Algebraic & Geometric Topology 17 (2017) 2413–2428

Epimorphisms between 2–bridge knot groups and their crossing numbers

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Suppose that there exists an epimorphism from the knot group of a 2–bridge knot K onto that of another knot K'. We study the relationship between their crossing numbers $c(K)$ and $c(K')$. More specifically, it is shown that $c(K)$ is greater than or equal to $3c(K')$, and we estimate how many knot groups a 2–bridge knot group maps onto. Moreover, we formulate the generating function which determines the number of 2–bridge knot groups admitting epimorphisms onto the knot group of a given 2–bridge knot.

57M25; 57M27

1 Introduction

Let K be a knot and $G(K)$ the knot group, namely, the fundamental group of the exterior of K in S^3. We denote by $c(K)$ the crossing number of K. Recently, many authors have studied epimorphisms between knot groups. One of the main goals of their papers was Simon's conjecture: every knot group maps onto at most finitely many knot groups. For example, Boileau, Boyer, Reid and Wang [4] showed that Simon's conjecture is true for 2–bridge knots. Finally, Agol and Liu [2] proved that Simon's conjecture holds for all knots.

In Kitano and Suzuki [12] and Horie, Kitano, Matsumoto and Suzuki [10], the existence and nonexistence of a meridional epimorphism between knot groups of prime knots with up to 11 crossings are determined completely. We say that a homomorphism from $G(K)$ to $G(K')$ is *meridional* if a meridian of $G(K)$ is sent to a meridian of $G(K')$; see also Cha and Suzuki [7]. This result raises the following question: if there exists an epimorphism from $G(K)$ onto $G(K')$, then is $c(K)$ greater than or equal to $c(K')$? This question is also mentioned in Kitano and Suzuki [13]. If the answer is affirmative, then we obtain another proof for Simon's conjecture. This paper gives a partial affirmative answer for this question. That is to say, if there exists an epimorphism from the knot group of a 2–bridge knot K onto that of another knot K', then $c(K)$ is greater than or equal to $3c(K')$.

Published: 3 August 2017 DOI: 10.2140/agt.2017.17.2413

In order to prove this result, we make use of the Ohtsuki–Riley–Sakuma construction [18]; these authors established a systematic construction of epimorphisms between 2–bridge knot groups. Additionally, Garrabrant, Hoste and Shanahan [9] gave necessary and sufficient conditions for any set of 2–bridge knots to have an upper bound with respect to the Ohtsuki–Riley–Sakuma construction. Conversely, it is shown that all epimorphisms between 2–bridge knot groups arise from the Ohtsuki–Riley–Sakuma construction, as a consequence of Agol's result announced in [1]. Aimi, Lee and Sakuma [3] give another proof for this result.

In this paper, we consider the crossing numbers of 2–bridge knots whose knot groups admit epimorphisms onto a 2–bridge knot group. By using this result, we estimate how many knot groups a 2–bridge knot group maps onto. Furthermore, we formulate the generating function which determines the number of 2–bridge knots K admitting epimorphisms from $G(K)$ onto the knot group of a given 2–bridge knot.

Throughout this paper, we do not distinguish a knot from its mirror image, since their knot groups are isomorphic and we discuss epimorphisms between knot groups. The numberings of the knots with up to 10 and 11 crossings follow Rolfsen's book [19] and the web page KnotInfo [6] by Cha and Livingston, respectively

2 2–bridge knot and continued fraction expansion

In this section, we recall some well-known results on 2–bridge knots. See [5; 17] in detail, for example.

A 2–bridge knot corresponds to a rational number $r = q/p \in \mathbb{Q}$; we denote the knot by $K(q/p)$. Schubert classified 2–bridge knots as follows.

Theorem 2.1 (Schubert) *Let $K(q/p)$ and $K(q'/p')$ be 2–bridge knots. These knots are equivalent if and only if the following conditions hold:*

(1) $p = p'$.

(2) *Either* $q \equiv \pm q' \pmod{p}$ *or* $qq' \equiv \pm 1 \pmod{p}$.

By using this theorem, it is sufficient to consider $r \in \mathbb{Q} \cap (0, \frac{1}{2})$. Note that $K(0)$ is the trivial link and that $K(\frac{1}{2})$ is the Hopf link. A rational number $q/p \in \mathbb{Q} \cap (0, \frac{1}{2})$ can be expressed as a continued fraction expansion

$$\frac{q}{p} = [a_1, a_2, \ldots, a_{m-1}, a_m] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{a_{m-1} + \cfrac{1}{a_m}}}}},$$

where $a_1 > 0$. Note that a rational number admits many continued fraction expansions. For example, we have $\frac{29}{81} = [3, -5, 4, 1, -2] = [2, 1, 3, 1, 5]$. It is easy to see that the following properties are satisfied. First, we can delete zeros in a continued fraction expansion by using the equation

$$[a_1, a_2, \dots, a_{i-2}, a_{i-1}, 0, a_{i+1}, a_{i+2}, \dots, a_m] \\ = [a_1, a_2, \dots, a_{i-2}, a_{i-1} + a_{i+1}, a_{i+2}, \dots, a_m].$$

If we consider a 2–bridge knot, we may assume that $a_1, a_m \neq \pm 1$, since

$$[a_1, a_2, \dots, a_{m-1}, \pm 1] = [a_1, a_2, \dots, a_{m-1} \pm 1]$$

and $K([a_1, a_2, \dots, a_m])$ is equivalent to $K([a_m, a_{m-1}, \dots, a_1])$ up to mirror image. Moreover, the euclidean algorithm allows us to take a continued fraction expansion such that all a_i in $[a_1, a_2, \dots, a_m]$ are positive.

If a rational number r is expressed as $[a_1, a_2, \dots, a_m]$ with $a_i > 0$ and $a_1, a_m \geq 2$, then the continued fraction expansion is called *standard*. By the above arguments, we can always take the standard continued fraction expansion of the rational number r for a 2–bridge knot $K(r)$. Furthermore, the standard continued fraction expansion gives us the unique continued fraction expansion of the rational number which corresponds to a 2–bridge knot in the following sense. Let $K(q/p)$ and $K(q'/p')$ be 2–bridge knots. Suppose that these rational numbers are written as the standard continued fraction expansions $q/p = [a_1, a_2, \dots, a_m]$ and $q'/p' = [a'_1, a'_2, \dots, a'_{m'}]$. It is known that $K(q/p)$ and $K(q'/p')$ are equivalent up to mirror image if and only if

$$(a_1, a_2, \dots, a_m) = (a'_1, a'_2, \dots, a'_{m'}) \text{ or } (a'_{m'}, a'_{m'-1}, \dots, a'_1).$$

Thistlethwaite [21], Kauffman [11] and Murasugi [15; 16] independently proved the first Tait conjecture. Hence, we can determine the crossing number of a 2–bridge knot by using the standard continued fraction expansion. Namely, the crossing number for the standard continued fraction $[a_1, a_2, \dots, a_m]$ is given by

$$c(K([a_1, a_2, \dots, a_m])) = \sum_{i=1}^{m} a_i.$$

3 Epimorphisms between 2–bridge knot groups

We have the following remarkable result about epimorphisms between 2–bridge knot groups: an epimorphism between 2–bridge knot groups is always meridional. Moreover, the rational numbers for these 2–bridge knots have the following relationship.

Theorem 3.1 (Ohtsuki, Riley and Sakuma [18], Agol [1], Aimi, Lee and Sakuma [3]) *Let $K(r), K(\widetilde{r})$ be 2–bridge knots, where $r = [a_1, a_2, \dots, a_m]$. If there exists an epimorphism $\varphi\colon G(K(\widetilde{r})) \to G(K(r))$, then φ is meridional and $\widetilde{r}$ can be written as*

$$(*)\qquad \widetilde{r} = [\varepsilon_1 \boldsymbol{a}, 2c_1, \varepsilon_2 \boldsymbol{a}^{-1}, 2c_2, \varepsilon_3 \boldsymbol{a}, 2c_3, \varepsilon_4 \boldsymbol{a}^{-1}, 2c_4, \dots, \varepsilon_{2n} \boldsymbol{a}^{-1}, 2c_{2n}, \varepsilon_{2n+1} \boldsymbol{a}],$$

where $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$, $\boldsymbol{a}^{-1} = (a_m, a_{m-1}, \dots, a_1)$, $\varepsilon_i = \pm 1$ $(\varepsilon_1 = 1)$, and $c_i \in \mathbb{Z}$.

Remark If a rational number $\widetilde{r}$ is expressed in the form $(*)$, then we say that $\widetilde{r}$ has an expansion of *type* $2n+1$ with respect to $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$.

(1) In this paper, we do not need to consider an expression of type $2n$ with respect to $\boldsymbol{a}$, since $K([\varepsilon_1 \boldsymbol{a}, 2c_1, \dots, 2c_{2n-1}, \varepsilon_{2n} \boldsymbol{a}^{-1}])$ is a 2–bridge link.

(2) If $c_i = 0$ and $\varepsilon_i \cdot \varepsilon_{i+1} = -1$, then

$$\begin{aligned}\widetilde{r} &= [\dots, \varepsilon_{i-1} \boldsymbol{a}^{\pm 1}, 2c_{i-1}, \varepsilon_i \boldsymbol{a}^{\mp 1}, 0, \varepsilon_{i+1} \boldsymbol{a}^{\pm 1}, 2c_{i+1}, \varepsilon_{i+2} \boldsymbol{a}^{\mp 1}, \dots] \\ &= [\dots, \varepsilon_{i-1} \boldsymbol{a}^{\pm 1}, 2c_{i-1}, 0, 2c_{i+1}, \varepsilon_{i+2} \boldsymbol{a}^{\mp 1}, \dots] \\ &= [\dots, \varepsilon_{i-1} \boldsymbol{a}^{\pm 1}, 2(c_{i-1} + c_{i+1}), \varepsilon_{i+2} \boldsymbol{a}^{\mp 1}, \dots].\end{aligned}$$

It follows that $\widetilde{r}$ has type $2n-1$. Hence we do not deal with the case $c_i = 0$, $\varepsilon_i \cdot \varepsilon_{i+1} = -1$.

Example 3.2 For example, we consider a 2–bridge knot $K(\frac{5}{27})$. The rational number $\frac{5}{27}$ has continued fraction expansions

$$\tfrac{5}{27} = [5, 2, 2] = [3, 0, 3, -2, 3].$$

The second expression implies that the crossing number of $K(\frac{5}{27})$ is 9. The last expression is of type 3 with respect to $\boldsymbol{a} = (3)$. Therefore the knot group $G(K(\frac{5}{27}))$ admits an epimorphism onto the trefoil knot group $G(3_1) = G(K(\frac{1}{3})) = G(K([3]))$. Similarly, we have

$$\tfrac{1}{9} = [9] = [3, 0, 3, 0, 3], \qquad \tfrac{19}{45} = [2, 2, 1, 2, 2] = [3, -2, 3, -2, 3].$$

It follows that there exist epimorphisms from $G(K(\frac{1}{9}))$ and $G(K(\frac{19}{45}))$ onto the trefoil knot group.

The previous papers [12] and [10] determined all the pairs of prime knots with up to 11 crossings which admit meridional epimorphisms between their knot groups. The results in those works coincide with the above examples. Note that $K(\frac{1}{9}) = 9_1$, $K(\frac{5}{27}) = 9_6$, and $K(\frac{19}{45}) = 9_{23}$.

In general, even if $[a_1,\dots,a_m]$ is the standard continued fraction expansion, and $\widetilde{r}$ is of type $2n+1$ with respect to $(a_1,\dots,a_m)$, this expansion of $\widetilde{r}$ may not be standard. However, we can get the standard continued fraction expansion and then determine the crossing number of $K(\widetilde{r})$.

Theorem 3.3 *Let $[a_1,\dots,a_m]$ be the standard continued fraction expansion. Suppose that a rational number $\widetilde{r}$ has a continued fraction expansion of the form $(*)$ of type $2n+1$ with respect to $\boldsymbol{a}=(a_1,\dots,a_m)$. Then the crossing number of $K(\widetilde{r})$ is given by*

$$c(K(\widetilde{r}))=(2n+1)|\boldsymbol{a}|+\sum_{i=1}^{2n}(2|c_i|-\psi(i)-\bar{\psi}(i)),$$

where $|\boldsymbol{a}|=\sum_{i=1}^m a_i$ and

$$\psi(i)=\begin{cases}1 & \text{if } \varepsilon_i\cdot c_i<0,\\ 0 & \text{if } \varepsilon_i\cdot c_i\geq 0,\end{cases}\qquad \bar{\psi}(i)=\begin{cases}1 & \text{if } c_i\cdot\varepsilon_{i+1}<0,\\ 0 & \text{if } c_i\cdot\varepsilon_{i+1}\geq 0.\end{cases}$$

Note that

$$\sum_{i=1}^{2n}(\psi(i)+\bar{\psi}(i))$$

is the number of sign changes. To prepare for Theorem 3.3, we prove the following lemma. Namely, negative integers in a continued fraction expansion can be changed into positive integers.

Lemma 3.4 *Let $a_1,\dots,a_k,b_1,\dots,b_l,c_1,\dots,c_m$ be integers. We have four cases:*

(1) *If $l\geq 2$, then*

$$\begin{aligned}&[a_1,\dots,a_k,-b_1,-b_2,\dots,-b_{l-1},-b_l,c_1,\dots,c_m]\\ &\quad=[a_1,\dots,,a_{k-1},a_k-1,1,b_1-1,b_2,\dots,b_{l-1},b_l-1,1,c_1-1,c_2,\dots,c_m].\end{aligned}$$

(2) *If $l=1$ and $b_1\geq 2$, then*

$$\begin{aligned}&[a_1,\dots,a_k,-b_1,c_1,\dots,c_m]\\ &\qquad=[a_1,\dots,,a_{k-1},a_k-1,1,b_1-2,1,c_1-1,c_2,\dots,c_m].\end{aligned}$$

(3) *If $l\geq 2$, then*

$$[a_1,\dots,a_k,-b_1,\dots,-b_l]=[a_1,\dots,,a_{k-1},a_k-1,1,b_1-1,b_2,\dots,b_l].$$

(4) *If $l=1$ and $b_1\geq 2$, then*

$$[a_1,\dots,a_k,-b_1]=[a_1,\dots,,a_{k-1},a_k-1,1,b_1-1].$$

Proof Recall the matrix representation of a continued fraction expansion (see for instance [14]). For a continued fraction $[x_1, x_2, \dots, x_m]$, we define p, q by

$$\begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_m & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p & * \\ q & * \end{pmatrix}.$$

It is known that we have an equality

$$[x_1, x_2, \dots, x_m] = \frac{q}{p}.$$

We will prove (1) by using the above matrix representation on both sides of the equation. Let A, C be the matrices defined by

$$A = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_m & 1 \\ 1 & 0 \end{pmatrix},$$

respectively, and define B by

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} b_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_{l-1} & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrix representation of the left-hand side of (1) is

$$\begin{aligned}
&\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -b_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -b_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \\
&\qquad \cdot \begin{pmatrix} -b_{l-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -b_l & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_m & 1 \\ 1 & 0 \end{pmatrix} \\
&= A \begin{pmatrix} -a_k b_1 + 1 & a_k \\ -b_1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^l B_{11} & (-1)^{l+1} B_{12} \\ (-1)^{l+1} B_{21} & (-1)^l B_{22} \end{pmatrix} \begin{pmatrix} -b_l c_1 + 1 & -b_l \\ c_1 & 1 \end{pmatrix} C \\
&= A \begin{pmatrix} a_k b_1 - 1 & a_k \\ b_1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^l B_{11} & (-1)^l B_{12} \\ (-1)^l B_{21} & (-1)^l B_{22} \end{pmatrix} \begin{pmatrix} b_l c_1 - 1 & b_l \\ c_1 & 1 \end{pmatrix} C \\
&= (-1)^l A \begin{pmatrix} a_k b_1 - 1 & a_k \\ b_1 & 1 \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} b_l c_1 - 1 & b_l \\ c_1 & 1 \end{pmatrix} C \\
&= (-1)^l \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_k - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&\qquad \cdot \begin{pmatrix} b_1 - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_{l-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_l - 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&\qquad \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_m & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

The last expression is $(-1)^l$ times the matrix representation of the right-hand side of (1). Therefore the rational numbers on both sides of (1) coincide.

Next, we examine the matrix representation of the left-hand side of (2):

$$\begin{aligned}
&\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -b_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_m & 1 \\ 1 & 0 \end{pmatrix} \\
&\qquad = A \begin{pmatrix} -a_k b_1 c_1 + a_k + c_1 & -a_k b_1 + 1 \\ -b_1 c_1 + 1 & -b_1 \end{pmatrix} C \\
&\qquad = (-1) A \begin{pmatrix} a_k b_1 c_1 - a_k - c_1 & a_k b_1 - 1 \\ b_1 c_1 - 1 & b_1 \end{pmatrix} C \\
&\qquad = (-1) \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_k - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_1 - 2 & 1 \\ 1 & 0 \end{pmatrix} \\
&\qquad\qquad \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_m & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

The last expression is also (-1) times the matrix representation of the right-hand side of (2). Hence these continued fraction expressions represent the same rational number.

A similar proof works for (3) and (4). □

Example 3.5 Suppose that the rational number $\frac{29}{81}$ is expressed as $[3,-5,4,1,-2]$. The above arguments show that

$$[3,-5,4,1,-2] = [2,1,3,1,3,0,1,1] = [2,1,3,1,4,1] = [2,1,3,1,5].$$

The last expression is the standard continued fraction expansion, and then the crossing number of $K(\frac{29}{81})$ is

$$c(K(\tfrac{29}{81})) = c(K([3,-5,4,1,-2])) = c(K([2,1,3,1,5])) = 12.$$

In Lemma 3.4, if all a_i, b_i, c_i are positive, then the integers on the right-hand sides of the equations are positive or zero. Hence, we can obtain the standard continued fraction expansion and determine the crossing number of a 2–bridge knot.

Corollary 3.6 *Let a_i, b_i, c_i be positive integers. If $l \neq 1$ or $b_1 \geq 2$, then we have*

(1) $c(K([a_1,\dots,a_k,-b_1,\dots,-b_l,c_1,\dots,c_m])) = \sum_{i=1}^{k} a_i + \sum_{i=1}^{l} b_i + \sum_{i=1}^{m} c_i - 2,$

(2) $c(K([a_1,\dots,a_k,-b_1,\dots,-b_l])) = \sum_{i=1}^{k} a_i + \sum_{i=1}^{l} b_i - 1.$

Corollary 3.6 suggests how to determine the crossing number without using the explicit standard continued fraction expansion. To be precise, it is sufficient to compute the sum of the absolute values in a continued fraction expansion and to count the number of sign changes. In the above example, the signs of components in $[3,-5,4,1,-2]$ are changed three times. Then the crossing number is

$$c(K([3,-5,4,1,-2])) = |3|+|-5|+|4|+|1|+|-2|-3 = 12.$$

These arguments prove Theorem 3.3.

Proof of Theorem 3.3 The sum of the absolute values of components in $\tilde{r}$ is

$$(2n+1)|\boldsymbol{a}| + \sum_{i=1}^{2n}(2|c_i|).$$

By Lemma 3.4, if the signs in a continued fraction expansion of $\tilde{r}$ are changed k times, then the crossing number of $K(\tilde{r})$ is decreased by k from the above value. The number of sign changes in $\tilde{r}$ is

$$\sum_{i=1}^{2n}(\psi(i)+\bar{\psi}(i))$$

by definition. Since $\tilde{r}$ is an expression of type $2n+1$ with respect to standard $\boldsymbol{a}$, we can apply Corollary 3.6. Therefore this completes the proof. □

We define $\bar{c}_i$ to be $2|c_i|-\psi(i)-\bar{\psi}(i)$. Then $\bar{c}_i$ is not negative.

Proposition 3.7 *Suppose that $\tilde{r}$ is as above. Then $\bar{c}_i \geq 0$ for $1 \leq i \leq 2n$.*

Proof If $c_i \neq 0$, then $2|c_i| \geq 2$. On the other hand, $\psi(i)$ and $\bar{\psi}(i)$ are 0 or 1, and then we get $\bar{c}_i \geq 0$. If $c_i = 0$, then $\psi(i) = 0$ and $\bar{\psi}(i) = 0$ by definition. Therefore $\bar{c}_i = 0$ in this case. □

4 Simon's conjecture

Simon's conjecture for 2–bridge knots is proved in [4], and for all knots in [2], as mentioned in Section 1. In this section, we investigate how many knot groups a 2–bridge knot group maps onto.

Let $\mathrm{EK}(n)$ be the maximal number of knots whose knot groups a 2–bridge knot group with n crossings admits epimorphisms onto. Theorem 3.3 and Proposition 3.7 imply the following, which is one of the main results in this paper. It gives us a rough estimate of $\mathrm{EK}(n)$.

Theorem 4.1 *Let $K(\widetilde{r})$ be a 2–bridge knot. If there exists an epimorphism from $G(K(\widetilde{r}))$ onto the knot group of another knot K, then*

$$c(K(\widetilde{r})) \geq 3c(K).$$

In particular, all the 2–bridge knots K with up to 8 crossings are minimal, that is to say, if $G(K)$ admits an epimorphism onto a knot group $G(K')$, then K' is equivalent to K or the trivial knot.

Proof By [4, Corollary 1.3] and [20, Proposition 2.4], if $G(K(\widetilde{r}))$ admits an epimorphism onto $G(K)$, then K is also a 2–bridge knot or the trivial knot. In the case that K is the trivial knot, the desired inequality obviously holds.

Next, we assume that K is a 2–bridge knot and that r is the corresponding rational number. Take the standard continued fraction expansion $[a_1, a_2, \dots, a_m]$ of r. Then $\widetilde{r}$ has an expansion of type $2n+1$ with respect to $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$. By Theorem 3.3, we have

$$\begin{aligned} c(K(\widetilde{r})) &= (2n+1)|\boldsymbol{a}| + \sum_{i=1}^{2n} (2|c_i| - \psi(i) - \overline{\psi}(i)) \\ &= (2n+1)c(K) + \sum_{i=1}^{2n} \overline{c}_i \\ &\geq (2n+1)c(K) \qquad \text{(by Proposition 3.7)} \\ &\geq 3c(K). \end{aligned}$$

Furthermore, since a nontrivial knot has at least 3 crossings, all the 2–bridge knots with up to 8 crossings are minimal. □

Remark The previous paper [12] shows that there are seven knots with less than 9 crossings whose knot groups admit epimorphisms onto the trefoil knot group. To be precise, they are the 3–bridge knots 8_5, 8_{10}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}. So the inequality of Theorem 4.1 does not hold for 3–bridge knots.

Ernst and Sumners [8] determined the number $\mathrm{TK}(n)$ of 2–bridge knots in terms of the crossing number $n \geq 3$ as follows:

$$\mathrm{TK}(n) = \begin{cases} \frac{1}{3}(2^{(n-3)} + 2^{(n-4)/2}) & \text{if } n \equiv 0 \pmod 4, \\ \frac{1}{3}(2^{(n-3)} + 2^{(n-3)/2}) & \text{if } n \equiv 1 \pmod 4, \\ \frac{1}{3}(2^{(n-3)} + 2^{(n-4)/2} - 1) & \text{if } n \equiv 2 \pmod 4, \\ \frac{1}{3}(2^{(n-3)} + 2^{(n-3)/2} + 1) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Then we can estimate $\mathrm{EK}(n)$ by using Theorem 4.1:

$$\mathrm{EK}(n) \leq \sum_{k=3}^{\lfloor n/3 \rfloor} \mathrm{TK}(k).$$

These numbers are obtained as shown in the following table:

n	9–11	12–14	15–17	18–20	21–23	24–26	27–29	30–32	33–35	36–38	39–41
$\sum_{k=3}^{\lfloor n/3 \rfloor} \mathrm{TK}(k)$	1	2	4	7	14	26	50	95	186	362	714

In particular, we obtain that the knot groups of 2–bridge knots with 12, 13 or 14 crossings map onto at most two knot groups, which are the trefoil knot group $G(3_1)$ and the figure eight knot group $G(4_1)$. On the other hand, Garrabrant, Hoste and Shanahan studied an upper bound for a set of 2–bridge knots with respect to epimorphisms between their knot groups. We recall their arguments more precisely. Let $\boldsymbol{a} = (a_1, a_2, \dots, a_{2n})$ be a vector such that

(1) each a_i is in $\{-2, 0, 2\}$,

(2) $a_1 \neq 0$ and $a_{2n} \neq 0$,

(3) if $a_i = 0$, then $a_{i-1} = a_{i+1} = \pm 2$.

For such an $\boldsymbol{a}$, we call $[a_1, a_2, \dots, a_{2n}]$ an *even standard continued fraction expansion*. If we consider $\boldsymbol{a} = (a_1, a_2, \dots, a_{2n})$ up to the equivalence relations $\boldsymbol{a} = \pm \boldsymbol{b}$ and $\boldsymbol{a} = \pm \boldsymbol{b}^{-1}$, where $\boldsymbol{b}^{-1}$ is $\boldsymbol{b}$ read backwards, then a 2–bridge knot can be expressed uniquely as $K([a_1, a_2, \dots, a_{2n}])$ by using an even standard continued fraction expansion:

Proposition 4.2 (Garrabrant, Hoste and Shanahan [9]) *Let $[a_1, a_2, \dots, a_{2n}]$ and $[b_1, b_2, \dots, b_{2n}]$ be even standard continued fraction expansions of the same length. If a 2–bridge knot group admits epimorphisms onto $G(K([a_1, a_2, \dots, a_{2n}]))$ and $G(K([b_1, b_2, \dots, b_{2n}]))$, then $(a_1, a_2, \dots, a_{2n}) = (b_1, b_2, \dots, b_{2n})$.*

For example, the trefoil is $3_1 = K([2, -2])$ and the figure eight knot is $4_1 = K([2, 2])$. Since the lengths of these even standard continued fraction expansions are the same, there does not exist a 2–bridge knot whose knot group admits epimorphisms onto $G(3_1)$ and $G(4_1)$ simultaneously, by Proposition 4.2. Similarly, a 2–bridge knot group maps onto the knot group of at most one of $\{5_1, 5_2, 6_1, 6_2, 6_3\}$, since

$$5_1 = K([2, -2, 2, -2]), \quad 5_2 = K([2, -2, 0, -2]),$$
$$6_1 = K([2, 0, 2, 2]), \quad 6_2 = K([2, 2, -2, 2]), \quad 6_3 = K([2, -2, -2, 2]).$$

In order to extend this argument, we consider the relationship between the length of an even standard continued fraction expansion and the crossing number.

Proposition 4.3 *Let $[a_1, a_2, \ldots, a_{2n}]$ be an even standard continued fraction expansion. Then the crossing number $c(K([a_1, a_2, \ldots, a_{2n}]))$ satisfies the inequalities*

$$2n+1 \leq c(K([a_1, a_2, \ldots, a_{2n}])) \leq 4n.$$

Proof First of all, we delete zeros in $(a_1, a_2, \ldots, a_{2n})$ as before:

$$[a_1, a_2, \ldots, a_{2n}] = [a'_1, a'_2, \ldots, a'_{2n'}],$$

where $a'_i \in 2\mathbb{Z} \setminus \{0\}$. Let ℓ be the number of zeros in $(a_1, a_2, \ldots, a_{2n})$. Then we have $2\ell = 2n - 2n'$ and

$$\sum_{i=1}^{2n'} (|a'_i| - 2) = 2\ell.$$

It follows that

$$\sum_{i=1}^{2n'} |a'_i| = 2\ell + 4n' = 2n + 2n'.$$

By the same argument as in the proof of Theorem 3.3, we obtain

$$c(K([a_1, a_2, \ldots, a_{2n}])) = c(K([a'_1, a'_2, \ldots, a'_{2n'}])) = \sum_{i=1}^{2n'} |a'_i| - k = \sum_{i=1}^{2n} |a_i| - k,$$

where k is the number of sign changes in $(a'_1, a'_2, \ldots, a'_{2n'})$. Note that $0 \leq k \leq 2n' - 1$. (If all a'_i are positive or negative, then $k = 0$. If $a'_i \cdot a'_{i+1} < 0$ for all i $(0 \leq i \leq 2n' - 1)$, then $k = 2n' - 1$.) Since $|a_i| \leq 2$, we have

$$\sum_{i=1}^{2n} |a_i| - k \leq 4n.$$

Moreover, we obtain

$$\begin{aligned} \sum_{i=1}^{2n'} |a'_i| - k &= 2n + 2n' - k \\ &\geq 2n + 2n' - (2n' - 1) \\ &= 2n + 1. \end{aligned}$$

This completes the proof. □

By Proposition 4.2, if two distinct 2–bridge knots K, K' have even standard continued fraction expansions of the same length, then there does not exist a 2–bridge knot whose knot group maps onto $G(K)$ and $G(K')$. Combined with Proposition 4.2 and Proposition 4.3, we can estimate $\mathrm{EK}(n)$ more precisely.

Theorem 4.4 *The number* $\mathrm{EK}(n)$ *satisfies*

$$\mathrm{EK}(n) \leq \left\lfloor \frac{n-3}{6} \right\rfloor.$$

Proof Let K be a 2–bridge knot with n crossings. If $G(K)$ admits an epimorphism onto $G(K')$, then the crossing number of K' is at most $\lfloor n/3 \rfloor$, by Theorem 4.1. Let $[a_1, a_2, \dots, a_{2m}]$ be the even standard continued fraction expansion of K', namely $K' = K([a_1, a_2, \dots, a_{2m}])$. By Proposition 4.2, $\mathrm{EK}(n)$ is less than or equal to the number of the lengths of even standard continued fraction expansions. By Proposition 4.3, we have

$$2m \leq \lfloor n/3 \rfloor - 1.$$

Hence we obtain

$$\mathrm{EK}(n) \leq \left\lfloor \frac{\lfloor n/3 \rfloor - 1}{2} \right\rfloor = \left\lfloor \frac{n-3}{6} \right\rfloor. \qquad \square$$

For example, the knot group of a 2–bridge knot with 50 crossings maps onto at most seven distinct knot groups. Actually, we can get the precise number $\mathrm{EK}(n)$ for $n \leq 30$ by computer program:

$$\mathrm{EK}(n) = \begin{cases} 0 & \text{if } n = 3, 4, 5, 6, 7, 8, \\ 1 & \text{if } n = 9, 10, 11, 12, 13, 14, 18, 19, 20, 24, \\ 2 & \text{if } n = 15, 16, 17, 21, 22, 23, 25, 26, 27, 28, 29, 30. \end{cases}$$

In particular, $\mathrm{EK}(n)$ is less than 3 for all $n \leq 30$. On the other hand, it is easy to see that $G(K([45]))$ maps onto $G(K([3]))$, $G(K([5])$, $G(K([9]))$ and $G(K([15]))$. It follows that $\mathrm{EK}(45) \geq 4$.

Problem Does there exist a 2–bridge knot with less than 45 crossings whose knot group maps onto three (or four) distinct knot groups? In general, determine $\mathrm{EK}(n)$ explicitly for all $n \geq 31$.

5 The generating function

As shown in Example 3.2, there exist three distinct 2–bridge knots with 9 crossings whose knot groups admit epimorphisms onto the trefoil knot group. In this section, we generalize this result. Namely, for a given 2–bridge knot $K(r)$, we determine the number of 2–bridge knots $K(\widetilde{r})$ which admit epimorphisms $\varphi\colon G(K(\widetilde{r})) \to G(K(r))$, in terms of $c(K(\widetilde{r}))$.

Theorem 5.1 *For a given rational number* r*, we take the standard continued fraction expansion* $[a_1, a_2, \dots, a_m]$ *of* r *and define the generating function* f *as follows:*

(1) *If* $(a_1, a_2, \dots, a_m) \neq (a_m, \dots, a_2, a_1)$, *then*

$$f(r) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} 2^{2n} \binom{2n+k-1}{k} t^{(2n+1)c(K(r))+k}.$$

(2) *If* $(a_1, a_2, \dots, a_m) = (a_m, \dots, a_2, a_1)$, *then*

$$f(r) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} g(n,k) t^{(2n+1)c(K(r))+k},$$

where

$$g(n,k) = \begin{cases} 2^{2n-1}\binom{2n+k-1}{k} & \text{for } k \text{ odd,} \\ 2^{2n-1}\binom{2n+k-1}{k} + 2^{n-1}\binom{n+k/2-1}{k/2} & \text{for } k \text{ even.} \end{cases}$$

Here $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. *Then the number of 2–bridge knots* $K(\widetilde{r})$ *which admit epimorphisms* $\varphi\colon G(K(\widetilde{r})) \to G(K(r))$ *is the coefficient of* $t^{c(K(\widetilde{r}))}$ *in* $f(r)$.

Proof We will count the number of 2–bridge knots with $(2n+1)c(K(r))+k$ crossings which correspond to rational numbers of the form $(*)$. The crossing number $c(K(r))$ is $\sum_{i=1}^{m} a_i$. Compared with Theorem 3.3, we have

$$k = \sum_{i=1}^{2n} \overline{c_i} = \sum_{i=1}^{2n} 2|c_i| - \psi(i) - \overline{\psi}(i),$$

where $\overline{c_i} \geq 0$ by Proposition 3.7.

Suppose that $\overline{c_i} = j$ (≥ 0). Then $(\varepsilon_i \boldsymbol{a}^{\pm 1}, 2c_i, \varepsilon_{i+1} \boldsymbol{a}^{\mp 1})$, which is a part of $\widetilde{r}$, has the following possibilities:

(1) if j is even and $\varepsilon_i = 1$, then

$$(\varepsilon_i \boldsymbol{a}^{\pm 1}, 2c_i, \varepsilon_{i+1} \boldsymbol{a}^{\mp 1}) = (\boldsymbol{a}^{\pm 1}, j, \boldsymbol{a}^{\mp 1}) \text{ or } (\boldsymbol{a}^{\pm 1}, -(j+2), \boldsymbol{a}^{\mp 1});$$

(2) if j is even and $\varepsilon_i = -1$, then

$$(\varepsilon_i \boldsymbol{a}^{\pm 1}, 2c_i, \varepsilon_{i+1} \boldsymbol{a}^{\mp 1}) = (-\boldsymbol{a}^{\pm 1}, -j, -\boldsymbol{a}^{\mp 1}) \text{ or } (-\boldsymbol{a}^{\pm 1}, j+2, -\boldsymbol{a}^{\mp 1});$$

(3) if j is odd and $\varepsilon_i = 1$, then

$$(\varepsilon_i \boldsymbol{a}^{\pm 1}, 2c_i, \varepsilon_{i+1} \boldsymbol{a}^{\mp 1}) = (\boldsymbol{a}^{\pm 1}, j+1, -\boldsymbol{a}^{\mp 1}) \text{ or } (\boldsymbol{a}^{\pm 1}, -(j+1), -\boldsymbol{a}^{\mp 1});$$

(4) if j is odd and $\varepsilon_i = -1$, then

$$(\varepsilon_i \boldsymbol{a}^{\pm 1}, 2c_i, \varepsilon_{i+1} \boldsymbol{a}^{\mp 1}) = (-\boldsymbol{a}^{\pm 1}, j+1, \boldsymbol{a}^{\mp 1}) \text{ or } (-\boldsymbol{a}^{\pm 1}, -(j+1), \boldsymbol{a}^{\mp 1}).$$

Therefore $(\varepsilon_i \boldsymbol{a}^{\pm 1}, 2c_i, \varepsilon_{i+1}\boldsymbol{a}^{\mp 1})$ always has two possibilities. Besides, there are $\binom{2n+k-1}{k}$ cases for $(\overline{c}_1, \dots, \overline{c}_{2n})$, namely

$$(\overline{c}_1, \overline{c}_2, \dots, \overline{c}_{2n}) = (k, 0, \dots, 0), \quad (k-1, 1, \dots, 0), \quad \dots, \quad (0, 0, \dots, 0, k).$$

Hence there are $2^{2n}\binom{2n+k-1}{k}$ 2–bridge knots with $(2n+1)c(K(r)) + k$ crossings, and we get the generating function of (1).

In the case when $(a_1, \dots, a_m) = (a_m, \dots, a_1)$, we see

$$K([\varepsilon_1 \boldsymbol{a}, 2c_1, \varepsilon_2 \boldsymbol{a}^{-1}, \dots, 2c_{2n}, \varepsilon_{2n+1}\boldsymbol{a}]) = K([\varepsilon_{2n+1}\boldsymbol{a}, 2c_{2n}, \dots, \varepsilon_2 \boldsymbol{a}^{-1}, 2c_1, \varepsilon_1 \boldsymbol{a}]).$$

It implies that if $\widetilde{r}$ is not symmetric, that is, if $\widetilde{r}$ is not in the form

$$[\varepsilon_1 \boldsymbol{a}, 2c_1, \dots, 2c_n, \varepsilon_{n+1}\boldsymbol{a}^{\pm 1}, 2c_n, \dots, 2c_1, \varepsilon_1 \boldsymbol{a}],$$

we counted the same knot twice. Then the number of knots is

$$\frac{1}{2}\left(2^{2n}\binom{2n+k-1}{k} - 2^n\binom{n+k/2-1}{k/2}\right) + 2^n\binom{n+k/2-1}{k/2} = 2^{2n-1}\binom{2n+k-1}{k} + 2^{n-1}\binom{n+k/2-1}{k/2}.$$

Notice that if k is odd, then $\widetilde{r}$ must not be symmetric. As we saw in Section 2, if the standard continued fraction expansions are not the same, then the 2–bridge knots are not equivalent. We can get the standard fraction expansion of $\widetilde{r}$ by Lemma 3.4. It shows that these knots which are obtained by the Ohtsuki–Riley–Sakuma construction are not equivalent. This completes the proof. □

Example 5.2 First, we apply Theorem 5.1 to the trefoil knot. The generating function for the trefoil $K(\frac{1}{3}) = K([3])$ is

$$f\left(\tfrac{1}{3}\right) = 3t^9 + 4t^{10} + 7t^{11} + 8t^{12} + 11t^{13} + 12t^{14} + 25t^{15} + 48t^{16} + 103t^{17} + 180t^{18} + 309t^{19} + 472t^{20} + 743t^{21} + 1180t^{22} + 2045t^{23} + 3584t^{24} + 6391t^{25} + \cdots.$$

Then the number of 2–bridge knots with 9 crossings whose knot groups admit epimorphisms onto the trefoil knot group is the coefficient of t^9, which is 3. These 2–bridge knots are 9_1, 9_6, 9_{23}, as shown in Example 3.2.

Similarly, as shown in [12], there are four distinct 2–bridge knots with 10 crossings whose knot groups admit epimorphisms onto the trefoil knot group, namely 10_5, 10_9, 10_{32}, 10_{40}; as shown in [10], there are seven distinct such 2–bridge knots with 11 crossings, namely $11a_{117}$, $11a_{175}$, $11a_{176}$, $11a_{203}$, $11a_{236}$, $11a_{306}$, $11a_{355}$.

Another example shows the generating function for $5_2 = K(\frac{3}{7}) = K([2,3])$:

$$f(\tfrac{3}{7}) = 4t^{15} + 8t^{16} + 12t^{17} + 16t^{18} + 20t^{19} + 24t^{20} + 28t^{21} + 32t^{22} + 36t^{23} + 40t^{24} + 60t^{25} + 112t^{26} + 212t^{27} + 376t^{28} + 620t^{29} + 960t^{30} + \cdots .$$

Acknowledgments

The author wishes to express his thanks to Makoto Sakuma and Takayuki Morifuji for helpful communications. He also thanks the referee for some useful comments. This work is partially supported by KAKENHI grant No. 16K05159 and 15H03618 from the Japan Society for the Promotion of Science.

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Received: 15 June 2016 Revised: 1 October 2016