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Untwisting information from Heegaard Floer homology

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The unknotting number of a knot is the minimum number of crossings one must change to turn that knot into the unknot. We work with a generalization of the unknotting number due to Mathieu–Domergue, which we call the untwisting number. The p–untwisting number is the minimum number (over all diagrams of a knot) of full twists on at most $2p$ strands of a knot, with half of the strands oriented in each direction, necessary to transform that knot into the unknot. In previous work, we showed that the unknotting and untwisting numbers can be arbitrarily different. In this paper, we show that a common route for obstructing low unknotting number, the Montesinos trick, does not generalize to the untwisting number. However, we use a different approach to get conditions on the Heegaard Floer correction terms of the branched double cover of a knot with untwisting number one. This allows us to obstruct several 10– and 11–crossing knots from being unknotted by a single positive or negative twist. We also use the Ozsváth–Szabó τ invariant and the Rasmussen s invariant to differentiate between the p– and q–untwisting numbers for certain $p,q>1$.

57M25, 57M27; 57R58

1 Introduction

It is a natural knot-theoretic question to seek to measure "how knotted up" a knot is. One such "knottiness" measure is given by the *unknotting number* $u(K)$, the minimum number of crossings, taken over all diagrams of K, one must change to turn K into the unknot. By a *crossing change* we shall mean one of the two local moves on a knot diagram given in Figure 1.

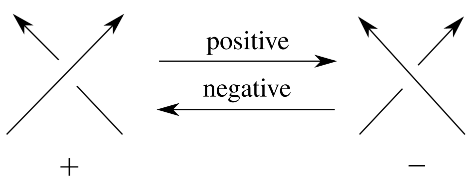

Figure 1: A positive and negative crossing change

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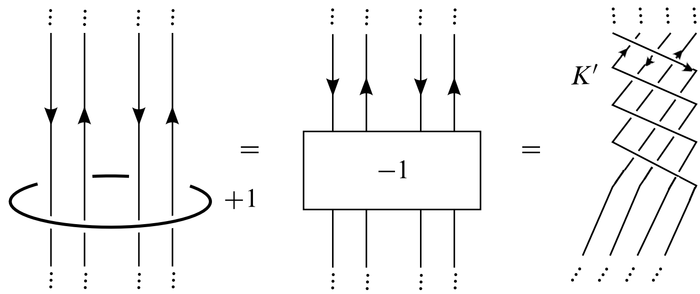

Figure 2: A left-handed, or positive, generalized crossing change

This invariant is quite simple to define but has proven itself very difficult to master. Fifty years ago, Milnor conjectured that the unknotting number for the (p,q)–torus knot was $\frac{1}{2}(p-1)(q-1)$; only in 1993, in two celebrated papers [6; 7], did Kronheimer and Mrowka prove this conjecture true. Hence, it is desirable to look at variants of the unknotting number which may be more tractable. One natural variant (due to Murakami [12]) is the *algebraic unknotting number* $u_a(K)$, the minimum number of crossing changes necessary to turn a given knot into an Alexander polynomial-one knot. Alexander polynomial-one knots are significant because they "look like the unknot" to *classical invariants*, knot invariants derived from the Seifert matrix. It is obvious that $u_a(K) \leq u(K)$ for any knot K, and there exist knots such that $u_a(K) < u(K)$ (for instance, the Whitehead double of any nontrivial knot).

In [9], Mathieu and Domergue defined another generalization of the unknotting number. In [8], Livingston worked with this definition. He described it as follows:

> "One can think of performing a crossing change as grabbing two parallel strands of a knot with opposite orientation and giving them one full twist. More generally, one can grab $2k$ parallel strands of K with k of the strands oriented in each direction and give them one full twist."

Following Livingston, we call such a twist a *generalized crossing change*. We describe in [4] how a crossing change may be encoded as a ±1–surgery on a nullhomologous unknot $U \subset S^3 - K$ bounding a disk D such that $D \cap K = 2$ points. From this perspective, a generalized crossing change is a relaxing of the previous definition to allow $D \cap K = 2k$ points for any k, provided $\operatorname{lk}(K,U) = 0$; see Figure 2. In particular, any knot can be unknotted by a finite sequence of generalized crossing changes.

One may then naturally define the *untwisting number* $\operatorname{tu}(K)$ to be the minimum length, taken over all diagrams of K, of a sequence of generalized crossing changes beginning at K and resulting in the unknot. By $\operatorname{tu}_p(K)$, we will denote the minimum number of generalized crossing changes on $2p$ or fewer strands, with p strands oriented in each direction, needed to unknot K. Notice that $\mathrm{tu}_1 = u$ and that

$$\mathrm{tu} \leq \cdots \leq \mathrm{tu}_{p+1} \leq \mathrm{tu}_p \leq \cdots \leq \mathrm{tu}_1 = u.$$

The *algebraic untwisting number* $\mathrm{tu}_a(K)$ is the minimum number of generalized crossing changes, taken over all diagrams of K, needed to transform K into an Alexander polynomial-one knot. It is clear that $\mathrm{tu}_a(K) \leq \mathrm{tu}(K)$ for all knots K. In [4], we showed that, in fact, $\mathrm{tu}_a(K) = u_a(K)$ for all knots K; hence the unknotting and untwisting numbers are "algebraically the same". However, we also showed that tu and u can be arbitrarily different in general: there exists a family of knots $\{S_p^q\}$ such that $(u - \mathrm{tu}_q)(S_p^q) \geq p - 1$ for all $p, q \geq 2$.

Since the family $\{S_p^q\}$ consists of $(p,1)$–cables of (untwisted) Whitehead doubles, most members of this family have very high crossing number. In this paper, we compare the unknotting and untwisting numbers for several 10– and 11–crossing knots with signature 0. In order to do this, we will develop an obstruction to a knot with signature 0 having untwisting number 1. This will require the methods of Heegaard Floer homology, specifically the d–*invariants* or *Heegaard Floer correction terms* of a 3–manifold.

In [19], Ozsváth and Szabó develop an unknotting number-1 obstruction using d–invariants. This obstruction relies on the *Montesinos trick*, which allows them to construct a definite 4–manifold with boundary the branched double cover $\Sigma(K)$ of an unknotting number-1 knot K. In Section 3, we give an infinite family of knots which have untwisting number 1 but which do not satisfy the Montesinos trick, eliminating that route toward a d–invariant obstruction:

Theorem 1.1 *There exists an infinite family $\{K_n\}_{n>1}$ of knots such that $\mathrm{tu}(K_n) = 1$ for all n, but $\Sigma(K_n)$ is not a half-integer surgery on any knot in S^3 for any n.*

In Section 4, we get around the failure of the Montesinos trick for untwisting number-1 knots by porting the machinery used by Owens and Strle in [16] and Nagel and Owens in [13] as an obstruction to low untwisting number:

Theorem 1.2 *Let K be a knot with signature $\sigma(K)$ which can be unknotted by p positive and n negative generalized crossing changes. Then $Y = \Sigma(K)$, the branched double cover of K, bounds a smooth, definite 4–manifold W with $b_2(W) = 2n + 2p$ and signature $2n - 2p + \sigma(K)$. Moreover, $H_2(W;\mathbb{Z})$ contains n classes of self-intersection $+2$ and p classes of self-intersection -2 which span a primitive sublattice; in other words, the quotient of $H_2(W;\mathbb{Z})$ by this sublattice is torsion-free.*

Once we have constructed a definite 4–manifold W with $\partial W = \Sigma(K)$, the next step is to apply a result of Ozsváth and Szabó to get conditions that the d–invariants of $\Sigma(K)$ must satisfy. These invariants are easily computable for alternating K via the *Goeritz matrix* associated to K. These computations are discussed further in Section 4. We successfully obstruct several 10–crossing knots from being unknotted by a single positive and/or negative generalized crossing change, though these untwisting numbers cannot be computed using the methods available prior to the development of Heegaard Floer homology:

Theorem 1.3 *The knots 10_{68} and 10_{96} have untwisting number 2, the knots 10_{22}, 10_{34}, 10_{35}, 10_{87} and 10_{90} cannot be unknotted by a single positive generalized crossing change, and the knot 10_{48} cannot be unknotted by a single negative generalized crossing change.*

Similarly, we apply these obstructions to all 11–crossing knots with signature 0, algebraic unknotting number 1, and unknotting number 2 to get the following:

Theorem 1.4 *The knots $11a_{37}, 11a_{103}, 11a_{169}$, $11a_{214}$ and $11a_{278}$ have untwisting number 2.*

Finally, we showed in [4] that there can be arbitrarily large gaps between the p–untwisting number and the 1–untwisting number (which by definition equals the unknotting number) for several families of knots. However, we had not yet been able to distinguish between tu_p and tu_q for $p, q > 1$.

In Section 6, we use invariants coming from Heegaard Floer homology (the Ozsváth–Szabó τ invariant) and Khovanov homology (the Rasmussen s invariant) to give lower bounds on the p–untwisting number for arbitrary p via the following theorem. While visiting Mark Powell at the Max Planck Institute, he suggested this theorem and outlined a proof similar to the proof of Powell and coauthors T Cochran, S Harvey, and A Ray that the τ and s invariants give lower bounds for their bipolar metrics (to appear in a future paper). The referee suggested a simpler approach involving the 4–genus, detailed in Section 6.

Theorem 1.5 *Let K be a knot which can be converted to the unknot via n generalized crossing changes, where for every i, the i^{th} generalized crossing change is performed on $2p_i$ strands. Then*

$$|\tau(K)| \leq \sum_{i=1}^{n} p_i^2 \quad \text{and} \quad \tfrac{1}{2}|s(K)| \leq \sum_{i=1}^{n} p_i^2.$$

This allows us to show that there exist $p, q > 1$ such that the difference between the p– and q–untwisting numbers of several families of knots can be made arbitrarily large:

Example 1.6 Let K_{p^3} denote the $(p^3, 1)$–cable of a knot K with genus 1 and $u(K) = 1 = \tau(K)$ (one example of such a K is the right-handed trefoil knot). We know from [4, Section 5] that $\mathrm{tu}_{p^3}(K_{p^3}) = 1$. We may use Theorem 1.5 to show that

$$\mathrm{tu}_p(K_{p^3}) - \mathrm{tu}_{p^3}(K_{p^3}) \xrightarrow{p\to\infty} \infty.$$

Convention In this paper, all manifolds are assumed to be smooth, compact, orientable and connected, and all surfaces in manifolds are assumed to be smoothly embedded. When homology groups are given without specifying coefficients, they are assumed to have coefficients in $\mathbb{Z}$.

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2 Preliminaries

2.1 Dehn surgery

In this section, we will describe the operation of Dehn surgery on knots.

Definition 2.1 Let $K \subset S^3$ be an oriented knot, let N be a closed tubular neighborhood of K, and consider the preferred framing for N (see [20, Definition 2E8]) in which the longitude L is oriented in the same way as K and the meridian M has linking number $+1$ with K. We may write any simple closed curve $J \subset \partial N$ in terms of the homology basis $\{\lambda = [L],\ \mu = [M]\}$:

$$[J] = q\lambda + p\mu \in H_1(\partial N).$$

The *result of* (p/q)*–surgery on* K is the 3–manifold

$$S^3_{p/q}(K) := (S^3 - \mathring{N}) \cup_h (S^1 \times D^2),$$

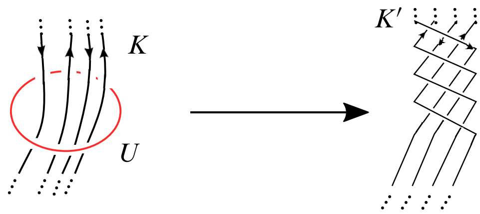

Figure 3: Performing $+1$–surgery on an unknot U gives the knot K a left-handed twist.

where $h: \partial(S^1 \times D^2) \to \partial N$ is a homeomorphism taking $* \times S^1$ onto a curve J of class $[J] = p\mu + q\lambda$ in $H_1(\partial N)$. By convention, we indicate that surgery is to be performed on K by writing the ratio p/q next to a diagram of K.

If $U \subset S^3 \setminus K$ is an unknot such that $\operatorname{lk}(K, U) = 0$, we define a *generalized crossing change diagram for* K to be a diagram of the link $K \sqcup U$ with the number ± 1 written next to U, indicating that U is to have ± 1–surgery performed on it.

There is an orientation-preserving homeomorphism Φ of the manifold $M := S^3_{\pm 1}(U)$ resulting from ± 1–surgery on U with S^3. However, $K' := \Phi(K) \subset S^3$ may have a different knot type than K. (Note that the knot type of K' does not depend on the choice of homeomorphism Φ since any two orientation-preserving homeomorphisms of S^3 are isotopic.) In particular, if D is a disk bounded by U such that $2p$ strands of K pass through D in straight segments, then each of the $2p$ straight pieces is replaced by a helix which screws through a neighborhood of D in the right- (respectively, left-) hand sense; see Figure 3.

The process of performing ± 1–surgery on an unknot U in a generalized crossing change diagram for a knot K, mapping the resulting manifold to S^3 via an orientation-preserving homeomorphism Φ, then erasing $\Phi(U)$ from the resulting diagram of $\Phi(K) \sqcup \Phi(U)$ is called a $\pm$ *generalized crossing change* on K. Now, it can be easily verified that performing a $-$ generalized crossing change on the knot K on the left side of Figure 4 transforms the crossing labeled $+$ into the crossing labeled $-$. The inverse process of introducing an unknot labeled with a $+1$ to the right side of Figure 4 and performing a $+$ generalized crossing change in the resulting generalized crossing change diagram transforms the crossing labeled $-$ into the crossing labeled $+$.

2.2 The untwisting number

In a generalized crossing change diagram for K consisting of a diagram of K and an unknot U, we have that K must pass through U an even number of times, for otherwise $\operatorname{lk}(K, U) \neq 0$. If at most $2p$ strands of K pass through an unknot U in

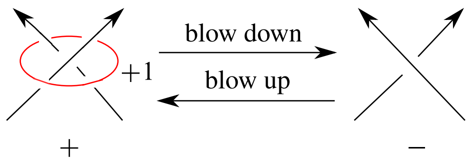

Figure 4: A crossing change is a 1–generalized crossing change.

a generalized crossing change diagram, we may call the associated $\pm$ generalized crossing change a *$\pm p$–generalized crossing change on K*.

The *untwisting number* $\mathrm{tu}(K)$ *of* K is the minimum length of a sequence of generalized crossing changes on K such that the result of the sequence is the unknot, where we allow ambient isotopy of the diagram in between generalized crossing changes. Note that by the reasoning on page 58 of [1], this definition is equivalent to taking the minimum length, over all diagrams of K, of a sequence of generalized crossing changes beginning with a fixed diagram of K such that the result of the sequence is the unknot, where we do not allow ambient isotopy of the diagram in between generalized crossing changes.

For $p = 1, 2, 3, \ldots$, we define the *p–untwisting number* $\mathrm{tu}_p(K)$ to be the minimum length of a sequence of $\pm p$–generalized crossing changes on K resulting in the unknot, where we allow ambient isotopy of the diagram in between generalized crossing changes. It follows immediately that we have the chain of inequalities

$$\mathrm{tu}(K) \le \cdots \le \mathrm{tu}_{p+1}(K) \le \mathrm{tu}_p(K) \le \cdots \le \mathrm{tu}_2(K) \le \mathrm{tu}_1(K) = u(K). \tag{2-1}$$

2.3 Heegaard Floer homology

In this section, we will recall some properties of Heegaard Floer homology, a set of invariants of 3–manifolds defined by Ozsváth and Szabó. For details, refer to their papers, in particular [17; 18; 19].

Let Y be an oriented rational homology 3–sphere. Recall that one can associate to Y a set $\mathrm{Spin}^c(Y)$ of *spin^c structures on Y*. In the case where $|H^2(Y;\mathbb{Z})|$ is odd, there is a canonical bijection $H^2(Y;\mathbb{Z}) \leftrightarrow \mathrm{Spin}^c(Y)$ under which $0 \in H^2(Y;\mathbb{Z})$ is sent to the unique spin structure on Y. In this way, we may give $\mathrm{Spin}^c(Y)$ a group structure inherited from that of $H^2(Y;\mathbb{Z})$.

Fix a spin^c structure $\mathfrak{s}$ on Y. Then the *(plus flavor of) Heegaard Floer homology* $\mathrm{HF}^+(Y,\mathfrak{s})$ is a $\mathbb{Q}$–graded abelian group with a $\mathbb{Z}[U]$–action, where multiplication by U lowers the grading by 2. Associated to $\mathfrak{s}$ is a *d–invariant* $d(Y,\mathfrak{s}) \in \mathbb{Q}$ which satisfies the symmetry condition $d(Y,\mathfrak{s}) = -d(-Y,\mathfrak{s})$. The correction terms are useful for obstructing the existence of a 4–manifold with boundary Y:

Theorem 2.2 (Ozsváth and Szabó [17]) *Let X be a negative-definite 4–manifold with boundary Y and intersection form represented by a matrix Q, and let $\mathfrak{s}$ be any spin^c structure on X. Let $c_1(\mathfrak{s})$ denote the first Chern class associated to $\mathfrak{s}$. Then*

$$\begin{aligned} &\tfrac{1}{4}(c_1(\mathfrak{s})^2 + b_2(X)) \le d(Y, \mathfrak{s}|_Y), \\ &\tfrac{1}{4}(c_1(\mathfrak{s})^2 + b_2(X)) \equiv d(Y, \mathfrak{s}|_Y) \bmod 2. \end{aligned} \tag{2-2}$$

Following [17], we now show how to check this obstruction in practice. In addition to the assumptions of Theorem 2.2, suppose for simplicity that $\pi_1(X) = 1$ and that $|H^2(Y;\mathbb{Z})|$ is odd. (This will always be true for the examples in this paper.) Let $r = b_2(X)$, the second Betti number of X. It is straightforward to see that $H_2(X;\mathbb{Z})$ is free of rank r in this case. Choose a basis $\{x_i\}_{i=1}^r$ for $H_2(X;\mathbb{Z})$ and let $Q = (Q_{ij})$ be a negative-definite $r \times r$ matrix representing the intersection pairing of X in this basis; then $\det Q = |H^2(Y;\mathbb{Z})|$. The dual basis $\{x^i\}_{i=1}^r$ for $H^2(X;\mathbb{Z})$ given by the universal coefficient theorem defines an isomorphism $H^2(X;\mathbb{Z}) \cong \mathbb{Z}^r$. Under this isomorphism, the set $\{c_1(\mathfrak{s}) \mid \mathfrak{s} \in \mathrm{Spin}^c(X)\} \subset H^2(X;\mathbb{Z})$ of first Chern classes of spin^c structures on X is sent to the set of characteristic covectors $\mathrm{Char}(Q)$ for Q. (Recall that a *characteristic covector* for an $r \times r$ matrix Q is a vector $\xi = (\xi_1, \dots, \xi_r) \in \mathbb{Z}^r$ such that $\xi_i \equiv Q_{ii} \bmod 2$ for $i = 1, \dots, r$.) In our basis, the square of the first Chern class of the spin^c structure corresponding to a characteristic covector ξ is given by $\xi^T Q^{-1} \xi$.

Define a function $m_Q\colon \mathbb{Z}^r / Q(\mathbb{Z}^r) \to \mathbb{Q}$ by

$$m_Q(g) = \max\bigl\{\tfrac{1}{4}(\xi^T Q^{-1} \xi + r) \mid \xi \in \mathrm{Char}(Q),\ [\xi] = g\bigr\},$$

where $[\xi]$ is the image of $\xi \in \mathbb{Z}^r$ under the projection to $\mathbb{Z}^r / Q(\mathbb{Z}^r)$. In computing m_Q, it is enough to consider characteristic covectors $\xi = (\xi_1, \dots, \xi_r)$ with $-Q_{ii} \ge \xi_i \ge Q_{ii}$; if, say, $\xi_i < Q_{ii}$, subtracting twice the i^{th} column of Q from ξ shows that $\xi^T Q^{-1} \xi$ is not maximal. Then we may simplify the conditions (2-2) as follows:

Theorem 2.3 (Ozsváth and Szabó) *Let Y be a rational homology 3–sphere which is the boundary of a simply connected, negative-definite 4–manifold X, with $|H^2(Y;\mathbb{Z})|$ odd. If the intersection pairing of X is represented in a basis by the matrix Q, then there is a group isomorphism*

$$\phi\colon \mathbb{Z}^r / Q(\mathbb{Z}^r) \to \mathrm{Spin}^c(Y)$$

such that for all $g \in \mathbb{Z}^r / Q(\mathbb{Z}^r)$,

$$\begin{aligned} &m_Q(g) \le d(Y, \phi(g)) \\ &m_Q(g) \equiv d(Y, \phi(g)) \bmod 2. \end{aligned} \tag{2-3}$$

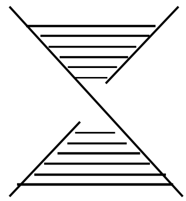

Figure 5: Crossing conventions for negative-definite Goeritz matrices of alternating knots

Under the assumptions of the theorem, we say that the 4–manifold X bounded by Y is *sharp* if equality holds in (2-3). In this case, we may compute the correction terms for Y using the values of m_Q. Moreover, if a rational homology sphere Y bounds a *positive*-definite 4–manifold X, we may compute the correction terms for Y by applying Theorem 2.3 to $-Y$.

If K is an alternating knot, we may compute the d–invariants of $\Sigma(K)$ using the negative-definite *Goeritz matrix* computed from an alternating diagram of K as follows. Consider a regular projection of K into a plane $\mathbb{R}^2 \subset \mathbb{R}^3 = S^3 \setminus \{\infty\}$. Color the regions of $\mathbb{R}^2 \setminus K$ alternately black and white so that the n white regions $X_1, \dots, X_n$ are separated by crossings of the type depicted in Figure 5.

For $0 \le i, j \le n$, where d is the number of double points incident to X_i and X_j, define

$$g_{ij} = \begin{cases} d, & i \neq j, \\ -\sum_{k \neq i} g_{ik}, & i = j. \end{cases}$$

Let $G' = (g_{ij})$. Then the *negative-definite Goeritz matrix* G associated to K is the $n \times n$ symmetric integer matrix obtained from G' by deleting the 0^{th} row and column of G'. It is shown in [19, Proposition 3.2] that G represents the intersection pairing of a sharp 4–manifold with boundary $\Sigma(K)$; thus, the correction terms for $\Sigma(K)$ are given by the values of m_G.

3 Failure of the Montesinos trick

The "Montesinos trick" relates crossing changes downstairs on K to surgery upstairs on $\Sigma(K)$, the branched double cover of K. We use the convention that the determinant of a knot is given by $|\Delta_K(-1)|$, where Δ_K is the Alexander polynomial for the knot K.

Theorem 3.1 [11] *If $u(K) = 1$, then $\Sigma(K) \cong S^3_{\pm D/2}(C)$ for some other knot $C \subset S^3$, where here D is the determinant of K.*

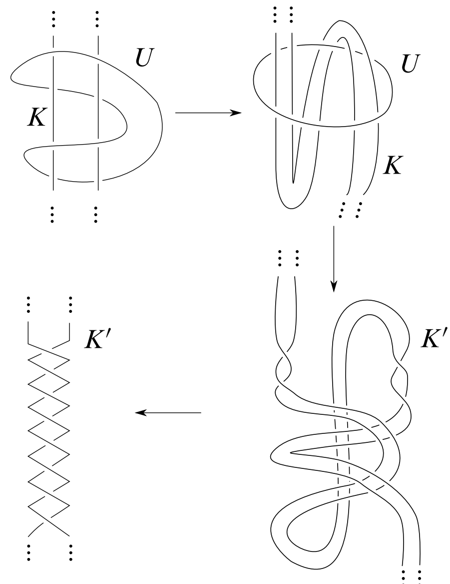

Figure 6: The (local) effect of performing a $+$ generalized crossing change on the unknot U

We show that this theorem does not generalize to untwisting number-1 knots:

Theorem 3.2 *There exists an infinite family $\{K_n\}$ of knots such that, for all $n \geq 1$, $\mathrm{tu}(K_n) = 1$, but $\Sigma(K_n)$ is not a half-integer surgery on any knot in S^3.*

In order to prove Theorem 3.2, we will need two main ingredients. The first is the following lemma:

Lemma 3.3 *The effect of performing a $+$ generalized crossing change on the unknot U in the local picture given in Figure 6 is to add -4 full twists to the knot K.*

Proof See Figure 6. The intermediate steps are left to the reader. □

Our second ingredient is the following theorem of McCoy [10]:

Theorem 3.4 *Let K be an alternating knot. Then the following are equivalent:*

(1) $u(K) = 1$;

(2) *the branched double cover $\Sigma(K)$ can be obtained by half-integer surgery on some knot in S^3;*

(3) *in every minimal diagram of K, there exists a crossing which can be changed to unknot that diagram.*

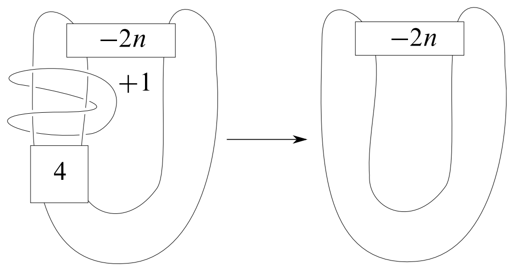

Figure 7: The knots K_n, together with the $+1$–generalized crossing change that unknots them. Here, positive (resp. negative) numbers in boxes denote right-handed (resp. left-handed) full twists.

Proof of Theorem 3.2 Fix an orientation on K_n. The generalized crossing change pictured in Figure 7 introduces -4–twists on the left side of K_n, which undo the 4–twists already present. Hence, $\mathrm{tu}(K_n) = 1$ for all n. Moreover, if $n > 1$, then K_n is a minimal diagram of an alternating knot. One can easily see that K_n cannot be unknotted by any single crossing change in this diagram. By Theorem 3.4, the branched double cover $\Sigma(K_n)$ cannot be obtained by half-integer surgery on any knot in S^3, and moreover, $u(K_n) > 1$. □

Note 3.5 The first knot in this family is $K_2 = 12a_{1166}$. The unknotting number of $12a_{1166}$ is listed as "not known" in the KnotInfo tables, but is either 1 or 2. By Theorem 3.4, we must have that $\mathrm{tu}(12a_{1166}) = 1 < 2 = u(12a_{1166})$.

Question 3.6 *Does there exist a knot K with $\mathrm{tu}(K) = 1$ such that $\Sigma(K)$ is not a surgery on* any *knot in S^3?*

4 Heegaard Floer-theoretic obstructions to untwisting number 1

Although the Montesinos trick does not hold for knots with untwisting number 1, we can still get obstructions to a knot K being unknotted by a single positive or negative generalized crossing change using techniques similar to those of Owens and Strle in [16] and Nagel and Owens in [13] together with Theorem 2.2.

In order to apply Theorem 2.2, we first compute a Goeritz matrix G for K and, from G, the function m_G as in Theorem 2.2. The image of $\mathbb{Z}^r/G(\mathbb{Z}^r)$ under m_G, where G is an $r \times r$ matrix, is the set of d–invariants for Y. We construct the 4–manifold W as in [13, Proposition 2.3] using the propositions below, then compute the m_Q and show that no isomorphism satisfying both conditions of (2-2) exists.

Proposition 4.1 *Let K be an oriented knot in S^3, and suppose that K can be unknotted by p positive and n negative generalized crossing changes. Then K bounds a disk Δ in a manifold $C \cong B^4 \#_n \mathbb{CP}^2 \#_p \overline{\mathbb{CP}^2}$ with $[\Delta] = 0 \in H_2(C, \partial C)$ and $\pi_1(C \setminus \Delta) = \mathbb{Z}$, generated by a meridian of K.*

Proof Suppose that K is an oriented knot in S^3 and that K can be unknotted by p positive and n negative generalized crossing changes. Then there is a sequence of knots

$$K := K_{p+n} \xrightarrow{\epsilon_{p+n}} K_{p+n-1} \xrightarrow{\epsilon_{p+n-1}} \cdots \xrightarrow{\epsilon_2} K_1 \xrightarrow{\epsilon_1} K_0 := U \tag{4-1}$$

for which K_i is obtained from K_{i+1} by a single generalized crossing change of sign $\epsilon_{i+1} \in \{\pm 1\}$ for $i = 1, \dots, p+n$, with precisely p of the ϵ_i equal to $+1$ and n of the ϵ_i equal to -1, and U is the unknot. Reversing our point of view, there is a sequence of knots

$$U := K_0 \xrightarrow{-\epsilon_1} K_1 \xrightarrow{-\epsilon_2} \cdots \xrightarrow{-\epsilon_{p+n-1}} K_{p+n-1} \xrightarrow{-\epsilon_{p+n}} K_{p+n} =: K \tag{4-2}$$

for which K_i is obtained from K_{i-1} by a single generalized crossing change of sign $-\epsilon_i$ for $i = 1, \dots, p+n$ and U is the unknot.

Consider U to be embedded in $\partial B^4 = S^3$. Since U is an unknot in S^3, it bounds an embedded disk $D \subset S^3$. We push D into B^4 to get a disk $\Delta_0 \subset B^4$ such that $\Delta_0 \cap \partial B^4 = U$ and $\pi_1(B^4 \setminus \Delta_0) = \mathbb{Z}$, where the latter is generated by a meridian of U.

Now, we build a 4–manifold C in which K bounds a disk Δ as follows. Let $C_0 := B^4$. We now build C from C_0 by sequentially thickening the boundary of C_0 and attaching 2–handles to the new boundary. First, we thicken the boundary $S_0 := \partial B^4$ to $S_0 \times [0, 1]$, obtaining a new 4–manifold B_0. We denote the disk $\Delta_0 \cup (U \times I) \subset B_0$ by Δ_1. The first generalized crossing change can be realized via the attachment of a $-\epsilon_1$–framed 2–handle h_1 along an unknot $U_1 \subset S_0 \times \{1\}$ with $\mathrm{lk}(U \times \{1\}, U_1) = 0$. There is a unique orientation-preserving diffeomorphism from the new boundary S_1 resulting from this handle attachment to S^3, and after this diffeomorphism $U \times \{1\}$ is isotopic to K_1. We denote by C_1 the new 4–manifold resulting from this handle attachment. Since attaching a ± 1–framed 2–handle to the boundary of a 4–manifold along an unknot results in connect-summing a $\pm\mathbb{CP}^2$, we have that $C_1 \cong C_0 \# {-\epsilon_1} \mathbb{CP}^2 = B^4 \# {-\epsilon_1} \mathbb{CP}^2$ (here $\pm\mathbb{CP}^2$ denotes $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$, respectively). Note that Δ_1 is still a disk in C_1 and that $\partial \Delta_1 = K_1$.

Attaching a 2–handle generally adds a relation to the fundamental group of the resulting manifold, where the relation is given by the attaching map. Since the attaching circle U_1 of h_1 is trivial in $H_1\big((S_0 \times \{1\}) \setminus (U \times \{1\})\big) \cong \mathbb{Z}\langle \mu_0 \rangle$, where μ_0 is a meridian of $U \times \{1\} \subset S_0 \times \{1\}$, it is also trivial in $\pi_1(B_0 \setminus \Delta_1) \cong \mathbb{Z}\langle \mu_0 \rangle$. Thus, we get that $\pi_1(C_1 \setminus \Delta_1) \cong \mathbb{Z}$ as well, generated by a meridian μ_1 of K_1.

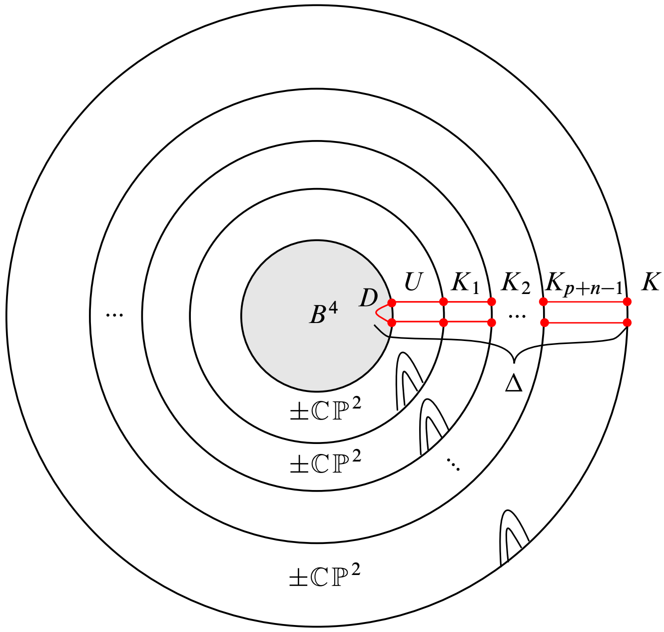

Figure 8: The construction of a manifold C in which K bounds a disk Δ

We continue in this way to iteratively get 4–manifolds $C_1, \dots, C_{p+n}$ so that C_{i+1} is obtained from C_i by adding a collar $\partial C_i \times [i, i+1]$ to ∂C_i and attaching a $-\epsilon_{i+1}$–framed 2–handle h_{i+1} to $\partial C_i \times \{i+1\}$. At each stage, the attaching circle $U_{i+1} \subset S_i \times \{i+1\}$ of h_{i+1} is trivial in

$$H_1\big((S_i \times \{i+1\}) \setminus (K_i \times \{i+1\})\big) \cong \mathbb{Z}\langle \mu_i \rangle,$$

where μ_i is a meridian of $K_i \times \{i+1\}$. Hence, U_{i+1} is trivial in $\pi_1(B_i \setminus \Delta_{i+1}) \cong \mathbb{Z}\langle \mu_i \rangle$. The end result of this process is a 4–manifold $C := C_{p+n} \cong B^4 \#_n \mathbb{CP}^2 \#_p \overline{\mathbb{CP}^2}$ in which $K := K_{p+n}$ bounds a disk $\Delta := \Delta_{p+n}$ such that $\pi_1(C \setminus \Delta) \cong \mathbb{Z}$, generated by a meridian μ_{p+n} of $K = K_{p+n}$.

We now consider the nondegenerate intersection form $H_2(C, \partial C) \times H_2(C) \to \mathbb{Z}$ in order to show that $[\Delta] = 0 \in H_2(C, \partial C)$. Since $H_2(C) \cong \mathbb{Z}^{p+n}$ is generated by the $\mathbb{CP}^1$ factors $\mathbb{CP}^1_1, \dots, \mathbb{CP}^1_{p+n}$, where $\mathbb{CP}^1_i$ is a generator of the second homology of the i^{th} connect-summed copy of $\pm\mathbb{CP}^2$, we know that an element $a \in H_2(C, \partial C)$ is 0 if and only if $a \cdot [\mathbb{CP}^1_i] = 0$ for all $i = 1, \dots, p+n$.

Let d_i denote the disk bounded by the unknot U_i, and let D_i denote the second D^2 factor in the i^{th} 2–handle attached to C. Then $\mathbb{CP}^1_i$ is homologous to

$$\big(d_{i-1} \times \{i - \tfrac{1}{2}\}\big) \cup \big(U_i \times [i - \tfrac{1}{2}, i]\big) \cup (* \times D_i).$$

The only intersections of Δ with $\mathbb{CP}^1_i$ come from the intersections of K_{i-1} with d_i. Since $\mathrm{lk}(K_{i-1}, U_i) = 0$ for all i, we have that $[K_{i-1}]\cdot[d_i] = 0$ for all i. Therefore, $[\Delta] = 0 \in H_2(C, \partial C)$. This completes the proof of the proposition. □

Next, we prove a generalization of [13, Proposition 2.3]:

Proposition 4.2 *Let K be a knot in $S^3 = \partial B^4$, and suppose K bounds a properly embedded disk Δ in $C := B^4 \#_n \mathbb{CP}^2 \#_p \overline{\mathbb{CP}^2}$ such that $[\Delta] = 0 \in H_2(C, \partial C)$ and $\pi_1(C \setminus \Delta) = \mathbb{Z}$, generated by a meridian of K. Then there exists an oriented 4–manifold W with boundary $\partial W = \Sigma(K)$, the branched double cover of K, such that*

(1) *W is simply connected;*

(2) *$H_2(W;\mathbb{Z}) \cong \mathbb{Z}^{2(p+n)}$;*

(3) *the signature of W is $\sigma(W) = 2(n-p) + \sigma(K)$;*

(4) *there exist $p+n$ pairwise disjoint classes in $H_2(W;\mathbb{Z})$ represented by p surfaces of self-intersection -2 and n surfaces of self-intersection $+2$ which span a primitive sublattice; in other words, the quotient of $H_2(W;\mathbb{Z})$ by this sublattice is torsion-free.*

Proof Since $\pi_1(C \setminus \Delta) = \mathbb{Z}$ with generator the meridian of K, we may take the double cover $W = \Sigma_2(C, \Delta)$ of C branched along Δ, and by definition, we have $\partial W = \Sigma_2(K)$.

(1) Let $p\colon (\widetilde{C \setminus N(\Delta)}) \to C \setminus N(\Delta)$ denote the two-fold, unbranched cover of the complement of an open tubular neighborhood of Δ in C. Since $\pi_1(C \setminus \Delta) \cong \mathbb{Z}$, we have that $\pi_1(\widetilde{C \setminus \Delta}) \cong \mathbb{Z}$ as well. The branched cover W may be recovered from $\widetilde{C \setminus N(\Delta)}$ by gluing back a closed neighborhood $\overline{N(\Delta)} \cong D^2 \times \Delta$ along $p^{-1}(\partial \overline{N(\Delta)}) \cong S^1 \times \Delta$. A straightforward application of the Seifert–van Kampen theorem to $W = \widetilde{C \setminus \Delta} \cup_{p^{-1}(\partial \overline{N(\Delta)})} \overline{N(\Delta)}$ shows that $\pi_1(W) = 1$.

(2) We will need the following claim.

Claim The Euler characteristic of W is $\chi(W) = 2(p+n) + 1$.

Proof of claim By a standard Mayer–Vietoris argument, we may show that

$$H_i(C) = \begin{cases} \mathbb{Z}, & i = 0, \\ 0, & i = 1, 3, \\ \mathbb{Z}^{p+n}, & i = 2, \\ 0, & i = 4, \end{cases}$$

where $H_4(C)=0$ because C is a manifold with boundary. Thus, $\chi(C)=1+p+n$. We have that

$$\chi(C)=\chi(C\setminus\Delta)+\chi(\Delta)=\chi(C\setminus\Delta)+1.$$

Therefore, the double cover $\widetilde{C\setminus\Delta}$ of $C\setminus\Delta$ has Euler characteristic $2(\chi(C)-1)$. Since $W=\widetilde{C\setminus\Delta}\cup_{p^{-1}(\partial\overline{N(\Delta)})}\overline{N(\Delta)}$ as above, we have that

$$\chi(W)=2(\chi(C)-1)+1=2(p+n)+1. \qquad \square$$

Now, since $H_1(W;\mathbb{Z})=0$, the universal coefficient theorem together with the long exact cohomology sequence for $(X,\partial X)$ implies that $H^1(W,\partial W;\mathbb{Z})=0$ as well. By Poincaré–Lefschetz duality, we have that $H_3(W;\mathbb{Z})=0$ as well. Note that $H_4(W;\mathbb{Z})=0$ since W is a manifold with boundary. Now the Euler characteristic of W is

$$2p+2n+1=\chi(W)=1+b_2(W).$$

Therefore, $b_2(W)=2(p+n)$, and $H_2(W;\mathbb{Z})$ is free abelian of rank $2(p+n)$.

(3) Our proof follows the proof of [3, Theorem 3.7]. Let F_{-K} be a connected Seifert surface of the knot $-K$ with interior pushed into $-B^4$. Then the manifold $(\widehat{C},F):=(C,\Delta)\cup_{(S^3,K)}(-B^4,F_{-K})$ is closed. Let $\widehat{W}$ denote the double cover $\Sigma_2(\widehat{C},F)$ of $\widehat{C}$ branched over $F:=\Delta\cup_K F_{-K}$. Then $\widehat{W}=W\cup_{\Sigma_2(K)}X_K$, where X_K is the double cover $\Sigma_2(F_{-K})$ of $-B^4$ branched along F_{-K}. By [21; 5], the signature of X_K is $-\sigma(K)$. Applying Novikov additivity, we get that

$$\sigma(\widehat{W})=\sigma(W)+\sigma(X_K).$$

Furthermore, the G–signature theorem [2, Lemma 2.1] tells us that

$$\sigma(\widehat{W})=2\sigma(\widehat{C})-\tfrac{1}{2}([F]\cdot[F]).$$

Since in this case $[\Delta]=0\in H_2(C,\partial C)$, we have that $[F]\cdot[F]=0$ so

$$\sigma(W)=2\sigma(C)+\sigma(K).$$

Since $\sigma(C)=n-p$, we get that $\sigma(W)=2(n-p)+\sigma(K)$.

(4) We let S_i be a smoothly embedded surface representing the generator of $H_2(-\epsilon_i\mathbb{CP}^2_i)$, the i^{th} summand of C. We define x_i to be the homology class of the two-fold cover $\widehat{S}_i\subset W$ of S_i branched over $\Delta\cap S_i$, which is a subset of W. Since the S_i are pairwise disjoint downstairs, the $\widehat{S}_i$ are also pairwise disjoint. We show that the x_i have self-intersection $-2\epsilon_i$.

Let S_i^+ be a push-off of S_i. Then $S_i\cdot S_i^+=-\epsilon_i$. We make the disk Δ disjoint from the (codimension-2) intersection points $S_i\cap S_i^+$. In the branched cover, denote the

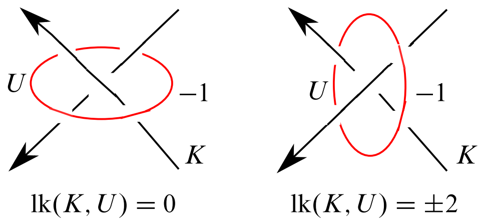

Figure 9: No matter the sign of the crossing to be changed, Nagel and Owens [13] may perform only -1–generalized crossing changes in order to do so.

preimage of S_i by T_i and the preimage of S_i^+ by T_i^+. Then T_i^+ is also a push-off of T_i. The intersection points of T_i and T_i^+ are the preimages of the intersection points of S_i and S_i^+; since the points of $S_i \cap S_i^+$ are disjoint from the branch set, there are geometrically two intersection points of T_i and T_i^+. Furthermore, the orientations upstairs give the same signs of intersection as downstairs. Therefore, $T_i \cdot T_i^+ = -2\epsilon_i$.

The proof of [13, Proposition 2.3] applies to our case to show that these classes span a primitive sublattice. This completes the proof of the proposition. □

Remark 4.3 The proof of Proposition 4.2 is very similar to the proof of [13, Proposition 2.3], with the caveat that Nagel and Owens use only -1–generalized crossing changes in order to unknot K, no matter the signs of the crossings of K that need to be changed (see Figure 9). The diagram on the right side of Figure 9 is not a generalized crossing change diagram, since $\mathrm{lk}(K, U) \neq 0$. Therefore, we must assume that K can be unknotted only by positive generalized crossing changes.

From Propositions 4.1 and 4.2, we derive a theorem analogous to [13, Theorem 1], but requiring the additional condition that the signature of the knot K is 0:

Theorem 4.4 *Let $K \subset S^3$ be an oriented knot with signature 0 which can be unknotted by p generalized crossing changes, all of sign $+1$. Then the double cover $Y := \Sigma(K)$ of S^3 branched along K bounds a smooth, simply connected, negative-definite 4–manifold W with $H_2(W;\mathbb{Z}) \cong \mathbb{Z}^{2p}$. Moreover, $H_2(W;\mathbb{Z})$ contains p pairwise disjoint homology classes of self-intersection -2 which span a primitive sublattice.*

Proof By Proposition 4.1, K bounds a disk Δ in a manifold $C \cong B^4 \#_p \mathbb{CP}^2$ such that $[\Delta] = 0 \in H_2(C, \partial C)$ and $\pi_1(C \setminus \Delta) = \mathbb{Z}\langle \mu \rangle$, where μ is a meridian of K. By Proposition 4.2, the double cover $W := \Sigma_2(C, \Delta)$ of C branched over Δ is simply connected, has $H_2(W;\mathbb{Z}) \cong \mathbb{Z}^{2p}$, and contains p pairwise disjoint homology classes of self-intersection -2 which span a primitive sublattice. Moreover, the signature of W is $\sigma(W) = -2p + \sigma(K) = -2p$, so W is negative definite. □

Note 4.5 If instead K can be unknotted using n generalized crossing changes, all of sign -1, Theorem 4.4 applied to $-K$ shows that the double cover $-Y$ of S^3 branched along $-K$ bounds a smooth negative-definite 4–manifold W with $b_1(W)=0$, $b_2(W)=2n$, and such that $H_2(W;\mathbb{Z})$ contains n pairwise disjoint surface classes of self-intersection -2 which span a primitive sublattice.

In the rest of this paper, we will say $\mathrm{tu}(K)=\pm 1$ if K can be unknotted by a single $\pm$ generalized crossing change. If $\sigma(K)=0$ and $\mathrm{tu}(K)=\pm 1$, we can always get a negative-definite 4–manifold W bounding $\pm\Sigma(K)$: if K can be unknotted by a positive generalized crossing change, then we get a negative-definite W bounding $+\Sigma(K)$, and if K can be unknotted by a negative generalized crossing change, then we get a negative-definite W bounding $-\Sigma(K)$. Moreover, the intersection form on W is represented by a definite 2×2 matrix Q. For an $n\times n$ matrix M, we denote by Γ_M the group $\mathbb{Z}^n/M(\mathbb{Z}^n)$. With this terminology established, we may state the following corollary of Proposition 4.1, which simplifies our computations:

Corollary 4.6 *Let K be an alternating knot such that $\mathrm{tu}(K)=\pm 1$ and $\sigma(K)=0$. We use the convention that $\det K=|\Delta_K(-1)|>0$. Let G be the negative-definite Goeritz matrix obtained from an alternating diagram for $\pm K$. Then there exists a negative-definite matrix of the form*

$$Q=\begin{pmatrix}-\frac{1}{2}(\det K+1) & 1\\ 1 & -2\end{pmatrix}$$

such that $\pm Y=\pm\Sigma(K)$ bounds a negative-definite 4–manifold with intersection form Q. Moreover, there is an isomorphism $\phi\colon \Gamma_Q\to\Gamma_G$ such that, for all $g\in\Gamma_Q$,

$$m_Q(g)\le m_G(\phi(g)), \tag{4-3}$$

$$m_Q(g)\equiv m_G(\phi(g)) \bmod 2. \tag{4-4}$$

Proof By Theorem 4.4, $\pm Y$ bounds a negative-definite 4–manifold with intersection form represented by

$$P=\begin{pmatrix}a & b\\ b & -2\end{pmatrix}$$

for some $a,b\in\mathbb{Z}$. By Theorem 2.2, there must exist isomorphisms

$$\Gamma_P \underset{\cong}{\xrightarrow{\phi}} \mathrm{Spin}^c(\pm Y)\cong H^2(Y;\mathbb{Z}) \underset{\cong}{\xrightarrow{\mathrm{PD}}} H_1(Y;\mathbb{Z}),$$

where the isomorphism labeled "PD" is from Poincaré duality and the order of $H_1(Y;\mathbb{Z})$ is equal to $\det K$. The matrix P presents the group $\mathbb{Z}/(\det P)\mathbb{Z}$. Therefore, we must have $\det P = \pm \det K$. Since $\det K$ is odd, we have that

$$b^2 \equiv -2a - b^2 = \det P \equiv \det K \equiv 1 \mod 2,$$

and hence b is odd. Therefore, we can use simultaneous row and column operations to change P into a matrix of form

$$Q = \begin{pmatrix} a & 1 \\ 1 & -2 \end{pmatrix}.$$

Since Q is negative definite, $\det Q \geq 0$, so we must have $\det Q = +\det K$. Therefore, $a = -\frac{1}{2}(\det K + 1)$. It follows from Theorem 3.1 that $m_Q(g) \leq m_G(g)$ and that the two are congruent modulo 2. The corollary follows. □

Note 4.7 Ozsváth and Szabó used a similar process to obstruct knots from having unknotting number 1 in [19], although their isomorphisms ϕ were also required to satisfy a "symmetry" condition which is not necessarily satisfied in our case. In [19, Corollary 1.3], Ozsváth and Szabó computed the m_Q and m_G for various knots to determine whether there exist isomorphisms ϕ of the type given in Corollary 4.6. The only knot with signature 0 which had its unknotting number determined by Ozsváth and Szabó for which the untwisting number was unknown and for which the "symmetry" condition was not necessary is 10_{68}. In this way, we get from their computations that $\mathrm{tu}(10_{68}) = 2 = u(10_{68})$, even though $u_a(10_{68}) = 1$.

5 Examples

In this section, we will prove Theorems 1.3 and 1.4 using Corollary 4.6. Following Ozsváth and Szabó in [19], we will refer to an isomorphism ϕ satisfying (4-3) as a *positive matching* and an isomorphism ϕ satisfying (4-4) as an *even matching*. We obstruct the existence of positive, even matchings for each of the cases listed in Theorem 1.3. We illustrate the proof that $\mathrm{tu}(10_{68}) = 2$; the remaining knots are obstructed from having untwisting number $+1$ and/or -1 similarly.

Example 5.1 Although Ozsváth and Szabó have already verified in [19] that $\Sigma(10_{68})$ cannot bound a 4–manifold with intersection form

$$Q = \begin{pmatrix} -29 & 1 \\ 1 & -2 \end{pmatrix},$$

as it would have to if $\mathrm{tu}(10_{68}) = 1$, we replicate the computation below. The knot 10_{68} has $\sigma(10_{68}) = 0$, $\det 10_{68} = 57$ and Goeritz matrix

$$G = \begin{pmatrix} -4 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 \end{pmatrix}.$$

The values of m_G mod 2 are

0	98/57	50/57	28/19	86/57	56/57	36/19	14/57	2/57	24/19
110/57	2/57	30/19	32/57	56/57	16/19	8/57	50/57	20/19	2/3
98/57	4/19	8/57	86/57	6/19	32/57	14/57	26/19	110/57	110/57
26/19	14/57	32/57	6/19	86/57	8/57	4/19	98/57	2/3	20/19
50/57	8/57	16/19	56/57	32/57	30/19	2/57	110/57	24/19	2/57
14/57	36/19	56/57	86/57	28/19	50/57	98/57.			

If $\Sigma(10_{68})$ bounded a 4–manifold W as in Corollary 4.6, the matrix

$$Q = \begin{pmatrix} a & 1 \\ 1 & -2 \end{pmatrix}$$

representing the intersection form on W would have determinant equal to $-2a - 1 = \det(10_{68}) = 57$, so that $a = -29$ and

$$Q = \begin{pmatrix} -29 & 1 \\ 1 & -2 \end{pmatrix}.$$

In this case, the values of m_Q mod 2 are

0	112/57	106/57	32/19	82/57	64/57	14/19	16/57	100/57	22/19
28/57	100/57	18/19	4/57	64/57	2/19	58/57	106/57	12/19	4/3
112/57	10/19	58/57	82/57	34/19	4/57	16/57	8/19	28/57	28/57
8/19	16/57	4/57	34/19	82/57	58/57	10/19	112/57	4/3	12/19
106/57	58/57	2/19	64/57	4/57	18/19	100/57	28/57	22/19	100/57
16/57	14/19	64/57	82/57	32/19	106/57	112/57.			

These lists are not identical (in particular, there is a $112/57$ in the m_Q list but not in the m_G list), so there are no even matchings here and $\mathrm{tu}(10_{68}) \neq +1$.

The Goeritz matrix for -10_{68} is

$$G' = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -5 & 3 \\ 0 & 3 & -6 \end{pmatrix};$$

the values of $m_{G'}$ are

0	4/57	16/57	12/19	-50/57	-14/57	10/19	-32/57	28/57	-6/19
-56/57	28/57	2/19	-8/57	-14/57	-4/19	-2/57	16/57	-24/19	-2/3
4/57	-20/19	-2/57	-50/57	-30/19	-8/57	-32/57	-16/19	-56/57	-56/57
-16/19	-32/57	-8/57	-30/19	-50/57	-2/57	-20/19	4/57	-2/3	-24/19
16/57	-2/57	-4/19	-14/57	-8/57	2/19	28/57	-56/57	-6/19	28/57
-32/57	10/19	-14/57	-50/57	12/19	16/57	4/57.			

Using a Python program, we check all possible isomorphisms ϕ and find that there are no positive, even matchings between the values of m_Q and the values of $m_{G'}$. Therefore, $\mathrm{tu}(10_{68}) \neq -1$. Since $u(10_{68}) = 2$, we must have that $\mathrm{tu}(10_{68}) = 2$ as well.

6 Ozsváth–Szabó τ invariant and Rasmussen s invariant obstructions to p–untwisting number

In this section, we investigate p–generalized crossing changes for fixed p in order to prove Theorem 1.5.

Every p–generalized crossing change consists of $p(p-1) + p^2 = p(2p-1)$ standard crossing changes. Thus, for every positive integer p and every knot $K \subset S^3$, if $\mathrm{tu}_p(K) \leq n$, then there is an unknotting sequence consisting of $pn(2p-1)$ crossing changes such that

$$u(K) \leq p(2p-1)\,\mathrm{tu}_p(K),$$

whence

$$|\tau(K)| \leq u(K) \leq p(2p-1)\,\mathrm{tu}_p(K).$$

Thus, it is possible to use the τ invariant to get lower bounds on tu_p for all p. These bounds may be useful in distinguishing tu_p from tu_q for $p \neq q$. However, we may obtain a stronger bound using the smooth 4–genus as follows. While visiting Mark Powell at the Max Planck Institute, he suggested this theorem and outlined a somewhat more complicated proof. It is similar to the proof of Powell and coauthors T Cochran, S Harvey, and A Ray that the τ and s invariants give lower bounds for their bipolar metrics (to appear in a future paper). The following, simpler proof involving the 4–genus was suggested by the referee.

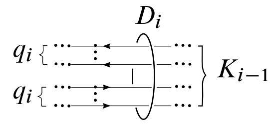

Figure 10: The result of the isotopy on D_i and the strands of K_{i-1}. We call the strands on the top *left-oriented* and those on the bottom *right-oriented*.

Theorem 6.1 *If K can be unknotted by k generalized crossing changes, where the i^{th} change is performed on $2q_i$ strands, then*

$$g_4(K) \leq \sum_{i=1}^{k} q_i^2.$$

Proof Suppose that K may be unknotted via k generalized crossing changes. Then there is a sequence of k generalized crossing changes taking K to U,

$$K = K_0 \xrightarrow{q_1\text{–gcc}} K_1 \xrightarrow{q_2\text{–gcc}} \cdots \xrightarrow{q_{k-1}\text{–gcc}} K_{k-1} \xrightarrow{q_k\text{–gcc}} K_k = U,$$

for which K_i is obtained from K_{i-1} by a single q_i–generalized crossing change for $i = 1, \ldots, k$. Let D_i be the disk bounded by the unknot U_i on which the i^{th} q_i–generalized crossing change is performed.

First, note that we can isotope D_i so that the strands of K_{i-1} pass through it as in Figure 10. The strands passing through D_i are oriented in two different ways; we separate the q_i strands of each orientation as in the figure. Let us arbitrarily call one group of q_i strands (say, the ones on the top of the figure) "left-oriented" and the other group "right-oriented". Hence, we may assume without loss of generality that we have a local picture as in Figure 10.

A q_i–generalized crossing change can be undone by changing $q_i(2q_i - 1)$ crossings; one changes precisely one crossing between the i^{th} and j^{th} strands (s_i and s_j) for each $1 \leq i < j \leq 2q_i$. Since q_i of the strands are oriented in one direction and q_i in the other, q_i^2 of these crossing changes occur between strands oriented in opposite directions and $q_i(q_i - 1)$ occur between strands oriented in the same direction (see Figure 11 for an illustration in the case of a 4–generalized crossing change). Thus, q_i^2 of the crossing changes have one sign, and $q_i^2 - q_i$ have the other sign. Therefore, K can be unknotted by changing P positive crossings and N negative crossings, where

$$\max\{P, N\} \leq \sum_{i=1}^{k} q_i^2.$$

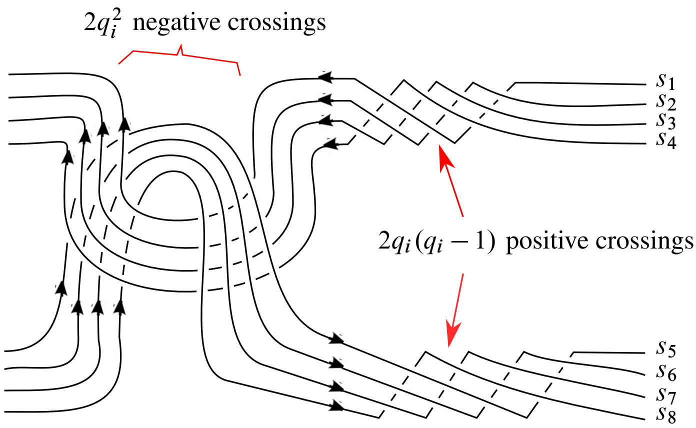

Figure 11: Two sets of four strands twisted around each other at a positive 4–generalized crossing change

However, it is well known, for instance by the argument in the third paragraph of the introduction of [15], that if K can be unknotted by changing P positive crossings and N negative crossings, then $g_4(K) \leq \max\{P, N\}$. □

Since the Ozsváth–Szabó τ invariant and Rasmussen s invariant give lower bounds on the slice genus of any knot, we immediately get the following:

Corollary 6.2 *Let K be a knot which can be converted to the unknot via k generalized crossing changes, where the i^{th} generalized crossing change is performed on $2q_i$ strands for $i = 1, \dots, k$. Then*

$$|\tau(K)| \leq \sum_{i=1}^{k} q_i^2 \quad \text{and} \quad \tfrac{1}{2}|s(K)| \leq \sum_{i=1}^{k} q_i^2.$$

This corollary gives rise to a method for distinguishing $\mathrm{tu}_q(K)$ from $\mathrm{tu}_p(K)$ for some $p, q > 1$. Suppose that $\mathrm{tu}_q(K) \leq n$. Then there exists an untwisting sequence for K consisting of n generalized crossing changes on $2p_i$ strands each, where $i = 1, \dots, n$ and $p_i \leq q$ for all i. Applying the corollary, we get that

$$|\tau(K)| \leq \sum_{i=1}^{n} p_i^2 \leq \sum_{i=1}^{n} q^2 = nq^2,$$

so we must have

$$n \geq \frac{|\tau(K)|}{q^2},$$

and similarly for $\frac{1}{2}|s(K)|$ in place of $|\tau(K)|$. We thus obtain the following obstruction to $\mathrm{tu}_q(K) = n$:

Corollary 6.3 *For all integers $q \geq 1$ and all knots $K \subset S^3$,*

$$\operatorname{tu}_q(K) \geq \frac{|\tau(K)|}{q^2} \quad \text{and} \quad \operatorname{tu}_q(K) \geq \frac{|s(K)|}{2q^2}.$$

Note 6.4 The above obstruction shows that $|\tau(K)| \leq p^2 \cdot \operatorname{tu}_p(K)$ for all K, which is stronger than the obstruction $|\tau(K)| \leq p(2p-1)\operatorname{tu}_p(K)$ given by representing a p–generalized crossing change as $p(2p-1)$ standard crossing changes.

Example 6.5 Let K_{p^3} denote the $(p^3, 1)$–cable of a knot K with $u(K) = 1 = \tau(K) = g(K)$ (one example is the right-handed trefoil knot). We know from [4, Section 5.1] that $\operatorname{tu}_{p^3}(K_{p^3}) = 1$ and that $\tau(K_{p^3}) = p^3$. However, the above result shows that

$$\operatorname{tu}_p(K_{p^3}) \geq \frac{|\tau(K_{p^3})|}{p^2} = p$$

for all integers $p \geq 1$. Hence

$$\operatorname{tu}_p(K_{p^3}) - \operatorname{tu}_{p^3}(K_{p^3}) = \operatorname{tu}_p(K_{p^3}) - 1 \geq p - 1 \xrightarrow{p \to \infty} \infty.$$

References

[1] **C C Adams**, *The knot book*, W. H. Freeman, New York (1994) MR

[2] **A J Casson**, **C M Gordon**, *On slice knots in dimension three*, from “Algebraic and geometric topology, Part 2” (R J Milgram, editor), Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., Providence, RI (1978) 39–53 MR

[3] **T D Cochran**, **W B R Lickorish**, *Unknotting information from* 4*–manifolds*, Trans. Amer. Math. Soc. 297 (1986) 125–142 MR

[4] **K Ince**, *The untwisting number of a knot*, Pacific J. Math. 283 (2016) 139–156 MR

[5] **L H Kauffman**, **L R Taylor**, *Signature of links*, Trans. Amer. Math. Soc. 216 (1976) 351–365 MR

[6] **P B Kronheimer**, **T S Mrowka**, *Gauge theory for embedded surfaces, I*, Topology 32 (1993) 773–826 MR

[7] **P B Kronheimer**, **T S Mrowka**, *Gauge theory for embedded surfaces, II*, Topology 34 (1995) 37–97 MR

[8] **C Livingston**, *The slicing number of a knot*, Algebr. Geom. Topol. 2 (2002) 1051–1060 MR

[9] **Y Mathieu**, **M Domergue**, *Chirurgies de Dehn de pente* ± 1 *sur certains nœuds dans les* 3*–variétés*, Math. Ann. 280 (1988) 501–508 MR

[10] **D McCoy**, *Alternating knots with unknotting number one*, Adv. Math. 305 (2017) 757–802 MR

[11] **J M Montesinos**, *Three-manifolds as 3–fold branched covers of S^3*, Quart. J. Math. Oxford Ser. 27 (1976) 85–94 MR

[12] **H Murakami**, *Algebraic unknotting operation*, from "Proceedings of the Second Soviet-Japan Joint Symposium of Topology", volume 8 (1990) 283–292 MR

[13] **M Nagel**, **B Owens**, *Unlinking information from 4–manifolds*, Bull. Lond. Math. Soc. 47 (2015) 964–979 MR

[14] **B Owens**, *Unknotting information from Heegaard Floer homology*, Adv. Math. 217 (2008) 2353–2376 MR

[15] **B Owens**, *On slicing invariants of knots*, Trans. Amer. Math. Soc. 362 (2010) 3095–3106 MR

[16] **B Owens**, **S Strle**, *Immersed disks, slicing numbers and concordance unknotting numbers*, Comm. Anal. Geom. 24 (2016) 1107–1138

[17] **P Ozsváth**, **Z Szabó**, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003) 179–261 MR

[18] **P Ozsváth**, **Z Szabó**, *On the Floer homology of plumbed three-manifolds*, Geom. Topol. 7 (2003) 185–224 MR

[19] **P Ozsváth**, **Z Szabó**, *Knots with unknotting number one and Heegaard Floer homology*, Topology 44 (2005) 705–745 MR

[20] **D Rolfsen**, *Knots and links*, Mathematics Lecture Series 7, Publish or Perish, Berkeley, CA (1976) MR

[21] **O J Viro**, *Branched coverings of manifolds with boundary, and invariants of links, I*, Izv. Akad. Nauk SSSR Ser. Mat. 37 (1973) 1241–1258 MR In Russian; translated in Math. USSR-Izvestiya 7 (1973) 1239–1256

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