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A categorification of the Alexander polynomial in embedded contact homology

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Given a transverse knot K in a three-dimensional contact manifold (Y,α), Colin, Ghiggini, Honda and Hutchings defined a hat version $\widehat{\mathrm{ECK}}(K,Y,\alpha)$ of embedded contact homology for K and conjectured that it is isomorphic to the knot Floer homology $\widehat{\mathrm{HFK}}(K,Y)$.

We define here a full version $\mathrm{ECK}(K,Y,\alpha)$ and generalize the definitions to the case of links. We prove then that if $Y=S^3$, then ECK and $\widehat{\mathrm{ECK}}$ categorify the (multivariable) Alexander polynomial of knots and links, obtaining expressions analogous to that for knot and link Floer homologies in the minus and, respectively, hat versions.

57M27, 57R17, 57R58

Introduction

Given a 3–manifold Y, Ozsváth and Szabó [29] defined topological invariants of Y, indicated by $\mathrm{HF}^\infty(Y)$, $\mathrm{HF}^+(Y)$, $\mathrm{HF}^-(Y)$ and $\widehat{\mathrm{HF}}(Y)$. These groups are the *Heegaard Floer homologies of Y* in the respective versions.

Moreover, Ozsváth and Szabó [28] and Rasmussen [33] proved that any homologically trivial knot K in Y induces a "knot filtration" on the Heegaard Floer chain complexes. The first pages of the associated spectral sequences (in each version) are topological invariants of K: these are bigraded homology groups $\mathrm{HFK}^\infty(K,Y)$, $\mathrm{HFK}^+(K,Y)$, $\mathrm{HFK}^-(K,Y)$ and $\widehat{\mathrm{HFK}}(K,Y)$, called *Heegaard Floer knot homologies* (in the respective versions).

These homologies are powerful invariants for the couple (K,Y). For instance, in [28] and [33], it was proved that $\widehat{\mathrm{HFK}}(K,S^3)$ *categorifies the Alexander polynomial* Δ_K *of* K; ie

$$\chi(\widehat{\mathrm{HFK}}(K,S^3)) \doteq \Delta(K),$$

where $\doteq$ means that the two sides are equal up to sign change and multiplication by a monic monomial, and χ denotes the *graded Euler characteristic*.

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This was the first categorification of the Alexander polynomial; a second one (in Seiberg–Witten–Floer homology) was discovered later by Kronheimer and Mrowka [23].

Ozsváth and Szabó [31] developed a similar construction for any link L in S^3 and got invariants $\mathrm{HFL}^-(L, S^3)$ and $\widehat{\mathrm{HFL}}(L, S^3)$ for L, which they called *Heegaard Floer link homologies*. Now these homologies come with an additional $\mathbb{Z}^n$ degree, where n is the number of connected components of L. Ozsváth and Szabó proved moreover that $\mathrm{HFL}^-(L, S^3)$ *categorifies the multivariable Alexander polynomial of* L, which is a generalization of the classic Alexander polynomial. They found in particular that

$$\chi(\mathrm{HFL}^-(L, S^3)) \doteq \begin{cases} \Delta_L(t_1, \dots, t_n) & \text{if } n > 1, \\ \Delta_L(t)/(1-t) & \text{if } n = 1, \end{cases} \tag{1}$$

and

$$\chi(\widehat{\mathrm{HFL}}(L, S^3)) \doteq \begin{cases} \Delta_L \cdot \prod_{i=1}^n (t_i^{1/2} - t_i^{-1/2}) & \text{if } n > 1, \\ \Delta_L(t) & \text{if } n = 1. \end{cases} \tag{2}$$

In the series of papers [5; 6; 7; 8; 9], Colin, Ghiggini and Honda prove the equivalence between Heegaard Floer homology and *embedded contact homology* for three-manifolds. The last one is another Floer homology theory, first defined by Hutchings, which associates to a contact manifold (Y, α) two graded modules $\mathrm{ECH}(Y, \alpha)$ and $\widehat{\mathrm{ECH}}(Y, \alpha)$.

Theorem 0.1 (Colin, Ghiggini and Honda [5]–[9])

$$\mathrm{HF}^+(-Y) \cong \mathrm{ECH}(Y, \alpha),$$
$$\widehat{\mathrm{HF}}(-Y) \cong \widehat{\mathrm{ECH}}(Y, \alpha),$$

where $-Y$ *is the manifold* Y *with the inverted orientation.*

In light of Theorem 0.1, it is a natural problem to find an embedded contact counterpart of Heegaard Floer knot homology. In analogy with the *sutured Heegaard Floer theory* developed by Juhász [22], Colin, Ghiggini, Honda and Hutchings [10, Sections 6–7] define a sutured version of embedded contact homology. This can be thought of as a version of embedded contact homology for manifolds with boundary. In particular, given a knot K in a contact three-manifold (Y, ξ), using sutures they define a *hat version* $\widehat{\mathrm{ECK}}(K, Y, \alpha)$ *of embedded contact knot homology*.

Roughly speaking, this is the hat version of ECH for the contact manifold with boundary $(Y \setminus \mathcal{N}(K), \alpha)$, where $\mathcal{N}(K)$ is a suitable thin tubular neighborhood of K in Y and α is a contact form satisfying specific compatibility conditions with K. In [10, Conjecture 1.5], the following conjecture is stated:

Conjecture 0.2 $\widehat{\mathrm{ECK}}(K, Y, \alpha) \cong \widehat{\mathrm{HFK}}(-K, -Y)$.

In the present paper, we first define a *full version of embedded contact knot homology* $\mathrm{ECK}(K, Y, \alpha)$ for knots K in any contact three-manifold (Y, ξ) endowed with a (suitable) contact form α for ξ. Moreover, we generalize the definitions to the case of links L with more than one component to obtain homologies $\mathrm{ECK}(L, Y, \alpha)$ and $\widehat{\mathrm{ECK}}(L, Y, \alpha)$. We state then the following:

Conjecture 0.3 For any link L in Y, there exists a contact form α for which

$$\widehat{\mathrm{ECK}}(L, Y, \alpha) \cong \widehat{\mathrm{HFK}}(-L, -Y),$$
$$\mathrm{ECK}(L, Y, \alpha) \cong \mathrm{HFK}^{+}(-L, -Y).$$

We remark that $\mathrm{ECK}(L, Y, \alpha)$ (as well as $\widehat{\mathrm{ECK}}(L, Y, \alpha)$) is defined as the first page of a spectral sequence arising from a filtration induced by L on a suitable chain complex for $\mathrm{ECH}(Y)$. In light of the last conjecture, this fact is interesting because the analogous filtration for $\mathrm{HFK}^{-}(L, Y)$ is useful to study link surgery formulae in Heegaard Floer (see for example Ozsváth and Szabó [32] and Manolescu and Ozsváth [25]), and one can expect to find similar relations in ECH.

Next we compute the graded Euler characteristics of the ECK homologies for knots and links in homology three-spheres, and we prove the following:

Theorem 0.4 *Let L be an n–component link in a homology three-sphere Y. Then there exists a contact form α such that*

$$\chi(\mathrm{ECK}(L, Y, \alpha)) \doteq \mathrm{ALEX}(Y \setminus L).$$

Here $\mathrm{ALEX}(Y \setminus L)$ is the *Alexander quotient* of the complement of L in Y. The theorem is proved using Fried's dynamic reformulation of ALEX [14]. Classical relations between $\mathrm{ALEX}(S^3 \setminus L)$ and Δ_L imply the following result:

Theorem 0.5 *Let L be any n–component link in S^3. Then there exists a contact form α for which*

$$\chi(\mathrm{ECK}(L, S^3, \alpha)) \doteq \begin{cases} \Delta_L(t_1, \dots, t_n) & \text{if } n > 1, \\ \Delta_L(t)/(1-t) & \text{if } n = 1, \end{cases}$$

and

$$\chi(\widehat{\mathrm{ECK}}(L, S^3, \alpha)) \doteq \begin{cases} \Delta_L(t_1, \dots, t_n) \cdot \prod_{i=1}^{n} (1 - t_i) & \text{if } n > 1, \\ \Delta_L(t) & \text{if } n = 1. \end{cases}$$

This implies that the homology ECK *is a categorification of the multivariable Alexander polynomial.* A straightforward consequence is:

Corollary 0.6 *In S^3, Conjectures 0.2 and 0.3 hold at the level of Euler characteristics.*

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1 Review of embedded contact homology

1.1 Preliminaries

This subsection is devoted to recall some basic notions about contact geometry, holomorphic curves, Morse–Bott theory and open books.

1.1.1 Contact geometry A (cooriented) *contact form* on a three-dimensional oriented manifold Y is a 1–form α on Y such that $\alpha \wedge d\alpha$ is a positive volume form. A *contact structure* is a smooth plane field ξ on Y such that there exists a contact form α for which $\xi = \ker \alpha$. The *Reeb vector field* of α is the (unique) vector field R_α determined by the equations $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$. A *simple Reeb orbit* is a closed oriented orbit of $R = R_\alpha$; ie it is the image δ of an embedding $S^1 \hookrightarrow Y$ such that R_P is positively tangent to δ for any $P \in \delta$. A *Reeb orbit* is an m–fold cover of a simple Reeb orbit, with $m \geq 1$. The form α determines an *action* $\mathcal{A}$ on the set of its Reeb orbits defined by $\mathcal{A}(\gamma) = \int_\gamma \alpha$. By definition, $\mathcal{A}(\gamma) > 0$ for any nonempty orbit γ.

A basic result in contact geometry asserts that the flow of the Reeb vector field (abbreviated *Reeb flow*) $\phi = \phi_R$ preserves ξ, that is, $(\phi_t)_*(\xi_P) = \xi_{\phi_t(P)}$ for any $t \in \mathbb{R}$; see [15, Chapter 1]. Given a Reeb orbit δ, there exists $T \in \mathbb{R}^+$ such that $(\phi_T)_*(\xi_P) = \xi_P$ for any $P \in \delta$; if T is the smallest possible, the isomorphism $\mathfrak{L}_\delta := (\phi_T)_*\colon \xi_P \to \xi_P$ is called the (*symplectic*) *linearized first return map* of R in P.

The orbit δ is called *nondegenerate* if 1 is not an eigenvalue of $\mathfrak{L}_\delta$. There are two types of nondegenerate Reeb orbits, *elliptic* and *hyperbolic*: δ is elliptic if the eigenvalues of $\mathfrak{L}_\delta$ are on the unit circle and is hyperbolic if they are real. In the last case, we can make a further distinction: δ is called *positive* or *negative* hyperbolic if the eigenvalues are both positive or negative, respectively.

Definition 1.1 The *Lefschetz sign* of a nondegenerate Reeb orbit δ is

$$\epsilon(\delta) := \operatorname{sign}(\det(\mathbb{1} - \mathfrak{L}_\delta)) \in \{+1, -1\}.$$

Observation 1.2 It is easy to check that $\epsilon(\delta) = +1$ if δ is elliptic or negative hyperbolic and $\epsilon(\delta) = -1$ if δ is positive hyperbolic.

To any nondegenerate orbit δ and a trivialization τ of $\xi|_\delta$, we can associate also the *Conley–Zehnder (CZ) index* $\mu_\tau(\delta) \in \mathbb{Z}$ of δ with respect to τ; see for example [20, Section 3.2] or [12]. Even if $\mu_\tau(\delta)$ depends on τ, its parity depends only on δ, and

$$(-1)^{\mu_\tau(\delta)} = -\epsilon(\delta).$$

Definition 1.3 Given $X \subseteq Y$, we will indicate by $\mathcal{P}(X)$ the set of simple Reeb orbits of α contained in X. An *orbit set* (or *multiorbit*) in X is a formal finite product $\gamma = \prod_i \gamma_i^{k_i}$, where $\gamma_i \in \mathcal{P}(X)$ and $k_i \in \mathbb{N}$ is the *multiplicity* of γ_i in γ, with $k_i \in \{0, 1\}$ whenever γ_i is hyperbolic. The set of multiorbits in X will be denoted by $\mathcal{O}(X)$.

Note that the empty set is a legitimate orbit set, and it will be indicated by $\varnothing$. An orbit set $\gamma = \prod_i \gamma_i^{k_i}$ determines the homology class $[\gamma] = \sum_i k_i[\gamma_i] \in H_1(Y)$ (unless stated otherwise, all homology groups will be taken with integer coefficients). Moreover, the action of γ is defined by $\mathcal{A}(\gamma) = \sum_i k_i \int_{\gamma_i} \alpha$.

1.1.2 Holomorphic curves We recall here some definitions and properties about holomorphic curves in dimension 4. We refer the reader to [26] and [27] for the general theory and to [20] and [5; 7; 8; 9] for an approach which is more specialized to our context.

Let X be an oriented even-dimensional manifold. An *almost complex structure* on X is an isomorphism $J\colon TX \to TX$ such that $J(T_PX) = T_PX$ and $J^2 = -\mathrm{id}$. If (X_1, J_1) and (X_2, J_2) are two even-dimensional manifolds endowed with an almost complex structure, a map $u\colon (X_1, J_1) \to (X_2, J_2)$ is *pseudoholomorphic* if it satisfies the *Cauchy–Riemann equation*

$$du \circ J_1 = J_2 \circ du.$$

Definition 1.4 A *pseudoholomorphic curve* in a four-dimensional manifold (X, J) is a pseudoholomorphic map $u\colon (F, j) \to (X, J)$, where (F, j) is a (possibly disconnected) Riemann surface.

In this paper, we will be particularly interested in pseudoholomorphic curves (that sometimes we will simply call holomorphic curves) in "symplectizations" of contact three-manifolds. Given (Y, α), consider the four-manifold $\mathbb{R} \times Y$. Call s the $\mathbb{R}$–coordinate and let $R = R_\alpha$ be the Reeb vector field of α. The almost complex structure J on $\mathbb{R} \times Y$ is *adapted to* α if

(1) J is s–invariant;

(2) $J(\xi) = \xi$ and $J(\partial_s) = R$ at any point of $\mathbb{R} \times Y$;

(3) $J|_\xi$ is compatible with $d\alpha$; ie $d\alpha(\cdot, J\cdot)$ is a Riemannian metric on ξ.

For us, a holomorphic curve u in the symplectization of (Y, α) is a holomorphic curve $u\colon (\dot{F}, j) \to (\mathbb{R} \times Y, J)$, where

(i) J is adapted to α;

(ii) $(\dot{F}, j)$ is a Riemann surface obtained from a closed surface F by removing a finite number of points (called *punctures*);

(iii) for any puncture x there exists a neighborhood $U(x) \subset F$ such that $U(x) \setminus \{x\}$ is mapped by u asymptotically to a cover of a cylinder $\mathbb{R} \times \delta$ over an orbit δ of R in a way that $\lim_{y \to x} \pi_{\mathbb{R}}(u(y)) = \pm\infty$, where $\pi_{\mathbb{R}}$ is the projection on the $\mathbb{R}$–factor of $\mathbb{R} \times Y$.

We say that x is a *positive puncture* of u if in the last condition above the limit is $+\infty$: in this case the orbit δ is a *positive end* of u. If otherwise, the limit is $-\infty$, and we say x is a *negative puncture* and δ is a *negative end* of u.

By condition (iii) above, u near a puncture x determines a cover of the Reeb orbit δ corresponding to x: the number of sheets of this cover is the *local x–multiplicity of δ in u*. The sum of the x–multiplicities over all the punctures x associated to δ is the *(total) multiplicity* of δ in u.

If γ and γ' are the orbit sets determined by the sets of all positive and, respectively, negative ends of u counted with multiplicity, then we say that u is a *holomorphic curve from γ to γ'*.

Example 1.5 A *cylinder over an orbit set γ* of Y is the holomorphic curve $\mathbb{R} \times \gamma \subset \mathbb{R} \times Y$.

Observation 1.6 Note that if there exists a holomorphic curve u from γ to γ', then $[\gamma] = [\gamma'] \in H_1(Y, \mathbb{Z})$.

Theorem 1.7 [27, Lemma 2.4.1] *Let $u\colon (F, j) \to (\mathbb{R} \times Y, J)$ be a nonconstant holomorphic curve in (X, J). Then the critical points of $\pi_{\mathbb{R}} \circ u$ are isolated. In particular, if π_Y denotes the projection $\mathbb{R} \times Y \to Y$, then $\pi_Y \circ u$ is transverse to R_α away from a set of isolated points.*

From now on, if u is a map with image in $\mathbb{R} \times Y$, we will set $u_{\mathbb{R}} := \pi_{\mathbb{R}} \circ u$ and $u_Y := \pi_Y \circ u$.

Holomorphic curves also enjoy the following property, which will be essential for us; see for example Gromov [17] and, for the noncompact case, Siefring [34].

Theorem 1.8 (positivity of intersection) *Let u and v be two distinct holomorphic curves in a four-manifold (W, J). Then $\#(\mathrm{Im}(u) \cap \mathrm{Im}(v)) < \infty$. Moreover, if P is an intersection point between $\mathrm{Im}(u)$ and $\mathrm{Im}(v)$, then its contribution m_P to the algebraic intersection number $\langle \mathrm{Im}(u), \mathrm{Im}(v) \rangle$ is strictly positive, and $m_P = 1$ if and only if u and v are embeddings near P that intersect transversely in P.*

When the almost complex structure does not play an important role or is understood, it will be omitted from the notation.

1.1.3 Morse–Bott theory The Morse–Bott theory in contact geometry was first developed by Bourgeois [3]. We present in this subsection some basic notions and applications, mostly as presented in [5].

Definition 1.9 A *Morse–Bott (MB) torus* in a 3–dimensional contact manifold (Y, α) is an embedded torus T in Y foliated by a family γ_t, $t \in S^1$, of Reeb orbits, all in the same class in $H_1(T)$, that are nondegenerate in the Morse–Bott sense. Here this means the following: given any $P \in T$ and a positive basis (v_1, v_2) of ξ_P where $v_2 \in T_P(T)$ (so that v_1 is transverse to $T_P(T)$), then the differential of the first return map of the Reeb flow on ξ_P is of the form

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

for some $a \neq 0$. T is a *positive* or *negative* MB torus if $a > 0$ or $a < 0$, respectively.

We say that α is a *Morse–Bott contact form* if all the Reeb orbits of α are either isolated and nondegenerate or come in S^1–families foliating MB tori.

As explained in [3] and [5, Section 4], it is possible to modify the Reeb vector field in a small neighborhood of a MB torus T preserving only two orbits, say e and h, of the S^1–family of Reeb orbits associated to T.

Moreover, for any fixed $L > 0$, the perturbation can be done in such a way that e and h are the only orbits in a neighborhood of T with action less then L.

If T is a positive (respectively, negative) MB torus and τ is the trivialization of ξ along the orbits given pointwise by the basis (v_1, v_2) above, then one can make the MB perturbation in a way that h is positive hyperbolic with $\mu_\tau(h) = 0$ and e is elliptic with $\mu_\tau(e) = 1$ (respectively, $\mu_\tau(e) = -1$).

The orbits e and h can be seen as the only two critical points of a Morse function $f_T \colon S^1 \to \mathbb{R}$ defined on the S^1–family of Reeb orbits foliating T and with maximum corresponding to the orbit with higher CZ index. Often MB tori will be implicitly given with such a function.

Observation 1.10 It is important to remark that, before the perturbation, T is foliated by Reeb orbits of α and so these are nonisolated. Moreover, the form of the differential of the first return map of the flow of ξ implies that these orbits are also degenerate. After the perturbation, T contains only two isolated and nondegenerate orbits, but other orbits are created in a neighborhood of T, and these orbits can be nonisolated and degenerate. See Figure 1 for a picture of a MB perturbation.

Proposition 1.11 [3, Section 3] *For any MB torus T and any $L \in \mathbb{R}$, there exists a MB perturbation of T such that, with the exception of e and h, all the periodic orbits in a neighborhood of T have action greater then L.*

A torus T foliated by Reeb orbits all in the same class of $H_1(T)$ (like, for example, a Morse–Bott torus) can be used to obtain constraints about the behavior of a holomorphic curve near T.

Following [5, Section 5], if γ is any of the Reeb orbits in T, we can define the *slope of T* as the equivalence class $s(T)$ of $[\gamma] \in H_1(T, \mathbb{R}) - \{0\}$ up to multiplication by positive real numbers.

Let $T \times [-\epsilon, \epsilon]$ be a neighborhood of $T = T \times \{0\}$ in Y with coordinates (ϑ, t, y) such that $(\partial_\vartheta, \partial_t)$ is a positive basis for $T(T)$ and ∂_y is directed as a positive normal vector to T.

Suppose that $u\colon (F, j) \to (\mathbb{R} \times Y,\ J)$ is a holomorphic curve in the symplectization of (Y, α); by Theorem 1.7, there exist at most finitely many points in $T \times [-\epsilon, \epsilon]$ where $u_Y(F)$ is not transverse to R_α. Then, if $T_y := T \times \{y\}$ and $u(F)$ intersects $\mathbb{R} \times T_y$, we can associate a slope $s_{T_y}(u)$ to $u_Y(F) \cap T_y$ for any $y \in [-\epsilon, \epsilon]$: this is defined exactly like $s(T)$, where $u_Y(F) \cap T_y$ is considered with the orientation induced by $\partial(u_Y(F) \cap (T \times [-\epsilon, y]))$.

Observation 1.12 If u has no ends in $T \times [y, y']$, then

$$\partial(u_Y(F) \cap T \times [y, y']) = u_Y(F) \cap T_{y'} - u_Y(F) \cap T_y,$$

and $s_{T_y}(u) = s_{T'_y}(u)$.

The following lemma is a consequence of the positivity of intersection in dimension four; see [5, Lemma 5.2.3].

Lemma 1.13 (blocking lemma) *Let T be linearly foliated by Reeb trajectories with slope $s = s(T)$ and u a holomorphic curve as above.*

(1) *If u is homotopic, by a compactly supported homotopy, to a map whose image is disjoint from $\mathbb{R} \times T$, then $u_Y(F) \cap T = \varnothing$.*

(2) *Let $T \times [-\epsilon, \epsilon]$ be a neighborhood of $T = T \times \{0\}$. Suppose that for some $y \in [-\epsilon, \epsilon] \setminus \{0\}$, u has no ends in $T \times (0, y]$ if $y \in (0, \epsilon]$ or in $T \times [y, 0)$ if $y \in [-\epsilon, 0)$. If $s_{T_y}(u) = \pm s(T)$, then u has an end which is a Reeb orbit in T.*

Let now x be a puncture of F whose associated end is an orbit γ in T; if there exists a neighborhood $U(x)$ of x in F such that $u_Y(U(x) \setminus \{x\}) \cap T = \varnothing$, then γ is a *one-sided end* of u in x. This is equivalent to requiring that $u_Y(U(x))$ is contained either in $T \times (-\epsilon, 0)$ or in $T \times (0, \epsilon)$.

Lemma 1.14 (trapping lemma [5, Lemma 5.3.2]) *If T is a positive (respectively, negative) MB torus and $\gamma \subset T$ is a one-sided end of u associated to the puncture x, then x is positive (respectively, negative).*

Definition 1.15 Let α be a Morse–Bott contact form on the three-manifold Y, and J a regular almost complex structure on $\mathbb{R} \times Y$. Suppose that any MB torus T in (Y, α) comes with a fixed a Morse function f_T. Let $\mathcal{P}(Y)$ be the set of simple Reeb orbits in Y minus the set of the orbits which correspond to some regular point of some f_T.

A *Morse–Bott building in* (Y, α) is a disjoint union of objects u of one of the following two types:

(1) u is the submanifold of a MB torus T corresponding to a gradient flow line of f_T: in this case, the positive (respectively, negative) end of u is the positive (respectively, negative) end of the flow line.

(2) u is a union of curves $\widetilde{u} \cup u_1 \cup \cdots \cup u_n$ of the following kind: $\widetilde{u}$ is a J–holomorphic curve in $\mathbb{R} \times Y$ with n ends $\{\delta_1, \ldots, \delta_n\}$ corresponding to regular values of some $\{f_{T_1}, \ldots, f_{T_n}\}$. Then, for each i, the curve $\widetilde{u}$ is augmented by a gradient flow trajectory u_i of f_{T_i}: u_i goes from the maximum ϵ_i^+ of f_{T_i} to δ_i if δ_i is a positive end and goes from δ_i to the minimum ϵ_i^- of f_{T_i} if δ_i is a negative end. The ends of u are obtained from the ends of $\widetilde{u}$ by substituting each δ_i with the respective ϵ_i^+ or ϵ_i^-.

A Morse–Bott building is *nice* if the $\widetilde{u}$ above has at most one connected component which is not a cover of a trivial cylinder.

1.2 ECH for closed three-manifolds

We briefly review here Hutchings' original definitions of $\mathrm{ECH}(Y, \alpha)$ and $\widehat{\mathrm{ECH}}(Y, \alpha)$ for a closed contact three-manifold (Y, α).

Assume that α is nondegenerate (ie that any Reeb orbit of α is nondegenerate). For a fixed $\Gamma \in H_1(Y)$, define $\mathrm{ECC}(Y, \alpha, \Gamma)$ to be the free $\mathbb{Z}_2$–module generated by the orbit sets of Y in the homology class Γ, and set

$$\mathrm{ECC}(Y,\alpha) = \bigoplus_{\Gamma \in H_1(Y)} \mathrm{ECC}(Y,\alpha,\Gamma).$$

This is the ECH *chain group of* (Y,α). The ECH–*differential* ∂^{ECH} (called simply ∂ when no risk of confusion occurs) is defined in [19] in terms of holomorphic curves in the symplectization $(\mathbb{R}\times Y, d\alpha, J)$ of (Y,α) as follows.

Given $\gamma,\delta \in \mathcal{O}(Y)$, let $\mathcal{M}(\gamma,\delta)$ be the set of (possibly disconnected) holomorphic curves $u\colon (\dot{F}, j) \to (\mathbb{R}\times Y, J)$ from γ to δ, where $(\dot{F}, j)$ is a punctured compact Riemannian surface. It is clear that u determines a relative homology class $[\mathrm{Im}(u)]$ in $H_2(\mathbb{R}\times Y;\gamma,\delta)$ and that if such a curve exists, then $[\gamma]=[\delta]\in H_1(Y)$.

If $\xi = \ker(\alpha)$ and a trivialization τ of $\xi|_{\gamma\cup\delta}$ is given, then to any surface $C \subset \mathbb{R}\times Y$ with $\partial C = \gamma - \delta$, it is possible to associate an ECH–*index*

$$I(C) := c_\tau(C) + Q_\tau(C) + \mu_\tau^I(\gamma,\delta),$$

which depends only on the relative homology class of C. Here,

- $c_\tau(C) := c_1(\xi|_C, \tau)$ is the *first relative Chern class* of C;
- $Q_\tau(C)$ is the τ*–relative intersection paring* of $\mathbb{R}\times Y$ applied to C;
- $\mu_\tau^I(\gamma,\delta) := \sum_i \sum_{j=1}^{k_i} \mu_\tau(\gamma_i^j) - \sum_i \sum_{j=1}^{k_i} \mu_\tau(\delta_i^j)$, where μ_τ is the Conley–Zehnder index defined in Section 1.1.1.

We refer the reader to [20] for the details about these quantities. If u is a holomorphic curve from γ to δ, set $I(u) = I(\mathrm{Im}(u))$ (well-defined up to approximating $\mathrm{Im}(u)$ with a surface in the same homology class).

Define $\mathcal{M}_k(\gamma,\delta) := \{u \in \mathcal{M}(\gamma,\delta) \mid I(u) = k\}$. The ECH–differential is then defined on the generators of $\mathrm{ECC}(Y,\alpha)$ by

$$\partial^{\mathrm{ECH}}(\gamma) = \sum_{\delta\in\mathcal{O}(Y)} \#(\mathcal{M}_1(\gamma,\delta)/\mathbb{R})\cdot\delta, \tag{3}$$

where we quotient $\mathcal{M}_1(\gamma,\delta)$ by the $\mathbb{R}$–action on the curves given by the translation in the $\mathbb{R}$–direction in $\mathbb{R}\times Y$. In [20, Section 5], Hutchings proves that $\mathcal{M}_1(\gamma,\delta)/\mathbb{R}$ is a compact 0–dimensional manifold, so $\partial^{\mathrm{ECH}}(\gamma)$ is well-defined.

The *(full) embedded contact homology of* (Y,α) is

$$\mathrm{ECH}_*(Y,\alpha) := H_*(\mathrm{ECC}(Y,\alpha), \partial^{\mathrm{ECH}}).$$

It turns out that these groups do not depend on either the choices J in the symplectization or the contact form for ξ.

If $\gamma = \prod_i \gamma_i^{k_i}$ is a generator of $\mathrm{ECC}(Y,\alpha)$, set

$$\epsilon(\gamma) = \prod\nolimits_i \epsilon(\gamma_i)^{k_i},$$

where $\epsilon(\gamma_i)$ is the Lefschetz sign of the simple orbit γ_i. Note that $\epsilon(\gamma)$ is given by the parity of the number of positive hyperbolic simple orbits in γ.

If u is a holomorphic curve from γ to δ, by simple computations it is possible to prove the *index parity formula* (see for example Section 3.4 in [20])

$$(-1)^{I(u)} = \epsilon(\gamma)\epsilon(\delta). \tag{4}$$

It follows then that the Lefschetz sign endows embedded contact homology with a well-defined absolute $\mathbb{Z}/2$–grading.

Fix now a generic point $(0,z) \in \mathbb{R} \times Y$. Given two orbit sets γ and δ, let

$$U_z\colon \mathrm{ECC}_*(Y,\alpha) \to \mathrm{ECC}_{*-2}(Y,\alpha)$$

be the map defined on the generators by

$$U_z(\gamma) = \sum_{\delta \in \mathcal{O}(Y)} \#\bigl\{u \in \mathcal{M}_2(\gamma,\delta) \mid (0,z) \in \mathrm{Im}(u)\bigr\} \cdot \delta.$$

Hutchings proves that U_z is a chain map that counts only a finite number of holomorphic curves and that this count does not depend on the choice of z. So it makes sense to define the map $U := U_z$ for any z as above. This is called the *U–map*.

The *hat version of embedded contact homology of* (Y,α) is defined as the homology $\widehat{\mathrm{ECH}}(Y,\alpha)$ of the mapping cone of the U–map. By this, we mean that $\widehat{\mathrm{ECH}}(Y,\alpha)$ is defined to be the homology of the chain complex

$$\mathrm{ECC}_{*-1}(Y,\alpha) \oplus \mathrm{ECC}_*(Y,\alpha)$$

with differential defined by the matrix

$$\begin{pmatrix} -\partial_{*-1} & 0 \\ U & \partial_* \end{pmatrix},$$

where the elements of the complex are thought as columns. Also, $\widehat{\mathrm{ECH}}(Y,\alpha)$ has the relative and the absolute gradings above.

We end this section by stating the following result; see for example [20, Section 1.4].

Theorem 1.16 *Let ξ be a contact structure on Y and α a contact form with* $\ker\alpha = \xi$. *Then the homology class* $[\varnothing] \in \mathrm{ECH}(Y,\alpha)$ *of the empty orbit set $\varnothing$ depends only on ξ, and it is called the* ECH–contact invariant of ξ.

1.3 ECH for manifolds with torus boundary

In order to define ECH for contact three-manifolds (N, α) with nonempty boundary, some compatibility between α and ∂N should be assumed. In this paper, we are particularly interested in three-manifolds whose boundary is a collection of disjoint tori.

In [5, Section 7], Colin, Ghiggini and Honda analyze this situation when ∂N is connected. If $\mathcal{T} = \partial N$ is homeomorphic to a torus, then they prove that the ECH–complex, and the differential can be defined almost as in the closed case, provided that $R = R_\alpha$ is tangent to $\mathcal{T}$ and that α is nondegenerate in $\operatorname{int}(N)$.

If the flow of $R|_{\mathcal{T}}$ is irrational, they define $\mathrm{ECH}(N, \alpha) = \mathrm{ECH}(\operatorname{int}(N), \alpha)$, and if it is rational, they consider the case of $\mathcal{T}$ Morse–Bott and do a MB perturbation of α near $\mathcal{T}$; this gives two Reeb orbits h and e on $\mathcal{T}$, and since α is now a MB contact form, the ECH–differential counts MB buildings.

1.3.1 Contact forms If Y is a closed 3–manifold and $K \subset Y$ is an oriented null-homologous knot, let $\mathcal{N}$ be a closed tubular neighborhood of K, and define N to be the closure of $Y \setminus \mathcal{N}$. Fix a neighborhood $[0,2] \times T^2$ of $\partial\mathcal{N} = \{1\} \times T^2$ in Y with $[0,1] \times T^2 \subset N$, and let $V \subset \mathcal{N}$ be the solid torus with core K and $\partial V = \{2\} \times T^2$. Obviously, $\mathcal{N} = ([1,2] \times T^2) \cup V$.

Identify now $V \setminus K$ with $[2,3) \times T^2$, and fix coordinates $(y, \vartheta, t) \in [0,3) \times T^2 \cong \mathcal{N} \setminus K$ such that the natural projection $[0,3) \times T^2 \twoheadrightarrow K$ sends (y, ϑ, t) to ϑ, and for any given y_0 and $t_0 \in [0,1]/\langle 0 \sim 1\rangle$, the push off $\{(y_0, \vartheta, t_0) \mid \vartheta \in K\}$ of K has linking number 0 with K in Y (ie it gives the Seifert framing of K). Note in particular that each strip $\{t = t_0\}$ can be seen as the intersection between $\mathcal{N} \setminus K$ and some Seifert surface for K and with the inherited orientation, so that (y, ϑ, t_0) is a positive coordinate system and any $\{(y_0, \vartheta_0, t) \mid t \in [0,1]/\langle 0 \sim 1\rangle$ is a positive meridian for K.

Definition 1.17 We say that the contact form α on Y is adapted to K if there exists a tubular neighborhood $\mathcal{N}$ of K as before such that

(1) α is a Morse–Bott contact form which is nondegenerate in $\operatorname{int}(N)$;

(2) the Reeb flow R_α is positively transverse to each strip $\{t = t_0\}$ in $\mathcal{N} \setminus K$;

(3) all the tori $y_0 \times T^2$ for $y_0 \in [1,3)$ are linearly foliated by Reeb trajectories of α;

(4) $T_1 := \{1\} \times T^2$ and $T_2 := \{2\} \times T^2$ are respectively negative and positive MB tori foliated by Reeb orbits which are meridians of K;

(5) R_α is transverse to the disks of the form $\{\vartheta = \vartheta_0\} \cap \operatorname{int}(V)$;

(6) K is a Reeb orbit.

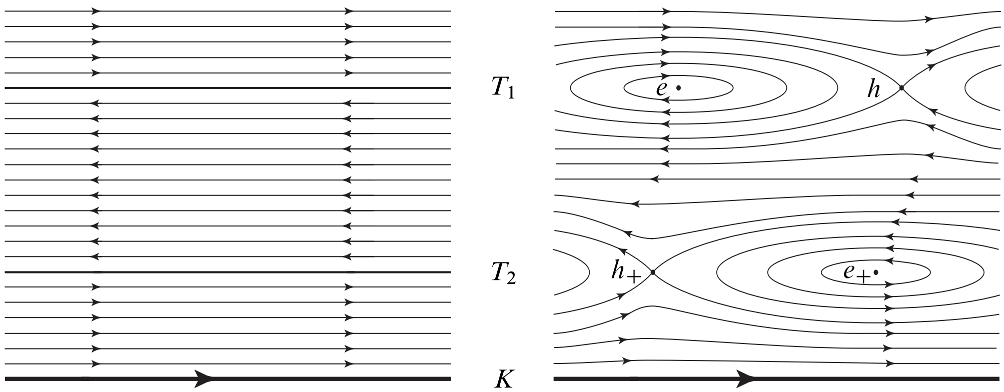

Figure 1: Reeb dynamic before and after a MB perturbation of the tori T_1 and T_2. Both pictures take place in a strip $\{t = t_0\} \subset \mathcal{N} \setminus K$. Each flow line represents an invariant subset of $\{t = t_0\}$ under the Reeb flow near K; the orientation gives the direction in which any point is mapped under the first return map of the flow.

The families of Reeb orbits in T_1 and T_2 can be perturbed into two pairs of Reeb orbits (e, h) and (e_+, h_+), where e and e_+ are elliptic with CZ index -1 and $+1$ respectively, and h and h_+ are positive hyperbolic, both with CZ index 0; see Figure 1.

Definition 1.18 A contact form α is adapted to a Seifert surface S for K if the R_α is positively transverse to $\mathrm{int}(S)$.

The proof of the following lemma is given in Sections 9.2 and 10.3 of [5]; compare also Section 7.2 of [10].

Lemma 1.19 [5] *Given a null-homologous knot K and a contact structure ξ on Y, there exists a contact form α for ξ and a genus minimizing Seifert surface S for K such that*

(1) *α is adapted to K;*
(2) *α is adapted to S.*

It is important to remark that the proof of (1) is obtained by locally modifying a given contact form near K using the Darboux–Weinstein neighborhood theorem; see for example [15]. Moreover, the (perturbed) contact form compatible with K obtained in [5] can be arranged to have all the orbits in $\mathcal{N} \setminus K$ that have arbitrarily large linking number with K, with the exception of the four relevant orbits e, h, e_+ and h_+.

Example 1.20 Let (K, S, ϕ) be an *open book decomposition* of Y, where S is the *page*, ϕ the *monodromy* and $K = \partial S$ the (not necessarily connected) *binding* of the open book. Let α be a *contact form adapted* to (K, S, ϕ) obtained via the Thurston–

Winkelnkemper construction [36]. Then α is compatible with S, and it can be easily adapted also to K; see for example [5, Section 9.3]. The strips $\{t = t_0\} \subset \mathcal{N} \setminus K$ can be obtained as intersections of the pages of the open book with $\mathcal{N} \setminus K$, and the flow depicted in Figure 1 is a dynamical representation of the restriction of ϕ to a strip.

1.3.2 The relative ECH With the notation above, if α is a contact form on Y which is compatible with K, in [5], the authors define relative versions $\mathrm{ECH}(N, \partial N, \alpha)$ and $\widehat{\mathrm{ECH}}(N, \partial N, \alpha)$ of embedded contact homology groups, and if α is also compatible with a Seifert surface S for K, they prove that

$$\mathrm{ECH}(N, \partial N, \alpha) \cong \mathrm{ECH}(Y, \alpha), \tag{5}$$

$$\widehat{\mathrm{ECH}}(N, \partial N, \alpha) \cong \widehat{\mathrm{ECH}}(Y, \alpha). \tag{6}$$

The notation suggests that these new homology groups are obtained by counting only orbits in N and quotienting by orbits on ∂N. Let us see the definition of these homologies in more detail.

In [5], the authors prove that it is possible to define the ECH–chain groups without taking into account the orbits in $\mathrm{int}(V)$ and in $T^2 \times (1, 2)$, so that the only interesting orbits in $\mathcal{N}(K)$ are the four orbits above (plus, obviously, the empty orbit set). Moreover, the only curves counted by the (restriction of the) ECH–differential ∂ have projection on Y as depicted in Figure 2. These curves give the relations

$$\partial(e) = 0, \quad \partial(h) = 0, \quad \partial(h_+) = e + \varnothing, \quad \partial(e_+) = h. \tag{7}$$

Note that the two holomorphic curves from h to e, as well as the two from e_+ to h_+, cancel each other since we work with coefficients in $\mathbb{Z}/2$.

Observation 1.21 The compactification of the projection of the holomorphic curve that limits to the empty orbit set is topologically a disk with boundary h_+, which should be seen as a cylinder closing on some point of K. This curve contribute to the "$\varnothing$ part" of the third equation of (7), which gives $[e] = [\varnothing]$ in ECH–homology. In the rest of this manuscript, the fact that this disk is the only ECH index-1 connected holomorphic curve that crosses K will be essential.

Convention From now on, we will use the following notation. If (Y, α) is understood, given a submanifold $X \subset Y$ and a set of Reeb orbits $\{\gamma_1, \dots, \gamma_n\} \subset \mathcal{P}(Y \setminus X)$, we will denote by $\mathrm{ECC}^{\gamma_1,\dots,\gamma_n}(X, \alpha)$ the free $\mathbb{Z}/2$–module generated by orbit sets in $\mathcal{O}(X \sqcup \{\gamma_1, \dots, \gamma_n\})$.

Unless stated otherwise, the group $\mathrm{ECC}^{\gamma_1,\dots,\gamma_n}(X, \alpha)$ will come with the natural restriction of the ECH–differential of $\mathrm{ECC}(Y, \alpha)$, still denoted by ∂^{ECH}; if this restriction is

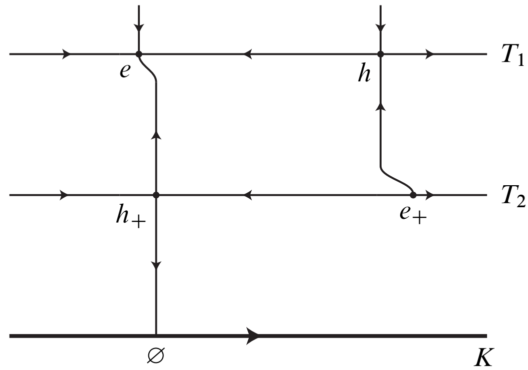

Figure 2: Orbits and holomorphic curves near K. Here the marked points denote the simple Reeb orbits, and the flow lines represent projections of the holomorphic curves counted by ∂^{ECH}. The two flow lines arriving from the top on e and h are depicted only to remember that, by the trapping lemma, holomorphic curves can only arrive at T_1.

still a differential, the associated homology is

$$\text{ECH}^{\gamma_1,\dots,\gamma_n}(X,\alpha) := H_*(\text{ECC}^{\gamma_1,\dots,\gamma_n}(X,\alpha), \partial^{\text{ECH}}).$$

This notation is not used in [5], where the authors introduced a specific notation for each relevant ECH–group. In particular, with their notation,

$$\begin{aligned}\text{ECC}^{\flat}(N,\alpha) &= \text{ECC}^{e}(\text{int}(N),\alpha),\\ \text{ECC}^{\sharp}(N,\alpha) &= \text{ECC}^{h}(\text{int}(N),\alpha),\\ \text{ECC}^{\natural}(N,\alpha) &= \text{ECC}^{h_+}(N,\alpha).\end{aligned}$$

As mentioned before, even if there are other Reeb orbits in $\mathcal{N}$, it is possible to define chain complexes for the ECH homology of (Y,α) only taking into account the orbits $\{e, h, e_+, h_+\}$.

The blocking and trapping lemmas and the relations above imply that the restriction of the full ECH–differential of Y to the chain group $\text{ECH}^{e_+,h_+}(N,\alpha)$ is given by

$$\partial(e_+^a h_+^b \gamma) = e_+^{a-1} h_+^b h\gamma + e_+^a h_+^{b-1}(1+e)\gamma + e_+^a h_+^b \partial\gamma, \tag{8}$$

where $\gamma \in \mathcal{O}(N)$ and where a term in the sum is meant to be zero if it contains some elliptic orbit with negative total multiplicity or a hyperbolic orbit with total multiplicity not in $\{0,1\}$; see [5, Section 9.5]. We remark that the blocking lemma implies also that $\partial\gamma \in \mathcal{O}(N)$.

The further restriction of the differential to $\mathrm{ECH}^{h_+}(N,\alpha)$ is then given by

$$\partial(h_+^b\gamma) = h_+^{b-1}(1+e)\gamma + h_+^b\partial\gamma. \tag{9}$$

Combining the computations of Sections 8 and 9 of [5] gives the following result.

Theorem 1.22 *Suppose that α is adapted to K and there exists a Seifert surface S for K such that α is adapted to S. Then*

$$\mathrm{ECH}(Y,\alpha) \cong \mathrm{ECH}^{e_+,h_+}(N,\alpha), \tag{10}$$

$$\widehat{\mathrm{ECH}}(Y,\alpha) \cong \mathrm{ECH}^{h_+}(N,\alpha). \tag{11}$$

Observation 1.23 It is important to remark that the empty orbit set is always taken into account as a generator of the groups above. This implies that if orbit sets with h_+ are considered, ∂^{ECH} counts also the holomorphic plane that contributes to the third of relations (7). Later we will give the definition of another differential, that we will call ∂^{ECK}, which is obtained from ∂^{ECH} by simply deleting that disk.

Define now the *relative embedded contact homology groups of* $(N,\partial N)$ by

$$\mathrm{ECH}(N,\partial N,\alpha) = \mathrm{ECH}^e(\mathrm{int}(N),\alpha)/\langle[e\gamma]\sim[\gamma]\rangle,$$
$$\widehat{\mathrm{ECH}}(N,\partial N,\alpha) = \mathrm{ECH}(N,\alpha)/\langle[e\gamma]\sim[\gamma]\rangle.$$

Since h_+ does not belong to the complexes $\mathrm{ECC}^e(\mathrm{int}(N),\alpha)$ and $\mathrm{ECC}(N,\alpha)$, the blocking lemma implies that the ECH–differentials count only holomorphic curves in N. This "lack" is balanced in the quotient by the equivalence relation

$$[e\gamma]\sim[\gamma]. \tag{12}$$

The reason behind this claim lies in the third of the relations (7). Indeed one can prove the following lemma; see Lemma 9.7.1 in [5].

Lemma 1.24 $\mathrm{ECH}^{e_+,h_+}(N,\alpha) \cong \mathrm{ECH}^{e_+}(N,\alpha)/\langle[e\gamma]\sim[\gamma]\rangle$.

Similarly, the fourth relation of (7) indicates why we can avoid considering h in the full $\mathrm{ECH}(Y,\alpha)$:

Lemma 1.25 [5, Lemma 9.9.1] $\mathrm{ECH}^{e_+}(N,\alpha) \cong \mathrm{ECH}^e(\mathrm{int}(N),\alpha)$.

Since $\partial(e\gamma) = e\partial(\gamma)$, the differential is compatible with the equivalence relation. So, instead of taking the quotient by $[e\gamma]\sim[\gamma]$ of the homology, we could take the homology of the quotient of the chain groups under the relation $e\gamma\sim\gamma$, and we would obtain the same homology groups. We will use this fact later. Note moreover that $[e^k] = [\varnothing]$ for every k. Equations (5) and (6) follow then from last two lemmas and Theorem 1.22.

1.4 ECH for knots

Let K be a homologically trivial knot in (Y, α). In this subsection, we recall the definition of a hat version of contact homology for the triple (K, Y, α). This was first defined by Colin, Ghiggini, Honda and Hutchings in [10, Section 7] as a particular case of *sutured* contact homology. On the other hand, following [5, Section 10], it is possible to proceed without dealing directly with sutures; we follow this approach here.

Let S be a Seifert surface for K. By standard arguments in homology, it is easy to compute that

$$H_1(Y \setminus K) \to H_1(Y) \times \mathbb{Z}, \quad [a] \mapsto \big(i_*[a], \langle a, S\rangle\big), \tag{13}$$

is an isomorphism. Here $i\colon Y \setminus K \to Y$ is the inclusion and $\langle a, S\rangle$ denotes the intersection number between a and S: this is a homological invariant of the pair (a, S) and is well-defined up to a slight perturbation of S (to make it transverse to a). Note that a preferred generator of $\mathbb{Z}$ is given by the homology class of a meridian for K, positively oriented with respect to the orientations of S and Y.

Example 1.26 If Y is a homology three-sphere, the number $\langle a, S\rangle$ depends only on a and K. This is the *linking number between a and K*, usually denoted by $\mathrm{lk}(a, K)$.

If $\gamma = \prod_i \gamma_i^{k_i}$ is a finite formal product of closed curves in $Y \setminus K$, then $\langle \gamma, S\rangle = \sum_i k_i \langle \gamma_i, S\rangle$.

Example 1.27 If (K, S, ϕ) is an open book decomposition of Y, α is an adapted contact form (in the sense of Thurston and Winkelnkemper) and $\gamma \in \mathcal{O}(Y \setminus K)$ is the orbit set $\prod_i \gamma_i^{k_i}$, then each γ_i is a periodic orbit of the diffeomorphism ϕ with *degree* d_i, and $\langle \gamma, S\rangle = \sum_i k_i d_i$.

Proposition 1.28 (see Proposition 7.1 in [10]) *Suppose that K is an orbit of R_α and let S be any Seifert surface for K. If γ and δ are two orbit sets in $Y \setminus K$ and $u\colon (F, j) \to (\mathbb{R} \times Y, J)$ is a holomorphic curve from γ to δ, then*

$$\langle \gamma, S\rangle \geq \langle \delta, S\rangle.$$

If α is a contact form adapted to K, a choice of (a homology class for) the Seifert surface S for K defines then a *knot filtration* on the chain complex $(\mathrm{ECC}^{h_+}(N, \alpha), \partial^{\mathrm{ECH}})$

for $\widehat{\mathrm{ECH}}(Y,\alpha)$, where N is the complement of a neighborhood $\mathcal{N}(K)$ of K in which the only "interesting" orbits and holomorphic curves are those represented in Figure 2.

Let $\mathrm{ECC}_d^{h+}(N,\alpha)$ be the free submodule of $\mathrm{ECC}^{h+}(N,\alpha)$ generated by orbit sets γ in $\mathcal{O}(N \sqcup \{h_+\})$ such that $\langle \gamma, S\rangle = d$. Define moreover

$$\mathrm{ECC}_{\le d}^{h+}(N,\alpha) := \bigoplus_{j\le d} \mathrm{ECC}_j^{h+}(N,\alpha).$$

Definition 1.29 The *knot filtration* induced by K is the exhaustive filtration of the module $\mathrm{ECC}^{h+}(N,\alpha)$ given by

$$\cdots \subseteq \mathrm{ECC}_{\le d-1}^{h+}(N,\alpha) \subseteq \mathrm{ECC}_{\le d}^{h+}(N,\alpha) \subseteq \mathrm{ECC}_{\le d+1}^{h+}(N,\alpha) \subseteq \cdots .$$

The *filtration degree* of a generator γ of $\mathrm{ECC}_d^{h+}(N,\alpha)$ is the integer d.

Corollary 1.30 *The* ECH*–differential respects the knot filtration.*

Proof The result follows by Proposition 1.28 applied to the MB buildings counted by ∂^{ECH}, which immediately implies that, for any $d \in \mathbb{Z}$,

$$\partial^{\mathrm{ECH}}(\mathrm{ECC}_{\le d}^{h+}(N,\alpha)) \subseteq \mathrm{ECC}_{\le d}^{h+}(N,\alpha). \qquad \square$$

If α is also adapted to S, in [5, Section 10.3], the authors prove that the filtration above induces a spectral sequence whose page ∞ is isomorphic to $\mathrm{ECH}^{h+}(N,\alpha) \cong \widehat{\mathrm{ECH}}(Y,\alpha)$ and whose page 0 is the chain complex

$$\bigoplus_d \big(\mathrm{ECC}_d^{h+}(N,\alpha), \partial_d^{\mathrm{ECK}}\big), \tag{14}$$

where $\mathrm{ECC}_d^{h+}(N,\alpha) \cong \mathrm{ECC}_{\le d}^{h+}(N,\alpha)/\mathrm{ECC}_{\le d-1}^{h+}(N,\alpha)$ and $\partial_d^{\mathrm{ECK}}$ is the map on $\mathrm{ECC}_d^{h+}(N,\alpha)$ induced by ∂^{ECH} on the quotient; ie it is the part of $\partial^{\mathrm{ECH}}|_{\mathrm{ECC}_d^{h+}(N,\alpha)}$ that strictly preserves the filtration degree.

Observation 1.31 The proof of Proposition 1.28 implies that the holomorphic curves counted by ∂^{ECH} that strictly decrease the degree are exactly the curves that intersect K. So we can interpret ∂^{ECK} as the restriction of ∂^{ECH} (given by (8)) to the count of curves that do not cross a thin neighborhood of K. This is indeed the proper ECH–differential of the manifold $Y \setminus \mathrm{int}(V(K))$ (and not the restriction of the ECH–differential of Y to the orbit sets in $Y \setminus \mathrm{int}(V(K))$).

Note that, by definition of $\mathrm{ECC}^{h+}(N,\alpha)$, all the holomorphic curves contained in $\mathbb{R}\times N$ strictly preserve the filtration degree. In fact the only holomorphic curve that contributes to $\partial^{\mathrm{ECH}}|_{\mathrm{ECC}^{h_+}(N,\alpha)}$ and decreases the degree (by 1) is the plane from h_+ to $\varnothing$. Then (9) gives

$$\partial(h_+^d\gamma)=h_+^{d-1}e\gamma+h_+^d\partial\gamma, \tag{15}$$

where $\gamma\in\mathcal{O}(N)$ and any term is meant to be zero if it contains some orbit with total multiplicity that is negative or, if the orbit is hyperbolic, not in $\{0,1\}$.

Definition 1.32 The *hat version of embedded contact (knot) homology* of the triple (K,Y,α) is

$$\widehat{\mathrm{ECK}}_*(K,Y,\alpha):=H_*(\mathrm{ECC}^{h_+}(N,\alpha),\partial^{\mathrm{ECK}}).$$

Observation 1.33 In order to define $\widehat{\mathrm{ECK}}(K,Y,\alpha)$, we supposed that α is compatible with S. This hypothesis is not present in the original definition (via sutures) in [10, Section 7.2]. Indeed, without this condition we can still apply all the arguments above and define the knot filtration on $\mathrm{ECC}^{h_+}(N,\alpha)$ exactly in the same way. Page 1 of the spectral sequence is again the well-defined homology in the definition above, and page ∞ is still isomorphic to $\mathrm{ECH}^{h_+}(N,\alpha)$. The only difference is that now we do not know that $\mathrm{ECH}^{h_+}(N,\alpha)\cong\widehat{\mathrm{ECH}}(Y,\alpha)$ since, in Theorem 1.22, the hypothesis that α is adapted to S is assumed.

This homology comes naturally with a further relative degree inherited by the filtered degree: if $\widehat{\mathrm{ECK}}_{*,d}(K,Y,\alpha):=H_*(\mathrm{ECC}_d^{h_+}(N,\alpha),\partial_d^{\mathrm{ECK}})$ then

$$\widehat{\mathrm{ECK}}_*(K,Y,\alpha)=\bigoplus_d\widehat{\mathrm{ECK}}_{*,d}(K,Y,\alpha).$$

Sometimes, in analogy with Heegaard Floer, we call this degree the *Alexander degree*.

Example 1.34 Suppose that (K,S,ϕ) is an open book decomposition of Y and that α is an adapted contact form. Since any nonempty Reeb orbit in $Y\setminus K$ has strictly positive intersection number with S,

$$\widehat{\mathrm{ECK}}_{*,0}(K,Y,\alpha)\cong\langle[\varnothing]\rangle_{\mathbb{Z}/2}.$$

This is the ECH–analogue of the fact that if K is fibered, then

$$\widehat{\mathrm{HFK}}_{*,-g}(K,Y)\cong\langle[c]\rangle_{\mathbb{Z}/2},$$

where g is the genus of K and c is the associated contact element; see Ozsváth and Szabó [30].

Observation 1.35 The Alexander degree can be considered as an absolute degree only once a relative homology class in $H_2(Y,K)$ for S has been fixed since the function $\langle\cdot,S\rangle$ defined on $H_1(Y\setminus K)$ changes if $[S]$ varies. On the other hand, if

$[\gamma]=[\delta]\in H_1(Y\setminus K)$ and $F\subset Y$ is a surface such that $\partial F=\gamma-\delta$, computations analogous to those in the proof of Proposition 1.28 imply that

$$\langle\gamma,S\rangle-\langle\delta,S\rangle=\langle F,K\rangle,$$

and the Alexander degree, considered as a relative degree, does not depend on the choice of a homology class for S. Obviously, if $H_2(Y)=0$, the Alexander degree can be lifted to an absolute degree.

In [10] the authors conjectured that their sutured embedded contact homology is isomorphic to sutured Heegaard Floer homology. For knot complements, their conjecture becomes the following:

Conjecture 1.36 [10] For a homologically trivial knot K in Y,

$$\widehat{\mathrm{ECK}}(K,Y,\alpha)\cong\widehat{\mathrm{HFK}}(-K,-Y),$$

where α is a contact form on Y adapted to K.

2 Generalizations of $\widehat{\mathrm{ECK}}$

Let K be a homologically trivial knot in a contact three-manifold (Y,α). As recalled in Section 1.4, if α is adapted to K, a choice of a Seifert surface S for K induces a filtration on the chain complex $(\mathrm{ECC}^{h+}(N,\alpha),\partial^{\mathrm{ECH}})$, where $\mathrm{int}(N)$ is homeomorphic to $Y\setminus K$. Moreover, if α is also adapted to S, the homology of $(\mathrm{ECC}^{h+}(N,\alpha),\partial^{\mathrm{ECH}})$ is isomorphic to $\widehat{\mathrm{ECH}}(Y,\alpha)$, and the first page of the spectral sequence associated to the filtration is the hat version of embedded contact knot homology $\widehat{\mathrm{ECK}}(K,Y,\alpha)$. In this section, we generalize the knot filtration in two natural ways.

In Section 2.1, we extend to the chain complex $(\mathrm{ECC}^{h+,e+}(N,\alpha),\partial^{\mathrm{ECH}})$ the filtration induced by K. This filtration is defined in a way analogous to the hat case. We define the *full version of embedded contact knot homology* of (K,Y,α) to be the first page $\mathrm{ECK}(K,Y,\alpha)$ of the associated spectral sequence.

In Section 2.2, we generalize the knot filtration to n–component links L. The resulting homologies, defined in a similar way to the case of knots, are the *full and hat versions of embedded contact knot homologies of* (L,Y,α), which will be still denoted by $\mathrm{ECK}(L,Y,\alpha)$ and, respectively, $\widehat{\mathrm{ECK}}(L,Y,\alpha)$. Similarly to Heegaard Floer link homology, these homologies come endowed with an *Alexander (relative)* $\mathbb{Z}^n$*–degree*.

2.1 The full ECK

Let K be a homologically trivial knot in a contact three-manifold (Y,α) and suppose that α is adapted to K in the sense of Section 1.3. Recall in particular that there exist two concentric neighborhoods $V(K) \subset \mathcal{N}(K)$ of K whose boundaries are MB tori $T_1 = \partial\mathcal{N}(K)$ and $T_2 = \partial V(K)$ foliated by orbits of R_α in the homology class of meridians for K. These two families of orbits are modified into the two couples of orbits $\{e, h\}$ and, respectively, $\{e_+, h_+\}$.

Consider the chain complex $(\mathrm{ECC}^{e_+,h_+}(N,\alpha), \partial^{\mathrm{ECH}})$ where we recall that $N = Y \setminus \mathrm{int}(\mathcal{N}(K))$, $\mathrm{ECC}^{e_+,h_+}(N,\alpha) = \langle \mathcal{O}(N \sqcup \{h_+, e_+\})\rangle_{\mathbb{Z}/2}$ and ∂^{ECH} is the ECH–differential (obtained by restricting the differential on $\mathrm{ECC}(Y,\alpha)$) given by (8).

A Seifert surface S for K induces an *Alexander degree* $\langle \cdot, S\rangle$ on the generators of $\mathrm{ECC}^{h_+,e_+}(N,\alpha)$ exactly as in the case of $\mathrm{ECC}^{h_+}(N,\alpha)$. Let $\mathrm{ECC}_d^{h_+,e_+}(N,\alpha)$ be the submodule of $\mathrm{ECC}^{h_+,e_+}(N,\alpha)$ generated by the $\gamma \in \mathcal{O}(N) \sqcup \{h_+, e_+\}$ with $\langle \gamma, S\rangle = d$. If

$$\mathrm{ECC}_{\le d}^{h_+,e_+}(N,\alpha) := \bigoplus_{j \le d} \mathrm{ECC}_j^{h_+,e_+}(N,\alpha),$$

we have the exhaustive filtration

$$\cdots \subseteq \mathrm{ECC}_{\le d-1}^{h_+,e_+}(N,\alpha) \subseteq \mathrm{ECC}_{\le d}^{h_+,e_+}(N,\alpha) \subseteq \mathrm{ECC}_{\le d+1}^{h_+,e_+}(N,\alpha) \subseteq \cdots$$

of $\mathrm{ECC}^{h_+,e_+}(N,\alpha)$. Proposition 1.28 again implies that ∂^{ECH} preserves the filtration. Let

$$\partial_d^{\mathrm{ECK}} \colon \mathrm{ECC}_d^{h_+,e_+}(N,\alpha) \to \mathrm{ECC}_d^{h_+,e_+}(N,\alpha)$$

be the part of ∂^{ECH} that strictly preserves the filtration degree d, that is, the differential induced by $\partial^{\mathrm{ECH}}|_{\mathrm{ECC}_{\le d}^{h_+,e_+}(N,\alpha)}$ on the quotient

$$\mathrm{ECC}_{\le d}^{h_+,e_+}(N,\alpha)/\mathrm{ECC}_{\le d-1}^{h_+,e_+}(N,\alpha) = \mathrm{ECC}_d^{h_+,e_+}(N,\alpha).$$

Set

$$\partial^{\mathrm{ECK}} := \bigoplus_d \partial_d^{\mathrm{ECK}} \colon \mathrm{ECC}^{e_+,h_+}(N,\alpha) \to \mathrm{ECC}^{e_+,h_+}(N,\alpha).$$

Definition 2.1 We define the *full embedded contact knot homology of* (K, Y, α) by

$$\mathrm{ECK}(K,Y,\alpha) := H_*(\mathrm{ECC}^{e_+,h_+}(N,\alpha), \partial^{\mathrm{ECK}}).$$

Note that, as in the hat case, the only holomorphic curves counted by ∂^{ECH} that do not strictly respect the filtration degree are the curves that contain the plane from h_+ to $\varnothing$; see Observation 1.31. Recalling the expression of ∂^{ECH} given in (8), it follows that ∂^{ECK} is given by

$$\partial^{\mathrm{ECK}}(e_+^a h_+^b \gamma) = e_+^{a-1} h_+^b h\gamma + e_+^a h_+^{b-1} e\gamma + e_+^a h_+^b \partial\gamma, \tag{16}$$

where $\gamma \in \mathcal{O}(N)$ and any term is meant to be 0 if it contains an orbit with total multiplicity that is negative or, if the orbit is hyperbolic, not in $\{0, 1\}$.

Again the homology comes with an *Alexander degree*, which is well-defined once the homology class for S is fixed, and induces the natural splitting

$$\mathrm{ECK}_*(K,Y,\alpha) \cong \bigoplus_{d\in\mathbb{Z}} \mathrm{ECK}_{*,d}(K,Y,\alpha), \tag{17}$$

where

$$\mathrm{ECK}_{*,d}(K,Y,\alpha) := H_*(\mathrm{ECC}_d^{h+,e+}(N,\alpha), \partial_d^{\mathrm{ECK}}).$$

Lemma 2.2 *If $\mathcal{N}(K)$ is a neighborhood of K as above, then*

$$\mathrm{ECK}(K,Y,\alpha) \cong \mathrm{ECH}(Y \setminus \mathcal{N}(K),\alpha).$$

Proof Reasoning as in Lemma 1.24, it is easy to prove that

$$\begin{aligned}\mathrm{ECK}(K,Y,\alpha) &\cong H_*(\mathrm{ECC}^{e+,h+}(N,\alpha), \partial^{\mathrm{ECK}})\\ &\cong H_*(\mathrm{ECC}^{e,h+}(\mathrm{int}(N),\alpha), \partial^{\mathrm{ECK}})\\ &\cong H_*(\mathrm{ECC}(\mathrm{int}(N),\alpha), \partial^{\mathrm{ECK}})\\ &\cong \mathrm{ECH}(\mathrm{int}(N),\alpha),\end{aligned}$$

which follows from the fact that $\partial^{\mathrm{ECK}}(\gamma) = \partial^{\mathrm{ECH}}(\gamma)$ for all $\gamma \in \mathcal{O}(N)$. □

Observation 2.3 Note that so far we only assumed that α is compatible with K, while we did not assume the condition

(♠) α is compatible with a Seifert surface S for K.

As remarked in Observation 1.33, we cannot prove Theorem 1.22 without (♠), and so we do not know if the spectral sequence whose 0–page is the ECK–chain complex limits to $\mathrm{ECH}(Y,\alpha)$. On the other hand, this spectral sequence is in any case well-defined, and so is $\mathrm{ECK}(K,Y,\alpha)$. Even if, in light of Lemma 1.19, we could assume (♠) here without restrictions on K, we prefer to avoid it in the general definition of $\mathrm{ECK}(K,Y,\alpha)$ in order to consider a wider class of contact forms.

In analogy with Conjecture 1.36 we state:

Conjecture 2.4 For any knot K in Y and any contact form α on Y adapted to K,

$$\mathrm{ECK}(K,Y,\alpha) \cong \mathrm{HFK}^+(-K,-Y).$$

2.2 The generalization to links

In this subsection, we extend the definitions of ECK and $\widehat{\mathrm{ECK}}$ to the case of homologically trivial links with more than one component. A (*strongly*) *homologically trivial*

n–link in Y is a disjoint union of n knots, each of which is homologically trivial in Y. Suppose that

$$L = K_1 \sqcup \cdots \sqcup K_n$$

is a homologically trivial n–link in Y. We say that a contact form α on Y is *adapted to* L if it is adapted to K_i for each i.

Lemma 2.5 *For any link L and contact structure ξ on Y there exists a contact form compatible with ξ which is adapted to L.*

Proof The proof of part (1) of Lemma 1.19 is local near the knot K and can then be applied recursively to each K_i. □

Fix $L = K_1 \sqcup \cdots \sqcup K_n$ homologically trivial and α an adapted contact form. Since α is adapted to each K_i, there exist pairwise disjoint tubular neighborhoods

$$V(K_i) \subset \mathcal{N}(K_i)$$

of K_i where α behaves exactly like in the neighborhoods $V(K) \subset \mathcal{N}(K)$ in Section 1.3. In particular, for each i, the tori $T_{i,1} := \partial\mathcal{N}(K_i)$ and $T_{i,2} := \partial V(K_i)$ are MB and foliated by families of orbits of R_α in the homology class of a meridian of K_i. We will consider these two families as perturbed into two pairs $\{e_i, h_i\}$ and $\{e_i^+, h_i^+\}$ in the usual way. Let

$$V(L) := \bigsqcup_i V(K_i) \quad \text{and} \quad \mathcal{N}(L) := \bigsqcup_i \mathcal{N}(K_i),$$

and set

$$N := Y \setminus \operatorname{int}(\mathcal{N}(L)).$$

Define moreover $\overline{e} := \bigsqcup_i e_i$, and let $\overline{h}$, $\overline{e}_+$ and $\overline{h}_+$ be similarly defined.

Consider $\mathrm{ECC}^{\overline{e}_+,\overline{h}_+}(N,\alpha)$ endowed with the restriction ∂^{ECH} of the ECH differential of (Y,α), and let $\mathrm{ECH}^{\overline{e}_+,\overline{h}_+}(N,\alpha)$ be the associated homology.

Lemma 2.6 $\mathrm{ECH}^{\overline{e}_+,\overline{h}_+}(N,\alpha)$ *is well-defined and the curves counted by ∂^{ECH} inside each $\mathcal{N}(K_i)$ are given by analogous expressions to those in* (7).

Proof The blocking and trapping lemmas can be applied locally near each component of ∂N and the proofs of Lemmas 7.1.1 and 7.1.2 in [5] work immediately in this context too. This imply that the homology of $(\mathrm{ECC}(N,\alpha), \partial^{\mathrm{ECH}})$ is well-defined.

Again the blocking and trapping lemmas, together with the local homological arguments in Lemmas 9.5.1 and 9.5.3 in [5], imply that the only holomorphic curves counted by ∂^{ECH} inside each $\mathcal{N}(K_i)$ are as required (see Figure 2), and so $\mathrm{ECH}^{\overline{e}_+,\overline{h}_+}(N,\alpha)$ is well-defined. □

An explicit formula for ∂^{ECH} can be obtained by generalizing (8) in the obvious way.

For each $i \in \{1, \dots, n\}$, fix a (homology class for a) Seifert surface S_i for K_i. These surfaces are not necessarily pairwise disjoint, and it is even possible that $S_i \cap K_j \neq \varnothing$ for some $i \neq j$. Consider then the *Alexander* $\mathbb{Z}^n$*–degree* on $\mathrm{ECH}^{\overline{e}_+,\overline{h}_+}(N,\alpha)$ given by the function

$$\mathrm{ECC}^{\overline{e}_+,\overline{h}_+}(N,\alpha) \to \mathbb{Z}^n, \quad \gamma \mapsto (\langle \gamma, S_1\rangle, \dots, \langle \gamma, S_n\rangle). \tag{18}$$

Define a partial ordering on $\mathbb{Z}^n$ by

$$(a_1, \dots, a_n) \le (b_1, \dots, b_n) \qquad \Longleftrightarrow \qquad a_i \le b_i \quad \text{for all } i.$$

Proposition 1.28 applied to each K_i implies that if γ and δ are two orbit sets in $\mathcal{O}(N \sqcup \{\overline{e}_+, \overline{h}_+\})$, then for any k,

$$\mathcal{M}_k(\gamma,\delta)/\mathbb{R} \neq 0 \implies (\langle \delta, S_1\rangle, \dots, \langle \delta, S_n\rangle) \le (\langle \gamma, S_1\rangle, \dots, \langle \gamma, S_n\rangle).$$

This implies that ∂^{ECH} does not increase the Alexander degree, which induces a $\mathbb{Z}^n$–filtration on $(\mathrm{ECC}^{\overline{e}_+,\overline{h}_+}(N,\alpha), \partial^{\mathrm{ECH}})$. Like in the previous subsection, we are interested in the part of ∂^{ECH} that strictly respects the filtration degree. This can be defined again in terms of quotients as follows.

Let $d \in \mathbb{Z}^n$ and let $\mathrm{ECC}_d^{\overline{e}_+,\overline{h}_+}(N,\alpha)$ be the submodule of $\mathrm{ECC}^{\overline{e}_+,\overline{h}_+}(N,\alpha)$ freely generated by orbit sets $\gamma \in \mathcal{O}(N \sqcup \{\overline{e}_+, \overline{h}_+\})$ such that

$$(\langle \gamma, S_1\rangle, \dots, \langle \gamma, S_n\rangle) = d.$$

Define

$$\mathrm{ECC}_{\le d}^{\overline{e}_+,\overline{h}_+}(N,\alpha) := \bigoplus_{j \le d} \mathrm{ECC}_d^{\overline{e}_+,\overline{h}_+}(N,\alpha),$$

and let $\mathrm{ECC}_{<d}^{\overline{e}_+,\overline{h}_+}(N,\alpha)$ be similarly defined.

Define the *full* ECK*–differential in degree* d to be the map

$$\partial_d^{\mathrm{ECK}} \colon \mathrm{ECC}_d^{\overline{e}_+,\overline{h}_+}(N,\alpha) \to \mathrm{ECC}_d^{\overline{e}_+,\overline{h}_+}(N,\alpha)$$

induced by $\partial^{\mathrm{ECH}}|_{\mathrm{ECC}_{\le d}^{\overline{e}_+,\overline{h}_+}(N,\alpha)}$ on the quotient

$$\mathrm{ECC}_{\le d}^{\overline{e}_+,\overline{h}_+}(N,\alpha)/\mathrm{ECC}_{<d}^{\overline{e}_+,\overline{h}_+}(N,\alpha) \cong \mathrm{ECC}_d^{\overline{e}_+,\overline{h}_+}(N,\alpha).$$

Define then the *full* ECK*–differential* by

$$\partial^{\mathrm{ECK}} := \bigoplus_d \partial_d^{\mathrm{ECK}} \colon \mathrm{ECC}^{\overline{e}_+,\overline{h}_+}(N,\alpha) \to \mathrm{ECC}^{\overline{e}_+,\overline{h}_+}(N,\alpha).$$

Observation 2.7 Observing the form of ∂^{ECH}, it is easy again to see that the only holomorphic curves that are counted by ∂^{ECH} and not by ∂^{ECK} are the ones containing a holomorphic plane from some h_i^+ to $\varnothing$.

Definition 2.8 The *full embedded contact knot homology of* (L, Y, α) is

$$\mathrm{ECK}(L, Y, \alpha) := H_*(\mathrm{ECC}^{\bar{e}_+,\bar{h}_+}(N,\alpha), \partial^{\mathrm{ECK}}).$$

The fact that $\mathrm{ECK}(L, Y, \alpha)$ is well-defined is a direct consequence of the good definition of $\mathrm{ECH}^{\bar{e}_+,\bar{h}_+}(N,\alpha)$ and the fact that ∂^{ECH} respects the Alexander filtration. Note that we have again a natural splitting

$$\mathrm{ECK}_*(L, Y, \alpha) = \bigoplus_{d\in\mathbb{Z}^n} \mathrm{ECK}_{*,d}(L, Y, \alpha), \tag{19}$$

where

$$\mathrm{ECK}_{*,d}(L, Y, \alpha) = H_*(\mathrm{ECC}_d^{\bar{e}_+,\bar{h}_+}(N,\alpha), \partial_d^{\mathrm{ECK}}).$$

The proof of the following lemma is the same of that of the analogous Lemma 2.2 for knots applied to each component of L.

Lemma 2.9 *If $\mathcal{N}(L)$ is a neighborhood of L as above, then*

$$\mathrm{ECK}(L, Y, \alpha) \cong \mathrm{ECH}(Y \setminus \mathcal{N}(L), \alpha).$$

Consider now the submodule $\mathrm{ECC}^{\bar{h}_+}(N,\alpha)$ of $\mathrm{ECC}^{\bar{e}_+,\bar{h}_+}(N,\alpha)$ endowed with the restriction of ∂^{ECH}. Observe that its homology $\mathrm{ECH}^{\bar{h}_+}(N,\alpha)$ is well-defined. Filtering $(\mathrm{ECC}^{\bar{h}_+}(N,\alpha), \partial^{\mathrm{ECH}})$ by the Alexander degree, for any $d \in \mathbb{Z}^n$, we can define $\mathrm{ECC}_d^{\bar{h}_+}(N,\alpha)$ with differential

$$\partial_d^{\mathrm{ECK}}\colon \mathrm{ECC}_d^{\bar{h}_+}(N,\alpha) \to \mathrm{ECC}_d^{\bar{h}_+}(N,\alpha).$$

Definition 2.10 The *hat version of embedded contact knot homology of* (L, Y, α) is

$$\widehat{\mathrm{ECK}}(L, Y, \alpha) := H_*(\mathrm{ECC}^{\bar{h}_+}(N,\alpha), \partial^{\mathrm{ECK}}).$$

Observation 2.7 and a splitting like the one in (19) hold also for $\widehat{\mathrm{ECK}}(L, Y, \alpha)$. Moreover, it is easy to see that if L has only one connected component, we get the same theories of Sections 1.4 and 2.1.

Conjecture 2.11 If L is a link in Y and α is any contact form on Y adapted to L, then

$$\mathrm{ECK}(L, Y, \alpha) \cong \mathrm{HFK}^+(-L, -Y),$$
$$\widehat{\mathrm{ECK}}(L, Y, \alpha) \cong \widehat{\mathrm{HFK}}(L, Y).$$

Convention In order to simplify the notation in the rest of the paper, we will indicate the ECH chain groups for the knot embedded contact homology groups of links and knots by

$$\mathrm{ECC}(L, Y, \alpha) := \mathrm{ECC}^{\bar{e}_+,\bar{h}_+}(N,\alpha),$$
$$\widehat{\mathrm{ECC}}(L, Y, \alpha) := \mathrm{ECC}^{\bar{h}_+}(N,\alpha).$$

These groups will implicitly come endowed with the differential ∂^{ECK}.

We end this section by saying some words about a further generalization of ECK to *weakly homologically trivial* links. We say that $L \subset Y$ is a weakly homologically trivial (or simply *weakly trivial*) n–component link if there exist surfaces with boundary $S_1, \ldots, S_m \subset Y$ with $m \leq n$ and such that $\partial S_i \cap \partial S_j = \varnothing$ if $i \neq j$ and $\bigsqcup_{i=1}^m \partial S_i = L$. Also, here we do not require that S_i or even ∂S_i is disjoint from S_j for $j \neq i$. Clearly, L is a strongly trivial link if and only if it is weakly trivial with $m = n$.

If L is a weakly trivial link with $m \lneq n$, we cannot in general define a homology with a filtered n–degree. In fact, there exists $S \in \{S_1, \ldots, S_m\}$ such that ∂S has more than one connected component. Suppose for instance that $\partial S = K_1 \sqcup K_2$. The arguments behind Proposition 1.28 imply that if $u\colon (F, j) \to (\mathbb{R} \times Y, J)$ is a holomorphic curve from γ to δ, then

$$\langle \gamma, S\rangle - \langle \delta, S\rangle = \langle \mathrm{Im}(u), \mathbb{R} \times (K_1 \sqcup K_2)\rangle \geq 0.$$

So in this case, we can still apply the arguments above and get well-defined ECH invariants for L. However, this time they will come only with a filtered (relative) $\mathbb{Z}^m$–degree on the generators γ of an ECH complex of Y, which is given by the m–tuple $(\langle \gamma, S_1\rangle, \ldots, \langle \gamma, S_m\rangle)$.

Example 2.12 Let (L, S, ϕ) be an open book decomposition of Y with possibly disconnected boundary. Using a (connected) page of (L, S, ϕ) to compute the Alexander degree, the generators of the chain complex for $\mathrm{ECK}_d(L, Y, \alpha)$ are d–periodic orbits of the diffeomorphism ϕ for any $d \in \mathbb{Z}$.

3 Euler characteristics

In this section, we compute the graded Euler characteristics of the embedded contact homology groups for knots and links in homology three-spheres Y with respect to suitable contact forms. The computations will be done in terms of the Lefschetz zeta function of the flow of the Reeb vector field.

Before proceeding, we briefly recall what the graded Euler characteristic is. Given a collection of chain complexes

$$(C, \partial) = \{(C_{*,(i_1,\ldots,i_n)}, \partial_{(i_1,\ldots,i_n)})\}_{(i_1,\ldots,i_n)\in\mathbb{Z}^n},$$

where $*$ denotes a relative homological degree, its *graded Euler characteristic* is

$$\chi(C) = \sum_{i_1,\ldots,i_n} \chi(C_{*,(i_1,\ldots,i_n)})\, t_1^{i_1} \cdots t_n^{i_n} \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}],$$

where $\chi(C_{*,(i_1,\dots,i_n)})$ is the standard Euler characteristic of $C_{*,(i_1,\dots,i_n)}$, and the t_j are formal variables. By definition, $\chi(C)$ is a Laurent polynomial, and the properties of the standard Euler characteristic imply

$$\chi(C) = \chi(H(C,\partial)).$$

In this case, the homology $H(C,\partial)$ is a *categorification* of $\chi(C)$.

The most important result of this section relates the Euler characteristic of ECK homologies of a link in S^3 with its multivariable Alexander polynomial.

Theorem 3.1 *Let L be any n–link in S^3. Then there exists a contact form α adapted to L such that*

$$\chi(\mathrm{ECK}(L,S^3,\alpha)) \doteq \begin{cases} \Delta_L(t_1,\dots,t_n) & \text{if } n>1, \\ \Delta_L(t)/(1-t) & \text{if } n=1, \end{cases} \tag{20}$$

and

$$\chi(\widehat{\mathrm{ECK}}(L,S^3,\alpha)) \doteq \begin{cases} \Delta_L\cdot\prod_{i=1}^n(1-t_i) & \text{if } n>1, \\ \Delta_L(t) & \text{if } n=1. \end{cases} \tag{21}$$

This theorem implies that ECK *categorifies the Alexander polynomial of knots and links in S^3*. This is the third known categorification of this kind, after the ones obtained in Heegaard Floer homology by Ozsváth and Szabó [28; 31] and Rasmussen [33] and in Seiberg–Witten–Floer homology by Kronheimer and Mrowka [24; 23].

An immediate consequence of Theorem 3.1 and Equations (1) and (2) is:

Corollary 3.2 *For any link L in S^3, there exists a contact form α such that*

$$\chi(\mathrm{ECK}(L,S^3,\alpha)) \doteq \chi(\mathrm{HFL}^+(-L,-S^3)),$$
$$\chi(\widehat{\mathrm{ECK}}(L,S^3,\alpha)) \doteq \chi(\widehat{\mathrm{HFL}}(-L,-S^3)).$$

This corollary implies that Conjecture 2.11 (which generalizes Conjectures 1.36 and 2.4) holds for links in S^3 at least at the level of Euler characteristic.

A key ingredient to prove Theorem 3.1 is the dynamical formulation of the Alexander quotient given by Fried [14].

3.1 A dynamical formulation of the Alexander polynomial

Given any link $L = K_1\sqcup\cdots\sqcup K_n$ in S^3, we can associate to it its *multivariable Alexander polynomial*

$$\Delta_L(t_1,\dots,t_n)\in\mathbb{Z}[t_1^{\pm1},\dots,t_n^{\pm1}]/\langle\pm t_1^{a_1}\cdots t_n^{a_n}\rangle$$

with $a_i \in \mathbb{Z}$. The quotient means that the Alexander polynomial is well-defined only up to multiplication by monomials of the form $\pm t_1^{a_1} \cdots t_n^{a_n}$.

A slightly simplified version is the *(classical) Alexander polynomial* $\Delta_L(t)$ defined by setting $t_1 = \cdots = t_n = t$, ie

$$\Delta_L(t) := \Delta_L(t, \dots, t).$$

If L is a knot, the two notions obviously coincide.

There are many possible definitions of the Alexander polynomial Δ_L. In this section, we give a formulation of Δ_L in terms of the dynamics of suitable vector fields in $S^3 \setminus L$.

The fact that the Alexander polynomial is related to dynamical properties of its complement in S^3 originates with the study of fibrations of S^3. For example, A'Campo [1] studied the *twisted Lefschetz zeta function* of the monodromy of an open book decomposition (S, ϕ) of S^3 associated to a Milnor fibration of a complex algebraic singularity. More generally, if (K, S, ϕ) is any open book decomposition of S^3, one can easily prove that

$$\Delta_K(t) \doteq \det(\mathbb{1} - t\phi_*^1),$$

where $\mathbb{1}$ and ϕ_*^1 are the identity map and, respectively, the application induced by ϕ, on $H_1(S, \mathbb{Z})$. The basic idea in this context is to express the right-hand side of the above equation in terms of traces of iterations of ϕ_*^1, then to apply the Lefschetz fixed point theorem to get expressions in terms of periodic points (ie periodic orbits) for the flow of some vector field in $S^3 \setminus K$ whose first return on a page is ϕ.

Suppose now that L is not a fibered link, so its complement is not globally fibered over S^1, and let R be a vector field in $S^3 \setminus L$. If one wants to apply arguments as above, it is necessary to decompose $S^3 \setminus L$ in "fibered-like" pieces with respect to R, in which it is possible to define at least a local first return map of the flow ϕ_R of R. Obviously, some condition on R is required. For example, Franks [13] considers *Smale vector fields*, ie vector fields with one-dimensional and hyperbolic chain recurrent set; see [35].

Here we are more interested in the approach used by Fried [14]. Consider a three-dimensional manifold X. Any *abelian cover* $\widetilde{X} \xrightarrow{\pi} X$ with deck-transformation group isomorphic to a fixed abelian group G is uniquely determined by the choice of a class $\rho = \rho(\pi) \in H^1(X, G) \cong \mathrm{Hom}(H_1(X, \mathbb{Z}), G)$. Here ρ is determined by the following property: for any $[\gamma] \in H_1(X)$, if $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ is any lifting of the loop $\gamma\colon [0,1] \to X$, then $\rho([\gamma])$ is determined by $\rho([\gamma])(\widetilde{\gamma}(0)) = \widetilde{\gamma}(1)$.

Since the correspondence between abelian covers and cohomology classes is bijective, with abuse of notation sometimes we will refer to an abelian cover directly by identifying it with the corresponding ρ.

Example 3.3 The *universal abelian cover* of X is the abelian cover with deck-transformation group $G = H_1(X, \mathbb{Z})$ and corresponding to $\rho = \mathrm{id}$.

Example 3.4 Let $L = K_1 \sqcup \cdots \sqcup K_n$ be an n–component link in a three-manifold Y such that K_i is homologically trivial for any i, and fix a Seifert surface S_i for K_i. Let moreover μ_i be a positive meridian for K_i. If $i\colon Y \setminus L \hookrightarrow Y$ is the inclusion, the isomorphism

$$H_1(Y \setminus L) \to H_1(Y) \oplus \mathbb{Z}_{[\mu_1]} \oplus \cdots \oplus \mathbb{Z}_{[\mu_n]}, \quad [\gamma] \mapsto (i_*([\gamma]), \langle \gamma, S_1 \rangle, \dots, \langle \gamma, S_n \rangle),$$

gives rise naturally to the abelian cover

$$\rho_L \in \mathrm{Hom}(H_1(Y \backslash L, \mathbb{Z}),\ \mathbb{Z}^n)$$

of $Y \setminus L$ defined by

$$\rho_L([\gamma]) = (\langle \gamma, S_1 \rangle, \dots, \langle \gamma, S_n \rangle).$$

Setting $t_i = [\mu_i] \in H_1(Y \setminus L, \mathbb{Z})$, we can regard $\rho_L([\gamma])$ as a monomial in the t_i:

$$\rho_L([\gamma]) = t_1^{\langle \gamma, S_1 \rangle} \cdots t_n^{\langle \gamma, S_n \rangle}.$$

In the rest of the paper, we will often use this notation. Note finally that if Y is a homology three-sphere, ρ_L coincides with the universal abelian cover of $Y \setminus L$.

If R is a vector field on X satisfying some compatibility condition with ρ (and with ∂X if this is nonempty), Fried relates the Reidemeister–Franz torsion of $(X, \partial X)$ with the *twisted Lefschetz zeta function* of the flow ϕ_R.

3.1.1 Twisted Lefschetz zeta function of flows Let R be a vector field on X and γ a closed isolated orbit of ϕ_R. Pick any point $x \in \gamma$ and let D be a small disk transverse to γ such that $D \cap \gamma = \{x\}$. With this data it is possible to define the Lefschetz sign of γ exactly like we did in Section 1.1.1 for orbits of Reeb vector fields associated to a contact structure ξ, but using now $T_x D$ instead of ξ_x. Indeed, it is possible to prove that the Lefschetz sign of γ does not depend on the choice of x and D, and that it is an invariant $\epsilon(\gamma) \in \{-1, 1\}$ of ϕ_R near γ.

Definition 3.5 The *local Lefschetz zeta function* of ϕ_R near γ is the formal power series $\zeta_\gamma(t) \in \mathbb{Z}[\![t]\!]$ defined by

$$\zeta_\gamma(t) := \exp\biggl(\sum_{i \geq 1} \epsilon(\gamma^i) \frac{t^i}{i}\biggr).$$

Let now $\widetilde{X} \xrightarrow{\pi} X$ be an abelian cover with deck-transformation group G, and let $\rho = \rho(\pi) \in H^1(X, G)$. Suppose that all the periodic orbits of ϕ_R are isolated.

Definition 3.6 We define the *ρ–twisted Lefschetz zeta function of ϕ_R* by

$$\zeta_\rho(\phi_R) := \prod_\gamma \zeta_\gamma(\rho([\gamma])),$$

where the product is taken over the set of simple periodic orbits of ϕ_R.

When ρ is understood, we will write directly $\zeta(\phi_R)$ and we will call it the twisted Lefschetz zeta function of ϕ_R.

We remark that in [14], the author defines $\zeta_\rho(\phi_R)$ in a slightly different way, and then he proves in Theorem 2 that the two definitions coincide.

Convention Suppose that $\rho \in H^1(X, \mathbb{Z}^n)$ is an abelian cover of X and chose a generator $(t_1, \dots, t_n)$ of $\mathbb{Z}^n$. Then, with a similar notation to that of Example 3.4, we will often identify $\zeta_\rho(\phi_R)$ with an element of $\mathbb{Z}[\![t_1^{\pm 1}, \dots, t_n^{\pm 1}]\!]$.

3.1.2 Torsion and flows Fried [14] relates the Reidemeister torsion of an abelian cover ρ of a (not necessarily closed) three-manifold X with the twisted Lefschetz zeta function of certain flows. In particular, in Section 5, he considers a kind of torsion that he calls the *Alexander quotient* and denotes it by $\mathrm{ALEX}_\rho(X)$: the reason for the "quotient" comes from the fact that Fried uses a definition of Reidemeister torsion only up to the choice of a sign (this is the "refined Reidemeister torsion" of Turaev [38]), while $\mathrm{ALEX}_\rho(X)$ is defined up to an element in the abelian group of deck transformations of ρ.

In fact, one can check that $\mathrm{ALEX}_\rho(X)$ is exactly the Reidemeister–Franz torsion τ considered by Ozsváth and Szabó [31]. In particular, when X is the complement of an n–component link L in S^3 and ρ is the universal abelian cover of X, then

$$\mathrm{ALEX}(S^3 \setminus L) \doteq \begin{cases} \Delta_L(t_1, \dots, t_n) & \text{if } n > 1, \\ \Delta_L(t)/(1-t) & \text{if } n = 1, \end{cases} \tag{22}$$

where we removed $\rho = \mathrm{id}_{H_1(S^3 \setminus L, \mathbb{Z})}$ from the notation; see [14, Section 8] and [38].

Since the notation "τ" is ambiguous, we follow Fried [14] and we refer to the Reidemeister–Franz torsion as the Alexander quotient, indicated by $\mathrm{ALEX}_\rho(X)$.

In order to relate $\mathrm{ALEX}_\rho(X)$ to the twisted Lefschetz zeta function of the flow ϕ_R of a vector field R, Fried assumes some hypotheses on R. The first condition that R must satisfy is *circularity*.

Definition 3.7 A vector field R on X is *circular* if there exists a C^1 map $\theta\colon X \to S^1$ such that $d\theta(R) > 0$.

If $\partial X = \varnothing$, this is equivalent to say that R admits a global cross section. Intuitively, the circularity condition on R allows us to define a kind of first return map of ϕ_R. Suppose that R circular, and consider $S^1 \cong \mathbb{R}/\mathbb{Z}$ with $\mathbb{R}$–coordinate t. The cohomology class

$$u_\theta := \theta^*([dt]) \in H^1(X, \mathbb{Z})$$

is then well-defined.

Definition 3.8 Given an abelian cover $\widetilde{X} \xrightarrow{\pi} X$ with deck-transformation group G, let $\rho = \rho(\pi) \in H^1(X, G)$ be the corresponding cohomology class. A circular vector field R on X is *compatible* with ρ if there exists a homomorphism $v\colon G \to \mathbb{R}$ such that $v \circ \rho = u_\theta$, where θ and u_θ are as above.

Example 3.9 The universal abelian cover corresponds to $\rho = \mathrm{id}\colon H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$, and it is compatible with any circular vector field on X.

The following theorem is not the most general result in [14], but it will be enough for our purposes.

Theorem 3.10 [14, Theorem 7] *Let X be a three-manifold and $\rho \in H^1(X, G)$ an abelian cover. Let R be a nonsingular, circular and nondegenerate vector field on X compatible with ρ. Suppose moreover that, if $\partial X \neq \varnothing$, then R is transverse to ∂X and pointing out of X. Then*

$$\mathrm{ALEX}_\rho(X) \doteq \zeta_\rho(\phi_R),$$

where the equivalence $\doteq$ is up to multiplication by $\pm g$ for any $g \in G$.

An immediate consequence is the following:

Corollary 3.11 *If L is an n–component link in S^3, let $\mathcal{N}(L)$ be a tubular neighborhood of L, and let $N = S^3 \setminus \mathcal{N}(L)$. Let R be a nonsingular circular vector field on N, transverse to ∂N and pointing out of N. Then*

$$\zeta(\phi_R) \doteq \begin{cases} \Delta_L(t_1, \dots, t_n) & \text{if } n > 1, \\ \Delta_L(t)/(1-t) & \text{if } n = 1. \end{cases} \tag{23}$$

3.2 Results

In the next subsections, we prove Theorem 3.1, which will be obtained as a consequence of the following more general result. Recall that an n–link $L \subset Y$ determines the abelian cover $\rho_L \in H^1(Y \setminus L, \mathbb{Z}^n)$ of $Y \setminus L$ given in Example 3.4. When Y is a homology three-sphere, we have

$$\rho_L \equiv \mathbb{1}\colon H_1(Y \setminus L) \to H_1(Y \setminus L) \cong \mathbb{Z}^n.$$

In order to simplify the notations, we remove ρ_L from the notation of the Alexander quotient and of the twisted Lefschetz zeta function:

$$\mathrm{ALEX}(Y \setminus L) := \mathrm{ALEX}_{\mathbb{1}}(Y \setminus L), \quad \zeta(\phi) := \zeta_{\mathbb{1}}(\phi).$$

Let $(t_1, \dots, t_n)$ be a basis for $H_1(Y \setminus L)$, where $[\mu_i] = t_i$ for μ_i a positively oriented meridian of K_i.

Theorem 3.12 *Let L be an n–link in a homology three-sphere Y. Then there exists a contact form α such that*

$$\chi(\mathrm{ECK}(L, Y, \alpha)) \doteq \mathrm{ALEX}(Y \setminus L).$$

The proofs of Theorems 3.1 and 3.12 will be carried out in two main steps: in Section 3.3, we will prove the theorems in the case of fibered links, while the general case will be treated in Section 3.4.

3.3 Fibered links

In this subsection, we prove Theorems 3.1 and 3.12 for fibered links. Let (L, S, ϕ) be an open book decomposition of a homology three-sphere Y, and let α be an adapted contact form on Y. In particular, with our definition, α is also adapted to L.

In order to prove the theorems above, we want to express the Euler characteristic $\chi(\mathrm{ECK}(L, Y, \alpha))$ in terms of the twisted Lefschetz zeta function of the Reeb flow ϕ_R of $R = R_\alpha$ and then apply Theorem 3.10. The first thing that one should do is then to check if ϕ_R and ρ_L satisfy the hypotheses of that theorem. Unfortunately, this is not the case. The needed properties are, in fact, that R is

(1) nonsingular and circular;

(2) compatible with ρ_L;

(3) nondegenerate;

(4) transverse to $\partial V(L)$ and pointing out of $Y \setminus \mathring{V}(L)$, where $\mathring{V}(L) = \mathrm{int}(V(L))$.

In our situation, only properties (1) and (2) are satisfied. Indeed, by the definition of open book decomposition, there is a natural fibration $\theta\colon Y \setminus \mathring{V}(L) \to S^1 \cong \mathbb{R}/\mathbb{Z}$ such that the surfaces $\theta^{-1}(t)$ are the pages of the open book. The fact that α is adapted to (L, S, ϕ) implies that R is always positively transverse to the pages. This evidently implies that $d\theta(R) > 0$, so R is circular. The fact that R is compatible with ρ_L (that coincides with the universal abelian cover of $Y \setminus \mathring{V}(L)$) comes from Example 3.9.

On the other hand, properties (3) and (4) above are not satisfied. Indeed, after the MB perturbation of T_2, the vector field R is tangent to $\partial V(L)$ on $\overline{e}_+$ and $\overline{h}_+$. Moreover, as observed in Section 1.1.3, the MB perturbations near the two tori T_1 and T_2 may create degenerate orbits. We will then perturb R to get a new vector field R'. This vector field will be defined in $Y \setminus V'(L)$, where $V'(L) \subset \mathring{V}(L)$ is an open tubular neighborhood of L defined by $V'(L) = V'(K_1) \sqcup \cdots \sqcup V'(K_n)$, where, using the coordinates of Section 1.3.1, $\partial(V'(K_i)) = \{y = 2.5\}$.

Lemma 3.13 *There exists a (noncontact) vector field R' such that*

(i) *R' coincides with R outside a neighborhood of $\mathcal{N}(L)$;*

(ii) *R' satisfies properties* (1)–(4) *above with $V(L)$ replaced by $V'(L)$;*

(iii) *the only periodic orbits of R' in $\mathcal{N}(V) \setminus V'(L)$ are the four sets of nondegenerate orbits $\overline{e}, \overline{h}, \overline{e}_+, \overline{h}_+$.*

Observe that property (i) implies that the twisted Lefschetz zeta functions of the restrictions of the flows ϕ_R and $\phi_{R'}$ to $Y \setminus \mathcal{N}(K)$ coincide, while property (ii) allows us to apply Theorem 3.10 to $\phi_{R'}$.

Proof A perturbation of R into an R' satisfying the conditions (i)–(iii) can be obtained in more than one way. An example is pictured in Figure 3; see also Figure 1. We briefly explain how it is obtained. Since the modification of R is nontrivial only inside disjoint neighborhoods of each K_i, we will describe it only for a fixed component K of L. The characterization of the a perturbation will be presented in terms of perturbation of the lines in a page S of (L, S, ϕ) that are invariant under the first return map ϕ of ϕ_R: we will refer to these curves as *ϕ–invariant lines on S*. Note that these curves are naturally oriented by the flow.

Outside a neighborhood of $\partial V'$, one can see this perturbation in terms of a perturbation of ϕ into another monodromy ϕ', and R' is the vector field ∂_t in $Y \setminus V'(L) \cong S \times [0,1]/\langle (x,1) \sim (\phi'(x), 0)\rangle$, where t is the coordinate of $[0,1]$.

Observe first that the only periodic orbit in the (singular) ϕ–invariant line a_1 containing h (in correspondence to the singularity) is exactly h. Similarly, the only periodic orbit in the ϕ–invariant singular flow line a_2 containing h_+ is precisely h_+. Denote by $A_i \subset Y$ the mapping torus of $(a_i, \phi|_{a_i})$, $i = 1, 2$. We modify R separately inside the regions of $(Y \setminus V'(K)) \setminus (A_1 \sqcup A_2)$ as follows.

In the region containing e (and with boundary A_1), the set of ϕ–invariant lines (the elliptic lines in Figure 3 (left)) is perturbed in a set of ϕ'–invariant spiral-like lines (Figure 3 (right)), each of which is negatively asymptotic to a_1 and positively

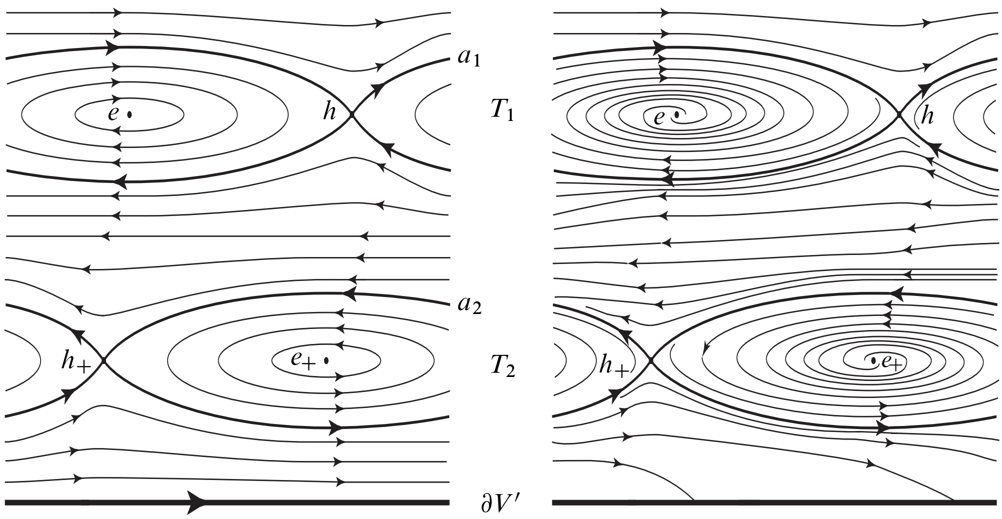

Figure 3: The dynamics of the vector fields R and R' near $\mathcal{N}(V) \setminus V'(L)$. Each oriented line represents an invariant subset of a page of (L, S, ϕ) under the first return map ϕ (left) and ϕ' (right); the invariant lines a_1 and a_2 are stressed. The situation at the left is the same depicted in Figure 1.

asymptotic to e. It is easy to see that after the perturbation, the only periodic orbit in the interior of this region is e. Moreover, we can arrange the perturbation such in a way that the differential $\mathfrak{L}_e^{R'}$ of the first return map on S of $\phi_{R'}$ along e coincides, up to a positive factor smaller then 1, with $\mathfrak{L}_e^{R}$, so that the Lefschetz sign $\epsilon(e)$ of e is still $+1$.

A similar perturbation is done in the region of $(Y \setminus V'(K)) \setminus (A_1 \sqcup A_2)$ containing e_+ in such a way that e_+ is the only periodic orbit of the perturbed vector field R', with still $\epsilon(e_+) = +1$.

The perturbation in the region between A_1 and A_2 is done by slightly pushing the monodromy in the positive y–direction in such a way that the set of ϕ–invariant lines is perturbed into a set of ϕ'–invariant lines, each of which is negatively asymptotic to a_1 and positively asymptotic to a_2 (in particular, there can not exist periodic orbits in this region).

A similar perturbation is done also inside the region between A_2 and $\partial V'(K)$, but in this case each ϕ'–invariant line is negatively asymptotic to a_2 and intersects $\partial V'(K)$ pointing out of the three-manifold.

Finally, we leave $R' = R$ in the rest of the manifold, where R was supposed having only isolated and nondegenerate periodic orbits.

Note that the two bases of eigenvectors of $\mathfrak{L}_h^R$ and $\mathfrak{L}_{h_+}^R$ are contained in the tangent spaces of the curves a_1 and a_2, and since $\phi_R = \phi_{R'}$ on these curves, the Lefschetz signs of the two orbits are preserved by the perturbation.

It is easy to convince ourselves that R' satisfies the properties (i)–(iii) above. □

Set $\zeta = \zeta_{\mathbb{1}}$. Since the Lefschetz zeta function of a flow depends only on its periodic orbits and their signs, we have the following:

Corollary 3.14 *If R' is obtained from R as above, then*

$$\zeta(\phi_{R'}) = \zeta(\phi_{R'}|_{(Y\setminus\mathcal{N}(K)\sqcup\{\bar{e},\bar{h},\bar{e}_+,\bar{h}_+\})}) = \zeta(\phi_R|_{(Y\setminus\mathcal{N}(K))}) \cdot \prod_{\gamma\in\{\bar{e},\bar{h},\bar{e}_+,\bar{h}_+\}} \zeta_\gamma([\gamma]),$$

where $[\gamma]$ is the homology class of γ in $H_1(Y\setminus\mathcal{N}(K))$.

Now we want to compute more explicitly the twisted Lefschetz zeta function $\zeta(\phi_{R'})$. Let us begin with the local Lefschetz zeta function of the simple orbits; see Definition 3.5.

Lemma 3.15 *Let γ be an orbit of R or R'. Then*

$$\zeta_\gamma(t) = \begin{cases} (1-t)^{-1} = 1+t+t^2+\cdots & \text{if } \gamma \text{ elliptic}, \\ 1-t & \text{if } \gamma \text{ positive hyperbolic}, \\ 1+t & \text{if } \gamma \text{ negative hyperbolic}. \end{cases} \tag{24}$$

Proof This is just matter of replacing the Lefschetz signs given in Observation 1.2. For example, if γ is positive hyperbolic, then all the iterates are also positive hyperbolic, $\epsilon(\gamma^i) = -1$ for every $i > 0$, and

$$\zeta_\gamma(t) = \exp\biggl(\sum_{i\geq 1} -\frac{t^i}{i}\biggr) = \exp(\log(1-t)) = 1-t. \qquad \square$$

Observation 3.16 Note that the equations above are exactly the generating functions given by Hutchings in [20, Section 2].

Let μ_i be a positive meridian of K_i for $i \in \{1,\dots,n\}$, and set $t_i = [\mu_i] \in H_1(Y\setminus K)$; fix moreover a Seifert surface S_i for each K_i. Recall that, for a given $X \subset Y$, we denote by $\mathcal{P}(X)$ the set of simple Reeb orbits contained in X.

Corollary 3.17 *The twisted Lefschetz zeta function of $\phi_R|_{(Y\setminus\mathcal{N}(L))}$ is*

$$\zeta(\phi_R|_{(Y\setminus\mathcal{N}(L))}) = \prod_{\gamma\in\mathcal{P}(Y\setminus\mathcal{N}(L))} \zeta_\gamma([\gamma]),$$

where $\zeta_\gamma([\gamma])$ *is determined as follows:*

$$\zeta_\gamma(\rho_L(\gamma)) = \left(1 - \prod_{i=1}^{n} t_i^{\langle \gamma, S_i \rangle}\right)^{-1} = \sum_{l=0}^{\infty} \left(\prod_{i=1}^{n} t_i^{\langle \gamma, S_i \rangle}\right)^{l} \quad (\gamma \text{ elliptic}),$$

$$\zeta_\gamma(\rho_L(\gamma)) = 1 - \prod_{i=1}^{n} t_i^{\langle \gamma, S_i \rangle} \quad (\gamma \text{ positive hyperbolic}),$$

$$\zeta_\gamma(\rho_L(\gamma)) = 1 + \prod_{i=1}^{n} t_i^{\langle \gamma, S_i \rangle} \quad (\gamma \text{ negative hyperbolic}).$$

Proof of Theorem 3.12 for fibered links To finish the proof, it remains essentially to prove that

$$\chi(\mathrm{ECC}(L, Y, \alpha)) = \zeta(\phi_R|_{(Y \setminus \mathcal{N}(L))}) \cdot \prod_{\gamma \in \{\bar{e}, \bar{h}, \bar{e}_+, \bar{h}_+\}} \zeta_\gamma([\gamma]). \tag{25}$$

This is easy to verify recursively on the set of simple orbits. Suppose $\delta = \prod_j \delta_j^{k_j}$ is an orbit set and let γ be an orbit such that $\gamma \neq \delta_j$ for any j. Then the set of all multiorbits that we can build using δ and γ can be expressed via the product formulae

$$\begin{array}{ll} \delta \cdot \{\varnothing, \gamma, \gamma^2, \dots\} & \text{if } \gamma \text{ is elliptic,} \\ \delta \cdot \{\varnothing, \gamma\} & \text{if } \gamma \text{ is hyperbolic.} \end{array} \tag{26}$$

As remarked in Section 1.2, the index parity formula (4) implies that the Lefschetz sign endows the ECH–chain complex with an absolute degree, and it coincides with the parity of the ECH–index. Then the contribution to the graded Euler characteristic of $\delta \cdot \gamma^l$, for any l ($l \in \mathbb{N}$ if γ is elliptic and $l \in \{0, 1\}$ if γ is hyperbolic), is

$$\epsilon(\delta) \prod_{i=1}^{n} t_i^{\langle \delta, S_i \rangle} \cdot \left(\epsilon(\gamma) \prod_{i=1}^{n} t_i^{\langle \gamma, S_i \rangle}\right)^{l}.$$

Substituting the last formula in (26), the total contribution of the product formulae to the Euler characteristic are

- $\epsilon(\delta) \prod_{i=1}^{n} t_i^{\langle \delta, S_i \rangle} \cdot \sum_{l=0}^{\infty} \left(\prod_{i=1}^{n} t_i^{\langle \gamma, S_i \rangle}\right)^{l}$ if γ is elliptic,
- $\epsilon(\delta) \prod_{i=1}^{n} t_i^{\langle \delta, S_i \rangle} \cdot \left(1 - \prod_{i=1}^{n} t_i^{\langle \gamma, S_i \rangle}\right)$ if γ is positive hyperbolic,
- $\epsilon(\delta) \prod_{i=1}^{n} t_i^{\langle \delta, S_i \rangle} \cdot \left(1 + \prod_{i=1}^{n} t_i^{\langle \gamma, S_i \rangle}\right)$ if γ is negative hyperbolic,

that is,

$$\epsilon(\delta) \prod_{i=1}^{n} t_i^{\langle \delta, S_i \rangle} \cdot \zeta_\gamma([\gamma]).$$

Starting from $\delta = \varnothing$, (25) follows by induction on the set of the simple Reeb orbits in $(Y \setminus \mathcal{N}(L)) \sqcup \{\overline{e}, \overline{h}, \overline{e}_+, \overline{h}_+\}$. The theorem follows then by applying Corollary 3.14 and Theorem 3.10 to the flow of R'. □

Proof of Theorem 3.1 for fibered links Theorem 3.12 and (22) imply (20) immediately. To prove the result in the hat version, we reason again at the level of chain complexes. Recall that if $N := Y \setminus \mathring{\mathcal{N}}(L)$, by the definition of the ECK–chain complexes,

$$\begin{aligned} \mathrm{ECC}(L, Y, \alpha) &= \mathrm{ECC}^{\overline{e}_+, \overline{h}_+}(N, \alpha) \\ &= \mathrm{ECC}^{\overline{h}_+}(N, \alpha) \otimes \bigotimes_{i=1}^{n} \langle \varnothing, e_i^+, (e_i^+)^2, \dots \rangle \\ &= \widehat{\mathrm{ECC}}(L, Y, \alpha) \otimes \bigotimes_{i=1}^{n} \langle \varnothing, e_i^+, (e_i^+)^2, \dots \rangle, \end{aligned}$$

where the second line comes from the product formula (26) and the fact that e_i^+ is elliptic for any i. Taking the graded Euler characteristics as above, we have

$$\begin{aligned} \chi(\mathrm{ECC}(L, Y, \alpha)) &= \chi(\widehat{\mathrm{ECC}}(L, Y, \alpha)) \cdot \prod_{i=1}^{n} \zeta_{e_i^+}([e_i^+]) \\ &= \chi(\widehat{\mathrm{ECC}}(L, Y, \alpha)) \cdot \prod_{i=1}^{n} \frac{1}{1-t_i}, \end{aligned}$$

where the last equality comes from the fact that $[e_i^+] = [\mu_i] = t_i \in H_1(Y \setminus L)$. If $Y = S^3$, then the last equation and (20) evidently imply (21). □

Observation 3.18 (symplectic Floer homology) If (L, S, ϕ) is an open book decomposition of Y, one can think of $\mathrm{ECK}(L, Y, \alpha)$ and $\widehat{\mathrm{ECK}}(L, Y, \alpha)$ as invariants of the pair (S, ϕ) and the adapted α. It is interesting to note that the Euler characteristic of $\mathrm{ECK}_1(L, Y, \alpha)$ with respect to the surface S (see Example 2.12) coincides with the sum of the Lefschetz signs of the Reeb orbits of period 1 in the interior of S, ie the Lefschetz number $\Lambda(\phi)$ of ϕ.

In fact, given Y (not necessarily an homology three-sphere) we can say even more about this fact by relating $\mathrm{ECK}_1(L, Y, \alpha)$ to the *symplectic Floer homology* $\mathrm{SH}(S, \phi)$ of (S, ϕ), whose Euler characteristic is precisely $\Lambda(\phi)$. Here we are considering the version of $\mathrm{SH}(S, \phi)$ for surfaces with boundary that is slightly rotated by ϕ in the positive direction, with respect to the orientation induced by S on ∂S; see for example Cotton-Clay [11].

Combining the definition of ECK, the relation between the *periodic Floer homology* PFH and ECH for a mapping torus (see Theorem 3.6.1 in [7]) and between PFH and SH (see for example Hutchings and Sullivan [21]) one can easily prove that

$$\mathrm{ECK}_1(L, Y, \alpha) \cong \mathrm{SH}(S, \phi), \tag{27}$$

where the degree of $\mathrm{ECK}(L, Y, \alpha)$ is computed with respect to the Alexander degree induced by S. We remark that an analogous result for HFK is currently unknown.

3.4 The general case

The first approach that one could use to attempt to apply Theorem 3.10 to a general link $L \subset Y$ is to look for a contact form on Y that is compatible with L and whose Reeb vector field is circular outside a neighborhood of L. Unfortunately we will not be able to find such a contact form. The basic idea to solve the problem consists of two steps:

Step 1 Find a contact form α on Y which is compatible with L and for which there exists a finite decomposition $Y \setminus L = \bigsqcup_i X_i$ for which $R = R_\alpha$ is circular in each X_i.

Step 2 Apply repeatedly the *Torres formula* for links to get the result.

As we will see, the Torres formula is a classical result which explains how to compute the Alexander polynomial of any sublink of a given link L starting from the Alexander polynomial of L.

3.4.1 Preliminary The key ingredient for Step 1 of our strategy is the following result; see Baker, Etnyre and Van Horn-Morris [2] and Guyard [18] for slightly different proofs.

Proposition 3.19 *Let $L = K_1 \sqcup \cdots \sqcup K_n \subset Y$ be an n–component link and let ξ be any fixed contact structure on Y. Then there exists an m–component link $L' \subset Y$ with $m \geq n$ and such that*

(1) $L' = L \sqcup K_{n+1} \sqcup \cdots \sqcup K_m$;

(2) *L' is fibered and the associated open book supports ξ.*

Proof The proof makes a deep use of the proof of the Giroux correspondence between open book decompositions and contact structures; see Giroux [16] and Colin [4]. Given a contact structure ξ on Y, Giroux explicitly constructs an open book decomposition of Y that supports a contact form α such that $\ker(\alpha) = \xi$. Such decomposition is built starting from a cellular decomposition $\mathcal{D}$ of Y that is compatible (in a specific sense) with ξ: for us it is important that, up to taking a refinement, any cellular decomposition of Y can be made compatible with ξ by an isotopy.

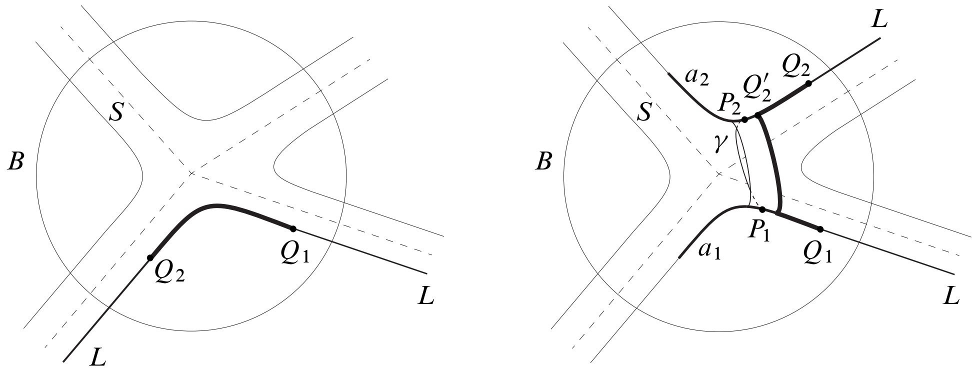

Figure 4: Making L contained in ∂S in $\mathcal{N}(\mathcal{D}^0)$: easy case (left) and general (right). The dotted lines are 1–simplexes in $\mathcal{D}^1$, while the bold segments from Q_1 to Q_2 represent the push-offs of L in $\mathcal{N}(\mathcal{D}^0)$.

Using the simplicial approximation theorem, it is possible to choose a triangulation $\mathcal{D}$ of Y in such a way that, up to isotopy, L is contained in $\mathcal{D}^1$, where $\mathcal{D}^i$ denotes the i–skeleton of $\mathcal{D}$. Up to taking a refinement, we can then suppose that $\mathcal{D}$ is adapted to ξ.

Let $\mathcal{N}(\mathcal{D}^1)$ be a tubular neighborhood of $\mathcal{D}^1$. Suppose that $\mathcal{N}(\mathcal{D}^0) \subset \mathcal{N}(\mathcal{D}^1)$ is a tubular neighborhood of $\mathcal{D}^0$ such that $\mathcal{N}(\mathcal{D}^1) \setminus \mathcal{N}(\mathcal{D}^0)$ is homeomorphic to a tubular neighborhood of $\mathcal{D}^1 \setminus \mathcal{N}(\mathcal{D}^0)$. The 0–page S of the associated open book built via the proof of Giroux satisfies then the following properties:

(1) $S \subset \mathcal{N}(\mathcal{D}^1)$, $L' := \partial S \subset \partial\mathcal{N}(\mathcal{D}^1)$ and $\mathcal{D}^1 \subset \mathrm{int}(S)$;

(2) $S \cap (\mathcal{N}(\mathcal{D}^1) \setminus \mathcal{N}(\mathcal{D}^0))$ is a disjoint union of strips which are diffeomorphic to $(\mathcal{D}^1 \setminus \mathcal{N}(\mathcal{D}^0)) \times [-1, 1]$, with $\mathcal{D}^1 \setminus \mathcal{N}(\mathcal{D}^0)$ corresponding to $(\mathcal{D}^1 \setminus \mathcal{N}(\mathcal{D}^0)) \times \{0\}$.

These properties imply that $L \subset \mathrm{int}(S)$ and that it is possible to push $L \setminus \mathcal{N}(\mathcal{D}^0)$ inside S to make it contained in ∂S. Note that in each strip composing $S \setminus \mathcal{N}(\mathcal{D}^0)$, we have only one possible choice for the direction in which to push $L \setminus \mathcal{N}(\mathcal{D}^0)$ to ∂S in such a way that the orientation of L coincides with that of ∂S.

We would like to extend this isotopy also to $L \cap \mathcal{N}(\mathcal{D}^0)$ to make the whole of L contained in ∂S. Suppose that B is a connected component (homeomorphic to a ball) of $\mathcal{N}(\mathcal{D}^0)$. In particular, we suppose that $B \cap S$ is connected. Then $L \cap \partial B$ consists of two points Q_1 and Q_2. The extension is done differently in the following two cases (see Figure 4):

Easy case This is when Q_1 and Q_2 belong to the same connected component of $\partial S \cap B$. The isotopy is then extended to B by pushing $L \cap B$ to $\partial S \cap B$ inside $S \cap B$; see Figure 4 (left).

General case If Q_1 and Q_2 belong to (the boundary of) different connected components a_1 and a_2 of $\partial S \cap B$, we proceed as follows: Let P_i be a point in the interior of a_i for $i = 1, 2$. Let γ be a simple arc in $S \cap B$ from P_1 to P_2 (there exists only one choice for γ up to isotopy). Let S' be obtained by positive Giroux stabilization of S along γ; see Figure 4 (right). Now we can connect Q_1 with a_2 by an arc in $\partial S'$ crossing the belt sphere of the 1–handle of the stabilization once; let Q_2' be the end point of this arc. Since a Giroux stabilization is compatible with the orientation of ∂S, the points Q_2' and Q_2 are in the same connected component of $a \setminus \{P_2\}$, so we can connect them inside $\partial S \cap B$, and we are done.

Pushing L to ∂S (and changing L and S as before where necessary) gives a link $\overline{L}$ that is contained in ∂S. To see that $\overline{L}$ is isotopic to L, we have to prove that, for any B as before, the two kinds of push-offs we use do not change the isotopy class of L.

Clearly, the isotopy class of L is preserved in the easy case. For the general case, it suffices to show that substituting the arc $L \cap S \cap B$ from Q_1 to Q_2 with an arc crossing the belt sphere of the handle once does not change the isotopy class of L. This is equivalent to proving that if γ is the path of the Giroux stabilization and $\overline{\gamma} = \gamma \cup c$, where c is the core curve of the handle, then $\overline{\gamma}$ bounds a disk in $Y \setminus L$. This can be proved for example by using the particular kind of Heegaard diagrams used in [7]. Observe that if b is the cocore of the handle, then $\overline{\gamma}$ is isotopic in S to $b \cup \phi'(b)$, where ϕ' is the monodromy on S' given by the Giroux stabilization. We finish by observing that $b \cup \phi'(b)$ is isotopic, up to a small perturbation near ∂S, to an attaching curve of a Heegaard diagram of Y. □

We now recall the Torres formula that we will use in the second step of our proof of Theorem 3.12. Since we need to consider the Alexander quotient as a polynomial, when we need to highlight its variables $t_1, \dots, t_k$, we will indicate them as subscripts and write $\mathrm{ALEX}_{t_1,\dots,t_k}$ instead of ALEX.

Theorem 3.20 (Torres formula) *Let $L = K_1 \sqcup \dots \sqcup K_n$ be an n–link in a homology three-sphere Y, K_{n+1} a knot in $Y \setminus L$ and $L' = L \sqcup K_{n+1}$. Let S_i be a Seifert surface for K_i for $i \in \{1, \dots, n+1\}$. Then*

$$\mathrm{ALEX}_{t_1,\dots,t_n,1}(Y \setminus L') \doteq \mathrm{ALEX}_{t_1,\dots,t_n}(Y \setminus L) \cdot \left(1 - \prod_{i=1}^{n} t_i^{\langle K_{n+1}, S_i \rangle}\right),$$

where $\mathrm{ALEX}_{t_1,\dots,t_n,1}(Y \setminus L')$ indicates the polynomial $\mathrm{ALEX}_{t_1,\dots,t_{n+1}}(Y \setminus L')$ evaluated at $t_{n+1} = 1$.

We refer the reader to Torres [37, Theorem 3] for the original proof. See also Franks [13, Theorem 6.4] for a proof making use of techniques of dynamics and Turaev [38, Section 1.4] for a generalization of the formula to links in any three-manifold.

Observation 3.21 One can see the condition $t_{n+1}=1$ from a purely topological point of view. Imagine taking the manifold $Y\setminus L'$ and then gluing back K_{n+1}. The effect on $H_1(Y\setminus L')$ is that the generator $[\mu_{n+1}]$ is killed, and now the homology class of a loop $\gamma\subset Y\setminus L'$ is determined only by the numbers $\langle\gamma,S_i\rangle$ for $S_i\in\{1,\dots,n\}$ (ie by $\rho_L(\gamma)$).

3.4.2 Proof of the result in the general case

Proof of Theorem 3.12 Let $L=K_1\sqcup\cdots\sqcup K_n$ be a given link in Y. Proposition 3.19 implies that there exists an open book decomposition (L',S,ϕ) of Y with binding

$$L'=L\sqcup K_{n+1}\sqcup\cdots\sqcup K_m$$

for some $m\geq n$. Let α be a contact form on Y adapted to (L',S,ϕ). Let $R=R_\alpha$ be its Reeb vector field. As remarked in Section 3.3, and using the same notation, R is circular in $Y\setminus\mathring{V}'(L')$, where we recall that $V'(L)$ is an union of tubular neighborhoods $V'(K_i)\subsetneq V(K_i)$, for $i\in\{1,\dots,m\}$, of L.

Since α is also adapted to L', each $\mathring{V}(K_i)$ is, by definition, foliated by concentric tori, which in turn are linearly foliated by Reeb orbits that intersect positively a meridian disk for K_i in $V(K_i)$. Now, we can choose α in such a way that for each $i\in\{n+1,\dots,m\}$, the tori contained in $V'(K_i)$ are foliated by orbits of R with fixed irrational slope. This condition can be achieved by applying the Darboux–Weinstein theorem in $V(K_i)$ to make $\alpha|_{V'(K_i)}$ like in Example 6.2.3 of [5]. It follows that for each $i\in\{n+1,\dots,m\}$, the only closed orbit of R in $V'(K_i)$ is K_i. Define $U(L')=\bigsqcup_{i=1}^m U(K_i)$, where

$$U(K_i)=\begin{cases}V(K_i) & \text{if } i\in\{1,\dots,n\},\\ V'(K_i) & \text{if } i\in\{n+1,\dots,m\}.\end{cases}$$

We have

$$\begin{aligned}\chi(\mathrm{ECC}(L,Y,\alpha)) &= \zeta_{\rho_L}(\phi_R|_{Y\setminus V(L)})\\ &= \zeta_{\rho_L}(\phi_R|_{Y\setminus U(L')})\cdot\prod_{i=n+1}^m\ \prod_{\gamma\in\mathcal{P}(V'(K_i))}\zeta_\gamma(\rho_L([\gamma]))\\ &= \zeta_{\rho_L}(\phi_R|_{Y\setminus U(L')})\cdot\prod_{i=n+1}^m\zeta_{K_i}(\rho_L([K_i]))\\ &= \zeta_{\rho_{L'}}(\phi_R|_{Y\setminus U(L')})|_{t_1,\dots,t_n,1,\dots,1}\cdot\prod_{i=n+1}^m\zeta_{K_i}(\rho_L([K_i]))\\ &\doteq \mathrm{ALEX}_{t_1,\dots,t_n,1,\dots,1}(Y\setminus L')\cdot\prod_{i=n+1}^m\zeta_{K_i}(\rho_L([K_i]))\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{ALEX}_{t_1,\dots,t_n,1,\dots,1}(Y \setminus L') \cdot \prod_{i=n+1}^{m} \Bigl(1 - \prod_{j=1}^{n} t_j^{\langle K_i, S_j \rangle}\Bigr)^{-1} \\
&= \mathrm{ALEX}_{t_1,\dots,t_n}(Y \setminus L),
\end{aligned}
$$

where

- line 2 follows from reasoning as in the proof of (25);
- line 3 holds since K_i, for $i \in \{n+1, \dots, m\}$, is the only Reeb orbit of α in $V'(K_i)$;
- line 4 comes from the idea in Observation 3.21: ρ_L and $\rho_{L'}$ coincide on the generators t_i of $H_1(Y \setminus L)$ for $i \in \{1, \dots, n\}$, and $t_i = [\mu_i] = 1 \in H_1(Y \setminus L)$ for $i \in \{n+1, \dots, m\}$;
- line 5 holds since, up to a slight perturbation of R near each $\partial U(K_i)$ to make it nondegenerate and transverse to the boundary like in the proof in Section 3.3, $\rho_{L'}$ and $R|_{Y \setminus U(L')}$ satisfy the hypothesis of Theorem 3.10;
- line 6 is due to the fact that the K_i are elliptic;
- line 7 is obtained by applying repeatedly the Torres formula on the components $K_{n+1}, \dots, K_m$. □

The proof of Theorem 3.1 works then exactly as in the fibered case.

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