsilon, b+\epsilon] \times [a-\epsilon, a+\epsilon))\big)^*.\end{aligned} \tag{32}$$

In view of the tameness of f,

$$\big(\mathbb{F}^f_{n-r}((b-\epsilon, b+\epsilon] \times [a-\epsilon, a+\epsilon))\big)^* = (\hat{\delta}^f_{n-r}(b,a))^*. \tag{33}$$

Putting together equalities (27)–(33), one derives the result for f as above. In view of Theorem 4.2 and the fact that locally polynomial maps are dense in the space of all continuous maps when X is a smooth manifold, the result holds as stated.

A comment The hypothesis of compact ANR can be replaced by ANR with total homology of finite dimension and proper map by *homologically proper map*, which means that for I a closed interval, the total homology of $f^{-1}(I)$ has finite dimension. All results remain unchanged with essentially the same proof. An interesting situation when such a generalization is relevant is the case of the absolute value of the complex polynomial function f when restricted to the complement of its zeros, which will be treated in future work, but can be easily reduced to the case of a proper map considered above.

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