Non-L–space integral homology 3–spheres with no nice orderings

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We give infinitely many examples of non-L–space irreducible integer homology 3–spheres whose fundamental groups do not have nontrivial $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ representations.

1 Introduction

Before stating the main result, I will review some definitions. A rational homology 3–sphere Y is called an *L–space* if rk $\widehat{\mathrm{HF}}(Y) = |H_1(Y;\mathbb{Z})|$, ie its Heegaard Floer homology is minimal. An L–space does not admit any coorientable taut foliation, by Bowden [1], Kazez and Roberts [13] and Ozsváth and Szabó [15]. A nontrivial group G is called left-orderable if there exists a strict total ordering of G invariant under left multiplication. Boyer, Gordon and Watson conjectured in [2] that an irreducible rational homology 3–sphere is a non-L–space if and only if its fundamental group is left-orderable. A stronger conjecture states that for an irreducible $\mathbb{Q}$–homology 3–sphere, being a non-L–space, having left-orderable fundamental group and admitting a coorientable taut foliation are the same (see eg Culler and Dunfield [5]).

To show the fundamental group $\pi_1(Y)$ of a 3–manifold Y is orderable, it is most common to consider $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ representations of $\pi_1(Y)$. In fact, in many cases, $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ representations are sufficient to define an order on $\pi_1(Y)$; see [5]. However, Theorem 1 in this paper shows that, even in the case of non-L–space integral homology spheres, orders coming from $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ are not enough to prove the conjecture of Boyer, Gordon and Watson.

It is conjectured that any integer homology 3–sphere different from the 3–sphere admits an irreducible representation in $\mathrm{SU}_2(\mathbb{C})$ (see eg Kirby's problem list [14, Problem 3.105]). Zentner [20] showed that if one enlarges the target group to $\mathrm{SL}_2(\mathbb{C})$, then every such integral homology 3–sphere has an irreducible representation. By contrast, I will give examples where there are no irreducible $\mathrm{PSL}_2(\mathbb{R})$ representations. Let $\mathcal{M}$ be the manifold $m137$ — see Callahan, Hildebrand and Weeks [3] — and $\mathcal{M}(1, n)$ be the integral homology sphere obtained by $(1, n)$ Dehn fillings on $\mathcal{M}$. The main result of this paper states:

Theorem 1 *For all $n \ll 0$, the manifold $\mathcal{M}(1, n)$ is a hyperbolic integral homology 3–sphere where*

   (a) $\pi_1(\mathcal{M}(1, n))$ *does not have a nontrivial* $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ *representation;*

   (b) $\mathcal{M}(1, n)$ *is not an L–space.*

This means that we can not produce an order on $\pi_1(\mathcal{M}(1, n))$ simply by pulling back the action of $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ on $\mathbb{R}$.

Section 2 is devoted to proving Theorem 1(a). Let $X_0(\mathcal{M})$ be the component of the $\mathrm{SL}_2(\mathbb{C})$ character variety of $\mathcal{M}$ containing the character of an irreducible representation (see Culler and Shalen [7] for the definition). Here is an outline of the approach. Let $X_{0,\mathbb{R}}(\mathcal{M})$ be the real points of $X_0(M)$. Define $[\rho] \in X_{0,\mathbb{R}}(\mathcal{M})$ and denote by s the trace of $\rho(\lambda)$, where λ is the homological longitude of $\mathcal{M}$. The proof is divided into two parts. In the first part, I show that points on the $|s| < 2$ components of $X_{0,\mathbb{R}}(\mathcal{M})$ all correspond to $\mathrm{SU}_2(\mathbb{C})$ representations, while points on the $|s| > 2$ components correspond to $\mathrm{SL}_2(\mathbb{R})$ representations. In the second part, I show that $\mathrm{SL}_2(\mathbb{R})$ representations of $\pi_1(\mathcal{M})$ give rise to no $\mathrm{SL}_2(\mathbb{R})$ representations of $\pi_1(\mathcal{M}(1, n))$ when $n \ll 0$. This part of the proof is basically analyzing real solutions to the A–polynomial of $\mathcal{M}$ under the relation $\mu\lambda^n = 1$ given by $(1, n)$ Dehn filling, where μ is a choice of meridian of $\partial\mathcal{M}$.

In Section 3, by applying techniques in the paper by Rasmussen and Rasmussen [17] and Gillespie [11], I show that none of the $(1, n)$ Dehn fillings on $m137$ is an L–space, completing the proof of Theorem 1.

2 $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ representations

I will prove Theorem 1(a) in this section.

SnapPy [6] gives us the following presentation of the fundamental group of $\mathcal{M} = m137$:

$$\pi_1(\mathcal{M}) = \langle \alpha, \beta \mid \alpha^3\beta^2\alpha^{-1}\beta^{-3}\alpha^{-1}\beta^2 \rangle.$$

The peripheral system of $\mathcal{M}$ can be represented as

$$\{\mu, \lambda\} = \{\alpha^{-1}\beta^2\alpha^4\beta^2, \alpha^{-1}\beta^{-1}\} = \{\beta^2\lambda^{-1}\beta^{-3}\lambda^{-1}\beta^2, \lambda\},$$

where λ is the homological longitude and μ is a choice of meridian. Then we can rewrite the fundamental group as

(2-1) $$\pi_1(\mathcal{M}) = \langle \lambda, \beta \mid \beta^{-1}\lambda^{-1}\beta^{-1}\lambda^{-1}\beta^2\lambda = \lambda\beta^{-2}\lambda^{-1}\beta^2 \rangle,$$

and the meridian becomes $\mu = \beta^2\lambda^{-1}\beta^{-3}\lambda^{-1}\beta^2$ under this presentation.

Remark The triangulation of $m137$ we used (included in [9]) to get these presentations is different from SnapPy's default triangulation. We got it by performing random Pachner moves on the default triangulation in SnapPy. In particular, our notations for longitude and meridian in the peripheral system are meridian and longitude, respectively, in SnapPy's default notations.

We will first look at irreducible $\mathrm{SL}_2(\mathbb{C})$ representations of the fundamental group of $\mathcal{M}$ before we look at those of Dehn fillings of $\mathcal{M}$. Denote by $X(\mathcal{M})$ the $\mathrm{SL}_2(\mathbb{C})$ character variety of $\mathcal{M}$, that is, the geometric invariant theory quotient

$$\mathrm{Hom}(\pi_1(\mathcal{M}), \mathrm{SL}_2(\mathbb{C}))/\!/\mathrm{SL}_2(\mathbb{C}).$$

It is an affine variety [7]. Suppose $\rho\colon \pi_1(\mathcal{M}) \longrightarrow \mathrm{SL}_2(\mathbb{C})$ is a representation of the fundamental group of $\mathcal{M}$. Recall that a representation ρ of G in $\mathrm{SL}_2(\mathbb{C})$ is irreducible if the only subspaces of $\mathbb{C}^2$ invariant under $\rho(G)$ are $\{0\}$ and $\mathbb{C}^2$ [7]. This is equivalent to saying that ρ can't be conjugated to a representation by upper triangular matrices. Otherwise ρ is called reducible. We will call a character irreducible (reducible) if the corresponding representation is irreducible (reducible).

First, I determine which components of $X(\mathcal{M})$ contain characters of irreducible representations. Computation with SnapPy [6] shows that the Alexander polynomial $\Delta_{\mathcal{M}}$ of $m137$ is 1, which has no root. So there are no reducible nonabelian representations [4, Section 6.1]. Therefore all the reducible representations are abelian. Since $H_1(\mathcal{M}) = \mathbb{Z}$, there is only one such component and it is parametrized by the image of β and is isomorphic to $\mathrm{Hom}(\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))/\!/\mathrm{SL}_2(\mathbb{C}) \simeq \mathbb{C}$. Moreover, it is disjoint from any component of $X(\mathcal{M})$ containing the character of an irreducible representation [4, Section 6.2]. For more details, we refer the readers to Tillmann's note [19], where he studied $m137$ as an example.

If an abelian representation of $\pi_1(\mathcal{M})$ induces an abelian representation of $\pi_1(\mathcal{M}(1, n))$ then it factors through the abelianization $\mathrm{ab}(\pi_1(\mathcal{M}(1, n))) = 1$. So they correspond to trivial $\mathrm{SL}_2(\mathbb{C})$ representations and we don't need to worry about them.

Now we consider components of $X(\mathcal{M})$ that contain the character of an irreducible representation. We have:

Lemma 2 *There is a single component $X_0(\mathcal{M})$ of $X(\mathcal{M})$ containing an irreducible character. The functions $s = \operatorname{tr} \rho(\lambda) = \operatorname{tr} \rho(\alpha^{-1}\beta^{-1}) = \operatorname{tr} \rho(\alpha\beta)$ and $t = \operatorname{tr} \rho(\beta)$ give complete coordinates on $X_0(\mathcal{M})$, which is the curve in $\mathbb{C}^2$ cut out by*

$$(-2 - 3s + s^3)t^4 + (4 + 4s - s^2 - s^3)t^2 - 1 = 0.$$

Moreover, $w := \operatorname{tr} \rho(\lambda\beta) = \operatorname{tr} \rho((\lambda\beta)^{-1}) = t - 1/(t(s+1))$.

Proof Let $X_0(\mathcal{M})$ be $X(\mathcal{M}) - \{$reducible characters$\}$. From the discussion above, we know that all the reducible characters form a single component of $X(\mathcal{M})$ and this component is disjoint from any other component of $X(\mathcal{M})$. So $X_0(\mathcal{M})$ is Zariski closed. We will show later that $X_0(\mathcal{M})$ is actually an irreducible algebraic variety, as claimed in the lemma.

Suppose $[\rho] \in X_0(\mathcal{M})$. So ρ is an irreducible representation. By conjugating ρ if necessary, we can assume that ρ has the form

$$\rho(\lambda) = \begin{pmatrix} z & 1 \\ 0 & 1/z \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} x & 0 \\ y & 1/x \end{pmatrix}.$$

From the relator of $\pi_1(\mathcal{M})$ in (2-1) we have $\rho(\beta)^{-1}\rho(\lambda)^{-1}\beta^{-1}\rho(\lambda)^{-1}\rho(\beta)^2\rho(\lambda) = \rho(\lambda)\rho(\beta)^{-2}\rho(\lambda)^{-1}\rho(\beta)^2$. Comparing the entries of the matrices on both sides, we get four equations. These four equations together with $s = z + 1/z$, $t = x + 1/x$ and $w = zx + z^{-1}x^{-1} + y$ form a system $\mathcal{S}$ which defines $X_0(\mathcal{M})$. By computing a Gröbner basis of this system, SageMath [18] gives the following generators of the radical ideal $I = I(X_0(\mathcal{M}))$:

(2-2) $$stw - t^2 - w^2 - s + 2,$$

(2-3) $$t^3 - w^3 + st - sw - 2t + w,$$

(2-4) $$st^2 - tw - w^2 - s + 1,$$

(2-5) $$sw^3 - s^2t + s^2w - t^2w - tw^2 + st - sw + t.$$

Subtracting (2-4) from (2-2), we get

(2-6) $$w = t - \frac{1}{t(s+1)}.$$

Eliminating w, we get a defining equation for $X_0(\mathcal{M})$:

(2-7) $$\begin{aligned} 0 &= (-2 - 3s + s^3)t^4 + (4 + 4s - s^2 - s^3)t^2 - 1 \\ &= (s-2)(s+1)^2t^4 - (s-2)(s+2)(s+1)t^2 - 1. \end{aligned}$$

Thus, we can think of $X_0(\mathcal{M})$ as living in $\mathbb{C}^2$.

To prove the lemma, we must show that $X_0(M)$ is irreducible or, equivalently, the polynomial $P(s,t) := (s-2)(s+1)^2 t^4 - (s-2)(s+2)(s+1)t^2 - 1$ in (2-7) does not factor in $\mathbb{C}[s,t]$. Assume $P(s,t)$ factors. Suppose it factors as

$$(at^2 + bt + c)\left(dt^2 + et - \frac{1}{c}\right) = adt^4 + (ae + bd)t^3 + \left(cd - \frac{a}{c} + be\right)t^2 + \left(ce - \frac{b}{c}\right)t - 1,$$

where $a, b, d, e \in \mathbb{C}[s]$ and $c \in \mathbb{C} - \{0\}$. Setting the coefficients of t and t^3 to be 0, we get $b = c^2 e$ and $ae = -c^2 de$. If $e \neq 0$, then $a = -c^2 d$. But this is impossible as $ad = (s-2)(s+1)^2$ is a polynomial in s of odd degree. So $e = 0$ and it follows that $b = 0$. Comparing the coefficients of t^2 and t^4, we get

(2-8) $$ad = (s-2)(s+1)^2,$$

(2-9) $$cd - \frac{a}{c} = -(s-2)(s+2)(s+1).$$

So $\text{degree}(a) + \text{degree}(d) = 3$ and $\max\{\text{degree}(a), \text{degree}(d)\} \geq 3$, which implies exactly one of a and d has degree 3 and the other has degree 0. Without loss of generality, we can assume that $\text{degree}(a) = 3$ and $\text{degree}(d) = 0$. Multiplying both sides of (2-9) by c, we get $a = c^2 d + c(s-2)(s+2)(s+1)$. So the coefficient of s^3 in a is c. Comparing with the coefficient of s^3 in (2-8), we see that $d = 1/c$. Eliminating a and d gives us an equality $1 + (s-2)(s+2)(s+1) = (s-2)(s+1)^2$, which does not hold.

Otherwise, suppose $P(s,t)$ factors as

$$(at + c)\left(bt^3 + dt^2 + et - \frac{1}{c}\right) = abt^4 + (ad + cb)t^3 + (cd + ae)t^2 + \left(ce - \frac{a}{c}\right)t - 1,$$

where $a, b, d, e \in \mathbb{C}[s]$ and $c \in \mathbb{C} - \{0\}$. Setting the coefficients of t and t^3 to be 0, we get $a = c^2 e$ and $b = ced$. Comparing the coefficients of t^2 and t^4, we get

(2-10) $$c^3 de^2 = (s-2)(s+1)^2,$$

(2-11) $$cd + c^2 e^2 = -(s-2)(s+2)(s+1).$$

So $\text{degree}(d) + 2\,\text{degree}(e) = 3$ and $\max\{\text{degree}(d), 2\,\text{degree}(e)\} \geq 3$, which implies $\text{degree}(d) = 3$ and $\text{degree}(e) = 0$. Comparing the coefficients of s^3 in (2-10) and (2-11), we know that $c^2 e^2 = -1$. Plugging into (2-10), we get $cd = (s-2)(s+1)^2$, which when plugging into (2-11) implies $c^2 e^2 = -(s+1)(s-2)$, a contradiction. So $P(s,t)$ is irreducible over $\mathbb{C}$. Therefore, $X_0(\mathcal{M})$ has only one component.   □

To find irreducible $\text{SL}_2(\mathbb{R})$ representations of $\pi_1(\mathcal{M})$, we need to check all real points on $X_0(\mathcal{M})$, which correspond to real solutions of (2-7). Notice that (2-7) has no solutions when $s = -1$ or 2, so (2-7) is a quadratic equation in t^2. In order for t to

be real, t^2 has to be real and nonnegative. Then first we need the discriminant to be nonnegative. That is,

$$\Delta_1 = (s+1)^2(s-2)(s^3 + 2s^2 - 4s - 4) \geq 0.$$

So $s \in U := (-\infty, p_1] \cup [p_2, p_3] \cup (2, \infty)$, where $p_1 \approx -2.9032$, $p_2 \approx -0.8061$ and $p_3 \approx 1.7093$ are the three roots of the cubic polynomial $s^3 + 2s^2 - 4s - 4$.

The following lemma will help us determine when a $SL_2(\mathbb{C})$ representation of $\pi_1(\mathcal{M})$ can be conjugated into $SL_2(\mathbb{R})$ by simply checking where it lies on the character variety.

Lemma 3 *The set of real points $X_{0,\mathbb{R}}(\mathcal{M}) = X_0(\mathcal{M}) \cap \mathbb{R}^2$ of $X_0(\mathcal{M})$ has 6 connected components:*

- *Points on the two components with $|s| < 2$ correspond to $SU_2(\mathbb{C})$ representations.*
- *Points on the four components with $|s| > 2$ correspond to $SL_2(\mathbb{R})$ representations.*

Remark The above lemma shows that, in our case, the absolute value of one character being smaller than 2 implies that the representation is $SU_2(\mathbb{C})$. But, in general, this is not true.

To prove this lemma, we need to determine when $[\rho] \in X_{0,\mathbb{R}}(\mathcal{M})$ corresponds to $\rho \in SU_2(\mathbb{C})$ and when it corresponds to $\rho \in SL_2(\mathbb{R})$. It can't be in both because otherwise it would be reducible [5, Lemma 2.10] and we know $X_0(\mathcal{M})$ contains only irreducible characters. The tool we use is a reformulation of Proposition 3.1 in [12], which states that given three angles $\theta_i \in [0, \pi]$, $i = 1, 2, 3$, there exist three $SU_2(\mathbb{C})$ matrices C_i satisfying $C_1 C_2 C_3 = I$ with eigenvalues $\exp(\pm i\theta_i)$, respectively, if and only if these angles satisfy

$$(2\text{-}12) \qquad |\theta_1 - \theta_2| \leq \theta_3 \leq \min\{\theta_1 + \theta_2, 2\pi - (\theta_1 + \theta_2)\}.$$

We want to rewrite the above inequality in terms of traces of C_1, C_2 and C_3. We have the following lemma:

Lemma 4 *Suppose $t_1, t_2, t_3 \in (-2, 2)$ are the traces of $C_1, C_2, C_3 \in SL_2(\mathbb{C})$ which satisfy $C_1 C_2 C_3 = I$. Then C_1, C_2 and C_3 are simultaneously conjugate in $SU_2(\mathbb{C})$ if and only if*

$$(2t_3 - t_1 t_2)^2 \leq (4 - t_1^2)(4 - t_2^2).$$

Proof Suppose $t_1 = 2\cos\theta_1$, $t_2 = 2\cos\theta_2$ and $t_3 = 2\cos\theta_3$ with $\theta_1, \theta_2, \theta_3 \in [0, \pi]$.

If $0 \leq \theta_1 + \theta_2 \leq \pi$, then the inequality (2-12) becomes $|\theta_1 - \theta_2| \leq \theta_3 \leq \theta_1 + \theta_2$. Taking cosines, we get $\cos(\theta_1 + \theta_2) \leq \cos\theta_3 \leq \cos(\theta_1 - \theta_2)$.

If $\pi \leq \theta_1 + \theta_2 \leq 2\pi$, then the inequality becomes $|\theta_1 - \theta_2| \leq \theta_3 \leq 2\pi - (\theta_1 + \theta_2)$. Taking cosines, we also get $\cos(\theta_1 + \theta_2) \leq \cos \theta_3 \leq \cos(\theta_1 - \theta_2)$.

Using the relations $t_1 = 2\cos \theta_1$, $t_2 = 2\cos \theta_2$ and $t_3 = 2\cos \theta_3$, we get in both cases that

$$\frac{t_1 t_2}{4} - \sqrt{\left(1 - \frac{t_1^2}{4}\right)\left(1 - \frac{t_2^2}{4}\right)} \leq \frac{t_3}{2} \leq \frac{t_1 t_2}{4} + \sqrt{\left(1 - \frac{t_1^2}{4}\right)\left(1 - \frac{t_2^2}{4}\right)}.$$

Then

$$-\sqrt{\left(1 - \frac{t_1^2}{4}\right)\left(1 - \frac{t_2^2}{4}\right)} \leq \frac{t_3}{2} - \frac{t_1 t_2}{4} \leq \sqrt{\left(1 - \frac{t_1^2}{4}\right)\left(1 - \frac{t_2^2}{4}\right)}.$$

So we have

$$\left| \frac{t_3}{2} - \frac{t_1 t_2}{4} \right| \leq \sqrt{\left(1 - \frac{t_1^2}{4}\right)\left(1 - \frac{t_2^2}{4}\right)}.$$

Squaring both sides and simplifying, we get

$$(2t_3 - t_1 t_2)^2 \leq (4 - t_1^2)(4 - t^2),$$

as desired. $\qquad \square$

With the criterion of Lemma 4 in hand, we now can prove Lemma 3.

Proof of Lemma 3 The six components correspond to $s \in (-\infty, p_1] \cup [p_2, p_3] \cup (2, \infty)$ and $t \in (-\infty, 0) \cup (0, \infty)$.

Set $C_1 = \rho(\lambda)$, $C_2 = \rho(\beta)$ and $C_3 = \rho(\beta^{-1}\lambda^{-1}) = \rho((\lambda\beta)^{-1})$. Then $t_1 = s$, $t_2 = t$ and $t_3 = w$. Applying Lemma 4 we have

$$(2\text{-}13) \qquad\qquad (2w - st)^2 \leq (4 - s^2)(4 - t^2).$$

Plugging (2-6) into (2-13) and simplifying,

$$(s - 2)^2 t^2 + \frac{4(s - 2)}{s + 1} + \frac{4}{t^2(s + 1)^2} \leq (4 - s^2)(4 - t^2).$$

Multiplying both sides by $t^2(s + 1)^2$, we get

$$(s + 1)^2(s - 2)^2 t^4 + 4(s - 2)(s + 1)t^2 + 4 \leq (4 - s^2)(s + 1)^2(4 - t^2)t^2,$$

which simplifies to

$$-(s + 1)^2(s - 2)t^4 + (s^2 + 3s + 3)(s - 2)(s + 1)t^2 + 1 \leq 0.$$

Plugging in (2-7), we get

$$(s + 1)^3(s - 2)t^2 \leq 0,$$

which always holds when $s \in (p_2, p_3) \approx (-0.8061, 1.7093) \subset (-2, 2)$.

So, points on $X_{0,\mathbb{R}}(\mathcal{M})$ correspond to $SU_2(\mathbb{C})$ representations if and only if $|s| < 2$ and correspond to $SL_2(\mathbb{R})$ representations if and only if $|s| > 2$. $\square$

Proof of Theorem 1(a) Lemma 3 tells us a $SL_2(\mathbb{C})$ representation ρ of $m137$ is real if and only if eigenvalues of $\rho(\lambda)$ are real. Moreover, the condition $\mu\lambda^n = 1$ forces the eigenvalues of $\rho(\mu)$ to also be real in this case. So we could restrict our attention to $|s| > 2$ and look at the A–polynomial instead (see eg [4] for the definition of the A–polynomial). Recall that z is an eigenvalue of $\rho(\lambda)$. Denote by m the eigenvalue of $\rho(\mu)$ which shares its eigenvector with z. The A–polynomial of $m137$ is computed by SageMath [18] as

$$(z^4 + 2z^5 + 3z^6 + z^7 - z^8 - 3z^9 - 2z^{10} - z^{11}) + m^2(-1 - 3z - 2z^2 - z^3$$
$$+ 2z^4 + 4z^5 + z^6 + 4z^7 + z^8 + 4z^9 + 2z^{10} - z^{11} - 2z^{12} - 3z^{13} - z^{14})$$
$$+ m^4(-z^3 - 2z^4 - 3z^5 - z^6 + z^7 + 3z^8 + 2z^9 + z^{10}).$$

Write $A = -1 - 2z - 3z^2 - z^3 + z^4 + 3z^5 + 2z^6 + z^7 = (z-1)(z^2 + z + 1)^3$ and $B = 1 + 3z + 2z^2 + z^3 - 2z^4 - 4z^5 - z^6 - 4z^7 - z^8 - 4z^9 - 2z^{10} + z^{11} + 2z^{12} + 3z^{13} + z^{14}$. So the A–polynomial can be simplified as $-z^4 A - Bm^2 + z^3 Am^4$. We are interested in the real solutions of

(2-14) $-z^4 A - Bm^2 + z^3 Am^4 = 0.$

Now consider the $(1, n)$ Dehn filling on $m137$. Then we are adding an extra relation $\mu\lambda^n = 1$, which is $\rho(\mu)\rho(\lambda)^n = I$ under the representation ρ, ie

$$\rho(\mu) = \rho(\lambda)^{-n} = \begin{pmatrix} z^{-n} & * \\ 0 & z^n \end{pmatrix}.$$

Restricting to $\partial\mathcal{M}$ gives us the relation $m = z^{-n}$.

When n is negative, we shall write $n' = -n$. So we have $m = z^{n'}$. Plugging into (2-14) and dividing both sides by z^4, we get

(2-15) $-A - Bz^{2n'-4} + Az^{4n'-1} = 0.$

We will show the following lemma is true, completing the proof of Theorem 1(a).

Lemma 5 *Equation* (2-15) *has no real solutions when n' is large enough.*

Proof Define $F(z) = A(z^{4n'-1} - 1) - Bz^{2n'-4}$. We'll show $F(z) > 0$.

First notice that $A = 0$ only when $z = 1$. And $A > 0$ when $z > 1$ while $A < 0$ when $z < 1$. The polynomial B has 6 real roots, which are all simple: -2.3396, -1.4121, -0.7082, -0.4274, 0.8684, 1.1516 (rounded to the fourth digit).

As we saw earlier, the domain for s is

$$U := (-\infty, p_1) \cup [p_2, p_3] \cup (2, \infty) \approx (-\infty, -2.9032] \cup [-0.8061, 1.7093] \cup (2, \infty).$$

So the $|s| > 2$ condition restricts s to $(-\infty, p_1] \cup (2, \infty)$. Then

$$z \in V := (-\infty, -2.5038] \cup [-0.3994, 0) \cup (0, 1) \cup (1, \infty).$$

Notice that $z^7 A(1/z) = -A(z)$ and $z^{14} B(1/z) = B(z)$. Interchanging z with $1/z$ in $F(z)$ gives us $F(1/z) = A(1/z)(z^{-(4n'-1)} - 1) - B(1/z)z^{-(2n'-4)} = F(z)/z^{4n'+6}$. So we can assume $|z| < 1$.

Case 1 $(0.8684 \le z < 1)$ In this case, we have $A(z) < 0$, $B(z) \le 0$ and $z^{4n'-1} - 1 < 0$. So $F(z) > 0$.

Case 2 $(-0.3994 \le z < 0.8684$ and $z \ne 0)$ In this case, we have $A(z) < C_5 < 0$ and $C_6 > B(z) > 0$ for some constants C_5 and C_6. When n' is large enough, we have $|C_5| \times |(z^{4n'-1} - 1)| > C_6 z^{2n'-4}$. So $A(z^{4n'-1} - 1) = |A| \times |(z^{4n'-1} - 1)| > B z^{2n'-4}$ and it follows that $F(z) > 0$.

Therefore, when $n' = -n$ is large enough, we always have $F(z) > 0$ on the domain V. So (2-15) has no real solution when $n' \gg 0$. □

It follows from the above lemma that (2-14) has no real solution when $n \ll 0$ and thus the equality $\rho(\mu)\rho(\lambda)^n = I$ does not hold for $n \ll 0$.

From all the discussion above, we can now conclude that $\mathcal{M}(1, n)$ has no nontrivial $SL_2(\mathbb{R})$ representation and thus no nontrivial $PSL_2(\mathbb{R})$ representation for $n \ll 0$. Since the first Betti number of $\mathcal{M}(1, n)$ is 0, the lift of a trivial $PSL_2(\mathbb{R})$ representation of $\pi_1(\mathcal{M}(1, n))$ into $\widetilde{PSL}_2(\mathbb{R})$ will be trivial. So all representations of $\pi_1(\mathcal{M}(1, n))$ into $\widetilde{PSL}_2(\mathbb{R})$ are trivial for $n \ll 0$, proving Theorem 1(a). □

In contrast, when n is positive there are examples of nontrivial $SL_2(\mathbb{R})$ representations.

Plugging $m = z^{-n}$ into (2-14) and multiplying both sides by z^{4n-3}, we get

$$-A + B z^{2n-3} + A z^{4n+1} = 0.$$

Similarly define $G(z) = A(z^{4n+1} - 1) + B z^{2n-3}$. Since $G(1) = -4$ and $G(0.8684) > 0$, $G(z)$ must have at least one root in $[0.8684, 1)$. So $\pi_1(\mathcal{M}(1, n))$ has at least one nontrivial $SL_2(\mathbb{R})$ representation for any $n > 0$. They lift to $\widetilde{PSL}_2(\mathbb{R})$ representations, since the Euler number of any representation of an integral homology sphere vanishes [10, Section 6].

3   No L–space fillings

In this section, I will prove Theorem 1(b) using results from Gillespie [11], which are based on Rasmussen and Rasmussen [17]. In fact, I will show that none of the nonlongitudinal fillings of $m137$ is an L–space. The homology groups in this section are all homology with integral coefficients.

Suppose Y is a compact connected 3–manifold with a single torus as boundary. I will follow Gillespie's notation [17]. Define the set of slopes on ∂Y as

$$Sl(Y) = \{a \in H_1(\partial Y) \mid a \text{ is primitive}\}/\pm 1.$$

Define the set of L–space filling slopes of Y

$$\mathcal{L}(Y) = \{a \in Sl(Y) \mid Y(a) \text{ is an L–space}\}.$$

Moreover, Y is said to have genus 0 if $H_2(Y, \partial Y)$ is generated by a surface of genus 0.

We will use [11, Theorem 1.2], which is stated as:

Theorem 6   *The following are equivalent:*

 (1) $\mathcal{L}(Y) = Sl(Y) - \{l\}$.

 (2) Y *has genus* 0 *and has an L–space filling.*

Proof of Theorem 1(b)   Let $l \in Sl(\mathcal{M})$ be the homological longitude. In our case l can be taken to be $[\lambda]$. I will show that none of the $(1, n)$ fillings to $\mathcal{M}$ is an L–space.

I will find one non-L–space filling first. SnapPy [6] shows that $(1, -1)$ filling on the knot 8_{20} complement with homological framing is homeomorphic to $m011(2, 3)$, which is also homeomorphic to $\mathcal{M}(1, -3)$. Ozsváth and Szabó showed that if some $(1, p)$ Dehn filling of a knot complement in S^3 with homological framing is an L–space, then the Alexander polynomial of the knot has coefficients ± 1 [16, Corollary 1.3]. We can compute with SnapPy [6] that the Alexander polynomial of 8_{20} is $x^4 - 2x^3 + 3x^2 - 2x + 1$. So $\mathcal{M}(1, -3)$ is not an L–space. Therefore,

$$-3l + [\mu] \notin \mathcal{L}(\mathcal{M}) \neq Sl(\mathcal{M}) - \{l\} \ni -3l + [\mu],$$

By Theorem 6, either $\mathcal{M}$ has no L–space fillings or $\mathcal{M}$ has positive genus.

The manifold $\mathcal{M}$ can be viewed as the complement of a knot K in $S^2 \times S^1$ [8]. This knot K intersects each S^2 three times. So $[K] \neq 0$ in $H_1(S^2 \times S^1; \mathbb{Z})$. It follows that $H_2(\mathcal{M}, \partial \mathcal{M})$ is generated by genus 0 surface $(S^2 \times \{P\}) \cap \mathcal{M}$ for generic point P on K. So $\mathcal{M}$ has genus 0, which forces $\mathcal{M}$ to have no L–space filling. Therefore, none of the integral homology spheres $\mathcal{M}(1, n)$ is an L–space. $\square$

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