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An algebraic model for rational SO(3)–spectra

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Greenlees established an equivalence of categories between the homotopy category of rational SO(3)–spectra and the derived category $d\mathcal{A}(\mathrm{SO}(3))$ of a certain abelian category. In this paper we lift this equivalence of homotopy categories to the level of Quillen equivalences of model categories. Methods used in this paper provide the first step towards obtaining an algebraic model for the toral part of rational G–spectra, for any compact Lie group G.

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1 Introduction

Modelling the category of rational G–spectra This paper is a contribution to the study of G–equivariant cohomology theories and gives a complete analysis for one class of theories, namely rational SO(3)–equivariant cohomology theories. To start with, let G be a compact Lie group. Recall that G–equivariant cohomology theories are represented by G–spectra, so the category of G–equivariant cohomology theories is equivalent to the homotopy category of G–spectra. The category of G–spectra is quite complicated, with rich structures coming from two sources: topology and the group actions, and one cannot expect a complete analysis of either cohomology theories or spectra integrally.

For a compact Lie group G, the category of rational G–spectra is the category of G–spectra, but with the model structure that is a left Bousfield localisation of the stable model structure at the rational sphere spectrum; see for example Barnes [1, Section 2.2]. Thus the weak equivalences are maps which become isomorphisms after applying the rational homotopy group functors, ie $\pi_*^H(-)\otimes\mathbb{Q}$ for all closed subgroups H in G.

Rationalising the category of G–spectra reduces topological complexity, simplifying it greatly. At the same time interesting equivariant behaviour remains. In order to understand this behaviour, we try to find a purely algebraic description of the category, that is an algebraic model category which is Quillen equivalent to the category of rational G–spectra. As a result, the homotopy category of the algebraic model is equivalent to the rational stable G–homotopy category via triangulated equivalences. Moreover all the homotopy information, such as homotopy limits, in both is the same.

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The conjecture by Greenlees states that for any compact Lie group G there is a nice graded abelian category $\mathcal{A}(G)$ such that the category $d\mathcal{A}(G)$ of differential objects in $\mathcal{A}(G)$ with a certain model structure is Quillen equivalent to the category of rational G–spectra:

$$G\text{–}\mathrm{Sp}_{\mathbb{Q}} \simeq_Q d\mathcal{A}(G).$$

If we find such $d\mathcal{A}(G)$ we say that $d\mathcal{A}(G)$ is an *algebraic model* for rational G–spectra.

Existing work There are several examples of specific Lie groups G for which an algebraic model has been given. Firstly, when G is trivial, it was shown in Shipley [22, Theorem 1.1] that rational spectra are monoidally Quillen equivalent to chain complexes of $\mathbb{Q}$–modules. An algebraic model for rational G–spectra for finite G is described in Schwede and Shipley [21, Example 5.1.2] and simplified in Barnes [2] and Kędziorek [16]. An algebraic model for rational torus-equivariant spectra was presented in Greenlees and Shipley [11], whereas a slightly different approach in Barnes, Greenlees, Kędziorek and Shipley [6] gives a *symmetric monoidal* algebraic model for $SO(2)$. This was recently used by Barnes [4] to provide an algebraic model for rational $O(2)$–spectra.

However, there is no algebraic model known for the whole category of rational G–spectra for an arbitrary compact Lie group G. A first step in this direction, a model for rational G–spectra over an exceptional subgroup (see Definition 5.1) for any compact Lie group G, was provided in [16]. This result is used in Section 5.

The group SO(3) The group $SO(3)$ is the group of rotations of $\mathbb{R}^3$. This is the natural next candidate to analyse on the way to understanding the behaviour of $d\mathcal{A}(G)$ for an arbitrary compact Lie group G. Notice that $SO(3)$ is significantly more complicated than all groups considered so far, since it is the first group where the maximal torus is not normal in the whole group. Dealing with this complication shows a method to provide an algebraic model for a part of rational G–spectra called *toral* for any compact Lie group G. The toral part models those G–spectra whose geometric isotropy is a set of subgroups of the maximal torus and corresponds to cohomology theories with toral support. We discuss this further in Remark 3.29.

Main result Let G be $SO(3)$. In this paper we work with orthogonal G–spectra; see Mandell and May [18, Definition 2.6]. By Barnes [3, Theorem 4.4], the category of rational $SO(3)$–orthogonal spectra splits into three parts: toral, dihedral and exceptional. This uses idempotents of the rational Burnside ring $\mathsf{A}(SO(3))_{\mathbb{Q}}$ (see Section 2.3), and reflects a similar splitting at the level of homotopy categories.

The toral part models rational $SO(3)$–spectra with geometric isotropy in the family of subconjugates of the maximal torus $SO(2)$ in $SO(3)$. The dihedral part models rational SO(3)–spectra with geometric isotropy in the collection of subgroups $\mathcal{D}$, which consists of all dihedral subgroups of order greater than 4 and $O(2)$. The last part, which we call the exceptional part, models rational SO(3)–spectra with geometric isotropy in the collection of subgroups $\mathcal{E}$, which consists of all remaining subgroups (see Section 2.1). Thus we are able to work with each of these three parts separately to obtain an algebraic model for rational SO(3)–spectra.

The main result of this paper is as follows.

Main Theorem *There is a zig-zag of Quillen equivalences between rational* SO(3)*–orthogonal spectra and the algebraic category* $d\mathcal{A}(\mathrm{SO}(3))$.

The category $d\mathcal{A}(\mathrm{SO}(3))$, which we call the *algebraic model for rational* SO(3)*–spectra*, is a product of three parts, which reflects the splitting of the category of rational SO(3)–spectra

$$d\mathcal{A}(\mathrm{SO}(3)) \cong d\mathcal{A}(\mathrm{SO}(3),\mathcal{T}) \times \mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D})) \times \prod_{(H),H\in\mathcal{E}} \mathsf{Ch}(\mathbb{Q}[W_{\mathrm{SO}(3)}H]).$$

Here $d\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ is the *algebraic model for the toral part* described in Section 3.2, $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D}))$ is the *algebraic model for the dihedral part* described in Section 4.1 and $\mathsf{Ch}(\mathbb{Q}[W_{\mathrm{SO}(3)}H])$ is the *algebraic model for the rational* SO(3)*–spectra over an exceptional subgroup* H discussed in Section 5.1. Since $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ is a graded abelian category we use the notation $d\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ for differential objects in there. We use the notation $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D}))$ for differential graded objects (ie chain complexes) in $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$, since $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$ doesn't have a grading.

The Main Theorem follows from Proposition 2.6 and Theorems 3.36, 4.11 and 5.4.

Contribution of this paper The new idea in this paper concerns the toral part in Section 3. Since the maximal torus is not normal in SO(3) the algebraic model for the toral part gets more complicated than that for the torus (see Greenlees [9] and Barnes, Greenlees, Kędziorek and Shipley [6]) or $O(2)$ (see Barnes [4]). To control these complications we use the following method. We consider the restriction–coinduction adjunction between the toral part of rational SO(3)–spectra and the toral part of rational $O(2)$–spectra. Here $O(2)$ is the normaliser of the maximal torus in SO(3). This adjunction is a Quillen adjunction, but not a Quillen equivalence.

However, the cellularisation principle of Greenlees and Shipley [12] (see Section 2.2.2 for the definition of cellularisation) gives a Quillen equivalence between the toral part of rational SO(3)–spectra and a certain cellularisation of the toral part of rational $O(2)$–spectra; see Theorem 3.28. Now it is enough to cellularise the algebraic model for the toral part of rational $O(2)$–spectra and simplify this category (see Section 3.4) to obtain the model for the toral part of rational SO(3)–spectra.

The idea of using the restriction–coinduction adjunction between the toral part of rational G–spectra and the toral part of rational $N_G\mathbb{T}$–spectra (where $\mathbb{T}$ is the maximal torus in G) together with the cellularisation principle allows one to provide an algebraic model for the toral part of rational G–spectra, for any compact Lie group G; see Barnes, Greenlees and Kędziorek [5].

The method to obtain the algebraic model for the dihedral part of rational SO(3)–spectra is a slight alteration of the method for the dihedral part for rational $O(2)$–spectra from [4] and is presented in Section 4.2. Some changes in the proof from [4] are needed to take into account the fact that our dihedral part excludes subgroups conjugate to D_2 and D_4 (for reasons explained in Section 2.1), whereas the dihedral part of $O(2)$–spectra contains them. However, the idea of the proof remains the same.

Finally, an algebraic model of the exceptional part is an application of the methods from Kędziorek [16]. We point out that this is the only part of the paper that considers monoidal structures and gives a monoidal algebraic model.

Outline of the paper This paper is structured as follows. In Section 2 we present some general results about subgroups of SO(3), its rational Burnside ring $\mathsf{A}(\mathrm{SO}(3))_{\mathbb{Q}}$ and the idempotents used to split the category of rational SO(3)–spectra into three parts: toral, dihedral and exceptional (Proposition 2.6). Section 3 is the heart of this paper. It contains the description of the algebraic model for the toral part of rational SO(3)–spectra. It also presents Quillen equivalences used in obtaining this algebraic model from the algebraic model for toral rational $O(2)$–spectra. Section 4 contains the algebraic model for the dihedral part. Finally, in Section 5 we recall the results from [16] to give an algebraic model for the exceptional part of rational SO(3)–spectra.

Notation We will stick to the convention of drawing the left adjoint above (or to the left of) the right one in any adjoint pair. We use the notation $G-\mathrm{Sp}$ for the category of G–equivariant orthogonal spectra.

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2 General results for SO(3)

We start this part by considering the closed subgroups of SO(3) in Section 2.1. We discuss the space $\mathcal{F}(G)/G$, which is the orbit space of all closed subgroups with finite index in their normaliser, where the topology is induced from the Hausdorff metric; see [17, Section V.2]. In Section 2.2 we recall two ways of changing a given stable model structure: left Bousfield localisation at an object and cellularisation. We will use these techniques repeatedly throughout the paper. In Section 2.3 we discuss the idempotents of the rational Burnside ring $\mathsf{A}(\mathrm{SO}(3))_{\mathbb{Q}}$ and the induced splitting of rational SO(3)–orthogonal spectra. The main part of Section 2.3 consists of the analysis of two adjunctions: the induction–restriction and restriction–coinduction adjunctions in relation to localisations of categories of equivariant spectra at idempotents.

2.1 Closed subgroups of SO(3)

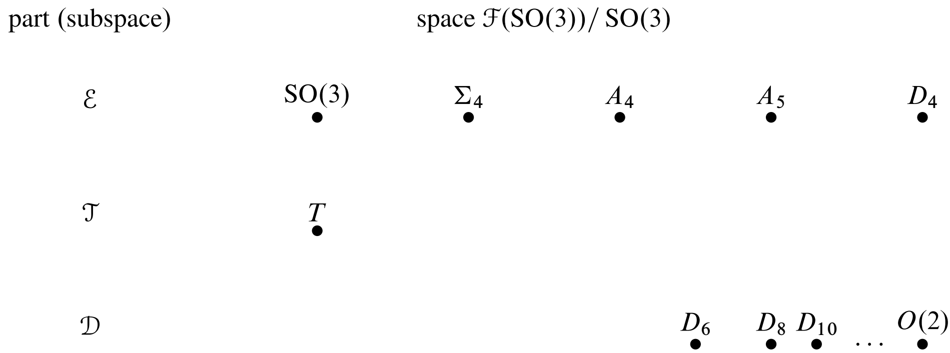
Recall that $\mathrm{SO}(3)$ is the group of rotations of $\mathbb{R}^3$. We choose a maximal torus T in $\mathrm{SO}(3)$ with rotation axis the z–axis. We divide the closed subgroups of G into three types: *toral* $\mathcal{T}$, *dihedral* $\mathcal{D}$ and *exceptional* $\mathcal{E}$. This division is motivated by the choice of idempotents in the rational Burnside ring for $\mathrm{SO}(3)$ that we will use to split the category of rational SO(3)–spectra.

The toral part consists of all tori in $\mathrm{SO}(3)$ and all cyclic subgroups of these tori. Note that for any natural number n there is one conjugacy class of subgroups from the toral part of order n in $\mathrm{SO}(3)$.

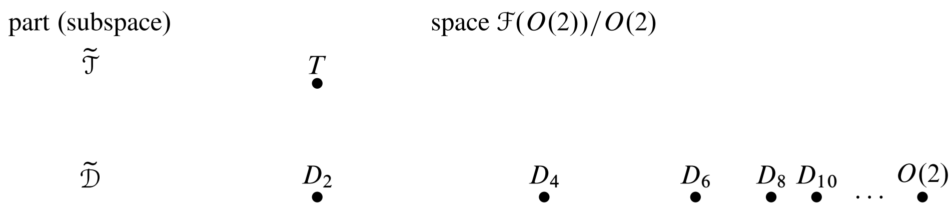
The dihedral part consists of all dihedral subgroups D_{2n} (dihedral subgroups of order $2n$) of $\mathrm{SO}(3)$ where n is greater than 2, together with all subgroups $O(2)$. Note that $O(2)$ is the normaliser for itself in $\mathrm{SO}(3)$. Moreover, there is only one conjugacy class of a dihedral subgroup D_{2n} for each n greater than 2, and the normaliser of D_{2n} in $\mathrm{SO}(3)$ is D_{4n} for $n > 2$. Notice that we excluded subgroups in the conjugacy classes of D_2 and D_4 from this part. Conjugates of D_2 are excluded from the dihedral part, since D_2 is conjugate to C_2 in $\mathrm{SO}(3)$ and that subgroup is already taken into account in the toral part. Conjugates of D_4 are excluded from the dihedral part since their normalisers in $\mathrm{SO}(3)$ are Σ_4 (symmetries of a cube), thus their Weyl groups Σ_4/D_4 are of order 6, whereas all other finite dihedral subgroups D_{2n}, $n > 2$ have Weyl groups of order 2. For simplicity we decided to treat D_4 separately and put it into the exceptional part.

There are five conjugacy classes of subgroups which we call exceptional, namely $\mathrm{SO}(3)$ itself, the rotation group Σ_4 of a cube, the rotation group A_4 of a tetrahedron, the rotation group A_5 of a dodecahedron and the dihedral group D_4 of order 4. Normalisers of these exceptional subgroups are as follows: Σ_4 is equal to its normaliser, A_5 is equal to its normaliser and the normaliser of A_4 is Σ_4, as is the normaliser of D_4.

Consider the space $\mathcal{F}(\mathrm{SO}(3))/\,\mathrm{SO}(3)$ of conjugacy classes of subgroups of $\mathrm{SO}(3)$ with finite index in their normalisers. The topology on this space is induced by the Hausdorff metric. We will use this space for choosing idempotents of the rational Burnside ring in Section 2.3. The topology on $\mathcal{E}$ is discrete, $\mathcal{T}$ consists of one point T and $\mathcal{D}$ forms a sequence of points converging to $O(2)$, as shown in the following diagram:



Before we go any further we recall the space $\mathcal{F}(O(2))/O(2)$. It consists of two parts: toral and dihedral. To distinguish between these parts and their analogues for SO(3) we choose the notation $\widetilde{\mathcal{T}}$ for the toral part of $O(2)$ and $\widetilde{\mathcal{D}}$ for the dihedral part of $O(2)$ (note that in [4] the notation without tilde was used for the toral and dihedral parts of $O(2)$). We will stick to this new notation convention throughout the paper. The toral part is just one point T corresponding to the maximal torus and all its subgroups. The dihedral part corresponds to all dihedral subgroups together with $O(2)$ and we present it below:



The only difference in the dihedral parts for $O(2)$ and $\mathrm{SO}(3)$ is captured by the fact that the dihedral part for $O(2)$ is a disjoint union of $\mathcal{D}$ and two points (corresponding to D_2 and D_4, respectively). At a first glance the toral part for $\mathrm{SO}(3)$ looks the same as the toral part for $O(2)$. However, for $\mathrm{SO}(3)$ it contains information about D_2 (since D_2 is conjugate to C_2 in $\mathrm{SO}(3)$), whereas for $O(2)$ it does not. These differences will become significant in Section 2.3.

2.2 Left Bousfield localisation and cellularisation

In this section we briefly recall two ways of changing a given stable model structure: left Bousfield localisation at an object and cellularisation. We will repeatedly use them in the rest of the paper.

2.2.1 Left Bousfield localisation at an object For details on left Bousfield localisation at an object we refer the reader to [18, Section IV.6]. We recall the following result:

Theorem 2.1 [18, Chapter IV, Theorem 6.3] *Suppose E is a cofibrant object in G–Sp or a cofibrant based G–space. Then there exists a new model structure on the category G–Sp, where a map $f\colon X \to Y$ is*

- *a weak equivalence if it is an E–equivalence, ie $\mathrm{Id}_E \wedge f\colon E \wedge X \to E \wedge Y$ is a weak equivalence;*
- *a cofibration if it is a cofibration with respect to the stable model structure;*
- *a fibration if it has the right lifting property with respect to all trivial cofibrations.*

The E–fibrant objects Z are the E–local objects, ie those such that $[f, Z]^G\colon [Y, Z]^G \to [X, Z]^G$ is an isomorphism for all E–equivalences f. E–fibrant approximation gives Bousfield localisation $\lambda\colon X \to L_E X$ of X at E.

We use the notation $L_E(G\text{–Sp})$ for the model category described above and will refer to it as a *left Bousfield localisation of the category of G–spectra at E*. If E and F are cofibrant objects in G–Sp then the localisation first at E and then at F is the same model category as the localisation at $E \wedge F$ (and $F \wedge E$).

Recall that an E–equivalence between E–local objects is a weak equivalence (see [13, Theorems 3.2.13 and 3.2.14]).

In this paper we use the above definition with $X \in G$–Sp of the form $eS_{\mathbb{Q}}$ (for various e) where e is an idempotent of a rational Burnside ring $A(G)_{\mathbb{Q}}$ and $S_{\mathbb{Q}}$ is a rational sphere spectrum (see [2, Section 5] for construction of the rational sphere spectrum $S_{\mathbb{Q}}$). Since we use idempotents of a rational Burnside ring, all our localisations are smashing (see [19] for definition of a smashing localisation). Thus they preserve homotopically compact generators (see Definition 2.5) since the fibrant replacement preserves infinite coproducts.

2.2.2 Cellularisation A cellularisation of a model category is a right Bousfield localisation at a set of objects. Such a localisation exists by [13, Theorem 5.1.1] whenever the model category is right proper and cellular. When we are in a stable context the results of [7] can be used.

In this section we recall the notion of cellularisation when $\mathcal{C}$ is a stable model category and some basic definitions and results.

Definition 2.2 Let $\mathcal{C}$ be a stable model category and K a stable set of objects of $\mathcal{C}$, ie a set such that the class of K–cellular objects of $\mathcal{C}$ is closed under desuspension (note that the class is always closed under suspension). We call K a set of *cells*. We say that a map $f\colon A \to B$ of $\mathcal{C}$ is a *K–cellular equivalence* if the induced map

$$[k, f]^{\mathcal{C}}_*\colon [k, A]^{\mathcal{C}}_* \to [k, B]^{\mathcal{C}}_*$$

is an isomorphism of graded abelian groups for each $k \in K$. An object $Z \in \mathcal{C}$ is said to be *K–cellular* if

$$[Z, f]^{\mathcal{C}}_*\colon [Z, A]^{\mathcal{C}}_* \to [Z, B]^{\mathcal{C}}_*$$

is an isomorphism of graded abelian groups for any K–cellular equivalence f.

Definition 2.3 A *right Bousfield localisation* or *cellularisation* of $\mathcal{C}$ with respect to a set of objects K is a model structure $K-\text{cell}-\mathcal{C}$ on $\mathcal{C}$ such that

- the weak equivalences are K–cellular equivalences,
- the fibrations of $K-\text{cell}-\mathcal{C}$ are the fibrations of $\mathcal{C}$,
- the cofibrations of $K-\text{cell}-\mathcal{C}$ are defined via left lifting property.

By [13, Theorem 5.1.1], if $\mathcal{C}$ is a right proper, cellular model category and K a set of objects in $\mathcal{C}$, then the cellularisation of $\mathcal{C}$ with respect to K, $K-\text{cell}-\mathcal{C}$, exists and is a right proper model category. The cofibrant objects of $K-\text{cell}-\mathcal{C}$ are called K–cofibrant and are precisely the K–cellular and cofibrant objects of $\mathcal{C}$.

The cellularisation of a proper, cellular, stable model category at a stable set of cofibrant objects K is very well behaved (see [7, Theorem 5.9]), in particular it is proper, cellular and stable. Left properness follows from [7, Proposition 5.8].

There is another important property we will often want the cells to satisfy, which makes right localisation behave in an even more tractable manner; see [7, Section 9]. This property is variously called smallness, compactness or finiteness. We choose to call it *homotopical compactness*, since there are several different meanings of compactness in the literature.

Definition 2.4 [21, Definition 2.1.2] An object X in a stable model category $\mathcal{C}$ is *homotopically compact* if for any family of objects $\{A_i\}_{i\in I}$ the canonical map

$$\bigoplus_{i\in I}[X, A_i]^{\mathcal{C}} \to \Big[X, \coprod_{i\in I} A_i\Big]^{\mathcal{C}}$$

is an isomorphism in the homotopy category of $\mathcal{C}$.

Recall that a homotopy category of a stable model category is triangulated; see Definition 7.1.1 of [14]. In this setting we can make the following definition after Definition 2.1.2 of [21].

Definition 2.5 Let $\mathcal{C}$ be a triangulated category with infinite coproducts. A full triangulated subcategory of $\mathcal{C}$ (with shift and triangles induced from $\mathcal{C}$) is called *localising* if it is closed under coproducts in $\mathcal{C}$. A set $\mathcal{P}$ of objects of $\mathcal{C}$ is called a *set of generators* if the only localising subcategory of $\mathcal{C}$ containing objects of $\mathcal{P}$ is the whole of $\mathcal{C}$. An object of a stable model category is called a generator if it is a generator when considered as an object of the homotopy category.

Using [21, Lemma 2.2.1] it is routine to check that if K consists of homotopically compact objects of $\mathcal{C}$ then K is a set of generators for $K-\text{cell}-\mathcal{C}$. Hence we know a set of generators for each of our cellularisations.

Notice that derived functors of both left and right Quillen equivalences preserve homotopically compact objects.

2.3 Idempotents, splitting and reductions

By the results of tom Dieck [8, Propositions 5.6.4 and 5.9.13] there is an isomorphism of rings

$$\mathsf{A}(\mathrm{SO}(3))_{\mathbb{Q}} = C(\mathcal{F}(\mathrm{SO}(3))/\,\mathrm{SO}(3), \mathbb{Q}).$$

Here $\mathsf{A}(\mathrm{SO}(3))_{\mathbb{Q}}$ is the rational Burnside ring for SO(3) and $C(\mathcal{F}(\mathrm{SO}(3))/\,\mathrm{SO}(3), \mathbb{Q})$ denotes the ring of continuous functions on the orbit space $\mathcal{F}(\mathrm{SO}(3))/\,\mathrm{SO}(3)$ with values in the discrete space $\mathbb{Q}$.

Thus it is clear that idempotents of the rational Burnside ring of SO(3) correspond to the characteristic functions on subspaces of the orbit space $\mathcal{F}(\mathrm{SO}(3))/\,\mathrm{SO}(3)$ discussed in Section 2.1 which are both open and closed.

In this paper we use the following idempotents in the rational Burnside ring of SO(3): $e_{\mathcal{T}}$ corresponding to the characteristic function of the toral part $\mathcal{T}$, ie the conjugacy class of the torus T; $e_{\mathcal{D}}$ corresponding to the characteristic function of the dihedral part $\mathcal{D}$; and $e_{\mathcal{E}}$ corresponding to the characteristic function of the exceptional part $\mathcal{E}$. Since $\mathcal{E}$ is a disjoint union of five points, it is in fact a sum of five idempotents, one for every (conjugacy class of a) subgroup in the exceptional part: $e_{\mathrm{SO}(3)}$, e_{Σ_4}, e_{A_4}, e_{A_5} and e_{D_4}. We use a simplified notation e_H to mean $e_{(H)_{\mathrm{SO}(3)}}$ here.

Analogously, we will use the notation $e_{\widetilde{\mathcal{T}}}$ for the idempotent in the rational Burnside ring of $O(2)$ corresponding to the toral part $\widetilde{\mathcal{T}}$ and $e_{\widetilde{\mathcal{D}}}$ for the idempotent corresponding to the dihedral part $\widetilde{\mathcal{D}}$ of $O(2)$.

For an idempotent $e \in \mathsf{A}(\mathrm{SO}(3))_{\mathbb{Q}}$ and a rational sphere spectrum $S_{\mathbb{Q}}$ (see [2, Section 5] for the construction) we define $eS_{\mathbb{Q}}$ to be the homotopy colimit (a mapping telescope) of the diagram

$$S_{\mathbb{Q}} \xrightarrow{e} S_{\mathbb{Q}} \xrightarrow{e} S_{\mathbb{Q}} \xrightarrow{e} \cdots .$$

We ask for this spectrum to be cofibrant either by choosing a good construction of homotopy colimit, or by cofibrantly replacing the result in the stable model structure for SO(3)–spectra. Now, by [18, Chapter IV, Theorem 6.3] (see also Theorem 2.1) the following left Bousfield localisations exist:

$$L_{e_{\mathcal{T}} S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp}), \quad L_{e_{\mathcal{D}} S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp}), \quad L_{e_{\mathcal{E}} S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp}).$$

Also, $L_{e_H S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp})$ exists for any exceptional subgroup $H \in \mathcal{E}$.

The first step on the way towards an algebraic model for rational SO(3)–spectra is to split this category using the above idempotents of the Burnside ring $\mathsf{A}(\mathrm{SO}(3))_{\mathbb{Q}}$. By [3, Theorem 4.4] we get the following decomposition.

Proposition 2.6 *The adjunction*

$$\begin{array}{c} \mathrm{SO}(3){-}\mathrm{Sp}_{\mathbb{Q}} \\ \triangle \big\downarrow \big\uparrow \Pi \\ L_{e_{\mathcal{T}} S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp}) \times L_{e_{\mathcal{D}} S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp}) \times L_{e_{\mathcal{E}} S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp}) \end{array}$$

is a strong monoidal Quillen equivalence, where $\mathrm{SO}(3){-}\mathrm{Sp}_{\mathbb{Q}}$ *denotes the category of rational* $\mathrm{SO}(3)$ *orthogonal spectra, the left adjoint is the diagonal functor and the right adjoint is the product.*

The main idea is to relate each of these localised categories to corresponding ones for simpler groups. Thus we recall that an inclusion $i\colon H \to G$ of a subgroup H into a group G induces two adjoint pairs at the level of orthogonal spectra, induction–restriction and restriction–coinduction (see [18, Section V.2]):

$$G{-}\mathrm{Sp} \; \overset{G_+ \wedge_H -}{\underset{F_H(G_+,-)}{\leftleftarrows}} \!\xrightarrow{\;i^*\;}\; N{-}\mathrm{Sp}$$

These are both Quillen pairs with respect to the usual stable model structures on both sides. On the way to obtaining an algebraic model for rational SO(3)–spectra we will relate both the toral and dihedral parts of this category to the corresponding parts for rational $O(2)$–spectra. The natural choice of adjunction between SO(3)–spectra and

$O(2)$–spectra would be the induction and restriction functors. However, this turns out not to be a Quillen adjunction between the toral parts, as we discuss below.

Proposition 2.7 *Suppose $e_{\mathcal{T}}$ is the idempotent in $\mathsf{A}(\mathrm{SO}(3))_{\mathbb{Q}}$ corresponding to the characteristic function of the toral part $\mathcal{T}$ (ie all subconjugates of the maximal torus of $\mathrm{SO}(3)$) and $e_{\widetilde{\mathcal{T}}}$ is the idempotent in $\mathsf{A}(O(2))_{\mathbb{Q}}$ corresponding to the characteristic function of the toral part $\widetilde{\mathcal{T}}$ (ie all subconjugates of the maximal torus of $O(2)$). Then*

$$i^*\colon L_{e_{\mathcal{T}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–Sp}) \rightleftarrows L_{e_{\widetilde{\mathcal{T}}}S_{\mathbb{Q}}}(O(2)\text{–Sp}) \ :\mathrm{SO}(3)_+\wedge_{O(2)} -$$

is not a Quillen adjunction.

Proof It is enough to show that $\mathrm{SO}(3)_+\wedge_{O(2)}-$ does not preserve acyclic cofibrations. This argument is the same as the one in [16, Proposition 4.5], since D_2 is conjugate to C_2 in $\mathrm{SO}(3)$ and thus $i^*(e_{\mathcal{T}})\neq e_{\widetilde{\mathcal{T}}}$. □

Although the adjunction above does not behave well with respect to these model structures, the one with restriction and coinduction does, as is shown in Proposition 2.12 below.

Proposition 2.8 *Suppose $e_{\mathcal{D}}$ is the idempotent of $\mathsf{A}(\mathrm{SO}(3))_{\mathbb{Q}}$ corresponding to all dihedral subgroups of order greater than 4 and all subgroups isomorphic to $O(2)$. Then*

$$i^*\colon L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–Sp}) \rightleftarrows L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2)\text{–Sp}) \ :\mathrm{SO}(3)_+\wedge_{O(2)} -$$

is a Quillen adjunction.

Proof The proof follows the same pattern as the proof of [16, Proposition 4.4]. It was a Quillen adjunction before localisation by [18, Chapter V, Proposition 2.3] so the left adjoint preserves cofibrations. It preserves acyclic cofibrations as $\mathrm{SO}(3)_+\wedge_{O(2)}-$ preserved acyclic cofibrations before localisation and we have a natural (in $O(2)$–spectra X) isomorphism

$$(\mathrm{SO}(3)_+\wedge_{O(2)} X)\wedge e_{\mathcal{D}}S_{\mathbb{Q}} \cong \mathrm{SO}(3)_+\wedge_{O(2)} (X\wedge i^*(e_{\mathcal{D}}S_{\mathbb{Q}})). \qquad \square$$

It turns out that the other adjunction — restriction and coinduction adjunction — gives a Quillen pair under general conditions on localisations.

Lemma 2.9 [16, Lemma 4.6] *Suppose G is any compact Lie group, $i\colon H\to G$ is an inclusion of a subgroup and V is an open and closed set in $\mathcal{F}(G)/G$. Then the adjunction*

$$i^*\colon L_{e_V S_{\mathbb{Q}}}(G\text{–Sp}) \rightleftarrows L_{i^*(e_V)S_{\mathbb{Q}}}(H\text{–Sp}) \ :F_H(G_+,-)$$

is a Quillen pair. We use the notation e_V here for the idempotent corresponding to the characteristic function on V.

In the next sections we will repeatedly use this lemma, mainly in situations where after a further localisation of the right-hand side we get a Quillen equivalence. To prepare for that, we distinguish the following two cases.

Corollary 2.10 *Let $\mathcal{D}$ denote the dihedral part of $\mathrm{SO}(3)$ and $e_{\mathcal{D}}$ the corresponding idempotent. Let $i\colon O(2) \to \mathrm{SO}(3)$ be an inclusion. Then*

$$i^*\colon L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–Sp}) \rightleftarrows L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2)\text{–Sp}) \;: F_{O(2)}(\mathrm{SO}(3)_+, -)$$

is a Quillen adjunction.

Remark 2.11 Note that the idempotent on the right-hand side $i^*(e_{\mathcal{D}})$ corresponds to the dihedral part of $O(2)$ excluding all subgroups D_2 and D_4. Thus $i^*(e_{\mathcal{D}}) = i^*(e_{\mathcal{D}})e_{\widetilde{\mathcal{D}}}$.

Proposition 2.12 *Let $i\colon O(2) \to \mathrm{SO}(3)$ be an inclusion. Then*

$$i^*\colon L_{e_{\mathcal{T}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–Sp}) \rightleftarrows L_{e_{\widetilde{\mathcal{T}}}S_{\mathbb{Q}}}(O(2)\text{–Sp}) \;: F_{O(2)}(\mathrm{SO}(3)_+, -)$$

is a strong monoidal Quillen adjunction, where the idempotent on the right-hand side corresponds to the family of all subgroups of $O(2)$ subconjugate to a maximal torus $\mathrm{SO}(2)$ in $O(2)$.

Proof This follows from Lemma 2.9 and the composition of Quillen adjunctions

$$L_{e_{\mathcal{T}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–Sp}) \underset{F_{O(2)}(\mathrm{SO}(3)_+,-)}{\overset{i^*}{\rightleftarrows}} L_{i^*(e_{\mathcal{T}})S_{\mathbb{Q}}}(O(2)\text{–Sp}) \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} L_{e_{\widetilde{\mathcal{T}}}S_{\mathbb{Q}}}(O(2)\text{–Sp}) .$$

Note that $i^*(e_{\mathcal{T}}S_{\mathbb{Q}})$ has nontrivial geometric fixed points not only for all cyclic subgroups of $O(2)$ and $\mathrm{SO}(2)$, but also for D_2, as D_2 is conjugate to C_2 in $\mathrm{SO}(3)$. To ignore that and take into account only the toral part we use the fact that $e_{\widetilde{\mathcal{T}}}i^*(e_{\mathcal{T}}) = e_{\widetilde{\mathcal{T}}}$, which implies that the identity adjunction above is a Quillen pair. □

3 The toral part

In this section we use results from [6] and [4] to obtain an algebraic model for the toral part of rational $\mathrm{SO}(3)$–spectra. The first paper establishes a zig-zag of symmetric monoidal Quillen equivalences between rational $\mathrm{SO}(2)$–spectra, while the second one lifts this comparison to one compatible with the $W = O(2)/\mathrm{SO}(2)$–action to obtain an algebraic model for the toral part of rational $O(2)$–spectra.

We begin by describing the category $d\mathcal{A}(O(2),\widetilde{\mathcal{T}})$ in Section 3.1 and $d\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ in Section 3.2. Then we proceed to establish the comparison between the toral part of rational SO(3)–orthogonal spectra and its algebraic model, $d\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$.

3.1 Categories $\mathcal{A}(O(2),\widetilde{\mathcal{T}})$ and $d\mathcal{A}(O(2),\widetilde{\mathcal{T}})$

Before we are ready to describe the category $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ we have to introduce the category $\mathcal{A}(O(2),\widetilde{\mathcal{T}})$. We give a short description of $\mathcal{A}(O(2),\widetilde{\mathcal{T}})$ as a category on the objects of $\mathcal{A}(\mathrm{SO}(2))$ with W–action. Recall that $W = O(2)/\mathrm{SO}(2)$ is the group of order 2.

Material in this section is based on [9] and [4, Section 3].

Definition 3.1 Let $\mathcal{F}$ denote the family of all finite cyclic subgroups in $O(2)$. Then we define a ring in the category of graded $\mathbb{Q}[W]$–modules

$$\mathcal{O}_{\mathcal{F}} := \prod_{H\in\mathcal{F}} \mathbb{Q}[c_H]$$

where each c_H has degree -2 and w (the nontrivial element of W) acts on each c_H by -1. For simplicity we set $c := c_1$.

We use the notation $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ for the colimit

$$\operatorname{colim}_k \mathcal{O}_{\mathcal{F}}[c^{-1}, c_{C_2}^{-1}, \dots, c_{C_k}^{-1}]$$

of localisations, where the maps in the colimit are the inclusions. $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is an $\mathcal{O}_{\mathcal{F}}$–module using the inclusion

$$\mathcal{O}_{\mathcal{F}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}.$$

Notice that we can perform a similar construction on the ring $\widetilde{\mathcal{O}_{\mathcal{F}}} := (1-e_1)\mathcal{O}_{\mathcal{F}}$ and call it $\widetilde{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}$, where e_1 is the projection on the first factor in the ring $\mathcal{O}_{\mathcal{F}}$. Then another way to define $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is as $\mathbb{Q}[c,c^{-1}]\times\widetilde{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}$. This last description of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ will be useful when we compare this model to the one for the toral part of rational SO(3)–spectra.

Definition 3.2 An object of $\mathcal{A}(O(2),\widetilde{\mathcal{T}})$ consists of a triple (M,V,β) where M is an $\mathcal{O}_{\mathcal{F}}$–module in $\mathbb{Q}[W]$–modules, V is a graded rational vector space with a W–action and β is a map of $\mathcal{O}_{\mathcal{F}}$–modules (in $\mathbb{Q}[W]$–modules)

$$\beta\colon M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$$

such that

($\star$) $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} \beta$ is an isomorphism of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$–modules in $\mathbb{Q}[W]$–modules.

A morphism between two such objects $(\alpha, \phi)\colon (M, V, \beta) \to (M', V', \beta')$ consists of a map of $\mathcal{O}_{\mathcal{F}}$–modules $\alpha\colon M \to M'$ and a map of graded $\mathbb{Q}[W]$–modules such that the relevant square commutes.

Instead of modules over $\mathcal{O}_{\mathcal{F}}$ in $\mathbb{Q}[W]$–modules we can consider modules over $\mathcal{O}_{\mathcal{F}}[W]$ in $\mathbb{Q}$–modules, where $\mathcal{O}_{\mathcal{F}}[W]$ is a group ring with a twisted W–action (namely $wc_H = -c_H w$). We will use this description in the next section. Similarly, $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}[W]$ denotes a group ring with a twisted W–action.

Definition 3.3 An object of $dA(O(2), \widetilde{\mathcal{T}})$ is an object of $\mathcal{A}(O(2), \widetilde{\mathcal{T}})$ equipped with a differential, or in other words it consists of a triple (M, V, β) where M is an $\mathcal{O}_{\mathcal{F}}$–module in $\mathsf{Ch}(\mathbb{Q}[W])$, V is an object of $\mathsf{Ch}(\mathbb{Q}[W])$ and β is a map of $\mathcal{O}_{\mathcal{F}}$–modules (in $\mathsf{Ch}(\mathbb{Q}[W])$)

$$\beta\colon M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$$

such that

($\star$) $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} \beta$ is an isomorphism of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$–modules in $\mathsf{Ch}(\mathbb{Q}[W])$.

A morphism in this category is a morphism in $\mathcal{A}(O(2), \widetilde{\mathcal{T}})$ which commutes with the differentials.

We proceed to discuss the properties of the category $dA(O(2), \widetilde{\mathcal{T}})$. Firstly, all limits and colimits exist in $dA(O(2), \widetilde{\mathcal{T}})$, by an argument analogous to [6, Definition 2.2.1].

The existence of a model structure on $dA(O(2), \widetilde{\mathcal{T}})$ follows from [9, Appendix B].

Theorem 3.4 *There is a stable, proper model structure on the category $dA(O(2), \widetilde{\mathcal{T}})$ where the weak equivalences are the homology isomorphisms. The cofibrations are the injections and the fibrations are defined via the right lifting property. We call this model structure the* ***injective model structure****.*

The existence of another, monoidal, model category structure on $d(\mathcal{A}(O(2), \widetilde{\mathcal{T}}))$ was established in [4]. However, since we are not considering monoidality of the algebraic model in this paper, the injective model structure on $dA(O(2), \widetilde{\mathcal{T}})$ is enough for our purposes.

3.2 Categories $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ and $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$

Looking at the toral parts of the spaces of subgroups of $\mathrm{SO}(3)$ and $O(2)$ we see that the stabiliser of the trivial subgroup is connected in $\mathrm{SO}(3)$, while it is not in $O(2)$.

This is a consequence of the fact that the maximal torus is not normal in SO(3) and it is the main ingredient capturing the difference between the algebraic models for the toral part of rational SO(3)–spectra and the toral part of rational $O(2)$–spectra.

Let us denote by $\mathcal{F}_{SO(3)}$ the family of all finite cyclic subgroups in SO(3). Then we use the simplified notation $\mathcal{O}_{\overline{\mathcal{F}}} := \mathcal{O}_{\mathcal{F}_{SO(3)}}$, by which we mean a graded ring

$$\mathbb{Q}[d] \times \prod_{(H)\in\mathcal{F}_{SO(3)}, H\neq 1} \mathbb{Q}[c_{(H)}]$$

where d is in degree -4 and all $c_{(H)}$ are in degree -2. The nontrivial element $w \in W$ acts on it by fixing d and sending $c_{(H)}$ to $-c_{(H)}$ for all subgroups $H \in \overline{\mathcal{F}}_{SO(3)}$, $H \neq 1$.

We define the ring $\mathcal{O}_{\overline{\mathcal{F}}}[W]$ as a product of $\mathbb{Q}[d]$ (with trivial W–action) and a group ring $(1-e_1)\mathcal{O}_{\overline{\mathcal{F}}}[W]$ with the twisted W–action, that is $wc_{(H)} = -c_{(H)}w$ for $H \in \mathcal{F}_{SO(3)}$, $H \neq 1$.

Recall that c was the element of the first factor of the ring $\mathcal{O}_{\mathcal{F}}$ (see Definition 3.1). There is an adjunction

$$\mathbb{Q}\text{–mod} \underset{(-)^W}{\overset{\text{Triv}}{\rightleftarrows}} \mathbb{Q}[W]\text{–mod}$$

where $(\mathbb{Q}[c])^W = \mathbb{Q}[d]$ (recall that $\mathbb{Q}[c]$ is the $\mathbb{Q}[W]$–module with W–action given by $wc = -c$). Thus using for example [20, Section 3.3] we have the adjunction

$$\mathbb{Q}[d]\text{–mod in } \mathbb{Q}\text{–mod} \underset{(-)^W}{\overset{\mathbb{Q}[c]\otimes_{\mathbb{Q}[d]}-}{\rightleftarrows}} \mathbb{Q}[c]\text{–mod in } \mathbb{Q}[W]\text{–mod}.$$

This extends to give the following result.

Proposition 3.5 *There is an adjunction*

$$\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} -\colon \mathcal{O}_{\overline{\mathcal{F}}}[W]\text{–mod} \rightleftarrows \mathcal{O}_{\mathcal{F}}[W]\text{–mod} :(-)^W \times \mathrm{Id}.$$

Proof The unit of this adjunction is the identity and the counit is the natural inclusion. □

We can compose this adjunction with the usual restriction–induction adjunction

$$\mathcal{O}_{\mathcal{F}}[W]\text{–mod} \underset{\text{res}}{\overset{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes_{\mathcal{O}_{\mathcal{F}}}-}{\rightleftarrows}} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}[W]\text{–mod}$$

to get the adjunction

$$\mathcal{O}_{\overline{\mathcal{F}}}[W]\text{–mod} \underset{U}{\overset{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes_{\mathcal{O}_{\overline{\mathcal{F}}}}-}{\rightleftarrows}} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}[W]\text{–mod} \tag{3-1}$$

in $\mathbb{Q}$–modules.

We define the category $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ as follows.

Definition 3.6 An object in $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ consists of a triple (M,V,β) where M is an $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–module in $\mathbb{Q}$–modules, V is a graded rational vector space with a W–action and β is a map of $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–modules

$$\beta\colon M \to U(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$$

such that the adjoint (using (3-1)) satisfies

($\star$) $\quad \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ is an isomorphism of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}[W]$–modules.

A morphism between two such objects $(\alpha,\phi)\colon (M,V,\beta) \to (M',V',\beta')$ consists of a map of $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–modules $\alpha\colon M \to M'$ and a map of graded $\mathbb{Q}[W]$–modules such that the relevant square commutes.

Notice that the condition on the map β implies that the image of $e_1 M$ must lie in $(\mathbb{Q}[c,c^{-1}]\otimes V)^W$, ie in W–fixed points. From now on we will abuse the notation slightly and leave out the functor U (3-1) in the codomain of β in $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$.

Remark 3.7 There are no idempotents in the category $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$; however, the category of $\mathcal{O}_{\overline{\mathcal{F}}}$–modules can be split, for example as $\mathbb{Q}[d]$–mod $\times\, (1-e_1)\mathcal{O}_{\overline{\mathcal{F}}}$–mod. We will use that property in the proof of Proposition 3.9.

Definition 3.8 An object of $d\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ consists of an $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–module M equipped with a differential and a chain complex of $\mathbb{Q}[W]$–modules V together with a map of $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–modules $\gamma\colon M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ which commutes with differentials. A differential on a $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–module M consists of maps $d_n\colon M_n \to M_{n-1}$ such that $d_{n-1}\circ d_n = 0$ and $\overline{c}d_n = d_{n-2}\overline{c}$, where $\overline{c}$ consists of elements $c_{(H)}$ on the H–factor, for all $(H)\in\overline{\mathcal{F}}$, $H\neq 1$, and 0 on the first factor, and where $\overline{d}d_n = d_{n-4}\overline{d}$; here $\overline{d}$ is d on the first factor and 0 everywhere else in the product.

A morphism in this category is a morphism in $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ which commutes with the differentials.

We proceed to study the adjunction relating $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ and $\mathcal{A}(O(2),\widetilde{\mathcal{T}})$.

Proposition 3.9 *We have the following adjunction, where the adjoints are defined in the proof:*

$$F\colon \mathcal{A}(\mathrm{SO}(3),\mathcal{T}) \rightleftarrows \mathcal{A}(O(2),\widetilde{\mathcal{T}}) \;:R.$$

Proof Take $X = (\gamma\colon M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ in $d\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$. Then define

$$F(X) := (\overline{\gamma}\colon \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V),$$

where $\overline{\gamma}$ is the adjoint of γ (since $\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} -$ is a left adjoint from $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–modules to $\mathcal{O}_{\mathcal{F}}[W]$–modules; see Proposition 3.5). It is easy to see that this construction gives an object in $\mathcal{A}(O(2), \widetilde{\mathcal{T}})$, ie that it satisfies the condition $(\star)$ from Definition 3.2. Since $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} \overline{\gamma}$ agrees with $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} \gamma$ in $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}[W]$–modules, it is an isomorphism by condition $(\star)$ from Definition 3.6.

Now take $Y = (\delta\colon N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ in $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$. Then define

$$R(Y) := \big(\delta \circ i\colon (e_1 N)^W \times (1-e_1)N \to N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U\big),$$

where i is the inclusion.

To see that $R(Y) \in \mathcal{A}(\mathrm{SO}(3), \mathcal{T})$, we show the adjoint condition $(\star)$ from Definition 3.6 holds for $\delta \circ i$.

Thus we want to show that

$$\overline{\delta \circ i}\colon \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} ((e_1 N)^W \times (1-e_1)N) \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$$

is an isomorphism of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}[W]$ modules.

Notice that we have a natural map

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} (\varepsilon_N)\colon \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} ((e_1 N)^W \times (1-e_1)N) \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} (N)$$

where ε is the counit of the adjunction from Proposition 3.5.

After applying e_1, the map $e_1\varepsilon_N$ is an isomorphism for finitely generated modules N. Since every module is a colimit of finitely generated ones and $\otimes$ commutes with colimits, $e_1\varepsilon_N$ is an isomorphism for any N. Since ε_N is an isomorphism away from e_1 it is an isomorphism. To complete the argument notice that the diagram

$$\begin{array}{ccc}
\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} ((e_1 N)^W \times (1-e_1)N) & \xrightarrow{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} (\varepsilon_N)} & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} (N) \\
& \searrow^{\overline{\delta \circ i}} & \downarrow \overline{\delta} \\
& & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U
\end{array}$$

commutes, where $\overline{\delta}$ is the adjoint of δ (see Proposition 3.5).

It is easy to see that this is an adjoint pair, since the unit is the identity and the counit is the pair of maps $(\varepsilon, \mathrm{Id})$ and the identity on graded $\mathbb{Q}[W]$–modules. Here ε is the counit of the adjunction in Proposition 3.5. □

Proposition 3.10 *All small limits and colimits exist in* $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$.

Proof Suppose we have a diagram of objects $M_i \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V_i$ in $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ indexed by a category I. The colimit of this diagram is

$$\mathrm{colim}_i\, M_i \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (\mathrm{colim}_i\, V_i).$$

If the diagram is finite, than the limit is formed in $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ in a similar way:

$$\lim_i M_i \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (\lim_i V_i).$$

To construct infinite limits in a category $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ we use the same method as in [6, Definition 2.2.1]. However, since we don't use the construction of infinite limits anywhere in this paper, we skip the technicalities.

Verifying that these constructions define limits and colimits in $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ is routine. □

Let $g\mathbb{Q}[W]$–mod denote the category of graded $\mathbb{Q}[W]$–modules. Recall that an $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–module M is $\overline{\mathcal{F}}$–finite if it is a direct sum of its submodules $e_{(H)}M$:

$$M = \bigoplus_{(H)\in\overline{\mathcal{F}}} e_{(H)}M,$$

and let tors–$\mathcal{O}_{\overline{\mathcal{F}}}[W]^f$–mod denote the category of $\overline{\mathcal{F}}$–finite torsion $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–modules. We define two functors relating $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ to some simpler categories, which will allow us to create two classes of injective objects in $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$.

Definition 3.11 Define the functor $e\colon g\mathbb{Q}[W]\text{–mod} \to \mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ by

$$e(V) := (P \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V),$$

where

$$e_1 P = \mathbb{Q}[d, d^{-1}] \otimes V^+ \oplus \Sigma^2 \mathbb{Q}[d, d^{-1}] \otimes V^- \quad \text{and} \quad (1-e_1)P = (1-e_1)\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V.$$

Here V^+ is the W–fixed part of V, V^- is the -1 eigenspace and Σ is the suspension. The structure map is essentially just an inclusion.

Define a functor $f\colon$ tors–$\mathcal{O}_{\overline{\mathcal{F}}}[W]^f$–mod $\to \mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ by

$$f(N) := (N \to 0).$$

The domain for this functor was chosen so that $f(N) \in \mathcal{A}(\mathrm{SO}(3), \mathcal{T})$, that is, it satisfies condition $(\star)$ from Definition 3.6.

Proposition 3.12 *For any object $X = (\gamma\colon M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ in $\mathcal{A} = \mathcal{A}(\mathrm{SO}(3), \mathcal{T})$, any V in $\mathbb{Q}[W]$–mod and any N in tors–$\mathcal{O}_{\overline{\mathcal{F}}}[W]^f$–mod, we have natural isomorphisms*

$$\mathrm{Hom}_{\mathcal{A}}(X, e(V)) = \mathrm{Hom}_{\mathbb{Q}[W]}(U, V),$$
$$\mathrm{Hom}_{\mathcal{A}}(X, f(N)) = \mathrm{Hom}_{\mathcal{O}_{\overline{\mathcal{F}}}[W]}(M, N).$$

Remark 3.13 This proposition implies that an object $e(V)$ is injective for any V and that if N is an injective $\overline{\mathcal{F}}$–finite torsion $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–module then $f(N)$ is also injective.

Lemma 3.14 *The category $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ is a (graded) abelian category of injective dimension 1. Moreover it is split, ie every object X of $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ has a splitting $X = X_+ \oplus X_-$ such that $\mathrm{Hom}(X_\delta, Y_\epsilon) = 0$ and $\mathrm{Ext}(X_\delta, Y_\epsilon) = 0$ if $\delta \neq \epsilon$ and $(\Sigma X)_+ = \Sigma(X_-)$ and $(\Sigma X)_- = \Sigma(X_+)$.*

Proof The category $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ is enriched in abelian groups and by construction of all limits and colimits we can conclude that it is an abelian category.

For an object $X = (\gamma\colon M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ we construct the injective resolution of length 1 as follows. Let $TM := \ker \gamma$, which is torsion. Thus, since $\mathbb{Q}[d]$ and all $\mathbb{Q}[c_{(H)}][W]$ are of injective dimension 1, there is an injective resolution of $\overline{\mathcal{F}}$–finite torsion $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–modules

$$0 \to TM \to I' \to J' \to 0,$$

where I', J' are injective $\overline{\mathcal{F}}$–finite torsion $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–modules.

Let us use simplified notation below. Let P denote the $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–module from the definition of $e(V)$ (see Definition 3.11).

If Q is the image of γ then $J'' = P/Q$ is divisible and an $\overline{\mathcal{F}}$–finite torsion $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–module and hence injective. We form a diagram of $\mathcal{O}_{\overline{\mathcal{F}}}[W]$–modules

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TM & \longrightarrow & M & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I' & \longrightarrow & I' \oplus P & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J' & \longrightarrow & J' \oplus J'' & \longrightarrow & J'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the middle vertical column is obtained using the horseshoe lemma (see for example [23, Lemma 2.2.8]), since left and right vertical columns are injective resolutions of TM and Q, respectively. Thus we get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & I' \oplus P & \longrightarrow & J' \oplus J'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V & \longrightarrow & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

which is the required resolution of $\gamma\colon M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ in $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$.

Finally, the splitting is given by taking the even- and odd-graded parts. This satisfies the required conditions since the resolution above of an object X_δ is entirely in parity δ. □

3.3 Model category $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$

In this section we will concentrate on the model category $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ and we will investigate its properties. First notice that all constructions from the previous section (limits and colimits, adjoints F and R) pass naturally to the category $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$.

By the results of the previous section and [9, Proposition 4.1.3] we can construct the derived category of $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ by taking objects with differential in $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ and inverting the homology isomorphisms.

Theorem 3.15 *There is an injective model structure on the category $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ where weak equivalences are homology isomorphisms and cofibrations are monomorphisms.*

Proof Since the category $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ is abelian of injective dimension 1 we can use the construction from [9, Appendix A]. □

We call $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ with the injective model structure the *algebraic model for toral rational* SO(3)*–spectra*.

To show that the injective model structure is right proper in Proposition 3.19 we need to introduce a class of objects in $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ called *wide spheres*. This class generalises the images of representation spheres from rational SO(3)–spectra in $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$, hence the name.

Definition 3.16 Define $\underline{c}^{2n}$ to be an element of the form $(c^{2n}, c^{2n}, c^{2n}, \dots)$ in $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$. Notice that we can view $\underline{c}^{2n}$ as an element of the form $(d^n, c^{2n}, c^{2n}, \dots)$ in $\mathcal{O}_{\mathcal{F}}$ if $n > 0$.

For $n > 0$ define $\underline{c}^{2n+1}$ to be an element of the form $(c^{2n+1}, c^{2n+1}, c^{2n+1}, c^{2n+1}, \dots)$ in $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$.

Definition 3.17 A *wide sphere* in $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ is an object $P=(S\to\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes T)$ where T is a graded $\mathbb{Q}[W]$–module which is finitely generated as a $\mathbb{Q}$–module on elements $t_1,\dots,t_d$, where every t_i is either W–fixed or W acts on t_i by -1 and $\deg(t_i)=k_i$. The module S is an $\mathcal{O}_{\overline{\mathcal{F}}}$–submodule of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes T$ generated by elements $\underline{c}^{a_i}\otimes t_1,\dots,\underline{c}^{a_d}\otimes t_d$ where a_i is either even if t_i is W–fixed or odd if W acts on t_i by -1, and an element $\sum_{i=1}^{d}\sigma_i\otimes t_i$ of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes T$. It is also required that the structure map be the inclusion. We denote by $\mathcal{P}$ the set of isomorphism classes of wide spheres.

We want to show that there are enough wide spheres in $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$, ie for any $X\in\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ there exists an epimorphism from some coproduct of wide spheres to X.

Proposition 3.18 *There are enough wide spheres in* $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$.

Proof We need to show that for any object $X=(\beta\colon N\to\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes U)$ in $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ and any $n\in N$ there exists a wide sphere P and a map $P\to X$ such that n is in the image and for any $u\in U$ there exists a wide sphere $\overline{P}$ and a map $\overline{P}\to X$ such that u is in the image. Since the adjoint of β is an isomorphism it is enough to show the above condition for any $n\in N$.

Take $X=(\beta\colon N\to\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes U)$ in $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ and $n\in N$. Then $\beta(n)=\sum_{i=1}^{d}\sigma_i\otimes t_i$. We may assume that for every i, either t_i is W–fixed or W acts on t_i by -1. Then notice that since $e_1\beta(n)$ is W–fixed, $e_1\sigma_i$ will be of the form c^{2k} if t_i was W–fixed or c^{2k+1} if W acts on t_i by -1 (k is some integer here).

For each i, there exist $p_i\in N$ such that $\beta(p_i)=\underline{c}^{2b_i}\otimes t_i$ if t_i was W–fixed or $\beta(p_i)=\underline{c}^{2b_i+1}\otimes t_i$ if W acts on t_i by -1. Set $f=(\underline{c})^{2b_1+\cdots+2b_d}$. We may assume that the b_i were large enough that $\sigma_i\underline{c}^{2b_1+\cdots+2b_d}/\underline{c}^{2b_i}$ is in $\mathcal{O}_{\overline{\mathcal{F}}}$ if t_i was W–fixed and $\sigma_i\underline{c}^{-1}\underline{c}^{2b_1+\cdots+2b_d}/\underline{c}^{2b_i}$ is in $\mathcal{O}_{\overline{\mathcal{F}}}$ if W acts on t_i by -1.

Now we have

$$\begin{aligned}\beta\Big(\sum\nolimits^{+}\sigma_i\underline{c}^{2b_1+\cdots+2b_d}/\underline{c}^{2b_i}\,p_i+\sum\nolimits^{-}\sigma_i\underline{c}^{-1}\underline{c}^{2b_1+\cdots+2b_d}/\underline{c}^{2b_i}\,p_i\Big)&=\sum_{i=1}^{d}\sigma_i f\otimes t_i\\&=\beta(fn),\end{aligned}$$

where $\sum^{+}$ denotes the sum over all t_i which are W–fixed and $\sum^{-}$ denotes the sum over all the others.

Since the adjoint of β is an isomorphism there exists an element $\underline{c}^{2b}$ such that

$$\underline{c}^{2b}\Big(\sum\nolimits^{+}\sigma_i\underline{c}^{2b_1+\cdots+2b_d}/\underline{c}^{2b_i}\,p_i+\sum\nolimits^{-}\sigma_i\underline{c}^{-1}\underline{c}^{2b_1+\cdots+2b_d}/\underline{c}^{2b_i}\,p_i\Big)=\underline{c}^{2b}fn.$$

We take $\underline{c}^{2b}$ to be the smallest such element.

We take a wide sphere $P = (S \to \mathcal{E}^{-1}_{\mathcal{O}_\mathcal{F}} \otimes T)$ where T is a $\mathbb{Q}$–vector space generated by t_i for $i = 1, \dots, d$, $\deg(t_i) = k_i$ and S is an $\mathcal{O}_\mathcal{F}$ submodule of $\mathcal{E}^{-1}_{\mathcal{O}_\mathcal{F}} \otimes T$ generated by $\sum_{i=1}^d \sigma_i \otimes t_i$ and $\underline{c}^{2b} f \otimes t_i$ if t_i is W–fixed and $\underline{c}^{2b-1} f \otimes t_i$ if W acts on t_i by -1. The structure map is the inclusion.

To finish the proof we get a map from P to X by sending $\sum_{i=1}^d \sigma_i \otimes t_i$ to n and $\underline{c}^{2b} f \otimes t_i$ to $\underline{c}^{2b}\underline{c}^{2b_1+\dots+2b_d}/\underline{c}^{2b_i} p_i$ if t_i is W–fixed and $\underline{c}^{2b-1} f \otimes t_i$ to $\underline{c}^{2b-1}\underline{c}^{2b_1+\dots+2b_d}/\underline{c}^{2b_i} p_i$ if W acts on t_i by -1.

The elements $\underline{c}^{2b}$ and f are needed to ensure that the relation between n and the p_i's after applying β is replicated in the wide sphere. □

Proposition 3.19 *The injective model structure on $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ is proper.*

Proof Since cofibrations are the monomorphism it is left proper. To show that it is right proper, notice that among trivial cofibrations there are maps $0 \to D^n \otimes P$, for any $P \in \mathcal{P}$, where $D^n \otimes P$ denotes an object built from P and ΣP with the differential being the identity map from the suspension of P to P. Recall that $\mathcal{P}$ denotes the set of isomorphism classes of wide spheres. Since there are enough wide spheres, the fibrations are in particular surjections. Right properness follows from the fact that in $\mathbb{Q}[W]$–mod and $\mathcal{O}_\mathcal{F}[W]$–mod pullbacks along surjections of homology isomorphisms are homology isomorphisms. □

Corollary 3.20 *The category $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ is a Grothendieck category.*

Proof Directed colimits are exact in $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$, since they are in R–modules, for any ring R. Thus it remains to show that there is a (categorical) generator. We take $J := \bigoplus_{P \in \mathcal{P}} P$, where $\mathcal{P}$ is the set of all wide spheres. By Proposition 3.18, $\mathrm{Hom}(J, -)$ is faithful and thus J is a categorical generator. □

Next we define a set of objects which will be generators for the homotopy category of $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ with the injective model structure. Before we were considering categorical generators, but from now on the meaning of the word *generator* is as in Definition 2.5. Recall that if $\beta\colon M \to \mathcal{E}^{-1}\mathcal{O}_\mathcal{F} \otimes V$ is an object in $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$, then M is in particular a module over $\mathcal{O}_\mathcal{F}[W]$ (which is an infinite product over conjugacy classes of cyclic subgroups in $\mathrm{SO}(3)$; see beginning of Section 3.2).

Definition 3.21 We define a set $\mathcal{K}$ in $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ to consist of all suspensions and desuspensions of the following objects:

- For the trivial subgroup, we take
$$\sigma_1 := (\mathbb{Q}_1 \to 0),$$
where $\mathbb{Q}$ is at the place indexed by the trivial subgroup and all other factors are 0.
- For every $H \in \overline{\mathcal{F}}$, $H \neq 1$, we take
$$\sigma_H := (\mathbb{Q}[W]_{(H)} \to 0),$$
where $\mathbb{Q}[W]$ is at the place indexed by the conjugacy class of a subgroup H and all other factors are 0.
- For the torus, we take
$$\sigma_T := (M \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W]),$$
where $e_1 M = \mathbb{Q}[d] \oplus \Sigma^2\mathbb{Q}[d]$, $(1-e_1)M = (1-e_1)\mathcal{O}_{\mathcal{F}}$. Here the map is the inclusion.

It remains to show that the set of cells $\mathcal{K}$ is a set of generators for the injective model structure on $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$.

Theorem 3.22 *The set $\mathcal{K}$ is a set of homotopically compact generators for the category $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ with the injective model structure.*

Proof First note that
$$\sigma_T = (\mathcal{O}_{\overline{\mathcal{F}}} \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}) \oplus (N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \widetilde{\mathbb{Q}}),$$
where $e_1 N = \Sigma^2\mathbb{Q}[d]$ and $(1-e_1)N = (1-e_1)\mathcal{O}_{\mathcal{F}} \otimes \widetilde{\mathbb{Q}}$ (here $\widetilde{\mathbb{Q}}$ denotes $\mathbb{Q}$ with the action of w by -1), and both structure maps are inclusions. We call the first summand S^0 and the second σ_T^-. Therefore it is enough to show that all suspensions and desuspensions of σ_1, σ_H, σ_T^-, S^0 for all $H \in \mathcal{F}$, $H \neq 1$ form a set of generators. We will call this set $\mathcal{L}$.

All cells are homotopically compact since they are compact and fibrant replacement commutes with direct sums.

We will show that if $[\sigma, X]^{\mathcal{A}}_* = 0$ for all $\sigma \in \mathcal{L}$ then $H_*(X) = 0$ and thus X is weakly equivalent to 0. By Lemma 3.14, [9, Lemma 4.2.4] and [4, Theorem 3.8] we can use the following Adams short exact sequence to calculate the maps in the derived category of $\mathcal{A} = \mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ from X to Y in $d\mathcal{A}$:
$$0 \to \mathrm{Ext}_{\mathcal{A}}(\Sigma H_*(X), H_*(Y)) \to [X, Y]^{\mathcal{A}}_* \to \mathrm{Hom}_{\mathcal{A}}(H_*(X), H_*(Y)) \to 0.$$

Observe that for every $X \in d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$, where

$$X = (\gamma\colon P \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V),$$

we have the fibre sequence

$$\hat{X} \to X \to e(V),$$

where $e(V)$ is the functor described before Proposition 3.12 and $\hat{X}$ is the fibre of the map $X \to e(V)$.

By definition, the structure map of $e(V)$ is an inclusion, and thus it is a torsion–free object. To simplify the notation, let

$$E\bar{\mathcal{F}}_+ = (\Sigma^{-2}\mathbb{Q}[d, d^{-1}]/\mathbb{Q}[d] \to 0) \oplus \bigoplus_{\substack{(H)\in\bar{\mathcal{F}}\\ H\neq 1}} ((\Sigma^{-2}\mathbb{Q}[c_{(H)}, c_{(H)}^{-1}]/\mathbb{Q}[c_{(H)}]) \to 0).$$

We call the H–summand in the above formula α_H. Then

$$\hat{X} \simeq E\bar{\mathcal{F}}_+ \otimes X.$$

Now observe that every summand α_H in $E\bar{\mathcal{F}}_+$ is built as a sequential colimit from suspensions of $\alpha_H^n = (\mathbb{Q}[c_{(H)}]/c_{(H)}{}^n \to 0)$ and inclusions, or if it is the first summand α_1 it is built as a sequential colimit of $\alpha_1^n = (\mathbb{Q}[d]/d^n \to 0)$ and inclusions, and thus

$$[\sigma_K, \hat{X}]_*^{\mathcal{A}} = [\sigma_K, E\bar{\mathcal{F}}_+ \otimes X]_*^{\mathcal{A}} \cong \Big[\sigma_K, \bigoplus_{(H)} (\alpha_H \otimes X)\Big]_*^{\mathcal{A}} \cong \bigoplus_i [\sigma_K, \alpha_H \otimes X]_*^{\mathcal{A}},$$

where the last isomorphism follows since σ_K is a homotopically compact object. For all H, α_H^n is a strongly dualisable object (by [9, Corollary 2.3.7 and Lemma 2.4.3]), and thus we can proceed:

$$\begin{aligned} [\sigma_K, \alpha_H \otimes X]_*^{\mathcal{A}} &\cong [\sigma_K, \mathrm{colim}_n\, \alpha_H^n \otimes X]_*^{\mathcal{A}} \\ &\cong \mathrm{colim}_i [\sigma_K, \mathrm{Hom}(D(\alpha_H^n), X)]_*^{\mathcal{A}} \\ &\cong \mathrm{colim}_i [D(\alpha_H^n) \otimes \sigma_K, X], \end{aligned} \tag{3-2}$$

since $D(\alpha_H^n) \otimes \sigma_K = 0$ if $K \neq H$ and every $D(\alpha_H^n) \otimes \sigma_H$ is finitely built from σ_H and by assumption $[\sigma, X] = 0$ for all $\sigma \in \mathcal{L}$, we have that $[D(\alpha_H^n) \otimes \sigma_H, X] = 0$ and thus $[\sigma_H, \hat{X}]_*^{\mathcal{A}} = 0$ for all $H \in \bar{\mathcal{F}}$.

Now take X to be an object in $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ and assume that $[\sigma, X]_*^{\mathcal{A}} = 0$ for all $\sigma \in \mathcal{L}$. By the calculation above it follows that $[\sigma_H, \hat{X}]_*^{\mathcal{A}} = 0$ for all $H \in \mathcal{F}$.

From the Adams short exact sequence we get that

$$\mathrm{Hom}_{\mathcal{A}}(H_*(\sigma_H), H_*(\hat{X})) = \mathrm{Hom}_{\mathcal{A}}(\sigma_H, H_*(\hat{X})) = e_{(H)} H_*(\hat{X}) = 0.$$

Since $H_*(\widehat{X}) = \bigoplus_{(H)\in\overline{\mathcal{F}}} e_H H_*(\widehat{X})$ we conclude that $\widehat{X}$ is weakly equivalent to 0 and thus $[S^0, \widehat{X}]^{\mathcal{A}}_* = 0$ and $[\sigma_T^-, \widehat{X}]^{\mathcal{A}}_* = 0$.

Now, by the fibre sequence and the fact that every fibre sequence induces a long exact sequence on $[E,-]$ we deduce that $[\sigma, e(V)]^{\mathcal{A}}_* = 0$ for every $\sigma \in \mathcal{L}$. From the Adams short exact sequence it follows that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(H_*(S^0), H_*(e(V))) &= \mathrm{Hom}_{\mathcal{A}}(S^0, H_*(e(V))) = H_*^+(e(V)) = 0,\\ \mathrm{Hom}_{\mathcal{A}}(H_*(\sigma_T^-), H_*(e(V))) &= \mathrm{Hom}_{\mathcal{A}}(\sigma_T^-, H_*(e(V)) = H_*^-(e(V)) = 0,\end{aligned}$$

where $H_*^+(e(V))$ is the W–fixed part of $H_*(e(V))$ and $H_*^-(e(V))$ denotes the -1 eigenspace. Since $H_*(e(V)) = H_*^+(e(V)) \oplus H_*^-(e(V))$ we get that $e(V)$ is weakly equivalent to 0. Since the fibre sequence induces a long exact sequence in homology we conclude that $H_*(X) = 0$ and thus X is weakly equivalent to 0, which finishes the proof. □

We finish this section by relating $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ and $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$.

Lemma 3.23 *The adjunction*

$$d\mathcal{A}(\mathrm{SO}(3), \mathcal{T}) \underset{R}{\overset{F}{\rightleftarrows}} d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$$

is a Quillen pair when we equip both categories with the injective model structures, where F and R are defined as in the proof of Proposition 3.9.

Proof The left adjoint is exact, so it preserves cofibrations (monomorphisms) and homology isomorphisms. □

Theorem 3.24 *The adjunction*

$$d\mathcal{A}(\mathrm{SO}(3), \mathcal{T}) \underset{R}{\overset{F}{\rightleftarrows}} F(\mathcal{K})\text{–cell–}d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$$

is a Quillen equivalence, where $\mathcal{K}$ is given in Definition 3.21.

Proof We cellularise the left-hand side of the adjunction in Lemma 3.23 at the set $\mathcal{K}$ and the right one at $F(\mathcal{K})$. The left-hand side is then just $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ by Theorem 3.22. Thus to use the cellularisation principle [12, Theorem 2.1] we need to prove that the derived unit is an isomorphism for every element of $\mathcal{K}$. Since the right adjoint preserves all weak equivalences it is enough to show that the categorical unit is a weak equivalence. However, we already know that the unit of this adjunction is the identity (it was shown in the proof of Proposition 3.9). It remains to show that the elements of the set $F(\mathcal{K})$ are homotopically compact in $d\mathcal{A}(O(2),\widetilde{\mathcal{T}})$ with the injective model structure. This follows from the fact that R preserves coproducts (notice that one component of R is taking W–fixed points and over $\mathbb{Q}$ this is isomorphic to taking W–orbits; the other components of R are identities). This finishes the proof. □

In the next section we will compare the cells coming from the topological generators (see Proposition 3.27) with the ones used for cellularising $d\mathcal{A}(O(2),\widetilde{\mathcal{T}})$. For these two sets of cells to agree we now change the set of cells used for cellularising $d\mathcal{A}(O(2),\widetilde{\mathcal{T}})$. We introduce the following Quillen self-equivalence (which is also an equivalence of categories) of $d\mathcal{A}(O(2),\widetilde{\mathcal{T}})$ with the injective model structure. Use the notation $\widetilde{\mathbb{Q}}$ for the $\mathbb{Q}[W]$–module $\mathbb{Q}$ with nontrivial W–action. We denote by $-\otimes\widetilde{\mathbb{Q}}$ a self-adjoint functor on $d\mathcal{A}(O(2),\widetilde{\mathcal{T}})$ defined as

$$-\otimes\widetilde{\mathbb{Q}}(\beta\colon M\to\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes V):=(\beta\otimes\widetilde{\mathbb{Q}}\colon M\otimes\widetilde{\mathbb{Q}}\to\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes(V\otimes\widetilde{\mathbb{Q}}).$$

Thus, below, we use the notation $\widetilde{F}$ to denote $-\otimes\widetilde{\mathbb{Q}}\circ F$ and $\widetilde{R}$ to denote $R\circ-\otimes\widetilde{\mathbb{Q}}$.

The final result of this section follows from Theorem 3.24.

Corollary 3.25 *The following is a Quillen equivalence, where $\mathcal{K}$ is as in Definition 3.21 and $d\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ is considered with the injective model structure:*

$$d\mathcal{A}(\mathrm{SO}(3),\mathcal{T})\underset{\widetilde{R}}{\overset{\widetilde{F}}{\rightleftarrows}}\widetilde{F}(\mathcal{K})\text{–cell–}d\mathcal{A}(O(2),\widetilde{\mathcal{T}}).$$

Remark 3.26 Let us calculate the cells from $\widetilde{F}(\mathcal{K})$ (ignoring suspensions as they work in the same way in both categories):

$$\widetilde{F}(\sigma_1)=\widetilde{F}(\mathbb{Q}_1\to 0)=\widetilde{\mathbb{Q}}\oplus\Sigma^{-2}\mathbb{Q}\to 0,$$

where c sends $\widetilde{\mathbb{Q}}$ to $\mathbb{Q}$ (both copies of $\mathbb{Q}$ are in the place corresponding to the trivial subgroup) and

$$\widetilde{F}(\sigma_{(H)})=\widetilde{F}(\mathbb{Q}[W]_{(H)}\to 0)=\mathbb{Q}[W]_{H},\to 0$$

where the left $\mathbb{Q}[W]$ is in the place corresponding to (H) and the resulting $\mathbb{Q}[W]$ is in the place corresponding to H. This holds for all $(H)\in\overline{\mathcal{F}}$ except for $H=1$. For the torus we have

$$\widetilde{F}(\sigma_{(T)})=\widetilde{F}(M\to\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes\mathbb{Q}[W])=\Sigma^2\widetilde{\mathbb{Q}}+\mathcal{O}_{\mathcal{F}}\otimes\mathbb{Q}[W]\to\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes\mathbb{Q}[W],$$

where c acts on $\widetilde{\mathbb{Q}}$ in degree 2 ($\widetilde{\mathbb{Q}}$ is in the place corresponding to the trivial subgroup) by sending it to $\mathbb{Q}\subseteq\mathbb{Q}[W]$ in degree 0 and the map is the inclusion.

3.4 Restriction to the toral part of rational $O(2)$–spectra

The idea for the comparison is to restrict the toral part of rational SO(3)–spectra to the toral part of rational $O(2)$–spectra using the functor i^* as a left adjoint. Recall that the adjunction $(\mathrm{SO}(3)_+ \wedge_{O(2)} -, i^*)$ is not a Quillen pair for the model categories localised at the idempotents corresponding to the toral parts; see Proposition 2.7.

We use the proof from [4] giving an algebraic model for the toral part of rational $O(2)$–spectra, cellularising every step of the zig-zag of Quillen equivalences presented there. This way we obtain an algebraic model for the toral part of rational $O(2)$–spectra cellularised at the derived images of generators for $L_{e_{\mathcal{T}} S_{\mathbb{Q}}}(\mathrm{SO}(3)\mathrm{-Sp})$. This gives an algebraic model; however, it is not very explicit. We finish this section by simplifying this category in Theorem 3.35 and showing that it is Quillen equivalent to $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ with the injective model structure.

We start by establishing generators for the toral part of rational SO(3)–spectra. We used the notation $\mathcal{K}$ in Definition 3.21 for the generators on the algebraic side. We will use the notation K for the generators on the topological side. We will end this section by showing that the derived images of the topological generators $\mathrm{im}(K)$ are precisely the algebraic generators $\mathcal{K}$ in $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$.

Proposition 3.27 *A set K consisting of all suspensions and desuspensions of one* SO(3)*–spectrum*

$$\sigma_n = \mathrm{SO}(3)_+ \wedge_{C_n} e_{C_n} S^0$$

for every natural $n > 0$ and an SO(3)*–spectrum* $\mathrm{SO}(3)/\mathrm{SO}(2)_+$ *is a set of cofibrant homotopically compact generators for the category* $L_{e_{\mathcal{T}} S_{\mathbb{Q}}}(\mathrm{SO}(3)\mathrm{-Sp})$.

Proof First consider a set L consisting of all suspensions and desuspensions of one SO(3)–spectrum $\mathrm{SO}(3)/C_{n+}$ for every natural $n > 0$ and an SO(3)–spectrum $\mathrm{SO}(3)/\mathrm{SO}(2)_+$. All objects in L are homotopically compact in $L_{e_{\mathcal{T}} S_{\mathbb{Q}}}(\mathrm{SO}(3)\mathrm{-Sp})$ since they are in $\mathrm{SO}(3)\mathrm{-Sp}$ and fibrant replacement in $L_{e_{\mathcal{T}} S_{\mathbb{Q}}}(\mathrm{SO}(3)\mathrm{-Sp})$ commutes with coproducts. This is a set of generators for $L_{e_{\mathcal{T}} S_{\mathbb{Q}}}(\mathrm{SO}(3)\mathrm{-Sp})$ by [18, Chapter IV, Proposition 6.7]. Since

$$\mathrm{SO}(3)/C_{n+} = \bigvee_{C_m \subseteq C_n} \sigma_m,$$

which is a consequence of [9, Lemma 2.1.5], the set K is a set of homotopically compact generators for $L_{e_{\mathcal{T}} S_{\mathbb{Q}}}(\mathrm{SO}(3)\mathrm{-Sp})$. □

Next we restrict to the toral part of rational $O(2)$–spectra.

Theorem 3.28 *The adjunction*

$$i^*\colon L_{e_{\mathcal{T}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–}\mathrm{Sp}) \rightleftarrows i^*(K)\text{–cell–}L_{e_{\tilde{\mathcal{T}}}S_{\mathbb{Q}}}(O(2)\text{–}\mathrm{Sp}) \;:F_{O(2)}(\mathrm{SO}(3)_+,-)$$

is a Quillen equivalence, where the idempotent on the right-hand side corresponds to the family of all cyclic subgroups of $O(2)$.

Proof The fact that this is a Quillen adjunction follows from Proposition 2.12 and the cellularisation principle [12, Theorem 2.1] for K and $i^*(K)$. Since K was a set of generators for the category $L_{e_{\mathcal{T}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–}\mathrm{Sp})$, the cellularisation with respect to K will not change this model structure.

All cells from K are homotopically compact and cofibrant by Proposition 3.27. We need to check that their images with respect to i^* are homotopically compact in $L_{e_{\tilde{\mathcal{T}}}S_{\mathbb{Q}}}(O(2)\text{–}\mathrm{Sp})$, ie suspension spectra of $\mathrm{SO}(3)/C_{n+}$ for all n and $\mathrm{SO}(3)/\mathrm{SO}(2)_+$ as toral $O(2)$–spectra. It is enough to show that they are homotopically compact as $O(2)$–spectra, which follows from the fact that a smooth, compact G–manifold admits a structure of a finite G–CW complex [15, Theorem I] and a suspension spectrum of a finite G–CW complex is homotopically compact. It thus follows that the images of the summands σ_n are also homotopically compact and cofibrant in $L_{e_{\tilde{\mathcal{T}}}S_{\mathbb{Q}}}(O(2)\text{–}\mathrm{Sp})$.

It remains to show that the components of the derived unit maps at generators are weak equivalences. For this, it is enough to check the induced map on the level of homotopy categories. This is equivalent to showing that the derived functor Li^* is an isomorphism on hom-sets. This holds by [10, Theorem 6.1], which states that if $X \cong e_{\mathcal{T}}X$ then Li^* is an isomorphism

$$[X,Y]^{\mathrm{SO}(3)} \to e_{\tilde{\mathcal{T}}}[i^*X, i^*Y]^{O(2)},$$

which implies that

$$\begin{aligned} Li^*\colon [X,Y]^{L_{e_{\mathcal{T}}}\mathrm{SO}(3)} &\cong [e_{\mathcal{T}}X, e_{\mathcal{T}}Y]^{\mathrm{SO}(3)} \\ &\to e_{\tilde{\mathcal{T}}}[i^*(e_{\mathcal{T}}X), i^*(e_{\mathcal{T}}Y)]^{O(2)} \cong [i^*X, i^*Y]^{L_{e_{\tilde{\mathcal{T}}}}O(2)} \end{aligned}$$

is an isomorphism, where the superscript $L_{e_{\mathcal{T}}}\mathrm{SO}(3)$ was used to mean the homotopy category of $L_{e_{\mathcal{T}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–}\mathrm{Sp})$. Similarly, the superscript $L_{e_{\tilde{\mathcal{T}}}}O(2)$ was used to mean the homotopy category of $L_{e_{\tilde{\mathcal{T}}}S_{\mathbb{Q}}}(O(2)\text{–}\mathrm{Sp})$. Thus the adjunction is a Quillen equivalence. □

Remark 3.29 The result above generalises to any compact Lie group G. The restriction–coinduction adjunction is a Quillen equivalence between the toral part of rational G–spectra and a certain cellularisation of the toral part of rational N–spectra, where N is the normaliser of the maximal torus in G. This is used in [5] to provide an algebraic model for the toral part of rational G–spectra for any compact Lie group G.

Since the Quillen equivalence above provides a link between the toral part of rational SO(3)–spectra and the toral part of rational $O(2)$–spectra we use the result of [4].

Theorem 3.30 [4, Corollary 4.22] *There is a zig-zag of Quillen equivalences between $L_{e_{\widetilde{\mathcal{T}}} S_{\mathbb{Q}}}(O(2)\text{–Sp})$ and $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$, where $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$ is considered with the dualisable model structure.*

To provide an algebraic model for rational SO(3)–spectra we need to cellularise every step of the zig-zag from [4, Section 4] with respect to derived images of $i^*(K)$ from Theorem 3.28. Cellularisation preserves Quillen equivalences and gives the following result.

Theorem 3.31 *There is a zig-zag of Quillen equivalences between $L_{e_{\mathcal{T}} S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–Sp})$ and $\mathrm{im}(K)\text{–cell–}d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$, where $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$ is considered with the dualisable model structure. Here $\mathrm{im}(K)$ denotes the derived image under the zig-zag of Quillen equivalences described in* [4, Section 4] *of the set of cells K described in Proposition 3.27.*

Theorem 3.31 already gives an algebraic model for the toral part of rational SO(3)–spectra. However, it is not easy to work with a cellularisation of a model category. Thus we show that the model above is Quillen equivalent to the simpler, algebraic category $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ described in Section 3.2. To do this, we first switch to the cellularisation of the injective model structure.

Lemma 3.32 *The identity adjunction between*

$$\mathrm{im}(K)\text{–cell–}d\mathcal{A}(O(2), \widetilde{\mathcal{T}}) \quad \text{and} \quad \mathrm{im}(K)\text{–cell–}d\mathcal{A}(O(2), \widetilde{\mathcal{T}}),$$

where one $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$ is equipped with the dualisable model structure and the other is equipped with the injective model structure, is a Quillen equivalence.

Proof The result follows from the fact that the identity adjunction was a Quillen equivalence between $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$ with the dualisable model structure and $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$ with the injective model structure. □

Lemma 3.33 *The set of homology of elements of $\mathrm{im}(K)$ consists of the same objects as $\widetilde{F}(\mathcal{K})$, where $\mathcal{K}$ is the set described in Definition 3.21 and $\mathrm{im}(K)$ denotes the derived image under the zig-zag of Quillen equivalences described in* [4, Section 4] *of the set of cells K described in Proposition 3.27.*

Proof First we show that, for every $n>1$, σ_n is weakly equivalent in $L_{e_{\widetilde{\mathfrak{T}}}S_{\mathbb{Q}}}(O(2)\text{–Sp})$ to $O(2)\wedge_{C_n} e_{C_n}S^0$. The map is induced by the inclusion of $O(2)$ into $\mathrm{SO}(3)$ and we will show that it induces an isomorphism on all π_*^H for $H\in\widetilde{\mathfrak{T}}$. We will use the notation $N=O(2)$ and $G=\mathrm{SO}(3)$ below. We have

$$\begin{aligned}\pi_*^H(N\wedge_{C_n} e_{C_n}S^0) &= [N/H_+, F_{C_n}(N_+, S^{L_N(C_n)}\wedge e_{C_n}S^0)]^N\\ &= [N/H_+, S^{L_N(C_n)}\wedge e_{C_n}S^0]^{C_n}.\end{aligned}$$

Here $L_N(C_n)$ is the tangent C_n–representation at the identity coset of N/C_n and thus is the 1–dimensional trivial representation. Since the codomain has only geometric fixed points for $H=C_n$ we get a nonzero result only for $H=C_n$:

$$[\Phi^{C_n}(N/C_{n+}), \Phi^{C_n}(S^{L_N(C_n)})] = [S^1\vee S^1, S^1] = \Sigma(\mathbb{Q}[W]).$$

Similarly we have

$$\begin{aligned}\pi_*^H(G\wedge_{C_n} e_{C_n}S^0) &= [G/H_+, F_{C_n}(G_+, S^{L_G(C_n)}\wedge e_{C_n}S^0)]^G\\ &= [G/H_+, S^{L_G(C_n)}\wedge e_{C_n}S^0]^{C_n},\end{aligned}$$

and since the codomain has only geometric fixed points for $H=C_n$ we get a nonzero result only for $H=C_n$:

$$[\Phi^{C_n}(G/C_{n+}), \Phi^{C_n}(S^{L_G(C_n)})] = [S^1\vee S^1, S^1] = \Sigma(\mathbb{Q}[W]).$$

Notice that $L_G(C_n)$ is 3–dimensional, but it has a 1–dimensional C_n–fixed subspace.

The images of the cells in $\mathcal{A}(O(2),\widetilde{\mathfrak{T}})$ are therefore

$$\mathrm{im}(G\wedge_{C_n} e_{C_n}S^0) = \mathrm{im}(N\wedge_{C_n} e_{C_n}S^0) = (\Sigma\mathbb{Q}[W]_{C_n}\to 0)$$

by [9, Example 5.8.1], where $\Sigma\mathbb{Q}[W]$ is in the place C_n.

Now we will use the functors $\pi_*^{\mathcal{A}}$ described in [9, Theorem 5.6.1 and Lemma 5.6.2]. Since $\mathrm{SO}(3)_+$ is free we get

$$\begin{aligned}\pi_*^{\mathcal{A}}(\mathrm{SO}(3)_+) &= (\pi_*^T(\mathrm{SO}(3)_+)\to 0)\\ &= (\pi_*(\Sigma\,\mathrm{SO}(3)/T_+)\to 0)\\ &= (\pi_*(\Sigma S(\mathbb{R}^3)_+)\to 0)\\ &= (\Sigma^3\widetilde{\mathbb{Q}}\oplus\Sigma\mathbb{Q}\to 0),\end{aligned}$$

where $\Sigma^3\widetilde{\mathbb{Q}}\oplus\Sigma\mathbb{Q}$ is in the place corresponding to the trivial subgroup 1 and c sends $\widetilde{\mathbb{Q}}$ in degree 3 to $\mathbb{Q}$ in degree 1.

Finally, $\mathrm{SO}(3)/T_+ = S(\mathbb{R}^3)_+$ is built as an $O(2)$–space from the cells

$$N/T_+ \vee N/D_{2+} \cup N_+ \wedge e^1.$$

Thus the cofibre sequence

$$N_+ \to N/T_+ \vee N/D_{2+} \to G/T_+$$

gives the long exact sequence

$$\cdots \to (\Sigma\mathbb{Q}[W] \to 0) \to (\mathcal{O}_{\mathcal{F}}[W] \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W]) \oplus (\Sigma\mathbb{Q} \to 0) \to \mathrm{im}(G/T_+) \to \cdots$$

and hence

$$\mathrm{im}(G/T_+) = \Sigma^2\widetilde{\mathbb{Q}} + \mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W] \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W],$$

where c acts on $\widetilde{\mathbb{Q}}$ in degree 2 ($\widetilde{\mathbb{Q}}$ is in the place corresponding to the trivial subgroup) by sending it to $\mathbb{Q} \subseteq \mathbb{Q}[W]$ in degree 0 and the map is the inclusion.

These images are exactly the cells (up to suspension) in $\widetilde{F}(\mathcal{K})$ (see Remark 3.26), which finishes the proof. □

Remark 3.34 It remains to show that the derived images in $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$ of generators described in Definition 3.21 are formal, that is, they are weakly equivalent to their homology in $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$. We claim it's clear for $(\Sigma\mathbb{Q}[W]_{C_n} \to 0)$, where $\Sigma\mathbb{Q}[W]$ is in the place C_n. It is also clear for $(\Sigma^3\widetilde{\mathbb{Q}} \oplus \Sigma\mathbb{Q} \to 0)$, where $\Sigma^3\widetilde{\mathbb{Q}} \oplus \Sigma\mathbb{Q}$ is in the place corresponding to the trivial subgroup 1 and c sends $\widetilde{\mathbb{Q}}$ in degree 3 to $\mathbb{Q}$ in degree 1.

To show that $A = (\Sigma^2\widetilde{\mathbb{Q}} + \mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W] \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W])$ is formal (where c acts on $\widetilde{\mathbb{Q}}$ in degree 2 ($\widetilde{\mathbb{Q}}$ is in the place corresponding to the trivial subgroup) by sending it to $\mathbb{Q} \subseteq \mathbb{Q}[W]$ in degree 0 and the structure map is the inclusion) we proceed as follows. Suppose $X = (N \to \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V) \in d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$ such that $H_*(X) \cong A$. We want to construct a map $A \to X$ in $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$ which is a weak equivalence. We proceed in two parts, using the fact that $\mathbb{Q}[W] \cong \mathbb{Q} \oplus \widetilde{\mathbb{Q}}$ and a $\mathbb{Q}[W]$–map from $\mathbb{Q}[W]$ is determined by the image of $1 \in \mathbb{Q}$ and the image of $1 \in \widetilde{\mathbb{Q}}$.

First, we choose an anti-fixed cycle x in e_1N representing 1 in $\Sigma^2\widetilde{\mathbb{Q}}$. This determines $c(x) \in \mathbb{Q}$ which represents 1 in homology of e_1N (it also determines all higher powers of c applied to x). Now we choose a fixed cycle $\bar{x} \in (1-e_1)N$ in degree 0 representing $\bar{1}$ in homology (where $\bar{1}$ is 1 on all places of the infinite product except the first one, where it's 0); $\bar{x}$ is fixed by W. It follows that $(c(x), \bar{x})$ is a cycle in N representing constant (and fixed by W) 1 in $H_0(N)$. The element $(c(x), \bar{x})$ maps to an element $1 \otimes b \in \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$, which represents $1 \otimes 1$ in degree 0 of $H_*(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V) \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes H_*(V) \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W]$.

Second, we choose an anti-fixed cycle y in N in degree 0 representing a constant element 1 in $H_0(N)$ which is W–anti-fixed. Element y maps into an element $1 \otimes k \in \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ representing the anti-fixed $1 \otimes 1$ in degree 0 of $H_*(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V) \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes H_*(V) \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W]$. The choices of $x, \overline{x}$ and y determine a map in $d\mathcal{A}(O(2), \widetilde{\mathcal{T}})$ which is clearly a homology isomorphism.

Theorem 3.35 *The adjunction*

$$\widetilde{F}\colon \mathrm{d}\mathcal{A}(\mathrm{SO}(3), \mathcal{T}) \rightleftarrows \mathrm{im}(K)\text{–cell–}d\mathcal{A}(O(2), \widetilde{\mathcal{T}}) :\widetilde{R}$$

defined after Theorem 3.24 is a Quillen equivalence, where both categories (before cellularisation on the right) are equipped with the injective model structure. Here $\mathrm{im}(K)$ *denotes the derived image under the zig-zag of Quillen equivalences described in* [4, Section 4] *of the set of cells* K *described in Proposition 3.27.*

Proof It is enough to show that $\mathrm{im}(K)$ consists of the same objects (up to a weak equivalence) as $\widetilde{F}(\mathcal{K})$, where $\mathcal{K}$ is the set described in Definition 3.21, which we established in Lemma 3.33 and Remark 3.34. The result follows then from Corollary 3.25. □

We summarise the results of this section.

Theorem 3.36 *There is a zig-zag of Quillen equivalences between* $L_{e_{\mathcal{T}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–Sp})$ *and* $d\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$.

4 The dihedral part

The algebraic model for the dihedral part of rational SO(3)–spectra is almost identical to the algebraic model of the dihedral part of rational $O(2)$–spectra presented in [4, Section 5]. The difference comes from two things. First, in SO(3) every dihedral subgroup of order 2, namely D_2, is conjugate to cyclic subgroups C_2 and thus is already taken into account in the toral part. Second, the normaliser of D_4 in SO(3) is a subgroup Σ_4. For those reasons we exclude subgroups conjugate to D_2 and subgroups conjugate to D_4 from the dihedral part $\mathcal{D}$. Excluding D_2 and D_4 from the dihedral part $\mathcal{D}$ allows us to deduce that the information captured by subgroups of SO(3) that are in $\mathcal{D}$ is the same as that captured by subgroups of $O(2)$ that are in $\widetilde{\mathcal{D}} \setminus \{D_2, D_4\}$; see Proposition 4.8. This leads to the reduction of the dihedral part of rational SO(3)–spectra to the (part of the) dihedral part of rational $O(2)$–spectra in Theorem 4.9.

We know from [10] that the model for the homotopy category of the dihedral part of rational SO(3)–spectra is of the form of certain sheaves over an orbit space for $\mathcal{D}$, denoted further by $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$. Section 4.1 discusses this category as well as the category of chain complexes in $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$; $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D}))$. In Section 4.2 we present the comparison between the dihedral part of rational SO(3)–spectra and its algebraic model $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D}))$.

4.1 Categories $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$ and $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D}))$

First we recall the construction of $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$ (see also [10]), then we present the model structure on $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D}))$ and recall a set of homotopically compact generators for this model category.

Material in this section is based on [4, Section 5.1]. There is a slight difference between the definition of $\mathcal{A}(O(2),\widetilde{\mathcal{D}})$ presented there ($\mathcal{A}(O(2),\mathcal{D})$ is the notation used in [4] for this category) and $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$ below, namely we start indexing modules from $k=3$, which corresponds to $D_6=D_{2k}$. Indexing in [4] starts from 1.

Let W be the group of order two.

Definition 4.1 Define a category $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$ as follows.

An object M consists of a $\mathbb{Q}$–module M_∞, a collection $M_k\in\mathbb{Q}[W]$–mod for $k>2$ and a map (called the germ map) of $\mathbb{Q}[W]$–modules $\sigma_M\colon M_\infty\to\operatorname{colim}_{n>2}\prod_{k\geqslant n}M_k$, where the W–action on M_∞ is trivial.

A map $f\colon M\to N$ in $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$ consists of a map $f_\infty\colon M_\infty\to N_\infty$ of $\mathbb{Q}$–modules and a collection of maps of $\mathbb{Q}[W]$–modules $f_k\colon M_k\to N_k$ which commute with germ maps σ_M and σ_N:

$$\begin{array}{ccc} M_\infty & \xrightarrow{\sigma_M} & \operatorname{colim}_{n>2}\prod_{k\geqslant n}M_k \\ {\scriptstyle f_\infty}\downarrow & & \downarrow{\scriptstyle \operatorname{colim}_{n>2}\prod_{k\geqslant n}f_k} \\ N_\infty & \xrightarrow{\sigma_N} & \operatorname{colim}_{n>2}\prod_{k\geqslant n}N_k \end{array}$$

Definition 4.2 Define $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D}))$ to be the category of chain complexes in $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$ and $g\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$ to be the category of graded objects in $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$.

An object M of $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D}))$ consists of a rational chain complex M_∞, a collection of chain complexes of $\mathbb{Q}[W]$–modules M_k for $k>2$ and a germ map of chain complexes of $\mathbb{Q}[W]$–modules $\sigma_M\colon M_\infty\to\operatorname{colim}_{n>2}\prod_{k\geqslant n}M_k$, where the W–action on M_∞ is trivial.

Note that we used a chain complex notation here, unlike for the toral part, where we used $d\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$ to mean differential objects in $\mathcal{A}(\mathrm{SO}(3),\mathcal{T})$. The difference between these two is that $\mathcal{A}(\mathrm{SO}(3), \mathcal{D})$ is not a graded category, and we introduce a grading taking chain complexes in $\mathcal{A}(\mathrm{SO}(3), \mathcal{D})$. On the other hand, $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$ is already graded, and we are interested in differential objects in $\mathcal{A}(\mathrm{SO}(3), \mathcal{T})$.

Remark 4.3 Since the only difference between our definition of $\mathcal{A}(\mathrm{SO}(3), \mathcal{D})$ and the one for $\mathcal{A}(O(2), \widetilde{\mathcal{D}})$ lies in index k, all constructions for $\mathcal{A}(\mathrm{SO}(3), \mathcal{D})$ are analogous to the ones for $\mathcal{A}(O(2), \widetilde{\mathcal{D}})$ presented in [4].

It is helpful to consider several adjoint pairs involving the category $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3), \mathcal{D}))$. They are used to get a model structure on $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3), \mathcal{D}))$.

Definition 4.4 [4, Definition 5.9] Let $A \in \mathsf{Ch}(\mathbb{Q})$, $X \in \mathsf{Ch}(\mathbb{Q}[W])$ and $M \in \mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3), \mathcal{D}))$. For a natural number $k > 2$ we define the following functors:

- $i_k \colon \mathsf{Ch}(\mathbb{Q}[W]) \to \mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3), \mathcal{D}))$, given by $(i_k(X))_\infty = 0$ and $(i_k(X))_n = 0$ for $n \neq k$ and $(i_k(X))_k = X$.
- $p_k \colon \mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3), \mathcal{D})) \to \mathsf{Ch}(\mathbb{Q}[W])$, given by $p_k(M) = M_k$.
- $c \colon \mathsf{Ch}(\mathbb{Q}) \to \mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3), \mathcal{D}))$, given by $(cA)_k = A$, $(cA)_\infty = A$, and where σ_{cA} is the diagonal map into the product.

Then (i_k, p_k), (p_k, i_k) and $(c, \boxplus^W)$ are adjoint pairs, where the functor $\boxplus^W$ is given in [4, Definition 5.6].

The category $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3), \mathcal{D}))$ is bicomplete by [4, Lemma 5.7] so we can proceed to define a model structure on it.

Proposition 4.5 [4, Proposition 5.10] *There exists a model structure on the category* $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3), \mathcal{D}))$ *where f is a weak equivalence or fibration if f_∞ and each of the f_k are weak equivalences or fibrations, respectively. This model structure is cofibrantly generated and proper.*

We call this model structure the *projective model structure* on $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3), \mathcal{D}))$. By [4, Proposition 5.10] the generating cofibrations are of the form $cI_{\mathbb{Q}}$ and $i_k I_{\mathbb{Q}[W]}$ for $k \geqslant 3$ and generating acyclic cofibrations are of the form $cJ_{\mathbb{Q}}$ and $i_k J_{\mathbb{Q}[W]}$ for $k \geqslant 3$. Here $I_{\mathbb{Q}}$ and $J_{\mathbb{Q}}$ denote generating cofibrations and generating trivial cofibrations, respectively, for the projective model structure on $\mathsf{Ch}(\mathbb{Q})$, and $I_{\mathbb{Q}[W]}$, $J_{\mathbb{Q}[W]}$ denote generating cofibrations and generating trivial cofibrations, respectively, for the projective model structure on $\mathsf{Ch}(\mathbb{Q}[W])$ (for details see [14, Definition 2.3.3]).

We finish this section by giving a set of homotopically compact generators (recall Definitions 2.5 and 2.4) for $\mathcal{A}(\mathrm{SO}(3), \mathcal{D})$.

Lemma 4.6 [4, Lemma 5.11] *The set of objects $\mathcal{G}_a$ consisting of $i_k\mathbb{Q}[W]$ for $k \geqslant 3$ and $c\mathbb{Q}$ is a set of homotopically compact, cofibrant and fibrant generators for the category $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D}))$ with the projective model structure.*

4.2 Comparison

First we give homotopically compact, cofibrant generators for $L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp})$. We stick to the convention of writing e_H for $e_{(H)_{\mathrm{SO}(3)}}$.

Lemma 4.7 *The set*

$$\widehat{\mathcal{G}} := \{\mathrm{SO}(3)/O(2)_+\} \cup \{e_{D_{2n}}\mathrm{SO}(3)/D_{2n+} \mid n > 2\}$$

is a set of homotopically compact, cofibrant generators for $L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp})$.

Proof The proof is the same as the proof of [4, Lemma 5.14]. □

To finish the discussion about generators, we show that the restriction functor

$$i^*\colon L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp}) \to L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2){-}\mathrm{Sp})$$

preserves generators up to weak equivalence.

Proposition 4.8 *Recall that $i^*(e_{\mathcal{D}})$ is the idempotent in $\mathsf{A}(O(2))_{\mathbb{Q}}$ corresponding to the characteristic function on subgroups D_{2n} for $n > 2$ and $O(2)$.*

1. *The map $f\colon O(2)/O(2)_+ \to i^*(\mathrm{SO}(3)/O(2)_+)$ induced by inclusion $O(2) \to \mathrm{SO}(3)$ is a weak equivalence in $L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2){-}\mathrm{Sp})$.*
2. *The map $f_{2n}\colon e_{D_{2n}}O(2)/D_{2n+} \to i^*(e_{D_{2n}}\,\mathrm{SO}(3)/D_{2n+})$ for $n > 2$ induced by the inclusion $O(2) \to \mathrm{SO}(3)$ is a weak equivalence in $L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2){-}\mathrm{Sp})$.*

Proof To show that the map $f\colon O(2)/O(2)_+ \to i^*(\mathrm{SO}(3)/O(2)_+)$ is a weak equivalence in the given model structure, we need to show that $i^*(e_{\mathcal{D}})f$ is an equivariant rational π_*–isomorphism. Thus we need to check that for all subgroups $H \leq O(2)$ the H–geometric fixed points map

$$\Phi^H(i^*(e_{\mathcal{D}})f)\colon \Phi^H(i^*(e_{\mathcal{D}})O(2)/O(2)_+) \to \Phi^H(i^*(e_{\mathcal{D}})i^*(\mathrm{SO}(3)/O(2)_+))$$

is a nonequivariant rational π_*–isomorphism.

Since taking geometric fixed points commutes with smash product and suspensions, for every subgroup $H \notin (\widetilde{\mathcal{D}} \setminus \{D_2, D_4\})$, $\Phi^H(i^*(e_{\mathcal{D}})f)$ is a trivial map between trivial objects. For $H = O(2)$ the map is an identity on S^0 since $O(2)$ is its own normaliser in $\mathrm{SO}(3)$. For other $H \in (\widetilde{\mathcal{D}} \setminus \{D_2, D_4\})$ it is an identity on S^0 since, for each n, there is just one conjugacy class of D_{2n} subgroups in $O(2)$ (and if $g \in \mathrm{SO}(3)$ and $g \notin O(2)$ then $g^{-1} D_{2n} g \not\subset O(2)$).

Part (2) follows the same pattern, however the domain and codomain of the map f_{2n} are already local in $L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2)\text{–Sp})$, so $f \cong i^*(e_{\mathcal{D}}) f$. Since the idempotent used is $e_{D_{2n}}$ the only nontrivial geometric fixed points will be for the subgroup $H = D_{2n}$. The result follows from the fact that $N_{O(2)} D_{2n} = N_{\mathrm{SO}(3)} D_{2n}$, which implies that the map on geometric fixed points for D_{2n} is the identity on D_{4n}/D_{2n+}, and that finishes the proof. □

To give an algebraic model for the dihedral part of rational SO(3)–spectra we firstly use the restriction–coinduction adjunction in the next theorem to move to a certain part of rational $O(2)$–spectra. Then we show that this part of rational $O(2)$–spectra is a localisation of the dihedral part of rational $O(2)$–spectra from [4]. As a result, the method presented in [4] of obtaining an algebraic model for this part applies in our case almost verbatim.

Theorem 4.9 *Let $i\colon O(2) \to \mathrm{SO}(3)$ be an inclusion. Then the adjunction*

$$i^*\colon L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–Sp}) \rightleftarrows L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2)\text{–Sp}) : F_{O(2)}(\mathrm{SO}(3)_+, -)$$

is a Quillen equivalence. (Note that the idempotent on the right-hand side corresponds to the set of all dihedral subgroups of order greater than 4 together with $O(2)$.)

Proof This is a Quillen adjunction by Corollary 2.10 and moreover i^* is a right Quillen functor by Proposition 2.8.

We will use [14, Corollary 1.3.16(c)]. To show that this adjunction is a Quillen equivalence first notice that $F_{O(2)}(\mathrm{SO}(3)_+, -)$ preserves and reflects weak equivalences between fibrant objects. For any fibrant $X \in L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2)\text{–Sp})$ and $H \in \widetilde{\mathcal{D}} \setminus \{D_2, D_4\}$ we have natural isomorphisms

$$[\mathrm{SO}(3)/H_+, F_{O(2)}(\mathrm{SO}(3)_+, X)] \cong [i^* \mathrm{SO}(3)/H_+, X] \cong [O(2)/H_+, X],$$

where the second one follows from Proposition 4.8. Since weak equivalences between fibrant objects are detected by H–homotopy groups, $F_{O(2)}(\mathrm{SO}(3)_+, -)$ preserves and reflects weak equivalences between fibrant objects.

We need to show that the derived unit

$$Y \to F_{O(2)}(\mathrm{SO}(3)_+, \hat{f} i^*(Y))$$

is a weak equivalence on cofibrant objects in $L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp})$. It is enough to check that the induced map

$$[X,Y]^{L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3)-\mathrm{Sp})} \cong [X, e_{\mathcal{D}}Y]^{\mathrm{SO}(3)} \to [X, F_{O(2)}(\mathrm{SO}(3)_+, \hat{f}i^*(e_{\mathcal{D}}Y))]^{\mathrm{SO}(3)}$$

is an isomorphism for every generator X of $L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp})$ (see Lemma 4.7 for the set of generators). This map fits into the commuting diagram below:

$$\begin{array}{ccc} [X, e_{\mathcal{D}}Y]^{\mathrm{SO}(3)} & & \\ \downarrow & \overset{i^*}{\searrow} & \\ [X, F_{O(2)}(\mathrm{SO}(3)_+, \hat{f}i^*(e_{\mathcal{D}}Y))]^{\mathrm{SO}(3)} & \xrightarrow{\cong} & [i^*X, \hat{f}i^*(e_{\mathcal{D}}Y)]^{O(2)} \end{array}$$

Since the horizontal map is an isomorphism it is enough to show that i^* is an isomorphism on hom sets, where the domain is a generator for $L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp})$. We do this by using the second Quillen adjunction between these two categories, namely $(\mathrm{SO}(3)_+ \wedge_{O(2)} -, i^*)$.

Let η denote the categorical unit of the adjunction $(\mathrm{SO}(3)_+ \wedge_{O(2)} -, i^*)$. The map η on cofibrant generators is of the form

$$\eta_{e_H O(2)/H_+} \colon e_H O(2)/H_+ \to e_H i^*(\mathrm{SO}(3)/H_+),$$

induced by an inclusion $O(2) \to \mathrm{SO}(3)$. By Proposition 4.8 this is a weak equivalence in $L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2){-}\mathrm{Sp})$ for all H in $\mathcal{D}$ and thus $-\circ\eta$ induces an isomorphism in the homotopy category. We have the commuting diagram

$$\begin{array}{ccc} [e_H\,\mathrm{SO}(3)/H_+, e_{\mathcal{D}}Y]^{\mathrm{SO}(3)} & & \\ \cong\downarrow & \overset{i^*}{\searrow} & \\ [e_H O(2)/H_+, i^*(e_{\mathcal{D}}Y)]^{O(2)} & \xleftarrow[-\circ\eta]{} & [i^*(e_H\,\mathrm{SO}(3)/H_+), i^*(e_{\mathcal{D}}Y)]^{O(2)} \end{array}$$

where H above denotes a finite dihedral subgroup or $O(2)$ (when H is $O(2)$ we understand e_H as $e_{\mathcal{D}}$).

It follows that i^* is an isomorphism on hom sets and thus the derived unit of the adjunction where i^* is the left adjoint is a weak equivalence in $L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3){-}\mathrm{Sp})$, which finishes the proof. □

To obtain the algebraic model for rational SO(3)–spectra it is enough to get one for $L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2){-}\mathrm{Sp})$. We use the comparison method presented in [4] for the dihedral part of rational $O(2)$–spectra in this case.

Theorem 4.10 *There is a zig-zag of Quillen equivalences from* $L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2)\text{–Sp})$ *to* $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D}))$.

Proof Notice that $L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2)\text{–Sp})$ is a localisation of the dihedral part of rational $O(2)$–spectra $L_{e_{\widetilde{\mathcal{D}}}S_{\mathbb{Q}}}(O(2)\text{–Sp})$ at an idempotent $i^*(e_{\mathcal{D}})$, since $i^*(e_{\mathcal{D}})e_{\widetilde{\mathcal{D}}} = i^*(e_{\mathcal{D}})$. The set

$$\widetilde{\mathcal{G}} := \{O(2)/O(2)_+\} \cup \{e_{D_{2n}} O(2)/D_{2n+} \mid n > 2\}$$

is a set of homotopically compact, cofibrant generators for $L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2)\text{–Sp})$ by the same argument as in [4, Lemma 5.14].

Thus it is enough to use the proof of [4, Theorem 5.18] based on the tilting theorem of Schwede and Shipley [21, Theorem 5.1.1] restricted to the set of generators $\widetilde{\mathcal{G}}$ for $L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2)\text{–Sp})$ on one hand and the set of generators $\mathcal{G}_a$ (see Lemma 4.6) on the algebraic side. This shows that $L_{i^*(e_{\mathcal{D}})S_{\mathbb{Q}}}(O(2)\text{–Sp})$ is Quillen equivalent to the category $\mathsf{Ch}(\mathcal{A}(\mathrm{SO}(3),\mathcal{D}))$. □

Theorem 4.9 and Theorem 4.10 give the algebraic model for the dihedral part of rational SO(3)–spectra.

Theorem 4.11 *There is a zig-zag of Quillen equivalences between* $L_{e_{\mathcal{D}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–Sp})$ *and* $\mathcal{A}(\mathrm{SO}(3),\mathcal{D})$.

5 The exceptional part

The last part of rational SO(3)–spectra, $L_{e_{\mathcal{E}}S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–Sp})$, captures the behaviour of conjugacy classes of five subgroups: $\mathrm{SO}(3)$, Σ_4, A_4, A_5 and D_4; see Section 2.1.

Definition 5.1 [16, Definition 2.1] Recall that a subgroup H of G is *exceptional* if three conditions are satisfied:

- there is an idempotent $e_{(H)} \in \mathsf{A}(G)_{\mathbb{Q}}$ corresponding to the conjugacy class of H,
- the Weyl group $N_G H/H$ of H is finite, and
- H does not contain any subgroup K such that H/K is a (nontrivial) torus.

All subgroups in this part satisfy the definition above, hence the name *exceptional part*.

Recall that the stable model structure on G–spectra is a monoidal model structure satisfying the monoid axiom. Thus any left Bousfield localisation at a cofibrant object E

of a category of G–spectra is again a monoidal model category (by a straightforward check of the pushout-product axiom and the definition of E–weak equivalence). It also satisfies the monoid axiom, since $E \wedge -$ commutes with transfinite compositions and pushouts. By [3, Theorem 4.4] we have the following result.

Proposition 5.2 *There is a strong symmetric monoidal Quillen equivalence*

$$\triangle\colon L_{e_{\mathcal{E}} S_{\mathbb{Q}}} \mathrm{SO}(3)\text{–}\mathrm{Sp}_{\mathbb{Q}} \rightleftarrows \prod_{(H), H \in \mathcal{E}} L_{e_{(H)_{\mathrm{SO}(3)}} S_{\mathbb{Q}}} (\mathrm{SO}(3)\text{–}\mathrm{Sp}) \ :\Pi.$$

First we recall some details on what will be the building block of the algebraic model for the exceptional part, ie the category $\mathsf{Ch}(\mathbb{Q}[W_G H])$ of chain complexes of $\mathbb{Q}[W_G H]$–modules, and then we summarise the monoidal comparison from [16].

5.1 The category $\mathsf{Ch}(\mathbb{Q}[W])$

Suppose W is a finite group. The category of chain complexes of left $\mathbb{Q}[W]$–modules can be equipped with the projective model structure, where weak equivalences are homology isomorphisms and fibrations are levelwise surjections. This model structure is cofibrantly generated by [14, Section 2.3].

Note that $\mathbb{Q}[W]$ is not generally a commutative ring, however it is a Hopf algebra with cocommutative coproduct given by $\Delta\colon \mathbb{Q}[W] \to \mathbb{Q}[W] \otimes \mathbb{Q}[W]$, $g \mapsto g \otimes g$. This allows us to define an associative and commutative tensor product on $\mathsf{Ch}(\mathbb{Q}[W])$, namely tensor over $\mathbb{Q}$, where the W–action on the $X \otimes_{\mathbb{Q}} Y$ is diagonal. The unit is a chain complex with $\mathbb{Q}$ at the level 0 with trivial W–action and zeros everywhere else and it is cofibrant in the projective model structure. The monoidal product defined this way is closed, where the internal hom is given by a formula for an internal hom in $\mathbb{Q}$–modules with W–action given by conjugation.

By [2, Proposition 4.3] the category $\mathsf{Ch}(\mathbb{Q}[W])$ is a monoidal model category satisfying the monoid axiom.

5.2 Monoidal comparison

The following result is the main theorem of [16].

Theorem 5.3 *Suppose G is any compact Lie group. Then there is a zig-zag of symmetric monoidal Quillen equivalences from $L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–}\mathrm{Sp})$ of rational G–spectra over an exceptional subgroup H to $\mathsf{Ch}(\mathbb{Q}[W_G H])$ equipped with the projective model structure.*

We apply the result above for $G = \mathrm{SO}(3)$ to get the algebraic model for the exceptional part of rational $\mathrm{SO}(3)$–spectra.

Theorem 5.4 *There is a zig-zag of symmetric monoidal Quillen equivalences from* $L_{e_{\mathcal{E}} S_{\mathbb{Q}}}(\mathrm{SO}(3)\text{–Sp})$ *to* $\prod_{(H), H \in \mathcal{E}} \mathsf{Ch}(\mathbb{Q}[W_{\mathrm{SO}(3)} H])$

Proof This follows from Proposition 5.2 and Theorem 5.3. □

Below we present a short sketch of steps in the monoidal comparison for rational G–spectra over an exceptional subgroup to outline general ideas. We refer the reader to [16] for all the details.

Fix an exceptional subgroup H in G. First, using the restriction–coinduction adjunction, we move from the category $L_{e_{(H)_G} S_{\mathbb{Q}}}(G\text{–Sp})$ to the category $L_{e_{(H)_N} S_{\mathbb{Q}}}(N\text{–Sp})$, where N denotes the normaliser $N_G H$. The second step is to use the fixed point–inflation adjunction between $L_{e_{(H)_N} S_{\mathbb{Q}}}(N\text{–Sp})$ and $L_{e_1 S_{\mathbb{Q}}}(W\text{–Sp})$, where W denotes the Weyl group N/H. Recall that W is finite, as H is an exceptional subgroup of G. Next we use the restriction of universe to pass from $L_{e_1 S_{\mathbb{Q}}}(W\text{–Sp})$ to the category $\mathrm{Sp}[W]$ of rational orthogonal spectra with W–action. We then pass to symmetric spectra with W–action using the forgetful functor from orthogonal spectra and then to $H\mathbb{Q}$–modules with W–action in symmetric spectra. From here we use [22, Theorem 1.1] to get to $\mathsf{Ch}(\mathbb{Q})[W]$, the category of rational chain complexes with W–action, which is equivalent as a monoidal model category to $\mathsf{Ch}(\mathbb{Q}[W])$, the category of chain complexes of $\mathbb{Q}[W]$–modules. That gives an algebraic model which is compatible with the monoidal product, ie this zig-zag of Quillen equivalences induces a strong monoidal equivalence on the level of homotopy categories.

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