

The topological sliceness of 3–strand pretzel knots

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We give a complete characterization of the topological slice status of odd 3–strand pretzel knots, proving that an odd 3–strand pretzel knot is topologically slice if and only if it either is ribbon or has trivial Alexander polynomial. We also show that topologically slice even 3–strand pretzel knots, except perhaps for members of Lecuona’s exceptional family, must be ribbon. These results follow from computations of the Casson–Gordon 3–manifold signature invariants associated to the double branched covers of these knots.

[57M25](#); [57N70](#)

1 Introduction

In the years since Fox first posed the slice-ribbon conjecture (Problem 1.33 on Kirby’s list [14]), its validity has been established for several families of knots. The usual strategy is to give an explicit list of ribbon knots in the family and then to provide an obstruction to the smooth sliceness of all others in the family. An early example of this is the following classification of the smoothly slice rational knots due to Lisca.

Theorem 1.1 (Lisca [16]) *A rational knot is smoothly slice if and only if it is ribbon if and only if it is in \mathcal{R} .*

\mathcal{R} is an explicit family of rational knots known to be ribbon at least since the work of Casson and Gordon [4]. Lisca argues that if K is not in \mathcal{R} , then Donaldson’s diagonalization theorem obstructs $\Sigma_2(K)$ from smoothly bounding a rational homology ball, and hence obstructs K from being smoothly slice.

In a similar spirit, though with entirely different methods, we give an almost complete characterization of the topological sliceness of 3–strand pretzels via the computation of Casson–Gordon signatures corresponding to the double branched cover. In particular, we have the following complete characterization of topologically slice odd 3–strand pretzel knots. (Note that we call a pretzel knot $P(p_1, \dots, p_n)$ *odd* if all of its parameters p_i are odd and *even* if one parameter is even.)

Theorem 1.2 (Main Theorem A) *Let K be an odd 3–strand pretzel knot with non-trivial Alexander polynomial. Then K is topologically slice if and only if K is of the form $\pm P(p, q, -q)$ or $\pm P(1, q, -q - 4)$ for some odd $p, q \in \mathbb{N}$, in which case it is obviously ribbon.*

By work of Freedman in [9], every knot with trivial Alexander polynomial is topologically slice. The following result, originally proved by Fintushel and Stern, illustrates that this is far from true for 3–strand pretzel knots in the smooth category.

Theorem 1.3 (Fintushel and Stern [8]) *Let K be a nontrivial odd 3–strand pretzel knot with $\Delta_K(t) = 1$. Then K is not smoothly slice.*

Theorems 1.2 and 1.3 therefore together give an alternate proof of the following complete characterization of smoothly slice 3–strand pretzel knots given by Greene and Jabuka in [11]. Their arguments, like Lisca’s, are smooth in nature and rely on Donaldson’s theorem along with additional obstructions coming from Heegaard Floer homology.

Theorem 1.4 (Greene and Jabuka [11]) *Let K be an odd 3–strand pretzel knot. Then K is smoothly slice if and only if it is ribbon if and only if K is of the form $\pm P(p, q, -q)$ or $\pm P(1, q, -q - 4)$ for odd $p, q \in \mathbb{N}$.*

Note that both Lisca and Greene and Jabuka actually prove stronger results that completely characterize the order of rational knots and odd 3–strand pretzel knots in the smooth concordance group. Theorem 1.2 has the following nice corollary.

Corollary 1.5 *Let K be a genus-one alternating knot. Then K is topologically slice if and only if K is ribbon.*

Proof Let K be a genus-one alternating knot. Then by work of Stoimenow in [18], K either is an odd 3–strand pretzel knot with all parameters of the same sign (and hence has nonzero signature and is not even algebraically slice) or is rational. Therefore we may assume that K is a genus-one rational knot and hence (up to reflection) corresponds to the fraction $(4ab + 1)/(2a)$ for some $a, b > 0$; see for example Burde and Zieschang [2, Proposition 12.26]. Note that K has determinant $4ab + 1 > 1$ and hence does not have trivial Alexander polynomial. Therefore, since such knots can also be described as the 3–strand pretzel knot $P(1, 2a - 1, -(2b + 1))$, Theorem 1.2 implies that K is topologically slice if and only if it is ribbon. \square

We also consider the topological slice status of even 3–strand pretzel knots, and are able to use Casson–Gordon signatures to prove the following theorem, where for odd $a > 0$ we define P_a to be the even 3–strand pretzel knot $P(a, -a - 2, -(a + 1)^2/2)$.

Theorem 1.6 (Main Theorem B) *Let K be an even 3–strand pretzel knot that is not of the form $\pm P_a$ for $a \equiv 1, 11, 37, 47, 59 \pmod{60}$. Then K is topologically slice if and only if K is of the form $P(p, q, -q)$ for some even p and odd q , in which case it is obviously ribbon.*

The family $\{\pm P_a\}$ was first considered by Lecuona in [15]. Lecuona uses techniques analogous to those of Greene and Jabuka to describe the smooth sliceness of even 3–strand pretzel knots, except for this exceptional family $\{\pm P_a\}$. In fact, Lecuona’s results are much broader, essentially characterizing the smooth sliceness up to mutation of all even pretzel knots not in this exceptional family. It follows from work of Jabuka in [13] that the knots $\{\pm P_a\}$ are exactly the even 3–strand pretzel knots with trivial rational Witt class and determinant one.

Theorem 1.7 (Lecuona [15]) *Let K be an even 3–strand pretzel knot that is not of the form $\pm P_a$ for any $a \equiv 1, 11, 37, 47, 49, 59 \pmod{60}$. Then K is smoothly slice if and only if it is ribbon if and only if it is of the form $P(p, q, -q)$ for some even p and odd q .*

Lecuona conjectures that the (non)existence of a Fox–Milnor factorization for the Alexander polynomial obstructs even the algebraic sliceness of the $\{\pm P_a\}$ family. When combined with Theorem 1.6, this would imply an affirmative answer to the following conjecture.

Conjecture 1.8 *Let K be an even 3–strand pretzel knot. Then K is topologically slice if and only if K is ribbon.*

We conveniently summarize Theorems 1.2 and 1.6 in this (slightly weaker) statement:

Theorem 1.9 *Let K be a 3–strand pretzel knot with nontrivial determinant. Then K is topologically slice if and only if K is ribbon.*

Note that despite our almost complete understanding of topological sliceness for 3–strand pretzel knots, it remains open whether smoothly slice equals topologically slice for rational knots. Recent work of Feller and McCoy [7] shows that there are rational knots with distinct smooth and topological 4–genera.

A natural next question is the extent to which double branched cover Casson–Gordon signatures obstruct the topological sliceness of pretzel knots with more than three strands. However, several difficulties arise. First, pretzel knots with more than three strands have nontrivial mutations which often persist in concordance. (See the work of Herald, Kirk and Livingston [12] for examples.) However, even if we are willing to consider knots only up to mutation we cannot expect a complete answer from these techniques. In particular, there exist algebraically slice odd 5–strand pretzel knots with nontrivial Alexander polynomial but trivial determinant. (For example, consider $P(7, 11, 53, -5, -19)$.) There is no reason to believe that these knots are topologically slice, but there are also no double branched cover Casson–Gordon signatures to serve as sliceness obstructions.

Outline of the paper In Section 2, we provide background and basic results on Casson–Gordon signatures. In Section 3, we provide necessary results concerning the colored signatures of links. In Section 4, we prove Main Theorem A (Theorem 1.2), completely characterizing which odd 3–strand pretzel knots are topologically slice. Finally, in Section 5 we briefly outline the arguments used to prove Main Theorem B (Theorem 1.6), our result for even 3–strand pretzel knots.

2 Casson–Gordon signature invariants

Casson and Gordon associate to a knot K and a map $\chi: H_1(\Sigma_n(K)) \rightarrow \mathbb{Z}_d$ the invariant $\tau(K, n, \chi) \in L_0(\mathbb{Q}(\omega)(t)) \otimes \mathbb{Q}$. Note that $L_0(\mathbb{Q}(\omega)(t))$ is the Witt group of nonsingular Hermitian forms on finite-dimensional $\mathbb{Q}(\omega)(t)$ –modules, where $\omega = e^{2\pi i/d}$. These invariants obstruct the topological sliceness of K as follows.

Theorem 2.1 (Casson and Gordon [4]) *Let K be a topologically slice knot and n a prime power. Then there exists a square-root order subgroup $M \leq H_1(\Sigma_n(K))$, invariant under the action of the covering transformations, with the linking form of $\Sigma_n(K)$ vanishing on $M \times M$ (ie M is a **metabolizer** for the linking form) such that if χ is a prime-power order character with $\chi|_M = 0$, then $\tau(K, n, \chi) = 0$.*

While this is a powerful sliceness obstruction, $\tau(K, n, \chi)$ cannot generally be directly computed. Instead, as originated in [4], one relates the Witt class signature $\bar{\sigma}_1(\tau(K, n, \chi))$ to a simpler signature associated to any 3–manifold Y and character from $H_1(Y)$ to a cyclic group. We give the definition of this signature, following [3].

First, whenever $X_\chi \rightarrow X$ is a cyclic d –fold cover, perhaps branched, we let $\omega = e^{2\pi i/d}$ and define the χ –twisted homology of X to be the $\mathbb{Q}(\omega)$ vector space

$$H_*^\chi(X) := H_*(C_*(X_\chi) \otimes_{\mathbb{Z}[\mathbb{Z}_d]} \mathbb{Q}(\omega)) \cong H_*(X_\chi) \otimes_{\mathbb{Z}[\mathbb{Z}_d]} \mathbb{Q}(\omega).$$

We now let Y be a closed 3–manifold and $\chi: H_1(Y) \rightarrow \mathbb{Z}_d$ an onto homomorphism. The map χ induces a d –fold cyclic cover $Y_\chi \rightarrow Y$ with a canonical generator τ for the group of covering transformations. Suppose that there is some d –fold branched cyclic cover of 4–manifolds $W_\chi \rightarrow W$ with branch set a closed surface $F \subset \text{int}(W)$ such that $\partial(W_\chi \rightarrow W) = r(Y_\chi \rightarrow Y)$ for some $r \in \mathbb{N}$. Suppose also that the covering transformation $\tilde{\tau}$ of W_χ that induces rotation by $2\pi/d$ on the fibers of the normal bundle of the preimage of F in W_χ induces the canonical covering transformation τ on Y_χ . We can always choose either $F = \emptyset$ or $r = 1$ by bordism group considerations and an explicit description in [3], respectively, and all of our work will be in one of these

cases. The action of $\tilde{\tau}$ on $H := H_2(W_\chi, \mathbb{C})$ allows us to decompose H as the direct sum of eigenspaces $H_2^k(W_\chi)$ corresponding to eigenvalues ω^k for $k = 0, \dots, d - 1$. For $k > 0$, define $\epsilon_k(W_\chi)$ to be the signature of the intersection form of W_χ when restricted to $H_2^k(W_\chi)$. Note that $\epsilon_1(W_\chi)$ can be equivalently be defined as the signature of the twisted intersection form on $H_2^X(W) = H_2(W_\chi) \otimes_{\mathbb{Z}[\mathbb{Z}_d]} \mathbb{Q}(\omega)$.

Definition 2.2 With the above setup, the k^{th} Casson–Gordon signature of (Y, χ) is

$$\sigma_k(Y, \chi) = \frac{1}{r} \left(\sigma(W) - \epsilon_k(W_\chi) - \frac{2k(d-k)}{d^2} ([F] \cdot [F]) \right).$$

Those familiar with the definition of $\tau(K, n, \chi)$ should note that we generally have $\sigma_1(\Sigma_n(K), \chi) \neq \bar{\sigma}_1(\tau(K, n, \chi))$. However, we can bound the difference between $\sigma_1(\Sigma_n(K), \chi)$ and $\bar{\sigma}_1(\tau(K, n, \chi))$, in a straightforward extension of [4, Theorem 3].

Theorem 2.3 (Casson and Gordon [4]) *Let $\chi: H_1(\Sigma_n(K)) \rightarrow \mathbb{Z}_d$ be an onto homomorphism. Then*

$$|\sigma_1(\Sigma_n(K), \chi) - \bar{\sigma}_1(\tau(K, n, \chi))| \leq \dim H_1^X(\Sigma_n(K)) + 1.$$

Proof We follow the proof of [4, Theorem 3]. Let M_n denote the n –fold cyclic cover of the 3–manifold $S_0^3(K)$ obtained by doing 0–surgery along K . For convenience we let $\Sigma_n = \Sigma_n(K)$. Note that χ determines a map $H_1(M_n) \rightarrow \mathbb{Z}_d$, which by an abuse of notation we also refer to as χ . By the usual bordism group considerations, for some $r \in \mathbb{N}$ there is a compact 4–manifold W_n with boundary $r\Sigma_n$ such that χ extends over $H_1(W_n)$. Note that M_n can be obtained from Σ_n by a single 0–framed surgery along \tilde{K} , the preimage of K under the branched covering map. Therefore rM_n bounds a 4–manifold V_n obtained by attaching r 0–framed 2–handles to W_n . Let ν denote the nullity of the twisted intersection form on $H_2^X(V_n)$. The arguments of the proof of [4, Theorem 3] carry over verbatim to establish the inequality

$$|\sigma_1(M_n, \chi) - \bar{\sigma}_1(\tau(K, n, \chi))| \leq \frac{\nu}{r}.$$

Since our covers are unbranched, Definition 2.2 gives us

$$\begin{aligned} \sigma_1(\Sigma_n, \chi) &= \frac{1}{r} (\sigma(W_n) - \sigma(H_2^X(W_n))), \\ \sigma_1(M_n, \chi) &= \frac{1}{r} (\sigma(V_n) - \sigma(H_2^X(V_n))). \end{aligned}$$

By our construction of V_n from W_n , it is straightforward to verify that $\sigma(V_n) = \sigma(W_n)$ and that $H_2^X(V_n)$ has a codimension- r subspace which is isometric to $H_2^X(W_n)$. Note that by duality the intersection form on $H_2^X(V_n)$ has nullity equal to $r \dim H_1^X(\Sigma_n)$,

whereas by definition the intersection form on $H_2^X(W_n)$ has nullity ν . We thus have the following, which when combined with our previous inequality gives the desired result:

$$\begin{aligned} |\sigma_1(\Sigma_n, \chi) - \sigma_1(M_n, \chi)| &= \left| \frac{1}{r} [\sigma(W_n) - \sigma(H_2^X(W_n))] - \frac{1}{r} [\sigma(V_n) - \sigma(H_2^X(V_n))] \right| \\ &= \frac{1}{r} |\sigma(H_2^X(W_n)) - \sigma(H_2^X(V_n))| \\ &\leq \frac{1}{r} [r - (\nu - r \dim H_1^X(\Sigma_n))] \\ &= \dim H_1^X(\Sigma_n) + 1 - \frac{\nu}{r}. \quad \square \end{aligned}$$

The following corollary will be our main obstruction to topological sliceness.

Corollary 2.4 [4] *Suppose that K is a topologically slice knot and that $n = p^r$ is a prime power. Then there exists a metabolizer M for the linking form on $H_1(\Sigma_n(K))$ such that if χ is a character of prime-power order d vanishing on M , then for any $k = 1, \dots, d - 1$,*

$$|\sigma_k(\Sigma_n(K), \chi)| \leq \dim H_1^X(\Sigma_n(K)) + 1.$$

Proof Replacing χ with a nonzero multiple of itself permutes $\{\sigma_k(\Sigma_n(K), \chi)\}_{k=1}^{d-1}$ while preserving the property of vanishing on M , so Theorems 2.1 and 2.3 combine to give the desired result. \square

If the obstruction of Corollary 2.4 vanishes for characters from $H_1(\Sigma_2(K))$ to \mathbb{Z}_d , then we will refer to K as *CG-slice* at d . The following proposition is often convenient in recognizing that $\Sigma_n(K)_\chi$ is a rational homology sphere, and hence that the bound of Corollary 2.4 reduces to $|\sigma_1(\Sigma_n(K), \chi)| \leq 1$.

Proposition 2.5 (Casson and Gordon [3]) *Suppose that Y is a rational homology sphere with $H_1(Y, \mathbb{Z}_p)$ cyclic for some prime p . Then any cyclic p^n -fold cover of Y is also a rational homology sphere.*

In order to effectively apply this obstruction, we would like to be able to compute $\sigma_k(Y, \chi)$ from an arbitrary integral surgery description of Y .

Definition 2.6 Let K be an oriented knot, and A an embedded annulus such that $\partial A = K \sqcup -K'$ and $\text{lk}(K, K') = \lambda$. An λ -twisted a -cable of K is any oriented link L obtained as the union of $n = n_+ + n_-$ parallel copies of K in A such that n_+ are oriented with K , n_- opposite to K , and $n_+ - n_- = a$.

Let $L = \bigcup_{i=1}^n L_i$ be an oriented link in S^3 such that surgery along L with integer framings $\{\lambda_i\}_{i=1}^n$ gives Y . We refer to the meridian of component L_i as μ_i and let

$A = [a_{ij}]$ be the linking matrix of L . The following proposition is a generalization of [3, Lemma 3.1].

Proposition 2.7 (Gilmer [10]) *Let Y be obtained by integer surgery on L as above and $\chi: H_1(Y) \rightarrow \mathbb{Z}_d$ be an onto homomorphism. Let L_χ be a satellite of L obtained by replacing each L_i by a nonempty λ_i -twisted m_i -cable of L_i , such that $\chi(\mu_i) \equiv m_i \pmod d$. Then for any $0 < k < d$,*

$$\sigma_k(Y, \chi) = \sigma(A) - \sigma_{L_\chi}(\omega^k) - \frac{2k(d-k)}{d^2} \left(\sum_{i,j=1}^n m_i m_j a_{ij} \right).$$

In order to effectively apply Proposition 2.7 we will need to compute the Tristram–Levine signatures of cables of links. The techniques of colored signatures prove useful for this, as well as providing an independent means of computation for $\sigma_1(Y, \chi)$.

3 Colored signatures of colored links

A n -colored link is an oriented link L together with a surjective map assigning to each component of L a color in $\{1, 2, \dots, n\}$. We let L_i denote the sublink of L consisting of i -colored components, and call each L_i a *colored component*. A C -complex for a colored link L consists of a union of Seifert surfaces for the colored components of L which intersect only in a prescribed way (in “clasps”; see [5] for the precise definition).

The *colored signature* of L is a map $\sigma_L: (S^1)^n \rightarrow \mathbb{Z}$ that is defined via the C -complex in a way exactly analogous to the definition of the Tristram–Levine signatures in terms of a Seifert surface for a link. The colored signature shares many properties, including a 4-dimensional interpretation, with the ordinary signatures. We need the following results, due primarily to Cimasoni and Florens [5]:

Recovery of Tristram–Levine signatures Let L be a n -component, n -colored link, and call the underlying ordinary link L' . Then for any $\omega \in S^1 - \{1\}$, we have $\sigma_L(\omega, \dots, \omega) = \sigma_{L'}(\omega) + \sum_{i < j} \text{lk}(L_i, L_j)$.

Additivity Let $L' = L'_1 \cup \dots \cup L'_m$ and $L'' = L''_{m+1} \cup \dots \cup L''_{m+n}$ be colored links and L be the $(m+n-1)$ -colored link obtained by connected summing any component of L'_m with any component of L''_{m+1} . Then $\sigma_L(\omega_1, \dots, \omega_m, \dots, \omega_{m+n-1}) = \sigma_{L'}(\omega_1, \dots, \omega_m) + \sigma_{L''}(\omega_{m+1}, \dots, \omega_{m+n-1})$.

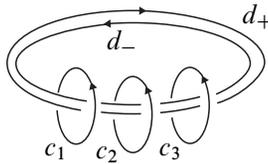
Behavior under reversal and mirroring The colored signature is invariant under global reversal of orientations. Also, letting \bar{L} denote the mirror of L we have $\sigma_{\bar{L}}(\omega_1, \dots, \omega_n) = -\sigma_L(\omega_1, \dots, \omega_n)$.

Behavior at 1 (Degtyarev, Florens and Lecuona [6]) Let L be an n -colored link and L' be the $(n-1)$ -colored link obtained by deleting the n^{th} colored component of L . Then $\sigma_L(\omega_1, \dots, \omega_{n-1}, 1) = \sigma_{L'}(\omega_1, \dots, \omega_{n-1})$.

Hopf link computation Let L be either Hopf link, considered as a 2-colored link. Then the colored signature function of L is identically 0.

We also need the following consequence of Degtyarev, Florens and Lecuona’s general description of the signature of a splice in [6].

Example 3.1 Let L be the following 5-colored link:



Let $\Phi(L)$ be the satellite of L obtained by replacing each component c_i with a coherently oriented torus link $T(a_i, p_i a_i)$ for $i = 1, 2, 3$. Observe that as an ordinary oriented link, L is isotopic to its mirror image in a way that swaps components d_+ and d_- and preserves all other components. It follows that $\sigma_L(\omega_0, \omega_0, \vec{\omega}) = 0$ for all $\omega_0 \in S^1$ and $\vec{\omega} \in (S^1)^3$. Let $\theta \in S^1$ be such that $\theta^{a_i} \neq 1$ for $i = 1, 2, 3$. Then [6, Theorem 2.2] and the above results imply that $\sigma_{\Phi(L)}(\theta) = \sum_{i=1}^3 \sigma_{T(a_i, p_i a_i)}(\theta)$.

Finally, in some cases colored signatures give us an alternate computational method for Casson–Gordon signatures.

Theorem 3.2 (Cimasoni and Florens [5]) Let Y be a 3-manifold obtained by surgery on a framed n -component link L with linking matrix $A = [a_{ij}]$. Let $\chi: H_1(Y) \rightarrow \mathbb{Z}_d$ be a character of prime-power order that takes the meridian of each component of L to a unit in \mathbb{Z}_d . Denote the lift of the image of the i^{th} meridian of L to $\{1, \dots, d-1\}$ by m_i . Consider L as a n -colored link, and let $\omega_\chi = (\omega^{m_1}, \dots, \omega^{m_n})$. Then

$$\sigma_1(Y, \chi) = \sigma(A) - \left(\sigma_L(\omega_\chi) - \sum_{i < j} a_{ij} \right) - \frac{2}{d^2} \left(\sum_{i, j} (d - m_i) m_j a_{ij} \right).$$

Note that in the case that every meridian is sent to 1 and $k = 1$, Theorems 2.7 and 3.2 both reduce to the original [3, Lemma 3.1].

4 Casson–Gordon signatures of 3-strand pretzels

We now give the outline of the proof of Theorem 1.2, deferring computations to later propositions.

Proof of Theorem 1.2 Suppose that K is an algebraically slice odd 3–strand pretzel knot with nontrivial Alexander polynomial. We will argue that either the Casson–Gordon signatures of $\Sigma_2(K)$ obstruct the topological sliceness of K or the knot K is in fact ribbon. Since K is algebraically slice, the ordinary signature of K vanishes, and so an easy computation from the standard genus-one Seifert surface for K shows that $pq + qr + pr < 0$; see also [13]. Also, $|H_1(\Sigma_2(K))| = -pq - qr - pr = D^2$ for some odd $D \in \mathbb{N}$. Note that since K is a genus-one algebraically slice knot with nontrivial Alexander polynomial, $D^2 \neq 1$ and hence D has prime divisors. Since $pq + pr + qr < 0$, the parameters p, q and r are not all of the same sign and so via reflection and the symmetries of 3–strand pretzel knots we can assume that $p, q > 0$ and $r < 0$.

In the following cases, the existence of a prime d that divides D and satisfies the given conditions implies that the Casson–Gordon signatures of $\Sigma_2(K)$ corresponding to characters to \mathbb{Z}_d obstruct the topological sliceness of K :

- Case 1** (Proposition 4.1) d divides p and q but not r .
- Case 2** (Proposition 4.3) d divides r and exactly one of p and q .
- Case 3** (Proposition 4.6) d divides all of p, q and r .
- Case 4** (Proposition 4.10) d divides D but none of p, q and r ; $p \not\equiv q \pmod d$; and $r \not\equiv -(4p + q)$ (assuming without loss of generality that $q > p$).
- Case 5** (Proposition 4.11) d divides D but none of p, q and $r = -(4p + q)$.
- Case 6** (Proposition 4.12) d divides D but none of p, q and r ; $p \equiv q \pmod d$; and $d \neq 3$.

Now suppose that there is no prime satisfying any of the above. It follows that p, q and r are relatively prime, $p \equiv q \pmod 3$, and D is a power of three. We show that in this case the Casson–Gordon signatures corresponding to characters of order 3 and 9 obstruct topological sliceness in Proposition 4.13. □

We now set up for our various computations. Note that if r equals one of $-p$ and $-q$, there is a single band move taking K to a 2–component unlink, and hence K is ribbon. So we suppose $r \neq -p, -q$. We start with the surgery diagram for $\Sigma_2(K)$ in Figure 1, with linking matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & p & 0 & 0 \\ 1 & 0 & q & 0 \\ 1 & 0 & 0 & r \end{bmatrix}$$

and $\sigma(A) = 0$. We refer to the meridians of each component by μ_0, μ_p, μ_q and μ_r according to their framings.

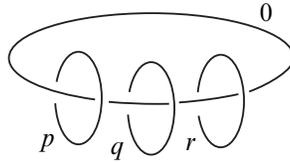


Figure 1: A surgery diagram L_0 for $\Sigma_2(P(p, q, r))$

Note that A is a presentation matrix for $H_1(\Sigma_2(K))$, and that it is straightforward to use row and column moves and obtain the smaller presentation matrix $A' = \begin{bmatrix} p+q & p \\ p & p+r \end{bmatrix}$. Let d be any prime dividing D and suppose that d does not divide all of p, q and r . Observe that this implies that some entry of A' is a unit in \mathbb{Z}_d , and hence by choosing this as our pivot entry and working over \mathbb{Z}_d we can use row and column moves to obtain $A'' = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$. Observe that A'' is a presentation matrix for $H_1(\Sigma_2(K), \mathbb{Z}_d)$, and so we see that $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is cyclic and hence every regular d^n -fold cyclic cover of $\Sigma_2(K)$ is a rational homology sphere (Proposition 2.5). In addition, when $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is cyclic any character $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$ will vanish on any metabolizer for the linking form; see [12, Lemma 8.2]. So we have the following:

Useful fact Suppose that $K = P(p, q, r)$ is topologically slice, d is a prime dividing $pq + qr + pr$ that does not divide all of p, q and r , and χ is any character $H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$. Then $|\sigma_1(\Sigma_2(K), \chi)| \leq 1$.

4.1 Cases 1 and 2: d divides some but not all of p, q and r

Proposition 4.1 (Case 1) Let $K = K(p, q, r)$, where

$$p, q > 0, \quad r < 0 \quad \text{and} \quad pq + pr + qr = -D^2.$$

Suppose that d is a prime that divides p and q but not r . Then the Casson–Gordon signatures of $\Sigma_2(K)$ associated to characters to \mathbb{Z}_d obstruct the topological sliceness of K .

Proof We start by manipulating our surgery description for $\Sigma_2(K)$. Slide the curves with framing p and q over the curve with framing r . Then convert the 0–framed 2–handle to a 1–handle, and cancel the 1–handle with the r –framed 2–handle. We end with a new surgery description for $\Sigma_2(K)$ with underlying link $L = T(2, 2r)$ and framings $p + r$ and $q + r$. The linking matrix of L is $A = \begin{bmatrix} p+r & r \\ r & q+r \end{bmatrix}$ and has $\sigma(A) = 0$. Note that if we consider the entries of $A \pmod d$ we get a presentation matrix for $H_1(\Sigma_2(K), \mathbb{Z}_d)$ with respect to basis $\{\mu_p, \mu_q\}$, which immediately implies that $H_1(\Sigma_2(K), \mathbb{Z}_d) \cong \mathbb{Z}_d$, with generator $\mu_p = -\mu_q$.

By our **useful fact**, it suffices to show that for some $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$ we have that $|\sigma_1(\Sigma_2(K), \chi)| > 1$. Define χ on $H_1(\Sigma_2(K))$ by $\chi(\mu_p) = \chi(-\mu_q) = 1$. So L_χ is the torus link $T(2, 2r)$ with strands oppositely oriented. Note that $\sigma_{L_\chi}(\omega^k) = -1$ for $0 < k < d$ and so we have by **Proposition 2.7** that

$$\sigma_k(\Sigma_2(K), \chi) = 1 - 2((p+r) - 2r + (q+r)) \frac{k(d-k)}{d^2} = 1 - 2\left(\frac{p+q}{d}\right) \left(\frac{k(d-k)}{d}\right).$$

Note that d divides p and q , so $p+q \geq 2d$. Note that $k(d-k) \geq (d-1)$ for all choices of $k = 1, \dots, d-1$. Since $d \geq 3$, we have

$$|\sigma_k(\Sigma_2(K), \chi)| \geq 2 \cdot 2 \cdot \left(1 - \frac{1}{3}\right) - 1 = \frac{8}{3} - 1 > 1. \quad \square$$

The above proof shows $\sigma_k(\Sigma_2(K), \chi) < -1$ for all choices of $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$ and $k = 1, \dots, d-1$, giving the following easy corollary.

Corollary 4.2 *For each odd prime s , let $K_s = P(p_s, q_s, r_s)$ be an odd 3–strand pretzel knot such that $p_s, q_s > 0$ are divisible by s ; $r_s < 0$ is not divisible by s ; and $p_s q_s + p_s r_s + q_s r_s = -s^2$. Then $\{K_s\}$ is a basis of algebraically slice knots for a \mathbb{Z}^∞ subgroup of the topological concordance group.*

Note that such K_s exist; for example, we can take $K_s = (s^2, s^2, -(s^2 + 1)/2)$. (Note since s is odd $s^2 + 1$ is equivalent to $2 \pmod 4$ and so this is an odd pretzel as desired.)

Proof Suppose that $K = \sum_{i=1}^n a_i K_{s_i}$ is topologically slice, where each a_i is nonzero. By reflecting K , we can assume without loss of generality that $a_1 > 0$. Since K is topologically slice and $H_1(\Sigma_2(K), \mathbb{Z}_{s_1})$ is nonzero, it follows from **Theorem 2.1** that there is some nontrivial character $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_{s_1}$ such that $\bar{\sigma}_1(\tau(K, 2, \chi)) = 0$. Observe that

$$H_1(\Sigma_2(K)) = \bigoplus_{i=1}^n (H_1(\Sigma_2(K_{s_i}))^{\oplus |a_i|}) = \bigoplus_{i=1}^n (\mathbb{Z}_{s_i}[t]/\langle t+1 \rangle)^{\oplus |a_i|}.$$

Note that χ is trivial on each $H_1(\Sigma_2(K_{s_i}))$ factor for $i \neq 1$, and that χ can be decomposed as $\chi = \bigoplus_{j=1}^{|a_1|} \chi_j$, where each $\chi_j: H_1(\Sigma_2(K_{s_1})) \rightarrow \mathbb{Z}_{s_1}$ and at least one χ_j is nontrivial. By the additivity of Casson–Gordon signatures,

$$\bar{\sigma}_1(\tau(K, 2, \chi)) = \sum_{j=1}^{|a_1|} \bar{\sigma}_1(\tau(K_{s_1}, 2, \chi_j)).$$

However, the proof of **Proposition 4.1** shows that $\sigma_1(\Sigma_2(K_{s_1}), \chi_j) < -1$ whenever χ_j is nontrivial, and that

$$|\bar{\sigma}_1(\tau(K_{s_1}, 2, \chi_j) - \sigma_1(\Sigma_2(K_{s_1}), \chi_j)| \leq 1.$$

It follows that $\bar{\sigma}_1(\tau(K, 2, \chi_j))$ is strictly negative whenever χ_j is nontrivial (and zero when χ_j is trivial), and so $\bar{\sigma}_1(\tau(K, 2, \chi)) < 0$, which is our desired contradiction. \square

Now we continue to the next case.

Proposition 4.3 (Case 2) *Let $K = K(p, q, r)$. Suppose that there exists a prime d that divides r and exactly one of p and q , but that $r \neq -p, -q$. Then the Casson–Gordon signatures of $\Sigma_2(K)$ associated to characters to \mathbb{Z}_d obstruct the topological sliceness of K .*

Proof The argument is exactly analogous to that of the proof of Proposition 4.1, except that we choose k to be $(d - 1)/2$; the details are left to the reader. \square

4.2 Case 3: d divides all of p, q and r

In this case, we have that $H_1(\Sigma_2(K), \mathbb{Z}_d) \cong \mathbb{Z}_d \oplus \mathbb{Z}_d$, and so there may be metabolizers $M \leq H_1(\Sigma_2(K))$ with nontrivial image in $H_1(\Sigma_2(K), \mathbb{Z}_d)$. For each such metabolizer we provide a character χ to \mathbb{Z}_d vanishing on M such that the corresponding Casson–Gordon signature has sufficiently large absolute value. We first determine what “sufficiently large” is in the context of Corollary 2.4.

Lemma 4.4 *Let $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$. Then $\dim H_1^\chi(\Sigma_2(K))$ is 1 if $\chi(\mu_p), \chi(\mu_q)$ and $\chi(\mu_r)$ are all nonzero and 0 otherwise.*

Proof By slight simplifications of the Wirtinger presentation, we obtain

$$\pi_1(S^3 - L_0) = \langle \mu_0, \mu_p, \mu_q, \mu_r : \mu_0\mu_p = \mu_p\mu_0, \mu_0\mu_q = \mu_q\mu_0, \mu_0\mu_r = \mu_r\mu_0 \rangle,$$

where μ_* is any meridian of the $*$ -framed curve, for $* = 0, p, q, r$. Note that the 0-framed longitudes of the surgery curves are given with respect to this generating set by $\lambda_0 = \mu_r\mu_q\mu_p$ and $\lambda_p = \lambda_q = \lambda_r = \mu_0$. Gluing in solid tori according to the surgery framings gives new relations

$$\lambda_0 = \mu_r\mu_q\mu_p = 1, \quad \mu_p^p\lambda_p = \mu_p^p\mu_0 = 1, \quad \mu_q^q\lambda_q = \mu_q^q\mu_0 = 1, \quad \mu_r^r\lambda_r = \mu_r^r\mu_0 = 1.$$

We therefore have the following presentation for $\pi_1(\Sigma_2(K))$, in which generators and relators correspond respectively to the 1- and 2-cells of a cell-complex structure (with a single 0-cell) on a space homotopy equivalent to $\Sigma_2(K)$:

$$\begin{aligned} \pi_1(\Sigma_2(K)) = \left\langle \mu_0, \mu_p, \mu_q, \mu_r : \begin{array}{l} [\mu_0, \mu_p] = [\mu_0, \mu_q] = [\mu_0, \mu_r] = 1, \\ \mu_r\mu_q\mu_p = \mu_p^p\mu_0 = \mu_q^q\mu_0 = \mu_r^r\mu_0 = 1 \end{array} \right\rangle \\ = \langle \mu_p, \mu_q, \mu_r : \mu_r\mu_q\mu_p = \mu_p^p\mu_q^{-q} = \mu_p^p\mu_r^{-r} = 1 \rangle. \end{aligned}$$

Any choice of $x, y, z \in \mathbb{Z}_d$ such that $x + y + z \equiv 0 \pmod d$ will define a character χ via $\mu_p \mapsto x, \mu_q \mapsto y$ and $\mu_r \mapsto z$. First suppose that none of x, y and z are equivalent to 0. Then by replacing χ with a nonzero multiple, which does not change the underlying cover, we may assume that $x = 1$.

We now follow the Reidemeister–Schreier algorithm to lift these 0–, 1–, and 2–cells to obtain a 2–complex with the same fundamental group as $\Sigma_2(K)_\chi$. Note that all subscripts are considered mod d . First, lift the single 0–cell to d 0–cells o_1, \dots, o_d . Note that μ_p has d lifts $\alpha_1, \dots, \alpha_d$, where α_i is a 1–cell from o_i to o_{i+1} ; μ_q has d lifts β_1, \dots, β_d , where β_i is a 1–cell from o_i to o_{i+y} ; and μ_r has d lifts $\gamma_1, \dots, \gamma_d$, where γ_i is a 1–cell from o_i to o_{i+z} . We similarly compute the attaching maps of the d lifts of each of the 2–cells. For example, the lifts of the 2–cell corresponding to the relator $\mu_r \mu_q \mu_p$ have attaching maps of the form $\gamma_i \beta_{z+i} \alpha_{y+z+i}$ for $i = 1, \dots, d$. Now contract along $\alpha_2, \dots, \alpha_d$ to obtain a complex with a single 0–cell, $(2d+1)$ 1–cells, and $(3d)$ 2–cells, with a corresponding presentation for $\pi_1(\Sigma_2(K)_\chi)$. Abelianizing gives a presentation for $H_1(\Sigma_2(K)_\chi)$ with generators $a, b_1, \dots, b_d, c_1, \dots, c_d$ and relations $a + b_1 + c_x = 0$; $b_k + c_{x+k-1} = 0$ for $k = 2, \dots, d$; and $(p/d)a = (q/d)(b_1 + \dots + b_d) = (r/d)(c_1 + \dots + c_d)$. This simplifies to

$$H_1(\Sigma_2(K)_\chi) = \left\langle a, b_1, \dots, b_d : \frac{p}{d}a = \frac{q}{d}(b_1 + \dots + b_d) = -\frac{r}{d}(b_1 + \dots + b_d + a) \right\rangle.$$

So as a \mathbb{Q} –module, $H_1(\Sigma_2(K)_\chi, \mathbb{Q})$ has generators b_1, \dots, b_d and single relation $(pq + pr + qr)(b_1 + \dots + b_d) = 0$. Note that the covering transformation of $\Sigma_2(K)_\chi$ sends b_i onto b_{i+1} for $i = 1, \dots, d-1$, and we have that $H_1(\Sigma_2(K)_\chi, \mathbb{Q})$ is a cyclic $\mathbb{Q}[\mathbb{Z}_d]$ –module with generator b_1 and relator $(pq + pr + qr)(1 + t + t^2 + \dots + t^{d-1})b_1$. Since $1 + \xi_d + \xi_d^2 + \dots + \xi_d^{d-1} = 0$, we have

$$H_1^\chi(\Sigma_2(K)) = H_1(\Sigma_2(K)_\chi, \mathbb{Q}) \otimes_{\mathbb{Q}[\mathbb{Z}_d]} \mathbb{Q}(\xi_d) \cong \mathbb{Q}(\xi_d).$$

When one of x, y and z is 0, an extremely similar argument shows that $\Sigma_2(K)_\chi$ is a rational homology sphere and so $\dim H_1^\chi(\Sigma_2(K)) = 0$. □

By considering the linking matrix A for L_0 with its entries taken mod d , we see that $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is generated as a \mathbb{Z}_d –module by the images of μ_p, μ_q and μ_r (which we continue to refer to as μ_p, μ_q and μ_r by a mild abuse of notation) and has single relation $\mu_p + \mu_q + \mu_r = 0$. Suppose that $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$ sends μ_p to a, μ_q to b and μ_r to c , where $0 < a, b, c < d$. We must have $\chi(\mu_0) \equiv 0$ and $a + b + c \equiv 0 \pmod d$. We will use [Proposition 2.7](#) to compute $\sigma_1(\Sigma_2(K), \chi)$, letting L_χ be the distant union of $T(a, pa), T(b, qb)$ and $T(c, rc)$, each with all strands coherently oriented, along with two incoherently oriented linking 0 strands parallel to λ_0 , as in [Figure 2](#).

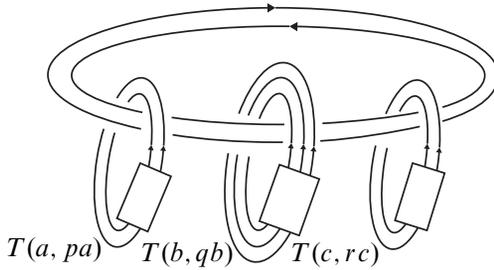


Figure 2: The link L_χ , pictured with $a = 2$, $b = 3$ and $c = 2$

Note, as computed in [Example 3.1](#), $\sigma_{L_\chi}(\omega) = \sigma_{T(a,pa)}(\omega) + \sigma_{T(b,qb)}(\omega) + \sigma_{T(c,rc)}(\omega)$. Also, Litherland’s formula in [\[17\]](#) for the Tristram–Levine signature of a torus link implies that $\sigma_{T(j,jkn)}(e^{2\pi i/n}) = -2j(j - 1)k$ for $0 < j < n$. While Litherland’s result is stated only for torus knots, it holds for torus links as well. In particular, the underlying computation in [\[1\]](#) of the signature of the Brieskorn manifold $V(p, q, r)_\delta = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^p + z_2^q + z_3^r = \delta\} \cap \mathbb{D}^6$ does not depend on any relative primeness of the parameters p , q and r .

Therefore, we have that

$$\begin{aligned} \sigma_1(\Sigma_2(K), \chi) &= 0 - \sigma_{L_\chi}(\omega) - 2(a^2 p + b^2 q + c^2 r) \left(\frac{d-1}{d^2} \right) \\ &= -\sigma_{T(a,pa)}(\omega) - \sigma_{T(b,qb)}(\omega) - \sigma_{T(c,rc)}(\omega) - 2(a^2 p + b^2 q + c^2 r) \left(\frac{d-1}{d^2} \right) \\ &= 2a(a - 1) \frac{p}{d} + 2b(b - 1) \frac{q}{d} + 2c(c - 1) \frac{r}{d} - 2(a^2 p + b^2 q + c^2 r) \left(\frac{d-1}{d^2} \right) \\ &= \frac{2}{d^2} (a(d - a)p + b(d - b)q + c(d - c)r). \end{aligned}$$

Unfortunately, we cannot conclude that $|\sigma_1(\Sigma_2(K), \chi)| > 1$ for all such choices of χ . For example, when $K = P(3 \cdot 7, 5 \cdot 7, -17 \cdot 7)$, $d = 7$, and χ sends μ_p to 2, μ_q to 4 and μ_r to 1, we have $|\sigma_1(\Sigma_2(K), \chi)| = \frac{8}{11}$. However, this choice of χ does not vanish on any metabolizer for the linking form $\lambda: H_1(\Sigma_2(K)) \times H_1(\Sigma_2(K)) \rightarrow \mathbb{Q}/\mathbb{Z}$, and so there is still some hope to obstruct the sliceness of K via double branched cover Casson–Gordon signatures.

Lemma 4.5 *Suppose M is a metabolizer for the linking form on $H_1(\Sigma_2(K))$ with nonzero image in $H_1(\Sigma_2(K), \mathbb{Z}_d)$. If $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$ vanishes on M and takes μ_p , μ_q and μ_r to nonzero elements of \mathbb{Z}_d , then $\sigma_1(\Sigma_2(K), \chi)$ is an integer that is divisible by 4.*

Proof For convenience, we write $p = dp'$, $q = dq'$ and $r = dr'$. Note we have assumed that M has nontrivial image in $H_1(\Sigma_2(K), \mathbb{Z}_d)$, and hence we can assume there is $\alpha = x\mu_p + y\mu_q \in M$ such that not both of x and y are equivalent to 0 mod d .

The linking form is given with respect to our $\mu_0, \mu_p, \mu_q, \mu_r$ generating set for $H_1(\Sigma_2(K))$ by $-A^{-1}$ (Gordon and Litherland). Direct computation shows that $\lambda(x\mu_p + y\mu_q, x\mu_p + y\mu_q) = (1/D^2)((q+r)x^2 - 2rxy + (p+r)y^2)$. Since $\alpha \in M$, we know D^2 and hence d^2 divides $(q+r)x^2 - 2rxy + (p+r)y^2$, and so we have

$$(*) \quad (q' + r')x^2 - 2r'xy + (p' + r')y^2 \equiv 0 \pmod{d}.$$

Now, let $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$ be a character vanishing on M . As usual, we write $a = \chi(\mu_p)$, $b = \chi(\mu_q)$ and $c = \chi(\mu_r)$, with $a + b + c \equiv 0 \pmod{d}$. Since $\chi(\alpha) = ax + by \equiv 0 \pmod{d}$, we can write $y = -a\bar{b}x$, and so neither x nor y is equivalent to 0 mod d . Substituting into $(*)$, we obtain

$$\begin{aligned} 0 &\equiv (q' + r')x^2 - 2r'xy + (p' + r')y^2 \\ &\equiv (q' + r')x^2 + 2r'a\bar{b}x^2 + (p' + r')a^2\bar{b}^2x^2 \\ &\equiv [a^2\bar{b}^2 p' + q' + (a\bar{b} + 1)^2 r']x^2 \pmod{d}. \end{aligned}$$

Multiplying through by (b^2/x^2) and recalling that $c^2 \equiv (a + b)^2 \pmod{d}$ gives us that $a^2 p' + b^2 q' + c^2 r' \equiv 0 \pmod{d}$. Finally, we can write

$$\begin{aligned} \frac{d^2}{2} \sigma_1(\Sigma_2(K), \chi) &= a(d - a)p + b(d - b)q + c(d - c)r \\ &= d(a(d - a)p' + b(d - b)q' + c(d - c)r') \\ &= d^2(p' + q' + r') - d(a^2 p' + b^2 q' + c^2 r'). \end{aligned}$$

Observe that the right side is divisible by d^2 , and hence $\sigma_1(\Sigma_2(K))$ is an integer. Also, since d is odd, $a(d - a)p + b(d - b)q + c(d - c)r$ is even for any choice of a , b and c and $\sigma_1(\Sigma_2(K), \chi)$ is divisible by 4. □

Proposition 4.6 (Case 3) *Let $K = P(p, q, r)$, with $p, q \neq -r$ and suppose that d is a prime dividing all of p, q and r . Then the Casson–Gordon signatures of $\Sigma_2(K)$ associated to characters to \mathbb{Z}_d obstruct the topological sliceness of K .*

Proof Suppose that K is CG-slice at d , for a contradiction. So there exists a metabolizer $M \leq H_1(\Sigma_2(K))$ such that any character χ_0 of prime-power order that vanishes on M has $|\sigma_1(\Sigma_2(K), k\chi_0)| \leq \dim H_1^X(\Sigma_2(K)) + 1$ for all $0 < k < d$. If there exists χ to \mathbb{Z}_d vanishing on M that takes any of μ_p, μ_q and μ_r to 0, then $\Sigma_2(K)_\chi$ is a rational homology sphere and arguments as in Cases 1 and 2 show that there is some k such that $|\sigma_1(\Sigma_2(K), k\chi)| > 1$.

So we can now assume that no such χ exists. In particular, this implies that the image of M in $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is nontrivial. So let $\chi_0: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$ be a nontrivial character vanishing on M and taking none of μ_p, μ_q and μ_r to 0. Since K is CG-slice, Corollary 2.4 and Lemma 4.4 combine to give us that $|\sigma_1(\Sigma_2(K), k\chi_0)| \leq 2$ for all k . Lemma 4.5 gives us that $\sigma_1(K, k\chi_0)$ is an integer divisible by 4 and so $\sigma_1(\Sigma_2(K), k\chi_0) = 0$.

Now, let χ be a multiple of χ_0 such that $\chi(\mu_p) = 1$ and $\chi(\mu_q) = b$, and so $\chi(\mu_r) = d - b - 1$. We therefore have

$$(1) \quad 0 = \frac{d^2}{2}\sigma_1(K, \chi) = (d - 1)p + b(d - b)q + (b + 1)(d - b - 1)r.$$

We split into cases depending on the value of b .

Case I ($0 < b < (d - 1)/2$) In this case, we have $(2\chi)(\mu_p) = 2$, $(2\chi)(\mu_q) = 2b$, and $(2\chi)(\mu_r) = d - 2b - 2$, so

$$(2) \quad 0 = \frac{d^2}{2}(\sigma_1(K, 2\chi)) = 2(d - 2)p + 2b(d - 2b)q + (2b + 2)(d - 2b - 2)r.$$

We then have that

$$\begin{aligned} \frac{1}{2}(2 \text{ eq}(1) - \text{eq}(2)) &= p + b^2q + (b + 1)^2r = 0, \\ \frac{1}{2d}(4 \text{ eq}(1) - \text{eq}(2)) &= p + bq + (b + 1)r = 0. \end{aligned}$$

It follows that $(b + 1)r = -(b - 1)q$ and finally that $p + q = 0$, which is our desired contradiction.

Case II ($b = (d - 1)/2$) In this case, (1) simplifies to show that $q + r = -4p/(d + 1)$. Also, $(2\chi)(\mu_p) = 2$ and $(2\chi)(\mu_q) = (2\chi)(\mu_r) = d - 1$, so

$$(3) \quad 0 = 2(d - 2)p + (d - 1)q + (d - 1)r.$$

Substituting our expression for $q + r$ into (3), we obtain that $(d^2 - 3d)p = 0$, and so $d = 3$. But this implies that $q + r = -p$, and hence that p is even, which is our desired contradiction.

Case III ($d/2 < b < d$) In this case, we have $(2\chi)(\mu_p) = 2$, $(2\chi)(\mu_q) = 2b - d$ and $(2\chi)(\mu_r) = 2d - 2b - 2$. Therefore

$$(4) \quad \begin{aligned} 0 &= \frac{d^2}{2}(\sigma_1(K, 2\chi)) \\ &= 2(d - 2)p + (2b - d)(2d - 2b)q + (2b - d + 2)(2d - 2b - 2)r. \end{aligned}$$

We then have that

$$\begin{aligned} \frac{1}{2}(2 \text{eq}(1) - \text{eq}(4)) &= p + (d - b)^2q + (d - b - 1)^2r = 0, \\ \frac{1}{2d}(4 \text{eq}(1) - \text{eq}(4)) &= p + (d - b)q + (d - b - 1)r = 0 \end{aligned}$$

It follows that $(d - b)q = -(d - b - 2)r$, and finally that $p + r = 0$, which is our desired contradiction. \square

4.3 Cases 4, 5 and 6: d divides $pq + pr + qr$ but not any of p, q, r

The link L_0 considered as a 4-colored link has identically 0 colored signature, since it is a connected sum of 2-colored Hopf links. Note that since d divides none of p, q and r , every nontrivial character χ to \mathbb{Z}_d has all of $\chi(\mu_p), \chi(\mu_q), \chi(\mu_r)$ and $\chi(\mu_0)$ nonzero, and so [Theorem 3.2](#) applies and we have the following simple formula for $\sigma_1(\Sigma_2(K), \chi)$.

Proposition 4.7 *Let $K = P(p, q, r)$ and suppose $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$ has $\chi(\mu_p), \chi(\mu_q), \chi(\mu_r)$ and $\chi(\mu_0)$ all nonzero. Let a, b, c and ϵ be the unique lifts of $\chi(\mu_p), \chi(\mu_q), \chi(\mu_r)$ and $\chi(\mu_0)$ to $\{1, \dots, d - 1\}$. Then*

$$\sigma_1(\Sigma_2(K), \chi) = 3 - \frac{2}{d^2} f(\chi),$$

where $f(\chi) := (d - \epsilon)(a + b + c) + (d - a)(ap + \epsilon) + (d - b)(bq + \epsilon) + (d - c)(cr + \epsilon)$.

Remark 4.8 Note that the parity of $a + b + c$ and of ϵ together determine the parity of $f(\chi)$; in particular, if $a + b + c$ is odd then ϵ and $f(\chi)$ have opposite parities. Also, when $a + b + c = d$ we have that

$$f(\chi) = d^2 + d\epsilon + a(d - a)p + b(d - b)q + (a + b)(d - (a + b))r.$$

Lemma 4.9 *Let $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$, where d divides none of p, q and r . Then $f(\chi)$ is divisible by d^2 .*

Proof First, recall that $H_1(\Sigma_2(K))$ is presented by linking matrix A , and so our a, b, c and ϵ values must satisfy

$$a + b + c \equiv ap + \epsilon \equiv bq + \epsilon \equiv cr + \epsilon \equiv 0 \pmod{d}.$$

We can rewrite $f(\chi)$ as

$$\begin{aligned} f(\chi) &= d[(a + b + c) + (ap + \epsilon) + (bq + \epsilon) + (cr + \epsilon)] \\ &\quad - [\epsilon(a + b + c) + a(ap + \epsilon) + b(bq + \epsilon) + c(cr + \epsilon)]. \end{aligned}$$

The first term can immediately be seen to be divisible by d^2 , and so it suffices to show that $g(\chi) = \epsilon(a + b + c) + a(ap + \epsilon) + b(bq + \epsilon) + c(cr + \epsilon)$ is also divisible by d^2 . Writing $ap + \epsilon = k_1d$, $bq + \epsilon = k_2d$ and $cr + \epsilon = k_3d$ for $k_1, k_2, k_3 \in \mathbb{Z}$, we have

$$\begin{aligned} g(\chi) &= a(ap + \epsilon + \epsilon) + b(bq + \epsilon + \epsilon) + c(cr + \epsilon + \epsilon) \\ &= \frac{k_1d - \epsilon}{p}(k_1d + \epsilon) + \frac{k_2d - \epsilon}{q}(k_2d + \epsilon) + \frac{k_3d - \epsilon}{r}(k_3d + \epsilon) \\ &= \frac{k_1^2d^2 - \epsilon^2}{p} + \frac{k_2^2d^2 - \epsilon^2}{q} + \frac{k_3^2d^2 - \epsilon^2}{r}. \end{aligned}$$

Note that since d is relatively prime to all of p, q and r , we can multiply through by pqr without changing the divisibility of $g(\chi)$ by d^2 . We therefore have the desired result, since

$$\begin{aligned} g(\chi)pqr &= (k_1^2d^2 - \epsilon^2)qr + (k_2^2d^2 - \epsilon^2)pr + (k_3^2d^2 - \epsilon^2)pq \\ &= d^2(k_1^2qr + k_2^2qr + k_3^2pr) - (pq + qr + pr)\epsilon^2. \quad \square \end{aligned}$$

Proposition 4.10 (Case 4) *Let $K = P(p, q, r)$ with p, q and r odd, $q \geq p > 0$, and $r < 0$, and let d be some prime dividing $pq + pr + qr$ which divides none of p, q and r . Suppose also that $r \neq -(4p + q)$ and that $p \not\equiv q \pmod d$. Then the Casson–Gordon signatures of $\Sigma_2(K)$ associated to characters to \mathbb{Z}_d obstruct the topological sliceness of K .*

Proof For the sake of contradiction, assume K is CG-slice at d . Since $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is cyclic, for any $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$ we must have

$$|\sigma_1(\Sigma_2(K), \chi)| = \left| 3 - \frac{2}{d^2} f(\chi) \right| \leq 1.$$

Note that the first equality comes from Proposition 4.7 in the above equation. Therefore, by Lemma 4.9 we have $f(\chi) = d^2$ or $2d^2$.

We will work with two characters. Note that our formula for $f(\chi)$ uses the unique integer lifts of $\chi(\mu_i)$ to $\{1, \dots, d - 1\}$, so we will be careful to only write $\chi(\mu_i) = x$ if $0 < x < d$. We define χ_1 to have $\chi_1(\mu_r) = 1$, and $\chi_2 = 2\chi_1$. It follows that $\chi_1(\mu_0)$ is the unique integer ϵ in $(0, d)$ such that $\epsilon + r \equiv 0 \pmod d$, $\chi_1(\mu_p)$ is the unique integer a in $(0, d)$ such that $\epsilon + ap \equiv 0 \pmod d$, and $\chi_1(\mu_q) = d - a - 1$. Note that $\chi_i(\mu_p) + \chi_i(\mu_q) + \chi_i(\mu_r) = d$, so by Remark 4.8, $f(\chi_i)$ has the opposite parity as $\chi_i(\mu_0)$ for $i = 1, 2$. We now define some convenient notation:

$$\left[\begin{matrix} x_1 \\ x_2 \end{matrix} \right]_y = \begin{cases} x_1 & \text{if } 0 < y < d/2, \\ x_2 & \text{if } d/2 < y < d \end{cases} \quad \text{and} \quad \left[\begin{matrix} x_1 \\ x_2 \end{matrix} \right]_{p(y)} = \begin{cases} x_1 & \text{if } y \text{ is even,} \\ x_2 & \text{if } y \text{ is odd.} \end{cases}$$

We therefore have

$$\chi_2(\mu_p) = \begin{bmatrix} 2a \\ 2a-d \end{bmatrix}_a, \quad \chi_2(\mu_q) = \begin{bmatrix} d-2a-2 \\ 2d-2a-2 \end{bmatrix}_a, \quad \chi_2(\mu_0) = \begin{bmatrix} 2\epsilon \\ 2\epsilon-d \end{bmatrix}_\epsilon,$$

$$f(\chi_1) = \begin{bmatrix} d^2 \\ 2d^2 \end{bmatrix}_{p(\epsilon)} \quad \text{and} \quad f(\chi_2) = \begin{bmatrix} d^2 \\ 2d^2 \end{bmatrix}_\epsilon.$$

(Note that if $a = (d - 1)/2$, then χ_1 sends both μ_p and μ_q to $(d - 1)/2$. But this implies that $p \equiv q \pmod d$, which we have assumed is not the case.)

We thus have the following two equations from our formulas for $f(\chi_1)$ and $f(\chi_2)$:

$$(5) \quad \begin{bmatrix} 0 \\ d^2 \end{bmatrix}_{p(\epsilon)} = d\epsilon + a(d-a)p + (a+1)(d-a-1)q + (d-1)r,$$

$$(6) \quad \begin{bmatrix} 0 \\ d^2 \end{bmatrix}_\epsilon = d\epsilon + \begin{bmatrix} a(d-2a)p + (a+1)(d-2a-2)q \\ (2a-d)(d-a)p + (2+2a-d)(d-a-1)q \end{bmatrix}_a + (d-2)r.$$

Consider eq(7) = eq(5) - eq(6) and eq(7) = (1/d)(2 eq(5) - eq(6)):

$$(7) \quad \begin{bmatrix} 0 \\ d^2 \end{bmatrix}_{p(\epsilon)} - \begin{bmatrix} 0 \\ d^2 \end{bmatrix}_\epsilon = \begin{bmatrix} a^2p + (a+1)^2q \\ (d-a)^2p + (d-a-1)^2q \end{bmatrix}_a + r,$$

$$(8) \quad \begin{bmatrix} 0 \\ 2d \end{bmatrix}_{p(\epsilon)} - \begin{bmatrix} 0 \\ d \end{bmatrix}_\epsilon = \epsilon + \begin{bmatrix} ap + (a+1)q \\ (d-a)p + (d-a-1)q \end{bmatrix}_a + r.$$

Note that the left side of (8) is even exactly when $\epsilon < d/2$, while the right side has the same parity as ϵ . So we can assume $\epsilon < d/2$ if and only if ϵ is even, and (7) and (8) simplify to the following:

$$(9) \quad 0 = \begin{bmatrix} a^2p + (a+1)^2q \\ (d-a)^2p + (d-a-1)^2q \end{bmatrix}_a + r,$$

$$(10) \quad \begin{bmatrix} 0 \\ d \end{bmatrix}_\epsilon = \epsilon + \begin{bmatrix} ap + (a+1)q \\ (d-a)p + (d-a-1)q \end{bmatrix}_a + r.$$

We can use (9) to see that if $a < d/2$ then $D = ap + (a + 1)q$ and if $a > d/2$ then $D = (d - a)p + (d - a - 1)q$. We will now split into cases, and show that each leads to a contradiction by using (9) to write r in terms of a, d, p and q and substituting this expression into (10). Note that since d divides D , we certainly have that $d \leq D$.

Case I ($a, \epsilon < d/2$) By combining (9) and (10) in this case, we see that we have $\epsilon = a^2(p + q) + a(q - p)$, and so

$$2a^2(p + q) < 2a^2(p + q) + 2a(q - p) = 2\epsilon < d \leq D = ap + (a + 1)q.$$

It follows that $(2a^2 - a)p + (2a^2 - a - 1)q < 0$, which gives the desired contradiction.

Case II ($a < d/2 < \epsilon$) In this case we have

$$0 < d - \epsilon = -a(a - 1)p - a(a + 1)q < 0.$$

Case III ($\epsilon < d/2 < a$) First, suppose $a = d - 2$. Then (9) implies that $r = -(4p + q)$, which we have assumed is not the case. So we can assume that $a < d - 2$, and so

$$D = (d - a)p + (d - a - 1)q < (d - a)(d - a - 1)p + (d - a - 1)(d - a - 2)q = \epsilon < d.$$

Case IV ($d/2 < a, \epsilon$) As in Case III, we can assume that $a < d - 2$, and so

$$0 < d - \epsilon = -(d - a)(d - a - 1)p - (d - a - 1)(d - a - 2)q < 0. \quad \square$$

Proposition 4.11 (Case 5) Suppose $K = P(p, q, r)$ for $r = -(4p + q)$. Suppose d is a prime that divides $pq + pr + qr$ but none of p, q and r . Then either $K = P(1, q, -(q + 4))$, in which case K is ribbon, or the Casson–Gordon signatures of $\Sigma_2(K)$ corresponding to characters to \mathbb{Z}_d obstruct the topological sliceness of K .

Note that $K = P(1, q, -(q + 4))$ is a 2–bridge knot. If we write $q = 2k + 1$, then K is a generalized twist knot corresponding to the fraction $-(4(k + 1)(k + 2) + 1)/(2(k + 1))$ and has been known to be ribbon at least since [3].

Proof Let χ be the character sending μ_p to $d - 2$, μ_q and μ_r to 1 and μ_0 to ϵ . Then $\chi' = \frac{1}{2}(d - 1)\chi$ sends μ_p to 1, μ_q and μ_r to $\frac{1}{2}(d - 1)$ and μ_0 to ϵ' . Arguments as in the proof of Proposition 4.10 show that if $p > 1$ then at least one of $|\sigma_1(\Sigma_2(K), \chi)|$ and $|\sigma_1(\Sigma_2(K), \chi')|$ is strictly larger than 1, and hence that K is not CG-slice at d . \square

Proposition 4.12 (Case 6) Suppose d divides $pq + pr + qr$ but none of p, q and r , $p \equiv q \pmod d$ and $d \neq 3$. Then the Casson–Gordon signatures of $\Sigma_2(K)$ associated to characters to \mathbb{Z}_d obstruct the topological sliceness of K .

Proof For $i = 1, 2$, consider the characters $\chi_i: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$ defined by $\chi_i(\mu_p) = \chi_i(\mu_q) = i$, $\chi_i(\mu_r) = d - 2i$ and $\chi_i(\mu_0) = \epsilon_i$. (Note that since $d \neq 3$ we have that $d - 2i > 0$ for $i = 1, 2$.) Arguments as in the proof of Proposition 4.10 show that at least one of $|\sigma_1(\Sigma_2(K), \chi_i)|$ is strictly larger than 1, and hence that K is not CG-slice at d . \square

Proposition 4.13 Suppose that $K = P(p, q, r)$ has p, q and r relatively prime, $|H_1(\Sigma_2(K))| = |pq + pr + qr| = 3^{2n}$ for some $n \in \mathbb{N}$, and $p \equiv q \pmod 3$. Then either K is ribbon or the Casson–Gordon signatures associated to characters of order 3 and 9 obstruct the topological sliceness of K .

Proof First, suppose that $n \geq 2$. Since p, q and r are pairwise relatively prime, $H_1(\Sigma_2(K))$ is cyclic, and any character to \mathbb{Z}_{3^n} will vanish on the unique metabolizer for the linking form. Proposition 2.5 implies that the associated covers are rational homology spheres, and so it suffices to find such a character χ with $|\sigma_1(\Sigma_2(K), \chi)| > 1$. The arguments of Propositions 4.10, 4.11 and 4.12 applied to $d = 9$ (according to whether $r = -(4p + q)$ and whether $p \equiv q \pmod{9}$) show that this is the case.

Now suppose that $n = 1$ and so $pq + pr + pq = -9$ and $r = -(pq + 9)/(p + q)$. A slight variation on our usual arithmetic arguments then implies that $\sigma_1(\Sigma_2(K), \chi) < -1$ for some $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_3$, and hence that K is not CG-slice at $d = 3$. \square

5 Topological sliceness of even 3–strand pretzel knots.

We now outline the proof of our argument that all topologically slice even 3–strand pretzel knots are either ribbon or in Lecuona’s family $\{\pm P_a\}$, leaving the details of arithmetic to the reader.

Theorem 5.1 *Let K be an even 3–strand pretzel knot. Suppose that K is topologically slice. Then, up to reflection, either $K = P(p, -p, q)$ for some $p, q \in \mathbb{N}$ (and K is ribbon) or $K = P_a = P(a, -a-2, -(a+1)^2/2)$ for some $a \equiv 1, 11, 37, 47, 59 \pmod{60}$.*

Proof Suppose that K is an algebraically slice even 3–strand pretzel. First, note that by Jabuka’s computation of the rational Witt classes of pretzel knots, we can assume that either $K = P(p, -p, q)$ for some odd p and even q or $K = P(-p, p \pm 2, q)$ for some odd p and even q such that $\det(K) = \pm 2q - p^2 \mp 2p = m^2 > 0$ [13, Theorem 1.11]. In the first case K is ribbon, and so we assume that we are in the second case. By the symmetries of 3–strand pretzel knots, we can also assume that up to reflection $K = P(-p, p + 2, q)$ for $p \in \mathbb{N}$. Then our condition that $\det(K) = 2q - p^2 - 2p > 0$ implies that $q > 0$ as well.

First, observe that if $\det(K) = 1$ then $q = (p + 1)^2/2$ and up to reflection K is an element of Lecuona’s family $\{P_a\}$. For $a \not\equiv 1, 11, 37, 47, 49, 59 \pmod{60}$, Theorem 4.5 of [15] states that K is not algebraically slice. When $a \equiv 49 \pmod{60}$, an argument analogous to the proof of [15, Theorem 4.5] shows that $\Delta_K(t)$ does not have a Fox–Milnor factorization and hence that K is not algebraically slice. (In particular, note that since $a \equiv 49 \pmod{60}$ we have that 5 divides $(a + 1)^2/4$ and 3 divides $a + 2$. Working mod 5, we have $\Delta_{P_a}(t) \equiv \prod_{1 \neq d|a} \Phi_d(t) \prod_{1 \neq d|a+2} \Phi_d(t)$, where $\Phi_d(t)$ denotes the d^{th} cyclotomic polynomial. Since $\Phi_3(t)$ is symmetric, irreducible mod 5, and relatively prime to each $\Phi_d(t)$ for $d \neq 3$ dividing a or $a + 2$, the desired result follows.)

So we can assume that $\det(K) = m^2 > 1$, and in particular that there is an (odd) prime d dividing $\det(K)$. Arguments as in the proof of [Proposition 4.1](#) show that $\Sigma_2(K)$ has a surgery presentation with underlying link the coherently oriented torus link $-T(2, 2p)$ and linking matrix $\begin{bmatrix} 2 & -p \\ -p & q-p \end{bmatrix}$. It follows that $H_1(\Sigma_2(K))$ is cyclic, and hence that $H_1(\Sigma_2(K), \mathbb{Z}_d)$ is certainly cyclic as well. It therefore suffices to show that there is a single $\chi: H_1(\Sigma_2(K)) \rightarrow \mathbb{Z}_d$ with $|\sigma_k(\Sigma_2(K), \chi)| > 1$ for some $1 \leq k < d$.

The construction of χ and computation of the corresponding Casson–Gordon signatures is extremely similar to the arguments of [Section 4](#), and therefore we only list the cases one must consider and leave the verification of the details to the interested reader. It is convenient to consider six cases, according to the values mod d of the parameters of K : $-p \equiv q \equiv 0$; $p + 2 \equiv q \equiv 0$; $-p \equiv 2q \not\equiv 0$; $p + 2 \equiv 2q \not\equiv 0$; $-p \equiv p + 2 \not\equiv 0$; and $-p$, $p + 2$ and q are mutually distinct and nonzero. \square

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References

- [1] **E Brieskorn**, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. 2 (1966) 1–14 [MR](#)
- [2] **G Burde, H Zieschang**, *Knots*, 2nd edition, de Gruyter Studies in Math. 5, de Gruyter, Berlin (2003) [MR](#)
- [3] **A J Casson, C M Gordon**, *On slice knots in dimension three*, from “Algebraic and geometric topology, 2” (R J Milgram, editor), Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., Providence, RI (1978) 39–53 [MR](#)
- [4] **A J Casson, C M Gordon**, *Cobordism of classical knots*, from “À la recherche de la topologie perdue” (L Guillou, A Marin, editors), Progr. Math. 62, Birkhäuser, Boston (1986) 181–199 [MR](#)
- [5] **D Cimasoni, V Florens**, *Generalized Seifert surfaces and signatures of colored links*, Trans. Amer. Math. Soc. 360 (2008) 1223–1264 [MR](#)
- [6] **A Degtyarev, V Florens, A G Lecuona**, *The signature of a splice*, Int. Math. Res. Not. 2017 (2017) 2249–2283 [MR](#)
- [7] **P Feller, D McCoy**, *On 2–bridge knots with differing smooth and topological slice genera*, Proc. Amer. Math. Soc. 144 (2016) 5435–5442 [MR](#)
- [8] **R Fintushel, R J Stern**, *Pseudofree orbifolds*, Ann. of Math. 122 (1985) 335–364 [MR](#)

- [9] **MH Freedman**, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982) 357–453 [MR](#)
- [10] **PM Gilmer**, *Configurations of surfaces in 4–manifolds*, Trans. Amer. Math. Soc. 264 (1981) 353–380 [MR](#)
- [11] **J Greene, S Jabuka**, *The slice-ribbon conjecture for 3–stranded pretzel knots*, Amer. J. Math. 133 (2011) 555–580 [MR](#)
- [12] **C Herald, P Kirk, C Livingston**, *Metabelian representations, twisted Alexander polynomials, knot slicing, and mutation*, Math. Z. 265 (2010) 925–949 [MR](#)
- [13] **S Jabuka**, *Rational Witt classes of pretzel knots*, Osaka J. Math. 47 (2010) 977–1027 [MR](#)
- [14] **R Kirby**, *Problems in low dimensional manifold theory*, from “Algebraic and geometric topology, 2” (R J Milgram, editor), Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., Providence, RI (1978) 273–312 [MR](#)
- [15] **A G Lecuona**, *On the slice-ribbon conjecture for pretzel knots*, Algebr. Geom. Topol. 15 (2015) 2133–2173 [MR](#)
- [16] **P Lisca**, *Lens spaces, rational balls and the ribbon conjecture*, Geom. Topol. 11 (2007) 429–472 [MR](#)
- [17] **RA Litherland**, *Signatures of iterated torus knots*, from “Topology of low-dimensional manifolds” (R A Fenn, editor), Lecture Notes in Math. 722, Springer (1979) 71–84 [MR](#)
- [18] **A Stoimenow**, *Knots of genus one or on the number of alternating knots of given genus*, Proc. Amer. Math. Soc. 129 (2001) 2141–2156 [MR](#)

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