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Translation surfaces and the curve graph in genus two

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Let S be a (topological) compact closed surface of genus two. We associate to each translation surface $(X,\omega) \in \Omega\mathcal{M}_2 = \mathcal{H}(2) \sqcup \mathcal{H}(1,1)$ a subgraph $\hat{\mathcal{C}}_{\mathrm{cyl}}$ of the curve graph of S. The vertices of this subgraph are free homotopy classes of curves which can be represented either by a simple closed geodesic or by a concatenation of two parallel saddle connections (satisfying some additional properties) on X. The subgraph $\hat{\mathcal{C}}_{\mathrm{cyl}}$ is by definition $\mathrm{GL}^+(2,\mathbb{R})$–invariant. Hence it may be seen as the image of the corresponding Teichmüller disk in the curve graph. We will show that $\hat{\mathcal{C}}_{\mathrm{cyl}}$ is always connected and has infinite diameter. The group $\mathrm{Aff}^+(X,\omega)$ of affine automorphisms of (X,ω) preserves naturally $\hat{\mathcal{C}}_{\mathrm{cyl}}$, we show that $\mathrm{Aff}^+(X,\omega)$ is precisely the stabilizer of $\hat{\mathcal{C}}_{\mathrm{cyl}}$ in $\mathrm{Mod}(S)$. We also prove that $\hat{\mathcal{C}}_{\mathrm{cyl}}$ is Gromov-hyperbolic if (X,ω) is completely periodic in the sense of Calta.

It turns out that the quotient of $\hat{\mathcal{C}}_{\mathrm{cyl}}$ by $\mathrm{Aff}^+(X,\omega)$ is closely related to McMullen's prototypes in the case that (X,ω) is a Veech surface in $\mathcal{H}(2)$. We finally show that this quotient graph has finitely many vertices if and only if (X,ω) is a Veech surface for (X,ω) in both strata $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$.

51H20; 54H15

1 Introduction

1.1 The curve complex

Let S be a compact surface. The *curve complex* of S is a simplicial complex whose vertices are free homotopy classes of essential simple closed curves on S, and k–simplices are defined to be the sets of (homotopy classes of) $k+1$ curves that can be realized pairwise disjointly on S. This complex was introduced by Harvey [20] in order to use its combinatorial structure to encode the asymptotic geometry of the Teichmüller space. It turns out that its geometry is intimately related to the geometry and topology of Teichmüller space; see eg Rafi [43]. The curve complex has now become a central subject in Teichmüller theory, low-dimensional topology, and geometric group theory. Note that this complex is quasi-isometric to its 1–skeleton, which is referred to as the *curve graph* of S. In this paper we will denote the curve graph by $\mathcal{C}(S)$.

The mapping class group $\mathrm{Mod}(S)$ naturally acts on the curve complex by isomorphisms. In most cases, all automorphisms of the curve complex are induced by elements of $\mathrm{Mod}(S)$; see Ivanov [24] and Luo [31]. Based on this relation, topological and

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combinatorial properties of the curve complex have been used to study the mapping class group, for example in Harer [19] and Bestvina and Fujiwara [2]. Masur and Minsky [33] showed that the curve graph (and the curve complex) is Gromov-hyperbolic; see also Bowditch [6]. A stronger result, that the hyperbolicity constant is independent of the surface S, has recently been proved simultaneously by several people: Aougab [1], Bowditch [7], Clay, Rafi and Schleimer [12], and Hensel, Przytycki and Webb [21]. Its boundary at infinity has been studied by Klarreich [27] and Hamenstädt [16]. Those results have led to numerous applications and a fast growing literature on the subject. In particular, the hyperbolicity of the curve graph has been exploited in the resolution of the ending lamination conjecture by Brock, Canary and Minsky [8]. For a nice survey on the curve complex and its applications we refer to Bowditch [5].

1.2 Teichmüller disks and translation surfaces

Another important notion in Teichmüller theory are the Teichmüller disks. These are isometric embeddings of the hyperbolic disk $\mathbb{H}$ in the Teichmüller space. Such a disk can be viewed as a complex geodesic generated by a quadratic differential q on a Riemann surface X. This quadratic differential defines a flat metric structure on X with conical singularities such that the holonomy of any closed curve on X belongs to the subgroup $\{\pm\mathrm{Id}\}\times\mathbb{R}^2$ of $\mathrm{Isom}(\mathbb{R}^2)$. If this quadratic differential is the square of a holomorphic 1–form ω on X, then the holonomy of any closed curve is a translation of $\mathbb{R}^2$, and we have a translation surface (X,ω).

Using the flat metric viewpoint, one can easily define a natural action of $\mathrm{GL}^+(2,\mathbb{R})$ on the space of translation surfaces as follows: given a matrix $A\in\mathrm{GL}^+(2,\mathbb{R})$ and an atlas $\{\phi_i \mid i\in I\}$ defining a translation surface structure, we get an atlas for another translation surface structure defined by $\{A\circ\phi_i \mid i\in I\}$. The Teichmüller disk generated by a holomorphic 1–form (X,ω) is precisely the projection into the Teichmüller space of its $\mathrm{GL}^+(2,\mathbb{R})$–orbit. Translations surfaces and their $\mathrm{GL}^+(2,\mathbb{R})$–orbit also arise in different contexts such as dynamics of billiards in rational polygons, interval exchange transformations and pseudo-Anosov homeomorphisms.

The importance of the $\mathrm{GL}^+(2,\mathbb{R})$–action is related to the fact that the $\mathrm{GL}^+(2,\mathbb{R})$–orbit closure of a translation surface encodes information on its geometric and dynamical properties. A remarkable illustration of this phenomenon is the famous Veech dichotomy, which states that if the stabilizer of (X,ω) for the action of $\mathrm{GL}^+(2,\mathbb{R})$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$, then the linear flow in any direction on X is either periodic or uniquely ergodic. By a result of Smillie (see [49; 46]) the stabilizer of (X,ω), denoted by $\mathrm{SL}(X,\omega)$, is a lattice in $\mathrm{SL}(2,\mathbb{R})$ if and only if the $\mathrm{GL}^+(2,\mathbb{R})$–orbit of (X,ω) is a closed subset of the moduli space. For more details on translation surfaces and related problems we refer to the excellent surveys by Masur and Tabachnikov [35] and Zorich [53].

The group $\mathrm{SL}(X,\omega)$ is closely related to the subgroup of the mapping class group that stabilizes the Teichmüller disk generated by (X,ω). This subgroup consists of elements of $\mathrm{Mod}(S)$ that are realized by homeomorphisms of X preserving the set of singularities (for the flat metric), and given by affine maps in local charts of the flat metric structure. This subgroup is denoted by $\mathrm{Aff}^+(X,\omega)$. There is a natural homomorphism from $\mathrm{Aff}^+(X,\omega)$ to $\mathrm{SL}(X,\omega)$ which associates to each element of $\mathrm{Aff}^+(X,\omega)$ its derivative. It is not difficult to see that this homomorphism is surjective and has finite kernel. The study of $\mathrm{Aff}^+(X,\omega)$ and $\mathrm{SL}(X,\omega)$ is a recurrent theme in the theory of dynamics in Teichmüller space; see eg McMullen [36], Hubert and Schmidt [23], Hubert and Lanneau [22], Möller [40] and Lehnert [30].

1.3 The flat metric and curve complex

Consider now the flat metric defined by a holomorphic 1–form ω on a (compact) Riemann surface X. By compactness, there exists a curve of minimal length in the free homotopy class of any essential simple closed curve. In general this curve of minimal length may not be a geodesic as it may contain some singularity in its interior. Nevertheless, following a result by Masur [32], we know that there are infinitely many curves that can be realized as simple closed geodesics for ω. Thus (X,ω) specifies a subset of vertices of $\mathcal{C}(S)$. Note that unlike the situation of hyperbolic surfaces, closed geodesics of minimal length are not unique in their homotopy class. They actually arise in family, that is, simple closed geodesics in the same homotopy class fill out a subset of X which is isometric to $(\mathbb{R}/c\mathbb{Z})\times(0,h)$. We will call such a subset a *geometric cylinder*, and the corresponding simple closed geodesics its *core curves*.

Mimicking the construction of the curve graph, we can add an edge between two vertices representing two cylinders if there exist two curves, one in each homotopy class, that can be realized disjointly (this condition is equivalent to requiring that the corresponding geodesics for the flat metric are disjoint). Thus, for each translation surface, we have a subgraph $\mathcal{C}_{\mathrm{cyl}}$ of the curve graph.

Let A be a matrix in $\mathrm{GL}^+(2,\mathbb{R})$, and consider the surface $(X',\omega') := A\cdot(X,\omega)$. Since the action of A preserves the affine structure, a geodesic on X corresponds to a geodesic on X' and vice-versa. Therefore, the subgraphs associated to (X',ω') and to (X,ω) are the same. This subgraph is actually associated to the Teichmüller disk generated by (X,ω). As $\mathcal{C}(S)$ can be viewed as the combinatorial model for the Teichmüller space, $\mathcal{C}_{\mathrm{cyl}}$ can be viewed as the counterpart of a Teichmüller disk in this setting. By definition, elements of $\mathrm{Aff}^+(X,\omega)$ preserve $\mathcal{C}_{\mathrm{cyl}}$ and act on $\mathcal{C}_{\mathrm{cyl}}$

by isomorphisms. As properties of the mapping class group can be studied via its action on the curve complex, one can expect the knowledge about the combinatorial and geometric structure of $\mathcal{C}_{\mathrm{cyl}}$ to be useful for the study of $\mathrm{Aff}^+(X,\omega)$.

1.4 Statement of results

The main purpose of this paper is to investigate $\mathcal{C}_{\mathrm{cyl}}$ when X is a surface of genus two. The reason for this restriction is the technical difficulties for the general cases. Hopefully, the results and techniques used in this situation may inspire further results in higher genera.

Recall that the moduli space of translation surfaces is naturally stratified by the zero orders of the 1–form ω (or equivalently, the cone angles at the singularities). In genus two, we have two strata: $\mathcal{H}(2)$ which contains pairs (X,ω) such that ω has a unique double zero, and $\mathcal{H}(1,1)$ which contains pairs (X,ω) such that ω has two simple zeros. Our first result shows that the geometry of $\mathcal{C}_{\mathrm{cyl}}$ does depend on the stratum of (X,ω).

Theorem A (Theorem 2.6) *If $(X,\omega)\in\mathcal{H}(2)$ then $\mathcal{C}_{\mathrm{cyl}}$ contains no triangles, but if $(X,\omega)\in\mathcal{H}(1,1)$ then $\mathcal{C}_{\mathrm{cyl}}$ always contains triangles.*

Note that a triangle in $\mathcal{C}_{\mathrm{cyl}}$ is a triple of simple closed pairwise disjoint curves that are simultaneously realized as core curves of three cylinders in (X,ω).

From its definition, the geometric structure of the subgraph $\mathcal{C}_{\mathrm{cyl}}$ depends very much on the flat metric of (X,ω). It is not difficult to see that $\mathcal{C}_{\mathrm{cyl}}$ is not connected in general; see Section 3. To get a nicer subgraph of $\mathcal{C}(S)$, we enlarge $\mathcal{C}_{\mathrm{cyl}}$ by adjoining to it the vertices of $\mathcal{C}(S)$ representing *degenerate cylinders*. Roughly speaking, a degenerate cylinder on X is a union of two saddle connections in the same direction such that there are deformations of (X,ω) on which this union is freely homotopic to the core curves of a geometric cylinder. We refer to Section 3 for a more precise definition. In particular, any degenerate cylinder is freely homotopic to a simple closed curve. Thus it corresponds to a vertex of $\mathcal{C}(S)$.

We define $\widehat{\mathcal{C}}^{(0)}_{\mathrm{cyl}}$ to be the set of vertices of $\mathcal{C}(S)$ that correspond to geometric cylinders and degenerate cylinders in (X,ω). We then define $\widehat{\mathcal{C}}^{(1)}_{\mathrm{cyl}}$ to be the set of the edges of $\mathcal{C}(S)$ both of whose endpoints belong to $\widehat{\mathcal{C}}^{(0)}_{\mathrm{cyl}}$. We thus get a subgraph $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ of $\mathcal{C}(S)$. By a slight abuse of notation, we will also call $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ the *cylinder graph* of (X,ω). Subsequently, this subgraph will be the main object of our investigation. We summarize the results concerning $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ in the following theorem:

Theorem B *For any $(X,\omega)\in\mathcal{H}(1,1)\sqcup\mathcal{H}(2)$, the subgraph $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ is connected and has infinite diameter. The subgroup of $\mathrm{Mod}(S)$ that stabilizes $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ is precisely $\mathrm{Aff}^+(X,\omega)$. Moreover, if (X,ω) is completely periodic in the sense of Calta, then $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ is Gromov-hyperbolic.*

Theorem B actually comprises several statements, which are proved in Corollary 4.2, Propositions 5.1 and 6.1 and Theorem 7.1. The contexts and precise statements will be given in the corresponding sections.

We finally consider the quotient of $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ by the action of $\mathrm{Aff}^+(X,\omega)$ in the case that (X,ω) is a Veech surface, that is, $\mathrm{SL}(X,\omega)$ is a lattice of $\mathrm{SL}(2,\mathbb{R})$.

Theorem C *Let $\mathcal{G}$ be the quotient of $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ by the group of affine automorphisms. Then $(X,\omega)\in\mathcal{H}(2)\sqcup\mathcal{H}(1,1)$ is a Veech surface if and only if $\mathcal{G}$ has finitely many vertices. For any Veech surface in $\mathcal{H}(2)$ the set of edges of $\mathcal{G}$ is also finite. There exist Veech surfaces in $\mathcal{H}(1,1)$ such that $\mathcal{G}$ has infinitely many edges.*

The statements of Theorem C are proved in Theorem 8.1 and Proposition 8.2.

1.5 Outline

In Section 2 we recall standard notions concerning translation surfaces. We show some geometric and topological features of translation surfaces of genus two. We end this section with the proof of Theorem A.

In Section 3, we introduce the notion of degenerate cylinders and define the cylinder graphs $\mathcal{C}_{\mathrm{cyl}}$ and $\widehat{\mathcal{C}}_{\mathrm{cyl}}$. We show that $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ is connected and has infinite diameter in Sections 4 and 5. These results follow from Theorem 4.1, which gives a bound on the distance in $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ using the intersection number.

Section 6 is devoted to the proof of the fact that the stabilizer subgroup of $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ in $\mathrm{Mod}(S)$ is precisely the group of affine automorphisms.

In Section 7 we show that if (X,ω) is completely periodic in the sense of Calta, then $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ is Gromov-hyperbolic. Our proof follows a strategy of Bowditch and uses a hyperbolicity criterion by Masur and Schleimer.

We give the proof of Theorem C in Section 8. Finally, in Section 9, we give the connection between the quotient graph $\mathcal{G}=\widehat{\mathcal{C}}_{\mathrm{cyl}}/\,\mathrm{Aff}^+$ and the set of prototypes for Veech surfaces in $\mathcal{H}(2)$, which were introduced by McMullen [37].

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2 Preliminaries

In this section we will prove some topological properties of saddle connections and cylinders on translation surfaces in genus two. The main result of this section is Theorem 2.6.

Let (X,ω) be a translation surface. A saddle connection on X is a geodesic segment whose endpoints are singularities, but which contains no singularities in its interior. A (geometric) cylinder of X is a subset C isometric to $(\mathbb{R}/c\mathbb{Z})\times(0,h)$, with $c,h\in\mathbb{R}_{>0}$, which is not properly contained in another subset with the same property. The parameter c is called the *circumference* and h the *width* or *height* of this cylinder.

The isometry from $(\mathbb{R}/c\mathbb{Z})\times(0,h)$ to C can be extended by continuity to a map from $(\mathbb{R}/c\mathbb{Z})\times[0,h]$ to X. We will call the images of $(\mathbb{R}/c\mathbb{Z})\times\{0\}$ and $(\mathbb{R}/c\mathbb{Z})\times\{h\}$ the boundary components of C. Each boundary component is a concatenation of some saddle connections. It may happen that the two boundary components coincide as subsets of X. We say that C is a *simple cylinder* if each of its boundary components is a single saddle connection. It is worth noticing that on a translation surface of genus two, every cylinder is invariant by the hyperelliptic involution. Therefore, the two boundary components of any cylinder contain the same number of saddle connections.

Throughout this paper, for any cycle $c\in H_1(X,\{\text{zeros of }\omega\};\mathbb{Z})$, we will use the notation $\omega(c):=\int_c\omega$, and for any saddle connection s, its euclidean length will be denoted by $|s|$. Let us start by the following elementary lemma.

Lemma 2.1 *Let (X,ω) be a translation surface in one of the hyperelliptic components $\mathcal{H}^{\text{hyp}}(2g-2)$ or $\mathcal{H}^{\text{hyp}}(g-1,g-1)$, and s be a saddle connection invariant by the hyperelliptic involution τ of X. We assume that s is not vertical. Then there exist a parallelogram $\boldsymbol{P}=(P_1P_2P_3P_4)$ in $\mathbb{R}^2$ and a locally isometric mapping $\varphi\colon \boldsymbol{P}\to X$ such that the following hold:*

(a) *The vertical lines through the vertices P_3 and P_4 intersect the diagonal $\overline{P_1P_2}$.*

(b) *The vertices of $\boldsymbol{P}$ are mapped to the singularities of X, and $\overline{P_1P_2}$ is mapped isometrically to s.*

(c) *The restriction of φ into $\operatorname{int}(\boldsymbol{P})$ is an embedding.*

(d) *Let $\eta>0$ be the length of the vertical segment from P_3 or P_4 to a point in $\overline{P_1P_2}$. Then for any vertical segment u in X from a singular point to a point in s, we have $|u|\geq\eta$, where $|u|$ is the euclidean length of u.*

We will call $\boldsymbol{P}$ the embedded parallelogram *associated to s.*

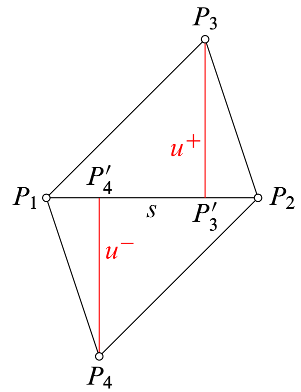

Figure 1: Here $s = \varphi(\overline{P_1 P_2})$, $u^+ = \varphi(\overline{P_3 P_3'})$, $u^- = \varphi(\overline{P_4 P_4'})$, and $\boldsymbol{P} = (P_1 P_2 P_3 P_4)$ is the embedded parallelogram associated to s

Remark 2.2

- Since s in invariant by τ, we must have $\tau(\varphi(\boldsymbol{P})) = \varphi(\boldsymbol{P})$.
- The sides of $\boldsymbol{P}$ are mapped to saddle connections on X. Even though the restriction of φ into $\mathrm{int}(\boldsymbol{P})$ is one-to-one, it may happen that φ maps the opposite sides of $\boldsymbol{P}$ to the same saddle connection.
- This lemma is also valid for translation surfaces in $\mathcal{H}(0)$ and $\mathcal{H}(0,0)$.

Proof of Lemma 2.1 We will only give the proof for the case $(X,\omega) \in \mathcal{H}^{\mathrm{hyp}}(2g-2)$, as the proof for $\mathcal{H}^{\mathrm{hyp}}(g-1,g-1)$ is the same. Using

$$U_- = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\},$$

we can assume that s is horizontal. Let Ψ_t be the vertical flow on X generated by the vertical vector field $(0,1)$; this flow moves regular points of X vertically, upward if $t > 0$.

Consider the vertical geodesic rays emanating from the unique zero x_0 of ω in direction $(0,-1)$. We claim that one of the rays in this direction must meet s. Indeed, if this is not the case, then $\Psi_t(s)$ does not contain x_0 for any $t \in \mathbb{R}_{>0}$, and it follows that one can embed a rectangle of infinite area into X. Let u^+ be a vertical geodesic segment of minimal length from x_0 to a point in s which is included in a ray in direction $(0,-1)$. Since s is invariant by τ, the segment $u^- := \tau(u^+)$ is vertical of minimal length from x_0 to a point in s which is included in a ray in direction $(0,1)$. Using the developing map, we can realize s as a horizontal segment $\overline{P_1 P_2} \subset \mathbb{R}^2$, u^+ (resp. u^-) as a vertical segment $\overline{P_3 P_3'}$ (resp. $\overline{P_4 P_4'}$) where $P_3', P_4' \in \overline{P_1 P_2}$; see Figure 1. We remark that the central symmetry fixing the midpoint of $\overline{P_1 P_2}$ exchanges $\overline{P_3 P_3'}$ and $\overline{P_4 P_4'}$.

Let $\boldsymbol{P}$ denote the parallelogram $(P_1 P_3 P_2 P_4)$. We define a map $\varphi\colon \boldsymbol{P} \to X$ as follows: for any point $M \in \boldsymbol{P}$, let M' be the orthogonal projection of M in $\overline{P_1 P_2}$, and t be the length of $\overline{MM'}$. Let $\widehat{M}'$ be the point in s corresponding to M' by the identification between $\overline{P_1 P_2}$ and s. We then define $\varphi(M) := \Psi_t(\widehat{M}')$ if M is above $\overline{P_1 P_2}$, and $\varphi(M) = \Psi_{-t}(\widehat{M}')$ if M is below $\overline{P_1 P_2}$. By definition, φ is a local isometry and maps the vertices of $\boldsymbol{P}$ to x_0.

Note that we have $|\overline{MM'}| \leq |\overline{P_3 P_3'}| = |\overline{P_4 P_4'}|$, and the equality only occurs when $M = P_3$ or $M = P_4$. Thus, for all $M \in \boldsymbol{P} \setminus \{P_1, P_2, P_3, P_4\}$, $\varphi(M)$ is a regular point in X; otherwise we would have a vertical segment from P_0 to a point in s of length smaller than $|u^+|$.

We now claim that $\varphi|_{\mathrm{int}(\boldsymbol{P})}$ is an embedding. Assume that there exist two points $M_1, M_2 \in \mathrm{int}(\boldsymbol{P})$ such that $\varphi(M_1) = \varphi(M_2)$. Set $\vec{v} := \overrightarrow{M_1 M_2}$; then for any $M, M' \in \boldsymbol{P}$ such that $\overrightarrow{MM'} = \vec{v}$, we have $\varphi(M) = \varphi(M')$. Since $\boldsymbol{P}$ is a parallelogram, there exists a vertex $P_i \in \{P_1, P_2, P_3, P_4\}$ and a point $M' \in \boldsymbol{P} \setminus \{P_1, P_2, P_3, P_4\}$ such that $\overrightarrow{P_i M'} = \vec{v}$, which implies that $\varphi(M') = x_0$, and we have a contradiction to the observation above.

It is now straightforward to verify that $\boldsymbol{P}$ and φ satisfy all the required properties. □

In what follows, by a *slit torus* we will mean a triple (X, ω, s) where X is an elliptic curve, ω a nonzero holomorphic 1–form and s an embedded geodesic segment (with respect to the flat metric defined by ω) on X. We consider the endpoints of s as marked points on X. Note that there is a unique involution of X that preserves s and permutes its endpoints. The following lemma is useful for us in the sequel.

Lemma 2.3 *Let (X, ω, s) be a slit torus and x_1, x_2 be the endpoints of s. Assume that the segment (slit) s is not vertical, that is, $|\mathrm{Re}\,\omega(s)| > 0$. Then there exists a pair of parallel simple closed geodesics c_1, c_2 with c_i passing through x_i such that $c_i \cap \mathrm{int}(s) = \varnothing$, and $|\mathrm{Re}\,\omega(c_i)| \leq |s|$. In particular, the geodesics c_1, c_2 cut X into two cylinders, one of which contains $\mathrm{int}(s)$. Moreover, any leaf of the vertical foliation intersecting c_i must intersect s, and if every leaf of the vertical foliation meets s, then we have $|\mathrm{Re}\,\omega(c_i)| > 0$.*

Proof We remark that a slit torus can be considered as hyperelliptic translation surface with the hyperelliptic involution being the unique one that preserves s and exchanges its endpoints. Let $\boldsymbol{P} = (P_1 P_2 P_3 P_4)$ be the parallelogram associated to s, and $\varphi\colon \boldsymbol{P} \to X$ the corresponding embedding defined as in Lemma 2.1. Since we have $\varphi(P_3) \in \{x_1, x_2\}$, either $\varphi(P_3) = \varphi(P_1)$ or $\varphi(P_3) = \varphi(P_2)$. It follows that one pair of opposite sides of $\boldsymbol{P}$ are mapped to a pair of parallel simple closed geodesics c_1, c_2 of X with c_i passing through x_i. The other pair of opposite sides of $\boldsymbol{P}$ are mapped

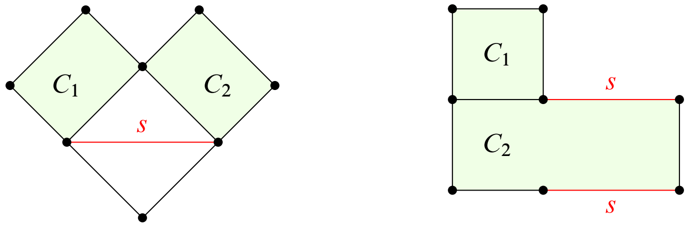

Figure 2: Configurations of C_1, C_2 with respect to s: none of C_1, C_2 contains s in its boundary (left) and s is contained in the boundary of C_2 (right).

to the same geodesic segment joining x_1 and x_2. Thus $\varphi(\boldsymbol{P})$ is a cylinder in X that contains s. Since X is a torus, the complement of $\varphi(\boldsymbol{P})$ is also a cylinder. It is straightforward to check that the pair $\{c_1, c_2\}$ satisfy all the required properties. □

We now turn to translation surfaces in genus two. Let (X,ω) be a translation surface in $\mathcal{H}(2) \sqcup \mathcal{H}(1,1)$. We denote by τ the hyperelliptic involution of X.

Lemma 2.4 *Let s_1, s_2 be a pair of saddle connections in X which are permuted by τ. If $(X,\omega) \in \mathcal{H}(2)$, then s_1 and s_2 bound a simple cylinder. If $(X,\omega) \in \mathcal{H}(1,1)$ then we have two cases:*

- *If s_i joins a zero of ω to itself, then s_1 and s_2 bound a simple cylinder.*
- *If s_i joins two different zeros of ω, then $s_1 \cup s_2$ decomposes X as a connected sum of two slit tori.*

Proof Since τ acts by $-\mathrm{Id}$ on $H_1(X,\mathbb{Z})$, s_1 and s_2 must be homologous. This lemma follows from an inspection on the configurations of rays originating from the zero(s) of ω in the same direction. □

Lemma 2.5 *Let (X,ω) be a surface in $\mathcal{H}(2)$ and s be a saddle connection in X invariant by the hyperelliptic involution τ. Then there exist two disjoint cylinders C_1, C_2 that do not intersect s, that is, $C_1 \cap C_2 = \varnothing$, and the core curves of C_1 and C_2 do not meet s. We remark that s may be contained in the boundary of C_1 or C_2. The possible configurations of C_1 and C_2 with respect to s are shown in Figure 2.*

Proof Without loss of generality, we can assume that s is horizontal. Let $\boldsymbol{P} = (P_1 P_3 P_2 P_4)$ be the embedded parallelogram associated to s, and $\varphi\colon \boldsymbol{P} \to X$ be the embedding map such that $s = \varphi(\overline{P_1 P_2})$; see Lemma 2.1. We choose the labeling of the vertices of $\boldsymbol{P}$ such that P_3 is the highest vertex, and P_4 is the lowest one. Throughout the proof, we will refer to Figure 3.

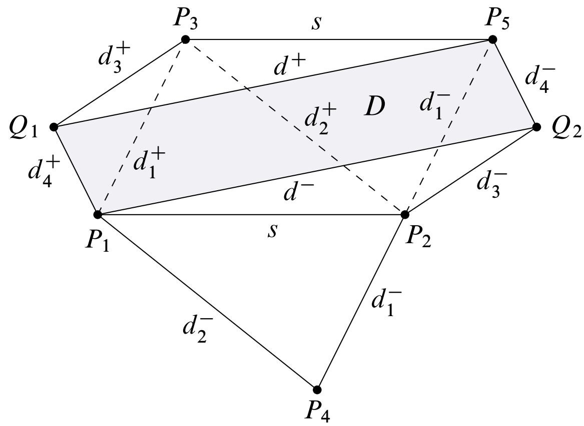

Figure 3: Finding a cylinder disjoint from s

Let $d_1^+ = \varphi(\overline{P_3 P_1}), d_2^+ = \varphi(\overline{P_3 P_2}), d_1^- = \varphi(\overline{P_4 P_2}), d_2^- = \varphi(\overline{P_4 P_1})$. We have $d_i^- = \tau(d_i^+)$. By Lemma 2.4, either $d_i^+ = d_i^-$ as subsets of X or the pair $d_i^\pm$ bound a simple cylinder. We remark that d_1^+ and d_2^+ cannot both be invariant by τ, otherwise we would have $X = \varphi(\boldsymbol{P})$, and X must be a torus. Thus we must only consider two cases:

(i) Both pairs $d_1^\pm$ and $d_2^\pm$ are respectively boundaries of two simple cylinders C_1, C_2 in X. In this case, it is not difficult to see that both C_1 and C_2 are disjoint from $\varphi(\boldsymbol{P})$, and $C_1 \cap C_2 = \varnothing$. We then get the configuration Figure 2 (left).

(ii) One of the pairs $d_1^\pm, d_2^\pm$ bound a simple cylinder, the other consist of a single saddle connection invariant by τ. In this case, $\varphi(\boldsymbol{P})$ is actually a simple cylinder. Without loss of generality, we can assume that the pair $d_1^\pm$ bound the cylinder $C = \varphi(\boldsymbol{P})$, and $d_2^+ = d_2^-$.

Let P_5 be the point in $\mathbb{R}^2$ such that the triangle $(P_3 P_5 P_2)$ is the image of $(P_1 P_2 P_4)$ by the translation by $\overrightarrow{P_1 P_3}$. Using the assumption that $d_2^+ = d_2^-$, that is, $\varphi(\overline{P_3 P_2}) = \varphi(\overline{P_1 P_4})$, we see that φ extends to a local isometric map from $\boldsymbol{P}' = (P_1 P_2 P_5 P_3)$ to X such that $\varphi(\boldsymbol{P}') = C$ and $\varphi|_{\mathrm{int}(\boldsymbol{P}')}$ is an embedding; see Figure 3.

Consider the horizontal rays emanating from the unique zero x_0 of ω to the outside of C. By the same argument as in Lemma 2.1, we see that one of the rays in direction $(1,0)$ reaches $d_1^+ = \varphi(\overline{P_3 P_1})$ from the outside of C. It follows that we can then extend φ to a convex hexagon $\boldsymbol{H} := (P_1 P_2 Q_2 P_5 P_3 Q_1)$, which is the union of $\boldsymbol{P}'$ and two triangles $(P_2 Q_2 P_5)$ and $(P_3 Q_1 P_1)$. Note that $(P_2 Q_2 P_5)$ and $(P_3 Q_1 P_1)$ are exchanged by the central symmetry fixing the midpoint of $\overline{P_2 P_3}$, and all the vertices of $\boldsymbol{H}$ are mapped to x_0.

Let $d_3^+ = \varphi(\overline{P_3Q_1})$, $d_4^+ = \varphi(\overline{Q_1P_1})$, $d_3^- = \varphi(\overline{P_2Q_2})$ and $d_4^- = \varphi(\overline{Q_2P_5})$. Again, for $i = 3, 4$, we have either $d_i^+ = d_i^-$ or the pair $d_i^\pm$ bound a simple cylinder. If $d_i^+ = d_i^-$ for both $i = 3, 4$, then $X = \varphi(\boldsymbol{H})$ and X must be a flat torus, so we have a contradiction. If both pairs $d_3^\pm, d_4^\pm$ are the boundaries of simple cylinders, then these cylinders are disjoint, and also disjoint from $\varphi(\boldsymbol{H})$. It follows that the total angle at x_0 is at least 8π (the total angle of $\boldsymbol{H}$ plus 4π), thus we have again a contradiction. We can then conclude that one of the pairs $d_3^\pm, d_4^\pm$ consists of a single saddle connection, and the other pair bounds a simple cylinder. Without loss of generality, we can assume that $d_3^\pm$ bounds a simple cylinder C_3, and $d_4^+ = d_4^- = d_4$. Note that C_3 must be disjoint from $\varphi(\boldsymbol{H})$, and in particular it is disjoint from s.

Let $d^+ = \varphi(\overline{Q_1P_5})$ and $d^- = \varphi(\overline{P_1Q_2})$; then the pair $d^\pm$ is the boundary of a cylinder D whose core curves cross $d_4^\pm$. If $\boldsymbol{H}$ is strictly convex then D is a simple cylinder, but if $\overline{P_2Q_2}$ is parallel to $\overline{P_1P_2}$ then D is not simple (in this case we actually have $\overline{D} = \varphi(\boldsymbol{H})$). Nevertheless, in both cases the core curves of D do not intersect s. Since D is contained in $\varphi(\boldsymbol{H})$, we have $C_3 \cap D = \varnothing$. Since both C_3 and D are disjoint from s, the lemma is proved. □

We are now ready to show the following theorem:

Theorem 2.6 (a) *On any $(X, \omega) \in \mathcal{H}(2)$, there always exist two disjoint simple cylinders. There cannot exist a triple of pairwise disjoint cylinders in X.*

(b) *On any $(X, \omega) \in \mathcal{H}(1,1)$, there always exists a triple of cylinders which are pairwise disjoint.*

Remark 2.7

- The cylinders in Theorem 2.6 are not necessarily parallel.
- There cannot exist more than three simple pairwise disjoint closed curves on S. Statement (b) means that given any holomorphic 1–form in $\mathcal{H}(1,1)$, there always exists a family of three disjoint (simple closed) curves, realized simultaneously as simple closed geodesics for the flat metric induced by this 1–form.
- The statement (a) of the theorem is a direct consequence of [42, Proposition A.1].

Proof of Theorem 2.6, case $\mathcal{H}(2)$ Lemma 2.5 almost proves the statement for $\mathcal{H}(2)$ except that it does not guarantee that both cylinders are simple. We will give here a proof by using [41, Lemma 2.1]. Let s be a saddle connection that is invariant by the hyperelliptic involution τ (one can find such a saddle connection by picking a geodesic segment of minimal length $\hat{s}$ joining a regular Weierstrass point of X to the unique zero of ω, then taking $s = \hat{s} \cup \tau(\hat{s})$). By [41, Lemma 2.1], there exists a simple cylinder C_1 that contains s. Cut off C_1 from X then identify the two geodesic segments on the boundary of the resulting surface, we obtain a flat torus (X', ω') with a marked geodesic segment s'.

We can consider (X', ω', s') as a slit torus. By Lemma 2.3, there exists a cylinder C' in X' that contains s' whose complement in X' is another cylinder C_2 disjoint from s'. By construction C_2 is a simple cylinder in X and disjoint from C_1, hence the first assertion follows.

For the second assertion, we observe that any triple of pairwise disjoint simple closed curves disconnect X into two three-holed spheres. If all the curves in this triple are simple closed geodesics (core curves of cylinders), then we get two flat surfaces with geodesic boundary. Since X has only one singularity, one of the surfaces has no singularities in its interior. But the Euler characteristic of a three-holed sphere is -1, thus we have a contradiction to the Gauss–Bonnet formula. We can then conclude that X can not contain three disjoint cylinders. □

Proof of Theorem 2.6, case $\mathcal{H}(1,1)$ By [41, Lemma 2.1], we know that there exists a simple cylinder C_0 on (X, ω) that is invariant by τ. Cut off C_0 and glue the two boundary components of the resulting surface; we obtain a surface $(\widehat{X}, \widehat{\omega}) \in \mathcal{H}(2)$ with a marked saddle connection $\hat{s}$. Note that $\hat{s}$ is invariant by the hyperelliptic involution of $\widehat{X}$. By Lemma 2.5, we know that there exist two cylinders C_1 and C_2 on $\widehat{X}$ disjoint from $\hat{s}$ such that $C_1 \cap C_2 = \varnothing$. It follows immediately that C_1 and C_2 are actually cylinders in X and disjoint from C_0, from which we get the desired conclusion. □

3 Degenerate cylinders and the cylinder graph

3.1 Cylinders and the curve graph

Each cylinder in a translation surface is filled by simple closed geodesics in the same free homotopy class. The following elementary lemma shows that two (freely) homotopic closed geodesics must belong to the same cylinder.

Lemma 3.1 *Let c_1 and c_2 be two simple closed geodesics in (X, ω) which are freely homotopic. Then c_1 and c_2 are contained in the same cylinder.*

Proof Since c_1, c_2 are freely homotopic, they are homologous, hence $\omega(c_1) = \omega(c_2)$. It follows that c_1 and c_2 are parallel, thus must be disjoint. The pair c_1, c_2 cut X into two components, one of which must be an annulus denoted by A; see Proposition A.11 of [9]. We have a flat metric on A induced by the flat metric of X. Let $\theta_1, \dots, \theta_k$ be the cone angles at the singularities in A. Since the boundary of A is geodesic, the Gauss–Bonnet formula gives

$$\sum_{1 \le i \le k} (2\pi - \theta_i) = 2\pi\chi(A) = 0.$$

Since any singularity on a translation surface has cone angle at least 4π, the equation above actually shows that A contains no singularities. Thus A is a flat annulus, which must be contained in a cylinder of X. Therefore, c_1 and c_2 are contained in the same cylinder. $\square$

Let S be a fixed topological compact closed surface of genus two. Let $\mathcal{C}(S)$ denote the curve graph of S. Let $\Omega\mathcal{T}_2$ be the abelian differential bundle over the Teichmüller space $\mathcal{T}_2$. Elements of $\Omega\mathcal{T}_2$ are equivalence classes of triples (X, ω, f), where X is a Riemann surface of genus two, ω is a holomorphic 1–form on X, and f is a homeomorphism from S to X; two triples (X, ω, f) and (X', ω', f') are identified if there exists an isomorphism $\varphi\colon X \to X'$ such that $\varphi^*\omega' = \omega$ and $f'^{-1} \circ \varphi \circ f\colon S \to S$ is isotopic to Id_S. The equivalence class of (X, ω, f) will be denoted by $[X, \omega, f]$.

Each element $[X, \omega, f]$ of $\Omega\mathcal{T}_2$ defines naturally a subgraph $\mathcal{C}_{\mathrm{cyl}}(X, \omega, f)$ of $\mathcal{C}(S)$. The vertices of this subgraph are free homotopy classes of the core curves of all cylinders on the translation surface (X, ω). The set $\mathcal{C}^{(1)}_{\mathrm{cyl}}(X, \omega, f)$ consists of the edges in $\mathcal{C}^{(1)}(S)$ both of whose endpoints belong to $\mathcal{C}^{(0)}_{\mathrm{cyl}}(X, \omega, f)$.

3.2 Degenerate cylinders

If C is a cylinder in X that fills X (ie $\overline{C} = X$), then C represents an isolated vertex in $\mathcal{C}_{\mathrm{cyl}}(X, \omega, f)$. This is because the core curve of any other cylinder in X must cross C. So in general $\mathcal{C}_{\mathrm{cyl}}(X, \omega, f)$ is not a connected graph. To fix this issue we introduce the notion of *degenerate cylinders*. Roughly speaking, a degenerate cylinder in X is a union of parallel saddle connections such that there exist deformations of (X, ω) where this union is freely homotopic to the core curves of a simple cylinder.

To be more precise, let x_0 be a singularity on a translation surface (X, ω). For any pair (r_1, r_2) of geodesic rays emanating from x_0, we will denote the counterclockwise angle from r_1 to r_2 by $\vartheta(r_1, r_2)$. If s is an oriented saddle connection from a singularity x_1 to a singularity x_2, then we denote by s^+ (resp. s^-) the intersection of s with a neighborhood of x_1 (resp. a neighborhood of x_2). This definition also makes sense when $x_1 = x_2$, in which case the orientation of s is to start in s^+ and end in s^-.

Definition 3.2 (degenerate cylinder) We will call the union of two saddle connections s_1, s_2 in $(X, \omega) \in \mathcal{H}(2) \sqcup \mathcal{H}(1,1)$ a *degenerate cylinder* if they are both invariant by the hyperelliptic involution, and up to an appropriate choice for the orientations of s_1 and s_2, we have

$$\vartheta(s_1^-, s_2^+) = \vartheta(s_1^+, s_2^-) = \pi.$$

In Figure 4, we represent the configurations of a degenerate cylinder at the singularities.

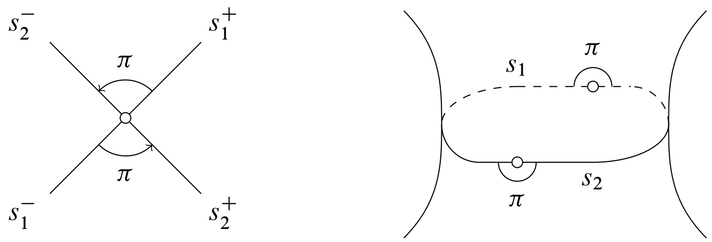

Figure 4: Configuration of a degenerate cylinder at the singularities for $\mathcal{H}(2)$ (left) and $\mathcal{H}(1,1)$ (right)

Remark 3.3

- If (X,ω) is in $\mathcal{H}(2)$, then a degenerate cylinder is not a simple curve: the zero of ω is its unique double point.
- If (X,ω) is in $\mathcal{H}(1,1)$, then the hyperelliptic involution τ of X permutes the zeros of ω, thus a saddle connection invariant by τ must connect the two zeros of ω. Therefore a degenerate cylinder must be a simple closed curve.

Examples Assume that $(X,\omega)\in\mathcal{H}(2)\sqcup\mathcal{H}(1,1)$ is horizontally periodic, and has a unique (geometric) horizontal cylinder C. If $(X,\omega)\in\mathcal{H}(2)$ then it has 3 horizontal saddle connections s_1,s_2,s_3, which are contained in the boundary of C; see Figure 5. Note that all of them are invariant by the hyperelliptic involution. By definition $s_1\cup s_2$, $s_2\cup s_3$ and $s_3\cup s_1$ are three degenerate cylinders. Similarly, if $(X,\omega)\in\mathcal{H}(1,1)$, then we have 4 horizontal saddle connections denoted by $s_1,\dots,s_4$ (see Figure 5) such that $s_i\cup s_{i+1}$ is a degenerate cylinder for $i=1,\dots,4$, with the convention $s_5=s_1$.

We will now prove some key properties of degenerate cylinders.

Lemma 3.4 *Let $s_1\cup s_2$ be a horizontal degenerate cylinder in $(X,\omega)\in\mathcal{H}(2)\sqcup\mathcal{H}(1,1)$. Then there exists in a neighborhood of (X,ω) a continuous family of translation surfaces $\{(X_t,\omega_t)\mid t\in[0,\epsilon)\}$ in the same stratum as (X,ω), with $\epsilon\in\mathbb{R}_{>0}$, such that*

- *$(X_0,\omega_0)=(X,\omega)$;*
- *for any $t\in(0,\epsilon)$, (X_t,ω_t) contains two saddle connections $s_{1,t}$ and $s_{2,t}$ corresponding to s_1 and s_2 and satisfying the following property: $s_{1,t}\cup s_{2,t}$ is freely homotopic to the core curves of a simple cylinder C_t in X_t;*
- *as $t\to 0$, the width of C_t decreases to zero.*

Moreover, for all $t\in(0,\epsilon)$, any vertical saddle connection (resp. regular geodesic) in (X,ω) corresponds to a vertical saddle connection (resp. regular geodesic) in (X_t,ω_t).

Proof Let us define a *half cylinder* to be the quotient $(\mathbb{R}\times[0,h])/\Gamma$, where $\Gamma\simeq\mathbb{Z}_2\ltimes\mathbb{Z}$ is generated by t: $(x,y)\mapsto(x+\ell,y)$ and s: $(x,y)\mapsto(-x,h-y)$. We will call h

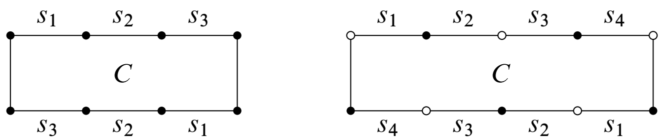

Figure 5: Degenerate cylinders on a horizontally periodic surface with a unique geometric horizontal cylinder for $\omega \in \mathcal{H}(2)$ (left) and $\omega \in \mathcal{H}(1,1)$ (right)

and ℓ the *width* and *circumference* of the half cylinder, respectively. We will refer to the projection of $(0,0)$ as the marked point on its boundary. Equivalently, a half cylinder is a closed disc equipped with a flat metric structure with geodesic boundary and two singularities of angle π in the interior.

Recall that all Riemann surfaces of genus two are hyperelliptic. Let $p\colon X \to \mathbb{CP}^1$ be the hyperelliptic double cover of X. There exists a meromorphic quadratic differential η on $\mathbb{CP}^1$ with at most simple poles such that $\omega^2 = p^*\eta$. Note that η has one zero and k poles, where $k = 5$ if $\omega \in \mathcal{H}(2)$, and $k = 6$ if $\omega \in \mathcal{H}(1,1)$. Let P_0 denote the unique zero of η, and $P_1, \dots, P_k$ its simple poles. Let Y be the flat surface defined by η on $\mathbb{CP}^1$. Observe that the cone angle of Y at P_0 is 3π if $\omega \in \mathcal{H}(2)$, and 4π if $\omega \in \mathcal{H}(1,1)$. The cone angle at P_i is π for $1, \dots, k$.

Since s_i, $i = 1, 2$, is invariant by τ, its projection in Y is a geodesic segment s'_i joining P_0 to a pole of η. By the definition of degenerate cylinder, one of the angles at P_0 specified by s'_1 and s'_2 is π. Let $\widehat{Y}$ be the flat surface obtained by slitting open Y along s'_1 and s'_2. By construction, $\widehat{Y}$ is a flat disc with $k-2$ singularities (of cone angle π) in its interior, and whose boundary is a geodesic loop c based at P_0. Note that P_0 is also a singular point of $\widehat{Y}$.

Let c denote the boundary of $\widehat{Y}$, and ℓ be the length of c. Fix an $\epsilon > 0$. For any $t \in (0, \epsilon)$, let $\widehat{C}_t$ be the half cylinder of circumference ℓ and width t. We can glue $\widehat{C}_t$ to $\widehat{Y}$ such that the marked point in the boundary of $\widehat{C}_t$ is identified with P_0. Let Y'_t denote the resulting flat surface. Observe that Y'_t corresponds to a meromorphic differential η'_t on $\mathbb{CP}^1$ which has a unique zero at P_0 and the same number of simple poles as η. It follows that the orienting double cover of $(\mathbb{CP}^1, \eta'_t)$ is an abelian differential (X_t, ω_t) in the same stratum as (X, ω). We also remark that the double cover of $\widehat{C}_t$ is a simple cylinder of width to t. We define (X_0, ω_0) to be (X, ω). It is now straightforward to check that the family $\{(X_t, \omega_t) \mid t \in [0, \epsilon)\}$ satisfies the properties in the statement of the lemma. □

As a byproduct of Lemma 3.4, we also have the following:

Lemma 3.5 *Let $s := s_1 \cup s_2$ be a degenerate horizontal cylinder in the surface $(X,\omega) \in \mathcal{H}(2) \sqcup \mathcal{H}(1,1)$.*

(i) *If $(X,\omega) \in \mathcal{H}(2)$, then there exist a pair of homologous saddle connections $r^{\pm}$ that cut out a slit torus containing s satisfying the following condition: any vertical leaf crossing $r^{\pm}$ must intersect s.*

(ii) *If $(X,\omega) \in \mathcal{H}(1,1)$, then either*
   (a) *there exist a pair of homologous saddle connections $r^{\pm}$ that cut out a slit torus containing s such that any vertical leaf crossing $r^{\pm}$ must intersect s, or*
   (b) *there are two simple cylinders C_1, C_2 disjoint from s such that any vertical leaf crossing C_1 or C_2 must intersect s.*

Proof Let us use the same notation as in the proof of Lemma 3.4. Recall that by slitting open Y along the projections of s_1 and s_2, we obtain a flat surface $\widehat{Y}$ whose boundary is a geodesic loop c based at P_0. One can construct a new flat surface homeomorphic to the sphere $\mathbb{CP}^1$ by "sewing up" c. This operation produces an extra singular point of angle π at the midpoint of c.

Let Y' denote the resulting surface. On Y', we have $k-1$ singularities of cone angles π and a singularity at P_0 of cone angle 2π if $\omega \in \mathcal{H}(2)$, or 3π if $\omega \in \mathcal{H}(1,1)$. The loop c corresponds to a segment c' on Y' joining P_0 to a singularity of angle π. Let (X',ω') be the orienting double cover of Y'. Then either $(X',\omega') \in \mathcal{H}(0,0)$ or $(X',\omega') \in \mathcal{H}(2)$. In both cases, c' gives rise to a saddle connection s' invariant by the hyperelliptic involution of X'. Note that by construction, we can identify $X' \setminus s'$ with $X \setminus s$.

Let $\varphi\colon \boldsymbol{P} \to X'$ be the embedded parallelogram associated to s' introduced in Lemma 2.1. By construction, φ maps the sides of $\boldsymbol{P}$ to saddle connections on X' which do not intersect s' in their interior. Thus those saddle connections correspond to some saddle connections on X. It follows that $\varphi(\boldsymbol{P}) \subset X'$ corresponds to a subsurface of X containing s. The conclusions of the lemma then follow from a careful inspection on the boundary of $\varphi(\boldsymbol{P})$. □

3.3 The cylinder graph

We now define a new subgraph $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$ of $\mathcal{C}(S)$ as follows: the vertices of $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$ are free homotopy classes of core curves of cylinders, or free homotopy classes of degenerate cylinders in X. Elements of $\widehat{\mathcal{C}}^{(1)}_{\mathrm{cyl}}(X,\omega,f)$ are the edges of $\mathcal{C}(S)$ both of whose endpoints are in $\widehat{\mathcal{C}}^{(0)}_{\mathrm{cyl}}(X,\omega,f)$.

Let $\boldsymbol{d}^{\mathcal{C}}$ denote the distance in $\mathcal{C}(S)$. Recall that by definition each edge of $\mathcal{C}(S)$ has length equal to one. Let a, b be two simple closed curves on S, and $[a]$, $[b]$ be their free homotopy classes, considered as vertices of $\mathcal{C}(S)$. We have

$$\boldsymbol{d}^{\mathcal{C}}([a],[b]) = \min\bigl\{\mathrm{leng}(\gamma) \mid \gamma \text{ is a path in } \mathcal{C}(S) \text{ from } [a] \text{ to } [b]\bigr\}.$$

We define a distance $\boldsymbol{d}$ in $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$ in the same manner, that is, every edge has length equal to one, and given $[a],[b] \in \widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$,

$$\boldsymbol{d}([a],[b]) = \min\bigl\{\mathrm{leng}(\gamma) \mid \gamma \text{ is a path in } \widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f) \text{ from } [a] \text{ to } [b]\bigr\}.$$

By convention, if there are no paths in $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$ from $[a]$ to $[b]$, then we define $\boldsymbol{d}([a],[b]) = \infty$. The subgraph $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$, called the *cylinder graph*, will be the main subject of our investigation in the remainder of this paper. To lighten notation, when (X,ω) and a marking mapping $f\colon S \to X$ are fixed, we will write $\mathcal{C}_{\mathrm{cyl}}$ and $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ instead of $\mathcal{C}_{\mathrm{cyl}}(X,\omega,f)$ and $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$.

Convention In the sequel, a "cylinder" could mean a usual geometric cylinder or a degenerate one. We will refer to usual geometric cylinders as *nondegenerate* cylinders. The term *core curve* will have the usual meaning for nondegenerate cylinder, for a degenerate one it just means the cylinder itself.

3.4 Intersection numbers

Let $\iota(\cdot,\cdot)$ denote the geometric intersection form on the set of free homotopy classes of simple closed curves on S. Let a, b be two simple closed curves in S, and $[a]$, $[b]$ their free homotopy classes, respectively. Recall that $[a]$ and $[b]$ are connected by an edge in $\mathcal{C}(S)$ if and only if $\iota([a],[b]) = 0$.

Assume now that a and b are simple closed geodesics in (X,ω). If a and b are parallel, then they do not have intersection, hence $\iota([a],[b]) = 0$. If they are not parallel, then they intersect transversally at every intersection point. By using the bigon criterion (see [14, Section 1.2.4]), it is not difficult to show that $\iota([a],[b]) = \#\{a \cap b\}$. However, if a or b is a degenerate cylinder then we must be a little more careful since in this case a or b may be not a simple curve (ie in $\mathcal{H}(2)$), and their intersections are not always transversal.

To deal with this complication, if a and b are core curves of two cylinders in X (possibly degenerate), we will fix some parametrizations $\alpha\colon \mathbb{S}^1 \to X$ for a, and $\beta\colon \mathbb{S}^1 \to X$ for b such that α and β are local homeomorphisms onto their images, and the restriction of α (resp. of β) to $\mathbb{S}^1 \setminus \alpha^{-1}(\{\text{singularities of } X\})$ (resp. to $\mathbb{S}^1 \setminus \beta^{-1}(\{\text{singularities of } X\})$) is one-to-one.

By an *intersection* of a and b, we will mean a pair $(t,t') \in \mathbb{S}^1 \times \mathbb{S}^1$ such that $\alpha(t) = \beta(t')$. This intersection is said to be *transversal* if there exist $\epsilon, \epsilon' > 0$ such that $a_1 := \alpha((t-\epsilon, t+\epsilon))$ and $b_1 := \beta((t'-\epsilon', t'+\epsilon'))$ are two simple arcs in X, a_1 intersects b_1 transversally at $p = \alpha(t) = \beta(t')$, and a_1 and b_1 have no other intersections. We denote by $a \cap b$ the set of intersections of a and b, and by $a \mathbin{\widehat{\cap}} b$ the subset of transversal intersections.

Lemma 3.6 *Let C and D be two cylinders on (X,ω) (both possibly degenerate) that are not parallel. Let c and d be respectively a core curve of C and a core curve of D. We denote by $[c]$ and $[d]$ the free homotopy classes of c and d, respectively. Let $c \mathbin{\widehat{\cap}} d$ denote the set of transversal intersections of c and d. Then we have*

$$\iota([c],[d]) = \#\{c \mathbin{\widehat{\cap}} d\}.$$

Since a nontransversal intersection of c and d can only occur at a singularity, it follows in particular that $\iota([c],[d]) = \#\{c \cap d\}$ if one of c and d is a regular geodesic.

Proof Let $\pi\colon \Delta = \{z \in \mathbb{C} : |z| < 1\} \to X$ denote the universal cover of X. The pull-back $\pi^*\omega$ of ω is a holomorphic 1–form, which defines a flat metric with cone singularities on Δ.

Fix a base point x for c and a base point y for d, which are not the singularities of X. Through any point in $\pi^{-1}(\{x\})$ (resp. any point in $\pi^{-1}(\{y\})$), there is a unique lift of c (resp. a unique lift d). Since c and d are not necessarily simple curves, a priori each lift of c and d may not be a simple arc. But this actually does not happen.

Claim 3.7 (i) *Each lift of c or of d is a simple arc in Δ.*

(ii) *Two lifts of c or of d can meet at at most one point (which is a nontransversal intersection).*

(iii) *A lift of c and a lift of d can meet at at most one point.*

Proof of the claim Since the argument for the three assertions are the same, we only give the proof of (iii). Let $\tilde{c}_0$ and $\tilde{d}_0$ be a lift of c and a lift of d in Δ, respectively. Let us assume that $\tilde{c}_0$ and $\tilde{d}_0$ intersect at two points. There exists then a disc $B \subset \Delta$ bounded by a subarc $c_0 \subset \tilde{c}_0$ and a subarc $d_0 \subset \tilde{d}_0$. Let p,q be the common endpoints of c_0 and d_0, and α and β be respectively the interior angles of B at p and q. Since c_0 and d_0 are geodesic segments for the flat metric on Δ, we have $\alpha > 0$ and $\beta > 0$ ($\alpha = 0$ or $\beta = 0$ means that c and d are parallel).

Let $p_1, \dots, p_r$ be the points in ∂B that correspond to the zeros of $\pi^*\omega$ and which are different from p,q. Let θ_i be the interior angle of B at p_i. By the definition of cylinders, we have $\theta_i \geq \pi$ for all $i = 1, \dots, r$. Let $x_1, \dots, x_s$ be the zeros of $\pi^*\omega$ in

$\mathrm{int}(B)$, and $\hat{\theta}_i$ be the angles at x_i. The Gauss–Bonnet formula gives (see, for instance, [48, Proposition 1])

$$\sum_{i=1}^{s}(2\pi-\hat{\theta}_i)+\sum_{i=1}^{r}(\pi-\theta_i)+2\pi-(\alpha+\beta)=2\pi\chi(B)=2\pi.$$

Since $\alpha+\beta>0$, $\pi-\theta_i\leq 0$ and $2\pi-\hat{\theta}_i<0$, we see that the equality above cannot be realized. Therefore, B cannot exist, which means that $\tilde{c}_0$ and $\tilde{d}_0$ can only meet at at most one point. □

Since nontransversal intersections of c and d can only occur at the singularities of X (zeros of ω), we can deform c and d slightly in a neighborhood of each zero of ω to get simple closed curves c' and d' in the same free homotopy classes as c and d, respectively, such that $\#\{c\mathbin{\widehat{\cap}}d\}=\#\{c'\cap d'\}$. Claim 3.7 then implies that any lift of c' in Δ intersects a lift of d' at at most one point and all the intersections are transversal. It follows from the bigon criterion (see eg [14, Proposition 1.7]) that

$$\iota([c],[d])=\#\{c'\cap d'\}=\#\{c\mathbin{\widehat{\cap}}d\}.$$

The lemma is then proved. □

Remark 3.8

- If C and D are not parallel, we can assume that C is horizontal and D is vertical. In the case both C and D are degenerate, to compute their intersection number, one can use Lemma 3.4 to get a deformation (X_t,ω_t) of (X,ω) in which C corresponds to a simple (horizontal) cylinder C_t. In X_t, D corresponds to a vertical cylinder D_t. Consequently, c is freely homotopic to a regular horizontal geodesic c_t in X_t, while d is freely homotopic to a core curve d_t of D_t. It follows from Lemma 3.6 that $\iota([c],[d])=\iota([c_t],[d_t])=\#\{c_t\cap d_t\}$.
- It may happen that two degenerate cylinders in the same direction have a positive intersection number.

4 Reducing numbers of intersection

In what follows, given two cylinders C,D in X, by $\iota(C,D)$ we will mean the geometric intersection number $\iota([c],[d])$, where c and d are some core curves of C and D, respectively. Our first goal is to estimate the distance in $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ using intersection numbers.

Theorem 4.1 *There exist two positive constants K_1, K_2 such that for any $[X,\omega,f]$ in $\Omega\mathcal{T}_2$, and any cylinders C and D in X (both possibly degenerate) considered as vertices of $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$, we have*

$$\boldsymbol{d}(C,D)\leq K_1\iota(C,D)+K_2. \tag{1}$$

As a direct consequence of inequality (1), we get the following:

Corollary 4.2 *The subgraph $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$ is connected.*

4.1 Reducing to simple cylinders

In what follows, we will fix a point $[X,\omega,f]\in\Omega\mathcal{T}_2$, and use the term "cylinder" to refer to both degenerate and nondegenerate cylinders. Our first step is to reduce the problem to the case where C and D are simple cylinders.

Lemma 4.3 *Let C be horizontal cylinder that does not fill X, ie $\overline{C}\neq X$, and D be a vertical cylinder. Assume that $\iota(C,D)>0$. Then there exists a simple cylinder C' such that $\boldsymbol{d}(C,C')\leq 1$ and $\iota(C',D)\leq\iota(C,D)$.*

Proof We first consider the case that C is nondegenerate. Let c be a core curve of C and d a core curve of D. Since c is a regular simple closed geodesic, by Lemma 3.6, we have $\iota(C,D)=\#\{c\cap d\}$. Obviously, we only need to consider the case that C is not simple.

If $(X,\omega)\in\mathcal{H}(2)$, then the complement of $\overline{C}$ is a simple cylinder C' whose boundary is a pair of homologous saddle connections contained in the boundary of C. In particular, C' is also horizontal, and we have $\iota(C,C')=0$, hence $\boldsymbol{d}(C,C')=1$. Any time d crosses C', it must cross C before returning to C'. Therefore, we have $\iota(C',D)\leq\iota(C,D)$.

If $(X,\omega)\in\mathcal{H}(1,1)$ then the complement of $\overline{C}$ is either (a) a horizontal simple cylinder, (b) two disjoint horizontal simple cylinders, or (c) a torus with a horizontal slit. In case (a) and case (b), the boundaries of the horizontal cylinders in the complement are contained in the boundary of C. Therefore, it suffices to choose one of them to be C'. In case (c), let (X',ω',s') be the slit torus which is the complement of $\overline{C}$. Note that the slit s' corresponds to a pair of homologous saddle connections in the boundary of C. By Lemma 2.3 we know that X' contains a simple cylinder C' disjoint from the slit s' such that any vertical line crossing C' must cross s'. Since C' is disjoint from C we have $\boldsymbol{d}(C,C')=1$. Any time d crosses C', it must cross the slit s' and hence C. Therefore, we also have $\iota(C',D)\leq\iota(C,D)$.

We now turn to the case that C is degenerate. If $(X,\omega)\in\mathcal{H}(2)$, from Lemma 3.5, we know that C is contained in a slit torus cut out by a pair of homologous saddle connections $r^{\pm}$ such that every vertical leaf crossing $r^{\pm}$ intersects C. Since $(X,\omega)\in\mathcal{H}(2)$, the complement of the slit torus is a simple cylinder C' bounded by $r^{\pm}$. Clearly, we have $\boldsymbol{d}(C,C')=1$. If the core curves of D are regular geodesics (that is, D is nondegenerate), then we can immediately conclude that $\iota(C',D)\leq\iota(C,D)$. When D is degenerate, we consider the deformations $\{(X_t,\omega_t)\mid t\in[0,\epsilon)\}$ of (X,ω) given by

Lemma 3.4. For $t \in (0, \epsilon)$, in (X_t, ω_t), D becomes a simple cylinder D_t, while the cylinders C and C' persist and have the same properties. Since $\iota(C', D) = \iota(C', D_t)$ and $\iota(C, D) = \iota(C, D_t)$, we also get $\iota(C', D) \le \iota(C, D)$.

The case $(X, \omega) \in \mathcal{H}(1, 1)$ also follows from similar arguments. $\square$

Lemma 4.4 *Assume that C is a horizontal cylinder that fills X, and D is a vertical cylinder. Then there exists a simple cylinder C' such that*

$$\boldsymbol{d}(C', C) = 2, \quad \iota(C', D) \le \iota(C, D).$$

Proof Let c be a core curve of C. If $(X, \omega) \in \mathcal{H}(2)$ then the complement of C is the union of three horizontal saddle connections s_1, s_2, s_3, all invariant by the hyperelliptic involution. We remark that the union of any two of these saddle connections is a degenerate cylinder. One can easily find a transverse simple cylinder C' containing s_1, disjoint from the union $s_2 \cup s_3$, whose core curves cross c once. Furthermore, we can choose C' such that the horizontal component of its core curves has length smaller than the length of c. Clearly, we have $\boldsymbol{d}(C, C') = 2$. Since any vertical geodesic crossing C' crosses also C, we have $\iota(C', D) \le \iota(C, D)$. Thus the lemma is proved for this case.

The case $(X, \omega) \in \mathcal{H}(1, 1)$ follows from the same arguments. $\square$

In what follows, a geodesic line on X that does not contain any singularity is called *regular*.

Lemma 4.5 *Let C be a horizontal cylinder and D be a vertical cylinder in X. If there exists a regular vertical leaf which does not cross C, then $\boldsymbol{d}(C, D) \le 2$.*

Proof Obviously we only need to consider the case that $\iota(C, D) > 0$. Assume that there is a regular vertical closed geodesic that does not intersect C. Then there exists another vertical cylinder D' which is disjoint from both C and D. Consequently, we have $\boldsymbol{d}(C, D) = 2$.

Assume now that there is an infinite regular vertical leaf that does not intersect C. The closure of this leaf is a subsurface X' of X bounded by some vertical saddle connections. Let s be a saddle connection in the boundary of X'. Note that s and $\tau(s)$ are homologous. Thus they decompose X into two subsurfaces X_1 and X_2 both invariant by τ. Since C is invariant by τ, it must be contained in one of the subsurfaces, say X_1. Since s and $\tau(s)$ are vertical, the core curves of D cannot cross s and $\tau(s)$, which means that D is also contained in one subsurface. Since we have assumed that $\iota(C, D) > 0$, D must be contained in X_1.

The subsurface X_2 must be either a slit torus or a surface in $\mathcal{H}(2)$ with a marked saddle connection. However, the latter case does not occur because it would imply that X_1 is a vertical simple cylinder containing both C and D, which is impossible. Now, by Lemma 2.3, one can find in the torus X_2 a simple cylinder C' that does not meet the slit. Since C' corresponds to a simple cylinder of X which is disjoint from both C and D, and we have $\boldsymbol{d}(C,D)=2$. The lemma is then proved. □

From Lemmas 4.3, 4.4, we know that if C is not simple then there exists a simple cylinder C' such that $\boldsymbol{d}(C,C')\leq 2$ and $\iota(C',D)\leq\iota(C,D)$. Consequently, we can find simple cylinders C', D' such that

$$\boldsymbol{d}(D,D')\leq 2, \quad \boldsymbol{d}(C,C')\leq 2, \quad \iota(C',D')\leq\iota(C,D).$$

It follows in particular that $\boldsymbol{d}(C,D)\leq\boldsymbol{d}(C',D')+4$. Therefore, in order to prove Theorem 4.1, we only need to prove (1) for the case that C and D are simple cylinders. Moreover, by Lemma 4.5, we can further assume that all the leaves of the foliation in the direction of D intersect $\overline{C}$. Thus, Theorem 4.1 is a consequence of the following:

Proposition 4.6 *Let C and D be two simple cylinders such that all the leaves of the foliation in the direction of D intersect $\overline{C}$. Then there always exists a simple cylinder C' such that*

$$\boldsymbol{d}(C',C)\leq 3, \quad \iota(C',D)<\iota(C,D). \tag{2}$$

To prove this proposition we will make use of the representation of translation surfaces as polygons in $\mathbb{R}^2$. In Appendix A, we give a uniform construction from symmetric polygons of translation surfaces in genus two satisfying the hypothesis of Proposition 4.6.

4.2 Proof of Proposition 4.6, case $\mathcal{H}(2)$

By using $\mathrm{GL}^+(2,\mathbb{R})$, we can assume that C is a horizontal cylinder, and D is vertical. From Proposition A.1(i), we can construct (X,ω) from a symmetric polygon $\boldsymbol{P}:=(P_0\cdots P_3Q_0\cdots Q_3)$ in $\mathbb{R}^2$. Note that by construction, the hyperelliptic involution of X lifts to the central symmetry fixing the midpoint of $\overline{P_0Q_0}$.

Let X_1, X_2 and Y be respectively the vertical projections of P_1, P_2 and Q_0 on $\overline{P_0P_3}$. Let x_1,x_2,x_3,y be respectively the lengths of $\overline{P_0X_1}$, $\overline{P_0X_2}$, $\overline{P_0P_3}$, $\overline{P_0Y}$. Clearly, we have $0\leq x_1\leq x_2\leq x_3$ and $0\leq y\leq x_3$. By cutting and regluing, we see that the cases $y=0$ ($Y\equiv P_0$) and $y=x_3$ ($Y\equiv P_3$) are equivalent. Therefore we can always suppose $0<y\leq x_3$.

By symmetry, we can assume that $|\overline{P_1X_1}|\geq|\overline{P_2X_2}|$; see Figure 6. Observe that the union of the projections of $(P_0P_1P_2)$ and $(Q_0Q_1Q_2)$ in X is a cylinder E which is disjoint from C. Similarly, the union of the projections of $(P_2P_3Q_0)$ and $(Q_2Q_3P_0)$

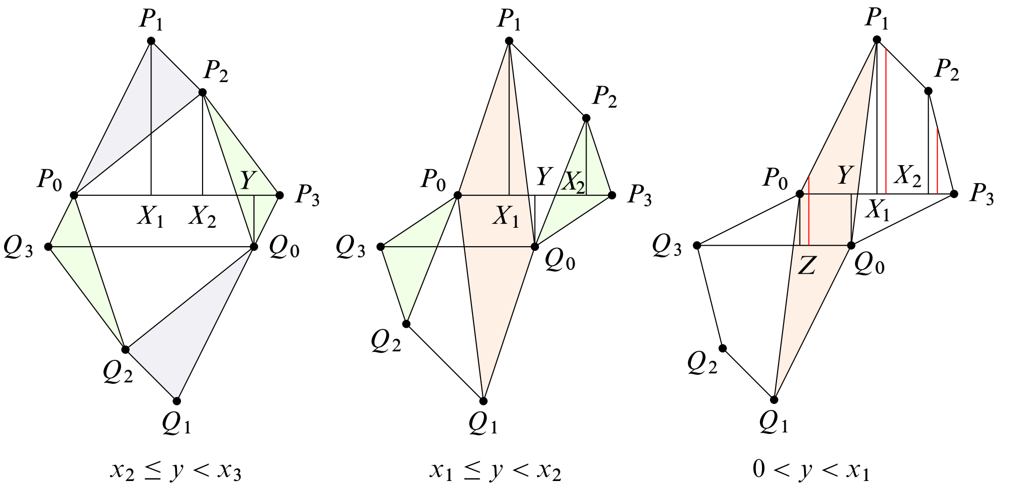

Figure 6: Finding simple cylinders having fewer intersections with D than C in the case $(X, \omega) \in \mathcal{H}(2)$. C is represented by the parallelogram $(P_0P_3Q_0Q_3)$; D is supposed to be vertical.

is also a cylinder F in X, which is disjoint from E. Observe that by assumption, E is always a simple cylinder, but F can be a degenerate one (that is, when both $\overline{P_2P_3}$ and $\overline{P_3Q_0}$ are vertical). Note that we have $\boldsymbol{d}(C, E) = 1$ and $\boldsymbol{d}(C, F) = 2$.

Let d be a core curve of D and $\widehat{d}$ be the preimage of d in $\boldsymbol{P}$. We remark that $\widehat{d}$ is a (finite) union of vertical segments with endpoints in the boundary of $\boldsymbol{P}$ and none of the vertices of $\boldsymbol{P}$ is contained in $\widehat{d}$. We first consider the generic case, where none of the sides of $\boldsymbol{P}$ is vertical. By assumption, we have

$$0 < x_1 < x_2 < x_3 \quad \text{and} \quad 0 < y < x_3.$$

We have three possibilities:

(a) $\boldsymbol{x_2 \le y < x_3}$ We observe that if a vertical line intersects $\overline{P_0P_2}$ or $\overline{P_2Q_0}$ then it must intersect $\overline{P_0X_2}$ or $\overline{X_2Y}$, respectively. Thus, we have

$$\#\{\widehat{d} \cap \overline{P_0P_3}\} \ge \#\{\widehat{d} \cap \overline{P_0P_2}\} + \#\{\widehat{d} \cap \overline{P_2Q_0}\}.$$

It follows that at least one of the following inequalities is true:

$$\#\{\widehat{d} \cap \overline{P_0P_2}\} < \#\{\widehat{d} \cap \overline{P_0P_3}\} \quad \Rightarrow \quad \iota(E, D) < \iota(C, D),$$
$$\#\{\widehat{d} \cap \overline{P_2Q_0}\} < \#\{\widehat{d} \cap \overline{P_0P_3}\} \quad \Rightarrow \quad \iota(F, D) < \iota(C, D).$$

Therefore, in this case, we can choose C' to be either E or F.

(b) $\boldsymbol{x_1 \le y < x_2}$ In this case, the parallelogram $(P_0P_1Q_0Q_1)$ is contained in $\boldsymbol{P}$, thus it projects to a simple cylinder G in X, which is disjoint from F. In particular, we have $\boldsymbol{d}(G,C) \leq 3$. We now observe that

$$\#\{\hat{d} \cap \overline{X_1X_2}\} = \#\{\hat{d} \cap \overline{P_1Q_0}\} + \#\{\hat{d} \cap \overline{P_2Q_0}\} \leq \#\{\hat{d} \cap \overline{P_0P_3}\}.$$

Therefore, at least one of the following inequalities is true $\iota(F,D) < \iota(C,D)$ or $\iota(G,D) < \iota(C,D)$. Hence we can choose C' to be either F or G.

(c) $\boldsymbol{0 < y < x_1}$ We will show that in this case $\iota(G,D) < \iota(C,D)$. Let Z be the vertical projection of P_0 to $\overline{Q_0Q_3}$. We choose a core curve d of D which is contained in the ϵ–neighborhood of the left boundary of D, with $\epsilon > 0$ small. The left boundary of D is a vertical saddle connection, thus it contains (the projection of) one of the following segments: $\overline{P_0Z}$, $\overline{P_1X_1}$, $\overline{P_2X_2}$. It follows that $\hat{d}$ contains a vertical segment $\hat{d}_0$ which is ϵ–close to one of $\overline{P_0Z}$, $\overline{P_1X_1}$, $\overline{P_2X_2}$ from the right. Observe that $\hat{d}_0$ always intersects $\overline{P_0P_3}$, but when ϵ is chosen to be small enough, $\hat{d}_0$ does not intersect $\overline{P_1Q_0}$. Since any vertical segment in $\boldsymbol{P}$ intersecting $\overline{P_1Q_0}$ must intersect $\overline{YX_1} \subset \overline{P_0P_3}$, it follows that $\iota(G,D) < \iota(C,D)$, and we can choose C' to be G.

It remains to show that the same arguments work in the degenerating situations, that is, when one of the sides of $\boldsymbol{P}$ is vertical. First, let us suppose that $\overline{P_2P_3}$ is vertical, ie $x_2 = x_3$.

- If $y = x_3$, then F becomes a degenerate cylinder. Clearly F and D are disjoint since they are both vertical. Therefore $\boldsymbol{d}(C,D) \leq \boldsymbol{d}(C,F) + 1 \leq 3$, hence we can choose C' to be D.
- If $0 < y < x_3$, then case (a) and case (b) follow from the same arguments. For case (c), we observe that the left boundary of D is not invariant by the hyperelliptic involution, and $\overline{P_2P_3}$ projects to an invariant saddle connection. Therefore $\hat{d}_0$ is either ϵ–close to $\overline{P_0Z}$ or $\overline{P_1X_1}$. Hence we can use the same argument to conclude that $\iota(G,D) < \iota(C,D)$ and we can choose C' to be G.

Other degenerations are easy to deal with in similar manner; details are left for the reader. □

4.3 Proof of Proposition 4.6, case $\mathcal{H}(1,1)$

Using the notation in Proposition A.1(ii), we know that (X,ω) is obtained from a decagon $\boldsymbol{P} := (P_0 \cdots P_4 Q_0 \cdots Q_4) \subset \mathbb{R}^2$. Our arguments depend on the properties of this decagon. We have three different models for $\boldsymbol{P}$ (see Figure 7): (I) both $\mathrm{int}(\overline{P_0P_2})$ and $\mathrm{int}(\overline{P_2P_4})$ are contained in $\mathrm{int}(\boldsymbol{P})$, (II) only one of $\mathrm{int}(\overline{P_0P_2})$ and $\mathrm{int}(\overline{P_2P_4})$ is contained in $\mathrm{int}(\boldsymbol{P})$, and (III) none of $\mathrm{int}(\overline{P_0P_2})$ and $\mathrm{int}(\overline{P_2P_4})$ is contained in $\mathrm{int}(\boldsymbol{P})$.

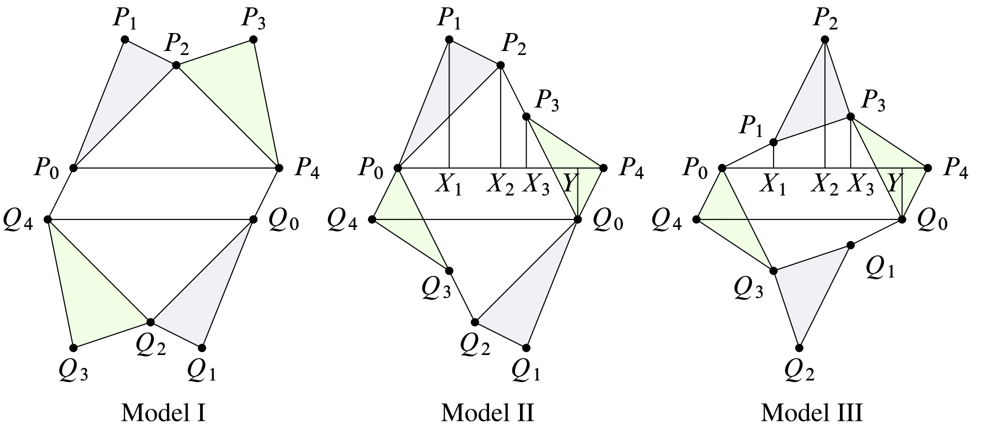

Figure 7: Finding a simple cylinder with fewer intersections with D than C in the case $(X,\omega)\in\mathcal{H}(1,1)$. C is represented by the parallelogram $(P_0P_4Q_0Q_4)$, D is supposed to be vertical.

Let X_1, X_2, X_3 and Y be respectively the vertical projections of P_1, P_2, P_3 and Q_0 on $\overline{P_0P_4}$. The lengths of $\overline{P_0X_i}$, $\overline{P_0P_4}$ and $\overline{P_0Y}$ are denoted by x_i, x_4 and y, respectively. As in the previous case, we have $0\le x_i\le x_{i+1}$, $i=1,2,3$, and $0<y\le x_4$. Let d be a core curve of D, and $\widehat{d}$ its preimage in $\boldsymbol{P}$.

4.3.1 Model I In this model, the sets $(P_0P_1P_2)\cup(Q_0Q_1Q_2)$ and $(P_2P_3P_4)\cup(Q_2Q_3Q_4)$ project to two disjoint simple cylinders in X which will be denoted by E and F, respectively. Note that $\boldsymbol{d}(C,E)=\boldsymbol{d}(C,F)=1$. Clearly, we have

$$\#\{\widehat{d}\cap\overline{P_0P_4}\}=\#\{\widehat{d}\cap\overline{P_0P_2}\}+\#\{\widehat{d}\cap\overline{P_2P_4}\}\quad\Rightarrow\quad\iota(C,D)=\iota(E,D)+\iota(F,D).$$

Therefore, we can pick C' to be E or F.

4.3.2 Model II By symmetry, we only need to consider the case that $\mathrm{int}(\overline{P_0P_2})\subset\mathrm{int}(\boldsymbol{P})$, and $\mathrm{int}(\overline{P_2P_4})\not\subset\mathrm{int}(\boldsymbol{P})$. Let E be the simple cylinder on X which is the projection of $(P_0P_1P_2)\cup(Q_0Q_1Q_2)$. Let F be the cylinder which is the projection of $(P_3P_4Q_0)\cup(Q_3Q_4P_0)$. We have $\boldsymbol{d}(C,E)=1$ and $\boldsymbol{d}(C,F)=2$.

We first consider the generic situation, that is, $0<x_i<x_{i+1}$, $i=1,2,3$, and $0<y<x_4$. Note that in this situation F is a simple cylinder. We have three cases: (a) $x_2\le y<x_4$, (b) $x_1\le y<x_2$ and (c) $0<y<x_1$. In all of these cases, one can find a simple cylinder having the desired property by the same arguments as the case that $(X,\omega)\in\mathcal{H}(2)$.

Consider now the degenerating situations: (1) $\overline{P_0P_1}$ is vertical, equivalently $x_1=0$; (2) $\overline{P_1P_2}$ is vertical, equivalently $x_1=x_2$; (3) $\overline{P_2P_3}$ is vertical, equivalently $x_2=x_3$; (4) $\overline{P_3P_4}$ is vertical, equivalently $x_3=x_4$; (5) $Y\equiv P_4$, equivalently $y=x_4$. If (4) or (5) does not occur then F is always a simple cylinder, hence the arguments above apply. If (4) and (5) hold then F is a vertical degenerate cylinder. Since F must be disjoint from D, we have $\boldsymbol{d}(C, D) \leq 3$. Therefore, we can choose C' to be D.

4.3.3 Model III In this case P_2 must be the highest point of $\boldsymbol{P}$, and $\overline{P_1 P_3}$ must be contained in $\boldsymbol{P}$. Consequently, the union $(P_1 P_2 P_3) \cup (Q_1 Q_2 Q_3)$ projects to a simple cylinder E in X. Let F denote the cylinder in X which is the projection of $(P_3 P_4 Q_0) \cup (Q_3 Q_4 P_0)$. We remark that $\boldsymbol{d}(C, E) = 1$ and $\boldsymbol{d}(C, F) = 2$. It is not difficult to see that the same arguments as the previous cases also allow us to get the desired conclusion. □

4.4 Proof of Theorem 4.1

By Lemmas 4.3 and 4.4, we know that there exist two simple cylinders C' and D' such that

$$\iota(C', D') \leq \iota(C, D) \quad \text{and} \quad \boldsymbol{d}(C, D) \leq \boldsymbol{d}(C', D') + 4.$$

It follows from Lemma 4.5 and Proposition 4.6 that $\boldsymbol{d}(C', D') \leq 3\iota(C', D') + 2$. Therefore

$$\boldsymbol{d}(C, D) \leq 3\iota(C, D) + 6.$$ □

5 Infinite diameter

In this section we prove the following proposition.

Proposition 5.1 *For any $(X, \omega) \in \mathcal{H}(2) \sqcup \mathcal{H}(1,1)$, the diameter of $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X, \omega, f)$ is infinite.*

The geometry of the curve complex is closely related to the Teichmüller space $\mathcal{T}(S)$. Recall that given a simple closed curve γ on S, for any $x \in \mathcal{T}(S)$ the extremal length $\mathrm{Ext}_x(\gamma)$ of γ is defined to be

$$\mathrm{Ext}_x(\gamma) = \sup_h |\gamma^*|_h^2,$$

where h ranges over the set of Riemannian metrics of area one in the conformal class of x, and $|\gamma^*|_h$ is the length of the shortest curve (with respect to h) in the homotopy class of γ. Alternatively, one can define $\mathrm{Ext}_x(\gamma)$ to be the inverse of the largest modulus of an annulus homotopic to γ on S. There is a natural coarse mapping Φ from $\mathcal{T}(S)$ to $\mathcal{C}(S)$ defined as follows: we assign to each $x \in \mathcal{T}(S)$ a curve of minimal x–extremal length on S. It is a well-known fact (see [33, Lemma 2.4]) that there is a universal constant c depending only the topology of S, such that the diameter of the subset of $\mathcal{C}(S)$ consisting of simple curves having minimal x–extremal length is at most c for any $x \in \mathcal{T}(S)$.

Teichmüller geodesics in $\mathcal{T}(S)$ through x are the projections of the lines $a_t \cdot q$, where q is a holomorphic quadratic differential on S equipped with the conformal structure x, and

$$a_t = \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}, \quad t \in \mathbb{R}.$$

It is proven in [33] that if $L_q\colon \mathbb{R} \to \mathcal{T}(S)$ is a Teichmüller geodesic, then $\Phi(L_q(\mathbb{R}))$ is an unparametrized quasigeodesic in $\mathcal{C}(S)$. It may happen that this quasigeodesic has finite diameter.

The curve graph $\mathcal{C}(S)$ has infinite diameter; see [33]. Klarreich [27] shows that the boundary at infinity $\partial_\infty \mathcal{C}(S)$ of $\mathcal{C}(S)$ can be identified with the space of topological minimal foliations $\mathcal{F}_{\min}(S)$ on S. Recall that a foliation on S is minimal if it has no leaf which is a simple closed curve, here we consider foliations up to isotopy and Whitehead moves. A characterization of sequences of curves converging to a foliation in $\partial_\infty \mathcal{C}(S)$ is given by Hamenstädt [16]. It follows from this result that if the vertical foliation of q are minimal then $\Phi \circ L_q([0,\infty))$ is a quasigeodesic of infinite diameter in $\mathcal{C}(S)$; see [17; 18].

Recall that a geometric (nondegenerate) cylinder on a translation surface is modeled by $\mathbb{R} \times (0,h)/((x,y) \sim (x+c,y))$, where $c > 0$ is its circumference and h is its width. Vorobets [50], developing Smillie's ideas in [45], showed the following:

Theorem 5.2 (Smillie and Vorobets) *Given any stratum $\mathcal{H}(\kappa)$ of translation surface, there exists a constant $K > 0$ depending on κ such that, on every translation surface of area one in $\mathcal{H}(\kappa)$, one can find a geometric cylinder of width bounded below by K.*

Proposition 5.1 follows easily from this and the results of Klarreich and Hamenstädt.

Proof of Proposition 5.1 Using the action of $\mathrm{GL}^+(2,\mathbb{R})$, we can always assume that $\mathrm{Area}(X,\omega) = 1$ and the vertical foliation of (X,ω) is minimal. Let $L\colon \mathbb{R} \to \mathcal{T}(S)$ be the Teichmüller geodesic defined by $q = \omega^2$. By the results of Klarreich and Hamenstädt, the quasigeodesic $\Phi \circ L(\mathbb{R}_{>0})$ has infinite diameter.

Denote by $\boldsymbol{d}^{\mathcal{C}}$ the distance in $\mathcal{C}(S)$, and by $\boldsymbol{d}$ the distance in $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$. For any pair (α,β) in $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$, we have $\boldsymbol{d}^{\mathcal{C}}(\alpha,\beta) \le \boldsymbol{d}(\alpha,\beta)$.

For each $t \in \mathbb{R}$, let $(X_t,\omega_t) := a_t \cdot (X,\omega)$. For any $R \in \mathbb{R}_{>0}$ there exist $t_1, t_2 \in (0,+\infty)$ such that $\boldsymbol{d}^{\mathcal{C}}(\Phi \circ L(t_1), \Phi \circ L(t_2)) \ge R$. Let $\alpha_i := \Phi \circ L(t_i)$. By Theorem 5.2 we know that there is a geometric cylinder C_i of width bounded below by K in (X_{t_i},ω_{t_i}). Let β_i be a core curve of C_i.

The extremal length of α_i in X_i is bounded by a universal constant $e_0(S)$; see eg [39, Lemma 2.1]. Thus by definition, the length of the shortest curve α_i^* in the homotopy class of α_i with respect to ω_{t_i} is bounded by $e_0(S)$. Since the width of C_i is at least K, we have $\#\{\alpha_i^* \cap \beta_i\} \leq e_0(S)/K$, which implies that $\iota([\alpha_i],[\beta_i]) \leq e_0(S)/K$. It is well known that the distance in $\mathcal{C}(S)$ is bounded by a linear function of the intersection number; see eg [33, Lemma 2.1] or [6, Lemma 1.1]. Thus there is a constant M depending only on S such that $\boldsymbol{d}^{\mathcal{C}}([\alpha_i],[\beta_i]) \leq M$. Therefore, we have

$$\boldsymbol{d}^{\mathcal{C}}([\beta_1],[\beta_2]) \geq \boldsymbol{d}^{\mathcal{C}}([\alpha_1],[\alpha_2]) - \boldsymbol{d}^{\mathcal{C}}([\alpha_1],[\beta_1]) - \boldsymbol{d}^{\mathcal{C}}([\alpha_2],[\beta_2]) \geq R - 2M.$$

Since $\boldsymbol{d}(C_1, C_2) = \boldsymbol{d}([\beta_1],[\beta_2]) \geq \boldsymbol{d}^{\mathcal{C}}([\beta_1],[\beta_2])$, the proposition follows. □

6 Automorphisms of the cylinder graph

Let $\mathrm{Aff}^+(X,\omega)$ denote the group of affine automorphisms of (X,ω). Recall that elements of $\mathrm{Aff}^+(X,\omega)$ are orientation-preserving homeomorphisms of X that preserve the zero set of ω, and are given by affine maps in local charts of the flat metric out side of this set; see [25; 35]. Note that the derivative of such a map (in local charts associated to the flat metric) is a constant matrix in $\mathrm{SL}(2,\mathbb{R})$. Thus we have a group homomorphism $D\colon \mathrm{Aff}^+(X,\omega) \to \mathrm{SL}(2,\mathbb{R})$ which associates to each element of $\mathrm{Aff}^+(X,\omega)$ its derivative. The image of D in $\mathrm{SL}(2,\mathbb{R})$ is called the *Veech group* of (X,ω) and usually denoted by $\mathrm{SL}(X,\omega)$. Note that the kernel of D is contained in the group $\mathrm{Aut}(X)$ of automorphisms of X, thus must be finite. The group $\mathrm{SL}(X,\omega)$ can also be viewed as the stabilizer of (X,ω) for the action of $\mathrm{SL}(2,\mathbb{R})$.

Given a point $[X,\omega,f] \in \Omega\mathcal{T}_2$, via the marking $f\colon S \to X$, one can identify $\mathrm{Aff}^+(X,\omega)$ with a subgroup of the mapping class group $\mathrm{Mod}(S)$ of S; see [35, Section 5]. An element of $\mathrm{Mod}(S)$ induces naturally an automorphism of the curve graph $\mathcal{C}(S)$. It is a well-known fact that every automorphism of $\mathcal{C}(S)$ arises from an element of $\mathrm{Mod}(S)$; see [24; 31]. Since an affine homeomorphism maps cylinders into cylinders, and saddle connections into saddle connections, it is clear that any element of $\mathrm{Aff}^+(X,\omega)$ induces an automorphism of the subgraph $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$. The aim of this section is to show the following.

Proposition 6.1 *Let ϕ be an element of* $\mathrm{Mod}(S)$ *which preserves the subgraph $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$, that is, $\phi(\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)) \subset \widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$. Then ϕ is induced by an affine automorphism in* $\mathrm{Aff}^+(X,\omega)$. *In particular, ϕ realizes an automorphism of $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$.*

Remark 6.2 Proposition 6.1 is equivalent to the following statement: if $\psi\colon X \to X$ is a homeomorphism such that for any regular simple closed geodesic or degenerate cylinder c, $\psi(c)$ is freely homotopic to the core curves of a cylinder (possibly degenerate) on X, then ψ is isotopic to an affine automorphism of (X,ω).

The proof of this proposition essentially follows from the arguments of [13, Lemma 22]. Before getting into the proof, let us recall some basic notions of Thurston's compactification of the Teichmüller space. Let $\mathcal{MF}(S)$ denote the space of *measured foliations* on S. The space of *projective measured foliations* denoted by $\mathcal{PMF}(S)$ is naturally the quotient of $\mathcal{MF}(S)$ by $\mathbb{R}_+^*$. Thurston showed that $\mathcal{PMF}(S)$ can be identified with the boundary of $\mathcal{T}(S)$. A foliation is *minimal* if none of its leaves is a closed curve. A (measured) foliation is *uniquely ergodic* if it is minimal and there exists a unique transverse measure up to scalar multiplication.

The set of (free homotopy classes of) simple closed curves in S (that is, the vertex set of $\mathcal{C}(S)$) is naturally embedded in $\mathcal{MF}(S)$ with the transverse measure being the counting measure of intersections. The geometric intersection number $\iota(\cdot\,,\,\cdot)$ defined on the set of pairs of simple closed curves extends to a continuous symmetric function $\iota\colon \mathcal{MF}(S)\times\mathcal{MF}(S)\to[0,+\infty)$ which satisfies $\iota(a\lambda,b\mu)=ab\iota(\lambda,\mu)$, for all $a,b\in[0,+\infty)$ and $\lambda,\mu\in\mathcal{MF}(S)$. It has been shown by Thurston that the set

$$\{(0,+\infty)\cdot\alpha \mid \alpha \text{ is a simple closed curve}\}$$

is dense in $\mathcal{MF}(S)$.

Two measured foliations are *topologically equivalent* if the corresponding topological foliations are the same up to isotopy and Whitehead moves.

Proposition 6.3 [44] *If λ is a minimal measured foliation, and $\iota(\lambda,\mu)=0$, then λ and μ are topologically equivalent.*

Measured foliations are a special case of more general objects called *geodesic currents* which were introduced by Bonahon; see [3; 4]. We refer to [13] for an introduction to this concept with more details. While the space of measure foliations is the completion of the set of *simple* closed curves, the space of geodesic currents, denoted by $\mathscr{C}(S)$, can be viewed as the completion of closed curves on S. In particular, we have a continuous extension of the intersection number function ι to $\mathscr{C}(S)\times\mathscr{C}(S)$. A characterization of measured foliations in the space of geodesic currents was given by Bonahon:

Proposition 6.4 [3, Proposition 4.8] *$\mathcal{MF}(S)$ is exactly the set of geodesic currents with zero self-intersection, that is,*

$$\mathcal{MF}(S)=\{\lambda\in\mathscr{C}(S)\mid\iota(\lambda,\lambda)=0\}.$$

We will also need the following important feature of geodesic currents, due to Bonahon:

Proposition 6.5 [4, Proposition 4] *Let α be a geodesic current with the following property: every geodesic in $\widetilde{S}$ transversely meets another geodesic in the support of α. Then the set $\beta\in\mathscr{C}(S)$ such that $\iota(\alpha,\beta)\le 1$ is compact in $\mathscr{C}(S)$.*

Note that if λ is a minimal foliation, then the corresponding geodesic current satisfies the hypothesis of Proposition 6.5.

Every holomorphic 1–form (X,ω) (or more generally every holomorphic quadratic differential) defines naturally two measured foliations on X. The leaves of these foliations are respectively vertical and horizontal geodesic lines with the transverse measures given by $|\mathrm{Re}\,\omega|$ and $|\mathrm{Im}\,\omega|$. It is also a well-known fact that, if λ and μ are two uniquely ergodic measured foliations jointly filling up S, that is, for any $\nu\in\mathcal{MF}(S)$, we have $\iota(\nu,\lambda)+\iota(\nu,\mu)>0$, then there is a unique Teichmüller geodesic g that joins $[\lambda]$ and $[\mu]$, where $[\lambda]$ and $[\mu]$ are the projections of λ and μ in $\mathcal{PMF}(S)$. As a consequence, assume that (X_1,ω_1) and (X_2,ω_2) are two holomorphic 1–forms that both satisfy the following condition: the vertical foliation of ω_i is topologically equivalent to λ, and the horizontal foliation is topologically equivalent to μ. Then there exists a diagonal matrix

$$A=\begin{pmatrix}\mathrm{e}^t & 0\\ 0 & \mathrm{e}^s\end{pmatrix}\in\mathrm{GL}^+(2,\mathbb{R})$$

such that $(X_2,\omega_2)=A\cdot(X_1,\omega_1)$.

Proof of Proposition 6.1 By definition, $\phi\cdot[X,\omega,f]=[X,\omega,f\circ\phi^{-1}]$. Equivalently, we can write $\phi\cdot[X,\omega,f]=[X',\omega',f']$, where $f'\colon S\to X'$ satisfies the following condition: there exists an isomorphism $\hat{\phi}\colon X'\to X$ such that $\hat{\phi}^*\omega=\omega'$, and $f\circ\phi^{-1}$ is isotopic to $\hat{\phi}\circ f'$. Using this identification, we have

$$\hat{\mathcal{C}}_{\mathrm{cyl}}(X',\omega',f')=\phi(\hat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)).$$

Thus, by assumption, we have $\hat{\mathcal{C}}_{\mathrm{cyl}}(X',\omega',f')\subset\hat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$.

Via the maps $f\colon S\to X$, $f'\colon S\to X'$, for any direction $\theta\in\mathbb{RP}^1$, we denote by ν^θ and ν'^θ the measured foliations on S corresponding to the vertical foliations defined by $\mathrm{e}^{i\theta}\omega$ and $\mathrm{e}^{i\theta}\omega'$, respectively. The leaves of ν^θ and ν'^θ are geodesic lines in the direction of $\pm(\pi/2-\theta)$. Observe that if $\{\theta_k\}$ is a sequence of angles converging to θ, then ν^{θ_k} converges to ν^θ, and ν'^{θ_k} converges to ν'^θ in $\mathcal{MF}(S)$.

It follows from a classical result of Kerckhoff, Masur and Smillie [26] that for almost all directions $\theta\in\mathbb{RP}^1$, ν^θ and ν'^θ are uniquely ergodic. Set

$$\mathcal{UE}(\omega):=\{[\nu^\theta]\in\mathcal{PMF}(S)\mid\nu^\theta\text{ is uniquely ergodic, }\theta\in\mathbb{RP}^1\}\subset\mathcal{PMF}(S).$$

We define $\mathcal{UE}(\omega')$ in the same manner.

We will show that $\mathcal{UE}(\omega')\subset\mathcal{UE}(\omega)$. Let θ be a direction such that ν'^θ is uniquely ergodic. Without loss of generality, we can assume that $\mathrm{Area}(X)=1$. For any $t\in\mathbb{R}$, set

$$(X_t'^{\theta}, \omega_t'^{\theta}) := \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix} \cdot (X', \mathrm{e}^{i\theta}\omega').$$

It follows from Theorem 5.2 that there exists a constant $R > 0$ such that for any $t \in \mathbb{R}$, $X_t'^{\theta}$ has a cylinder C_t' with circumference bounded by R. Let c_t' be a core curve of C_t', and consider the sequence $\{c_k'\}_{k\in\mathbb{N}}$. By definition, the length of c_k' with respect to $\omega_k'^{\theta}$, denoted by $\ell_{\omega_k'^{\theta}}(c_k')$, is bounded by R. Thus we have

$$\iota(\mathrm{e}^k \nu'^{\theta}, c_k') = \mathrm{e}^k \iota(\nu'^{\theta}, c_k') \leq \ell_{\omega_k'^{\theta}}(c_k') \leq R.$$

It follows that

$$\lim_{k\to+\infty} \iota(\nu'^{\theta}, c_k') = 0.$$

By Proposition 6.5, up to extracting a subsequence, we can assume that $\{c_k'\}$ converges to a geodesic current $\mu' \in \mathscr{C}(S)$. Since c_k' has zero self-intersection, it follows that $\iota(\mu', \mu') = 0$, hence $\mu' \in \mathcal{MF}(S)$ by Proposition 6.4. By continuity of ι, we have $\iota(\nu'^{\theta}, \mu') = 0$. Since ν'^{θ} is uniquely ergodic (so, in particular, it is minimal), it follows from Proposition 6.3 that μ' and ν'^{θ} are topologically equivalent. Hence μ' is also uniquely ergodic.

By definition, $\{c_k'\}_{k\in\mathbb{N}}$ are vertices of $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X', \omega', f')$. By assumption, we have $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X', \omega', f') \subset \widehat{\mathcal{C}}_{\mathrm{cyl}}(X, \omega, f)$. Therefore, $\{c_k'\}_{k\in\mathbb{N}}$ are also vertices of $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X, \omega, f)$, which means that c_k' is freely homotopic to either a simple closed geodesic or a degenerate cylinder in X. In particular, we see that each c_k' has a well-defined direction $\theta_k \in \mathbb{RP}^1$ with respect to ω. Again, by extracting a subsequence, we can assume that $\{\theta_k\}$ converges to $\hat{\theta}$. Thus, $\{\nu^{\theta_k}\}$ converges to $\nu^{\hat{\theta}}$. Since we have $\iota(\nu^{\theta_k}, c_k') = 0$, by continuity, it follows that $\iota(\nu^{\hat{\theta}}, \mu') = 0$. Since μ' is uniquely ergodic, so is $\nu^{\hat{\theta}}$, and we have $[\nu'^{\theta}] = [\mu'] = [\nu^{\hat{\theta}}] \in \mathcal{PMF}(S)$. We can then conclude that $\mathcal{UE}(\omega') \subset \mathcal{UE}(\omega)$.

Now pick a pair of projective uniquely ergodic measured foliations $([\lambda], [\mu]) \in \mathcal{UE}(\omega') \subset \mathcal{UE}(\omega)$ that jointly fill up S. There exist two matrices M and M' such that the vertical and horizontal foliations of $M \cdot [X, \omega, f]$ and $M' \cdot [X', \omega', f']$ are topologically equivalent to λ and μ, respectively. Since there is a unique Teichmüller geodesic joining $[\lambda]$ and $[\mu]$, there exists a diagonal matrix $A \in \mathrm{GL}^+(2, \mathbb{R})$ such that $M' \cdot [X', \omega', f'] = AM \cdot [X, \omega, f]$, implying that ϕ is represented by an affine automorphism of (X, ω). □

Remark 6.6 This proof actually works for translation surfaces in any genus with $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ replaced by the subgraph consisting of nondegenerate cylinders.

7 Hyperbolicity

A translation surface (X,ω) is said to be *completely periodic* (in the sense of Calta) if the direction of any nondegenerate cylinder in X is periodic, which means that whenever we find a simple closed geodesic on X, the surface decomposes as union of (finitely many) cylinders in the same direction; see [10; 11]. It follows from [10] and [38] that, in $\mathcal{H}(2)$, a surface is completely periodic if and only if it is a Veech surface. In $\mathcal{H}(1,1)$, a surface is completely periodic if and only if it is *an eigenform for a real multiplication of a quadratic order*. In particular, there are completely periodic surfaces in $\mathcal{H}(1,1)$ that are not Veech surfaces.

Let us denote by $\mathcal{E}_D$, where D is a natural number such that $D \equiv 0$ or $1 \mod 4$, the locus of eigenforms for the real multiplication by the quadratic order $\mathcal{O}_D$ in $\Omega\mathcal{M}_2$. Each $\mathcal{E}_D$ is a 3–dimensional irreducible (algebraic) subvariety of $\Omega\mathcal{M}_2$ which is invariant by the $\mathrm{SL}(2,\mathbb{R})$–action. The set of eigenforms in $\Omega\mathcal{M}_2$ is then (see [38])

$$\mathcal{E} = \bigcup_{D\equiv 0,1 \bmod 4} \mathcal{E}_D.$$

Even though complete periodicity is initially defined for directions of nondegenerate cylinders, it is not difficult to show that in the case of genus two, this property actually implies the periodicity for directions of degenerate cylinders; see Lemma B.1. Alternatively, one can also use the argument in [51] to get the same result in more general contexts; see [52]. In what follows, by a *completely periodic* surface we will mean a surface for which the direction of any cylinder (degenerate or not) is periodic. By Lemma B.1, this apparently new definition agrees with the usual one by Calta. Our goal in this section is to show the following theorem:

Theorem 7.1 *If $(X,\omega) \in \mathcal{H}(2) \sqcup \mathcal{H}(1,1)$ is completely periodic then $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$ is Gromov hyperbolic.*

To prove this, we will use Masur and Schleimer's hyperbolicity criterion (see also [7, Proposition 3.1] and [15]), and follow Bowditch's approach in [6].

Theorem 7.2 (Masur and Schleimer [34, Theorem 3.13]) *Suppose that $\mathcal{X}$ is a graph with all edge lengths equal to one. Then $\mathcal{X}$ is Gromov hyperbolic if there is a constant $M \geq 0$ such that for all unordered pairs of vertices x,y in $\mathcal{X}^0$, there is a connected subgraph $g_{x,y}$ containing x and y with the following properties:*

- (local) *If $d_{\mathcal{X}(x,y)} \leq 1$ then $g_{x,y}$ has diameter at most M.*
- (slim triangle) *For any $x,y,z \in \mathcal{X}^0$, the subgraph $g_{x,y}$ is contained in the M–neighborhood of $g_{x,z} \cup g_{z,y}$.*

Let us fix $[X,\omega,f]\in\Omega\mathcal{T}_2$, where $(X,\omega)\in\mathcal{E}$ and $\mathrm{Area}(X,\omega)=1$. We will write $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ instead of $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$. We know from Corollary 4.2 that $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ is connected, and by definition the edges of $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ have length equal to one. Let K be the constant in Theorem 5.2, and C be a cylinder of width bounded below by K in X. Note that the circumference of C is bounded above by $1/K$. Recall that from Theorem 4.1, we know that there are two constants K_1, K_2 such that for any pair of cylinders C, D in X, we have

$$\boldsymbol{d}(C,D)\le K_1\iota(C,D)+K_2,$$

where $\boldsymbol{d}$ is the distance in $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$, and $\iota(C,D)$ is the number of intersections of a core curve of C and a core curve of D.

7.1 Construction of subgraphs connecting pairs of vertices

We will now construct for each unordered pair of cylinders C, D a subgraph $\widehat{\mathcal{L}}_{C,D}$ of $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ that satisfies the conditions of Theorem 7.2 with a constant M which will be derived along the way.

Let us first consider the case that C and D are parallel. If C or D is nondegenerate then $\iota(C,D)=0$ hence $\boldsymbol{d}(C,D)=1$, which means that C and D are connected by an edge in $\widehat{\mathcal{C}}_{\mathrm{cyl}}$. We define $\widehat{\mathcal{L}}_{C,D}$ to be this edge. If both C and D are degenerate then it may happen that $\iota(C,D)>0$. Since (X,ω) is completely periodic, there is a nondegenerate cylinder E parallel to C and D. Since $\iota(C,E)=\iota(D,E)=0$, there are in $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ two edges connecting E to C and to D. In this case, we define $\widehat{\mathcal{L}}_{C,D}$ to be the union of these two edges.

Assume from now on that C and D are not parallel. By applying an appropriate element of $\mathrm{SL}(2,\mathbb{R})$, we can assume that C is horizontal, D is vertical, and C and D have the same circumference. For any $t\in\mathbb{R}$, set

$$a_t=\begin{pmatrix} e^t & 0\\ 0 & e^{-t}\end{pmatrix}\quad\text{and}\quad (X_t,\omega_t)=a_t\cdot(X,\omega).$$

For any saddle connection s in (X,ω), we will denote by $\ell_t(s)$ its Euclidean length in (X_t,ω_t). If E is a cylinder in (X,ω), then $c_t(E)$ and $w_t(E)$ are respectively its circumference and width in (X_t,ω_t).

For any $R\in\mathbb{R}_{>0}$, let $\mathcal{L}^*_{C,D}(t,R)$ denote the set of cylinders (possibly degenerate) of circumference bounded above by R in (X_t,ω_t). Note that this set is finite. Let us choose a constant L_1 such that

$$L_1>\max\{1/K,9\}, \tag{3}$$

and define

$$\mathcal{L}^*_{C,D}(L_1) = \bigcup_{t\in\mathbb{R}} \mathcal{L}^*_{C,D}(t, L_1).$$

We regard $\mathcal{L}^*_{C,D}(t, R)$ and $\mathcal{L}^*_{C,D}(L_1)$ as subsets of $\widehat{\mathcal{C}}^{(0)}_{\mathrm{cyl}}$. Observe that $\mathcal{L}^*_{C,D}(t, L_1)$ contains C when t tends to $-\infty$, and contains D when t tends to $+\infty$; therefore $\mathcal{L}^*_{C,D}$ contains C and D.

For each $t \in \mathbb{R}$, consider now the set $\mathcal{L}^*_{C,D}(t, 2L_1)$. From Theorem 5.2, $\mathcal{L}^*_{C,D}(t, 2L_1)$ contains a vertex corresponding to a cylinder $C_{0,t}$ of width bounded below by K. Set

$$M_1 := \max\{2(2K_1L_1/K + K_2), 2\}. \tag{4}$$

Then we have the following lemma:

Lemma 7.3 *As subset of $\widehat{\mathcal{C}}_{\mathrm{cyl}}$, $\mathcal{L}^*_{C,D}(t, 2L_1)$ has diameter bounded by M_1.*

Proof Let E be a cylinder in $\mathcal{L}^*_{C,D}(t, 2L_1)$. If $\iota(E, C_{0,t}) = 0$, then we have $\boldsymbol{d}(C_{0,t}, E) = 1$. Otherwise we have $K\iota(E, C_{0,t}) \le \ell_t(E) \le 2L_1$. Hence, from (1) we get

$$\boldsymbol{d}(C_{0,t}, E) \le 2K_1L_1/K + K_2,$$

and the lemma follows. □

Moreover, we have the following lemma as well:

Lemma 7.4 *Assume that the surface (X, ω) admits cylinder decompositions in both vertical and horizontal directions. Then there exists a constant $T > 0$ such that the following hold:*

- *If $t > T$, then $\mathcal{L}^*_{C,D}(t, 2L_1)$ only contains the vertical cylinders in (X, ω) and $\mathcal{L}^*_{C,D}(t, 2L_1)$ has diameter at most 2.*
- *If $t < -T$, then $\mathcal{L}^*_{C,D}(t, 2L_1)$ only contains the horizontal cylinders in (X, ω) and $\mathcal{L}^*_{C,D}(t, 2L_1)$ has diameter at most 2.*

Proof We only give the proof of the first assertion as the second one follows from the same argument. By assumption, X decomposes into the union of some nondegenerate vertical cylinders $D_1, \dots, D_k$. Let $w_t(D_i)$ denote the width of D_i in (X_t, ω_t). Let $w_t = \min\{w_t(D_i) \mid i = 1, \dots, k\}$. A nonvertical cylinder must cross one of D_i, thus its circumference is bounded below by w_t in (X_t, ω_t). Since we have $w_t = e^t w_0$; if t is large enough, any nonvertical cylinder has circumference at least $2L_1$ in (X_t, ω_t). Hence $\mathcal{L}^*_{C,D}(t, 2L_1)$ only contains the vertical cylinders. Since any vertical cylinder is of distance one from D_1 in $\widehat{\mathcal{C}}_{\mathrm{cyl}}$, $\mathcal{L}^*_{C,D}(t, 2L_1)$ has diameter at most two. □

Lemma 7.5 *If $t-\log(2) \le t' \le t+\log(2)$ then $\mathcal{L}^*_{C,D}(t',R) \subset \mathcal{L}^*_{C,D}(t,2R)$ for any $R \in \mathbb{R}_{>0}$. In particular, $C_{0,t'} \in \mathcal{L}^*_{C,D}(t,2L_1)$.*

Proof Let s be a saddle connection or a regular geodesic in $(X_{t'},\omega_{t'})$. Let $x+iy$ be the period of s in $(X_{t'},\omega_{t'})$. Note that $(X_t,\omega_t) = a_{t-t'}\cdot(X_{t'},\omega_{t'})$. Thus the period of s in (X_t,ω_t) is $(\mathrm{e}^{t-t'}x, \mathrm{e}^{t'-t}y)$. Therefore,

$$\ell_t(s) = \sqrt{\mathrm{e}^{2(t-t')}x^2 + \mathrm{e}^{2(t'-t)}y^2} \le 2\sqrt{x^2+y^2} = 2\ell_{t'}(s). \qquad \square$$

Set

$$\bar{\mathcal{L}}_{C,D}(2L_1) := \bigcup_{k\in\mathbb{Z}} \mathcal{L}^*_{C,D}(k\log(2),2L_1) \subset \widehat{\mathcal{C}}^{(0)}_{\mathrm{cyl}}.$$

It follows from Lemma 7.4 that if $n \in \mathbb{N}$ is large enough, then for any $m > n$, $\mathcal{L}^*_{C,D}(m,L_1) = \mathcal{L}^*_{C,D}(n,2L_1)$, and $\mathcal{L}^*_{C,D}(-m,2L_1) = \mathcal{L}^*_{C,D}(-n,2L_1)$. Therefore, the set $\bar{\mathcal{L}}_{C,D}(2L_1)$ is actually finite. For each unordered pair (x,y) of vertices in $\bar{\mathcal{L}}_{C,D}(2L_1)$, let $\Gamma(x,y)$ be a path of minimal length in $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ joining x to y. Set

$$\widehat{\mathcal{L}}_{C,D}(2L_1) = \bigcup_{x,y\in\bar{\mathcal{L}}_{C,D}(2L_1)} \Gamma(x,y).$$

As a direct consequence of Lemma 7.5, we get the following:

Corollary 7.6 (a) *If $x \in \mathcal{L}^*_{C,D}(t,2L_1)$ and $y \in \mathcal{L}^*_{C,D}(t',2L_1)$, then $\boldsymbol{d}(x,y) \le M_1(2+|t-t'|/\log(2))$.*

(b) *The set $\mathcal{L}^*_{C,D}(L_1)$ is contained in $\bar{\mathcal{L}}_{C,D}(2L_1)$ and $\bar{\mathcal{L}}_{C,D}(2L_1)$ is contained in the M_1–neighborhood of $\mathcal{L}^*_{C,D}(L_1)$.*

(c) *For any pair of vertices $(x,y) \in \mathcal{L}^*_{C,D}(L_1)\times\mathcal{L}^*_{C,D}(L_1)$, there is a path $\Gamma(x,y)$ in $\widehat{\mathcal{L}}_{C,D}(2L_1)$ from x to y of length equal to $\boldsymbol{d}(x,y)$.*

7.2 The local property for $\widehat{\mathcal{L}}_{C,D}$

We will now show that the subgraphs $\widehat{\mathcal{L}}_{C,D}(2L_1)$ constructed above satisfy the local property of Theorem 7.2.

Proposition 7.7 *There exists a constant M_2 such that if $(X,\omega) \in \mathcal{E}$ then for any pair of cylinders C, D in (X,ω) such that $\iota(C,D)=0$, we have* $\operatorname{diam}\widehat{\mathcal{L}}_{C,D}(2L_1) \le M_2$.

To prove this proposition, we make use of an elementary result on slit tori, Lemma B.3, and the fact that if C and D are not parallel, then there always exists a splitting of X into two subsurfaces, each of which contains one of C and D. Those auxiliary results are proved in Appendix B. The main technical difficulties arise when we have to deal with degenerate cylinders.

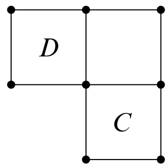

Figure 8: Disjoint simple cylinders on surfaces in $\mathcal{H}(2)$

Proof We split this proof into two cases: $(X,\omega)\in\mathcal{H}(2)$ and $(X,\omega)\in\mathcal{H}(1,1)$.

Case $(X,\omega)\in\mathcal{H}(2)$ If C and D are parallel then $\hat{\mathcal{L}}_{C,D}(2L_1)$ has diameter bounded by 2 and we have nothing to prove. Suppose from now on that C is horizontal, D is vertical, C and D have the same circumference equal to ℓ, and $\hat{\mathcal{L}}_{C,D}(2L_1)$ is the graph constructed above. Note that in this case (X,ω) is a Veech surface, thus both horizontal and vertical directions are periodic.

Case 1 One of C or D is nondegenerate. Assume that C is nondegenerate. Let c be a core curve of C and d a core curve of D. Note that c is a regular simple closed geodesic. By Lemma 3.6, the condition $\iota(C,D)=0$ implies that $c\cap d=\varnothing$. Clearly, C cannot fill X. If C is not simple then the complement of $\bar{C}$ is a horizontal simple cylinder C' whose boundary is contained in the boundary of C. Since D is disjoint from C, it must be contained in C'. But this is impossible since C' is horizontal and D is vertical. Therefore, C must be a simple cylinder.

The complement of C is then a slit torus with the slit corresponding to the boundary of C. We remark that a core curve of D must be disjoint from the interior of the slit, otherwise it would cross C entirely. Thus, we have in the slit torus an embedded square bounded by the boundary of D and the slit (which is actually the boundary of C); see Figure 8. By assumption, the length of the sides of this square is ℓ. Since this square has area less than one, we must have $\ell<1$. Therefore $C\in\mathcal{L}^*_{C,D}(t,L_1)$ for all $t\leq 0$, and $D\in\mathcal{L}^*_{C,D}(t,L_1)$ for any $t\geq 0$. Hence any $E\in\bar{\mathcal{L}}_{C,D}(2L_1)$ is of distance at most M_1 from C or from D. Thus $\operatorname{diam}\hat{\mathcal{L}}_{C,D}(2L_1)\leq 2M_1+1$.

Case 2 Both of C and D are degenerate. From Lemma 3.4, for any $\epsilon>0$ small enough, we can deform (X,ω) into another surface (X',ω') such that

- C corresponds to a simple horizontal cylinder C' in X' of width ϵ,
- D corresponds to a vertical cylinder in X'.

Since $\iota(C',D')=\iota(C,D)=0$, it follows from Lemma 3.6 that D' must be disjoint from C'. It follows in particular that D and D' have the same circumference ℓ. By construction C' has the same circumference as C, and

$$\operatorname{Area}(X',\omega')=\operatorname{Area}(X,\omega)+\epsilon\ell=1+\epsilon\ell.$$

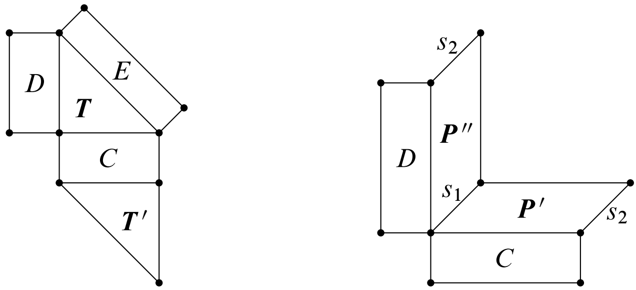

Figure 9: Disjoint cylinders on surfaces in $\mathcal{H}(1,1)$: one of C and D is simple

Applying the same arguments as above to (X',ω'), we see that X' contains an embedded square of size ℓ disjoint from C'. Therefore we have $\ell^2 < 1+\epsilon\ell$. Since ϵ can be chosen arbitrarily, we derive that $\ell \leq 1$. We can then conclude by the same arguments as the previous case.

Case $(X,\omega) \in \mathcal{H}(1,1)$ Again, we only have to consider the case that C and D are not parallel. Thus we can assume that C is horizontal and D is vertical. We first choose a positive real number $L > \sqrt{2}$ such that

$$L_1 \geq 3f(\sqrt{2}L), \tag{5}$$

where $f(x) = \sqrt{x^2 + 1/x^2}$; see Lemma B.3.

Case 1 One of C and D is a simple cylinder. By Lemma B.2, we need to consider two cases (see Figure 9):

(i) There is a simple cylinder E disjoint from $C \cup D$ and the complement of $C \cup D \cup E$ is the union of two triangles $\boldsymbol{T}, \boldsymbol{T}'$; see Figure 9 (left). Since we have

$$\mathrm{Area}(\boldsymbol{T}) + \mathrm{Area}(\boldsymbol{T}') = \ell^2 < \mathrm{Area}(X,\omega) = 1,$$

it follows that $\ell < 1$. Hence we can use the same argument as in the case $(X,\omega) \in \mathcal{H}(2)$ to conclude that $\operatorname{diam} \widehat{\mathcal{L}}_{C,D}(2L_1) \leq 2M_1 + 1$.

(ii) There is a pair of homologous saddle connections s_1, s_2 that decompose X into a connected sum of two slit tori, (X',ω',s') containing C and (X'',ω'',s'') containing D; see Figure 9 (right).

By construction, the complement of C in X' is an embedded parallelogram $\boldsymbol{P}'$ bounded by s_1, s_2 and the boundary of C. Similarly, the complement of D in X'' is also an embedded parallelogram $\boldsymbol{P}''$ bounded by s_1, s_2 and the boundary of D. If $\ell \leq 1$ then we can conclude using the argument above. Suppose that we have $\ell \geq 1$.

Let $\omega(s_i)=x+iy$. Since we have $\mathrm{Area}(\boldsymbol{P}')=|y|\ell$, and $\mathrm{Area}(\boldsymbol{P}'')=|x|\ell$, it follows that

$$\max\{|x|,|y|\}\leq 1/\ell\leq 1 \quad\text{and}\quad |s_i|=\sqrt{x^2+y^2}\leq\sqrt{2}/\ell\leq\sqrt{2}.$$

Set $A_1=\mathrm{Area}(X',\omega')$, $A_2=\mathrm{Area}(X'',\omega'')$; we have $A_1+A_2=1$. Without loss of generality, let us suppose that $A_1\geq\frac{1}{2}$. For any $t\in\mathbb{R}$, the period of s_i in (X_t,ω_t) is $(\mathrm{e}^t x,\mathrm{e}^{-t}y)$. Let (X'_t,ω'_t,s'_t) be the slit torus corresponding to (X',ω',s') in (X_t,ω_t). Recall that we have chosen $L>\sqrt{2}$ and L_1 satisfies (5). Let us choose a positive real number $L'\geq 1$ such that

$$L\geq\sqrt{L'^2+1}.$$

- For $0\leq t\leq\log(\ell L')$, we have $e^t|x|\leq L'$ and $\mathrm{e}^{-t}|y|\leq|y|\leq 1$, thus
$$\ell_t(s_1)\leq\sqrt{L'^2+1}\leq L.$$
Rescaling (X'_t,ω'_t,s'_t) by $1/\sqrt{A_1}$, we get a torus of area one with a slit of length bounded by $\sqrt{2}L$. Using Lemma B.3, we see that there exists in $(1/\sqrt{A_1})\cdot X'_t$ a cylinder E'_t disjoint from the slit of circumference bounded by L_1. Note that in X'_t, the circumference of E'_t is at most $\sqrt{A_1}L_1\leq L_1$. We have $\boldsymbol{d}(D,E'_t)=1$ and $E'_t\in\mathcal{L}^*_{C,D}(t,2L_1)$. Thus for any $E\in\mathcal{L}^*_{C,D}(t,2L_1)$ we have $\boldsymbol{d}(D,E)\leq M_1+1$.
- For $-\log(\ell L')\leq t\leq 0$, we have $\mathrm{e}^t|x|\leq|x|\leq 1$ and $\mathrm{e}^{-t}|y|\leq L'$, thus
$$\ell_t(s_i)\leq\sqrt{L'^2+1}\leq L.$$
The same argument as the previous case then shows that $\boldsymbol{d}(D,E)\leq M_1+1$, for any $E\in\mathcal{L}^*_{C,D}(t,2L_1)$.
- For $t\geq\log(\ell L')$, we have $\ell_t(D)=\mathrm{e}^{-t}\ell\leq 1/L'\leq 1\leq 2L_1$. Thus D is in $\mathcal{L}^*_{C,D}(t,2L_1)$, which implies that $\boldsymbol{d}(D,E)\leq M_1$ for any $E\in\mathcal{L}^*_{C,D}(t,2L_1)$.
- For $t\leq-\log(\ell L')$, we have $\ell_t(C)\leq 1/L'\leq 2L_1$, so for any $E\in\mathcal{L}^*_{C,D}(t,2L_1)$, $\boldsymbol{d}(C,E)\leq M_1$, which implies that $\boldsymbol{d}(D,E)\leq M_1+1$.

We can then conclude that for any $t\in\mathbb{R}$, and any $E\in\mathcal{L}^*_{C,D}(t,2L_1)$, we have $\boldsymbol{d}(D,E)\leq M_1+1$. Hence $\operatorname{diam}\widehat{\mathcal{L}}_{C,D}\leq 2(M_1+1)$.

Case 2 One of C,D is nondegenerate and not simple. Without loss of generality, we can assume that C is neither simple nor degenerate. Lemma 3.6 implies that D is disjoint from C. Since C is not simple, the complement of $\overline{C}$ is either (a) empty, (b) a horizontal simple cylinder, (c) the union of two simple horizontal cylinders, or (d) another horizontal cylinder whose closure is a slit torus. Since there exists a vertical cylinder disjoint from C (namely D), only (d) can occur. In this case, there is a pair of horizontal homologous saddle connection $\{s_1,s_2\}$ contained in the boundary of C that

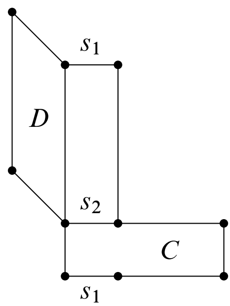

Figure 10: Disjoint cylinders on surfaces in $\mathcal{H}(1,1)$; C is neither simple nor degenerate.

decompose (X,ω) into the connected sum of two slit tori. Let (X',ω',s') be the slit torus which is the closure of C, and (X'',ω'',s'') be the other one that contains D; see Figure 10.

Let $x=|s_1|=|s_2|$. Observe that X'' contains a rectangle bounded by s_1,s_2 and the saddle connections bordering D. Therefore we have $x\ell\leq 1$, equivalently $0\leq x\leq 1/\ell$. By the same arguments as the previous case, we also get $\operatorname{diam}\widehat{\mathcal{L}}_{C,D}\leq 2(M+1)$.

Case 3 One of C and D is degenerate. Let us assume that C is degenerate. Using Lemma 3.4, we can find a family $(X_t,\omega_t), t\in[0,\epsilon)$, of surfaces in $\mathcal{H}(1,1)$ that are deformations of (X,ω), such that C corresponds to a simple horizontal cylinder C_t on X_t, for $t>0$, which has the same circumference. Note that the width of C_t is t. Therefore, $\operatorname{Area}(X_t,\omega_t)=\operatorname{Area}(X,\omega)+t\ell$.

By construction, D corresponds to a cylinder D_t on X_t which is disjoint from C_t (since we have $\iota(C_t,D_t)=\iota(C,D)=0$). By Lemma B.2 we know that either (i) (X_t,ω_t) contains two embedded triangles $\boldsymbol{T},\boldsymbol{T}'$ disjoint from C_t and D_t, or (ii) there is a splitting of (X_t,ω_t) into two slit tori (X'_t,ω'_t,s'_t) and (X''_t,ω''_t,s''_t) such that $C_t\subset X'_t$ and $D_t\subset X''_t$.

If (i) occurs, then we have $\operatorname{Area}(\boldsymbol{T})=\operatorname{Area}(\boldsymbol{T}')=\frac{1}{2}\ell^2\leq\frac{1}{2}$, which implies that $\ell\leq 1$. If (ii) occurs, then since the slits (s' and s'') are disjoint from C_t, they persist as we collapse C_t to get back (X,ω). Thus, we have the same splitting on (X,ω). In conclusion, we can use the same arguments as in Case 1 to handle this case. The proof of Proposition 7.7 is now complete. □

7.3 The slim triangle property for $\widehat{\mathcal{L}}_{C,D}$

We now prove that the subgraphs $\widehat{\mathcal{L}}_{C,D}(2L_1)$ satisfy the slim triangle property of Theorem 7.2. The idea of the proof can found in [6, Lemma 4.4]. To lighten notation, in what follows we will write $\widehat{\mathcal{L}}_{C,D}$ instead of $\widehat{\mathcal{L}}_{C,D}(2L_1)$.

Proposition 7.8 *There exists a constant M_3 such that for any triple of cylinders $\{C, D, E\}$ in (X,ω), we have that $\widehat{\mathcal{L}}_{C,D}$ is contained in the M_3–neighborhood of $\widehat{\mathcal{L}}_{C,E} \cup \widehat{\mathcal{L}}_{E,D}$ in $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$.*

Proof If C and D are parallel then $\widehat{\mathcal{L}}_{C,D}$ is contained in the 2–neighborhood of $\widehat{\mathcal{L}}_{C,E} \cup \widehat{\mathcal{L}}_{D,E}$. From now on we assume that C and D are not parallel.

By Corollary 7.6, we only need to show that $\mathcal{L}^*_{C,D}(L_1)$ is contained in the M_3–neighborhood of $\mathcal{L}^*_{C,E}(L_1) \cup \mathcal{L}^*_{E,D}(L_1)$. To define $\overline{\mathcal{L}}_{C,D}(2L_1)$ and $\widehat{\mathcal{L}}_{C,D}(2L_1)$ one needs to specify an origin for the time t by the condition that the circumferences of C and D are equal. On the other hand to define $\mathcal{L}^*_{C,D}(L_1)$, this normalization is not required. If E is parallel to C then $\mathcal{L}^*_{C,D}(L_1) = \mathcal{L}^*_{E,D}(L_1)$, and if E is parallel to D then $\mathcal{L}^*_{C,D}(L_1) = \mathcal{L}^*_{C,E}(L_1)$. In both of these cases we have nothing to prove.

Let us now assume that E is neither parallel to C nor to D. We can then renormalize (using $\mathrm{SL}(2,\mathbb{R})$) such that C is horizontal, D is vertical, and E has slope equal to 1. Recall that for any $t \in \mathbb{R}$, $(X_t,\omega_t) = a_t \cdot (X,\omega)$, $C_{0,t}$ is a cylinder of width bounded below by K in (X_t,ω_t), and the constant L_1 is chosen so that $L_1 > 1/K$; see (3).

Claim *If $t \le 0$ then $C_{0,t}$ is contained in the M_1–neighborhood of $\mathcal{L}^*_{C,E}(L_1)$.*

Proof of the claim Since (X,ω) is completely periodic, it decomposes into cylinders in both directions of C and E. Let us denote by $C = C_1,\dots,C_m$ the horizontal cylinders, and by $E = E_1,\dots,E_n$ the cylinders in the direction of E. As usual we denote by $\ell_t(C_i)$ (resp. $\ell_t(E_j)$) the circumference of C_i (resp. of E_j) in (X_t,ω_t). Let $u_i(t)$ be the width of C_i, and $v_j(t)$ be the width of E_j in (X_t,ω_t). We have

$$\begin{aligned} \ell_t(C_i) &= \mathrm{e}^t \ell(C_i), & u_i(t) &= \mathrm{e}^{-t} u_i, \\ \ell_t(E_j) &= \sqrt{\cosh(2t)}\,\ell(E_j), & v_j(t) &= \frac{v_j}{\sqrt{\cosh(2t)}}. \end{aligned}$$

Since (X,ω) has area 1, we also have

$$1 = \sum u_i \ell(C_i) = \sum v_j \ell(E_j). \tag{6}$$

Let x_j (resp. y_i) be the intersection number of a core curve of $C_{0,t}$ and a core curve of E_j (resp. of C_i). Since the circumference of $C_{0,t}$ is bounded by $1/K < L_1$, we have

$$\sum y_i u_i(t) = \mathrm{e}^{-t} \sum y_i u_i \le \ell(C_{0,t}) \le L_1 \quad \Rightarrow \quad \sum y_i u_i \le \mathrm{e}^t L_1. \tag{7}$$

Since the width of $C_{0,t}$ is bounded below by K, $x_j K \le \ell_t(E_j) = \sqrt{\cosh(2t)}\,\ell(E_j)$. Since $t \le 0$, it follows that

$$x_j \le \frac{\sqrt{\cosh(2t)}}{K} \ell(E_j) \le \frac{\mathrm{e}^{-t}}{K} \ell(E_j). \tag{8}$$

Let $(X',\omega') := U\cdot(X,\omega)$, where $U=\left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right)$. Let $\ell'(C_i)$ and u_i' (resp. $\ell'(E_j)$ and v_j') be the circumference and the width of C_i (resp. of E_j) in (X',ω'). Note that C_i is horizontal, and E_j is vertical in (X',ω'). Thus, $\ell'(C_i)=\ell(C_i)$, $u_i'=u_i$, and $\ell'(E_j)=\ell(E_j)/\sqrt{2}$, $v_j'=\sqrt{2}v_j$.

For any $s\in\mathbb{R}$, let $(X_s',\omega_s') := a_s\cdot(X',\omega')$. Let $\ell_s'(C_i)$ and $u_i'(s)$ (resp. $\ell_s'(E_j)$ and $v_j'(s)$) be the circumference and the width of C_i (resp. of E_j) in (X_s',ω_s').

Let $x+iy$ be the period of the core curves of $C_{0,t}$ in (X_s',ω_s'). From (8) we get

$$|x|=\sum x_j v_j'(s)=\mathrm{e}^s\sum x_j v_j'\le e^s\frac{\sqrt{2}\mathrm{e}^{-t}}{K}\sum \ell(E_j)v_j=\frac{\sqrt{2}\mathrm{e}^{s-t}}{K}. \tag{9}$$

From (7), we get

$$|y|=\sum y_i u_i'(s)=\mathrm{e}^{-s}\sum y_i u_i\le \mathrm{e}^{t-s}L_1. \tag{10}$$

Thus for $s=t$, the circumference of $C_{0,t}$ in (X_s',ω_s') is at most $\sqrt{3}L_1<2L_1$. Let ${C'}_{0,s}$ be a cylinder of width bounded below by K in (X_s',ω_s'). By Lemma 7.3, we have $\boldsymbol{d}({C'}_{0,s},C_{0,t})\le M_1$ which means that $C_{0,t}$ is contained in the M_1–neighborhood of $\mathcal{L}^*_{C,E}(L_1)$. □

It follows immediately from the claim that $\mathcal{L}^*_{C,D}(t,L_1)$ is contained in the $2M_1$–neighborhood of $\mathcal{L}^*_{C,E}(L_1)$ if $t\le 0$. By similar arguments, one can also show that $\mathcal{L}^*_{C,D}(t,L_1)$ is contained in the $2M_1$–neighborhood of $\mathcal{L}^*_{E,D}(L_1)$ if $t\ge 0$. Therefore, we can conclude that $\mathcal{L}^*_{C,D}(L_1)=\cup_{t\in\mathbb{R}}\mathcal{L}^*_{C,D}(t,L_1)$ is contained in the $2M_1$–neighborhood of $\mathcal{L}^*_{C,E}(L_1)\cup\mathcal{L}^*_{E,D}(L_1)$, which implies that $\widehat{\mathcal{L}}_{C,D}$ is contained in the $3M_1$–neighborhood of $\mathcal{L}^*_{C,E}(L_1)\cup\mathcal{L}^*_{E,D}(L_1)$. □

7.4 Proof of Theorem 7.1

From Proposition 7.7, and Proposition 7.8, we see that $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$ with the family of subgraphs $\widehat{\mathcal{L}}_{C,D}$ satisfies the two conditions of Theorem 7.2 with $M=\max\{M_2,M_3\}$. Therefore, $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$ is Gromov hyperbolic. □

8 The quotient by affine automorphisms

In this section we investigate the quotient of $\widehat{\mathcal{C}}_{\mathrm{cyl}}(X,\omega,f)$ by the group $\mathrm{Aff}^+(X,\omega)$. Our main focus is the case where (X,ω) is a Veech surface, that is, when $\mathrm{SL}(X,\omega)$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$. Throughout this section (X,ω) is a fixed translation surface in $\mathcal{H}(2)\sqcup\mathcal{H}(1,1)$, and $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ is the cylinder graph of (X,ω) with some marking map. We denote by $\mathscr{G}$ the quotient graph $\widehat{\mathcal{C}}_{\mathrm{cyl}}/\mathrm{Aff}^+(X,\omega)$, and by $\mathscr{V}$ and $\mathscr{E}$ the sets of vertices and edges of $\mathscr{G}$, respectively. Notice that an edge may join a vertex to itself (we then have a loop), and there may be more than one edge with the same endpoints. We use the notations $|\mathcal{V}|$ and $|\mathcal{E}|$ to designate the cardinalities of $\mathcal{E}$ and $\mathcal{V}$. We will show the following theorem:

Theorem 8.1 *Let (X,ω) be a surface in $\mathcal{H}(2)\sqcup\mathcal{H}(1,1)$. Then (X,ω) is a Veech surface if and only if $|\mathcal{V}|$ is finite.*

Theorem 8.1 does not mean, when (X,ω) is a Veech surface, that the quotient graph $\mathcal{G}$ is a finite graph, as we have the following:

Proposition 8.2 *If (X,ω) is Veech surface in $\mathcal{H}(2)$ then $\mathcal{G}$ is a finite graph, that is, $|\mathcal{V}|$ and $|\mathcal{E}|$ are both finite. There exist Veech surfaces in $\mathcal{H}(1,1)$ such that $|\mathcal{V}|<\infty$ but $|\mathcal{E}|=\infty$.*

8.1 Proof of Theorem 8.1

Recall that the $\mathrm{SL}(2,\mathbb{R})$–orbit of a Veech surface (X,ω) projects to an algebraic curve in $\mathcal{M}_2$ isomorphic to $\mathcal{X}:=\mathbb{H}/\mathrm{SL}(X,\omega)$; this curve is called a *Teichmüller curve*. The direction of any saddle connection on X is periodic, that is, X is decomposed into finitely many cylinders in this direction. Moreover, there is a parabolic element in $\mathrm{SL}(X,\omega)$ that fixes this direction. Thus each cylinder in X corresponds to a cusp in $\mathcal{X}$.

Let θ be a periodic direction for X. Let $C_1,\dots,C_k$ be the cylinders of X in the direction θ, and T_i be the Dehn twist about the core curves of C_i. Let γ be the generator of the parabolic subgroup of $\mathrm{SL}(X,\omega)$ that fixes θ. Then there exist some integers $m_1,\dots,m_k$ such that γ is the derivative of an element of $\mathrm{Aff}^+(X,\omega)$ isotopic to $T_1^{m_1}\circ\cdots\circ T_k^{m_k}$.

8.1.1 Proof that (X,ω) is Veech implies that $\mathcal{V}$ is finite If $(X,\omega)\in\mathcal{H}(2)$, then X has one or two cylinders in the direction θ. In the first case, we have three more degenerate ones, and in the second case there is no degenerate cylinder. Thus the total number of cylinders (degenerate or not) in a periodic direction is at most 4. If $(X,\omega)\in\mathcal{H}(1,1)$, then by similar arguments, we see that X has at most 5 cylinders in the direction θ. We have seen that θ corresponds to a cusp of $\mathcal{X}$. Since $\mathcal{X}$ has finitely many cusps, it follows that X has finitely many cylinders up to action of $\mathrm{Aff}^+(X,\omega)$. Therefore, $\mathcal{V}$ is finite. □

8.1.2 Proof that $\mathcal{V}$ is finite implies that (X,ω) is Veech In what follows, by an *embedded triangle* in X, we mean the image of a triangle $\boldsymbol{T}$ in the plane by a map $\varphi\colon \boldsymbol{T}\to X$ which is locally isometric, injective in the interior of $\boldsymbol{T}$, and which sends the vertices of $\boldsymbol{T}$ to the singularities of X. Note that φ maps a side of $\boldsymbol{T}$ to a concatenation of some saddle connections. By a slight abuse of notation, we will also denote by $\boldsymbol{T}$ the image of φ in X. To show that (X,ω) is a Veech surface, we will use the following characterization of Veech surfaces by Smillie and Weiss [47].

Theorem 8.3 (Smillie and Weiss) *(X,ω) is a Veech surface if and only if there exists an $\epsilon > 0$ such that the area of any embedded triangle $\boldsymbol{T}$ in X is at least ϵ.*

We now assume that $|\mathscr{V}|$ is finite. If v is a vertex of $\widehat{\mathcal{C}}_{\mathrm{cyl}}$, we denote by $\bar{v}$ its equivalence class in $\mathscr{V}$. Clearly, the group $\mathrm{Aff}^+(X,\omega)$ preserves the areas of the cylinders in X. Therefore, each element of $\mathscr{V}$ has a well-defined area (a degenerate cylinder has zero area). Since $\mathscr{V}$ is finite, we can write $\mathscr{V} = \{\bar{v}_1,\dots,\bar{v}_n\}$, where $n = |\mathscr{V}|$. Using $\mathrm{GL}^+(2,\mathbb{R})$, we can normalize so that $\mathrm{Area}(X,\omega) = 1$. Let $a_i = \mathrm{Area}(\bar{v}_i)$, and define

$$\begin{aligned}
\mathscr{A}_1 &= \{a_1,\dots,a_n\},\\
\mathscr{A}_2 &= \{|a_i - a_j| : a_i \neq a_j\},\\
\mathscr{A}_3 &= \{1-(a_i+a_j) : a_i + a_j < 1\},\\
\mathscr{A}_4 &= \{1-(a_i+a_j+a_k) : a_i+a_j+a_k < 1\}.
\end{aligned}$$

Set $\epsilon = \min\{\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3 \cup \mathscr{A}_4\}$. We will need the following lemma on slit tori.

Lemma 8.4 *Let $(\widehat{X},\widehat{\omega},\hat{s})$ be a slit torus. By a* cylinder *in $\widehat{X}$, we will mean a connected component of X that is cut out by a pair of parallel simple closed geodesics passing through the endpoints of $\hat{s}$.*

Assume that $\hat{s}$ is not parallel to any simple closed geodesic of $\widehat{X}$. Then there exists a sequence of cylinders $\{\widehat{C}_k\}_{k\in\mathbb{N}}$ such that $\widehat{C}_k$ is disjoint from the slit $\hat{s}$ for all $k \in \mathbb{N}$, and $\mathrm{Area}(\widehat{C}_k) \to \mathrm{Area}(\widehat{X})$ as $k \to +\infty$.

Proof Using $\mathrm{GL}^+(2,\mathbb{R})$, we can normalize so that $(\widehat{X},\widehat{\omega}) = (\mathbb{C}/(\mathbb{Z}\oplus i\mathbb{Z}), dz)$. The slit $\hat{s}$ is then represented by a segment $[0,(1+i\alpha)t]$, with $t \in (0,\infty)$ and $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. In this setting, each simple closed geodesic c of $\widehat{X}$ corresponds to a vector $p+iq$ with $p,q \in \mathbb{Z}$ and $\gcd(p,q) = 1$. Let c_1 and c_2 be the simple geodesics parallel to c which pass through the endpoints of $\hat{s}$. Note that c_1, c_2 cut $\widehat{X}$ into two cylinders. By [41, Lemma 4.1], we know that one of the two cylinders is disjoint from $\hat{s}$ if and only if

$$t\left|\det\begin{pmatrix} p & 1\\ q & \alpha\end{pmatrix}\right| = t|p\alpha - q| < 1.$$

Note that the quantity $t|p\alpha - q|$ is precisely the area of the cylinder that contains $\hat{s}$. Since α is an irrational number, one can find a sequence $\{(p_k,q_k)\}_{k\in\mathbb{N}}$ such that

$$\gcd(p_k,q_k) = 1, \quad t|\alpha p_k - q_k| < 1 \quad \text{and} \quad \lim_{k\to\infty} |\alpha p_k - q_k| = 0.$$

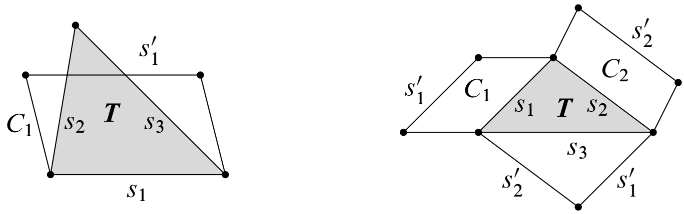

Figure 11: Embedded triangles in a surface in $\mathcal{H}(2)$ in Case 1 (left) and Case 2 (right)

For each (p_k, q_k) in this sequence, we have a cylinder $\widehat{C}_k$ in direction of $p_k + iq_k$ disjoint from $\hat{s}$ such that

$$\mathrm{Area}(\widehat{C}_k) = 1 - t|\alpha p_k - q_k|.$$

In particular, we have $\lim_{k\to\infty} \mathrm{Area}(\widehat{C}_k) = 1$, which proves the lemma. □

As a consequence of this lemma, we get the following.

Corollary 8.5 *Let (s_1, s_2) be a pair of homologous saddle connections in X that are exchanged by the hyperelliptic involution τ. If one of the connected components cut out by (s_1, s_2) is a slit torus, then the direction of s_1, s_2 is periodic.*

Proof If $(X, \omega) \in \mathcal{H}(2)$ then X is decomposed by (s_1, s_2) into a simple cylinder and a slit torus, if $(X, \omega) \in \mathcal{H}(1,1)$ then X is decomposed into two slit tori. Thus, it suffices to show that s_i is parallel to a closed geodesic in each slit torus. If this is not the case, then by Lemma 8.4, we can find in this slit torus a sequence of cylinders disjoint from the slit whose area converges to the area of the torus. Note that such cylinders are also cylinders of X. Thus their areas belong to $\mathcal{A}_1$. Since $\mathcal{A}_1$ is finite, it cannot contain a nonconstant converging sequence. Therefore, we can conclude that the direction of (s_1, s_2) is periodic. □

Let $\boldsymbol{T}$ be an embedded triangle in X. We will show that $\mathrm{Area}(\boldsymbol{T}) > \frac{1}{2}\epsilon$. We first remark that it suffices to consider the case where each side of $\boldsymbol{T}$ is a saddle connection, since otherwise there is another embedded triangle contained in $\boldsymbol{T}$ with this property. Let τ denote the hyperelliptic involution of X, and $\boldsymbol{T}' = \tau(\boldsymbol{T})$. Let s_1, s_2, s_3 be the sides of $\boldsymbol{T}$ and s'_i be the image of s_i by τ. The proof that $\mathrm{Area}(\boldsymbol{T}) > \frac{1}{2}\epsilon$ naturally splits into two cases depending on the stratum of (X, ω).

Case $(X, \omega) \in \mathcal{H}(2)$ We need to consider the following two situations:

Case 1 None of the sides of $\boldsymbol{T}$ is invariant by τ. From Lemma 2.4, s_i and s'_i bound a simple cylinder denoted by C_i. Let h_i be length of the perpendicular segment from the opposite vertex of s_i in $\boldsymbol{T}$ to s_i. If $\mathrm{int}(\boldsymbol{T}) \cap \mathrm{int}(C_1) \neq \varnothing$, then both s_2 and s_3 cross C_1

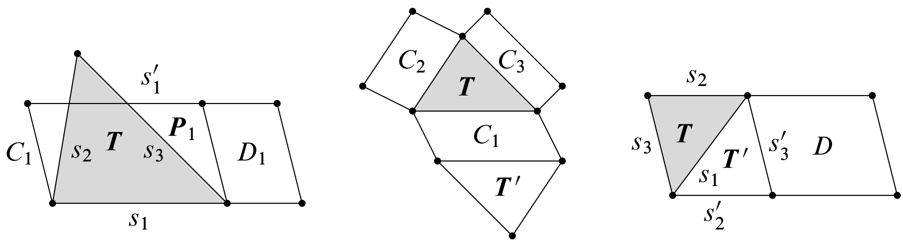

Figure 12: Embedded triangles in a surface in $\mathcal{H}(1,1)$; from left to right we have Cases 1–3.

entirely, which implies that the width of C_1 is at most h_1; see Figure 11 (left). It follows that $\mathrm{Area}(\boldsymbol{T}) \geq \frac{1}{2}\,\mathrm{Area}(C_1) > \min \frac{1}{2}\mathscr{A}_1$. The same arguments apply in the cases that $\mathrm{int}(\boldsymbol{T})$ intersects $\mathrm{int}(C_2)$ or $\mathrm{int}(C_3)$. If $\mathrm{int}(\boldsymbol{T})$ is disjoint from $\mathrm{int}(C_i)$, $i = 1,2,3$, then we have three disjoint cylinders in X (if $\mathrm{int}(C_i) \cap \mathrm{int}(C_j) \neq \varnothing$ then s_i must cross C_j entirely hence $\mathrm{int}(\boldsymbol{T}) \cap \mathrm{int}(C_j) \neq \varnothing$). Since $(X,\omega) \in \mathcal{H}(2)$, this situation cannot occur; see Theorem 2.6. Hence, we can conclude that $\mathrm{Area}(\boldsymbol{T}) \geq \frac{1}{2}\epsilon$ here.

Case 2 One of the sides of $\boldsymbol{T}$ is invariant by τ. In this case, the union of $\boldsymbol{T}$ and its image by τ is an embedded parallelogram; see Lemma 2.1. This means that there is a parallelogram $\boldsymbol{P}$ in the plane such that $\boldsymbol{T}$ is one of the two triangles cut out by a diagonal of $\boldsymbol{P}$, and there is a map $\varphi\colon \boldsymbol{P} \to X$ locally isometric, injective in $\mathrm{int}(\boldsymbol{T})$, mapping the vertices of $\boldsymbol{P}$ to the singularity of X. We remark that all the sides of $\boldsymbol{T}$ cannot be invariant by τ because this would imply that $X = \varphi(\boldsymbol{P})$ is a torus. If there are two sides of $\boldsymbol{T}$ that are invariant by τ, then $\varphi(\boldsymbol{P})$ is a simple cylinder in X, hence $\mathrm{Area}(\boldsymbol{T}) \geq \min \frac{1}{2}\mathscr{A}_1$. If there is only one side invariant by τ, then the complement of $\varphi(\boldsymbol{P})$ is the union of two disjoint simple cylinders C_1, C_2 (see Figure 11, right), which implies $\mathrm{Area}(\boldsymbol{P}) = 1 - (\mathrm{Area}(C_1) + \mathrm{Area}(C_2))$. Therefore, we have $\mathrm{Area}(\boldsymbol{T}) > \min \frac{1}{2}\mathscr{A}_3 \geq \frac{1}{2}\epsilon$. This completes the proof of Theorem 8.1 for the case $(X,\omega) \in \mathcal{H}(2)$.

Case $(X,\omega) \in \mathcal{H}(1,1)$ We consider the following situations:

Case 1 There exists i such that s_i' intersects $\mathrm{int}(\boldsymbol{T})$. Note that we must have $s_i' \neq s_i$. Let us assume that $i = 1$. Recall that s_1 and s_1' either bound a simple cylinder or decompose X into two tori. In the first case, the same argument as in the case $(X,\omega) \in \mathcal{H}(2)$ shows that $\mathrm{Area}(\boldsymbol{T}) \geq \min \frac{1}{2}\mathscr{A}_1$. For the second case, observe that the intersection of $\boldsymbol{T}$ with one of the slit tori consists of a domain bounded by s_1 and some subsegments of s_2, s_3 and s_1'; see Figure 12. Let $(X_1, \omega_1, \tilde{s}_1)$ denote this slit torus.

We can assume that s_1 is horizontal. By Corollary 8.5 we know that the horizontal direction is periodic for X_1, thus X_1 is the closure of a horizontal cylinder C_1. We remark that X_1 contains a transverse simple cylinder D_1 disjoint from $s_1 \cup s_1'$, whose core curves cross C_1 once. The complement of D_1 in X_1 is an embedded parallelogram $\boldsymbol{P}_1$ bounded by s_1, s_1' and the boundary of D_1. Clearly, we have $\operatorname{Area}(\boldsymbol{T}) \geq \frac{1}{2}\operatorname{Area}(\boldsymbol{P}_1)$. By definition, we have

$$\operatorname{Area}(\boldsymbol{P}_1) = \operatorname{Area}(C_1) - \operatorname{Area}(D_1) \geq \min \mathscr{A}_2.$$

Thus we have $\operatorname{Area}(\boldsymbol{T}) \geq \frac{1}{2}\epsilon$.

Case 2 None of s_i' intersects $\operatorname{int}(\boldsymbol{T})$, and $s_i' \neq s_i$, $i = 1,2,3$. It is not difficult to show that this case only happens when s_i and s_i' bound a simple cylinder C_i disjoint from $\operatorname{int}(\boldsymbol{T}) \cup \operatorname{int}(\boldsymbol{T}')$. Therefore, X is decomposed into the union of three cylinders C_1, C_2, C_3, and $\boldsymbol{T} \cup \boldsymbol{T}'$; see Figure 12. Thus in this case, we have

$$\operatorname{Area}(\boldsymbol{T}) = \tfrac{1}{2}\big(1 - (\operatorname{Area}(C_1) + \operatorname{Area}(C_2) + \operatorname{Area}(C_3))\big) \geq \min \tfrac{1}{2}\mathscr{A}_4 \geq \tfrac{1}{2}\epsilon.$$

Case 3 None of s_i' intersects $\operatorname{int}(\boldsymbol{T})$ and one of s_1, s_2, s_3 is invariant by τ. Assume that $s_1' = s_1$. It follows that $\boldsymbol{T} \cup \boldsymbol{T}'$ is an embedded parallelogram $\boldsymbol{P}$. If both (s_2, s_2') and (s_3, s_3') are the boundaries of some simple cylinders C_2 and C_3, respectively, then C_2 and C_3 are disjoint, and $C_2 \cup C_3$ is disjoint from $\boldsymbol{P}$. By construction we must have $X = \overline{\boldsymbol{P}} \cup \overline{C}_2 \cup \overline{C}_3$, which is impossible since $(X,\omega) \in \mathcal{H}(1,1)$. Therefore, we can assume that (s_2, s_2') decompose X into two slit tori. Let X_1 be the slit torus that contains $\boldsymbol{P}$. By Corollary 8.5, we know that the direction of (s_2, s_2') is periodic, which means that X_1 is the closure of a cylinder C. Observe that the complement of $\boldsymbol{P}$ in X_1 must be a cylinder D bounded by (s_3, s_3'); see Figure 12. Therefore,

$$\operatorname{Area}(\boldsymbol{T}) = \tfrac{1}{2}\operatorname{Area}(\boldsymbol{P}) = \tfrac{1}{2}(\operatorname{Area}(C) - \operatorname{Area}(D)) \geq \tfrac{1}{2}\min \mathscr{A}_2 \geq \tfrac{1}{2}\epsilon.$$

Case 4 None of s_i' intersects $\operatorname{int}(\boldsymbol{T})$ and two of s_1, s_2, s_3 are invariant by τ. In this case $\boldsymbol{T} \cup \boldsymbol{T}'$ is a simple cylinder. Therefore, $\operatorname{Area}(\boldsymbol{T}) \geq \min \frac{1}{2}\mathscr{A}_1 \geq \frac{1}{2}\epsilon$.

In all cases $\operatorname{Area}(\boldsymbol{T}) \geq \frac{1}{2}\epsilon$, thus Theorem 8.3 implies that (X,ω) is a Veech surface. □

8.2 Proof of Proposition 8.2

Case $(X,\omega) \in \mathcal{H}(2)$ We have shown that $\mathscr{V}$ is finite; it remains to show that $\mathscr{E}$ is also finite. Let v be a vertex of $\widehat{\mathcal{C}}_{\mathrm{cyl}}$, and C be the corresponding cylinder in X. We denote by $\bar{v}$ the equivalence class of v in $\mathscr{G}$. Using $\mathrm{SL}(2,\mathbb{R})$, we can suppose that C is horizontal.

If C is a nondegenerate cylinder, then we have three cases: (a) C is the unique horizontal cylinder, (b) X has two horizontal cylinders and C is not simple, and (c) C is a simple cylinder. In case (a), there are three edges in $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ that have v as an endpoint, those edges connect v to three degenerate cylinders contained in the boundary of C. In case (b), there is only one edge in $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ having v as an endpoint, this edge connects C

to the other horizontal simple cylinder. Thus in cases (a) and (b), there are only finitely many edges having $\bar{v}$ as an endpoint.

Assume now that we are in case (c). Let D be the other horizontal cylinder of X. Observe that the closure of D is a slit torus (X', ω', s') where s' corresponds to the boundary of C. Let d be a core curve of D, and e be a simple closed geodesic in X' disjoint from the slit s' and crossing d once. We consider $\{d, e\}$ as a basis of $H_1(X', \mathbb{Z})$. If C' is a cylinder in X disjoint from C, then C' must be entirely contained in $\overline{D}$. Thus the core curves of C' are determined by a unique element of $H_1(X', \mathbb{Z})$, and we can write $C' = md + ne$ with $m, n \in \mathbb{Z}$.

By assumption, a core curve c' of C' cannot cross the slit s'. The necessary and sufficient condition for this is that $|\omega'(c') \wedge \omega'(s')| \leq \mathrm{Area}(X') = \mathrm{Area}(D)$; see [41, Lemma 4.1]. But $|\omega'(c') \wedge \omega'(s')| = |n||\omega'(e) \wedge \omega'(s')|$. Thus we can conclude that $|n|$ is bounded by some constant n_0.

We have seen that $\mathrm{Aff}^+(X, \omega)$ contains an element $\phi = T_1^{m_1} \circ T_2^{m_2}$, where T_1 and T_2 are the Dehn twists about the core curves of C and D, respectively. Observe that ϕ fixes the vertices of $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ corresponding to C and D. The action of ϕ on the curves contained in $\overline{D}$ is given by

$$\phi(md + ne) = (m \pm m_2 n)d + ne.$$

Thus up to action of $\{\phi^k\}_{k \in \mathbb{Z}}$, any cylinder C' contained in $\overline{D}$ belongs to the equivalence class of a cylinder C'' also contained in $\overline{D}$ whose core curves are represented by $md + nc$ with $|n| \leq |n_0|$ and $|m| \leq |m_2 n| \leq |m_2||n_0|$. We can then conclude that there are finitely many edges in $\mathscr{E}$ which contain $\bar{v}$ as an endpoint.

It remains to consider the case that C is degenerate. In this case X has a unique nondegenerate cylinder in the horizontal direction, which contains C in its boundary. Note that the complement of C in X can be isometrically identified with a flat torus with an embedded geodesic segment removed. Therefore, the arguments above also hold in this case. Since we have proved that the set of vertices of $\mathscr{G}$ is finite, it follows that the set of edges of $\mathscr{G}$ is also finite. □

Case $(X, \omega) \in \mathcal{H}(1,1)$ Let (X, ω) be the surface constructed from 6 squares as shown in Figure 13. This surface has 3 horizontal cylinders denoted by C_1, C_2, C_3, where C_i is the cylinder with i squares. It has two vertical cylinders denoted by D_1 and D_2, where the core curves of D_1 cross C_1 and C_3. Let v be the vertex of $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ corresponding to C_1, and w be the vertex corresponding to C_2. The fact that $\mathscr{G}$ has finitely many vertices follows from Theorem 8.1. We will show that $\mathscr{G}$ has infinitely many edges.

Given a cylinder C on X, we denote by T_C the Dehn twist about the core curves of C. Observe that $f = T_{C_1}^6 \circ T_{C_2}^3 \circ T_{C_3}^2$ and $g = T_{D_1} \circ T_{D_2}^2$ are two elements of

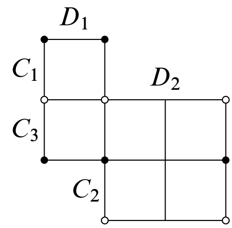

Figure 13: Example of a square-tiled surface in $\mathcal{H}(1,1)$

$\mathrm{Aff}^+(X,\omega)$ whose derivatives are

$$\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

respectively. If h is an element of $\mathrm{Aff}^+(X,\omega)$ that preserves the horizontal direction, then h must map a horizontal cylinder to a horizontal cylinder. Since C_1, C_2, C_3 have different circumferences, h must preserve each of them, which implies that $h = f^k$, $k \in \mathbb{Z}$. We derive in particular that there is no affine homeomorphism that maps C_2 to C_1.

For any $n \in \mathbb{N}$, let E_n be the image of C_2 by g^n. We remark that $E_n = T_{D_2}^{2n}(C_2)$, hence E_n is contained in the closure $\overline{D}_2$ of D_2. In particular, E_n is disjoint from C_1. Thus, there is an edge $\boldsymbol{e}_n$ in $\widehat{\mathcal{C}}_{\mathrm{cyl}}$ connecting v to the vertex w_n corresponding to E_n. By definition, all the vertices w_n belong to the equivalence class $\overline{w}$ of w in $\mathcal{G}$. We will show that the edges $\{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ are all distinct up to action of $\mathrm{Aff}^+(X,\omega)$, which means that there are infinitely many edges in $\mathcal{E}$ between $\bar{v}$ and $\overline{w}$.

Assume that there is an affine automorphism $h \in \mathrm{Aff}^+(X,\omega)$ such that $h(\boldsymbol{e}_{n_1}) = \boldsymbol{e}_{n_2}$, for some $n_1, n_2 \in \mathbb{N}$. If $h(w_{n_i}) = v$, then there is an element of $\mathrm{Aff}^+(X,\omega)$ that sends w to v, or equivalently C_2 to C_1. But we have already seen that such an element does not exist, thus this case cannot occur. Therefore, we must have $h(v) = v$ and $h(w_{n_1}) = w_{n_2}$. Since any element of $\mathrm{Aff}^+(X,\omega)$ preserving C_1 belongs to the subgroup generated by f, we derive that h also preserves C_2 and C_3. Observe that a core curve of E_{n_i} crosses C_2 $2n_i$ times. Therefore, if $n_1 \neq n_2$, then h cannot exist. We can then conclude that the projections of all the edges $\boldsymbol{e}_n$ are distinct in $\mathcal{G}$, which proves the proposition. □

9 Quotient graphs and McMullen's prototypes

By the works of McMullen [38; 37], we know that closed $\mathrm{GL}^+(2,\mathbb{R})$–orbits in $\mathcal{H}(2)$ are indexed by the discriminant D, that is, a natural number $D \in \mathbb{N}$ such that $D \equiv 0, 1 \mod 4$, together with the parity of the spin structure when $D \equiv 1 \mod 8$ and $D \neq 9$.

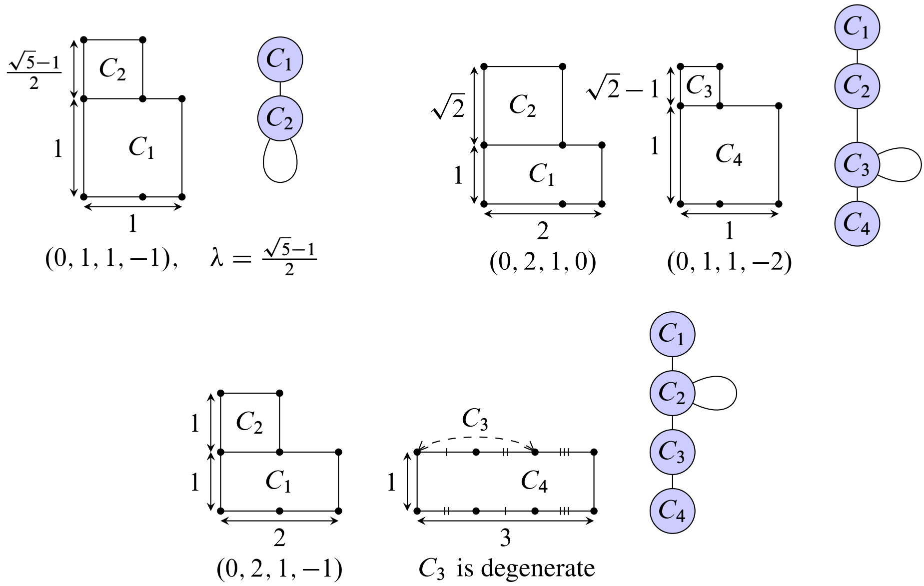

Figure 14: Examples of $\mathcal{G}$ for $D = 5$ (top left), $D = 8$ (top right) and $D = 9$ (bottom). For each two-cylinder decomposition, we provide the corresponding prototype (a, b, c, e). A loop at some vertex represents a butterfly move that does not change the prototype.

Let (X, ω) be an eigenform in $\mathcal{E}_D \cap \mathcal{H}(2)$ for some fixed D. Following [37], every two-cylinder decomposition of X is encoded by a quadruple of integers $(a, b, c, e) \in \mathbb{Z}^4$ called a *prototype* satisfying the following conditions:

$$(\mathcal{P}_D) \qquad \begin{array}{lll} b > 0, & c > 0, & \gcd(b, c) > a \geq 0, \\ D = e^2 + 4bc, & b > c + e, & \gcd(a, b, c, e) = 1. \end{array}$$

Set $\lambda = \frac{1}{2}(e + \sqrt{D})$. Up to action of $\mathrm{GL}^+(2, \mathbb{R})$, the decomposition of X consists of two horizontal cylinders. The first one is simple and represented by a square of size λ. The other one is nonsimple and represented by a parallelogram constructed from the vectors $(b, 0)$ and (a, c). Note that we always have $b > \lambda$.

The quotient graph $\mathcal{G}$ turns out to be closely related to the set of McMullen's prototypes. Namely, each prototype corresponds to a cluster of two vertices of $\mathcal{G}$ which represent the cylinders in the corresponding cylinder decomposition. Let C_1, C_2 be the cylinders in this decomposition, where C_1 is the simple one. Then the vertex corresponding to C_2 is only adjacent to the one corresponding to C_1 in $\mathcal{G}$. This is because any other cylinder of X must cross C_2.

On the other hand, if there is an edge in $\mathcal{G}$ between two vertices representing two simple cylinders which are not parallel, then the two cylinder decompositions are related by a "butterfly move"; see [37, Section 7] for the precise definitions. In other words, $\mathcal{G}$ can be viewed as a geometric object representing $\mathcal{P}_D$: each prototype is represented by two vertices connected by an edge, and all the other edges of $\mathcal{G}$ represent butterfly moves.

There is nevertheless a slight difference between the two notions. The set $\mathcal{P}_D$ only parametrizes two-cylinder decompositions of X, while in $\mathcal{G}$ we also have one-cylinder decompositions. If $\sqrt{D} \notin \mathbb{N}$, then any cylinder in X is contained in a two-cylinder decomposition. Thus, the set of prototypes exhausts all the equivalence classes of cylinders in X (hence it provides the complete list of cusps of the corresponding Teichmüller curve). But when D is a square (eg $D = 9$), we need to take into account one-cylinder decompositions as well as degenerate cylinders. In Figure 14, we draw the quotient cylinder graphs of surfaces corresponding to some small values of D.

Appendix A: Triangulations

In this section we construct triangulations of (X,ω) that are invariant by the hyperelliptic involution. The aim of these triangulations is to provide a preferred way to represent (X,ω) as a polygon in $\mathbb{R}^2$ when we have a horizontal simple cylinder on X. The results of this section are certainly not new and are known to most people in the field; see eg [49]. We present them here only for the sake of completeness.

In what follows, for any saddle connection s, we will denote by $\boldsymbol{h}(s)$ the length of the horizontal component of s, that is, $\boldsymbol{h}(s) = |\mathrm{Re}(\omega(s))|$. If Δ is a triangle bounded by the saddle connections s_1, s_2, s_3, we define $\boldsymbol{h}(\Delta) = \max\{\boldsymbol{h}(s_i) \mid i = 1,2,3\}$. Our main result in this section is the following:

Proposition A.1 *Let (X,ω) be a translation surface in $\mathcal{H}(2)\sqcup\mathcal{H}(1,1)$ having a simple horizontal cylinder C. Assume that every regular leaf of the vertical foliation of (X,ω) crosses C.*

(i) *If $(X,\omega)\in\mathcal{H}(2)$, then (X,ω) can be obtained by identifying the pairs of opposite sides of an octagon $\boldsymbol{P} = (P_0\cdots P_3 Q_0\cdots Q_3)\subset\mathbb{R}^2$* (see Figure 15), *where the vertices are labeled clockwise, such that the following hold:*

- $\overrightarrow{P_iP_{i+1}} = -\overrightarrow{Q_iQ_{i+1}}$, $i = 0,1,2$, *and* $\overrightarrow{P_3Q_0} = -\overrightarrow{Q_3P_0}$.
- *The diagonals $\overline{P_0P_3}$ and $\overline{Q_0Q_3}$ are horizontal, the parallelogram $(P_0P_3Q_0Q_3)$ is contained in $\boldsymbol{P}$ and projects to $C\subset X$.*
- *For $i = 1,2$, the vertical line through P_i (resp. Q_i) intersects $\overline{P_0P_3}$ (resp. $\overline{Q_0Q_3}$), and the vertical segment from P_i (resp. from Q_i) to the intersection is contained in $\boldsymbol{P}$.*

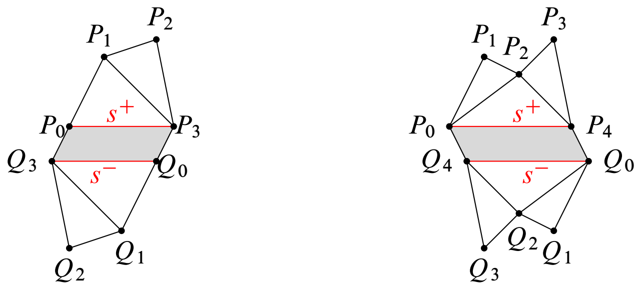

Figure 15: Representations of surfaces (X,ω) in $\mathcal{H}(2)$ (left) and $\mathcal{H}(1,1)$ (right) with symmetric polygons. The simple horizontal cylinder is represented by the highlighted parallelogram.

(ii) *If $(X,\omega) \in \mathcal{H}(1,1)$, then (X,ω) can be obtained by identifying the pairs of opposite sides of a decagon $\boldsymbol{P} = (P_0 \cdots P_4 Q_0 \cdots Q_4)$* (see Figure 15)*, where the vertices are labeled clockwise, such that the following hold:*

- $\overrightarrow{P_i P_{i+1}} = -\overrightarrow{Q_i Q_{i+1}}$, $i = 0, \dots, 3$, *and* $\overrightarrow{P_4 Q_0} = -\overrightarrow{Q_4 P_0}$.
- *The diagonals $\overline{P_0 P_4}$ and $\overline{Q_0 Q_4}$ are horizontal, the parallelogram $(P_0 P_4 Q_0 Q_4)$ is contained in $\boldsymbol{P}$ and projects to $C \subset X$.*
- *For $i = 1, 2, 3$, the vertical line through P_i (resp. Q_i) intersects $\overline{P_0 P_4}$ (resp. $\overline{Q_0 Q_4}$), and the vertical segment from P_i (resp. from Q_i) to the intersection is contained in $\boldsymbol{P}$.*

Proof Cut off C from X, and identify the geodesic segments in the boundary of the resulting surface, we then obtain either a slit torus (if $(X,\omega) \in \mathcal{H}(2)$) or a surface in $\mathcal{H}(2)$ with a marked saddle connection (if $(X,\omega) \in \mathcal{H}(1,1)$). Let (X',ω') denote the new surface, and s' the marked saddle connection. If (X',ω') is a slit torus, then there is a unique involution of X' that acts by $-\mathrm{Id}$ on $H_1(X',\mathbb{Z})$ and exchanges the endpoints of s'. By a slight abuse of notation, we will call this involution the hyperelliptic involution of X'. Thus, in both cases, s' is invariant by the hyperelliptic involution.

By assumption all the regular vertical leaves of X' intersect s'. Let $\{\Delta_i^{\pm} \mid i = 1, \dots, k\}$ be the triangulation of X' provided by Lemmas A.2 and A.3; if $(X',\omega') \in \mathcal{H}(0,0)$, $k = 2$, if $(X',\omega') \in \mathcal{H}(2)$, $k = 3$. We can represent C by a parallelogram in $\mathbb{R}^2$. The polygon $\boldsymbol{P}$ is obtained from this parallelogram by gluing successively the triangles $\Delta_1^+, \dots, \Delta_k^+$, then $\Delta_1^-, \dots, \Delta_k^-$. □

Lemma A.2 *Let (X,ω,s) be a slit torus. Let τ be the elliptic involution of X that exchanges the endpoints P_1, P_2 of s. Assume that all the leaves of vertical foliation*

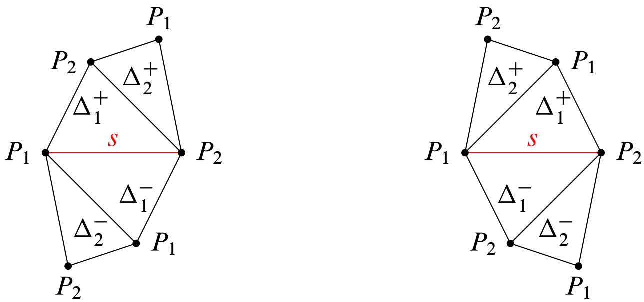

Figure 16: Triangulation of a slit torus

meet s. Then there exists a unique triangulation of X into four triangles $\Delta_1^\pm, \Delta_2^\pm$ with vertices in $\{P_1, P_2\}$, such that the following are satisfied:

- *Δ_i^+ and Δ_i^- are exchanged by τ.*
- *s is contained in both Δ_1^+ and Δ_1^-.*
- *For $i = 1, 2$, the union $\Delta_i^+ \cup \Delta_i^-$ is a cylinder in X.*
- *Δ_1^+ is adjacent to Δ_1^- and Δ_2^+, Δ_1^- is adjacent to Δ_1^+ and Δ_2^-.*
- *$\boldsymbol{h}(\Delta_1^\pm) = \boldsymbol{h}(s)$, and $\boldsymbol{h}(\Delta_2^\pm) = \boldsymbol{h}(c^+)$, where c^+ is the unique common side of Δ_2^+ and Δ_1^+.*

There are two possible configurations for this triangulation, shown in Figure 16.

Proof By Lemma 2.3, we know that there exists a pair of simple closed geodesics c^+, c^- passing through the endpoints of s that cut X into two cylinders satisfying $\boldsymbol{h}(c^\pm) \le \boldsymbol{h}(s)$. One of the cylinders cut out by $c^\pm$ contains s, we denote it by C_1, the other one is denoted by C_2. Note that we must have $\boldsymbol{h}(c^\pm) > 0$, otherwise there are vertical leaves that do not meet s. It is easy to see that we get the desired triangulation by adding some geodesic segments in C_1 and C_2 joining the endpoints of s. □

Lemma A.3 *Let (X, ω) be a surface in $\mathcal{H}(2)$ and s be a saddle connection on X, invariant by the hyperelliptic involution τ. We assume that s is horizontal and all the leaves of the vertical foliation meet s. Then we can triangulate X into six triangles $\Delta_i^\pm$, $i = 1, 2, 3$, whose sides are saddle connections, satisfying the following:*

- *$\tau(\Delta_i^+) = \Delta_i^-$, $i = 1, 2, 3$.*
- *Δ_1^+ and Δ_1^- contain s, and $\boldsymbol{h}(\Delta_1^\pm) = \boldsymbol{h}(s)$.*
- *Δ_2^+ has a unique common side with Δ_1^+ which will be denoted by a^+, and $\boldsymbol{h}(\Delta_2^+) = \boldsymbol{h}(a^+)$.*
- *Δ_3^+ either has a unique common side b^+ with Δ_1^+ and $\boldsymbol{h}(\Delta_3^+) = \boldsymbol{h}(b^+)$ or Δ_3^+ has a unique common side c^+ with Δ_2^+ and $\boldsymbol{h}(\Delta_3^+) = \boldsymbol{h}(c^+)$.*

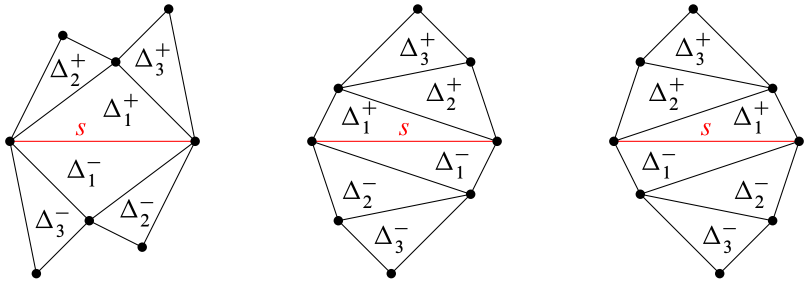

Figure 17: Triangulation of surfaces in $\mathcal{H}(2)$

This triangulation is unique. The configurations of the triangles $\Delta_i^{\pm}$, $i = 1, 2, 3$, are shown in Figure 17.

Proof From Lemma 2.1, we see that there exist a parallelogram $\boldsymbol{P} \subset \mathbb{R}^2$ and a locally isometric map $\varphi\colon \boldsymbol{P} \to X$ that maps a diagonal of $\boldsymbol{P}$ to s. By construction, $\varphi(\boldsymbol{P})$ is decomposed into two embedded triangles $\Delta_1^{\pm}$, where Δ_1^+ is the one above s, both of which satisfy $\boldsymbol{h}(\Delta_1^{\pm}) = \boldsymbol{h}(s) = |s|$. Note also that $\tau(\Delta_1^+) = \Delta_1^-$.

Let us denote the nonhorizontal sides of Δ_1^+ by a^+ and b^+, and their images by τ by a^- and b^-, respectively. If both of a^+ and b^+ are invariant by τ then $X = \varphi(\boldsymbol{P})$, which implies that X is a torus, and we have a contradiction. Therefore, we only have two cases:

(a) None of a^+, b^+ is invariant by τ. In this cases, by Lemma 2.4 the complement of $\varphi(\boldsymbol{P})$ is the disjoint union of two cylinders bounded by $a^{\pm}$ and $b^{\pm}$, respectively. Note that none of a^+ and b^+ is vertical, otherwise there would be vertical leaves that do not meet s. We can then triangulate the cylinders bounded by $a^{\pm}$ and $b^{\pm}$ in the same way as in Lemma A.2.

(b) One of a^+, b^+ is invariant by τ. We can assume that b^+ is invariant by τ. In this case, $\varphi(\boldsymbol{P})$ is a simple cylinder bounded by $a^{\pm}$. The complement of $\varphi(\boldsymbol{P})$ is then a slit torus (X_1, ω_1, s_1), where s_1 is the identification of $a^{\pm}$. From the assumption that all the vertical leaves meet s, we derive that $a^{\pm}$ are not vertical. Thus we can follow the same argument as in Lemma A.2 to get the desired triangulation. □

Appendix B: Cylinders and decompositions

In this section, we give the proofs of some lemmas which are used in Section 7.

Lemma B.1 *Let $(X, \omega) \in \mathcal{H}(2) \sqcup \mathcal{H}(1, 1)$ be a completely periodic surface in the sense of Calta. If C is a degenerate cylinder in X, then the direction of C is periodic, that is, X is decomposed into cylinders in the direction of C.*

Proof If (X,ω) is in $\mathcal{H}(2)$ then (X,ω) is a Veech surface, thus the direction of any saddle connection is periodic and we are done. Assume now that (X,ω) is in $\mathcal{H}(1,1)$. In $\mathcal{H}(1,1)$, we have a local action of $\mathbb{C}$ which only changes the relative periods and leaves the absolute periods invariant. Orbits of this local action are leaves of the kernel foliation. It is well known that the any eigenform locus is invariant by this local action.

Let us label the zeros of ω by x_1, x_2 and define the orientation of any path connecting x_1 and x_2 to be from x_1 to x_2. Using this local action of $\mathbb{C}$, we can collapse the two zeros of ω as follows. Let s be a saddle connection invariant by the hyperelliptic involution satisfying the following condition, which we will call condition $(\mathcal{S})$: if there exists another saddle connection s' joining x_1 and x_2 such that $\omega(s') = \lambda\omega(s)$ with $\lambda \in (0;+\infty)$, then we have $\lambda > 1$.

We can then reduce the length of s to zero by moving in the kernel foliation leaf of (X,ω), the resulting surface is an eigenform in $\mathcal{H}(2)$ having the same absolute periods as (X,ω). The condition on s implies that x_1 and x_2 do not collide before s is reduced to a point, for a proof of this fact, we refer to [28; 29]. We remark that the new surface in $\mathcal{H}(2)$ is a Veech surface.

Without loss of generality, we can assume that C is horizontal. By definition, C is the union of two saddle connections s_1, s_2 both invariant by the hyperelliptic involution, and up to a renumbering we have $\omega(s_1) \in \mathbb{R}_{>0}, \omega(s_2) \in \mathbb{R}_{<0}$.

Assume that neither of s_1, s_2 satisfies $(\mathcal{S})$, then there exist two other saddle connections s'_1, s'_2 such that $\omega(s'_i) = \lambda_i\omega(s_i)$, with $\lambda_i \in (0;1)$. This implies that there are four horizontal saddle connections on X. Since $(X,\omega) \in \mathcal{H}(1,1)$, there are at most 4 saddle connections in a fixed direction, and this maximal number is realized if and only if the direction is periodic. Thus, in this case we can conclude that X is horizontally periodic.

Let us now assume that one of s_1, s_2, say s_1, satisfies the condition $(\mathcal{S})$. We can then collapse x_1, x_2 along s_1 to get a Veech surface $(X_0,\omega_0) \in \mathcal{H}(2)$. Since $\omega(s_2) - \omega(s_1)$ is an absolute period, it stays unchanged along the collapsing procedure. Therefore, s_2 persists in X_0, and we have $\omega_0(s_2) = \omega(s_2) - \omega(s_1) \in \mathbb{R}$. In particular, (X_0,ω_0) has a horizontal saddle connection, and because (X_0,ω_0) is a Veech surface, it must be horizontally periodic. It follows that (X,ω) is also horizontally periodic. This completes the proof of the lemma. □

Lemma B.2 *Let $(X,\omega) \in \mathcal{H}(1,1)$. Let C be a horizontal (possibly degenerate) cylinder in X, and D be a vertical simple cylinder disjoint from C. Then either*

(a) *there is another simple cylinder E disjoint from $C \cup D$ such that the complement of $C \cup D \cup E$ is the union of two embedded triangles, or*

(b) *there exists a pair of homologous saddle connections s_1, s_2 that decompose X into two slit tori (X', ω', s') and (X'', ω'', s'') such that C is contained in X' and D is contained in X''.*

Proof We first consider the case that C is not degenerate. In this case, the complement of $\overline{C}$ in X is either (1) empty, (2) a horizontal simple cylinder, (3) the disjoint union of two horizontal simple cylinders, (4) a torus with a horizontal slit, or (5) a surface $(\widehat{X}, \widehat{\omega}) \in \mathcal{H}(2)$ with a marked horizontal saddle connection s. Since we have a vertical simple cylinder disjoint from C, only (4) and (5) can occur. In case (4), we automatically have two slit tori, one of which is the closure of C, and the other one must contain D. Therefore we get case (b) of the statement of the lemma.

Let us now assume that we are in case (5). In this case C must be a simple horizontal cylinder, and the saddle connection s in $\widehat{X}$ corresponds to the boundary of C. Note that s is invariant by the hyperelliptic involution $\hat{\tau}$ of $\widehat{X}$. Let $\varphi\colon \boldsymbol{P} \to \widehat{X}$ be the embedded parallelogram associated to s. Let $a^{\pm}$ and $b^{\pm}$ be the images by φ of the sides of $\boldsymbol{P}$, where $\hat{\tau}(a^+) = a^-$ and $\hat{\tau}(b^+) = b^-$. Note that D must be disjoint from $\varphi(\boldsymbol{P})$ since any vertical geodesic intersecting $\varphi(\mathrm{int}(\boldsymbol{P}))$ must intersect $\mathrm{int}(s)$, and hence C, but we have assumed that D is disjoint from C.

If $a^+ = a^-$ and $b^+ = b^-$ then $\widehat{X}$ must be a torus, and we have a contradiction. Therefore, we only have two cases:

- $\boldsymbol{a^+ \neq a^-}$ **and** $\boldsymbol{b^+ \neq b^-}$ In this case, the complement of $\varphi(\boldsymbol{P})$ is the disjoint union of two simple cylinders. Since D is contained in this union, D must be one of the two. Let us denote the other one by E. To obtain (X, ω) from $(\widehat{X}, \widehat{\omega})$, we need to slit open s and glue back C. Consequently, we see that (X, ω) has three disjoint simple cylinders C, D, E. The complement of $C \cup D \cup E$ is the union of two embedded triangles, which are the images of the triangles in $\boldsymbol{P}$ cut out by s. Thus, we get case (a) of the statement of the lemma.

- $\boldsymbol{a^+ = a^-}$ **and** $\boldsymbol{b^+ \neq b^-}$ In this case, $\varphi(\boldsymbol{P})$ is a simple cylinder bounded by $b^{\pm}$. The complement of $\varphi(\boldsymbol{P})$ is then a slit torus (X'', ω'', s'') with the slit s'' corresponding to $b^{\pm}$. We can view (X'', ω'', s'') as a subsurface of X. Observe that D must be contained in (X'', ω'') and disjoint from the slit s'', since otherwise a core curve of D must cross C. The complement of (X'', ω'', s'') is another slit torus (X', ω', s') which is obtained by slitting $\varphi(\boldsymbol{P})$ along s and gluing back C. Therefore, we get case (b) of the statement of the lemma.

Assume now C is degenerate. By Lemma 3.4, there exist deformations (X_t, ω_t), $t \in [0, \epsilon)$, of (X, ω) such that C corresponds to a simple horizontal cylinder C_t in X_t, which has the same circumference as C and height equal to t. By construction, D

corresponds to a simple vertical cylinder D_t in X_t which is disjoint from C_t. Observe that C_t and D_t satisfy case (5) above. Therefore, by the preceding arguments, the conclusion of the lemma is true for C_t and D_t. In either case, the corresponding decomposition of X_t persists as $t \to 0$, which implies that we have the same decomposition on (X, ω). □

In what follows, if $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two vectors in $\mathbb{R}^2$, we denote

$$u \wedge v := \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix},$$

and $|u|, |v|$ are the Euclidean norms of u and v, respectively.

Lemma B.3 *Given a constant $L > 0$, let*

$$L_1 := 3\max\{f(L), f(2\delta)\}, \tag{11}$$

where $f(x) = \sqrt{x^2 + 1/x^2}$, and $\delta := \left(\frac{3}{4}\right)^{\frac{1}{4}}$. Then for any slit torus (X, ω, s) with $\mathrm{Area}(X, \omega) = 1$, *and $|s| < L$, there exists in X a cylinder disjoint from s with area at least $\frac{1}{2}$ and circumference bounded above by L_1.*

Proof Let Λ be the lattice in $\mathbb{C}$ such that (X, ω) can be identified with $(\mathbb{C}/\Lambda, dz)$. Since Λ has covolume 1, there exists a vector $v \in \Lambda$ such that $|v| \leq \delta$. Define $u = \omega(s) \in \mathbb{C} \simeq \mathbb{R}^2$.

Let us first consider the case that $|u| \leq \frac{1}{2\delta}$. We then have

$$|u \wedge v| \leq |u|\,|v| \leq \tfrac{1}{2}.$$

The vector v corresponds to a simple closed geodesic c on X. The inequality above implies that there exist a pair of simple closed geodesics parallel to c cutting X into two cylinders, one of which contains s denoted by C, the other one denoted by C' consists of closed geodesics parallel to c that do not intersect s; see [41, Lemma 4.1] or [37, Theorem 7.2]. Note that the circumferences of both C and C' are $|v| \leq \delta$. Since $\mathrm{Area}(C) = |u \wedge v| \leq \frac{1}{2}$, we have $\mathrm{Area}(C') \geq \frac{1}{2}$. Thus C' has the required properties.

We can now turn to the case that $\frac{1}{2\delta} \leq |s| \leq L$. By definition, we have $f(|s|) \leq \frac{1}{3} L_1$. By multiplying ω by a complex number of modulus 1, which does not change the area of X and the length of s, we can assume that s is horizontal. From Lemma 2.1, we know that there exists a local isometry φ from a parallelogram $\boldsymbol{P} \subset \mathbb{R}^2$ into X such that a horizontal diagonal of $\boldsymbol{P}$ is mapped to s. Since X is a torus, $C := \varphi(\boldsymbol{P})$ is actually a cylinder in X. Let η be the distance from the highest point of $\boldsymbol{P}$ to its horizontal diagonal. By construction, we have $\mathrm{Area}(C) = \mathrm{Area}(\boldsymbol{P}) = \eta|s| \leq \mathrm{Area}(X, \omega) = 1$.

Thus $\eta \le 1/|s|$. Note that the boundary components of C are the images by φ of two opposite sides of $\boldsymbol{P}$. Hence the circumference of C is bounded by

$$\sqrt{|s|^2+\eta^2} \le f(|s|) \le \tfrac{1}{3}L_1.$$

Observe that the complement of C is another cylinder C' in X sharing the same boundary with C. If $\mathrm{Area}(C') \ge \frac{1}{2}$ then we are done. Let us consider the case that $\mathrm{Area}(C') < \frac{1}{2}$, which means that $\mathrm{Area}(C) > \frac{1}{2} > \mathrm{Area}(C')$. By cutting and pasting, we can also realize C as a parallelogram $\boldsymbol{Q} = (P_1P_2P_3P_4)$ with two horizontal sides $\overline{P_1P_2}$ and $\overline{P_4P_3}$ identified with s. Note that the distance between $\overline{P_1P_2}$ and $\overline{P_4P_3}$ is η. We can then realize C' as a parallelogram $\boldsymbol{Q}' = (P_2P_3P_5P_6)$ adjacent to $\boldsymbol{Q}$, where P_5 is contained in the horizontal stripe bounded by the lines supporting $\overline{P_1P_2}$ and $\overline{P_4P_3}$; see Figure 18. Let P_6' and P_5' be the intersections of the line supporting $\overline{P_5P_6}$ and the lines supporting $\overline{P_1P_2}$ and $\overline{P_4P_3}$, respectively.

Clearly we have $\mathrm{Area}(C') = \mathrm{Area}(\boldsymbol{Q}') = \mathrm{Area}((P_2P_3P'_5P'_6))$. Since $\mathrm{Area}(C') < \mathrm{Area}(C)$, we have $|\overline{P_2P_6'}| < |\overline{P_1P_2}|$, and $|\overline{P_1P_6'}| < 2|\overline{P_1P_2}| \le 2L$. If $P_6' \equiv P_6$, then X has a horizontal cylinder C_0 with circumference equal $|\overline{P_1P_6'}|$ and area equal 1. Clearly the core curves of C_0 do not intersect s, therefore C_0 has the required properties. If $P_6 \ne P_6'$, then by construction, $\overline{P_1P_5}$ and $\overline{P_4P_5}$ project to two simple closed geodesics in X, denoted by c_1 and c_2, respectively. These closed geodesics meet s only at one of its endpoints. Let d_1 and d_2 be respectively the simple closed geodesics parallel to c_1 and c_2 passing through the other endpoint of s. Observe that c_1 and d_1 (resp. c_2 and d_2) cut X into two cylinders, one of which contains s and will be denoted by C_1 (resp. C_2), and the other is denoted by C_1' (resp. C_2'). Now, we remark that

$$\mathrm{Area}(C_1) = |\overrightarrow{P_1P_5} \wedge \overrightarrow{P_1P_2}| \quad \text{and} \quad \mathrm{Area}(C_2) = |\overrightarrow{P_4P_5} \wedge \overrightarrow{P_4P_3}|.$$

Since

$$|\overrightarrow{P_1P_5} \wedge \overrightarrow{P_1P_2}| + |\overrightarrow{P_4P_5} \wedge \overrightarrow{P_4P_3}| = |\overrightarrow{P_1P_2} \wedge \overrightarrow{P_1P_4}| = \mathrm{Area}(C) \le 1,$$

we have either $\mathrm{Area}(C_1) \le \frac{1}{2}$ or $\mathrm{Area}(C_2) \le \frac{1}{2}$. Assume that $\mathrm{Area}(C_1) \le \frac{1}{2}$, so that $\mathrm{Area}(C_1') \ge \frac{1}{2}$. We have

$$|c_1| = |\overline{P_1P_5}| \le |\overline{P_1P_6'}| + |\overline{P_6'P_5}| \le \tfrac{2}{3}L_1 + \tfrac{1}{3}L_1 = L_1.$$

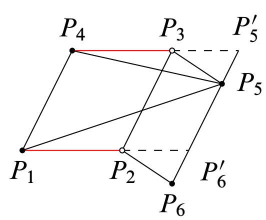

Figure 18: Cylinder with bounded circumference and area at least $\frac{1}{2}$ in a slit torus

Thus we can conclude that C_1' satisfies the statement of the lemma. In the case that $\mathrm{Area}(C_2) \leq \frac{1}{2}$, the same argument shows that the complement C_2' of C_2 has the required properties. The proof of the lemma is now complete. $\square$

References

[1] **T Aougab**, *Uniform hyperbolicity of the graphs of curves*, Geom. Topol. 17 (2013) 2855–2875 MR

[2] **M Bestvina**, **K Fujiwara**, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. 6 (2002) 69–89 MR

[3] **F Bonahon**, *Bouts des variétés hyperboliques de dimension 3*, Ann. of Math. 124 (1986) 71–158 MR

[4] **F Bonahon**, *The geometry of Teichmüller space via geodesic currents*, Invent. Math. 92 (1988) 139–162 MR

[5] **B H Bowditch**, *Hyperbolic 3–manifolds and the geometry of the curve complex*, from "European Congress of Mathematics" (A Laptev, editor), Eur. Math. Soc., Zürich (2005) 103–115 MR

[6] **B H Bowditch**, *Intersection numbers and the hyperbolicity of the curve complex*, J. Reine Angew. Math. 598 (2006) 105–129 MR

[7] **B H Bowditch**, *Uniform hyperbolicity of the curve graphs*, Pacific J. Math. 269 (2014) 269–280 MR

[8] **J F Brock**, **R D Canary**, **Y N Minsky**, *The classification of Kleinian surface groups, II: The ending lamination conjecture*, Ann. of Math. 176 (2012) 1–149 MR

[9] **P Buser**, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics 106, Birkhäuser, Boston (1992) MR

[10] **K Calta**, *Veech surfaces and complete periodicity in genus two*, J. Amer. Math. Soc. 17 (2004) 871–908 MR

[11] **K Calta**, **J Smillie**, *Algebraically periodic translation surfaces*, J. Mod. Dyn. 2 (2008) 209–248 MR

[12] **M Clay**, **K Rafi**, **S Schleimer**, *Uniform hyperbolicity of the curve graph via surgery sequences*, Algebr. Geom. Topol. 14 (2014) 3325–3344 MR

[13] **M Duchin**, **C J Leininger**, **K Rafi**, *Length spectra and degeneration of flat metrics*, Invent. Math. 182 (2010) 231–277 MR

[14] **B Farb**, **D Margalit**, *A primer on mapping class groups*, Princeton Mathematical Series 49, Princeton Univ. Press (2012) MR

[15] **R H Gilman**, *On the definition of word hyperbolic groups*, Math. Z. 242 (2002) 529–541 MR

[16] **U Hamenstädt**, *Train tracks and the Gromov boundary of the complex of curves*, from "Spaces of Kleinian groups" (Y N Minsky, M Sakuma, C Series, editors), London Math. Soc. Lecture Note Ser. 329, Cambridge Univ. Press (2006) 187–207 MR

[17] **U Hamenstädt**, *Geometry of the complex of curves and of Teichmüller space*, from "Handbook of Teichmüller theory, I" (A Papadopoulos, editor), IRMA Lect. Math. Theor. Phys. 11, Eur. Math. Soc., Zürich (2007) 447–467 MR

[18] **U Hamenstädt**, *Stability of quasi-geodesics in Teichmüller space*, Geom. Dedicata 146 (2010) 101–116 MR

[19] **J L Harer**, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. 84 (1986) 157–176 MR

[20] **W J Harvey**, *Boundary structure of the modular group*, from "Riemann surfaces and related topics: proceedings of the 1978 Stony Brook conference" (I Kra, B Maskit, editors), Ann. of Math. Stud. 97, Princeton Univ. Press (1981) 245–251 MR

[21] **S Hensel**, **P Przytycki**, **R C H Webb**, *1–slim triangles and uniform hyperbolicity for arc graphs and curve graphs*, J. Eur. Math. Soc. 17 (2015) 755–762 MR

[22] **P Hubert**, **E Lanneau**, *Veech groups without parabolic elements*, Duke Math. J. 133 (2006) 335–346 MR

[23] **P Hubert**, **T A Schmidt**, *Infinitely generated Veech groups*, Duke Math. J. 123 (2004) 49–69 MR

[24] **N V Ivanov**, *Automorphism of complexes of curves and of Teichmüller spaces*, Internat. Math. Res. Not. 1997 (1997) 651–666 MR

[25] **R Kenyon**, **J Smillie**, *Billiards on rational-angled triangles*, Comment. Math. Helv. 75 (2000) 65–108 MR

[26] **S Kerckhoff**, **H Masur**, **J Smillie**, *Ergodicity of billiard flows and quadratic differentials*, Ann. of Math. 124 (1986) 293–311 MR

[27] **E Klarreich**, *The boundary at infinity of the curve complex and the relative Teichmüller space*, preprint (1999) Available at `http://www.math.unicaen.fr/~levitt/klarreich.pdf`

[28] **M Kontsevich**, **A Zorich**, *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*, Invent. Math. 153 (2003) 631–678 MR

[29] **E Lanneau**, **D-M Nguyen**, *Connected components of Prym eigenform loci in genus three*, Math. Ann. (online publication April 2017) 41 pages

[30] **R Lehnert**, *On the critical exponent of infinitely generated Veech groups*, Math. Ann. (online publication August 2016) 42 pages

[31] **F Luo**, *Automorphisms of the complex of curves*, Topology 39 (2000) 283–298 MR

[32] **H Masur**, *Closed trajectories for quadratic differentials with an application to billiards*, Duke Math. J. 53 (1986) 307–314 MR

[33] **H A Masur**, **Y N Minsky**, *Geometry of the complex of curves, I: Hyperbolicity*, Invent. Math. 138 (1999) 103–149 MR

[34] **H Masur**, **S Schleimer**, *The geometry of the disk complex*, J. Amer. Math. Soc. 26 (2013) 1–62 MR

[35] **H Masur**, **S Tabachnikov**, *Rational billiards and flat structures*, from "Handbook of dynamical systems, 1A" (B Hasselblatt, A Katok, editors), North-Holland, Amsterdam (2002) 1015–1089 MR

[36] **C T McMullen**, *Teichmüller geodesics of infinite complexity*, Acta Math. 191 (2003) 191–223 MR

[37] **C T McMullen**, *Teichmüller curves in genus two: discriminant and spin*, Math. Ann. 333 (2005) 87–130 MR

[38] **C T McMullen**, *Dynamics of $\mathrm{SL}_2(\mathbb{R})$ over moduli space in genus two*, Ann. of Math. 165 (2007) 397–456 MR

[39] **Y N Minsky**, *Teichmüller geodesics and ends of hyperbolic 3–manifolds*, Topology 32 (1993) 625–647 MR

[40] **M Möller**, *Affine groups of flat surfaces*, from "Handbook of Teichmüller theory, II" (A Papadopoulos, editor), IRMA Lect. Math. Theor. Phys. 13, Eur. Math. Soc., Zürich (2009) 369–387 MR

[41] **D-M Nguyen**, *Parallelogram decompositions and generic surfaces in $\mathcal{H}^{\mathrm{hyp}}(4)$*, Geom. Topol. 15 (2011) 1707–1747 MR

[42] **D-M Nguyen**, *On the topology of $\mathcal{H}(2)$*, Groups Geom. Dyn. 8 (2014) 513–551 MR

[43] **K Rafi**, *Hyperbolicity in Teichmüller space*, Geom. Topol. 18 (2014) 3025–3053 MR

[44] **M Rees**, *An alternative approach to the ergodic theory of measured foliations on surfaces*, Ergodic Theory Dynamical Systems 1 (1981) 461–488 MR

[45] **J Smillie**, *The dynamics of billiard flows in rational polygons*, from "Dynamical systems, ergodic theory and applications" (Y G Sinai, editor), 2nd edition, Encyclopaedia of Mathematical Sciences 100, Springer (2000) 360–382 MR

[46] **J Smillie**, **B Weiss**, *Minimal sets for flows on moduli space*, Israel J. Math. 142 (2004) 249–260 MR

[47] **J Smillie**, **B Weiss**, *Characterizations of lattice surfaces*, Invent. Math. 180 (2010) 535–557 MR

[48] **M Troyanov**, *Prescribing curvature on compact surfaces with conical singularities*, Trans. Amer. Math. Soc. 324 (1991) 793–821 MR

[49] **W A Veech**, *Geometric realizations of hyperelliptic curves*, from "Algorithms, fractals, and dynamics" (Y Takahashi, editor), Plenum (1995) 217–226 MR

[50] **Y Vorobets**, *Periodic geodesics on generic translation surfaces*, from "Algebraic and topological dynamics" (S Kolyada, Y Manin, T Ward, editors), Contemp. Math. 385, Amer. Math. Soc. (2005) 205–258 MR

[51] **A Wright**, *Cylinder deformations in orbit closures of translation surfaces*, Geom. Topol. 19 (2015) 413–438 MR

[52] **A Wright**, *Translation surfaces and their orbit closures: an introduction for a broad audience*, EMS Surv. Math. Sci. 2 (2015) 63–108 MR

[53] **A Zorich**, *Flat surfaces*, from "Frontiers in number theory, physics, and geometry, I" (P Cartier, B Julia, P Moussa, P Vanhove, editors), Springer (2006) 437–583 MR

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