msp

*Algebraic & Geometric Topology* 17 (2017) 3057–3079

# The topological sliceness of 3–strand pretzel knots

ALLISON N MILLER

We give a complete characterization of the topological slice status of odd 3–strand pretzel knots, proving that an odd 3–strand pretzel knot is topologically slice if and only if it either is ribbon or has trivial Alexander polynomial. We also show that topologically slice even 3–strand pretzel knots, except perhaps for members of Lecuona's exceptional family, must be ribbon. These results follow from computations of the Casson–Gordon 3–manifold signature invariants associated to the double branched covers of these knots.

57M25; 57N70

## 1 Introduction

In the years since Fox first posed the slice-ribbon conjecture (Problem 1.33 on Kirby's list [14]), its validity has been established for several families of knots. The usual strategy is to give an explicit list of ribbon knots in the family and then to provide an obstruction to the smooth sliceness of all others in the family. An early example of this is the following classification of the smoothly slice rational knots due to Lisca.

**Theorem 1.1** (Lisca [16]) *A rational knot is smoothly slice if and only if it is ribbon if and only if it is in $\mathcal{R}$.*

$\mathcal{R}$ is an explicit family of rational knots known to be ribbon at least since the work of Casson and Gordon [4]. Lisca argues that if $K$ is not in $\mathcal{R}$, then Donaldson's diagonalization theorem obstructs $\Sigma_2(K)$ from smoothly bounding a rational homology ball, and hence obstructs $K$ from being smoothly slice.

In a similar spirit, though with entirely different methods, we give an almost complete characterization of the topological sliceness of 3–strand pretzels via the computation of Casson–Gordon signatures corresponding to the double branched cover. In particular, we have the following complete characterization of topologically slice odd 3–strand pretzel knots. (Note that we call a pretzel knot $P(p_1, \dots, p_n)$ *odd* if all of its parameters $p_i$ are odd and *even* if one parameter is even.)

**Theorem 1.2** (Main Theorem A) *Let $K$ be an odd 3–strand pretzel knot with nontrivial Alexander polynomial. Then $K$ is topologically slice if and only if $K$ is of the form $\pm P(p, q, -q)$ or $\pm P(1, q, -q-4)$ for some odd $p, q \in \mathbb{N}$, in which case it is obviously ribbon.*

*Published: 19 September 2017* DOI: 10.2140/agt.2017.17.3057

By work of Freedman in [9], every knot with trivial Alexander polynomial is topologically slice. The following result, originally proved by Fintushel and Stern, illustrates that this is far from true for 3–strand pretzel knots in the smooth category.

**Theorem 1.3** (Fintushel and Stern [8]) *Let $K$ be a nontrivial odd 3–strand pretzel knot with $\Delta_K(t) = 1$. Then $K$ is not smoothly slice.*

Theorems 1.2 and 1.3 therefore together give an alternate proof of the following complete characterization of smoothly slice 3–strand pretzel knots given by Greene and Jabuka in [11]. Their arguments, like Lisca's, are smooth in nature and rely on Donaldson's theorem along with additional obstructions coming from Heegaard Floer homology.

**Theorem 1.4** (Greene and Jabuka [11]) *Let $K$ be an odd 3–strand pretzel knot. Then $K$ is smoothly slice if and only if it is ribbon if and only if $K$ is of the form $\pm P(p, q, -q)$ or $\pm P(1, q, -q-4)$ for odd $p, q \in \mathbb{N}$.*

Note that both Lisca and Greene and Jabuka actually prove stronger results that completely characterize the order of rational knots and odd 3–strand pretzel knots in the smooth concordance group. Theorem 1.2 has the following nice corollary.

**Corollary 1.5** *Let $K$ be a genus-one alternating knot. Then $K$ is topologically slice if and only if $K$ is ribbon.*

**Proof** Let $K$ be a genus-one alternating knot. Then by work of Stoimenow in [18], $K$ either is an odd 3–strand pretzel knot with all parameters of the same sign (and hence has nonzero signature and is not even algebraically slice) or is rational. Therefore we may assume that $K$ is a genus-one rational knot and hence (up to reflection) corresponds to the fraction $(4ab+1)/(2a)$ for some $a, b > 0$; see for example Burde and Zieschang [2, Proposition 12.26]. Note that $K$ has determinant $4ab + 1 > 1$ and hence does not have trivial Alexander polynomial. Therefore, since such knots can also be described as the 3–strand pretzel knot $P(1, 2a-1, -(2b+1))$, Theorem 1.2 implies that $K$ is topologically slice if and only if it is ribbon. □

We also consider the topological slice status of even 3–strand pretzel knots, and are able to use Casson–Gordon signatures to prove the following theorem, where for odd $a > 0$ we define $P_a$ to be the even 3–strand pretzel knot $P(a, -a-2, -(a+1)^2/2)$.

**Theorem 1.6** (Main Theorem B) *Let $K$ be an even 3–strand pretzel knot that is not of the form $\pm P_a$ for $a \equiv 1, 11, 37, 47, 59 \bmod 60$. Then $K$ is topologically slice if and only if $K$ is of the form $P(p, q, -q)$ for some even $p$ and odd $q$, in which case it is obviously ribbon.*

The family $\{\pm P_a\}$ was first considered by Lecuona in [15]. Lecuona uses techniques analogous to those of Greene and Jabuka to describe the smooth sliceness of even 3–strand pretzel knots, except for this exceptional family $\{\pm P_a\}$. In fact, Lecuona's results are much broader, essentially characterizing the smooth sliceness up to mutation of all even pretzel knots not in this exceptional family. It follows from work of Jabuka in [13] that the knots $\{\pm P_a\}$ are exactly the even 3–strand pretzel knots with trivial rational Witt class and determinant one.

**Theorem 1.7** (Lecuona [15]) *Let $K$ be an even 3–strand pretzel knot that is not of the form $\pm P_a$ for any $a \equiv 1, 11, 37, 47, 49, 59 \bmod 60$. Then $K$ is smoothly slice if and only if it is ribbon if and only if it is of the form $P(p, q, -q)$ for some even $p$ and odd $q$.*

Lecuona conjectures that the (non)existence of a Fox–Milnor factorization for the Alexander polynomial obstructs even the algebraic sliceness of the $\{\pm P_a\}$ family. When combined with Theorem 1.6, this would imply an affirmative answer to the following conjecture.

**Conjecture 1.8** *Let $K$ be an even 3–strand pretzel knot. Then $K$ is topologically slice if and only if $K$ is ribbon.*

We conveniently summarize Theorems 1.2 and 1.6 in this (slightly weaker) statement:

**Theorem 1.9** *Let $K$ be a 3–strand pretzel knot with nontrivial determinant. Then $K$ is topologically slice if and only if $K$ is ribbon.*

Note that despite our almost complete understanding of topological sliceness for 3–strand pretzel knots, it remains open whether smoothly slice equals topologically slice for rational knots. Recent work of Feller and McCoy [7] shows that there are rational knots with distinct smooth and topological 4–genera.

A natural next question is the extent to which double branched cover Casson–Gordon signatures obstruct the topological sliceness of pretzel knots with more than three strands. However, several difficulties arise. First, pretzel knots with more than three strands have nontrivial mutations which often persist in concordance. (See the work of Herald, Kirk and Livingston [12] for examples.) However, even if we are willing to consider knots only up to mutation we cannot expect a complete answer from these techniques. In particular, there exist algebraically slice odd 5–strand pretzel knots with nontrivial Alexander polynomial but trivial determinant. (For example, consider $P(7, 11, 53, -5, -19)$.) There is no reason to believe that these knots are topologically slice, but there are also no double branched cover Casson–Gordon signatures to serve as sliceness obstructions.

**Outline of the paper** In Section 2, we provide background and basic results on Casson–Gordon signatures. In Section 3, we provide necessary results concerning the colored signatures of links. In Section 4, we prove Main Theorem A (Theorem 1.2), completely characterizing which odd 3–strand pretzel knots are topologically slice. Finally, in Section 5 we briefly outline the arguments used to prove Main Theorem B (Theorem 1.6), our result for even 3–strand pretzel knots.

## 2 Casson–Gordon signature invariants

Casson and Gordon associate to a knot $K$ and a map $\chi\colon H_1(\Sigma_n(K)) \to \mathbb{Z}_d$ the invariant $\tau(K,n,\chi) \in L_0(\mathbb{Q}(\omega)(t)) \otimes \mathbb{Q}$. Note that $L_0(\mathbb{Q}(\omega)(t))$ is the Witt group of nonsingular Hermitian forms on finite-dimensional $\mathbb{Q}(\omega)(t)$–modules, where $\omega = e^{2\pi i/d}$. These invariants obstruct the topological sliceness of $K$ as follows.

**Theorem 2.1** (Casson and Gordon [4]) *Let $K$ be a topologically slice knot and $n$ a prime power. Then there exists a square-root order subgroup $M \leq H_1(\Sigma_n(K))$, invariant under the action of the covering transformations, with the linking form of $\Sigma_n(K)$ vanishing on $M \times M$ (ie $M$ is a* ***metabolizer*** *for the linking form) such that if $\chi$ is a prime-power order character with $\chi|_M = 0$, then $\tau(K,n,\chi) = 0$.*

While this is a powerful sliceness obstruction, $\tau(K,n,\chi)$ cannot generally be directly computed. Instead, as originated in [4], one relates the Witt class signature $\overline{\sigma}_1(\tau(K,n,\chi))$ to a simpler signature associated to any 3–manifold $Y$ and character from $H_1(Y)$ to a cyclic group. We give the definition of this signature, following [3].

First, whenever $X_\chi \to X$ is a cyclic $d$–fold cover, perhaps branched, we let $\omega = e^{2\pi i/d}$ and define the $\chi$–twisted homology of $X$ to be the $\mathbb{Q}(\omega)$ vector space

$$H_*^\chi(X) := H_*(C_*(X_\chi) \otimes_{\mathbb{Z}[\mathbb{Z}_d]} \mathbb{Q}(\omega)) \cong H_*(X_\chi) \otimes_{\mathbb{Z}[\mathbb{Z}_d]} \mathbb{Q}(\omega).$$

We now let $Y$ be a closed 3–manifold and $\chi\colon H_1(Y) \to \mathbb{Z}_d$ an onto homomorphism. The map $\chi$ induces a $d$–fold cyclic cover $Y_\chi \to Y$ with a canonical generator $\tau$ for the group of covering transformations. Suppose that there is some $d$–fold branched cyclic cover of 4–manifolds $W_\chi \to W$ with branch set a closed surface $F \subset \operatorname{int}(W)$ such that $\partial(W_\chi \to W) = r(Y_\chi \to Y)$ for some $r \in \mathbb{N}$. Suppose also that the covering transformation $\tilde{\tau}$ of $W_\chi$ that induces rotation by $2\pi/d$ on the fibers of the normal bundle of the preimage of $F$ in $W_\chi$ induces the canonical covering transformation $\tau$ on $Y_\chi$. We can always choose either $F = \varnothing$ or $r = 1$ by bordism group considerations and an explicit description in [3], respectively, and all of our work will be in one of these cases. The action of $\widetilde{\tau}$ on $H := H_2(W_\chi, \mathbb{C})$ allows us to decompose $H$ as the direct sum of eigenspaces $H_2^k(W_\chi)$ corresponding to eigenvalues $\omega^k$ for $k = 0, \dots, d-1$. For $k > 0$, define $\epsilon_k(W_\chi)$ to be the signature of the intersection form of $W_\chi$ when restricted to $H_2^k(W_\chi)$. Note that $\epsilon_1(W_\chi)$ can be equivalently be defined as the signature of the twisted intersection form on $H_2^\chi(W) = H_2(W_\chi) \otimes_{\mathbb{Z}[\mathbb{Z}_d]} \mathbb{Q}(\omega)$.

**Definition 2.2** With the above setup, the $k^{\text{th}}$ Casson–Gordon signature of $(Y, \chi)$ is

$$\sigma_k(Y,\chi) = \frac{1}{r}\Big(\sigma(W) - \epsilon_k(W_\chi) - \frac{2k(d-k)}{d^2}([F]\cdot[F])\Big).$$

Those familiar with the definition of $\tau(K, n, \chi)$ should note that we generally have $\sigma_1(\Sigma_n(K), \chi) \neq \overline{\sigma}_1(\tau(K, n, \chi))$. However, we can bound the difference between $\sigma_1(\Sigma_n(K), \chi)$ and $\overline{\sigma}_1(\tau(K, n, \chi))$, in a straightforward extension of [4, Theorem 3].

**Theorem 2.3** (Casson and Gordon [4]) *Let $\chi$: $H_1(\Sigma_n(K)) \to \mathbb{Z}_d$ be an onto homomorphism. Then*

$$|\sigma_1(\Sigma_n(K),\chi) - \overline{\sigma}_1(\tau(K,n,\chi))| \le \dim H_1^\chi(\Sigma_n(K)) + 1.$$

**Proof** We follow the proof of [4, Theorem 3]. Let $M_n$ denote the $n$–fold cyclic cover of the 3–manifold $S_0^3(K)$ obtained by doing 0–surgery along $K$. For convenience we let $\Sigma_n = \Sigma_n(K)$. Note that $\chi$ determines a map $H_1(M_n) \to \mathbb{Z}_d$, which by an abuse of notation we also refer to as $\chi$. By the usual bordism group considerations, for some $r \in \mathbb{N}$ there is a compact 4–manifold $W_n$ with boundary $r\Sigma_n$ such that $\chi$ extends over $H_1(W_n)$. Note that $M_n$ can be obtained from $\Sigma_n$ by a single 0–framed surgery along $\widetilde{K}$, the preimage of $K$ under the branched covering map. Therefore $rM_n$ bounds a 4–manifold $V_n$ obtained by attaching $r$ 0–framed 2–handles to $W_n$. Let $\nu$ denote the nullity of the twisted intersection form on $H_2^\chi(V_n)$. The arguments of the proof of [4, Theorem 3] carry over verbatim to establish the inequality

$$|\sigma_1(M_n,\chi) - \overline{\sigma}_1(\tau(K,n,\chi))| \le \frac{\nu}{r}.$$

Since our covers are unbranched, Definition 2.2 gives us

$$\sigma_1(\Sigma_n,\chi) = \frac{1}{r}\big(\sigma(W_n) - \sigma(H_2^\chi(W_n))\big),$$
$$\sigma_1(M_n,\chi) = \frac{1}{r}\big(\sigma(V_n) - \sigma(H_2^\chi(V_n))\big).$$

By our construction of $V_n$ from $W_n$, it is straightforward to verify that $\sigma(V_n) = \sigma(W_n)$ and that $H_2^\chi(V_n)$ has a codimension-$r$ subspace which is isometric to $H_2^\chi(W_n)$. Note that by duality the intersection form on $H_2^\chi(V_n)$ has nullity equal to $r \dim H_1^\chi(\Sigma_n)$, whereas by definition the intersection form on $H_2^\chi(W_n)$ has nullity $\nu$. We thus have the following, which when combined with our previous inequality gives the desired result:

$$\begin{aligned}|\sigma_1(\Sigma_n,\chi)-\sigma_1(M_n,\chi)| &= \left|\frac{1}{r}\big[\sigma(W_n)-\sigma(H_2^\chi(W_n))\big]-\frac{1}{r}\big[\sigma(V_n)-\sigma(H_2^\chi(V_n))\big]\right| \\ &= \frac{1}{r}\big|\sigma(H_2^\chi(W_n))-\sigma(H_2^\chi(V_n))\big| \\ &\leq \frac{1}{r}\big[r-(\nu-r\dim H_1^\chi(\Sigma_n))\big] \\ &= \dim H_1^\chi(\Sigma_n)+1-\frac{\nu}{r}. \qquad \square\end{aligned}$$

The following corollary will be our main obstruction to topological sliceness.

**Corollary 2.4** [4] *Suppose that $K$ is a topologically slice knot and that $n = p^r$ is a prime power. Then there exists a metabolizer $M$ for the linking form on $H_1(\Sigma_n(K))$ such that if $\chi$ is a character of prime-power order $d$ vanishing on $M$, then for any $k = 1, \dots, d-1$,*

$$|\sigma_k(\Sigma_n(K),\chi)| \leq \dim H_1^\chi(\Sigma_n(K)) + 1.$$

**Proof** Replacing $\chi$ with a nonzero multiple of itself permutes $\{\sigma_k(\Sigma_n(K),\chi)\}_{k=1}^{d-1}$ while preserving the property of vanishing on $M$, so Theorems 2.1 and 2.3 combine to give the desired result. □

If the obstruction of Corollary 2.4 vanishes for characters from $H_1(\Sigma_2(K))$ to $\mathbb{Z}_d$, then we will refer to $K$ as *CG-slice* at $d$. The following proposition is often convenient in recognizing that $\Sigma_n(K)_\chi$ is a rational homology sphere, and hence that the bound of Corollary 2.4 reduces to $|\sigma_1(\Sigma_n(K),\chi)| \leq 1$.

**Proposition 2.5** (Casson and Gordon [3]) *Suppose that $Y$ is a rational homology sphere with $H_1(Y,\mathbb{Z}_p)$ cyclic for some prime $p$. Then any cyclic $p^n$–fold cover of $Y$ is also a rational homology sphere.*

In order to effectively apply this obstruction, we would like to be able to compute $\sigma_k(Y,\chi)$ from an arbitrary integral surgery description of $Y$.

**Definition 2.6** Let $K$ be an oriented knot, and $A$ an embedded annulus such that $\partial A = K \sqcup -K'$ and $\operatorname{lk}(K,K') = \lambda$. An *$\lambda$–twisted $a$–cable* of $K$ is any oriented link $L$ obtained as the union of $n = n_+ + n_-$ parallel copies of $K$ in $A$ such that $n_+$ are oriented with $K$, $n_-$ opposite to $K$, and $n_+ - n_- = a$.

Let $L = \bigcup_{i=1}^n L_i$ be an oriented link in $S^3$ such that surgery along $L$ with integer framings $\{\lambda_i\}_{i=1}^n$ gives $Y$. We refer to the meridian of component $L_i$ as $\mu_i$ and let

$A = [a_{ij}]$ be the linking matrix of $L$. The following proposition is a generalization of [3, Lemma 3.1].

**Proposition 2.7** (Gilmer [10]) *Let $Y$ be obtained by integer surgery on $L$ as above and $\chi\colon H_1(Y) \to \mathbb{Z}_d$ be an onto homomorphism. Let $L_\chi$ be a satellite of $L$ obtained by replacing each $L_i$ by a nonempty $\lambda_i$–twisted $m_i$–cable of $L_i$, such that $\chi(\mu_i) \equiv m_i \bmod d$. Then for any $0 < k < d$,*

$$\sigma_k(Y,\chi) = \sigma(A) - \sigma_{L_\chi}(\omega^k) - \frac{2k(d-k)}{d^2}\left(\sum_{i,j=1}^{n} m_i m_j a_{ij}\right).$$

In order to effectively apply Proposition 2.7 we will need to compute the Tristram–Levine signatures of cables of links. The techniques of colored signatures prove useful for this, as well as providing an independent means of computation for $\sigma_1(Y,\chi)$.

# 3 Colored signatures of colored links

A *$n$–colored link* is an oriented link $L$ together with a surjective map assigning to each component of $L$ a color in $\{1, 2, \dots, n\}$. We let $L_i$ denote the sublink of $L$ consisting of $i$–colored components, and call each $L_i$ a *colored component*. A *C-complex* for a colored link $L$ consists of a union of Seifert surfaces for the colored components of $L$ which intersect only in a prescribed way (in "clasps"; see [5] for the precise definition).

The *colored signature* of $L$ is a map $\sigma_L\colon (S^1)^n \to \mathbb{Z}$ that is defined via the C-complex in a way exactly analogous to the definition of the Tristram–Levine signatures in terms of a Seifert surface for a link. The colored signature shares many properties, including a 4–dimensional interpretation, with the ordinary signatures. We need the following results, due primarily to Cimasoni and Florens [5]:

**Recovery of Tristram–Levine signatures** Let $L$ be a $n$–component, $n$–colored link, and call the underlying ordinary link $L'$. Then for any $\omega \in S^1 - \{1\}$, we have $\sigma_L(\omega, \dots, \omega) = \sigma_{L'}(\omega) + \sum_{i<j} \operatorname{lk}(L_i, L_j)$.

**Additivity** Let $L' = L'_1 \cup \dots \cup L'_m$ and $L'' = L''_{m+1} \cup \dots \cup L''_{m+n}$ be colored links and $L$ be the $(m+n-1)$–colored link obtained by connected summing any component of $L'_m$ with any component of $L''_{m+1}$. Then $\sigma_L(\omega_1, \dots, \omega_m, \dots, \omega_{m+n-1}) = \sigma_L(\omega_1, \dots, \omega_m) + \sigma_{L''}(\omega_m, \dots, \omega_{m+n-1})$.

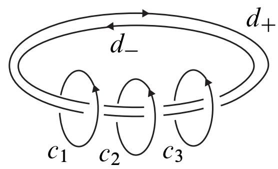
**Behavior under reversal and mirroring** The colored signature is invariant under global reversal of orientations. Also, letting $\overline{L}$ denote the mirror of $L$ we have $\sigma_{\overline{L}}(\omega_1, \dots, \omega_n) = -\sigma_L(\omega_1, \dots, \omega_n)$.

**Behavior at 1** (Degtyarev, Florens and Lecuona [6]) Let $L$ be an $n$–colored link and $L'$ be the $(n-1)$–colored link obtained by deleting the $n^{\text{th}}$ colored component of $L$. Then $\sigma_L(\omega_1, \dots, \omega_{n-1}, 1) = \sigma_{L'}(\omega_1, \dots, \omega_{n-1})$.

**Hopf link computation** Let $L$ be either Hopf link, considered as a 2–colored link. Then the colored signature function of $L$ is identically 0.

We also need the following consequence of Degtyarev, Florens and Lecuona's general description of the signature of a splice in [6].

**Example 3.1** Let $L$ be the following 5–colored link:




Let $\Phi(L)$ be the satellite of $L$ obtained by replacing each component $c_i$ with a coherently oriented torus link $T(a_i, p_i a_i)$ for $i = 1, 2, 3$. Observe that as an ordinary oriented link, $L$ is isotopic to its mirror image in a way that swaps components $d_+$ and $d_-$ and preserves all other components. It follows that $\sigma_L(\omega_0, \omega_0, \vec{\omega}) = 0$ for all $\omega_0 \in S^1$ and $\vec{\omega} \in (S^1)^3$. Let $\theta \in S^1$ be such that $\theta^{a_i} \neq 1$ for $i = 1, 2, 3$. Then [6, Theorem 2.2] and the above results imply that $\sigma_{\Phi(L)}(\theta) = \sum_{i=1}^{3} \sigma_{T(a_i, p_i a_i)}(\theta)$.

Finally, in some cases colored signatures give us an alternate computational method for Casson–Gordon signatures.

**Theorem 3.2** (Cimasoni and Florens [5]) *Let $Y$ be a 3–manifold obtained by surgery on a framed $n$–component link $L$ with linking matrix $A = [a_{ij}]$. Let $\chi\colon H_1(Y) \to \mathbb{Z}_d$ be a character of prime-power order that takes the meridian of each component of $L$ to a unit in $\mathbb{Z}_d$. Denote the lift of the image of the $i^{\text{th}}$ meridian of $L$ to $\{1, \dots, d-1\}$ by $m_i$. Consider $L$ as a $n$–colored link, and let $\omega_\chi = (\omega^{m_1}, \dots, \omega^{m_n})$. Then*

$$\sigma_1(Y, \chi) = \sigma(A) - \Big(\sigma_L(\omega_\chi) - \sum_{i<j} a_{ij}\Big) - \frac{2}{d^2}\Big(\sum_{i,j}(d - m_i) m_j a_{ij}\Big).$$

Note that in the case that every meridian is sent to 1 and $k = 1$, Theorems 2.7 and 3.2 both reduce to the original [3, Lemma 3.1].

# 4 Casson–Gordon signatures of 3–strand pretzels

We now give the outline of the proof of Theorem 1.2, deferring computations to later propositions.

**Proof of Theorem 1.2** Suppose that $K$ is an algebraically slice odd 3–strand pretzel knot with nontrivial Alexander polynomial. We will argue that either the Casson–Gordon signatures of $\Sigma_2(K)$ obstruct the topological sliceness of $K$ or the knot $K$ is in fact ribbon. Since $K$ is algebraically slice, the ordinary signature of $K$ vanishes, and so an easy computation from the standard genus-one Seifert surface for $K$ shows that $pq+qr+pr<0$; see also [13]. Also, $|H_1(\Sigma_2(K))| = -pq-qr-pr = D^2$ for some odd $D \in \mathbb{N}$. Note that since $K$ is a genus-one algebraically slice knot with nontrivial Alexander polynomial, $D^2 \neq 1$ and hence $D$ has prime divisors. Since $pq+pr+qr<0$, the parameters $p$, $q$ and $r$ are not all of the same sign and so via reflection and the symmetries of 3–strand pretzel knots we can assume that $p,q>0$ and $r<0$.

In the following cases, the existence of a prime $d$ that divides $D$ and satisfies the given conditions implies that the Casson–Gordon signatures of $\Sigma_2(K)$ corresponding to characters to $\mathbb{Z}_d$ obstruct the topological sliceness of $K$:

**Case 1** (Proposition 4.1) $d$ divides $p$ and $q$ but not $r$.

**Case 2** (Proposition 4.3) $d$ divides $r$ and exactly one of $p$ and $q$.

**Case 3** (Proposition 4.6) $d$ divides all of $p$, $q$ and $r$.

**Case 4** (Proposition 4.10) $d$ divides $D$ but none of $p$, $q$ and $r$; $p \not\equiv q \mod d$; and $r \neq -(4p+q)$ (assuming without loss of generality that $q>p$).

**Case 5** (Proposition 4.11) $d$ divides $D$ but none of $p$, $q$ and $r = -(4p+q)$.

**Case 6** (Proposition 4.12) $d$ divides $D$ but none of $p$, $q$ and $r$; $p \equiv q \mod d$; and $d \neq 3$.

Now suppose that there is no prime satisfying any of the above. It follows that $p$, $q$ and $r$ are relatively prime, $p \equiv q \mod 3$, and $D$ is a power of three. We show that in this case the Casson–Gordon signatures corresponding to characters of order 3 and 9 obstruct topological sliceness in Proposition 4.13. □

We now set up for our various computations. Note that if $r$ equals one of $-p$ and $-q$, there is a single band move taking $K$ to a 2–component unlink, and hence $K$ is ribbon. So we suppose $r \neq -p,-q$. We start with the surgery diagram for $\Sigma_2(K)$ in Figure 1, with linking matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & p & 0 & 0 \\ 1 & 0 & q & 0 \\ 1 & 0 & 0 & r \end{bmatrix}$$

and $\sigma(A)=0$. We refer to the meridians of each component by $\mu_0$, $\mu_p$, $\mu_q$ and $\mu_r$ according to their framings.

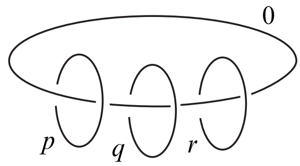



Figure 1: A surgery diagram $L_0$ for $\Sigma_2(P(p,q,r))$

Note that $A$ is a presentation matrix for $H_1(\Sigma_2(K)$, and that it is straightforward to use row and column moves and obtain the smaller presentation matrix $A' = \left[\begin{smallmatrix} p+q & p \\ p & p+r \end{smallmatrix}\right]$. Let $d$ be any prime dividing $D$ and suppose that $d$ does not divide all of $p$, $q$ and $r$. Observe that this implies that some entry of $A'$ is a unit in $\mathbb{Z}_d$, and hence by choosing this as our pivot entry and working over $\mathbb{Z}_d$ we can use row and column moves to obtain $A'' = \left[\begin{smallmatrix} 1 & 0 \\ 0 & * \end{smallmatrix}\right]$. Observe that $A''$ is a presentation matrix for $H_1(\Sigma_2(K), \mathbb{Z}_d)$, and so we see that $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is cyclic and hence every regular $d^n$–fold cyclic cover of $\Sigma_2(K)$ is a rational homology sphere (Proposition 2.5). In addition, when $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is cyclic any character $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ will vanish on any metabolizer for the linking form; see [12, Lemma 8.2]. So we have the following:

**Useful fact** Suppose that $K = P(p,q,r)$ is topologically slice, $d$ is a prime dividing $pq+qr+pr$ that does not divide all of $p$, $q$ and $r$, and $\chi$ is any character $H_1(\Sigma_2(K)) \to \mathbb{Z}_d$. Then $|\sigma_1(\Sigma_2(K), \chi)| \leq 1$.

### 4.1 Cases 1 and 2: $d$ divides some but not all of $p$, $q$ and $r$

**Proposition 4.1** (**Case 1**) *Let $K = K(p,q,r)$, where*

$$p, q > 0, \quad r < 0 \quad \text{and} \quad pq + pr + qr = -D^2.$$

*Suppose that $d$ is a prime that divides $p$ and $q$ but not $r$. Then the Casson–Gordon signatures of $\Sigma_2(K)$ associated to characters to $\mathbb{Z}_d$ obstruct the topological sliceness of $K$.*

**Proof** We start by manipulating our surgery description for $\Sigma_2(K)$. Slide the curves with framing $p$ and $q$ over the curve with framing $r$. Then convert the 0–framed 2–handle to a 1–handle, and cancel the 1–handle with the $r$–framed 2–handle. We end with a new surgery description for $\Sigma_2(K)$ with underlying link $L = T(2,2r)$ and framings $p+r$ and $q+r$. The linking matrix of $L$ is $A = \left[\begin{smallmatrix} p+r & r \\ r & q+r \end{smallmatrix}\right]$ and has $\sigma(A) = 0$. Note that if we consider the entries of $A$ mod $d$ we get a presentation matrix for $H_1(\Sigma_2(K), \mathbb{Z}_d)$ with respect to basis $\{\mu_p, \mu_q\}$, which immediately implies that $H_1(\Sigma_2(K), \mathbb{Z}_d) \cong \mathbb{Z}_d$, with generator $\mu_p = -\mu_q$.

By our useful fact, it suffices to show that for some $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ we have that $|\sigma_1(\Sigma_2(K),\chi)| > 1$. Define $\chi$ on $H_1(\Sigma_2(K))$ by $\chi(\mu_p) = \chi(-\mu_q) = 1$. So $L_\chi$ is the torus link $T(2,2r)$ with strands oppositely oriented. Note that $\sigma_{L_\chi}(\omega^k) = -1$ for $0 < k < d$ and so we have by Proposition 2.7 that

$$\sigma_k(\Sigma_2(K),\chi) = 1 - 2((p+r)-2r+(q+r))\frac{k(d-k)}{d^2} = 1 - 2\Big(\frac{p+q}{d}\Big)\Big(\frac{k(d-k)}{d}\Big).$$

Note that $d$ divides $p$ and $q$, so $p+q \geq 2d$. Note that $k(d-k) \geq (d-1)$ for all choices of $k = 1,\dots,d-1$. Since $d \geq 3$, we have

$$|\sigma_k(\Sigma_2(K),\chi)| \geq 2\cdot 2\cdot\big(1-\tfrac{1}{3}\big) - 1 = \tfrac{8}{3} - 1 > 1. \qquad \square$$

The above proof shows $\sigma_k(\Sigma_2(K),\chi) < -1$ for all choices of $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ and $k = 1,\dots,d-1$, giving the following easy corollary.

**Corollary 4.2** *For each odd prime $s$, let $K_s = P(p_s,q_s,r_s)$ be an odd 3–strand pretzel knot such that $p_s,q_s > 0$ are divisible by $s$; $r_s < 0$ is not divisible by $s$; and $p_sq_s + p_sr_s + q_sr_s = -s^2$. Then $\{K_s\}$ is a basis of algebraically slice knots for a $\mathbb{Z}^\infty$ subgroup of the topological concordance group.*

Note that such $K_s$ exist; for example, we can take $K_s = (s^2, s^2, -(s^2+1)/2)$. (Note since $s$ is odd $s^2+1$ is equivalent to $2 \bmod 4$ and so this is an odd pretzel as desired.)

**Proof** Suppose that $K = \sum_{i=1}^n a_i K_{s_i}$ is topologically slice, where each $a_i$ is nonzero. By reflecting $K$, we can assume without loss of generality that $a_1 > 0$. Since $K$ is topologically slice and $H_1(\Sigma_2(K),\mathbb{Z}_{s_i})$ is nonzero, it follows from Theorem 2.1 that there is some nontrivial character $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_{s_1}$ such that $\overline{\sigma}_1(\tau(K,2,\chi)) = 0$. Observe that

$$H_1(\Sigma_2(K)) = \bigoplus_{i=1}^n \big(H_1(\Sigma_2(K_{s_i}))^{\oplus|a_i|}\big) = \bigoplus_{i=1}^n \big(\mathbb{Z}_{s_i}[t]/\langle t+1\rangle\big)^{\oplus|a_i|}.$$

Note that $\chi$ is trivial on each $H_1(\Sigma_2(K_{s_i}))$ factor for $i \neq 1$, and that $\chi$ can be decomposed as $\chi = \bigoplus_{j=1}^{|a_1|} \chi_j$, where each $\chi_j\colon H_1(\Sigma_2(K_{s_1})) \to \mathbb{Z}_{s_1}$ and at least one $\chi_j$ is nontrivial. By the additivity of Casson–Gordon signatures,

$$\overline{\sigma}_1(\tau(K,2,\chi)) = \sum_{j=1}^{|a_1|} \overline{\sigma}_1(\tau(K_{s_1},2,\chi_j)).$$

However, the proof of Proposition 4.1 shows that $\sigma_1(\Sigma_2(K_{s_1}),\chi_j) < -1$ whenever $\chi_j$ is nontrivial, and that

$$\big|\overline{\sigma}_1(\tau(K_{s_1},2,\chi_j) - \sigma_1(\Sigma_2(K_{s_1}),\chi_j)\big| \leq 1.$$

It follows that $\overline{\sigma}_1(\tau(K,2,\chi_j))$ is strictly negative whenever $\chi_j$ is nontrivial (and zero when $\chi_j$ is trivial), and so $\overline{\sigma}_1(\tau(K,2,\chi)) < 0$, which is our desired contradiction. □

Now we continue to the next case.

**Proposition 4.3** (**Case 2**) *Let $K = K(p,q,r)$. Suppose that there exists a prime $d$ that divides $r$ and exactly one of $p$ and $q$, but that $r \neq -p, -q$. Then the Casson–Gordon signatures of $\Sigma_2(K)$ associated to characters to $\mathbb{Z}_d$ obstruct the topological sliceness of $K$.*

**Proof** The argument is exactly analogous to that of the proof of Proposition 4.1, except that we choose $k$ to be $(d-1)/2$; the details are left to the reader. □

## 4.2 Case 3: $d$ divides all of $p$, $q$ and $r$

In this case, we have that $H_1(\Sigma_2(K), \mathbb{Z}_d) \cong \mathbb{Z}_d \oplus \mathbb{Z}_d$, and so there may be metabolizers $M \leq H_1(\Sigma_2(K))$ with nontrivial image in $H_1(\Sigma_2(K), \mathbb{Z}_d)$. For each such metabolizer we provide a character $\chi$ to $\mathbb{Z}_d$ vanishing on $M$ such that the corresponding Casson–Gordon signature has sufficiently large absolute value. We first determine what "sufficiently large" is in the context of Corollary 2.4.

**Lemma 4.4** *Let $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$. Then $\dim H_1^{\chi}(\Sigma_2(K))$ is 1 if $\chi(\mu_p)$, $\chi(\mu_q)$ and $\chi(\mu_r)$ are all nonzero and 0 otherwise.*

**Proof** By slight simplifications of the Wirtinger presentation, we obtain

$$\pi_1(S^3 - L_0) = \langle \mu_0, \mu_p, \mu_q, \mu_r : \mu_0\mu_p = \mu_p\mu_0,\ \mu_0\mu_q = \mu_q\mu_0,\ \mu_0\mu_r = \mu_r\mu_0 \rangle,$$

where $\mu_*$ is any meridian of the $*$–framed curve, for $* = 0, p, q, r$. Note that the 0–framed longitudes of the surgery curves are given with respect to this generating set by $\lambda_0 = \mu_r\mu_q\mu_p$ and $\lambda_p = \lambda_q = \lambda_r = \mu_0$. Gluing in solid tori according to the surgery framings gives new relations

$$\lambda_0 = \mu_r\mu_q\mu_p = 1, \quad \mu_p^p\lambda_p = \mu_p^p\mu_0 = 1, \quad \mu_q^q\lambda_q = \mu_q^q\mu_0 = 1, \quad \mu_r^r\lambda_r = \mu_r^r\mu_0 = 1.$$

We therefore have the following presentation for $\pi_1(\Sigma_2(K))$, in which generators and relators correspond respectively to the 1– and 2–cells of a cell-complex structure (with a single 0–cell) on a space homotopy equivalent to $\Sigma_2(K)$:

$$\begin{aligned}\pi_1(\Sigma_2(K)) &= \left\langle \mu_0, \mu_p, \mu_q, \mu_r : \begin{array}{l}[\mu_0,\mu_p] = [\mu_0,\mu_q] = [\mu_0,\mu_r] = 1,\\ \mu_r\mu_q\mu_p = \mu_p^p\mu_0 = \mu_q^q\mu_0 = \mu_r^r\mu_0 = 1\end{array}\right\rangle \\ &= \left\langle \mu_p, \mu_q, \mu_r : \mu_r\mu_q\mu_p = \mu_p^p\mu_q^{-q} = \mu_p^p\mu_r^{-r} = 1 \right\rangle.\end{aligned}$$

Any choice of $x, y, z \in \mathbb{Z}_d$ such that $x + y + z \equiv 0 \bmod d$ will define a character $\chi$ via $\mu_p \mapsto x$, $\mu_q \mapsto y$ and $\mu_r \mapsto z$. First suppose that none of $x$, $y$ and $z$ are equivalent to $0$. Then by replacing $\chi$ with a nonzero multiple, which does not change the underlying cover, we may assume that $x = 1$.

We now follow the Reidemeister–Schreier algorithm to lift these 0–, 1–, and 2–cells to obtain a 2–complex with the same fundamental group as $\Sigma_2(K)_\chi$. Note that all subscripts are considered mod $d$. First, lift the single 0–cell to $d$ 0–cells $o_1, \dots, o_d$. Note that $\mu_p$ has $d$ lifts $\alpha_1, \dots, \alpha_d$, where $\alpha_i$ is a 1–cell from $o_i$ to $o_{i+1}$; $\mu_q$ has $d$ lifts $\beta_1, \dots, \beta_d$, where $\beta_i$ is a 1–cell from $o_i$ to $o_{i+y}$; and $\mu_r$ has $d$ lifts $\gamma_1, \dots, \gamma_d$, where $\gamma_i$ is a 1–cell from $o_i$ to $o_{i+z}$. We similarly compute the attaching maps of the $d$ lifts of each of the 2–cells. For example, the lifts of the 2–cell corresponding to the relator $\mu_r \mu_q \mu_p$ have attaching maps of the form $\gamma_i \beta_{z+i} \alpha_{y+z+i}$ for $i = 1, \dots, d$. Now contract along $\alpha_2, \dots, \alpha_d$ to obtain a complex with a single 0–cell, $(2d{+}1)$ 1–cells, and $(3d)$ 2–cells, with a corresponding presentation for $\pi_1(\Sigma_2(K)_\chi)$. Abelianizing gives a presentation for $H_1(\Sigma_2(K)_\chi)$ with generators $a, b_1, \dots, b_d, c_1, \dots, c_d$ and relations $a + b_1 + c_x = 0$; $b_k + c_{x+k-1} = 0$ for $k = 2, \dots, d$; and $(p/d)a = (q/d)(b_1 + \cdots + b_d) = (r/d)(c_1 + \cdots + c_d)$. This simplifies to

$$H_1(\Sigma_2(K)_\chi) = \left\langle a, b_1, \dots, b_d : \frac{p}{d} a = \frac{q}{d}(b_1 + \cdots + b_d) = -\frac{r}{d}(b_1 + \cdots + b_d + a) \right\rangle.$$

So as a $\mathbb{Q}$–module, $H_1(\Sigma_2(K)_\chi, \mathbb{Q})$ has generators $b_1, \dots, b_d$ and single relation $(pq + pr + qr)(b_1 + \cdots + b_d) = 0$. Note that the covering transformation of $\Sigma_2(K)_\chi$ sends $b_i$ onto $b_{i+1}$ for $i = 1, \dots, d-1$, and we have that $H_1(\Sigma_2(K)_\chi, \mathbb{Q})$ is a cyclic $\mathbb{Q}[\mathbb{Z}_d]$–module with generator $b_1$ and relator $(pq + pr + qr)(1 + t + t^2 + \cdots + t^{d-1}) b_1$. Since $1 + \xi_d + \xi_d^2 + \cdots + \xi_d^{d-1} = 0$, we have

$$H_1^\chi(\Sigma_2(K)) = H_1(\Sigma_2(K)_\chi, \mathbb{Q}) \otimes_{\mathbb{Q}[\mathbb{Z}_d]} \mathbb{Q}(\xi_d) \cong \mathbb{Q}(\xi_d).$$

When one of $x$, $y$ and $z$ is 0, an extremely similar argument shows that $\Sigma_2(K)_\chi$ is a rational homology sphere and so $\dim H_1^\chi(\Sigma_2(K)) = 0$. □

By considering the linking matrix $A$ for $L_0$ with its entries taken mod $d$, we see that $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is generated as a $\mathbb{Z}_d$–module by the images of $\mu_p, \mu_q$ and $\mu_r$ (which we continue to refer to as $\mu_p$, $\mu_q$ and $\mu_r$ by a mild abuse of notation) and has single relation $\mu_p + \mu_q + \mu_r = 0$. Suppose that $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ sends $\mu_p$ to $a$, $\mu_q$ to $b$ and $\mu_r$ to $c$, where $0 < a, b, c < d$. We must have $\chi(\mu_0) \equiv 0$ and $a + b + c \equiv 0 \bmod d$. We will use Proposition 2.7 to compute $\sigma_1(\Sigma_2(K), \chi)$, letting $L_\chi$ be the distant union of $T(a, pa)$, $T(b, qb)$ and $T(c, rc)$, each with all strands coherently oriented, along with two incoherently oriented linking 0 strands parallel to $\lambda_0$, as in Figure 2.

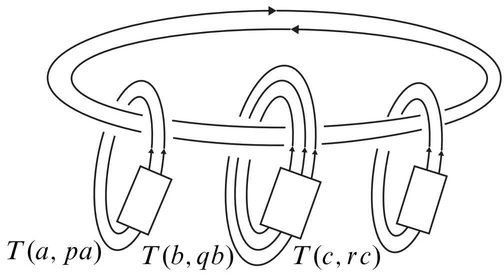



Figure 2: The link $L_\chi$, pictured with $a=2$, $b=3$ and $c=2$

Note, as computed in Example 3.1, $\sigma_{L_\chi}(\omega) = \sigma_{T(a,pa)}(\omega) + \sigma_{T(b,qb)}(\omega) + \sigma_{T(c,rc)}(\omega)$. Also, Litherland's formula in [17] for the Tristram–Levine signature of a torus link implies that $\sigma_{T(j,jkn)}(e^{2\pi i/n}) = -2j(j-1)k$ for $0 < j < n$. While Litherland's result is stated only for torus knots, it holds for torus links as well. In particular, the underlying computation in [1] of the signature of the Brieskorn manifold $V(p,q,r)_\delta = \{(z_1,z_2,z_3) \in \mathbb{C}^3 : z_1^p + z_2^q + z_3^r = \delta\} \cap \mathbb{D}^6$ does not depend on any relative primeness of the parameters $p$, $q$ and $r$.

Therefore, we have that

$$\begin{aligned}
&\sigma_1(\Sigma_2(K),\chi) \\
&\quad = 0 - \sigma_{L_\chi}(\omega) - 2(a^2p + b^2q + c^2r)\Big(\frac{d-1}{d^2}\Big) \\
&\quad = -\sigma_{T(a,pa)}(\omega) - \sigma_{T(b,qb)}(\omega) - \sigma_{T(c,rc)}(\omega) - 2(a^2p + b^2q + c^2r)\Big(\frac{d-1}{d^2}\Big) \\
&\quad = 2a(a-1)\frac{p}{d} + 2b(b-1)\frac{q}{d} + 2c(c-1)\frac{r}{d} - 2(a^2p + b^2q + c^2r)\Big(\frac{d-1}{d^2}\Big) \\
&\quad = \frac{2}{d^2}(a(d-a)p + b(d-b)q + c(d-c)r).
\end{aligned}$$

Unfortunately, we cannot conclude that $|\sigma_1(\Sigma_2(K),\chi)| > 1$ for all such choices of $\chi$. For example, when $K = P(3\cdot 7, 5\cdot 7, -17\cdot 7)$, $d=7$, and $\chi$ sends $\mu_p$ to 2, $\mu_q$ to 4 and $\mu_r$ to 1, we have $|\sigma_1(\Sigma_2(K),\chi)| = \frac{8}{11}$. However, this choice of $\chi$ does not vanish on any metabolizer for the linking form $\lambda\colon H_1(\Sigma_2(K)) \times H_1(\Sigma_2(K)) \to \mathbb{Q}/\mathbb{Z}$, and so there is still some hope to obstruct the sliceness of $K$ via double branched cover Casson–Gordon signatures.

**Lemma 4.5** *Suppose $M$ is a metabolizer for the linking form on $H_1(\Sigma_2(K))$ with nonzero image in $H_1(\Sigma_2(K),\mathbb{Z}_d)$. If $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ vanishes on $M$ and takes $\mu_p$, $\mu_q$ and $\mu_r$ to nonzero elements of $\mathbb{Z}_d$, then $\sigma_1(\Sigma_2(K),\chi)$ is an integer that is divisible by 4.*

**Proof** For convenience, we write $p = dp'$, $q = dq'$ and $r = dr'$. Note we have assumed that $M$ has nontrivial image in $H_1(\Sigma_2(K), \mathbb{Z}_d)$, and hence we can assume there is $\alpha = x\mu_p + y\mu_q \in M$ such that not both of $x$ and $y$ are equivalent to 0 mod $d$.

The linking form is given with respect to our $\mu_0$, $\mu_p$, $\mu_q$, $\mu_r$ generating set for $H_1(\Sigma_2(K))$ by $-A^{-1}$ (Gordon and Litherland). Direct computation shows that $\lambda(x\mu_p + y\mu_q, x\mu_p + y\mu_q) = (1/D^2)((q+r)x^2 - 2rxy + (p+r)y^2)$. Since $\alpha \in M$, we know $D^2$ and hence $d^2$ divides $(q+r)x^2 - 2rxy + (p+r)y^2$, and so we have

$$(*) \qquad (q' + r')x^2 - 2r'xy + (p' + r')y^2 \equiv 0 \bmod d.$$

Now, let $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ be a character vanishing on $M$. As usual, we write $a = \chi(\mu_p)$, $b = \chi(\mu_q)$ and $c = \chi(\mu_r)$, with $a + b + c \equiv 0 \bmod d$. Since $\chi(\alpha) = ax + by \equiv 0 \bmod d$, we can write $y = -a\overline{b}x$, and so neither $x$ nor $y$ is equivalent to 0 mod $d$. Substituting into $(*)$, we obtain

$$\begin{aligned} 0 &\equiv (q' + r')x^2 - 2r'xy + (p' + r')y^2 \\ &\equiv (q' + r')x^2 + 2r'a\overline{b}x^2 + (p' + r')a^2\overline{b}^2x^2 \\ &\equiv [a^2\overline{b}^2p' + q' + (a\overline{b} + 1)^2r']x^2 \quad \bmod d. \end{aligned}$$

Multiplying through by $(b^2/x^2)$ and recalling that $c^2 \equiv (a+b)^2 \bmod d$ gives us that $a^2p' + b^2q' + c^2r' \equiv 0 \bmod d$. Finally, we can write

$$\begin{aligned} \frac{d^2}{2}\sigma_1(\Sigma_2(K), \chi) &= a(d-a)p + b(d-b)q + c(d-c)r \\ &= d(a(d-a)p' + b(d-b)q' + c(d-c)r') \\ &= d^2(p' + q' + r') - d(a^2p' + b^2q' + c^2r'). \end{aligned}$$

Observe that the right side is divisible by $d^2$, and hence $\sigma_1(\Sigma_2(K))$ is an integer. Also, since $d$ is odd, $a(d-a)p + b(d-b)q + c(d-c)r$ is even for any choice of $a$, $b$ and $c$ and $\sigma_1(\Sigma_2(K), \chi)$ is divisible by 4. □

**Proposition 4.6** (**Case 3**) *Let $K = P(p,q,r)$, with $p, q \neq -r$ and suppose that $d$ is a prime dividing all of $p$, $q$ and $r$. Then the Casson–Gordon signatures of $\Sigma_2(K)$ associated to characters to $\mathbb{Z}_d$ obstruct the topological sliceness of $K$.*

**Proof** Suppose that $K$ is CG-slice at $d$, for a contradiction. So there exists a metabolizer $M \leq H_1(\Sigma_2(K))$ such that any character $\chi_0$ of prime-power order that vanishes on $M$ has $|\sigma_1(\Sigma_2(K), k\chi_0)| \leq \dim H_1^{\chi}(\Sigma_2(K)) + 1$ for all $0 < k < d$. If there exists $\chi$ to $\mathbb{Z}_d$ vanishing on $M$ that takes any of $\mu_p$, $\mu_q$ and $\mu_r$ to 0, then $\Sigma_2(K)_\chi$ is a rational homology sphere and arguments as in Cases 1 and 2 show that there is some $k$ such that $|\sigma_1(\Sigma_2(K), k\chi)| > 1$.

So we can now assume that no such $\chi$ exists. In particular, this implies that the image of $M$ in $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is nontrivial. So let $\chi_0\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ be a nontrivial character vanishing on $M$ and taking none of $\mu_p$, $\mu_q$ and $\mu_r$ to 0. Since $K$ is CG-slice, Corollary 2.4 and Lemma 4.4 combine to give us that $|\sigma_1(\Sigma_2(K), k\chi_0)| \leq 2$ for all $k$. Lemma 4.5 gives us that $\sigma_1(K, k\chi_0)$ is an integer divisible by 4 and so $\sigma_1(\Sigma_2(K), k\chi_0) = 0$.

Now, let $\chi$ be a multiple of $\chi_0$ such that $\chi(\mu_p) = 1$ and $\chi(\mu_q) = b$, and so $\chi(\mu_r) = d - b - 1$. We therefore have

$$0 = \frac{d^2}{2}\sigma_1(K, \chi) = (d-1)p + b(d-b)q + (b+1)(d-b-1)r. \tag{1}$$

We split into cases depending on the value of $b$.

**Case I $(0 < b < (d-1)/2)$** In this case, we have $(2\chi)(\mu_p) = 2$, $(2\chi)(\mu_q) = 2b$, and $(2\chi)(\mu_r) = d - 2b - 2$, so

$$0 = \frac{d^2}{2}(\sigma_1(K, 2\chi)) = 2(d-2)p + 2b(d-2b)q + (2b+2)(d-2b-2)r. \tag{2}$$

We then have that

$$\begin{aligned} \frac{1}{2}(2\,\mathrm{eq}(1) - \mathrm{eq}(2)) &= p + b^2q + (b+1)^2r = 0, \\ \frac{1}{2d}(4\,\mathrm{eq}(1) - \mathrm{eq}(2)) &= p + bq + (b+1)r = 0. \end{aligned}$$

It follows that $(b+1)r = -(b-1)q$ and finally that $p + q = 0$, which is our desired contradiction.

**Case II $(b = (d-1)/2)$** In this case, (1) simplifies to show that $q + r = -4p/(d+1)$. Also, $(2\chi)(\mu_p) = 2$ and $(2\chi)(\mu_q) = (2\chi)(\mu_r) = d - 1$, so

$$0 = 2(d-2)p + (d-1)q + (d-1)r. \tag{3}$$

Substituting our expression for $q + r$ into (3), we obtain that $(d^2 - 3d)p = 0$, and so $d = 3$. But this implies that $q + r = -p$, and hence that $p$ is even, which is our desired contradiction.

**Case III $(d/2 < b < d)$** In this case, we have $(2\chi)(\mu_p) = 2$, $(2\chi)(\mu_q) = 2b - d$ and $(2\chi)(\mu_r) = 2d - 2b - 2$. Therefore

$$\begin{aligned} 0 &= \frac{d^2}{2}(\sigma_1(K, 2\chi)) \\ &= 2(d-2)p + (2b-d)(2d-2b)q + (2b-d+2)(2d-2b-2)r. \end{aligned} \tag{4}$$

We then have that

$$\frac{1}{2}(2\,\mathrm{eq}(1) - \mathrm{eq}(4)) = p + (d-b)^2 q + (d-b-1)^2 r = 0,$$
$$\frac{1}{2d}(4\,\mathrm{eq}(1) - \mathrm{eq}(4)) = p + (d-b)q + (d-b-1)r = 0$$

It follows that $(d-b)q = -(d-b-2)r$, and finally that $p + r = 0$, which is our desired contradiction. □

### 4.3 Cases 4, 5 and 6: $d$ divides $pq + pr + qr$ but not any of $p$, $q$, $r$

The link $L_0$ considered as a 4–colored link has identically 0 colored signature, since it is a connected sum of 2–colored Hopf links. Note that since $d$ divides none of $p$, $q$ and $r$, every nontrivial character $\chi$ to $\mathbb{Z}_d$ has all of $\chi(\mu_p)$, $\chi(\mu_q)$, $\chi(\mu_r)$ and $\chi(\mu_0)$ nonzero, and so Theorem 3.2 applies and we have the following simple formula for $\sigma_1(\Sigma_2(K),\chi)$.

**Proposition 4.7** *Let $K = P(p,q,r)$ and suppose $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ has $\chi(\mu_p)$, $\chi(\mu_q)$, $\chi(\mu_r)$ and $\chi(\mu_0)$ all nonzero. Let $a$, $b$, $c$ and $\epsilon$ be the unique lifts of $\chi(\mu_p)$, $\chi(\mu_q)$, $\chi(\mu_r)$ and $\chi(\mu_0)$ to $\{1,\dots,d-1\}$. Then*

$$\sigma_1(\Sigma_2(K),\chi) = 3 - \frac{2}{d^2} f(\chi),$$

*where $f(\chi) := (d-\epsilon)(a+b+c) + (d-a)(ap+\epsilon) + (d-b)(bq+\epsilon) + (d-c)(cr+\epsilon)$.*

**Remark 4.8** Note that the parity of $a+b+c$ and of $\epsilon$ together determine the parity of $f(\chi)$; in particular, if $a+b+c$ is odd then $\epsilon$ and $f(\chi)$ have opposite parities. Also, when $a+b+c = d$ we have that

$$f(\chi) = d^2 + d\epsilon + a(d-a)p + b(d-b)q + (a+b)(d-(a+b))r.$$

**Lemma 4.9** *Let $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$, where $d$ divides none of $p$, $q$ and $r$. Then $f(\chi)$ is divisible by $d^2$.*

**Proof** First, recall that $H_1(\Sigma_2(K))$ is presented by linking matrix $A$, and so our $a$, $b$, $c$ and $\epsilon$ values must satisfy

$$a+b+c \equiv ap+\epsilon \equiv bq+\epsilon \equiv cr+\epsilon \equiv 0 \mod d.$$

We can rewrite $f(\chi)$ as

$$f(\chi) = d\big[(a+b+c) + (ap+\epsilon) + (bq+\epsilon) + (cr+\epsilon)\big] - \big[\epsilon(a+b+c) + a(ap+\epsilon) + b(bq+\epsilon) + c(cr+\epsilon)\big].$$

The first term can immediately be seen to be divisible by $d^2$, and so it suffices to show that $g(\chi)=\epsilon(a+b+c)+a(ap+\epsilon)+b(bq+\epsilon)+c(cr+\epsilon)$ is also divisible by $d^2$. Writing $ap+\epsilon=k_1d$, $bq+\epsilon=k_2d$ and $cr+\epsilon=k_3d$ for $k_1,k_2,k_3\in\mathbb{Z}$, we have

$$\begin{aligned}g(\chi)&=a(ap+\epsilon+\epsilon)+b(bq+\epsilon+\epsilon)+c(cr+\epsilon+\epsilon)\\&=\frac{k_1d-\epsilon}{p}(k_1d+\epsilon)+\frac{k_2d-\epsilon}{q}(k_2d+\epsilon)+\frac{k_3d-\epsilon}{r}(k_3d+\epsilon)\\&=\frac{k_1^2d^2-\epsilon^2}{p}+\frac{k_2^2d^2-\epsilon^2}{q}+\frac{k_3^2d^2-\epsilon^2}{r}.\end{aligned}$$

Note that since $d$ is relatively prime to all of $p$, $q$ and $r$, we can multiply through by $pqr$ without changing the divisibility of $g(\chi)$ by $d^2$. We therefore have the desired result, since

$$\begin{aligned}g(\chi)pqr&=(k_1^2d^2-\epsilon^2)qr+(k_2^2d^2-\epsilon^2)pr+(k_3^2d^2-\epsilon^2)pq\\&=d^2(k_1^2qr+k_2^2qr+k_3^2pr)-(pq+qr+pr)\epsilon^2.\end{aligned}$$ □

**Proposition 4.10** **(Case 4)** *Let $K=P(p,q,r)$ with $p$, $q$ and $r$ odd, $q\geq p>0$, and $r<0$, and let $d$ be some prime dividing $pq+pr+qr$ which divides none of $p$, $q$ and $r$. Suppose also that $r\neq-(4p+q)$ and that $p\not\equiv q \bmod d$. Then the Casson–Gordon signatures of $\Sigma_2(K)$ associated to characters to $\mathbb{Z}_d$ obstruct the topological sliceness of $K$.*

**Proof** For the sake of contradiction, assume $K$ is CG-slice at $d$. Since $H_1(\Sigma_2(K),\mathbb{Z}_d)$ is cyclic, for any $\chi\colon H_1(\Sigma_2(K))\to\mathbb{Z}_d$ we must have

$$|\sigma_1(\Sigma_2(K),\chi)|=\left|3-\frac{2}{d^2}f(\chi)\right|\leq 1.$$

Note that the first equality comes from Proposition 4.7 in the above equation. Therefore, by Lemma 4.9 we have $f(\chi)=d^2$ or $2d^2$.

We will work with two characters. Note that our formula for $f(\chi)$ uses the unique integer lifts of $\chi(\mu_i)$ to $\{1,\dots,d-1\}$, so we will be careful to only write $\chi(\mu_i)=x$ if $0<x<d$. We define $\chi_1$ to have $\chi_1(\mu_r)=1$, and $\chi_2=2\chi_1$. It follows that $\chi_1(\mu_0)$ is the unique integer $\epsilon$ in $(0,d)$ such that $\epsilon+r\equiv 0 \bmod d$, $\chi_1(\mu_p)$ is the unique integer $a$ in $(0,d)$ such that $\epsilon+ap\equiv 0 \bmod d$, and $\chi_1(\mu_q)=d-a-1$. Note that $\chi_i(\mu_p)+\chi_i(\mu_q)+\chi_i(\mu_r)=d$, so by Remark 4.8, $f(\chi_i)$ has the opposite parity as $\chi_i(\mu_0)$ for $i=1,2$. We now define some convenient notation:

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}_y=\begin{cases}x_1 & \text{if } 0<y<d/2,\\ x_2 & \text{if } d/2<y<d\end{cases}\quad\text{and}\quad\begin{bmatrix}x_1\\x_2\end{bmatrix}_{p(y)}=\begin{cases}x_1 & \text{if } y \text{ is even},\\ x_2 & \text{if } y \text{ is odd}.\end{cases}$$

We therefore have

$$\chi_2(\mu_p) = \begin{bmatrix} 2a \\ 2a-d \end{bmatrix}_a, \quad \chi_2(\mu_q) = \begin{bmatrix} d-2a-2 \\ 2d-2a-2 \end{bmatrix}_a, \quad \chi_2(\mu_0) = \begin{bmatrix} 2\epsilon \\ 2\epsilon-d \end{bmatrix}_\epsilon,$$
$$f(\chi_1) = \begin{bmatrix} d^2 \\ 2d^2 \end{bmatrix}_{p(\epsilon)} \quad \text{and} \quad f(\chi_2) = \begin{bmatrix} d^2 \\ 2d^2 \end{bmatrix}_\epsilon.$$

(Note that if $a = (d-1)/2$, then $\chi_1$ sends both $\mu_p$ and $\mu_q$ to $(d-1)/2$. But this implies that $p \equiv q \bmod d$, which we have assumed is not the case.)

We thus have the following two equations from our formulas for $f(\chi_1)$ and $f(\chi_2)$:

$$\begin{bmatrix} 0 \\ d^2 \end{bmatrix}_{p(\epsilon)} = d\epsilon + a(d-a)p + (a+1)(d-a-1)q + (d-1)r, \tag{5}$$

$$\begin{bmatrix} 0 \\ d^2 \end{bmatrix}_\epsilon = d\epsilon + \begin{bmatrix} a(d-2a)p + (a+1)(d-2a-2)q \\ (2a-d)(d-a)p + (2+2a-d)(d-a-1)q \end{bmatrix}_a + (d-2)r. \tag{6}$$

Consider $\mathrm{eq}(7) = \mathrm{eq}(5) - \mathrm{eq}(6)$ and $\mathrm{eq}(7) = (1/d)(2\,\mathrm{eq}(5) - \mathrm{eq}(6))$:

$$\begin{bmatrix} 0 \\ d^2 \end{bmatrix}_{p(\epsilon)} - \begin{bmatrix} 0 \\ d^2 \end{bmatrix}_\epsilon = \begin{bmatrix} a^2p + (a+1)^2q \\ (d-a)^2p + (d-a-1)^2q \end{bmatrix}_a + r, \tag{7}$$

$$\begin{bmatrix} 0 \\ 2d \end{bmatrix}_{p(\epsilon)} - \begin{bmatrix} 0 \\ d \end{bmatrix}_\epsilon = \epsilon + \begin{bmatrix} ap + (a+1)q \\ (d-a)p + (d-a-1)q \end{bmatrix}_a + r. \tag{8}$$

Note that the left side of (8) is even exactly when $\epsilon < d/2$, while the right side has the same parity as $\epsilon$. So we can assume $\epsilon < d/2$ if and only if $\epsilon$ is even, and (7) and (8) simplify to the following:

$$0 = \begin{bmatrix} a^2p + (a+1)^2q \\ (d-a)^2p + (d-a-1)^2q \end{bmatrix}_a + r, \tag{9}$$

$$\begin{bmatrix} 0 \\ d \end{bmatrix}_\epsilon = \epsilon + \begin{bmatrix} ap + (a+1)q \\ (d-a)p + (d-a-1)q \end{bmatrix}_a + r. \tag{10}$$

We can use (9) to see that if $a < d/2$ then $D = ap + (a+1)q$ and if $a > d/2$ then $D = (d-a)p + (d-a-1)q$. We will now split into cases, and show that each leads to a contradiction by using (9) to write $r$ in terms of $a$, $d$, $p$ and $q$ and substituting this expression into (10). Note that since $d$ divides $D$, we certainly have that $d \le D$.

**Case I** $(a, \epsilon < d/2)$ By combining (9) and (10) in this case, we see that we have $\epsilon = a^2(p+q) + a(q-p)$, and so

$$2a^2(p+q) < 2a^2(p+q) + 2a(q-p) = 2\epsilon < d \le D = ap + (a+1)q.$$

It follows that $(2a^2-a)p+(2a^2-a-1)q<0$, which gives the desired contradiction.

**Case II** $(a<d/2<\epsilon)$ In this case we have

$$0<d-\epsilon=-a(a-1)p-a(a+1)q<0.$$

**Case III** $(\epsilon<d/2<a)$ First, suppose $a=d-2$. Then (9) implies that $r=-(4p+q)$, which we have assumed is not the case. So we can assume that $a<d-2$, and so

$$D=(d-a)p+(d-a-1)q<(d-a)(d-a-1)p+(d-a-1)(d-a-2)q=\epsilon<d.$$

**Case IV** $(d/2<a,\epsilon)$ As in Case III, we can assume that $a<d-2$, and so

$$0<d-\epsilon=-(d-a)(d-a-1)p-(d-a-1)(d-a-2)q<0. \qquad \square$$

**Proposition 4.11** (**Case 5**) *Suppose $K=P(p,q,r)$ for $r=-(4p+q)$. Suppose $d$ is a prime that divides $pq+pr+qr$ but none of $p$, $q$ and $r$. Then either $K=P(1,q,-(q+4))$, in which case $K$ is ribbon, or the Casson–Gordon signatures of $\Sigma_2(K)$ corresponding to characters to $\mathbb{Z}_d$ obstruct the topological sliceness of $K$.*

Note that $K=P(1,q,-(q+4))$ is a 2–bridge knot. If we write $q=2k+1$, then $K$ is a generalized twist knot corresponding to the fraction $-(4(k+1)(k+2)+1)/(2(k+1))$ and has been known to be ribbon at least since [3].

**Proof** Let $\chi$ be the character sending $\mu_p$ to $d-2$, $\mu_q$ and $\mu_r$ to 1 and $\mu_0$ to $\epsilon$. Then $\chi'=\frac{1}{2}(d-1)\chi$ sends $\mu_p$ to 1, $\mu_q$ and $\mu_r$ to $\frac{1}{2}(d-1)$ and $\mu_0$ to $\epsilon'$. Arguments as in the proof of Proposition 4.10 show that if $p>1$ then at least one of $|\sigma_1(\Sigma_2(K),\chi)|$ and $|\sigma_1(\Sigma_2(K),\chi')|$ is strictly larger than 1, and hence that $K$ is not CG-slice at $d$. $\square$

**Proposition 4.12** (**Case 6**) *Suppose $d$ divides $pq+pr+qr$ but none of $p$, $q$ and $r$, $p\equiv q \bmod d$ and $d\neq 3$. Then the Casson–Gordon signatures of $\Sigma_2(K)$ associated to characters to $\mathbb{Z}_d$ obstruct the topological sliceness of $K$.*

**Proof** For $i=1,2$, consider the characters $\chi_i\colon H_1(\Sigma_2(K))\to\mathbb{Z}_d$ defined by $\chi_i(\mu_p)=\chi_i(\mu_q)=i$, $\chi_i(\mu_r)=d-2i$ and $\chi_i(\mu_0)=\epsilon_i$. (Note that since $d\neq 3$ we have that $d-2i>0$ for $i=1,2$.) Arguments as in the proof of Proposition 4.10 show that at least one of $|\sigma_1(\Sigma_2(K),\chi_i)|$ is strictly larger than 1, and hence that $K$ is not CG-slice at $d$. $\square$

**Proposition 4.13** *Suppose that $K=P(p,q,r)$ has $p$, $q$ and $r$ relatively prime, $|H_1(\Sigma_2(K)|=|pq+pr+qr|=3^{2n}$ for some $n\in\mathbb{N}$, and $p\equiv q \bmod 3$. Then either $K$ is ribbon or the Casson–Gordon signatures associated to characters of order 3 and 9 obstruct the topological sliceness of $K$.*

**Proof** First, suppose that $n \geq 2$. Since $p$, $q$ and $r$ are pairwise relatively prime, $H_1(\Sigma_2(K))$ is cyclic, and any character to $\mathbb{Z}_{3^n}$ will vanish on the unique metabolizer for the linking form. Proposition 2.5 implies that the associated covers are rational homology spheres, and so it suffices to find such a character $\chi$ with $|\sigma_1(\Sigma_2(K), \chi)| > 1$. The arguments of Propositions 4.10, 4.11 and 4.12 applied to $d = 9$ (according to whether $r = -(4p+q)$ and whether $p \equiv q \mod 9$) show that this is the case.

Now suppose that $n = 1$ and so $pq + pr + pq = -9$ and $r = -(pq+9)/(p+q)$. A slight variation on our usual arithmetic arguments then implies that $\sigma_1(\Sigma_2(K), \chi) < -1$ for some $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_3$, and hence that $K$ is not CG-slice at $d = 3$. □

# 5 Topological sliceness of even 3–strand pretzel knots.

We now outline the proof of our argument that all topologically slice even 3–strand pretzel knots are either ribbon or in Lecuona's family $\{\pm P_a\}$, leaving the details of arithmetic to the reader.

**Theorem 5.1** *Let $K$ be an even 3–strand pretzel knot. Suppose that $K$ is topologically slice. Then, up to reflection, either $K = P(p,-p,q)$ for some $p, q \in \mathbb{N}$ (and $K$ is ribbon) or $K = P_a = P(a, -a-2, -(a+1)^2/2)$ for some $a \equiv 1, 11, 37, 47, 59 \mod 60$.*

**Proof** Suppose that $K$ is an algebraically slice even 3–strand pretzel. First, note that by Jabuka's computation of the rational Witt classes of pretzel knots, we can assume that either $K = P(p,-p,q)$ for some odd $p$ and even $q$ or $K = P(-p, p \pm 2, q)$ for some odd $p$ and even $q$ such that $\det(K) = \pm 2q - p^2 \mp 2p = m^2 > 0$ [13, Theorem 1.11]. In the first case $K$ is ribbon, and so we assume that we are in the second case. By the symmetries of 3–strand pretzel knots, we can also assume that up to reflection $K = P(-p, p+2, q)$ for $p \in \mathbb{N}$. Then our condition that $\det(K) = 2q - p^2 - 2p > 0$ implies that $q > 0$ as well.

First, observe that if $\det(K) = 1$ then $q = (p+1)^2/2$ and up to reflection $K$ is an element of Lecuona's family $\{P_a\}$. For $a \not\equiv 1, 11, 37, 47, 49, 59 \mod 60$, Theorem 4.5 of [15] states that $K$ is not algebraically slice. When $a \equiv 49 \mod 60$, an argument analogous to the proof of [15, Theorem 4.5] shows that $\Delta_K(t)$ does not have a Fox–Milnor factorization and hence that $K$ is not algebraically slice. (In particular, note that since $a \equiv 49 \mod 60$ we have that 5 divides $(a+1)^2/4$ and 3 divides $a+2$. Working mod 5, we have $\Delta_{P_a}(t) \equiv \Pi_{1 \neq d | a} \Phi_d(t) \Pi_{1 \neq d | a+2} \Phi_d(t)$, where $\Phi_d(t)$ denotes the $d^{\text{th}}$ cyclotomic polynomial. Since $\Phi_3(t)$ is symmetric, irreducible mod 5, and relatively prime to each $\Phi_d(t)$ for $d \neq 3$ dividing $a$ or $a+2$, the desired result follows.)

So we can assume that $\det(K) = m^2 > 1$, and in particular that there is an (odd) prime $d$ dividing $\det(K)$. Arguments as in the proof of Proposition 4.1 show that $\Sigma_2(K)$ has a surgery presentation with underlying link the coherently oriented torus link $-T(2,2p)$ and linking matrix $\left[\begin{smallmatrix} 2 & -p \\ -p & q-p \end{smallmatrix}\right]$. It follows that $H_1(\Sigma_2(K))$ is cyclic, and hence that $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is certainly cyclic as well. It therefore suffices to show that there is a single $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_d$ with $|\sigma_k(\Sigma_2(K), \chi)| > 1$ for some $1 \le k < d$.

The construction of $\chi$ and computation of the corresponding Casson–Gordon signatures is extremely similar to the arguments of Section 4, and therefore we only list the cases one must consider and leave the verification of the details to the interested reader. It is convenient to consider six cases, according to the values mod $d$ of the parameters of $K$: $-p \equiv q \equiv 0$; $p+2 \equiv q \equiv 0$; $-p \equiv 2q \not\equiv 0$; $p+2 \equiv 2q \not\equiv 0$; $-p \equiv p+2 \not\equiv 0$; and $-p$, $p+2$ and $q$ are mutually distinct and nonzero. □

**Acknowledgements** I would like to thank Tye Lidman for suggesting Corollary 4.2, Lisa Piccirillo for her help with manipulating surgery descriptions, and my advisor Cameron Gordon for his advice, support and helpful comments. I would also like to thank the referee for thoughtful suggestions.

# References

[1] **E Brieskorn**, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. 2 (1966) 1–14 MR

[2] **G Burde**, **H Zieschang**, *Knots*, 2nd edition, de Gruyter Studies in Math. 5, de Gruyter, Berlin (2003) MR

[3] **A J Casson**, **C M Gordon**, *On slice knots in dimension three*, from "Algebraic and geometric topology, 2" (R J Milgram, editor), Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., Providence, RI (1978) 39–53 MR

[4] **A J Casson**, **C M Gordon**, *Cobordism of classical knots*, from "À la recherche de la topologie perdue" (L Guillou, A Marin, editors), Progr. Math. 62, Birkhäuser, Boston (1986) 181–199 MR

[5] **D Cimasoni**, **V Florens**, *Generalized Seifert surfaces and signatures of colored links*, Trans. Amer. Math. Soc. 360 (2008) 1223–1264 MR

[6] **A Degtyarev**, **V Florens**, **A G Lecuona**, *The signature of a splice*, Int. Math. Res. Not. 2017 (2017) 2249–2283 MR

[7] **P Feller**, **D McCoy**, *On 2–bridge knots with differing smooth and topological slice genera*, Proc. Amer. Math. Soc. 144 (2016) 5435–5442 MR

[8] **R Fintushel**, **R J Stern**, *Pseudofree orbifolds*, Ann. of Math. 122 (1985) 335–364 MR

[9] **M H Freedman**, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982) 357–453 MR

[10] **P M Gilmer**, *Configurations of surfaces in 4–manifolds*, Trans. Amer. Math. Soc. 264 (1981) 353–380 MR

[11] **J Greene**, **S Jabuka**, *The slice-ribbon conjecture for 3–stranded pretzel knots*, Amer. J. Math. 133 (2011) 555–580 MR

[12] **C Herald**, **P Kirk**, **C Livingston**, *Metabelian representations, twisted Alexander polynomials, knot slicing, and mutation*, Math. Z. 265 (2010) 925–949 MR

[13] **S Jabuka**, *Rational Witt classes of pretzel knots*, Osaka J. Math. 47 (2010) 977–1027 MR

[14] **R Kirby**, *Problems in low dimensional manifold theory*, from "Algebraic and geometric topology, 2" (R J Milgram, editor), Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., Providence, RI (1978) 273–312 MR

[15] **A G Lecuona**, *On the slice-ribbon conjecture for pretzel knots*, Algebr. Geom. Topol. 15 (2015) 2133–2173 MR

[16] **P Lisca**, *Lens spaces, rational balls and the ribbon conjecture*, Geom. Topol. 11 (2007) 429–472 MR

[17] **R A Litherland**, *Signatures of iterated torus knots*, from "Topology of low-dimensional manifolds" (R A Fenn, editor), Lecture Notes in Math. 722, Springer (1979) 71–84 MR

[18] **A Stoimenow**, *Knots of genus one or on the number of alternating knots of given genus*, Proc. Amer. Math. Soc. 129 (2001) 2141–2156 MR

*Department of Mathematics, University of Texas*
*Austin, TX, United States*

`amiller@math.utexas.edu`

`http://www.ma.utexas.edu/users/amiller/`

Received: 1 November 2016 Revised: 15 April 2017