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HOMFLY-PT homology for general link diagrams and braidlike isotopy

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Khovanov and Rozansky's categorification of the HOMFLY-PT polynomial is invariant under braidlike isotopies for any general link diagram and Markov moves for braid closures. To define HOMFLY-PT homology, they required a link to be presented as a braid closure, because they did not prove invariance under the other oriented Reidemeister moves. In this text we prove that the Reidemeister IIb move fails in HOMFLY-PT homology by using virtual crossing filtrations of the author and Rozansky. The decategorification of HOMFLY-PT homology for general link diagrams gives a deformed version of the HOMFLY-PT polynomial, $P^b(D)$, which can be used to detect nonbraidlike isotopies. Finally, we will use $P^b(D)$ to prove that HOMFLY-PT homology is not an invariant of virtual links, even when virtual links are presented as virtual braid closures.

57M25, 57M27

1 Introduction

Khovanov and Rozansky in [12] introduced a triply graded link homology theory categorifying the HOMFLY-PT polynomial. The construction given in [12] of Khovanov–Rozansky HOMFLY-PT homology, or briefly HOMFLY-PT homology, is an invariant of link diagrams up to braidlike isotopy (isotopies which locally resemble isotopies of a braid) and Markov moves for closed braid diagrams. However, Khovanov and Rozansky were not able to prove invariance under all oriented Reidemeister moves. In particular, they could not prove the Reidemeister IIb move, and in fact expected that it would fail in general. Because of this, they required that a link be presented as a braid closure so that HOMFLY-PT homology would be an invariant of links. In this text we will directly address this issue by proving the failure of the Reidemeister IIb move in HOMFLY-PT homology, and explore the consequences of this failure.

The framework of HOMFLY-PT homology can be extended to include the use of "virtual crossings", degree-4 vertices which are not actually positive or negative crossings. Virtual links (links with virtual crossings) were first introduced by Kauffman in [9]. The author and Rozansky in [1] proved that a filtration can be placed on the chain complex whose homology is HOMFLY-PT homology. The associated graded complex of this

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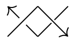
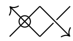
filtration is described using diagrams containing only virtual crossings. The filtration allows us to rewrite the chain complexes in an illuminating manner, allowing us to see new isomorphisms which would be difficult to see otherwise. Using the framework of virtual crossing filtrations we prove the following theorem.

Theorem 1.1 (see Theorem 4.11) *Let $\mathcal{H}(D)$ denote the* HOMFLY-PT *homology of the virtual link diagram D. Suppose D_1, D_2, and D_3 are oriented virtual link diagrams which are identical except in the neighborhood of a single point. Suppose in the neighborhood of that point, D_1 is* *, D_2 is* *, and D_3 is* *. Then $\mathcal{H}(D_1) \simeq \mathcal{H}(D_3)$ up to a grading shift, while $\mathcal{H}(D_1) \not\simeq \mathcal{H}(D_2)$ in general.*

In Section 4 we prove the above theorem and give an explicit example of a diagram of the unknot that does not have the HOMFLY-PT homology of the unknot (Example 4.12). Recall $\mathcal{H}(D)$ is a triply graded vector space. Suppose $d_{ijk} = \dim(\mathcal{H}(D)_{i,j,k})$. Then we can define the Poincaré series of $\mathcal{H}(D)$ as

$$\mathcal{P}(D) = \sum_{i,j,k\in\mathbb{Z}} d_{ijk} q^i a^j t^k. \tag{1-1}$$

Let $P(D)$ denote the HOMFLY-PT polynomial of the link diagram D. In [13], Murakami, Ohtsuki and Yamada introduced a state-sum formulation of the HOMFLY-PT polynomial commonly called the MOY construction. Their approach resolves a link diagram into a $\mathbb{Z}(q,a)$–linear combination of oriented planar 4–regular graphs. They give relations which evaluate each such planar graph as an element of $\mathbb{Z}(q,a)$. The resulting rational function from this process for any link diagram D is its HOMFLY-PT polynomial $P(D)$.

We now define a deformed HOMFLY-PT polynomial $P^b(D) = \mathcal{P}(D)|_{t=-1}$. In the case that D is presented as a braid closure then $P^b(D) = P(D)$. However, this is not true for general link diagrams. We collect known relations and properties of $P^b(D)$ into the following theorem.

Theorem 1.2 (see Theorem 5.1) *Let D be a link diagram. $P^b(D)$ is an invariant of link diagrams up to braidlike isotopy satisfying the skein relation*

$q P^b($  $) - q^{-1} P^b($  $) = (q - q^{-1}) P^b($  $).$

Furthermore, $P^b(D)$ satisfies the relations in Figure 1 in addition to the virtual MOY/Reidemeister moves and Z-moves (see Figures 14 and 15).

In Section 5 we use $P^b(D)$ to show that $\mathcal{H}(D)$ is not an invariant of virtual links, even when presented as a virtual braid closure, by showing it violates the virtual exchange move (see Kamada [8]).

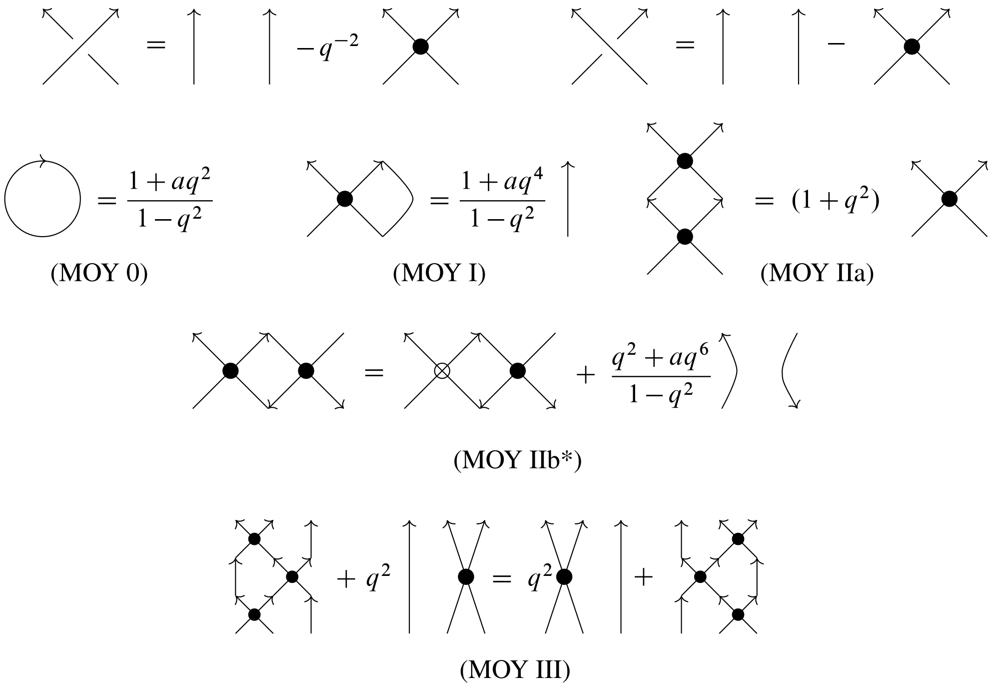


Figure 1: Relations for $P^b(D)$ (the notation $P^b(\cdot)$ omitted for readability)

We note that the relations in Figure 1 may not always be enough to determine $P^b(D)$, though in many examples the relations do suffice. We expect that in fact these relations will not compute $P^b(D)$ in general. The "nonbraidlike" MOY III diagram in Figure 2 does not split in a tractable manner and applying the MOYIIb relation introduces virtual crossings. Kauffman and Manturov in [10] construct an $\mathfrak{sl}_3$ specialization of the HOMFLY-PT polynomial for virtual links as formal $\mathbb{Z}[q,q^{-1}]$–linear combinations of directed graphs which are irreducible under MOY I, MOY IIa, MOY III, virtual MOY moves and Z-moves. Since our $P^b(D)$ for virtual braid closures specializes to their invariant, we expect that certain virtual MOY graphs will be irreducible with respect to the relations in Figure 1.

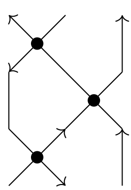
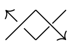
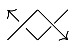
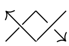


Figure 2: A "nonbraidlike" MOY III configuration

Recent research involving annular link homology by Auroux, Grigsby and Wehrli in the $\mathfrak{sl}_2$ case [4] and Queffelec and Rose in the $\mathfrak{sl}_n$ case [15] give some insight into why to expect this issue. We consider links in the thickened annulus as closed braids in a 3–ball D^3 with the braid axis ℓ removed from D^3. Annular link homology theories, that is homology theories of closed braids in the thickened annulus $D^3-\ell$, are normally constructed via the use of Hochschild homology on chain complexes of bimodules associated to braids. Hochschild homology $\mathrm{HH}(C)$ acts as a (horizontal) trace on the homotopy category of bimodules, but in general does not act like a Markov trace. In particular, if β_1 and β_2 are two braids which are Markov equivalent and $C(\beta_1)$ and $C(\beta_2)$ are their associated chain complexes of bimodules, then $\mathrm{HH}(C(\beta_1))$ is not necessarily homotopy equivalent to $\mathrm{HH}(C(\beta_2))$. This corresponds to the fact that even though the braid closures of β_1 and β_2 are isotopic as links in S^3, they may not be isotopic in $D^3-\ell$.

From this viewpoint, $\mathcal{H}(D)$ is an annular link invariant that happens to satisfy the Markov moves (that is, $\mathcal{H}(D)$ is a categorified Markov trace). This is why $\mathcal{H}(D)$ gives invariants of links in S^3 when D is presented as the closure of a braid. The Reidemeister IIb configuration  can only appear in a braid closure when the braid axis is between the two strands. In the case of annular invariants the braid axis is an obstruction to isotopy, and disallows the isotopy  $\sim$ . In an annular invariant the exchange move  $\sim$  is disallowed; however, this move preserves the isomorphism type of $\mathcal{H}(D)$.

Outline of the paper In Section 2 we review the definition of the HOMFLY-PT polynomial and the MOY construction of the HOMFLY-PT polynomial. We use nonstandard conventions in this text to illuminate the connections with HOMFLY-PT homology. In Section 3 we review the construction of HOMFLY-PT homology of links using closed braid diagrams. We also review some homological algebra, in particular properties of Koszul complexes. In Section 4 we explore the properties of HOMFLY-PT homology for general link diagrams. We introduce the role of virtual crossings in this framework and use virtual crossings as a tool to prove that HOMFLY-PT homology is not invariant under the Reidemeister IIb move. Finally, in Section 5 we explore the decategorification (Euler characteristic) of HOMFLY-PT homology and use it to prove that HOMFLY-PT homology cannot be extended to an invariant of virtual links.

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2 MOY construction of the HOMFLY-PT polynomial

We begin by recalling two constructions of the HOMFLY-PT polynomial for oriented links. Most of this material is well-known, but we introduce it with the purpose of setting our conventions for the sequel. The first construction is given by a skein relation and first appeared in [6]. The second construction, first introduced by Murakami, Ohtsuki and Yamada in [13], constructs the HOMFLY-PT polynomial in terms of a state-sum formula. It is this second construction which is categorified in the construction of Khovanov and Rozansky's HOMFLY-PT homology.

2.1 The HOMFLY-PT polynomial of an oriented link diagram

Let L denote a link in $\mathbb{R}^3$. In this text we will assume all links are oriented. Let D denote a link diagram of L, that is a regular projection of L onto a copy of $\mathbb{R}^2$. The HOMFLY-PT polynomial is an invariant of links which takes (oriented) link diagrams to elements of $\mathbb{Z}(q,a)$.

Definition 2.1 Let D be a link diagram and let O be a simple closed curve in the plane of the link diagram. We define the HOMFLY-PT *polynomial*, $P(D) \in \mathbb{Z}(q,a)$, via the following relations:

(1) $P(\varnothing) = 1$ and $P(O) = (1 + aq^2)/(1 - q^2)$.
(2) $P(D \sqcup O) = P(D)P(O)$.
(3) $qP(D_+) - q^{-1}P(D_-) = (q - q^{-1})P(D_0)$, where D_+, D_-, and D_0 are link diagrams which are the same except in the neighborhood of a single point where $D_+ =$ , $D_- =$ , and $D_0 =$ .
(4) If D and D' differ by a sequence of Reidemeister II and III moves (with any orientation), then $P(D) = P(D')$.

We will call a crossing which locally looks like  a *positive* crossing, and a crossing that locally looks like  a *negative crossing*. Let $f, g \in \mathbb{Z}(q,a)$ be nonzero. We will write $f \doteq g$ if $f = (-1)^i a^j q^k g$ for some $i, j, k \in \mathbb{Z}$. In other words, we write $f \doteq g$ if f/g is a unit in $\mathbb{Z}[q^{\pm 1}, a^{\pm 1}]$.

Theorem 2.2 (HOMFLY [6]; PT [14]) *Let D and D' be two link diagrams of a link L. Then $P(D) \doteq P(D')$. Furthermore, $P(D_2) = -q^{-2}P(D_1)$ and $P(D_3) = aq^2P(D_1)$, where D_1, D_2, D_3 are link diagrams which are the same except in the neighborhood of a single point where they are as in Figure 3.*

We will often denote the HOMFLY-PT polynomial of a link by $P(L)$, suppressing the choice of link diagram. In this case $P(L)$ is well defined up to a unit in $\mathbb{Z}[q^{\pm 1}, a^{\pm 1}]$.

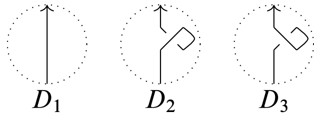


Figure 3: The diagrams D_1, D_2, and D_3 from Theorem 2.2

Remark 2.3 We are using nonstandard conventions for the HOMFLY-PT polynomial in this text. The HOMFLY-PT polynomial as defined here is not a polynomial, but rather is a rational function. One may choose a different normalization where both $P(D) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$ is honestly a (Laurent) polynomial and $P(D_1) = P(D_2) = P(D_3)$ where D_1, D_2, and D_3 are as in Figure 3. The choice of normalization here coincides with our conventions for HOMFLY-PT homology in the sequel.

2.2 The MOY construction of the HOMFLY-PT polynomial

Murakami, Ohtsuki and Yamada in [13] give a construction of the $\mathfrak{sl}_n$ polynomial, $P_n(L) \in \mathbb{Z}[q, q^{-1}]$, of a link L using evaluations of oriented colored trivalent plane graphs. These trivalent plane graphs correspond to the intertwiners between tensor powers of fundamental representations of $\mathcal{U}_q(\mathfrak{sl}_n)$. The $\mathfrak{sl}_n$ polynomial is actually a specialization of the HOMFLY-PT polynomial, that is $P_n(L)(q) = P(L)(q, a = q^{2-2n})$ in our conventions. We may adjust the MOY construction of the $\mathfrak{sl}_n$ polynomial to compute the HOMFLY-PT polynomial. We will replace the "wide edge" graph of [13] with a single degree-4 vertex (see Figure 4) which we will call a *MOY vertex.*

Recall an oriented graph is 4–*regular* if every vertex has degree 4, that is if each vertex has a total of 4 outgoing/incoming edges. The MOY state model of the HOMFLY-PT polynomial writes a link diagram as a formal $\mathbb{Z}(q, a)$–linear combination of planar, oriented, 4–regular graphs. The orientation locally at each vertex is the same as the orientation of the MOY vertex in Figure 4. We call such planar, oriented, 4–regular graphs *MOY graphs.*

We now define the MOY construction of the HOMFLY-PT polynomial. Let D be a link diagram. We can resolve any crossing c into either an oriented smoothing $\uparrow\uparrow$ or a MOY vertex  (with consistent orientation). To each resolution of c we associate a *weight*. If we smooth the crossing then the resolution has weight 0. If we replace the crossing with a MOY vertex, then the weight is -2 if the crossing was positive and 0

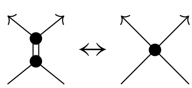


Figure 4: The MOY wide edge graph and our MOY vertex

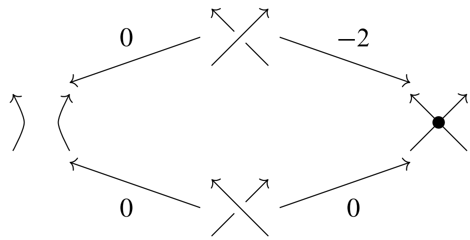


Figure 5: Resolution chart

if the crossing was negative. A resolution chart is given in Figure 5 for reference. We define a *state* σ of D as a choice of resolution for every crossing c in D. If D has n crossings, then it has 2^n possible states. We define the *weight of a state* $\mu(\sigma)$ to be the sum of the weights of the chosen resolutions of σ. Finally we will set $\nu(\sigma)$ to be the number of MOY vertices in σ.

Theorem 2.4 (Murakami, Ohtsuki and Yamada [13]) *The relations given in Figure 6 are sufficient to compute* $\overline{P}(D_\sigma)$ *as an element of* $\mathbb{Z}(q,a)$ *for any link diagram* D *and any state* σ.

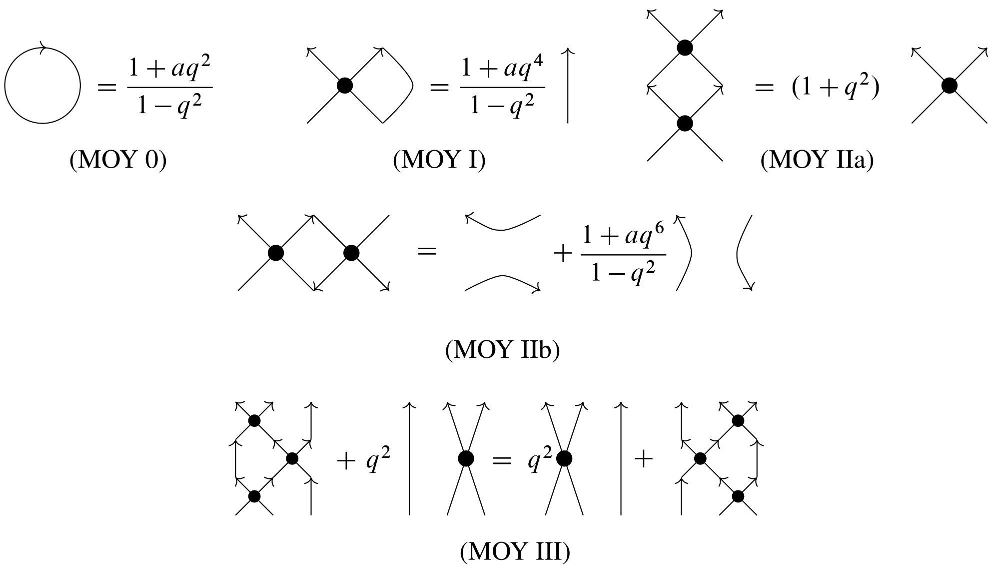


Figure 6: MOY relations (the notation $\overline{P}(\cdot)$ omitted for readability)

Definition 2.5 The *MOY polynomial,* $\overline{P}(D)$, is given by

$$\overline{P}(D) = \sum_\sigma (-1)^{\nu(\sigma)} q^{\mu(\sigma)} \overline{P}(D_\sigma). \tag{2-1}$$

Theorem 2.6 (Murakami, Ohtsuki and Yamada [13]) *Let D be an oriented link diagram. Then $\overline{P}(D) = P(D)$.*

Example 2.7 Using the relations in Figure 6, we compute $\overline{P}(D)$ for the diagram of the left-handed trefoil knot given in Figure 7. We leave it as an exercise to the reader to show

$$\overline{P}(D) = \left(\frac{1+aq^2}{1-q^2}\right)(q^2+aq^2+aq^6). \tag{2-2}$$

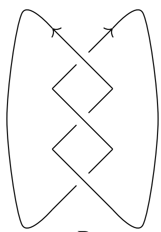


Figure 7: Diagram of the left-handed trefoil knot

3 HOMFLY-PT homology for closed braid diagrams

In this section we introduce the construction of Khovanov and Rozansky's HOMFLY-PT homology. The approach of this construction is to associate a chain complex of modules to every MOY graph and a *bicomplex* of modules to every link diagram. Our approach in this section is most similar to the approach of Rasmussen in [16] where we ignore his "$\mathfrak{sl}_n$" differential, as it is not needed in the construction of HOMFLY-PT homology.

3.1 Koszul complexes

Before introducing HOMFLY-PT homology, we recall some terminology and notation involving Koszul complexes. Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a $\mathbb{Z}$–graded commutative $\mathbb{Q}$–algebra and $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a $\mathbb{Z}$–graded R–module. It will be instructive to keep the example of $R = \mathbb{Q}[\boldsymbol{x}, \boldsymbol{y}]$ in mind, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are finite lists of variables (not necessarily of the same length). We define the grading shift functor $\bullet(k)$ by $M(k)_j = M_{j-k}$ for all $j \in \mathbb{Z}$. We will commonly use a nonstandard notation for grading shifts. In particular, we will set $q^k M := M(k)$ and say $\deg_q(x) = q^j$ if $x \in M_j$.

Definition 3.1 Let $p \in R$ be an element of degree k. The *Koszul complex* of p is defined as the chain complex

$$[p]_R = q^k R_1 \xrightarrow{p} R_0,$$

where p is used to denote the algebra endomorphism of R given by multiplication by p. Here $R_0 = R_1 = R$ and the subscript is simply used to denote the homological degree of the module. We will often write $[p] = [p]_R$ when there can be no confusion. Now let $\boldsymbol{p} = p_1, \dots, p_k$ be a sequence of elements in R. Then we define the *Koszul complex* of $\boldsymbol{p}$ as the complex

$$\begin{bmatrix} p_1 \\ \vdots \\ p_k \end{bmatrix} = [p_1] \otimes_R \cdots \otimes_R [p_k],$$

where $\otimes_R$ denotes the ordinary tensor product of chain complexes.

As a convention, we will call the homological grading in Koszul complexes the *Hochschild grading* and denote it by $\deg_a$. We write $\deg_a(x) = a^k$ to say that x is in Hochschild degree k and similarly write $a^k M$ to denote that M is being shifted k in Hochschild degree.

We say a sequence of elements $\boldsymbol{p} = p_1, \dots, p_k$ in R is a *regular sequence* if p_m is not a zero divisor in $R/(p_1, \dots, p_{m-1})$ for all $m = 1, \dots, k$. The following proposition is a standard fact in homological algebra and is proven in many introductory texts such as [18].

Proposition 3.2 *Let $\boldsymbol{p} = p_1, \dots, p_n$ be a regular sequence in R. Then the Koszul complex of $\boldsymbol{p}$ is a graded free R–module resolution of $R/(p_1, \dots, p_n)$.*

The notation we use for Koszul complexes is reminiscent of the notation for a column vector in $R^{\oplus n}$. Note that we will always use square brackets for Koszul complexes and round brackets for row vectors in $R^{\oplus n}$ to eliminate any confusion. Along these lines, we can look at "row operations" on Koszul complexes.

Proposition 3.3 *Let $\boldsymbol{p} = p_1, \dots, p_k$ be a sequence of elements in R, and let $\lambda \in \mathbb{Q}$. Then*

$$\begin{bmatrix} \vdots \\ p_i \\ \vdots \\ p_j \\ \vdots \end{bmatrix} \simeq \begin{bmatrix} \vdots \\ p_i + \lambda p_j \\ \vdots \\ p_j \\ \vdots \end{bmatrix}.$$

A homotopy equivalence of this form will be called a change of basis.

Proof We will omit grading shifts in the proof for clarity. We consider the map $\Phi\colon [p_i]\otimes_R[p_j]\to[p_i+\lambda p_j]\otimes_R[p_j]$ given by:

$$\begin{array}{ccccccc} [p_i]\otimes_R[p_j] & = & R & \xrightarrow{\begin{pmatrix}p_i\\p_j\end{pmatrix}} & R\oplus R & \xrightarrow{(-p_j\ \ p_i)} & R \\ \Big\downarrow{\scriptstyle\Phi} && \Big\downarrow{\scriptstyle 1} && \Big\downarrow{\scriptstyle\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}} && \Big\downarrow{\scriptstyle 1}\\ [p_i+\lambda p_j]\otimes_R[p_j] & = & R & \xrightarrow[\begin{pmatrix}p_i+\lambda p_j\\p_j\end{pmatrix}]{} & R\oplus R & \xrightarrow[(-p_j\ \ p_i+\lambda p_j)]{} & R\end{array}$$

This map is clearly invertible. □

3.2 Marked MOY graphs

A *marked MOY graph* is a MOY graph Γ (possibly with boundary) with markings such that the marks partition the graph into some combination of *elementary MOY graphs* as shown in Figure 8. We label the marks and the endpoints of the graph (if any) with variables. Typically, though not necessarily, we will label outgoing edges by variables y_i, incoming edges by variables x_i, and internal marks by variables t_i. An example of this process is given in Figure 8.

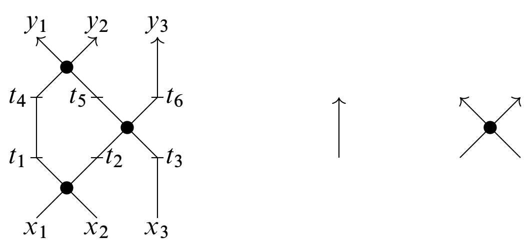


Figure 8: An example of a marked MOY graph and the elementary MOY graphs

To a marked MOY graph Γ, we will associate a collection of rings. Let $\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}$ denote the lists of incoming, outgoing, and internal variables respectively. We first define the *total ring* $E^t(\Gamma)$ of Γ as the polynomial ring $\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]$ containing all variables. We make this ring into a graded ring by setting $\deg_q(x_i)=\deg_q(y_i)=\deg_q(t_i)=q^2$. We call this grading the *internal* or *quantum grading*. We also suppose that all elements in $E^t(\Gamma)$ have Hochschild degree a^0. The other rings we will define will be subrings of $E^t(\Gamma)$. The *edge ring*, $E(\Gamma)$, is the polynomial ring of incoming and outgoing ("edge") variables $\mathbb{Q}[\boldsymbol{x},\boldsymbol{y}]$. The total ring $E^t(\Gamma)$ has a natural free $E(\Gamma)$–module structure. We also define the *incoming ring* (resp. *outgoing ring*) by $E^i(\Gamma) = \mathbb{Q}[\boldsymbol{x}]$ (resp. $E^o(\Gamma) = \mathbb{Q}[\boldsymbol{y}]$). Since $E(\Gamma) \cong E^i(\Gamma) \otimes_{\mathbb{Q}} E^o(\Gamma)$ as $\mathbb{Q}$–algebras, any $E(\Gamma)$–module can be considered as an $E^i(\Gamma)$–$E^o(\Gamma)$–bimodule. Note that if Γ does not have any boundary (eg if it is a resolution of a link diagram), then $E(\Gamma) \cong E^i(\Gamma) \cong E^o(\Gamma) \cong \mathbb{Q}$.

We will now define chain complexes $C(\Gamma)$ of free $E(\Gamma)$–modules associated to a marked MOY graph Γ. The chain modules of $C(\Gamma)$ will be direct sums of shifted copies of $E^t(\Gamma)$. We do this by first defining Koszul complexes associated to the elementary MOY graphs and then give rules for how gluing the graphs together affects the complexes associated to them. We will use the symbols $\uparrow$ and $\times$ to denote the elementary arc and vertex MOY graphs. To the arc, we associate the Koszul complex of modules over $E^t(\uparrow) = E(\uparrow) = \mathbb{Q}[x, y]$,

$$C(\uparrow) = [y - x]_{E(\uparrow)} = q^2 a E(\uparrow) \xrightarrow{y-x} E(\uparrow), \tag{3-1}$$

and to the vertex graph, we associate the Koszul complex of modules over $E^t(\times) = E(\times) = \mathbb{Q}[x_1, x_2, y_1, y_2]$,

$$\begin{aligned} C(\times) &= \begin{bmatrix} y_1 + y_2 - x_1 - x_2 \\ (y_1 - x_1)(y_1 - x_2) \end{bmatrix}_{E(\times)} \\ &= q^6 a^2 E(\times) \xrightarrow{A} q^4 a E(\times) \oplus q^2 a E(\times) \xrightarrow{B} E(\times), \end{aligned} \tag{3-2}$$

where

$$A = \begin{pmatrix} y_1 + y_2 - x_1 - x_2 \\ (y_1 - x_1)(y_1 - x_2) \end{pmatrix}, \quad B = \begin{pmatrix} -(y_1 - x_1)(y_1 - x_2) & y_1 + y_2 - x_1 - x_2 \end{pmatrix}.$$

Now suppose Γ is a marked MOY graph with edge ring E and total ring E^t. Also let Γ' be another marked MOY graph with edge ring E' and total ring E'^t. The disjoint union of these graphs $\Gamma \sqcup \Gamma'$ has edge ring $E'' \cong E \otimes_{\mathbb{Q}} E'$ and total ring $E''^t \cong E^t \otimes_{\mathbb{Q}} E'^t$. To the marked MOY graph $\Gamma \sqcup \Gamma'$ we will associate the complex of E''–modules $C(\Gamma \sqcup \Gamma') := C(\Gamma) \otimes_{\mathbb{Q}} C(\Gamma')$. A picture of the corresponding diagram is shown in Figure 9.

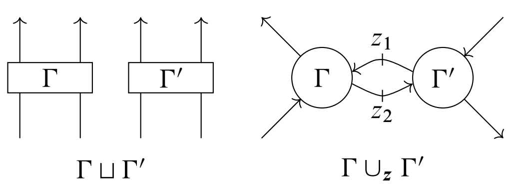


Figure 9: Examples of disjoint union and gluing of marked MOY graphs

Finally, we define a complex for when we glue two marked MOY graphs together. Let Γ and Γ' be two marked MOY graphs. We can glue outgoing edges of Γ to incoming edges of Γ' (or vice versa) to get a new marked MOY graph. First suppose only one pair of endpoints, one from each graph, are being glued together. Suppose that in both Γ and Γ' the endpoint being glued is labeled by the variable z (that is, $z \in E^i(\Gamma) \cap E^o(\Gamma')$ or $z \in E^i(\Gamma') \cap E^o(\Gamma)$). Then we define the new graph $\Gamma \cup_z \Gamma'$ by identifying the endpoints labeled by z and associate to $\Gamma \cup_z \Gamma'$ the complex

$$C(\Gamma \cup_z \Gamma') := C(\Gamma) \otimes_{\mathbb{Q}[z]} C(\Gamma'). \tag{3-3}$$

The edge ring of $\Gamma \cup_z \Gamma'$ is $E(\Gamma \cup_z \Gamma') = (E(\Gamma) \otimes_{\mathbb{Q}[z]} E(\Gamma'))/(z)$ and the total ring is $E^t(\Gamma \cup_z \Gamma') = E^t(\Gamma) \otimes_{\mathbb{Q}[z]} E^t(\Gamma')$. Note that after gluing, z is no longer in the edge ring as it is an internal variable. We may glue multiple edges at once in a similar manner. If $\boldsymbol{z} = z_1, \dots, z_n$ are the variables at the marked endpoints being identified, then we define $C(\Gamma \cup_{\boldsymbol{z}} \Gamma') := C(\Gamma) \otimes_{\mathbb{Q}[\boldsymbol{z}]} C(\Gamma')$. Similar to the case where we only identified one pair of edges, the edge ring of $\Gamma \cup_{\boldsymbol{z}} \Gamma'$ is given by $E(\Gamma \cup_{\boldsymbol{z}} \Gamma') = (E(\Gamma) \otimes_{\mathbb{Q}[\boldsymbol{z}]} E(\Gamma'))/(z_1, \dots, z_n)$ and the total ring is $E^t(\Gamma \cup_{\boldsymbol{z}} \Gamma') = E^t(\Gamma) \otimes_{\mathbb{Q}[\boldsymbol{z}]} E^t(\Gamma')$.

We can also describe disjoint union and gluing of marked MOY graphs in terms of Koszul complexes. Suppose $C(\Gamma)$ and $C(\Gamma')$ are given by the Koszul complexes

$$C(\Gamma) = \begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix}_{E^t(\Gamma)} \quad \text{and} \quad C(\Gamma') = \begin{bmatrix} p'_1 \\ \vdots \\ p'_n \end{bmatrix}_{E^t(\Gamma')}.$$

We can present $C(\Gamma \sqcup \Gamma')$ and $C(\Gamma \cup_{\boldsymbol{z}} \Gamma')$ as the Koszul complexes

$$C(\Gamma \sqcup \Gamma') = \begin{bmatrix} p_1 \\ \vdots \\ p_m \\ p_{1'} \\ \vdots \\ p'_n \end{bmatrix}_{E^t(\Gamma \sqcup \Gamma')} \quad \text{and} \quad C(\Gamma \cup_{\boldsymbol{z}} \Gamma') = \begin{bmatrix} p_1 \\ \vdots \\ p_m \\ p_{1'} \\ \vdots \\ p'_n \end{bmatrix}_{E^t(\Gamma \cup_z \Gamma')}. \tag{3-4}$$

Here the distinction comes from the difference in total and edge rings. $C(\Gamma \sqcup \Gamma')$ is a chain complex of free $E(\Gamma \sqcup \Gamma')$–modules with the chain modules as direct sums of shifted copies of $E^t(\Gamma \sqcup \Gamma')$. However $C(\Gamma \cup_{\boldsymbol{z}} \Gamma')$ is a chain complex of free $E(\Gamma \cup_{\boldsymbol{z}} \Gamma')$–modules with the chain modules as direct sums of shifted copies of $E^t(\Gamma \cup_{\boldsymbol{z}} \Gamma')$. We now give another useful technique for simplifying the complexes associated to marked MOY graphs, called *mark removal.*

Lemma 3.4 *Suppose that z is an internal variable of a marked MOY graph Γ and $C(\Gamma)$ is the Koszul complex of the sequence $\boldsymbol{p} = p_1, \dots, z - p_i, \dots, p_k$, where $p_1, \dots, p_k \in E(\Gamma)$. Let $\psi\colon E^t(\Gamma) \to E^t(\Gamma)/(z-p_i)$ be the quotient map identifying z with p_i. Then we have*

$$C(\Gamma) \simeq \psi\left(\begin{bmatrix} p_1 \\ \vdots \\ \widehat{z-p_i} \\ \vdots \\ p_k \end{bmatrix}\right) \simeq \begin{bmatrix} p_1 \\ \vdots \\ \widehat{z-p_i} \\ \vdots \\ p_k \end{bmatrix}_{E^t(\Gamma)/(z-p_i)}$$

as complexes of $E(\Gamma)$–modules, omitting the term $z - p_i$ from the sequence.

Various forms of this lemma are proven in other texts on HOMFLY-PT homology, such as the original work of Khovanov and Rozansky [12] or work of Rasmussen [16]. We refer the reader to Lemma 3.8 in [16] for this exact form, omitting the "backward" differentials of the matrix factorizations. Lemma 3.4 allows us to freely add or remove marks without changing the homotopy type of the complex (as a complex of $E(\Gamma)$–modules). This implies the following very useful statement.

Corollary 3.5 *Let Γ and Γ' be two marked MOY graphs whose underlying (unmarked) MOY graphs are the same (isomorphic as oriented graphs). Then $C(\Gamma) \simeq C(\Gamma')$ as complexes of modules over $E(\Gamma) = E(\Gamma')$.*

Example 3.6 Consider the marked MOY graph from Figure 10. The marks partition the MOY graphs into six elementary MOY graphs (three MOY vertices and three arcs) which are drawn in Figure 10.

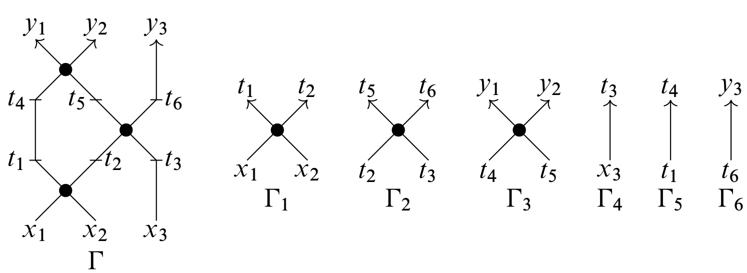


Figure 10: The marked MOY graph in Example 3.6 and its elementary MOY graphs

We can write Γ as $(\Gamma_1 \sqcup \Gamma_4) \cup_{t_1,t_2,t_3} (\Gamma_2 \sqcup \Gamma_5) \cup_{t_4,t_5,t_6} (\Gamma_3 \sqcup \Gamma_6)$, and therefore

$$C(\Gamma) = C(\Gamma_1 \sqcup \Gamma_4) \otimes_{\mathbb{Q}[t_1,t_2,t_3]} C(\Gamma_2 \sqcup \Gamma_5) \otimes_{\mathbb{Q}[t_4,t_5,t_6]} C(\Gamma_3 \sqcup \Gamma_6).$$

We can write $C(\Gamma)$, after some applications of mark removal to remove t_3, t_4, and t_6, as:

$$C(\Gamma) \simeq \begin{bmatrix} y_1 + y_2 - t_1 - t_5 \\ y_1 y_2 - t_1 t_5 \\ t_5 + y_3 - t_2 - x_3 \\ t_5 y_3 - t_2 x_3 \\ t_1 + t_2 - x_1 - x_2 \\ t_1 t_2 - x_1 x_2 \end{bmatrix}_{\mathbb{Q}[x_1,x_2,x_3,y_1,y_2,y_3,t_1,t_2,t_5]}$$

We invite the reader to finish the process of removing the internal variables t_1, t_2, and t_5 to get a *finite-rank* complex of $\mathbb{Q}[x_1, x_2, x_3, y_1, y_2, y_3]$–modules.

3.3 MOY braid graphs

A *MOY braid graph* is a graph formed by taking a braid and replacing every crossing with a MOY vertex, whose incoming and outgoing edges are consistent with the orientation of the braid. The complexes associated to MOY braid graphs and their "braid closures" satisfy the following local relations (as proven in [12; 16]):

Proposition 3.7 *Let Γ_0, Γ_{1a}, Γ_{1b}, Γ_{2a}, Γ_{2b}, Γ_{3a}, Γ_{3b}, Γ_{3c}, and Γ_{3d} be MOY graphs as in Figure 11. Then*

$$C(\Gamma_0) \simeq \bigoplus_{i=0}^{\infty} q^{2i}(\mathbb{Q} \oplus aq^2\mathbb{Q}), \tag{3-5}$$

$$C(\Gamma_{1a}) \simeq \bigoplus_{i=0}^{\infty} q^{2i}(C(\Gamma_{1b}) \oplus aq^4 C(\Gamma_{1b})), \tag{3-6}$$

$$C(\Gamma_{2a}) \simeq C(\Gamma_{2b}) \oplus q^2 C(\Gamma_{2b}), \tag{3-7}$$

$$C(\Gamma_{3a}) \oplus q^2 C(\Gamma_{3b}) \simeq q^2 C(\Gamma_{3c}) \oplus C(\Gamma_{3d}), \tag{3-8}$$

where $\simeq$ denotes homotopy equivalence over the corresponding edge rings.

To compare the isomorphisms in Proposition 3.7 to the relations in Figure 6 we introduce the notation of a "Laurent series shift functor". Suppose $F(q,a) \in \mathbb{N}[\![q^{\pm 1}, a^{\pm 1}]\!]$, that is

$$F(q,a) = \sum_{i,j \in \mathbb{Z}} c_{ij} q^i a^j, \quad c_{ij} \in \mathbb{N} \cup \{0\}.$$

Suppose M is a $\mathbb{Z}\times\mathbb{Z}$–graded R–module with grading shifts denoted by $q^i a^j$. Then

$$F(q,a)M := \bigoplus_{i,j \in \mathbb{Z}} q^i a^j M^{\oplus c_{ij}}. \tag{3-9}$$

We can write similar expressions for chain complexes C of $\mathbb{Z}\times\mathbb{Z}$–graded R–modules.

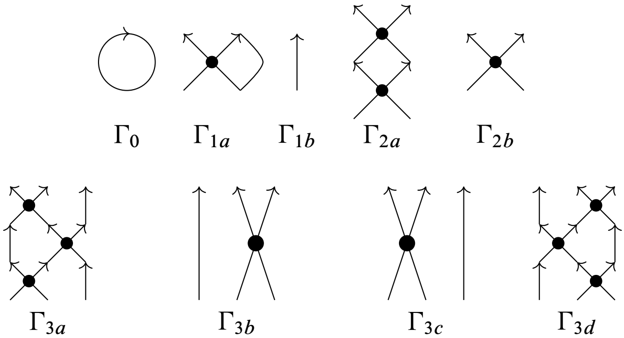


Figure 11: MOY graphs for Proposition 3.7

Let

$$F(q,a,t) = \sum_{i,j,k\in\mathbb{Z}} c_{ijk} q^i a^j t^k \in \mathbb{N}[\![q^{\pm 1}, a^{\pm 1}, t^{\pm 1}]\!],$$

and let the homological grading shift on C be denoted by t^k. Then

$$F(q,a,t)C := \bigoplus_{i,j,k\in\mathbb{Z}} q^i a^j t^k C^{\oplus c_{ijk}}. \tag{3-10}$$

The coefficients in MOY 0 and MOY I are not actually Laurent series, but rather rational functions. However, considering the rational function with Laurent polynomial numerator $F(q,a)$ as a geometric series, we can write the rational functions as a Laurent series

$$\frac{F(q,a)}{1-q^2} = F(q,a)\sum_{i=0}^{\infty} q^{2i}.$$

With this notation in mind, we can rewrite the isomorphism (3-5) as

$$\begin{aligned} C(\Gamma_0) &\simeq \bigoplus_{i=0}^{\infty} q^{2i}(\mathbb{Q}\oplus aq^2\mathbb{Q}) = \bigoplus_{i=0}^{\infty} q^{2i}(1+aq^2)\mathbb{Q} \\ &= (1+aq^2)\sum_{i=0}^{\infty} q^{2i}\mathbb{Q} = \frac{1+aq^2}{1-q^2}\mathbb{Q} \end{aligned} \tag{3-11}$$

and the isomorphism (3-6) as

$$\begin{aligned} C(\Gamma_{1a}) &\simeq \bigoplus_{i=0}^{\infty} q^{2i}(C(\Gamma_{1b})\oplus aq^4 C(\Gamma_{1b})) \\ &= (1+aq^4)\sum_{i=0}^{\infty} q^{2i} C(\Gamma_{1b}) = \frac{1+aq^4}{1-q^2} C(\Gamma_{1b}). \end{aligned} \tag{3-12}$$

We invite the reader to compare the rewritten relation (3-11) to (MOY 0) from Figure 6 and (3-12) to (MOY I). This comparison can be made for (3-7) to (MOY 2a) and (3-8) to (MOY 3) as well.

3.4 Khovanov–Rozansky HOMFLY-PT homology

We now have recalled the necessary tools to define Khovanov–Rozansky HOMFLY-PT homology, or briefly HOMFLY-PT homology. We first define two q–degree 0 maps $\chi_i\colon \uparrow\uparrow \to q^{-2}\overset{\bullet}{\times}$ and $\chi_o\colon \overset{\bullet}{\times} \to \uparrow\uparrow$. Set $E=\mathbb{Q}[x_1,x_2,y_1,y_2]$ to be the edge ring of both $\uparrow\uparrow$ and $\overset{\bullet}{\times}$. Then we define χ_i by

$$
\begin{array}{ccccccc}
\uparrow\uparrow & = & a^2q^4E & \xrightarrow{\begin{pmatrix} y_1-x_1\\ y_2-x_2\end{pmatrix}} & aq^2E\oplus aq^2E & \xrightarrow{(x_2-y_2\ \ y_1-x_1)} & E\\
\Big\downarrow \chi_i & & \Big\downarrow 1 & & \Big\downarrow \begin{pmatrix} y_1-x_2 & 0\\ 1 & 1\end{pmatrix} & & \Big\downarrow y_1-x_2\\
q^{-2}\overset{\bullet}{\times} & = & a^2q^4E & \xrightarrow[\begin{pmatrix} P_2\\ P_1\end{pmatrix}]{} & aE\oplus aq^2E & \xrightarrow[(-P_1\ \ P_2)]{} & q^{-2}E
\end{array}
\tag{3-13}
$$

and χ_o by

$$
\begin{array}{ccccccc}
\overset{\bullet}{\times} & = & a^2q^6E & \xrightarrow{\begin{pmatrix} P_2\\ P_1\end{pmatrix}} & aq^2E\oplus aq^4E & \xrightarrow{(-P_1\ \ P_2)} & E\\
\Big\downarrow \chi_o & & \Big\downarrow y_1-x_2 & & \Big\downarrow \begin{pmatrix} 1 & 0\\ -1 & y_1-x_2\end{pmatrix} & & \Big\downarrow 1\\
a\uparrow\uparrow & = & a^2q^4E & \xrightarrow[\begin{pmatrix} y_1-x_1\\ y_2-x_2\end{pmatrix}]{} & aq^2E\oplus aq^2E & \xrightarrow[(x_2-y_2\ \ y_1-x_1)]{} & E
\end{array}
\tag{3-14}
$$

Above, we set $P_1=y_1+y_2-x_1-x_2$ and $P_2=(y_1-x_1)(y_1-x_2)$ for the sake of legibility. We now define two bicomplexes of free E–modules for the positive crossing $\overset{+}{\times}$ and the negative crossing $\overset{-}{\times}$:

$$C(\overset{+}{\times}) := C(\uparrow\uparrow)\xrightarrow{\chi_i} tq^{-2}C(\overset{\bullet}{\times}), \tag{3-15}$$

$$C(\overset{-}{\times}) := t^{-1}C(\overset{\bullet}{\times})\xrightarrow{\chi_o} C(\uparrow\uparrow). \tag{3-16}$$

Note that we use the notation $t^kC(\Gamma)$ to mean that the complex for Γ sits in homological degree k. This is a different homological degree than our Hochschild degree we introduced earlier. We will simply denote this degree by $\deg_t$ and call it the *homological degree.* We will say that x has (total) degree $\deg(x) = q^i a^j t^k$ if it has quantum degree i, Hochschild degree k, and homological degree j. We will denote the differential in the complexes for the MOY graphs as d_g and the differentials in the complexes (built from χ_i and χ_o) associated to crossings as d_c. Both $C($  $)$ and $C($  $)$ are bicomplexes with commuting differentials d_g and d_c.

A *marked tangle diagram* is a tangle diagram with markings such that the marks partition the tangle diagram into arcs, positive crossings, and negative crossings. We label the marks and the endpoints (if any) by variables in a similar fashion to marked MOY graphs. We define rings associated to each marked tangle diagram τ in a similar manner to our constructions for marked MOY graphs.

Before defining a bicomplex for a tangle diagram, we recall a definition from homological algebra.

Definition 3.8 Let $C = (C_{\bullet\bullet}, d_h, d_v)$ and $C' = (C'_{\bullet\bullet}, d'_h, d'_v)$ be two bicomplexes. We define the *tensor product bicomplex* $C \otimes C' = ((C \otimes C')_{\bullet\bullet}, d_h^{\otimes}, d_v^{\otimes})$ as follows:

$$\begin{aligned}
(C \otimes C')_{mn} &= \bigoplus_{i+k=m, j+\ell=n} (C_{ij} \otimes C'_{k\ell}), \\
d_h^{\otimes}(x \otimes y) &= d_h(x) \otimes y + (-1)^i x \otimes d'_h(y) \quad \text{for } x \in C_{ij} \text{ and } y \in C'_{k\ell}, \\
d_v^{\otimes}(x \otimes y) &= d_v(x) \otimes y + (-1)^k x \otimes d'_v(y) \quad \text{for } x \in C_{ij} \text{ and } y \in C'_{k\ell}.
\end{aligned}$$

We can now build a bicomplex for any tangle diagram (and link diagram) in a similar manner to what we did in Section 3.3 for MOY graphs. To a disjoint union of (marked) tangles $\tau = \tau_1 \sqcup \tau_2$ we associate the bicomplex of $E(\tau)$–modules

$$C(\tau) := C(\tau_1) \otimes_{\mathbb{Q}} C(\tau_2).$$

Similarly if we are gluing two tangles τ_1 and τ_2 at the marked points $\boldsymbol{z} = z_1, \dots, z_k$ in such a way that the orientations are consistent, then we define a bicomplex of $E(\tau_1 \cup_{\boldsymbol{z}} \tau_2)$–modules

$$C(\tau_1 \cup_{\boldsymbol{z}} \tau_2) = C(\tau_1) \otimes_{\mathbb{Q}[\boldsymbol{z}]} C(\tau_2).$$

We omit the rest of the details in this case, and leave it to the reader to compare with the analogous conventions for marked MOY graphs. Now let $\beta \in \mathrm{Br}_n$ be a braid with n strands. We can mark β in such a way that we partition it into arcs and crossings of the form  or  and we label the endpoints and markings in a similar manner to our conventions for marked MOY graphs. Therefore we can use the rules of disjoint unions and gluing of tangles to write a bicomplex $C(\beta)$ of $E(\beta)$–modules. In this case, $E(\beta)=\mathbb{Q}[\boldsymbol{x},\boldsymbol{y}]$ where $|\boldsymbol{x}|=|\boldsymbol{y}|=n$.

We now describe the construction of the HOMFLY-PT homology of a link L. Suppose that $\beta\in\mathrm{Br}_n$ is a braid representative of L, that is L is the circular closure of β in $\mathbb{R}^3$. We will often use the notation L_β for the link diagram of the closure of β. Then we can describe the bicomplex $C(L_\beta)=C(\beta)\otimes_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y}]}C(1_n)$, where 1_n denotes the identity braid (oriented downwards) with the top endpoints labeled by $\boldsymbol{y}$ and the bottom endpoints labeled by $\boldsymbol{x}$. We refer the reader to Figure 12 for an example of this decomposition of a braid closure.

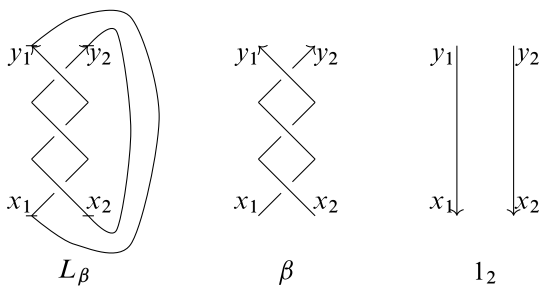


Figure 12: A link presented as a braid closure and the constituent tangles

Definition 3.9 Suppose L is a link with braid representative $\beta\in\mathrm{Br}_n$. The HOMFLY-PT homology of L_β is $\mathcal{H}(L_\beta)=H_{d_{c*}}(H_{d_g}(C(L_\beta)))$.

Remark 3.10 $\mathcal{H}(L_\beta)$, as defined above, arises as the E^2–page of a spectral sequence. Let L be an n–component link. It is easily shown that the E^∞–page of that spectral sequence is the homology of the n–component unlink (up to a grading shift). In particular, $H_{d_c}(C(L))$ is isomorphic to the E^∞–page.

Theorem 3.11 (Khovanov and Rozansky [12]) *Suppose $\beta\in\mathrm{Br}_n$ and $\beta'\in\mathrm{Br}_{n'}$ are two braid representatives of a link L. Then $\mathcal{H}(L_\beta)\cong\mathcal{H}(L_{\beta'})$ up to a grading shift. Furthermore, suppose the Poincaré series* (see (1-1)) *of $\mathcal{H}(L_\beta)$ is given by*

$$\mathcal{P}(L_\beta)=\sum_{i,j,k\in\mathbb{Z}} d_{i,j,k}q^i a^j t^k,\quad \text{where } d_{i,j,k}=\dim_{\mathbb{Q}}(\mathcal{H}(L_\beta))_{i,j,k}.$$

Then

$$\mathcal{P}(L_\beta)|_{t=-1}=\sum_{i,j,k\in\mathbb{Z}} d_{i,j,k}q^i a^j(-1)^k=P(L_\beta).$$

4 HOMFLY-PT homology for general link diagrams

In this section we study what happens when we consider general link diagrams in the construction of HOMFLY-PT homology. We will see that not all Reidemeister moves are respected, and that in general HOMFLY-PT homology is only an invariant up to braidlike isotopy.

4.1 Virtual crossings and marked MOY graphs

We start by introducing virtual crossings into the framework of (marked) MOY graphs. We will not fully discuss virtual knot theory here, but rather refer the reader to Kauffman [9]. Virtual crossings were first considered as a tool in HOMFLY-PT and $\mathfrak{sl}_n$ homologies by Khovanov and Rozansky in [11], and studied further by the author and Rozansky in [1].

A *virtual MOY graph* is a MOY graph where we allow the underlying graph to be nonplanar. Such a graph can always be drawn where the intersections forced by the projection onto the plane are transverse double points. An example of this is given in Figure 13. To the marked virtual crossing graph we associate the following complex of free $E(\otimes) = \mathbb{Q}[x_1, x_2, y_1, y_2]$–modules:

$$C(\otimes) = \begin{bmatrix} y_1 - x_2 \\ y_2 - x_1 \end{bmatrix}_{E(\otimes)} = q^4 E(\otimes) \xrightarrow{A} q^2 E(\otimes) \oplus q^2 E(\otimes) \xrightarrow{B} E(\otimes), \tag{4-1}$$

where

$$A = \begin{pmatrix} y_1 - x_2 \\ y_2 - x_1 \end{pmatrix}, \quad B = \begin{pmatrix} x_1 - y_2 & y_1 - x_2 \end{pmatrix}.$$

Note that $C(\otimes)$ resembles $C(\uparrow\ \uparrow)$ except for a transposition of x_1 and x_2 in the definition of the complexes. In this sense, we can think of a virtual crossing as being a permutation of strands with no additional crossing data or vertex at the intersection.

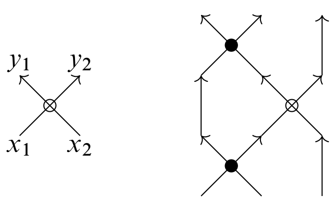
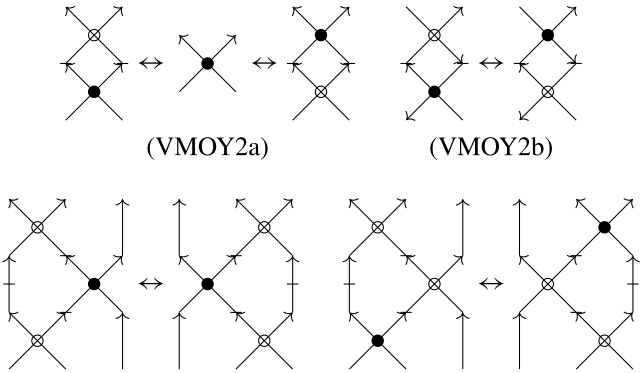


Figure 13: A (marked) virtual crossing and an example of a virtual MOY graph

Proposition 4.1 *The moves in Figure 14 preserve the homotopy equivalence type of $C(\Gamma)$.*



(VMOY2a) (VMOY2b)

(VMOY3a) (VMOY3b)

Figure 14: Virtual MOY moves (marks included for reference for Proposition 4.1)

Proof We begin with (VMOY2a). Let t_1 and t_2 be the variables associated to the marks (from left to right) in the diagram on the left for (VMOY2a). The left-hand side of (VMOY2a) is presented as the Koszul complex of the sequence

$$(y_2 - t_1, y_1 - t_2, t_1 + t_2 - x_1 - x_2, (t_1 - x_1)(t_1 - x_2)).$$

Let $w = y_1 + y_2 - x_1 - x_2$. Then

$$\begin{aligned}\begin{bmatrix} y_2 - t_1 \\ y_1 - t_2 \\ t_1 + t_2 - x_1 - x_2 \\ (t_1 - x_1)(t_1 - x_2) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]} &\simeq \begin{bmatrix} y_2 - t_1 \\ y_1 - t_2 \\ w \\ (t_1 - x_1)(t_1 - x_2) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]} \\ &\simeq \begin{bmatrix} y_1 - t_2 \\ w \\ (y_1 - x_1)(y_1 - x_2) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},t_2]} \\ &\simeq \begin{bmatrix} w \\ (y_1 - x_1)(y_1 - x_2) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y}]}.\end{aligned}$$

The first isomorphism is a change of basis and the other two isomorphisms are mark removals. The last term is the Koszul complex for $C($$)$. The second isomorphism in (VMOY2a) is proven by a similar argument. Next we prove (VMOY2b). For consistency, we label the bottom endpoints x_1 and x_2, the top endpoints y_1 and y_2 and the marks by t_1 and t_2 (reading from left to right). The left-hand side of (VMOY2b) can be written as

$$\begin{bmatrix} y_2 - t_1 \\ t_2 - y_1 \\ t_1 + x_1 - t_2 - y_2 \\ (t_1 - x_2)(t_1 - t_2) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]} \simeq \begin{bmatrix} y_2 + x_1 - y_1 - y_2 \\ (y_2 - x_2)(y_2 - t_2) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y}]},$$

where the isomorphism is given by removing the internal variables t_1 and t_2. Likewise, the right-hand side of (VMOY2b) can be written as

$$\begin{bmatrix} y_2 + t_2 - y_1 - t_1 \\ (y_2 - y_1)(y_2 - t_1) \\ t_1 - x_2 \\ x_1 - t_2 \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]} \simeq \begin{bmatrix} y_2 + x_1 - y_1 - y_2 \\ (y_2 - x_2)(y_2 - t_2) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y}]},$$

where again the isomorphism is given by removing the internal variables t_1 and t_2. This proves (VMOY2b).

Now we approach (VMOY3a). For both diagrams in (VMOY3a), label the top variables as y_1, y_2, y_3 and bottom variables as x_1, x_2, x_3 from left to right. Also label the marks as t_1, t_2, t_3 from bottom to top. The associated Koszul complex to the left-hand side is

$$\begin{bmatrix} y_1 - t_3 \\ y_2 - t_2 \\ t_2 - x_2 \\ t_1 - x_1 \\ y_3 + t_3 - t_1 - x_3 \\ (t_3 - t_1)(t_3 - x_3) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]} \simeq \begin{bmatrix} y_2 - x_2 \\ y_1 + y_3 - x_1 - x_3 \\ (y_1 - x_1)(y_3 - x_3) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y}]},$$

where the isomorphism is given by removing the internal variables t_1, t_2, t_3. Likewise, the associated Koszul complex to the right hand-side is given by

$$\begin{bmatrix} y_3 - t_3 \\ y_2 - t_2 \\ t_1 - x_3 \\ t_2 - x_2 \\ 2_1 + t_3 - t_1 - x_1 \\ (y_1 - x_1)(y_1 - t_1) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]} \simeq \begin{bmatrix} y_2 - x_2 \\ y_1 + y_3 - x_1 - x_3 \\ (y_1 - x_1)(y_3 - x_3) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y}]},$$

where the isomorphism is given by removing the internal variables t_1, t_2, t_3. This proves (VMOY3a).

The approach to proving (VMOY3b) is almost identical, but we include it for completeness. For both diagrams in (VMOY3b), label the top variables as y_1, y_2, y_3 and bottom variables as x_1, x_2, x_3 from left to right. Also label the marks as t_1, t_2, t_3 from bottom to top. The associated Koszul complex to the left-hand side is

$$\begin{bmatrix} y_1 - t_3 \\ y_2 - t_2 \\ y_3 - t_1 \\ t_3 - x_3 \\ t_2 + t_1 - x_1 - x_2 \\ (t_2 - x_1)(t_2 - x_2) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]} \simeq \begin{bmatrix} y_1 - x_3 \\ y_2 + y_3 - x_1 - x_2 \\ (y_2 - x_1)(y_2 - x_2) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y}]},$$

where the isomorphism is given by removing the internal variables t_1, t_2, t_3. Likewise, the associated Koszul complex to the right hand-side is given by

$$\begin{bmatrix} y_1 - t_1 \\ y_2 + y_3 - t_3 - t_2 \\ (y_2 - t_3)(y_2 - t_2) \\ t_3 - x_1 \\ t_2 - x_2 \\ t_1 - x_3 \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]} \simeq \begin{bmatrix} y_1 - x_3 \\ y_2 + y_3 - x_1 - x_2 \\ (y_2 - x_1)(y_2 - x_2) \end{bmatrix}_{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y}]},$$

where the isomorphism is given by removing the internal variables t_1, t_2, t_3. Therefore, (VMOY3b) is proven. □

For any *virtual link diagram* D, that is a link diagram with virtual crossings, we can repeat the procedure from Section 3.4 to build a bicomplex of $E(D)$–modules. We now record the additional "virtual" Reidemeister moves.

Proposition 4.2 *The moves in Figure 15 preserve the homotopy equivalence type of* $C(D)$*. The isomorphisms (VR1), (VR2a), (VR2b), (VR3), and (SVR) are called* virtual Reidemeister moves, *and the isomorphisms (Z1±) and (Z2±) are called Z-moves.*

The proofs of (VR1), (VR2a), (VR2b), and (VR3) follow the same outline (write Koszul complexes for both sides, and compare after mark removal) as the proof of Proposition 4.1. (SVR) and the Z-moves follow from resolving the single crossing and applying the moves (VR2a), (VR2b), (VMOY2a), (VMOY2b), and (VMOY3a). Note that our Koszul complexes for many of our diagrams are free resolutions of certain bimodules. On the level of these bimodules these moves are proven in other sources. (VR1) is proven in Lemma 6.5 of [1], (VR2a), (VR3), (SVR), (Z1+), and (Z1−) are proven in [17, Lemma 3.1, Lemma 3.2 and Theorem 3.4] and [1, Theorem 2.2]. Strictly speaking, a different move from the Z-moves, called "virtualization moves" are proven in these texts. However, the Z-moves follow via tensoring with a virtual crossing

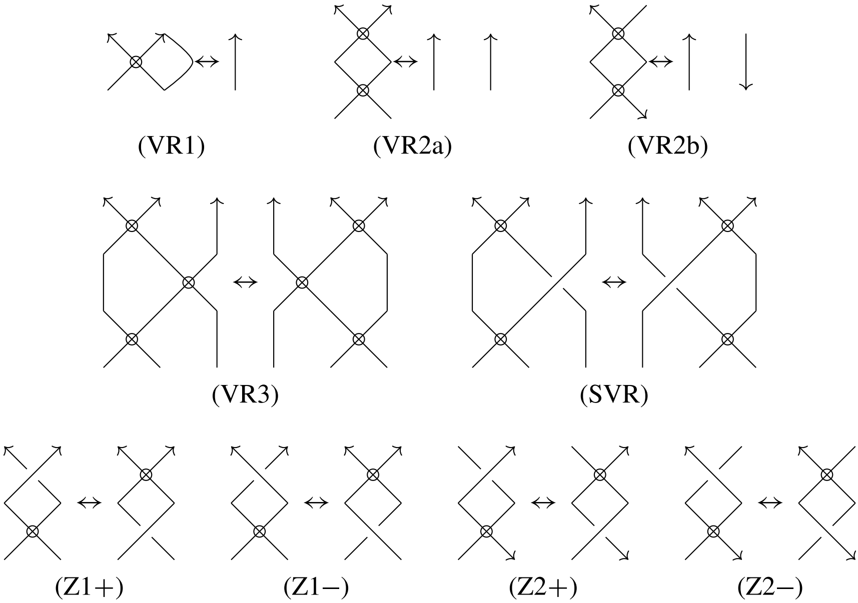


Figure 15: Virtual Reidemeister moves and Z-moves

and applying (VR2a). A little more needs to be said about the moves (Z2±). After resolving the crossing we apply (VMOY2b) and note, up to a relabeling of variables, the differential does not change.

4.2 Virtual filtrations of signed MOY graphs and a key lemma

Virtual crossing filtrations were first introduced by the author and Rozansky in [1]. We now introduce these filtrations in our current setting as an eventual tool in proving the failure of certain Reidemeister moves. In this text, we use virtual filtrations to prove a key lemma in the process of finding an explicit example of failure of Reidemeister IIb.

Definition 4.3 Let R be a commutative ring and let C and D be chain complexes of objects in an additive category. Let $\bullet[i]$ denote the homological shift functor given by $C_j[i]=C_{j-i}$. Define $\mathrm{Hom}^k(C,D)$ to be the $\mathbb{Z}$–module of chain maps $f\colon C\to D[-k]$ quotiented by the submodule of chain maps homotopic to the zero map.

In [11], Khovanov and Rozansky make the following observation.

Proposition 4.4 *There exists a unique map* $F\in\mathrm{Hom}^1(C(\uparrow\,\uparrow),q^2C(\otimes))$, *up to rescaling, such that* $\mathrm{Cone}(F)$ *is homotopy equivalent to* $C(\bullet)$. *Likewise there exists a unique map, up to rescaling,* $G\in\mathrm{Hom}^1(C(\otimes),q^2C(\uparrow\,\uparrow))$ *such that* $\mathrm{Cone}(G)$ *is homotopy equivalent to* $C(\bullet)$.

We will call the maps F and G *virtual saddle maps.* In our presentation of $C(\uparrow\uparrow)$ and $C(\otimes)$ we can write the virtual saddle maps explicitly. We give the following explicit presentation of the virtual saddle map G and leave it to the reader to do the same for the analogous map F:

$$
\begin{array}{ccccccc}
C(\otimes) = a^2q^4E & \xrightarrow{\begin{pmatrix} x_1-y_2 \\ x_2-y_1 \end{pmatrix}} & aq^2E \oplus aq^2E & \xrightarrow{(x_2-y_1 \;\; y_2-x_1)} & E \\
\Big\downarrow G & \searrow^{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} & & \searrow^{(1\;\;1)} & \\
q^2C(\uparrow\uparrow) = a^2q^6E & \xrightarrow[\begin{pmatrix} x_1-y_1 \\ x_2-y_2 \end{pmatrix}]{} & aq^4E \oplus aq^4E & \xrightarrow[(x_2-y_2 \;\; y_1-x_1)]{} & q^2E
\end{array}
$$

The mapping cone presentations give rise to filtrations. In particular, $\mathrm{Cone}(F)$ has $q^2C(\otimes)$ as a submodule and $C(\uparrow\uparrow)$ as the quotient $\mathrm{Cone}(F)/q^2C(\otimes)$. We call this filtration the *negative filtration* and denote it as $C_-(\bullet)$. Likewise $\mathrm{Cone}(G)$ has $q^2C(\uparrow\uparrow)$ as a subcomplex and $C(\otimes)$ as the quotient complex $\mathrm{Cone}(F)/q^2C(\uparrow\uparrow)$. We call this filtration the *positive filtration* and denote it as $C_+(\bullet)$.

We will often identify $C_-(\bullet)$ with $\mathrm{Cone}(F)$ (and $C_+(\bullet)$ with $\mathrm{Cone}(G)$) and consider the filtered complexes as mapping cones. This process simplifies the differential d_c so that it can be presented in the following manner (as proven in [1]).

Proposition 4.5 *The bicomplex $C(\times)$ is homotopy equivalent to the bicomplex $C(\uparrow\uparrow) \xrightarrow{\phi_i} tq^{-2}C_+(\bullet)$, where ϕ_i denotes the canonical inclusion of $C(\uparrow\uparrow)$ into* $\mathrm{Cone}(G)$*. Suppose $C(\uparrow\uparrow)$ has the trivial filtration. Then ϕ_i is a filtered map with respect to the filtration on $C_+(\bullet)$ and thus $C(\times)$ is a filtered bicomplex.*

In addition, $C(\times)$ is homotopy equivalent to the bicomplex $t^{-1}C(\bullet) \xrightarrow{\phi_o} C(\uparrow\uparrow)$, where ϕ_0 is the canonical projection of $C(\uparrow\uparrow)$ from $\mathrm{Cone}(F)$*. The projection ϕ_o is a filtered map with respect to the filtration on $C_-(\bullet)$ and thus $C(\times)$ is a filtered bicomplex.*

We can extend this filtration to any tangle or link diagram via the tensor product filtration. We will also refer to the given filtration on the bicomplex associated to a tangle as a *virtual filtration.* The following theorem was the main focus of [1].

Theorem 4.6 *Let β be a braid on n strands, and L_β denote its circular closure. Then the virtual filtration on $C(\beta)$ is invariant under Reidemeister IIa and is violated by at most two levels by Reidemeister III. Furthermore, the virtual filtration on $\mathcal{H}(L_\beta)$ is invariant under the Markov moves (up to a possible shift in filtration).*

We now focus on describing the filtrations on MOY graphs, which will be useful in proving Lemma 4.10. A *signed MOY graph* is a MOY graph where each vertex is marked by a sign $+$ or $-$. Suppose Γ is a signed MOY graph marked so that it is partitioned into graphs of the form and . Then we can define a filtration on $C(\Gamma)$. To each MOY vertex marked with a $+$ we associate $C_+(\;)$, and to each MOY vertex marked with a $-$ we associate $C_-(\;)$. We give the trivial filtration to $C(\;)$. Then the filtration on $C(\Gamma)$ is given by the tensor product filtration.

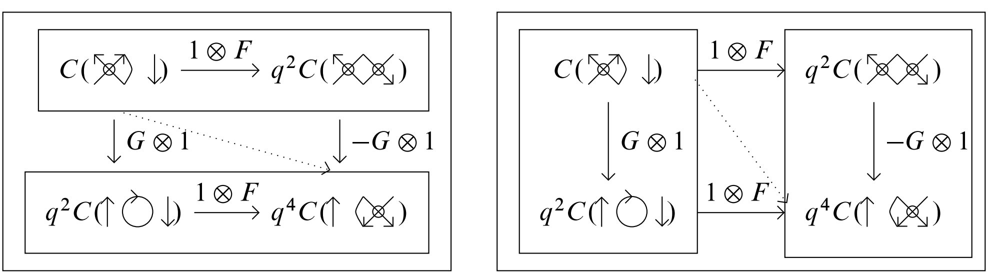
If we choose different sign assignments, then we receive homotopy equivalent complexes for $C(\Gamma)$, but not necessarily *filtered* homotopy equivalent complexes (eg $C_-(\;)$ and $C_+(\;)$ are not filtered homotopy equivalent). For this reason, if $\underline{\varepsilon}$ is a assignment of signs to each MOY vertex of Γ, then we will write $C_{\underline{\varepsilon}}(\Gamma)$ for the filtered complex we get from the above construction.

With this construction in mind, we may present every MOY graph as an *iterated mapping cone* of graphs with only virtual crossings and no MOY vertices. We will commonly use the alternate notation for mapping cones shown in Figure 16.

$$\boxed{A \xrightarrow{\;f\;} B} := \mathrm{Cone}(f\colon A \to B)$$

Figure 16: Alternate notation for mapping cones

Example 4.7 We now consider the signed MOY graph $\Gamma =$ whose left vertex is labeled by $+$ and right vertex is labeled by $-$. We can present $C_{+-}(\Gamma)$ as one of the two equal iterated mapping cones:



Equality of the above iterated mapping cones follows from the associativity of the mapping cone operation. Note that the dotted arrow is the zero map, but is drawn in the diagram above as a reminder that such a map may be needed after simplifying the above complex.

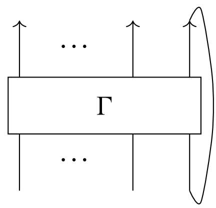


Figure 17: Partial braid closure of a virtual braidlike MOY graph

Next we state a few simplifications and tools which will be necessary in proving our key lemma. The following lemma is proven in [1].

Lemma 4.8 *Let* $\widehat{\otimes}$ *and* $\widehat{\uparrow\uparrow}$ *denote the partial braid closure of* $\otimes$ *and* $\uparrow\uparrow$ *respectively* (see Figure 17). *We write* $\widehat{G}\colon C(\widehat{\otimes}) \to q^2 C(\widehat{\uparrow\uparrow})$ *for the map induced by* G *under partial braid closure, and similarly for* $\widehat{F}$. *Then*

(1) $\operatorname{Cone}(\widehat{G}) \simeq \operatorname{Cone}(0) \simeq C(\uparrow) \oplus q^2 C(\uparrow \circlearrowright)$,

(2) $\operatorname{Cone}(\widehat{F}) \simeq \operatorname{Cone}\left(aq^2 C(\uparrow) \oplus \dfrac{1+aq^4}{1-q^2} C(\uparrow) \xrightarrow{(1\,0)} q^2 C(\uparrow)\right)$,

(3) $\operatorname{Cone}(\widehat{F}) \simeq \operatorname{Cone}(\widehat{G}) \simeq \dfrac{1+aq^4}{1-q^2} C(\uparrow)$.

Furthermore, the homotopy equivalences in (1) *and* (2) *are filtered.*

A proof of the following result can be found in other texts on link homology such as [5].

Proposition 4.9 *Consider the complex*

$$A \xrightarrow{\binom{\bullet}{\alpha}} B \oplus C \xrightarrow{\begin{pmatrix}\varphi & \lambda\\ \mu & \nu\end{pmatrix}} D \oplus E \xrightarrow{\binom{\bullet}{\varepsilon}} F,$$

where $\varphi\colon B \to D$ *is an isomorphism and all other maps are arbitrary up to the condition that* $d^2 = 0$. *Then there exists a homotopy equivalence:*

$$\begin{array}{ccccccc}
A & \xrightarrow{\binom{\bullet}{\alpha}} & B \oplus C & \xrightarrow{\begin{pmatrix}\varphi & \lambda\\ \mu & \nu\end{pmatrix}} & D \oplus E & \xrightarrow{(\bullet\ \varepsilon)} & F \\
{\scriptstyle (1)}\downarrow\ \uparrow{\scriptstyle (1)} & & {\scriptstyle (0\ 1)}\downarrow\ \uparrow{\scriptstyle \binom{-\varphi^{-1}\lambda}{1}} & & {\scriptstyle (-\mu\varphi^{-1}\ 1)}\downarrow\ \uparrow{\scriptstyle \binom{0}{1}} & & {\scriptstyle (1)}\downarrow\ \uparrow{\scriptstyle (1)} \\
A & \xrightarrow{(\alpha)} & C & \xrightarrow{(\nu-\mu\varphi^{-1}\lambda)} & E & \xrightarrow{(\varepsilon)} & F
\end{array}$$

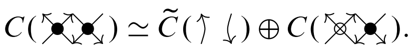
We call this homotopy equivalence Gaussian elimination.

We now look at the analogue of (MOYIIb) from Figure 6. This will be our key lemma in simplifying the Reidemeister IIb complex.

Lemma 4.10 *Let*

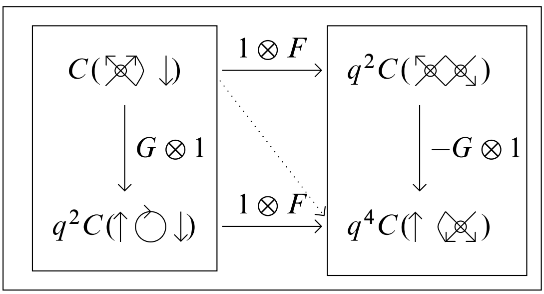
$$\widetilde{C}(\uparrow\ \downarrow) = \frac{q^2 + aq^6}{1 - q^2} C(\uparrow\ \downarrow).$$

Then



Proof Consider $C(\cdots)$ with the filtration described in Example 4.7. We then consider $C(\cdots)$ as the iterated mapping cone:

(4-2)

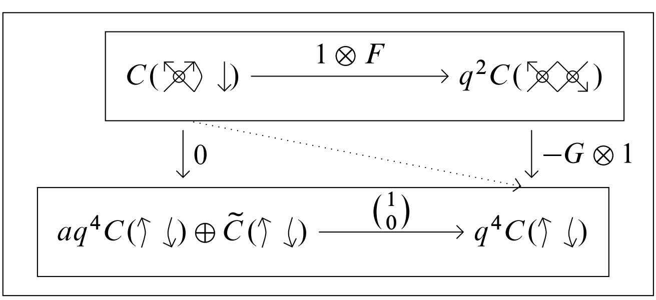


The maps

$$G \otimes 1\colon C(\cdots\ \downarrow) \to q^2 C(\uparrow \bigcirc \downarrow) \quad \text{and} \quad 1 \otimes F\colon q^2(C(\uparrow \bigcirc \downarrow)) \to q^4 C(\uparrow\ \cdots)$$

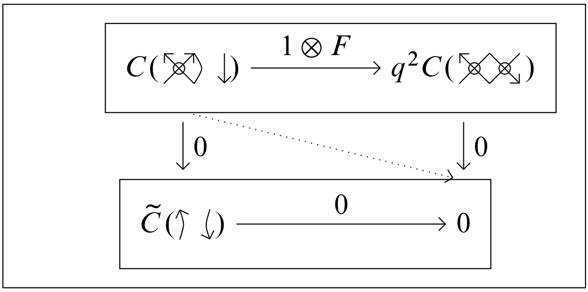
can be viewed as maps between partial braid closures $\widehat{G} \otimes 1_{\downarrow}$ and $1_{\uparrow} \otimes \widehat{F}$ respectively. We now apply isomorphisms (1) and (2) of Lemma 4.8 and associativity of the mapping cone operation to simplify the complex (4-2) to:

(4-3)



Note the dotted arrow may no longer be the zero map, but knowledge of the exact map will ultimately not be necessary. Now we apply Gaussian elimination to the bottom mapping cone in (4-3) to get:

(4-4)



The top mapping cone is isomorphic to $C(\otimes\bullet)$ by Proposition 4.5. Therefore $C(\bullet\bullet) \simeq \widetilde{C}(\uparrow\ \downarrow) \oplus C(\otimes\bullet)$ as desired. □

4.3 Failure of Reidemeister IIb for $\mathcal{H}(D)$

Now with the introduction of virtual crossings, we can study the exact outcome of allowing general link diagrams in the computation of $\mathcal{H}(D)$.

We now state the main result of this section.

Theorem 4.11 *The isomorphism $C(\times\times) \simeq tq^{-2}C(\otimes\times)$ holds.*

Proof We first write $C(\times\times)$ using (3-15) and (3-16):

$$C(\times\times) = t^{-1}C(\uparrow\ \bullet) \xrightarrow{A} C(\uparrow \circlearrowright \downarrow) \oplus q^{-2}C(\bullet\bullet) \xrightarrow{B} tq^{-2}C(\bullet\ \downarrow),$$

where the tensor product above is over the appropriate ring of internal variables and

$$A = \begin{pmatrix} 1 \otimes \chi_0 \\ \chi_i \otimes 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \chi_i \otimes 1 & -1 \otimes \chi_o \end{pmatrix}.$$

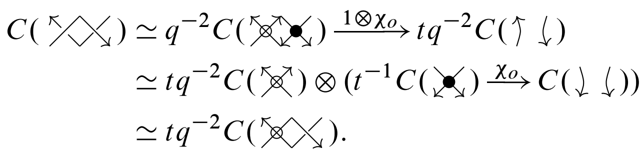
We use the isomorphisms from Proposition 3.7 and Lemma 4.10 to rewrite the complex as

$$q^{-2}\widetilde{C}(\uparrow\ \downarrow) \xrightarrow{D} \underline{q^{-2}\widetilde{C}(\uparrow\ \downarrow) \oplus C(\uparrow \circlearrowright \downarrow) \oplus q^{-2}C(\otimes\bullet)} \xrightarrow{F} C(\uparrow \circlearrowright \downarrow) \oplus q^{-2}C(\uparrow\ \downarrow),$$

where the underlined term is in homological degree 0 (omitting homological shifts for compactness of notation) and

$$D = \begin{pmatrix} 1 \\ * \\ * \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} * & 1 & 0 \\ * & * & -1 \otimes \chi_o \end{pmatrix}.$$

Note the maps $*$ are irrelevant to our discussion. Note above, we use the map $-1 \otimes \chi_o$ to denote the map $-1 \otimes \chi_o : q^{-2}C(\otimes\bullet) \to C(\otimes\ \downarrow) \simeq C(\uparrow\ \downarrow)$ (the isomorphism following from (VR1)). After performing Gaussian elimination on the complex above, we are left with



Example 4.12 We now give an example where the local failure of Reidemeister IIb gives a failure of isotopy invariance for a certain link diagram. Let D be the diagram for the unknot given in Figure 18, and let O denote the standard diagram of the unknot as a circle bounding a disc in $\mathbb{R}^2$. Then we have the following chain of isomorphisms

$$\mathcal{H}(D) \simeq t^2 q^{-4} \mathcal{H}(D') \simeq t^2 q^{-4} \mathcal{H}(D'') \simeq t^2 q^{-4} \mathcal{H}(D''').$$

The first isomorphism is given by applying Theorem 4.11 twice. The second isomorphism following from applying (Z2−) from Proposition 4.1. The last isomorphism follows from applying (VR2b) from Proposition 4.1. D''' is the diagram of the left-handed trefoil knot, and we computed $P(D''')$ in Example 2.7. This is enough to show that $\mathcal{H}(D) \not\simeq \mathcal{H}(O)$; however, it is an easy exercise to show that

$$\mathcal{H}(D''') = (aq^2 + t^2q^2 + t^2aq^4)\frac{1+aq^2}{1-q^2}\mathbb{Q} \not\simeq \frac{1+aq^2}{1-q^2}\mathbb{Q} \simeq \mathcal{H}(O).$$

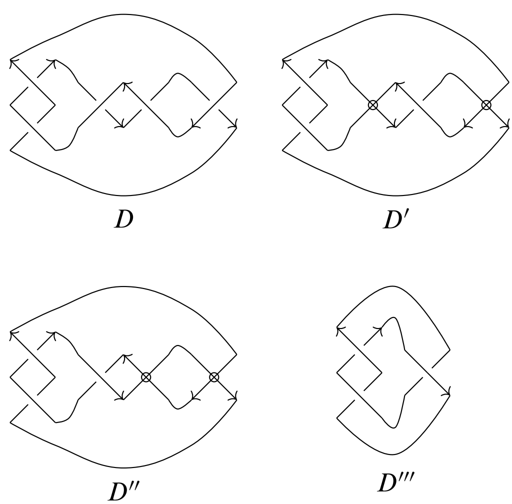


Figure 18: The failure of Reidemeister IIb for an unknot diagram. The above diagrams all have the same HOMFLY-PT homology up to a grading shift.

4.4 Braidlike isotopy and $\mathcal{H}(L)$

As we saw in Example 4.12, HOMFLY-PT homology for general link diagrams is *not* an isotopy invariant. However, it was proven in [12] that it was a *braidlike isotopy* invariant. We now carefully recall the definition of braidlike isotopy.

Definition 4.13 Two oriented link diagrams D and D' are said to represent *braidlike isotopic* links if they differ by a sequence of planar isotopies and the following moves in Figure 19. Such a sequence of moves will be called a *braidlike isotopy*.

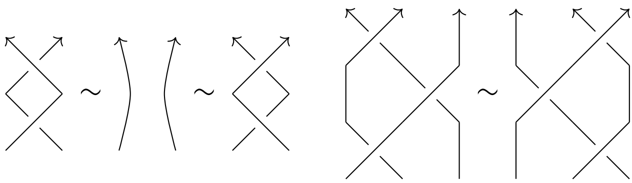


Figure 19: Braidlike Reidemeister moves

Theorem 4.14 (Khovanov and Rozansky [12]) *Let D and D' be two braidlike isotopic link diagrams. Then $\mathcal{H}(D) \simeq \mathcal{H}(D')$.*

Braidlike isotopy is an important notion in studying links in the solid torus. It is a well-known fact that two braid closures in the solid torus give isotopic links if and only if the braids are equivalent, or rather if they are braidlike isotopic. Audoux and Fiedler in [3] give a deformation of Khovanov homology which detects braidlike isotopy of links in $\mathbb{R}^3$. Their invariant decategorifies to a deformation of the Jones polynomial which can be computed using a Kauffman bracket-like relation. In the case of closed braid diagrams, their invariant corresponds with the homology theory studied in [2] by Asaeda, Przytycki and Sikora and the decategorification corresponds with the polynomial invariant studied by Hoste and Przytycki in [7].

We can now interpret Example 4.12 in the following manner: The unknot diagram D shown in Figure 18 is isotopic, but not braidlike isotopic to the standard unknot diagram O. In particular, there is no sequence of Reidemeister moves transforming D to O which does not contain the Reidemeister IIb move. In this sense, we see that $\mathcal{H}(D)$ can detect nonbraidlike isotopy. Note that $\mathcal{H}(D)$ is isomorphic to the homology of the left-handed trefoil knot (after two negative Reidemeister I moves), though clearly D is not a diagram for the left-handed trefoil knot. Therefore, this viewpoint of $\mathcal{H}(D)$ may not be useful in determining isotopy type of general diagrams, but can be very useful when we know the two diagrams are of the same isotopy type and we wish to determine if they are of the same braidlike isotopy type.

We can also see easily that the Poincaré series of $\mathcal{H}(D)$ and $\mathcal{H}(O)$ differ. Direct calculation shows that $\mathcal{P}(D) = (aq^2 + t^2q^2 + t^2aq^4)\mathcal{P}(O)$. This implies that we may be able to detect nonbraidlike isotopies on the level of the MOY calculus. As we will see in the next section, after a deformation of the MOY theory, this is indeed the case.

5 Decategorification of $\mathcal{H}(D)$ for general link diagrams

In this section we study the decategorification of $\mathcal{H}(D)$, which we will denote by $P^b(D)$. As we saw in Example 4.12, $P^b(D) \neq P(D)$ in general. In particular, when this occurs, this implies that D is not braidlike isotopic to a closed braid presentation of a link. We will end this section with a note on virtual links and give an explicit example of where $P^b(D)$ is not invariant under the virtual exchange move, which implies that $\mathcal{H}(D)$ cannot be extended to a virtual link invariant.

5.1 A deformation of the HOMFLY-PT polynomial

Let D be a link diagram and recall $\mathcal{P}(D)$ is defined as the Poincaré series of $\mathcal{H}(D)$. We define our deformed HOMFLY-PT polynomial as

$$P^b(D) = \mathcal{P}(D)|_{t=-1} \in \mathbb{Z}(q,a). \tag{5-1}$$

Theorem 5.1 *Let D and D' be two link diagrams which are braidlike isotopic. Then $P^b(D) = P^b(D')$. Furthermore, P^b satisfies the following skein relation:*

$$qP^b(\times_+) - q^{-1}P^b(\times_-) = (q - q^{-1})P^b(\uparrow\,\uparrow). \tag{5-2}$$

Proof The first statement is an immediate corollary of Theorem 4.14. For the second part of the statement, note that we have a map of homological degree 1 $\psi\colon tq^{-1}C(\times_-) \to qC(\times_+)$ given by:

$$\begin{array}{ccccc}
tq^{-1}C(\times_-) & = & q^{-1}C(\times_\bullet) & \xrightarrow{\chi_o} & tq^{-1}C(\uparrow\,\uparrow) \\
\downarrow{\scriptstyle\psi} & & & \searrow{\scriptstyle 1} & \\
qC(\times_+) & = & qC(\uparrow\,\uparrow) & \xrightarrow{\chi_i} & tq^{-1}C(\times_\bullet)
\end{array}$$

The mapping cone of ψ, after Gaussian elimination, is homotopy equivalent to

$$qC(\uparrow\,\uparrow) \xrightarrow{-\chi_o\chi_i} tq^{-1}C(\uparrow\,\uparrow),$$

and therefore

$$\mathrm{Cone}\big(tq^{-1}C(\times_-) \xrightarrow{\psi} C(\times_+)\big) \simeq \big(qC(\uparrow\,\uparrow) \xrightarrow{-\chi_o\chi_i} tq^{-1}C(\uparrow\,\uparrow)\big). \tag{5-3}$$

Properties of the Euler characteristic of a chain complex and (5-3) gives us the relation (5-2) as desired. □

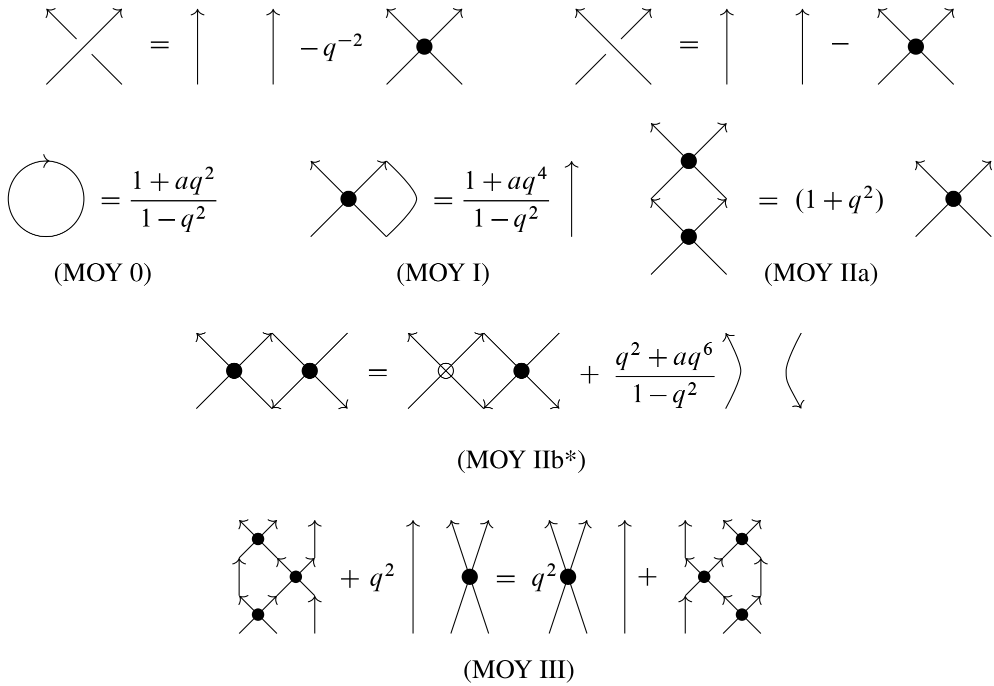
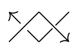
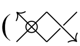


Figure 20: MOY relations for $P^b(D)$ and the deformed MOY IIb relation (the notation $P^b(\cdot)$ omitted for readability)

Corollary 5.2 *If D is a link diagram presented as a braid closure, then $P^b(D) = P(D)$. Equivalently if D is a link diagram for some link L and $P^b(D) \neq P(D)$, then D is not braidlike isotopic to a braid presentation of L.*

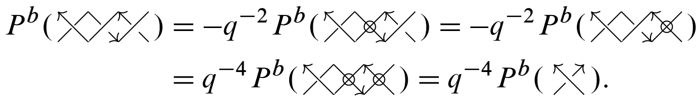
Corollary 5.3 *The relation $P^b($*  *$) = -q^{-2} P^b($*  *$)$ holds. We call this relation the deformed Reidemeister II relation and denote it by (RIIb*).*

We can also give a MOY-style construction for $P^b(D)$. In particular, we address the relation (MOY IIb) from Figure 6. As we saw in Lemma 4.10, the categorified MOY IIb relation does not hold as we would expect. However, we can decategorify the results in Proposition 3.7, Proposition 4.1, Theorem 4.11 and Lemma 4.10 in a natural way.

Proposition 5.4 *Let D be a link diagram. The braidlike Reidemeister moves, virtual MOY moves, virtual Reidemeister moves, and the relations in Figure 20 hold for $P^b(D)$.*

Example 5.5 Let D be the diagram of the $(2, 2k+1)$–torus knot given in Figure 21. First note that we can transform D to D' using (RIIb*) so that $P^b(D) = q^{-4} P^b(D')$.

More precisely, we use the fact that



The equalities above follow (from left to right) by (R2b*), (Z2+), (MOYIIb*), and (R2b*). D' is isotopic to the $(2, 2k-1)$–torus knot via a planar isotopy and a (braidlike) Reidemeister IIa move. Therefore via a straightforward calculation similar to that in Example 2.7,

$$P^b(D) = \frac{1+aq^2}{1-q^2}\left((a+1)\sum_{i=1}^{k-1} q^{-i} + q^{-4k}\right).$$

However, if $\widetilde{D}$ is a braid presentation of the $(2, 2k+1)$–torus knot, then

$$P^b(\widetilde{D}) = \frac{1+aq^2}{1-q^2}\left((a+1)\sum_{i=1}^{k} q^{-i-2} + q^{-4k-2}\right).$$

Therefore D is not braidlike isotopic to a braid presentation of the $(2, 2k+1)$–torus knot for all $k \geq 1$.

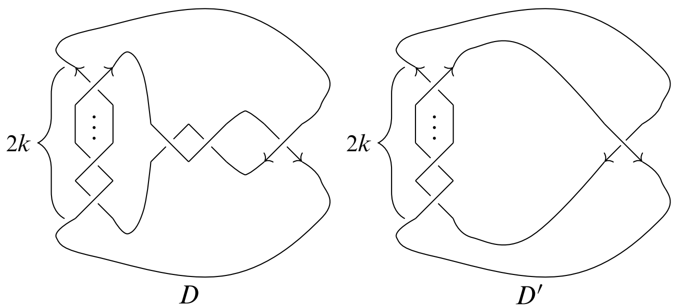


Figure 21: A nonbraidlike diagram D for the $(2, 2k+1)$–torus knot and a braidlike link diagram D' for the $(2, 2k-1)$–torus knot such that $P^b(D) = q^{-4} P^b(D')$.

5.2 An obstruction to extending HOMFLY-PT homology to virtual links

Finally we wish to present an argument showing that the current definition of $\mathcal{H}(D)$ cannot be extended to virtual links, even when they are presented as closures of virtual braids. We will ultimately use $P^b(D)$ to justify this statement.

A *virtual braid* is a braid in which we allow virtual crossings alongside positive and negative crossings. Two virtual braids are said to be equivalent if they differ by the moves from Figure 19 and the moves (VR2a), (VR3), and (SVR) from Figure 15. Kauffman in [9] proves that every virtual link can be presented as the closure of a virtual braid. There is also an analogue of the Markov theorem for virtual links.

Theorem 5.6 (Kamada [8]) *Let β and β' be two virtual braids. Their braid closures are equivalent virtual links if and only if they differ by a sequence of virtual braid equivalence moves (the braidlike Reidemeister moves, (VR2a), (VR3), and (SVR)), the Markov moves, and the virtual exchange move. The Markov moves and virtual exchange more are pictured in Figure 22.*

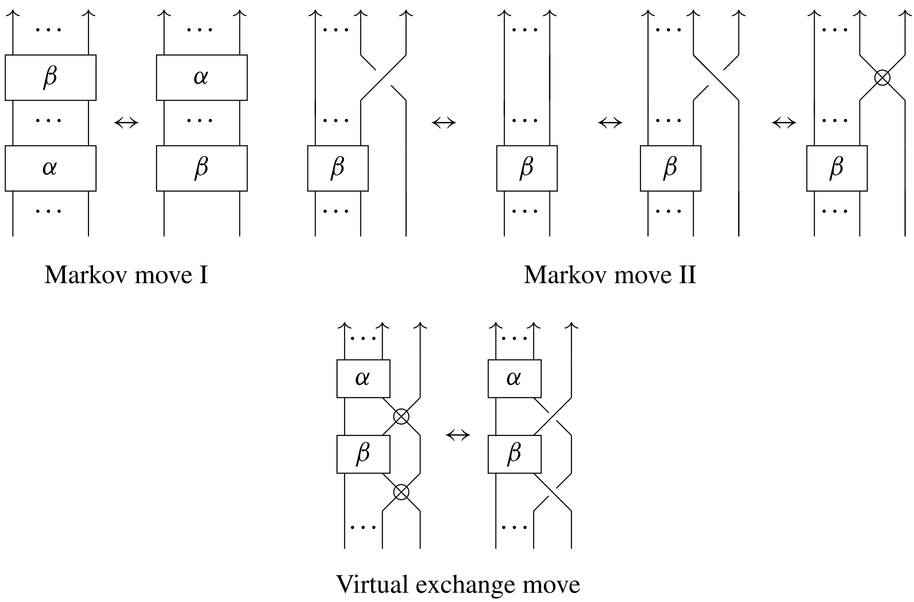


Figure 22: Markov moves for virtual links and the virtual exchange move. α and β are virtual braids.

Now we show by example that $P^b(D)$ is *not* invariant under the virtual exchange move. Therefore $\mathcal{H}(D)$ is not an invariant of virtual links, even when the links are presented as virtual braid closures.

Example 5.7 Let L be a connected sum of two virtual Hopf links as shown in Figure 23. β_1 and β_2 are two virtual braids whose closures are equivalent as virtual links to L. In particular, β_1 and β_2 are related by Markov move I and the virtual exchange move shown in Figure 22. Let D_1 be the braid closure of β_1 and D_2 be the

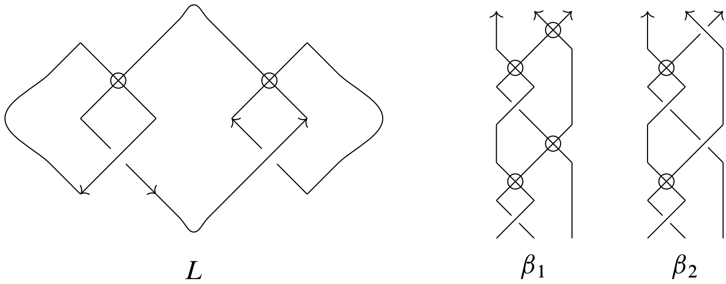


Figure 23: A connected sum of two virtual Hopf links, L, and two braid presentations of L

braid closure of β_2. Using the relations from Figure 20 we can directly compute that

$$P^b(D_1) = \frac{1+aq^2}{1-q^2}\left(1-q^{-2}\left(\frac{1+aq^4}{1-q^2}\right)\right)^2,$$

$$P^b(D_2) = \frac{1+aq^2}{1-q^2}\left(aq^2+2\left(\frac{1+aq^4}{1-q^2}\right)-q^{-2}\left(\frac{1+aq^4}{1-q^2}\right)^2\right).$$

It is easy to see that $P^b(D_2) \neq P^b(D_1)$. In particular,

$$P^b(D_2) - P^b(D_1) = q^2(1+a)\frac{1+aq^2}{1-q^2}.$$

Therefore, $P^b(D)$ is not an invariant of virtual links and thus neither is $\mathcal{H}(D)$.

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